

# Exact Controllability for a Refined Stochastic Wave Equation

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## Abstract

A widely used stochastic wave equation is the classical wave equation perturbed by a term of Itô's integral. We show that this equation is not exactly controllable even if the controls are effective everywhere in both the drift and the diffusion terms and also on the boundary. In some sense this means that some key feature has been ignored in this model. Then, based on a stochastic Newton's law, we propose a refined stochastic wave equation. By means of a new global Carleman estimate, we establish the exact controllability of our stochastic wave equation with three controls. Moreover, we give a result about the lack of exact controllability, which shows that the action of three controls is necessary. Our analysis indicates that, at least from the point of view of control theory, the new stochastic wave equation introduced in this paper is a more reasonable model than that in the existing literatures.

**Key Words.** Exact controllability, stochastic wave equation, stochastic Newton's law, global Carleman estimate, observability estimate.

**AMS subject classifications.** Primary, 93B05; Secondary, 60H15, 93B07, 35B45.

## 1 Introduction

Let  $T > 0$ ,  $G \subset \mathbb{R}^n$  ( $n \in \mathbb{N}$ ) be a given bounded domain with a  $C^2$  boundary  $\Gamma$ . Let  $\Gamma_0$  be a suitably chosen nonempty subset (to be given later) of  $\Gamma$ , and  $G_0 \subset G$  be a nonempty open subset. Write  $Q = (0, T) \times G$ ,  $\Sigma = (0, T) \times \Gamma$ ,  $\Sigma_0 = (0, T) \times \Gamma_0$  and  $Q_0 = (0, T) \times G_0$ . Let  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  be a complete filtered probability space with  $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ , the natural filtration generated by a one-dimensional standard Brownian motion  $\{W(t)\}_{t \geq 0}$ . More notations and assumptions used below will be given in Section 2.

Consider the following controlled stochastic wave equation:

$$\begin{cases} dy_t - \sum_{j,k=1}^n (\alpha^{jk} y_{x_j})_{x_k} dt = (a_1 \cdot \nabla y + a_2 y + g_1) dt + (a_3 y + g_2) dW(t) & \text{in } Q, \\ y = \chi_{\Sigma_0} h & \text{on } \Sigma, \\ y(0) = y_0, \quad y_t(0) = y_1 & \text{in } G, \end{cases} \quad (1.1)$$

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where the initial datum  $(y_0, y_1) \in L^2(G) \times H^{-1}(G)$ ,  $y$  is the state, and  $g_1, g_2 \in L_{\mathbb{F}}^{\infty}(0, T; H^{-1}(G))$  and  $h \in L_{\mathbb{F}}^2(0, T; L^2(\Gamma_0))$  are three controls.

The equation (1.1) is introduced to describe the vibration of strings and membranes perturbed by random forces, as well as the propagation of waves in random environment (e.g. [6, Chapter 2]). Let us recall below the derivation of (1.1) in one-dimensional spatial domain.

Consider the motion of a strand of DNA. Compared with its length, the diameter of a DNA molecule is so small that it can be viewed as a long elastic string. Usually, a DNA molecule floats in fluid. It is always hit by fluid molecules, just as a particle of pollen floating in fluid.

Without loss of generality, we assume the mass of the string per unit length is 1. Denote by  $y(t, x)$  the displacement of the strand (in  $\mathbb{R}^3$ ) at time  $t \in [0, +\infty)$  and position  $x \in [0, L]$  for some  $L > 0$ . There are mainly four kinds of forces acting on the string: an elastic force  $F_1(t, x)$ , a friction force  $F_2(t, x)$  due to viscosity of the fluid, an impact force  $F_3(t, x)$  from the flow of the fluid, and a random impulse  $F_4(t, x)$  from impacts of the fluid's molecules. By Newton's second law, we have that

$$\frac{d^2 y(t, x)}{dt^2} = F_1(t, x) + F_2(t, x) + F_3(t, x) + F_4(t, x). \quad (1.2)$$

Similar to the derivation of the deterministic wave equation, the elastic force  $F_1(t, x) = y_{xx}(t, x)$ . The friction depends on the nature of the fluid. For a fixed  $x$ , by the classical theory of Statistical Mechanics (e.g. [26, Chapter VI]), the random impulse  $F_4(t, x)$  at  $(t, x)$  can be approximated by a Gaussian white noise with a given spatial correlation matrix  $k(\cdot, \cdot, y)$ , depending on the fluid. More precisely, for  $x_1, x_2 \in [0, L]$  and  $0 \leq s \leq t < +\infty$ ,

$$\mathbb{E}(F_4(t, x_1)F_4(s, x_2)^{\top}) = k(x_1, x_2, y(t, x_1), y(t, x_2))\delta(t - s).$$

Here  $\delta(\cdot)$  is the usual Dirac delta function. Then, the equation (1.2) can be rewritten as the following stochastic wave equation:

$$dy_t(t, x) = y_{xx}(t, x)dt + F_2(t, x)dt + F_3(t, x)dt + \hat{k}(x, y(t, x))dW(t). \quad (1.3)$$

Here  $\hat{k}(x, y(t, x)) = k(x, x, y(t, x), y(t, x))$ . When  $y$  is small, we may assume that  $k$  is linear in  $y$ , that is,  $\hat{k}(x, y(t, x)) = k_1(t, x)y(t, x)$  for a suitable  $k_1(\cdot, \cdot)$ .

Many biological events are related to the motion of the DNA molecules. Hence, there is a strong motivation to control its motion. Clearly, one can introduce two kinds of controls. One is a force applied on the boundary, to control the displacement of the strand at the boundary point, the other is the force acted in the internal of the strand, which can be put in both the drift and the diffusion terms. These lead to a model like the control system (1.1).

Motivated by the above mentioned practical problem, we introduce the following notion of exact controllability for (1.1).

**Definition 1.1** *The system (1.1) is called exactly controllable at the time  $T$  if for any  $(y_0, y_1) \in L^2(G) \times H^{-1}(G)$  and  $(y'_0, y'_1) \in L^2_{\mathcal{F}_T}(\Omega; L^2(G)) \times L^2_{\mathcal{F}_T}(\Omega; H^{-1}(G))$ , one can find a triple of controls  $(g_1, g_2, h) \in L_{\mathbb{F}}^2(0, T; H^{-1}(G)) \times L_{\mathbb{F}}^2(0, T; H^{-1}(G)) \times L_{\mathbb{F}}^2(0, T; L^2(\Gamma_0))$  such that the corresponding solution  $y$  to the system (1.1) satisfies that  $(y(T), y_t(T)) = (y'_0, y'_1)$ .*

Since three controls are introduced in (1.1), one may guess that the desired exact controllability should be trivially correct. To “justify” this, let us recall that, in [27] the null controllability of the

following stochastic heat equation

$$\begin{cases} dp - \sum_{j,k=1}^n (a^{jk} p_{x_j})_{x_k} dt = (a_1 \cdot \nabla p + a_2 p + \chi_{G_0}(x) u_1) dt + (a_3 p + u_2) dW(t) & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(0) = p_0 & \text{in } G \end{cases} \quad (1.4)$$

was achieved by means of two controls  $u_1 \in L^2_{\mathbb{F}}(0, T; L^2(G_0))$  and  $u_2 \in L^2_{\mathbb{F}}(0, T; L^2(G))$ , where  $p_0 \in L^2(G)$  is the initial state. Further, one can easily prove the exact trajectory controllability for the equation (1.4) with the same type of controls (Note that, exactly for the same reason as that in the deterministic setting, one cannot expect the usual exact controllability for the stochastic heat equation). On the other hand, in [17, 18] the exact controllability of stochastic Schrödinger and transport equations were also obtained by a boundary control acted on the drift term (like  $h$  in (1.1)) and a distributed control imposed on the diffusion term.

Surprisingly, as we shall show in Theorem 2.1 (in Section 2) that, the exact controllability of (1.1) fails for any  $T > 0$  and  $\Gamma_0 \subset \Gamma$ , even if the controls  $g_1$  and  $g_2$  are acted everywhere on the domain  $G$  and  $\Gamma_0 = \Gamma$ . Note that, such kinds of controls are the strongest control actions that one can introduce into (1.1). Obviously, this differs significantly from the well-known controllability property of deterministic wave equations (See [12, 32, 33] and the rich references therein). Since (1.1) is a generalization of the classical wave equation to the stochastic setting, from the viewpoint of control theory, we believe that some key feature has been ignored in the derivation of the equation (1.1).

Motivated by the above-mentioned negative controllability result for (1.1), in what follows, we shall propose a refined model to describe the DNA molecule. For this purpose, we partially employ a dynamical theory of Brownian motions, developed in [23], to describe the motion of a particle perturbed by random forces. In our opinion, the essence of the theory in [23] is a stochastic Newton's law, at least in certain suitable sense.

According to [23, Chapter 11], we may suppose that

$$y(t, x) = \int_0^t v(s, x) ds + \int_0^t F(s, x, y(s)) dW(s). \quad (1.5)$$

Here  $v(\cdot, \cdot)$  is the expected velocity,  $F(\cdot, \cdot, \cdot)$  is the random perturbation from the fluid molecule. When  $y$  is small, one can assume that  $F(\cdot, \cdot, \cdot)$  is linear in the third argument, i.e.,

$$F(s, x, y(t, x)) = b_1(t, x) y(t, x) \quad (1.6)$$

for a suitable  $b_1(\cdot, \cdot)$ .

The acceleration at position  $x$  along the string at time  $t$  is  $v_t(t, x)$ . By Newton's law, it follows that

$$v_t(t, x) = F_1(t, x) + F_2(t, x) + F_3(t, x) + F_4(t, x). \quad (1.7)$$

Similar to the derivation of (1.3), we have

$$dv(t, x) = y_{xx}(t, x) dt + F_2(t, x) dt + F_3(t, x) dt + k_1(t, x) y(t, x) dW(t). \quad (1.8)$$

Combining (1.5), (1.6) and (1.8), we obtain that:

$$\begin{cases} dy = v dt + b_1(t, x) y dW(t) & \text{in } (0, T) \times (0, L), \\ dv = y_{xx} dt + F_2 dt + F_3 dt + k_1(t, x) y(t, x) dW(t) & \text{in } (0, T) \times (0, L). \end{cases} \quad (1.9)$$

Stimulated by (1.9), we consider the following controlled stochastic wave-like equation:

$$\left\{ \begin{array}{ll} dy = \hat{y}dt + (a_4y + f)dW(t) & \text{in } Q, \\ d\hat{y} - \sum_{j,k=1}^n (a^{jk}y_{x_j})_{x_k} dt = (a_1 \cdot \nabla y + a_2y + a_5g)dt + (a_3y + g)dW(t) & \text{in } Q, \\ y = \chi_{\Sigma_0}h & \text{on } \Sigma, \\ y(0) = y_0, \quad \hat{y}(0) = \hat{y}_0 & \text{in } G. \end{array} \right. \quad (1.10)$$

Here  $(y_0, \hat{y}_0) \in L^2(G) \times H^{-1}(G)$ ,  $(y, \hat{y})$  are the state, and  $f \in L^2_{\mathbb{F}}(0, T; L^2(G))$ ,  $g \in L^2_{\mathbb{F}}(0, T; H^{-1}(G))$  and  $h \in L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0))$  are three controls.

**Remark 1.1** *We put controls  $f$  and  $g$  in the diffusion terms to get the exact controllability. The first equation in (1.10) can be regarded as a family of stochastic differential equations with a parameter  $x \in G$ . One can put a control directly in the diffusion term. On the other hand, the second equation in (1.10) is a stochastic partial differential equation. Usually, if we put a control in the diffusion term, it may affect the drift term in one way or another. Here we assume that the effect is linear and in the form of “ $a_5gdt$ ” as that in the second equation of (1.10). One may consider more general cases, say to add a term like “ $a_6fdt$ ” into the first equation of (1.10). However, a detailed analysis is beyond the scope of this paper and will be investigated in future.*

**Definition 1.2** *The system (1.10) is called exactly controllable at time  $T$  if for any  $(y_0, \hat{y}_0) \in L^2(G) \times H^{-1}(G)$  and  $(y_1, \hat{y}_1) \in L^2_{\mathcal{F}_T}(\Omega; L^2(G)) \times L^2_{\mathcal{F}_T}(\Omega; H^{-1}(G))$ , one can find a triple of controls  $(f, g, h) \in L^2_{\mathbb{F}}(0, T; L^2(G)) \times L^2_{\mathbb{F}}(0, T; H^{-1}(G)) \times L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0))$  such that the corresponding solution  $(y, \hat{y})$  to (1.10) satisfies that  $(y(T), \hat{y}(T)) = (y_1, \hat{y}_1)$ .*

In this paper, we shall show that (1.10) is exactly controllable (See Theorem 2.2). Hence, from the viewpoint of controllability theory, the system (1.10) is a more reasonable model than (1.1). Noting that, we also introduce three controls into (1.10), which seems too many. However, we prove that none of these three controls can be ignored, and moreover the two internal controls  $f$  and  $g$  have to be effective everywhere in the domain  $G$  (See Theorem 2.3).

There exist many works on controllability of deterministic partial differential equations (PDEs for short). Contributions by D. L. Russell ([25]) and by J.-L. Lions ([12]) are classical in this field. Some recent progress can be found in [5, 32, 33]. In particular, one may find many works addressing the exact controllability problems for deterministic wave equations (See [2, 4, 12, 25, 29, 33] and the rich reference therein). However, people know very little about the controllability problems for stochastic PDEs. In this respect, we refer to [3, 8, 11, 15, 18, 27] for some results on the controllability of stochastic parabolic, complex Ginzburg-Landau, Kuramoto-Sivashinsky, Schrödinger and transport equations. To the best of our knowledge, there exists no nontrivial published result concerning the exact controllability of stochastic wave equation.

Compared with the deterministic situation, there are many new difficulties and phenomena appeared in the study of controllability problems for stochastic control systems, even for the systems governed by stochastic (ordinary) differential equations (e.g. [21, 24]). For example, it was shown in [21] that there exist no Kalman-type rank condition for the null controllability/approximate controllability for controlled stochastic differential equations. People will meet more obstacles and substantially extra difficulties in the study of controllability problems for stochastic PDEs. Some of them are as follows:

- Unlike the deterministic PDEs, the solution of a stochastic PDE is usually nondifferentiable with respect to the variable with noise (say, the time variable considered in this paper);
- The usual compactness embedding result does not remain true for the solution spaces related to stochastic PDEs;
- The diffusion term leads some difficulties in establishing observability estimate;
- The most essential difficulty is that, compared to their deterministic counterparts, stochastic PDEs themselves are much less-understood.

Generally speaking, one can find the following four main methods for solving the exact controllability problem of deterministic wave equations:

- The first one is based on the Ingham type inequality ([2]). This method works well for wave equations involved in some special domains, i.e., intervals and rectangles. However, it seems that it is very hard to be applied to equations in general domains.
- The second one is the classical Rellich-type multiplier approach ([12]). It is used to treat wave equations with time independent coefficients. However, it seems that it does not work for our problem since the coefficients of lower order terms are time dependent.
- The third one is the microlocal analysis approach ([4]). It is useful to solve controllability problems for several kinds of PDEs, such as wave equations, Schrödinger equations and plate equations. Further, it can give sharp sufficient conditions for the exact controllability of wave equations. However, there may be lots of obstacles needed to be surmounted if one wants to utilize this approach to study stochastic control problems (see remarks in Section 9 for more details).
- The last one is the global Carleman estimate ([29]). This approach has the advantage of being more flexible and allowing to address variable coefficients. Further, it is robust with respect to the lower order terms and can be used to get explicit bounds on the observability constant/control cost in terms of the potentials entering in it.

In recent years, Carleman estimate was also employed to study the controllability and observability problems for some stochastic PDEs (see [14, 16, 17, 27, 28, 31] and the references therein). *Nevertheless, as that in the deterministic setting, generally speaking, Carleman estimate works well only for single equation rather than system.*

In this paper, we borrow some idea from the proof of the observability estimate for stochastic wave equation (see [16, 31] for example). However, since (1.10) is a system (of stochastic equations) rather than a single stochastic wave equation, we cannot simply mimic the method in [16, 31] to solve our problem. To handle these troubles, we have to derive a completely new pointwise identity (see Lemma 6.1 in Section 6).

The rest of this paper is organized as follows. In Section 2, we present the main results. Section 3 is devoted to introducing the adjoint system of systems (1.1) and (1.10), and proving a hidden regularity of solutions to this system. In Section 4, we establish the well-posedness of the systems (1.1) and (1.10). In Section 5, we transform the exact controllability problem of (1.10) into the exact controllability problem of a backward stochastic wave-like equation. Section 6 is addressed to a fundamental identity for stochastic hyperbolic-like operators. In Section 7, we prove an observability estimate for a stochastic-wave like equation. Section 8 is devoted to proofs of the main results. At last, some further comments and open problems are given in Section 9.

## 2 Main results

We begin with some notations.

Denote by  $\mathbb{E}z$  the (mathematical) expectation of an integrable random variable  $z : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}$ . Let  $H$  be a Banach space. Denote by  $L_{\mathbb{F}}^2(0, T; H)$  the Banach space consisting of all  $H$ -valued and  $\mathbb{F}$ -adapted processes  $X(\cdot)$  such that  $\mathbb{E}(\|X(\cdot)\|_{L^2(0, T; H)}^2) < \infty$ ; by  $L_{\mathbb{F}}^\infty(0, T; H)$  the Banach space consisting of all  $H$ -valued and  $\mathbb{F}$ -adapted, essentially bounded processes; and by  $C_{\mathbb{F}}([0, T]; L^r(\Omega; H))$  the Banach space consisting of all  $H$ -valued and  $\mathbb{F}$ -adapted processes  $X(\cdot)$  such that  $X(\cdot) : [0, T] \rightarrow L_{\mathcal{F}_T}^r(\Omega; H)$  is continuous ( $r \in [1, \infty]$ ). Similarly, one can define  $C_{\mathbb{F}}^k([0, T]; L^r(\Omega; H))$  for any positive integer  $k$ . All of these spaces are endowed with their canonical norms.

In this paper, for simplicity, we use the notation  $y_{x_j} \triangleq \partial y(x) / \partial x_j$ , where  $x_j$  is the  $j$ -th coordinate of a generic point  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ . In a similar manner, we use notations  $z_{x_j}$ ,  $v_{x_j}$ , etc. for the partial derivatives of  $z$  and  $v$  with respect to  $x_j$ . Also, we denote by  $\nu(x) = (\nu^1(x), \dots, \nu^n(x))$  the unit outward normal vector of  $\Gamma$  at point  $x$ . In what follows, we use  $C$  to denote a generic positive constant depending on  $T$ ,  $G$  and  $\Gamma_0$  (unless otherwise stated), which may vary from line to line.

Let  $(a^{jk})_{1 \leq j, k \leq n} \in C^3(\overline{G}; \mathbb{R}^{n \times n})$  satisfying that  $a^{jk} = a^{kj}$  ( $j, k = 1, 2, \dots, n$ ) and for some constant  $s_0 > 0$ ,

$$\sum_{j, k=1}^n a^{jk} \xi^j \xi^k \geq s_0 |\xi|^2, \quad \forall (x, \xi) \triangleq (x, \xi^1, \dots, \xi^n) \in G \times \mathbb{R}^n. \quad (2.1)$$

Also we assume that

$$a_1 \in L_{\mathbb{F}}^\infty(0, T; W^{1, \infty}(G; \mathbb{R}^n)), \quad a_2, a_3, a_4 \in L_{\mathbb{F}}^\infty(0, T; L^\infty(G)), \quad a_5 \in L_{\mathbb{F}}^\infty(0, T; W_0^{1, \infty}(G)). \quad (2.2)$$

Let us first give the following negative controllability result for the system (1.1).

**Theorem 2.1** *The system (1.1) is not exactly controllable for any  $T > 0$  and  $\Gamma_0 \subset \Gamma$ .*

Next, we make the following additional assumptions on the coefficients  $(a^{jk})_{1 \leq j, k \leq n}$ :

**Condition 2.1** *There exists a positive function  $\varphi(\cdot) \in C^2(\overline{G})$  satisfying that:*

(1) *For some constant  $\mu_0 > 0$ , it holds that*

$$\sum_{j, k=1}^n \left\{ \sum_{j', k'=1}^n \left[ 2a^{jk'} (a^{j'k} \varphi_{x_{j'}})_{x_{k'}} - a_{x_{k'}}^{jk} a^{j'k'} \varphi_{x_{j'}} \right] \right\} \xi^j \xi^k \geq \mu_0 \sum_{j, k=1}^n a^{jk} \xi^j \xi^k, \quad (2.3)$$

$$\forall (x, \xi^1, \dots, \xi^n) \in \overline{G} \times \mathbb{R}^n.$$

(2) *There is no critical point of  $\varphi(\cdot)$  in  $\overline{G}$ , i.e.,*

$$\min_{x \in \overline{G}} |\nabla \varphi(x)| > 0. \quad (2.4)$$

The set  $\Gamma_0$  is as follows:

$$\Gamma_0 \triangleq \left\{ x \in \Gamma \mid \sum_{j, k=1}^n a^{jk} \varphi_{x_j}(x) \nu^k(x) > 0 \right\}. \quad (2.5)$$

It is easy to check that if  $\varphi(\cdot)$  satisfies Condition 2.1, then for any given constants  $\alpha \geq 1$  and  $\beta \in \mathbb{R}$ ,  $\tilde{\varphi} = \alpha\varphi + \beta$  still satisfies Condition 2.1 with  $\mu_0$  replaced by  $\alpha\mu_0$ . Therefore we may choose  $\varphi$ ,  $\mu_0$ ,  $c_0 > 0$ ,  $c_1 > 0$  and  $T$  such that the following condition holds:

**Condition 2.2**

- (1).  $\frac{1}{4} \sum_{j,k=1}^n a^{jk}(x) \varphi_{x_j}(x) \varphi_{x_k}(x) \geq R_1^2 \triangleq \max_{x \in \overline{G}} \varphi(x) \geq R_0^2 \triangleq \min_{x \in \overline{G}} \varphi(x), \quad \forall x \in \overline{G}. \quad (2.6)$
- (2).  $T > T_0 \triangleq 2R_1.$
- (3).  $\left(\frac{2R_1}{T}\right)^2 < c_1 < \frac{2R_1}{T}$  and  $c_1 < \min \left\{ 1, \frac{1}{16|a_5|_{L^\infty(0,T;L^\infty(G))}^4} \right\}.$
- (4).  $\mu_0 - 4c_1 - c_0 > \sqrt{R_1}.$

**Remark 2.1** As we have explained, since  $\sum_{j,k=1}^n a^{jk} \varphi_{x_j} \varphi_{x_k} > 0$ , and one can choose  $\mu_0$  in Condition 2.1 large enough, Condition 2.2 could be satisfied obviously. We put it here merely to emphasize the relationship among  $0 < c_0 < c_1 < 1$ ,  $\mu_0$  and  $T$ . In other words, once Condition 2.1 is fulfilled, Condition 2.2 can always be satisfied.

To be more clear, we give an example for the choice of  $\varphi$  when  $(a^{jk})_{1 \leq j,k \leq n}$  is the identity matrix. Let  $x_0 \in \mathbb{R}^n \setminus \overline{G}$  such that  $|x - x_0| \geq 1$  for all  $x \in G$  and  $\alpha_0 = \max_{x \in \overline{G}} |x - x_0|^2$ . Then for all  $\alpha \geq \max\{\alpha_0, 1\}$ ,

$$\alpha \geq \sqrt{\alpha} \max_{x \in \overline{G}} |x - x_0|.$$

Let  $\varphi(x) = \alpha|x - x_0|^2$ . Then the left hand side of (2.3) is reduced to

$$\sum_{j=1}^n \varphi_{x_j x_j} \xi_j^2 = 2\alpha|\xi|^2, \quad \forall (x, \xi^1, \dots, \xi^n) \in \overline{G} \times \mathbb{R}^n. \quad (2.7)$$

And, (2.3) holds with  $\mu_0 = 2\alpha$ . Further, it is clear that (2.4) is true and

$$\frac{1}{4} \sum_{j,k=1}^n a^{jk}(x) \varphi_{x_j}(x) \varphi_{x_k}(x) = \frac{1}{4} \sum_{j=1}^n \varphi_{x_j}(x)^2 = \alpha^2 |x - x_0|^2 \geq \max_{x \in \overline{G}} \varphi(x).$$

Hence, (2.6) holds. Next, one can choose  $T$  large enough such that the second and third inequalities in Condition 2.2 hold and  $c_1 < \min \left\{ 1, \frac{1}{16|a_5|_{L^\infty(0,T;L^\infty(G))}^4} \right\}$ . Let  $\alpha \geq \{\alpha_0, 1, 4c_1 + c_0\}$ . Then  $\mu_0 - 4c_1 - c_0 > \sqrt{R_1}$ .

**Remark 2.2** To ensure that (3) in Condition 2.2 holds, the larger of  $|a_5|_{L^\infty(0,T;L^\infty(G))}$  is given, the smaller of  $c_1$  and the longer of the time  $T$  we should choose. This is reasonable since  $a_5$  stands for the effect of the control in the diffusion term to the drift term. One needs time to get rid of such effect.

We have the following exact controllability result for the system (1.10).

**Theorem 2.2** Let Conditions 2.1 and 2.2 hold. Then, the system (1.10) is exactly controllable at time  $T$ .

As mentioned before, we introduce three controls in the system (1.10). At a first glance, it seems unreasonable, especially for that the controls in the diffusion term of (1.10) are acted on the whole domain  $G$ . One may ask whether localized controls are enough or the boundary control can be dropped. However, the answer is negative. More precisely, we have the following negative result.

**Theorem 2.3** *The system (1.10) is not exactly controllable at any time  $T > 0$  and  $\Gamma_0 \subset \Gamma$  provided that one of the following three conditions is satisfied:*

- 1)  $a_4 \in C_{\mathbb{F}}([0, T]; L^\infty(\Omega))$ ,  $G \setminus \overline{G_0} \neq \emptyset$  and  $f$  is supported in  $G_0$ ;
- 2)  $a_3 \in C_{\mathbb{F}}([0, T]; L^\infty(\Omega))$ ,  $G \setminus \overline{G_0} \neq \emptyset$  and  $g$  is supported in  $G_0$ ;
- 3)  $h = 0$ .

**Remark 2.3** *Although it is necessary to put controls  $f$  and  $g$  on the whole domain, one may suspect that Theorem 2.2 is trivial. For instance, one may give a possible “proof” of Theorem 2.2 as follows:*

Choosing  $f = -a_4y$  and  $g = -a_3y$ , then the system (1.10) becomes

$$\begin{cases} dy = \hat{y}dt & \text{in } Q, \\ d\hat{y} - \sum_{j,k=1}^n (a^{jk}y_{x_j})_{x_k} dt = (a_1 \cdot \nabla y + a_2y - a_5a_3y)dt & \text{in } Q, \\ y = \chi_{\Sigma_0}h & \text{on } \Sigma, \\ y(0) = y_0, \quad \hat{y}(0) = \hat{y}_0 & \text{in } G. \end{cases} \quad (2.8)$$

*This is a wave-like equation with random coefficients. If one regards the sample point  $\omega$  as a parameter, then for every given  $\omega \in \Omega$ , there is a control  $u(\cdot, \cdot, \omega)$  such that the solution to (2.8) fulfills  $(y(T, x, \omega), \hat{y}(T, x, \omega)) = (y_1(x, \omega), \hat{y}_1(x, \omega))$ . It is easy to see that the control constructed in this way belongs to  $L^2_{\mathcal{F}_T}(\Omega; L^2(0, T; L^2(\Gamma_0)))$ . However, we do not know whether it is adapted to the filtration  $\mathbb{F}$  or not. If it is not, then it means to determine the value of the control at present, one needs to use information in future, which is inadmissible in the stochastic context.*

### 3 Backward stochastic wave equations

In order to define solutions to both (1.1) and (1.10) in a suitable sense, we need to introduce the following “reference” equation:

$$\begin{cases} dz = \hat{z}dt + (b_5z + Z)dW(t) & \text{in } Q_\tau, \\ d\hat{z} - \sum_{j,k=1}^n (a^{jk}z_{x_j})_{x_k} dt = (b_1 \cdot \nabla z + b_2z + b_3Z + b_4\hat{Z})dt + \hat{Z}dW(t) & \text{in } Q_\tau, \\ z = 0 & \text{on } \Sigma_\tau, \\ z(\tau) = z^\tau, \quad \hat{z}(\tau) = \hat{z}^\tau & \text{in } G, \end{cases} \quad (3.1)$$

where  $\tau \in (0, T]$ ,  $Q_\tau \triangleq (0, \tau) \times G$ ,  $\Sigma_\tau \triangleq (0, \tau) \times \Gamma$ ,  $(z^\tau, \hat{z}^\tau) \in L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G) \times L^2(G))$  and

$$b_1 \in L^\infty_{\mathbb{F}}(0, T; W^{1,\infty}(G; \mathbb{R}^n)), \quad b_i \in L^\infty_{\mathbb{F}}(0, T; L^\infty(G)), \quad i = 2, 3, 4, \quad b_5 \in L^\infty_{\mathbb{F}}(0, T; W_0^{1,\infty}(G)).$$

For the convenience of the reader, we first recall the definition of the solution to (3.1).

**Definition 3.1** *A quadruple of stochastic processes  $(z, Z, \hat{z}, \hat{Z}) \in C_{\mathbb{F}}([0, \tau]; H_0^1(G)) \times L^2_{\mathbb{F}}(0, \tau; H_0^1(G)) \times C_{\mathbb{F}}([0, \tau]; L^2(G)) \times L^2_{\mathbb{F}}(0, \tau; L^2(G))$  is called a weak solution of the system (3.1) if for every  $\psi \in C_0^\infty(G)$  and a.e.  $(t, \omega) \in [0, \tau] \times \Omega$ , it holds that*

$$z^\tau(x) - z(t, x) = \int_t^\tau \hat{z}(s, x)ds + \int_t^\tau [b_5z(s, x) + Z(s, x)]dW(s) \quad (3.2)$$



and

$$\begin{aligned}
& \int_G \hat{z}^\tau(x) \psi(x) dx - \int_G \hat{z}(t, x) \psi(x) dx + \int_t^\tau \int_G \sum_{j,k=1}^n a^{jk}(x) z_{x_j}(s, x) \psi_{x_k}(x) dx ds \\
&= \int_t^\tau \int_G [b_1(s, x) \cdot \nabla z(s, x) + b_2(s, x) z(s, x) + b_3(s, x) Z(s, x) + b_4(s, x) \widehat{Z}(s, x)] \psi(x) dx ds \\
& \quad + \int_t^\tau \int_G \widehat{Z}(s, x) \psi(x) dx dW(s).
\end{aligned} \tag{3.3}$$

Let us recall the following well-posedness result for (1.10) (e.g. [1, 22]).

**Lemma 3.1** *For any  $(z^\tau, \hat{z}^\tau) \in L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G)) \times L^2_{\mathcal{F}_\tau}(\Omega; L^2(G))$ , the system (3.1) admits a unique solution  $(z, Z, \hat{z}, \widehat{Z})$ . Moreover,*

$$\begin{aligned}
& |z|_{C_{\mathbb{F}}([0, \tau]; H_0^1(G))} + |Z|_{L^2_{\mathbb{F}}(0, \tau; H_0^1(G))} + |\hat{z}|_{C_{\mathbb{F}}([0, \tau]; L^2(G))} + |\widehat{Z}|_{L^2_{\mathbb{F}}(0, \tau; L^2(G))} \\
& \leq C e^{Cr_1} (|z^\tau|_{L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G))} + |\hat{z}^\tau|_{L^2_{\mathcal{F}_\tau}(\Omega; L^2(G))}),
\end{aligned} \tag{3.4}$$

where

$$r_1 \triangleq |b_1|_{L^\infty_{\mathbb{F}}(0, T; W^{1, \infty}(G; \mathbb{R}^n))} + \sum_{i=2}^4 |b_i|_{L^\infty_{\mathbb{F}}(0, T; L^\infty(G))} + |b_5|_{L^\infty_{\mathbb{F}}(0, T; W_0^{1, \infty}(G))}.$$

We have the following hidden regularity for solutions to (3.1).

**Proposition 3.1** *Let  $(z^\tau, \hat{z}^\tau) \in L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G)) \times L^2_{\mathcal{F}_\tau}(\Omega; L^2(G))$ . Then the solution  $(z, Z, \hat{z}, \widehat{Z})$  of (3.1) satisfies  $\frac{\partial z}{\partial \nu}|_\Gamma \in L^2_{\mathbb{F}}(0, \tau; L^2(\Gamma))$ . Furthermore,*

$$\left| \frac{\partial z}{\partial \nu} \right|_{L^2_{\mathbb{F}}(0, \tau; L^2(\Gamma))} \leq C e^{Cr_1} (|z^\tau|_{L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G))} + |\hat{z}^\tau|_{L^2_{\mathcal{F}_\tau}(\Omega; L^2(G))}), \tag{3.5}$$

where the constant  $C$  is independent of  $\tau$ .

*Proof:* For any  $h \triangleq (h^1, \dots, h^n) \in C^1(\mathbb{R}_t \times \mathbb{R}_x^n; \mathbb{R}^n)$ , by Itô's formula and the first equation of (3.1), we have

$$\begin{aligned}
d(\hat{z}h \cdot \nabla z) &= d\hat{z}h \cdot \nabla z + \hat{z}h_t \cdot \nabla z dt + \hat{z}h \cdot \nabla dz + d\hat{z}h \cdot \nabla z \\
&= d\hat{z}h \cdot \nabla z + \hat{z}h_t \cdot \nabla z dt + \hat{z}h \cdot \nabla (\hat{z}dt + ZdW(t)) + d\hat{z}h \cdot \nabla z \\
&= d\hat{z}h \cdot \nabla z + \hat{z}h_t \cdot \nabla z dt + \frac{1}{2} \left[ \operatorname{div}(\hat{z}^2 h) - (\operatorname{div} h) \hat{z}^2 \right] + \hat{z}h \cdot \nabla ZdW(t) + d\hat{z}h \cdot \nabla z.
\end{aligned}$$

Hence, similar to the proofs of [10, Lemma 3.2] and [31, Proposition 3.2], it follows from a direct computation that

$$\begin{aligned}
& - \sum_{k=1}^n \left[ 2(h \cdot \nabla z) \sum_{j=1}^n a^{jk} z_{x_j} + h^k \left( \hat{z}^2 - \sum_{i,j=1}^n a^{ij} z_{x_i} z_{x_j} \right) \right]_{x_k} dt \\
&= 2 \left[ -d(\hat{z}h \cdot \nabla z) + \left( d\hat{z} - \sum_{j,k=1}^n (a^{jk} z_{x_j})_{x_k} dt \right) h \cdot \nabla z + \hat{z}h_t \cdot \nabla z dt - \sum_{i,j,k=1}^n a^{ij} z_{x_i} z_{x_k} h_{x_j}^k dt \right] \\
& \quad - (\operatorname{div} h) \hat{z}^2 dt + \sum_{j,k=1}^n z_{x_j} z_{x_k} \operatorname{div}(a^{jk} h) dt + 2d\hat{z}h \cdot \nabla z + 2\hat{z}h \cdot \nabla (b_5 z + Z) dW(t).
\end{aligned} \tag{3.6}$$

Since  $\Gamma \in C^2$ , one can find a vector field  $\xi = (\xi^1, \dots, \xi^n) \in C^1(\mathbb{R}^n; \mathbb{R}^n)$  such that  $\xi = \nu$  on  $\Gamma$  (See [12, p. 29]). Setting  $h = \xi$  in (3.6), integrating it in  $Q$ , and taking expectation on  $\Omega$ , we get that

$$\begin{aligned}
& -\mathbb{E} \int_{\Sigma_\tau} \sum_{k=1}^n \left[ 2(h \cdot \nabla z) \sum_{j=1}^n a^{jk} z_{x_j} + h^k \left( \hat{z}^2 - \sum_{i,j=1}^n a^{ij} z_{x_i} z_{x_j} \right) \right] \nu^k d\Gamma dt \\
& = -2\mathbb{E} \int_G \hat{z}^T h \cdot \nabla z^T dx + 2\mathbb{E} \int_G \hat{z}(0) h \cdot \nabla z(0) dx \\
& \quad + 2 \int_{Q_\tau} \left[ (b_1 \cdot \nabla z + b_2 z + b_3 Z + b_4 \hat{Z}) h \cdot \nabla z + \hat{z} h_t \cdot \nabla z - \sum_{j,k,l=1}^n a^{jk} z_{x_j} z_{x_l} h_{x_k}^l \right. \\
& \quad \left. - (\operatorname{div} h) \hat{z}^2 + \sum_{j,k=1}^n z_{x_k} z_{x_j} \operatorname{div} (a^{jk} h) + 2\hat{Z} h \cdot \nabla (b_5 z + Z) \right] dx dt \triangleq \mathcal{I}.
\end{aligned} \tag{3.7}$$

Noting that  $z = 0$  on  $(0, \tau) \times \Gamma$ , we have

$$\begin{aligned}
& \mathbb{E} \int_{\Sigma_\tau} \sum_{k=1}^n \left[ 2(h \cdot \nabla z) \sum_{j=1}^n a^{jk} z_{x_j} + h^k \left( \hat{z}^2 - \sum_{i,j=1}^n a^{ij} z_{x_i} z_{x_j} \right) \right] \nu^k d\Gamma dt \\
& = \mathbb{E} \int_{\Sigma_\tau} \left[ 2 \left( h \cdot \nu \frac{\partial z}{\partial \nu} \right) \sum_{j,k=1}^n a^{jk} \nu^k \nu^j \frac{\partial z}{\partial \nu} - \sum_{i,j,k=1}^n a^{ij} \nu^k \nu^i h^k \nu^j \left| \frac{\partial z}{\partial \nu} \right|^2 \right] d\Gamma dt \\
& = \mathbb{E} \int_{\Sigma_\tau} \sum_{j,k=1}^n a^{jk} \nu^k \nu^j \left| \frac{\partial z}{\partial \nu} \right|^2 d\Gamma dt \geq s_0 \mathbb{E} \int_{\Sigma_\tau} \left| \frac{\partial z}{\partial \nu} \right|^2 d\Gamma dt.
\end{aligned} \tag{3.8}$$

It follows from Lemma 3.1 that

$$|\mathcal{I}| \leq C e^{Cr_1} (|z^\tau|_{L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G))} + |\hat{z}^\tau|_{L^2_{\mathcal{F}_\tau}(\Omega; L^2(G))}).$$

This, together with (3.7) and (3.8), implies that (3.5) holds.  $\square$

**Remark 3.1** Proposition 3.1 shows that, solutions of (3.1) enjoy a better regularity on the boundary than the one provided by the classical trace theorem of Sobolev spaces. Such kind of result is called a hidden regularity (of the solution). There are many studies in this topic for deterministic PDEs (e.g. [13]).

## 4 Well-posedness of the systems (1.1) and (1.10)

In this section, we establish the well-posedness of systems (1.1) and (1.10). Throughout this section,  $\Gamma_0$  is any fixed open subset of  $\Gamma$ , which is not necessarily given by (2.5).

Systems (1.1) and (1.10) are nonhomogeneous boundary value problems. Like the deterministic ones (e.g. [12, 13]), their solutions are understood in the sense of transposition solution.

**Definition 4.1** A stochastic process  $y \in C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G))) \cap C_{\mathbb{F}}^1([0, T]; L^2(\Omega; H^{-1}(G)))$  is a transposition solution to (1.1) if for any  $\tau \in (0, T]$  and  $(z^\tau, \hat{z}^\tau) \in L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G)) \times L^2_{\mathcal{F}_\tau}(\Omega; L^2(G))$ , we have that

$$\begin{aligned}
& \mathbb{E} \langle y_t(\tau), z^\tau \rangle_{H^{-1}(G), H_0^1(G)} - \langle \hat{y}_0, z(0) \rangle_{H^{-1}(G), H_0^1(G)} - \mathbb{E} \langle y(\tau), \hat{z}^\tau \rangle_{L^2(G)} + \langle y_0, \hat{z}(0) \rangle_{L^2(G)} \\
& = \mathbb{E} \int_0^\tau \langle g_1, z \rangle_{H^{-1}(G), H_0^1(G)} dt + \mathbb{E} \int_0^\tau \langle g_2, Z \rangle_{H^{-1}(G), H_0^1(G)} dt - \mathbb{E} \int_0^\tau \int_{\Gamma_0} \frac{\partial z}{\partial \nu} h d\Gamma ds.
\end{aligned} \tag{4.1}$$

Here  $(z, Z, \hat{z}, \widehat{Z})$  solves (3.1) with

$$b_1 = -a_1, \quad b_2 = -\operatorname{div} a_1 + a_2, \quad b_3 = a_3, \quad b_4 = 0, \quad b_5 = 0.$$

**Definition 4.2** A pair of stochastic processes  $(y, \hat{y}) \in C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G))) \times C_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-1}(G)))$  is a transposition solution to (1.10) if for any  $\tau \in (0, T]$  and  $(z^\tau, \hat{z}^\tau) \in L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G)) \times L^2_{\mathcal{F}_\tau}(\Omega; L^2(G))$ , we have that

$$\begin{aligned} & \mathbb{E}\langle \hat{y}(\tau), z^\tau \rangle_{H^{-1}(G), H_0^1(G)} - \langle \hat{y}_0, z(0) \rangle_{H^{-1}(G), H_0^1(G)} - \mathbb{E}\langle y(\tau), \hat{z}^\tau \rangle_{L^2(G)} + \langle y_0, \hat{z}(0) \rangle_{L^2(G)} \\ &= -\mathbb{E} \int_0^\tau \langle f, \widehat{Z} \rangle_{L^2(G)} dt + \mathbb{E} \int_0^\tau \langle g, Z \rangle_{H^{-1}(G), H_0^1(G)} dt - \mathbb{E} \int_0^\tau \int_{\Gamma_0} \frac{\partial z}{\partial \nu} h d\Gamma ds. \end{aligned} \quad (4.2)$$

Here  $(z, Z, \hat{z}, \widehat{Z})$  solves (3.1) with

$$b_1 = -a_1, \quad b_2 = -\operatorname{div} a_1 + a_2 - a_3 a_5, \quad b_3 = a_3, \quad b_4 = -a_4, \quad b_5 = -a_5.$$

**Remark 4.1** When  $h = 0$ , both systems (1.1) and (1.10) are homogeneous boundary value problems. By the classical theory for stochastic evolution equations, (1.1) and (1.10) admit respectively a unique weak solution (e.g. [7, Chapter 6])  $y \in C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G))) \cap C_{\mathbb{F}}^1([0, T]; L^2(\Omega; H^{-1}(G)))$  and  $(y, \hat{y}) \in C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G))) \times C_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-1}(G)))$ . It follows from Itô's formula that these solutions are respectively transposition solutions to (1.1) and (1.10). Then, by the uniqueness of the transposition solution to (1.1) (resp. (1.10)), we know that the transposition solution to (1.1) (resp. (1.10)) is also the weak solution to (1.1) (resp. (1.10)).

We have the following well-posedness result for (1.10).

**Proposition 4.1** For each  $(y_0, \hat{y}_0) \in L^2(G) \times H^{-1}(G)$ , the system (1.10) admits a unique transposition solution  $(y, \hat{y})$ . Moreover,

$$\begin{aligned} & |(y, \hat{y})|_{C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G))) \times C_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-1}(G)))} \\ & \leq C e^{Cr_2} (|y_0|_{L^2(G)} + |\hat{y}_0|_{H^{-1}(G)} + |f|_{L^2_{\mathbb{F}}(0, T; L^2(G))} + |g|_{L^2_{\mathbb{F}}(0, T; H^{-1}(G))} + |h|_{L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0))}). \end{aligned} \quad (4.3)$$

Here

$$r_2 = |a_1|_{L^{\infty}_{\mathbb{F}}(0, T; W^{1, \infty}(G; \mathbb{R}^n))}^2 + \sum_{k=2}^4 |a_k|_{L^{\infty}_{\mathbb{F}}(0, T; L^{\infty}(G))}^2 + |a_5|_{L^{\infty}_{\mathbb{F}}(0, T; W_0^{1, \infty}(G))}^2. \quad (4.4)$$

*Proof: Uniqueness.* Assume that  $(y, \hat{y})$  and  $(\tilde{y}, \tilde{\hat{y}})$  are two transposition solutions of (1.10). It follows from Definition 4.2 that for any  $\tau \in (0, T]$  and  $(z^\tau, \hat{z}^\tau) \in L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G)) \times L^2_{\mathcal{F}_\tau}(\Omega; L^2(G))$ ,

$$\mathbb{E}\langle \hat{y}(\tau), z^\tau \rangle_{H^{-1}(G), H_0^1(G)} - \mathbb{E}\langle y(\tau), \hat{z}^\tau \rangle_{L^2(G)} = \mathbb{E}\langle \tilde{\hat{y}}(\tau), z^\tau \rangle_{H^{-1}(G), H_0^1(G)} - \mathbb{E}\langle \tilde{y}(\tau), \hat{z}^\tau \rangle_{L^2(G)}, \quad (4.5)$$

which implies that

$$(\hat{y}(\tau), y(\tau)) = (\tilde{\hat{y}}(\tau), \tilde{y}(\tau)), \quad \forall \tau \in (0, T].$$

Hence,  $(\hat{y}, y) = (\tilde{\hat{y}}, \tilde{y})$  in  $C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G))) \times C_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-1}(G)))$ .

**Existence.** Since  $\chi_{\Sigma_0} h \in L^2_{\mathbb{F}}(0, T; L^2(\Gamma))$ , there exists a sequence  $\{h_m\}_{m=1}^{\infty} \subset C_{\mathbb{F}}^2([0, T]; H^{3/2}(\Gamma))$  with  $h_m(0) = 0$  for all  $m \in \mathbb{N}$  such that

$$\lim_{m \rightarrow \infty} h_m = \chi_{\Sigma_0} h \quad \text{in } L^2_{\mathbb{F}}(0, T; L^2(\Gamma)). \quad (4.6)$$

For each  $m \in \mathbb{N}$ , we can find a  $\tilde{h}_m \in C_{\mathbb{F}}^2([0, T]; H^2(G))$  such that  $\tilde{h}_m|_{\Gamma} = h_m$  and  $\tilde{h}_m(0) = 0$ .

Consider the following equation:

$$\left\{ \begin{array}{ll} d\tilde{y}_m = (\hat{y}_m - \tilde{h}_{m,t})dt + [a_4(\tilde{y}_m + \tilde{h}_m) + f]dW(t) & \text{in } Q, \\ d\tilde{y}_m - \sum_{j,k=1}^n (a^{jk}\tilde{y}_{m,x_j})_{x_k} dt = (a_1 \cdot \nabla \tilde{y}_m + a_2 \tilde{y}_m + \zeta_m)dt + [a_3(\tilde{y}_m + \tilde{h}_m) + g]dW(t) & \text{in } Q, \\ \tilde{y}_m = \hat{y}_m = 0 & \text{on } \Sigma, \\ \tilde{y}_m(0) = y_0, \quad \hat{y}_m(0) = \hat{y}_0 & \text{in } G, \end{array} \right. \quad (4.7)$$

where  $\zeta_m = \sum_{j,k=1}^n (a^{jk}\tilde{h}_{m,x_j})_{x_k} + a_1 \cdot \nabla \tilde{h}_m + a_2 \tilde{h}_m$ . By the classical theory of stochastic evolution equations (e.g. [7, Chapter 6]), the system (4.7) admits a unique mild (also weak) solution  $(\tilde{y}_m, \hat{y}_m) \in C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G))) \times C_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-1}(G)))$ .

Let  $y_m = \tilde{y}_m + \tilde{h}_m$  and  $\hat{y}_m = \hat{y}_m$ . For any  $m_1, m_2 \in \mathbb{N}$ , by Itô's formula and integration by parts, we have that

$$\begin{aligned} & \mathbb{E}\langle \hat{y}_{m_1}(\tau), z^\tau \rangle_{H^{-1}(G), H_0^1(G)} - \langle \hat{y}_0, z(0) \rangle_{H^{-1}(G), H_0^1(G)} - \mathbb{E}\langle y_{m_1}(\tau), \hat{z}^\tau \rangle_{L^2(G)} + \langle y_0, \hat{z}(0) \rangle_{L^2(G)} \\ &= -\mathbb{E} \int_0^\tau \langle f, \hat{Z} \rangle_{L^2(G)} dt + \mathbb{E} \int_0^\tau \langle g, Z \rangle_{H^{-1}(G), H_0^1(G)} dt - \mathbb{E} \int_0^\tau \int_{\Gamma} \frac{\partial z}{\partial \nu} h_{m_1} d\Gamma ds \end{aligned} \quad (4.8)$$

and

$$\begin{aligned} & \mathbb{E}\langle \hat{y}_{m_2}(\tau), z^\tau \rangle_{H^{-1}(G), H_0^1(G)} - \langle \hat{y}_0, z(0) \rangle_{H^{-1}(G), H_0^1(G)} - \mathbb{E}\langle y_{m_2}(\tau), \hat{z}^\tau \rangle_{L^2(G)} + \langle y_0, \hat{z}(0) \rangle_{L^2(G)} \\ &= -\mathbb{E} \int_0^\tau \langle f, \hat{Z} \rangle_{L^2(G)} dt + \mathbb{E} \int_0^\tau \langle g, Z \rangle_{H^{-1}(G), H_0^1(G)} dt - \mathbb{E} \int_0^\tau \int_{\Gamma} \frac{\partial z}{\partial \nu} h_{m_2} d\Gamma ds. \end{aligned}$$

Consequently,

$$\mathbb{E}\langle \hat{y}_{m_1}(\tau) - \hat{y}_{m_2}(\tau), z^\tau \rangle_{H^{-1}(G), H_0^1(G)} - \mathbb{E}\langle y_{m_1}(\tau) - y_{m_2}(\tau), \hat{z}^\tau \rangle_{L^2(G)} = -\mathbb{E} \int_{\Sigma_\tau} \frac{\partial z}{\partial \nu} (h_{m_1} - h_{m_2}) d\Gamma ds. \quad (4.9)$$

Let us choose  $(z^\tau, \hat{z}^\tau) \in L_{\mathcal{F}_\tau}^2(\Omega; H_0^1(G)) \times L_{\mathcal{F}_\tau}^2(\Omega; L^2(G))$  such that

$$|z^\tau|_{L_{\mathcal{F}_\tau}^2(\Omega; H_0^1(G))} = 1, \quad |\hat{z}^\tau|_{L_{\mathcal{F}_\tau}^2(\Omega; L^2(G))} = 1$$

and

$$\begin{aligned} & \mathbb{E}\langle \hat{y}_{m_1}(\tau) - \hat{y}_{m_2}(\tau), z^\tau \rangle_{H^{-1}(G), H_0^1(G)} - \mathbb{E}\langle y_{m_1}(\tau) - y_{m_2}(\tau), \hat{z}^\tau \rangle_{L^2(G)} \\ & \geq \frac{1}{2} (|y_{m_1}(\tau) - y_{m_2}(\tau)|_{L_{\mathcal{F}_\tau}^2(\Omega; L^2(G))} + |\hat{y}_{m_1}(\tau) - \hat{y}_{m_2}(\tau)|_{L_{\mathcal{F}_\tau}^2(\Omega; H^{-1}(G))}). \end{aligned} \quad (4.10)$$

It follows from (4.9), (4.10) and Proposition 3.1 that

$$\begin{aligned} & |y_{m_1}(\tau) - y_{m_2}(\tau)|_{L_{\mathcal{F}_\tau}^2(\Omega; L^2(G))} + |\hat{y}_{m_1}(\tau) - \hat{y}_{m_2}(\tau)|_{L_{\mathcal{F}_\tau}^2(\Omega; H^{-1}(G))} \\ & \leq 2 \left| \mathbb{E} \int_{\Sigma_\tau} \frac{\partial z}{\partial \nu} (h_{m_1} - h_{m_2}) d\Gamma ds \right| \\ & \leq C |h_{m_1} - h_{m_2}|_{L_{\mathbb{F}}^2(0, T; L^2(\Gamma))} |(z^\tau, \hat{z}^\tau)|_{L_{\mathcal{F}_\tau}^2(\Omega; H_0^1(G)) \times L_{\mathcal{F}_\tau}^2(\Omega; L^2(G))} \\ & \leq C |h_{m_1} - h_{m_2}|_{L_{\mathbb{F}}^2(0, T; L^2(\Gamma))}, \end{aligned}$$

where the constant  $C$  is independent of  $\tau$ . Consequently, it holds that

$$|y_{m_1} - y_{m_2}|_{C_{\mathbb{F}}([0,T];L^2(\Omega;L^2(G)))} + |\hat{y}_{m_1} - \hat{y}_{m_2}|_{C_{\mathbb{F}}([0,T];L^2(\Omega;H^{-1}(G)))} \leq C|h_{m_1} - h_{m_2}|_{L_{\mathbb{F}}^2(0,T;L^2(\Gamma_0))}.$$

This concludes that  $\{(y_m, \hat{y}_m)\}_{m=1}^{\infty}$  is a Cauchy sequence in  $C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G))) \times C_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-1}(G)))$ . Denote by  $(y, \hat{y})$  the limit of  $\{(y_m, \hat{y}_m)\}_{m=1}^{\infty}$ . Letting  $m \rightarrow \infty$  in (4.8), we get that

$$\begin{aligned} & \mathbb{E}\langle \hat{y}(\tau), z^\tau \rangle_{H^{-1}(G), H_0^1(G)} - \langle \hat{y}_0, z(0) \rangle_{H^{-1}(G), H_0^1(G)} - \mathbb{E}\langle y(\tau), \hat{z}^\tau \rangle_{L^2(G)} + \langle y_0, \hat{z}(0) \rangle_{L^2(G)} \\ &= -\mathbb{E} \int_0^\tau \langle f, \hat{Z} \rangle_{L^2(G)} dt + \mathbb{E} \int_0^\tau \langle g, Z \rangle_{H^{-1}(G), H_0^1(G)} dt - \mathbb{E} \int_0^\tau \int_{\Gamma_0} \frac{\partial z}{\partial \nu} h d\Gamma ds. \end{aligned} \quad (4.11)$$

Thus,  $(y, \hat{y})$  is a transposition solution to (1.10).

Let us choose  $(z^\tau, \hat{z}^\tau) \in L_{\mathcal{F}_\tau}^2(\Omega; H_0^1(G)) \times L_{\mathcal{F}_\tau}^2(\Omega; L^2(G))$  such that

$$|z^\tau|_{L_{\mathcal{F}_\tau}^2(\Omega; H_0^1(G))} = 1, \quad |\hat{z}^\tau|_{L_{\mathcal{F}_\tau}^2(\Omega; L^2(G))} = 1$$

and

$$\mathbb{E}\langle \hat{y}(\tau), z^\tau \rangle_{H^{-1}(G), H_0^1(G)} - \mathbb{E}\langle y(\tau), \hat{z}^\tau \rangle_{L^2(G)} \geq \frac{1}{2}(|y(\tau)|_{L_{\mathcal{F}_\tau}^2(\Omega; L^2(G))} + |\hat{y}(\tau)|_{L_{\mathcal{F}_\tau}^2(\Omega; H^{-1}(G))}). \quad (4.12)$$

Combining (4.11), (4.12) and Proposition 3.1, we obtain that

$$\begin{aligned} & |y(\tau)|_{L_{\mathcal{F}_\tau}^2(\Omega; L^2(G))} + |\hat{y}(\tau)|_{L_{\mathcal{F}_\tau}^2(\Omega; H^{-1}(G))} \\ & \leq 2 \left( |\langle \hat{y}_0, z(0) \rangle_{H^{-1}(G), H_0^1(G)}| + |\langle y_0, \hat{z}(0) \rangle_{L^2(G)}| + \left| \mathbb{E} \int_0^\tau \langle f, \hat{Z} \rangle_{L^2(G)} dt \right| \right. \\ & \quad \left. + \left| \mathbb{E} \int_0^\tau \langle g, Z \rangle_{H^{-1}(G), H_0^1(G)} dt \right| + \left| \mathbb{E} \int_{\Sigma_\tau} \frac{\partial z}{\partial \nu} h d\Gamma ds \right| \right) \\ & \leq C e^{Cr_2} (|y_0|_{L^2(G)} + |\hat{y}_0|_{H^{-1}(G)} + |f|_{L_{\mathbb{F}}^2(0,T;L^2(G))} + |g|_{L_{\mathbb{F}}^2(0,T;H^{-1}(G))} + |h|_{L_{\mathbb{F}}^2(0,T;L^2(\Gamma_0))}) \\ & \quad \times |(z^\tau, \hat{z}^\tau)|_{L_{\mathcal{F}_\tau}^2(\Omega; H_0^1(G)) \times L_{\mathcal{F}_\tau}^2(\Omega; L^2(G))} \\ & \leq C e^{Cr_2} (|y_0|_{L^2(G)} + |\hat{y}_0|_{H^{-1}(G)} + |f|_{L_{\mathbb{F}}^2(0,T;L^2(G))} + |g|_{L_{\mathbb{F}}^2(0,T;H^{-1}(G))} + |h|_{L_{\mathbb{F}}^2(0,T;L^2(\Gamma_0))}), \end{aligned}$$

where the constant  $C$  is independent of  $\tau$ . Therefore, we have that

$$\begin{aligned} & |y|_{C_{\mathbb{F}}([0,T];L^2(\Omega;L^2(G)))} + |\hat{y}|_{C_{\mathbb{F}}([0,T];L^2(\Omega;H^{-1}(G)))} \\ & \leq C e^{Cr_2} (|y_0|_{L^2(G)} + |\hat{y}_0|_{H^{-1}(G)} + |f|_{L_{\mathbb{F}}^2(0,T;L^2(G))} + |g|_{L_{\mathbb{F}}^2(0,T;H^{-1}(G))} + |h|_{L_{\mathbb{F}}^2(0,T;L^2(\Gamma_0))}). \end{aligned}$$

This completes the proof of Proposition 4.1.  $\square$

Using the same argument as above, we have the following well-posedness result for (1.1) (Hence we omit its proof).

**Proposition 4.2** *For each  $(y_0, \hat{y}_0) \in L^2(G) \times H^{-1}(G)$ , the system (1.1) admits a unique transposition solution  $y$ . Furthermore,*

$$\begin{aligned} & |y|_{C_{\mathbb{F}}([0,T];L^2(\Omega;L^2(G))) \cap C_{\mathbb{F}}^1([0,T];L^2(\Omega;H^{-1}(G)))} \\ & \leq C e^{Cr_3} (|y_0|_{L^2(G)} + |y_1|_{H^{-1}(G)} + |g_1|_{L_{\mathbb{F}}^2(0,T;H^{-1}(G))} + |g_2|_{L_{\mathbb{F}}^2(0,T;H^{-1}(G))} + |h|_{L_{\mathbb{F}}^2(0,T;L^2(\Gamma_0))}). \end{aligned}$$

Here

$$r_3 = |a_1|_{L_{\mathbb{F}}^\infty(0,T;W^{1,\infty}(G;\mathbb{R}^n))}^2 + \sum_{k=2}^3 |a_2|_{L_{\mathbb{F}}^\infty(0,T;L^\infty(G))}^2.$$

## 5 A reduction of the exact controllability problem

Definition 4.2 is a natural generalization of the transposition solution from deterministic wave equations to the stochastic ones. Accordingly, one has to establish observability estimates for (3.1) to get the exact controllability of (1.10). But it is not so easy. In this section, we give a reduction of exact controllability problems for these systems, that is, we show that these problems can be transformed to exact controllability problems for backward stochastic wave equations.

Consider the following controlled backward stochastic wave equation:

$$\left\{ \begin{array}{ll} d\mathbf{y} = \hat{\mathbf{y}}dt + (a_4\mathbf{y} + \mathbf{Y})dW(t) & \text{in } Q, \\ d\hat{\mathbf{y}} - \sum_{j,k=1}^n (a^{jk}\mathbf{y}_{x_j})_{x_k} dt = (a_1 \cdot \nabla \mathbf{y} + a_2\mathbf{y} + a_5\hat{\mathbf{Y}})dt + (a_3\mathbf{y} + \hat{\mathbf{Y}})dW(t) & \text{in } Q, \\ \mathbf{y} = \chi_{\Sigma_0}\mathbf{h} & \text{on } \Sigma, \\ \hat{\mathbf{y}} = 0 & \text{on } \Sigma, \\ \mathbf{y}(T) = \mathbf{y}^T, \quad \hat{\mathbf{y}}(T) = \hat{\mathbf{y}}^T & \text{in } G. \end{array} \right. \quad (5.1)$$

Here  $(\mathbf{y}^T, \hat{\mathbf{y}}^T) \in L^2_{\mathcal{F}_T}(\Omega; L^2(G)) \times L^2_{\mathcal{F}_T}(\Omega; H^{-1}(G))$ ,  $(\mathbf{y}, \mathbf{Y}, \hat{\mathbf{y}}, \hat{\mathbf{Y}})$  are the state and  $\mathbf{h} \in L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0))$  is the control.

To define the solution to (5.1), we introduce the following (forward) equation:

$$\left\{ \begin{array}{ll} d\mathbf{z} = \hat{\mathbf{z}}dt + (\mathbf{f} - a_5\mathbf{z})dW(t) & \text{in } Q^\tau, \\ d\hat{\mathbf{z}} - \sum_{j,k=1}^n (a^{jk}\mathbf{z}_{x_j})_{x_k} dt = [-a_1 \cdot \nabla \mathbf{z} + (-\operatorname{div} a_1 + a_2 - a_3a_5)\mathbf{z} + a_3\mathbf{f} - a_4\hat{\mathbf{f}}]dt + \hat{\mathbf{f}}dW(t) & \text{in } Q^\tau, \\ \mathbf{z} = 0 & \text{on } \Sigma^\tau, \\ \mathbf{z}(\tau) = \mathbf{z}_\tau, \quad \hat{\mathbf{z}}(\tau) = \hat{\mathbf{z}}_\tau & \text{in } G. \end{array} \right. \quad (5.2)$$

Here  $Q^\tau \triangleq (\tau, T) \times G$ ,  $\Sigma^\tau \triangleq (\tau, T) \times \Gamma$ ,  $(\mathbf{z}_\tau, \hat{\mathbf{z}}_\tau) \in L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G)) \times L^2_{\mathcal{F}_\tau}(\Omega; L^2(G))$ ,  $\mathbf{f} \in L^2_{\mathbb{F}}(0, T; H_0^1(G))$  and  $\hat{\mathbf{f}} \in L^2_{\mathbb{F}}(0, T; L^2(G))$ .

Let us recall the following well-posedness result for (5.2) (e.g. [7, Chapter 6]).

**Lemma 5.1** *For any  $(\mathbf{z}_\tau, \hat{\mathbf{z}}_\tau) \in L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G)) \times L^2_{\mathcal{F}_\tau}(\Omega; L^2(G))$ ,  $\mathbf{f} \in L^2_{\mathbb{F}}(0, T; H_0^1(G))$  and  $\hat{\mathbf{f}} \in L^2_{\mathbb{F}}(0, T; L^2(G))$ , the system (5.2) admits a unique weak solution  $(\mathbf{z}, \hat{\mathbf{z}}) \in C_{\mathbb{F}}([\tau, T]; H_0^1(G)) \times C_{\mathbb{F}}([\tau, T]; L^2(G))$ . Moreover,*

$$\begin{aligned} & |\mathbf{z}|_{C_{\mathbb{F}}([\tau, T]; H_0^1(G))} + |\hat{\mathbf{z}}|_{C_{\mathbb{F}}([\tau, T]; L^2(G))} \\ & \leq Ce^{Cr_4} (|\mathbf{z}_\tau|_{L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G))} + |\hat{\mathbf{z}}_\tau|_{L^2_{\mathcal{F}_\tau}(\Omega; L^2(G))} + |\mathbf{f}|_{L^2_{\mathbb{F}}(0, T; H_0^1(G))} + |\hat{\mathbf{f}}|_{L^2_{\mathbb{F}}(0, T; L^2(G))}), \end{aligned} \quad (5.3)$$

where

$$r_4 \triangleq |a_1|_{L^\infty(0, T; W^{1, \infty}(G; \mathbb{R}^n))}^2 + \sum_{i=2}^5 |a_i|_{L^\infty(0, T; L^\infty(G))}^2 + \sum_{i=3}^5 |a_i|_{L^\infty(0, T; L^\infty(G))}^4 + |a_5|_{L^\infty(0, T; W_0^{1, \infty}(G))}^2$$

and the constant  $C$  is independent of  $\tau$ .

Similar to the proof of Proposition 3.1, we have

**Proposition 5.1** *The solution  $(\mathbf{z}, \hat{\mathbf{z}})$  to (5.2) satisfies  $\frac{\partial \mathbf{z}}{\partial \nu}|_{\Gamma} \in L^2_{\mathbb{F}}(\tau, T; L^2(\Gamma))$ . Furthermore,*

$$\left| \frac{\partial \mathbf{z}}{\partial \nu} \right|_{L^2_{\mathbb{F}}(\tau, T; L^2(\Gamma))} \leq C e^{C\tau_4} (|\mathbf{z}_{\tau}|_{L^2_{\mathcal{F}_{\tau}}(\Omega; H_0^1(G))} + |\hat{\mathbf{z}}_{\tau}|_{L^2_{\mathcal{F}_{\tau}}(\Omega; L^2(G))} + |\mathbf{f}|_{L^2_{\mathbb{F}}(0, T; H_0^1(G))} + |\hat{\mathbf{f}}|_{L^2_{\mathbb{F}}(0, T; L^2(G))}), \quad (5.4)$$

where the constant  $C$  is independent of  $\tau$ .

**Definition 5.1** *A quadruple of stochastic processes  $(\mathbf{y}, \mathbf{Y}, \hat{\mathbf{y}}, \hat{\mathbf{Y}}) \in C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G))) \times L^2_{\mathbb{F}}(0, T; L^2(G)) \times C_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-1}(G))) \times L^2_{\mathbb{F}}(0, T; H^{-1}(G))$  is a transposition solution to (5.1) if for every  $(\mathbf{z}_{\tau}, \hat{\mathbf{z}}_{\tau}) \in L^2_{\mathcal{F}_{\tau}}(\Omega; H_0^1(G)) \times L^2_{\mathcal{F}_{\tau}}(\Omega; L^2(G))$ ,  $\mathbf{f} \in L^2_{\mathbb{F}}(0, T; H_0^1(G))$  and  $\hat{\mathbf{f}} \in L^2_{\mathbb{F}}(0, T; L^2(G))$ , one has that*

$$\begin{aligned} & \mathbb{E}\langle \hat{\mathbf{y}}^T, \mathbf{z}(T) \rangle_{H^{-1}(G), H_0^1(G)} - \mathbb{E}\langle \hat{\mathbf{y}}(\tau), \mathbf{z}_{\tau} \rangle_{H^{-1}(G), H_0^1(G)} - \mathbb{E}\langle \mathbf{y}^T, \hat{\mathbf{z}}(T) \rangle_{L^2(G)} + \mathbb{E}\langle \mathbf{y}(\tau), \hat{\mathbf{z}}_{\tau} \rangle_{L^2(G)} \\ &= -\mathbb{E} \int_{\tau}^T \langle \mathbf{Y}, \hat{\mathbf{f}} \rangle_{L^2(G)} dt + \mathbb{E} \int_{\tau}^T \langle \hat{\mathbf{Y}}, \mathbf{f} \rangle_{H^{-1}(G), H_0^1(G)} dt - \mathbb{E} \int_{\tau}^T \int_{\Gamma_0} \frac{\partial \mathbf{z}}{\partial \nu} \mathbf{h} d\Gamma ds. \end{aligned} \quad (5.5)$$

Here  $(\mathbf{z}, \hat{\mathbf{z}})$  solves (5.2).

We have the following result:

**Proposition 5.2** *For each  $(\mathbf{y}^T, \hat{\mathbf{y}}^T) \in L^2_{\mathcal{F}_T}(\Omega; L^2(G)) \times L^2_{\mathcal{F}_T}(\Omega; H^{-1}(G))$ , the system (5.1) admits a unique transposition solution  $(\mathbf{y}, \mathbf{Y}, \hat{\mathbf{y}}, \hat{\mathbf{Y}})$ . Moreover,*

$$\begin{aligned} & |(\mathbf{y}, \mathbf{Y}, \hat{\mathbf{y}}, \hat{\mathbf{Y}})|_{C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G))) \times L^2_{\mathbb{F}}(0, T; L^2(G)) \times C_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-1}(G))) \times L^2_{\mathbb{F}}(0, T; H^{-1}(G))} \\ & \leq C e^{Cr_2} (|\mathbf{y}^T|_{L^2_{\mathcal{F}_T}(\Omega; L^2(G))} + |\hat{\mathbf{y}}^T|_{L^2_{\mathcal{F}_T}(\Omega; H^{-1}(G))} + |\mathbf{h}|_{L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0))}). \end{aligned} \quad (5.6)$$

Here  $r_2$  is given by (4.4).

*Proof:* The proof is similarly to that for Proposition 4.1. We give it here for the convenience of readers.

**Uniqueness.** Assume that  $(\mathbf{y}, \mathbf{Y}, \hat{\mathbf{y}}, \hat{\mathbf{Y}})$  and  $(\tilde{\mathbf{y}}, \tilde{\mathbf{Y}}, \tilde{\hat{\mathbf{y}}}, \tilde{\hat{\mathbf{Y}}})$  are two transposition solutions of (5.1). By Definition 5.1, for any  $\tau \in (0, T]$ ,  $(\mathbf{z}_{\tau}, \hat{\mathbf{z}}_{\tau}) \in L^2_{\mathcal{F}_{\tau}}(\Omega; H_0^1(G)) \times L^2_{\mathcal{F}_{\tau}}(\Omega; L^2(G))$ ,  $\mathbf{f} \in L^2_{\mathbb{F}}(0, T; H_0^1(G))$  and  $\hat{\mathbf{f}} \in L^2_{\mathbb{F}}(0, T; L^2(G))$ , we have

$$\begin{aligned} & \mathbb{E}\langle \hat{\mathbf{y}}(\tau), \mathbf{z}_{\tau} \rangle_{H^{-1}(G), H_0^1(G)} - \mathbb{E}\langle \mathbf{y}(\tau), \hat{\mathbf{z}}_{\tau} \rangle_{L^2(G)} + \mathbb{E} \int_{\tau}^T \langle \mathbf{Y}, \hat{\mathbf{f}} \rangle_{L^2(G)} dt - \mathbb{E} \int_{\tau}^T \langle \hat{\mathbf{Y}}, \mathbf{f} \rangle_{H^{-1}(G), H_0^1(G)} dt \\ &= \mathbb{E}\langle \tilde{\hat{\mathbf{y}}}(\tau), \mathbf{z}_{\tau} \rangle_{H^{-1}(G), H_0^1(G)} - \mathbb{E}\langle \tilde{\mathbf{y}}(\tau), \hat{\mathbf{z}}_{\tau} \rangle_{L^2(G)} + \mathbb{E} \int_{\tau}^T \langle \tilde{\mathbf{Y}}, \hat{\mathbf{f}} \rangle_{L^2(G)} dt - \mathbb{E} \int_{\tau}^T \langle \tilde{\hat{\mathbf{Y}}}, \mathbf{f} \rangle_{H^{-1}(G), H_0^1(G)} dt, \end{aligned}$$

which implies that

$$(\mathbf{y}(\tau), \hat{\mathbf{y}}(\tau)) = (\tilde{\mathbf{y}}(\tau), \tilde{\hat{\mathbf{y}}}(\tau)), \quad \forall \tau \in (0, T]$$

and

$$(\mathbf{Y}, \hat{\mathbf{Y}}) = (\tilde{\mathbf{Y}}, \tilde{\hat{\mathbf{Y}}}) \text{ in } L^2_{\mathbb{F}}(0, T; L^2(G)) \times L^2_{\mathbb{F}}(0, T; H^{-1}(G)).$$

Hence,  $(\mathbf{y}, \mathbf{Y}, \hat{\mathbf{y}}, \hat{\mathbf{Y}}) = (\tilde{\mathbf{y}}, \tilde{\mathbf{Y}}, \tilde{\hat{\mathbf{y}}}, \tilde{\hat{\mathbf{Y}}})$ .

**Existence.** Since  $\chi_{\Sigma_0} \mathbf{h} \in L^2_{\mathbb{F}}(0, T; L^2(\Gamma))$ , there exists a sequence  $\{\mathbf{h}_m\}_{m=1}^{\infty} \subset C_{\mathbb{F}}^2([0, T]; H^{3/2}(\Gamma))$  with  $\mathbf{h}_m(T) = 0$  for all  $m \in \mathbb{N}$  such that

$$\lim_{m \rightarrow \infty} \mathbf{h}_m = \chi_{\Sigma_0} \mathbf{h} \text{ in } L^2_{\mathbb{F}}(0, T; L^2(\Gamma)). \quad (5.7)$$





Let  $(\mathbf{z}_\tau, \hat{\mathbf{z}}_\tau) = (0, 0)$  and  $\tau = 0$  in (5.10). It follows from (5.10), (5.11) and Proposition 5.1 that

$$\begin{aligned} & |\mathbf{Y}_{m_1} - \mathbf{Y}_{m_2}|_{L^2_{\mathbb{F}}(0,T;L^2(G))} + |\hat{\mathbf{Y}}_{m_1} - \hat{\mathbf{Y}}_{m_2}|_{L^2_{\mathbb{F}}(0,T;H^{-1}(G))} \\ & \leq 2 \left| \mathbb{E} \int_{\Sigma_\tau} \frac{\partial \mathbf{z}}{\partial \nu} (\mathbf{h}_{m_1} - \mathbf{h}_{m_2}) d\Gamma ds \right| \\ & \leq C |\mathbf{h}_{m_1} - \mathbf{h}_{m_2}|_{L^2_{\mathbb{F}}(0,T;L^2(\Gamma))} |(\mathbf{z}_\tau, \hat{\mathbf{z}}_\tau)|_{L^2_{\mathcal{F}_\tau}(\Omega;H_0^1(G)) \times L^2_{\mathcal{F}_\tau}(\Omega;L^2(G))} \\ & \leq C |\mathbf{h}_{m_1} - \mathbf{h}_{m_2}|_{L^2_{\mathbb{F}}(0,T;L^2(\Gamma))}. \end{aligned}$$

This implies that

$$|\mathbf{Y}_{m_1} - \mathbf{Y}_{m_2}|_{L^2_{\mathbb{F}}(0,T;L^2(G))} + |\hat{\mathbf{Y}}_{m_1} - \hat{\mathbf{Y}}_{m_2}|_{L^2_{\mathbb{F}}(0,T;H^{-1}(G))} \leq C |\mathbf{h}_{m_1} - \mathbf{h}_{m_2}|_{L^2_{\mathbb{F}}(0,T;L^2(\Gamma))}.$$

Therefore,  $\{(\mathbf{Y}_m, \hat{\mathbf{Y}}_m)\}_{m=1}^\infty$  is a Cauchy sequence in  $L^2_{\mathbb{F}}(0, T; L^2(G)) \times L^2_{\mathbb{F}}(0, T; H^{-1}(G))$ . Denote by  $(\mathbf{y}, \mathbf{Y}, \hat{\mathbf{y}}, \hat{\mathbf{Y}})$  the limit of  $\{(\mathbf{y}_m, \mathbf{Y}_m, \hat{\mathbf{y}}_m, \hat{\mathbf{Y}}_m)\}_{n=1}^\infty$ . By letting  $m_1 \rightarrow \infty$  in (5.9), we conclude that

$$\begin{aligned} & \mathbb{E} \langle \hat{\mathbf{y}}^T, \mathbf{z}(T) \rangle_{H^{-1}(G), H_0^1(G)} - \mathbb{E} \langle \hat{\mathbf{y}}(\tau), \mathbf{z}_\tau \rangle_{H^{-1}(G), H_0^1(G)} - \mathbb{E} \langle \mathbf{y}^T, \hat{\mathbf{z}}(T) \rangle_{L^2(G)} + \mathbb{E} \langle \mathbf{y}(\tau), \hat{\mathbf{z}}_\tau \rangle_{L^2(G)} \\ & = -\mathbb{E} \int_\tau^T \langle \mathbf{Y}, \hat{\mathbf{f}} \rangle_{L^2(G)} dt + \mathbb{E} \int_\tau^T \langle \hat{\mathbf{Y}}, \mathbf{f} \rangle_{H^{-1}(G), H_0^1(G)} dt - \mathbb{E} \int_\tau^T \int_{\Gamma_0} \frac{\partial \mathbf{z}}{\partial \nu} \mathbf{h} d\Gamma ds. \end{aligned} \quad (5.12)$$

Thus,  $(\mathbf{y}, \mathbf{Y}, \hat{\mathbf{y}}, \hat{\mathbf{Y}})$  is a solution of (1.10).

Let us choose  $(\mathbf{z}_\tau, \hat{\mathbf{z}}_\tau) \in L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G)) \times L^2_{\mathcal{F}_\tau}(\Omega; L^2(G))$  such that

$$|\mathbf{z}_\tau|_{L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G))} = 1, \quad |\hat{\mathbf{z}}_\tau|_{L^2_{\mathcal{F}_\tau}(\Omega; L^2(G))} = 1$$

and

$$-\mathbb{E} \langle \hat{\mathbf{y}}(\tau), \mathbf{z}_\tau \rangle_{H^{-1}(G), H_0^1(G)} + \mathbb{E} \langle \mathbf{y}(\tau), \hat{\mathbf{z}}_\tau \rangle_{L^2(G)} \geq \frac{1}{2} (|\mathbf{y}(\tau)|_{L^2_{\mathcal{F}_\tau}(\Omega; L^2(G))} + |\hat{\mathbf{y}}(\tau)|_{L^2_{\mathcal{F}_\tau}(\Omega; H^{-1}(G))}). \quad (5.13)$$

Let  $\mathbf{f} = 0$  and  $\hat{\mathbf{f}} = 0$  in (5.12). Combining (5.12), (5.13) and Proposition 5.1, we obtain that

$$\begin{aligned} & |\mathbf{y}(\tau)|_{L^2_{\mathcal{F}_\tau}(\Omega; L^2(G))} + |\hat{\mathbf{y}}(\tau)|_{L^2_{\mathcal{F}_\tau}(\Omega; H^{-1}(G))} \\ & \leq 2 \left( |\mathbb{E} \langle \hat{\mathbf{y}}^T, \mathbf{z}(T) \rangle_{H^{-1}(G), H_0^1(G)}| + |\mathbb{E} \langle \mathbf{y}^T, \hat{\mathbf{z}}(T) \rangle_{L^2(G)}| + \left| \mathbb{E} \int_\tau^T \int_{\Gamma_0} \frac{\partial \mathbf{z}}{\partial \nu} \mathbf{h} d\Gamma ds \right| \right) \\ & \leq C e^{Cr_4} (|\mathbf{y}^T|_{L^2_{\mathcal{F}_T}(\Omega; L^2(G))} + |\hat{\mathbf{y}}^T|_{L^2_{\mathcal{F}_T}(\Omega; H^{-1}(G))} + |\mathbf{h}|_{L^2_{\mathbb{F}}(0,T;L^2(\Gamma_0))}), \end{aligned}$$

where the constant  $C$  is independent of  $\tau$ . Thus, we find that

$$\begin{aligned} & |\mathbf{y}|_{C_{\mathbb{F}}([0,T];L^2(\Omega;L^2(G)))} + |\hat{\mathbf{y}}|_{C_{\mathbb{F}}([0,T];L^2(\Omega;H^{-1}(G)))} \\ & \leq C e^{Cr_4} (|\mathbf{y}^T|_{L^2_{\mathcal{F}_T}(\Omega;L^2(G))} + |\hat{\mathbf{y}}^T|_{L^2_{\mathcal{F}_T}(\Omega;H^{-1}(G))} + |\mathbf{h}|_{L^2_{\mathbb{F}}(0,T;L^2(\Gamma_0))}). \end{aligned}$$

Let us choose  $\mathbf{f} \in L^2_{\mathbb{F}}(0, T; H_0^1(G))$  and  $\hat{\mathbf{f}} \in L^2_{\mathbb{F}}(0, T; L^2(G))$  such that

$$|\mathbf{f}|_{L^2_{\mathbb{F}}(0,T;H_0^1(G))} = 1, \quad |\hat{\mathbf{f}}|_{L^2_{\mathbb{F}}(0,T;L^2(G))} = 1$$

and

$$\mathbb{E} \int_\tau^T \langle \mathbf{Y}, \hat{\mathbf{f}} \rangle_{L^2(G)} dt - \mathbb{E} \int_\tau^T \langle \hat{\mathbf{Y}}, \mathbf{f} \rangle_{H^{-1}(G), H_0^1(G)} dt \geq \frac{1}{2} (|\mathbf{Y}|_{L^2_{\mathbb{F}}(0,T;L^2(G))} + |\hat{\mathbf{Y}}|_{L^2_{\mathbb{F}}(0,T;H^{-1}(G))}). \quad (5.14)$$

Let  $(\mathbf{z}_\tau, \hat{\mathbf{z}}_\tau) = (0, 0)$  and  $\tau = 0$  in (5.12). It follows from (5.12), (5.14) and Proposition 5.1 that

$$\begin{aligned} & |\mathbf{Y}|_{L^2_{\mathbb{F}}(0,T;L^2(G))} + |\widehat{\mathbf{Y}}|_{L^2_{\mathbb{F}}(0,T;H^{-1}(G))} \\ & \leq 2 \left( |\mathbb{E} \langle \hat{\mathbf{y}}^T, \mathbf{z}(T) \rangle_{H^{-1}(G), H_0^1(G)}| + |\mathbb{E} \langle \mathbf{y}^T, \hat{\mathbf{z}}(T) \rangle_{L^2(G)}| + \left| \mathbb{E} \int_{\tau}^T \int_{\Gamma_0} \frac{\partial \mathbf{z}}{\partial \nu} \mathbf{h} d\Gamma ds \right| \right) \\ & \leq C e^{Cr_4} (|\mathbf{y}^T|_{L^2_{\mathcal{F}_T}(\Omega;L^2(G))} + |\hat{\mathbf{y}}^T|_{L^2_{\mathcal{F}_T}(\Omega;H^{-1}(G))} + |\mathbf{h}|_{L^2_{\mathbb{F}}(0,T;L^2(\Gamma_0))}). \end{aligned}$$

This completes the proof.  $\square$

Now we give the definition of the exact controllability for the system (5.1).

**Definition 5.2** *The system (5.1) is called exactly controllable at time  $T$  if for any  $(\mathbf{y}^T, \hat{\mathbf{y}}^T) \in L^2_{\mathcal{F}_T}(\Omega;L^2(G)) \times L^2_{\mathcal{F}_T}(\Omega;H^{-1}(G))$  and  $(\mathbf{y}_0, \hat{\mathbf{y}}_0) \in L^2(G) \times H^{-1}(G)$ , one can find a control  $\mathbf{h} \in L^2_{\mathbb{F}}(0,T;L^2(\Gamma_0))$  such that the corresponding solution  $(\mathbf{y}, \hat{\mathbf{y}})$  of (5.1) satisfies that  $(\mathbf{y}(0), \hat{\mathbf{y}}(0)) = (\mathbf{y}_0, \hat{\mathbf{y}}_0)$ .*

It is clear that the following result holds.

**Proposition 5.3** *Let  $\tau = T$  in (3.1) and  $\tau = 0$  in (5.2). If  $(z, Z, \hat{z}, \widehat{Z})$  is a solution of (3.1), then  $(\mathbf{z}, \hat{\mathbf{z}}) = (z, \hat{z})$  is a solution of (5.2) with the initial data  $(\mathbf{z}_0, \hat{\mathbf{z}}_0) = (z(0), \hat{z}(0))$  and nonhomogeneous terms  $(\mathbf{f}, \hat{\mathbf{f}}) = (Z, \widehat{Z})$ . On the other hand, if  $(\mathbf{z}, \hat{\mathbf{z}})$  is a solution of (5.2), then  $(z, Z, \hat{z}, \widehat{Z}) = (\mathbf{z}, \mathbf{f}, \hat{\mathbf{z}}, \hat{\mathbf{f}})$  is a solution of (3.1) with the final data  $(z(T), \hat{z}(T)) = (\mathbf{z}(T), \hat{\mathbf{z}}(T))$ .*

By Proposition 5.3, and Definitions 4.2 and 5.1, we have the following result concerning the relationship between solutions of (1.10) and (5.1).

**Proposition 5.4** *If  $(y, \hat{y})$  is a transposition solution of (1.10), then  $(\mathbf{y}, \mathbf{Y}, \hat{\mathbf{y}}, \widehat{\mathbf{Y}}) = (y, f, \hat{y}, g)$  is a transposition solution of (5.1) with the final data  $(\mathbf{y}_T, \hat{\mathbf{y}}_T) = (y(T), \hat{y}(T))$ . On the other hand, if  $(\mathbf{y}, \mathbf{Y}, \hat{\mathbf{y}}, \widehat{\mathbf{Y}})$  is a transposition solution of (5.1), then  $(y, \hat{y}) = (\mathbf{y}, \hat{\mathbf{y}})$  is a transposition solution of (1.10) with the initial data  $(y(0), \hat{y}(0)) = (\mathbf{y}(0), \hat{\mathbf{y}}(0))$  and the nonhomogeneous terms  $(f, g) = (\mathbf{Y}, \widehat{\mathbf{Y}})$ .*

By Propositions 5.3 and 5.4, and by borrowing some idea from [24], we have the following fact:

**Proposition 5.5** *The system (1.10) is exactly controllable at time  $T$  if and only if the system (5.1) is exactly controllable at time  $T$ .*

*Proof: The “if” part.* Let  $(\mathbf{y}_0, \hat{\mathbf{y}}_0) \in L^2(G) \times H^{-1}(G)$  and  $(\mathbf{y}^T, \hat{\mathbf{y}}^T) \in L^2_{\mathcal{F}_T}(\Omega;L^2(G)) \times L^2_{\mathcal{F}_T}(\Omega;H^{-1}(G))$  be arbitrarily given. Since (5.1) is exactly controllable at time  $T$ , there is an  $\mathbf{h} \in L^2_{\mathbb{F}}(0,T;L^2(\Gamma_0))$  such that the corresponding solution  $(\mathbf{y}, \mathbf{Y}, \hat{\mathbf{y}}, \widehat{\mathbf{Y}})$  of (5.1) satisfies that  $(\mathbf{y}(T), \hat{\mathbf{y}}(T)) = (\mathbf{y}^T, \hat{\mathbf{y}}^T)$  and  $(\mathbf{y}(0), \hat{\mathbf{y}}(0)) = (\mathbf{y}_0, \hat{\mathbf{y}}_0)$ . Hence,  $(y, \hat{y}) = (\mathbf{y}, \hat{\mathbf{y}})$  is a solution of (1.10) with a triple of controls  $(f, g, h) = (\mathbf{Y}, \widehat{\mathbf{Y}}, \mathbf{h})$  such that  $(y(0), \hat{y}(0)) = (\mathbf{y}_0, \hat{\mathbf{y}}_0)$  and  $(y(T), \hat{y}(T)) = (\mathbf{y}^T, \hat{\mathbf{y}}^T)$ . Hence, the system (1.10) is exactly controllable at time  $T$ .

The proof for the “only if” part is similar.  $\square$

The following result shows that the exact controllability of (5.1) is equivalent to an observability estimate of (5.2) with  $\mathbf{f} = \hat{\mathbf{f}} = 0$ .

**Proposition 5.6** *The system (5.1) is exactly controllable at time  $T$  if and only if there is a constant  $C > 0$  such that for all  $(\mathbf{z}_0, \hat{\mathbf{z}}_0) \in H_0^1(G) \times L^2(G)$ , it holds that*

$$|(\mathbf{z}_0, \hat{\mathbf{z}}_0)|_{H_0^1(G) \times L^2(G)}^2 \leq C \mathbb{E} \int_0^T \int_{\Gamma_0} \left| \frac{\partial \mathbf{z}}{\partial \nu} \right|^2 d\Gamma dt, \quad (5.15)$$

where  $\mathbf{z}$  is the solution of (5.2) with  $\tau = 0$ ,  $\mathbf{f} = \hat{\mathbf{f}} = 0$ ,  $\mathbf{z}(0) = \mathbf{z}_0$  and  $\hat{\mathbf{z}}(0) = \hat{\mathbf{z}}_0$ .

**Remark 5.1** Compared with (3.1), (5.1) is a forward stochastic wave equations. Generally speaking, it is easier to establish an observability estimate for (5.1) than to prove an observability estimate for (3.1). This is why we introduce the reduction in this section.

*Proof of Proposition 5.6:* We use the classical duality argument, and divide the proof into two parts.

**The “if” part.** Since the system (5.1) is linear, we only need to show that for any  $(\mathbf{y}_0, \hat{\mathbf{y}}_0) \in L^2(G) \times H^{-1}(G)$ , there is a control  $\mathbf{h} \in L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0))$  such that the corresponding solution of (5.1) with  $(\mathbf{y}_T, \hat{\mathbf{y}}_T) = (0, 0)$  satisfies that  $(\mathbf{y}(0), \hat{\mathbf{y}}(0)) = (\mathbf{y}_0, \hat{\mathbf{y}}_0)$ .

Set

$$\mathcal{X} \triangleq \left\{ -\frac{\partial \mathbf{z}}{\partial \nu} \Big|_{\Gamma_0} \mid (\mathbf{z}, \hat{\mathbf{z}}) \text{ solves the equation (3.1) with some } (\mathbf{z}_0, \hat{\mathbf{z}}_0) \in H_0^1(G) \times L^2(G) \right\}.$$

Clearly,  $\mathcal{X}$  is a linear subspace of  $L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0))$ . Define a linear functional  $\mathcal{L}$  on  $\mathcal{X}$  as follows:

$$\mathcal{L} \left( -\frac{\partial \mathbf{z}}{\partial \nu} \Big|_{\Gamma_0} \right) = -\langle \hat{\mathbf{y}}_1, \mathbf{z}_0 \rangle_{H^{-1}(G), H_0^1(G)} + \langle \mathbf{y}_1, \hat{\mathbf{z}}_0 \rangle_{L^2(G)}.$$

By (5.15),  $\mathcal{L}$  is a bounded linear functional on  $\mathcal{X}$ . By the Hahn-Banach theorem, it can be extended to be a bounded linear functional on the space  $L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0))$ . For simplicity, we still use  $\mathcal{L}$  to denote this extension. By Riesz’s representation theorem, there is an  $\mathbf{h} \in L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0))$  such that

$$-\langle \hat{\mathbf{y}}_1, \mathbf{z}_0 \rangle_{H^{-1}(G), H_0^1(G)} + \langle \mathbf{y}_1, \hat{\mathbf{z}}_0 \rangle_{L^2(G)} = -\mathbb{E} \int_0^T \int_{\Gamma_0} \frac{\partial \mathbf{z}}{\partial \nu} \mathbf{h} d\Gamma dt. \quad (5.16)$$

We claim that the random field  $\mathbf{h}$  is the desired control. In fact, by the definition of the solution to (5.1), we have

$$-\langle \hat{\mathbf{y}}(0), \mathbf{z}_0 \rangle_{H^{-1}(G), H_0^1(G)} + \langle \mathbf{y}(0), \hat{\mathbf{z}}_0 \rangle_{L^2(G)} = -\mathbb{E} \int_0^T \int_{\Gamma_0} \frac{\partial \mathbf{z}}{\partial \nu} \mathbf{h} d\Gamma dt. \quad (5.17)$$

It follows from (5.16) and (5.17) that

$$-\langle \hat{\mathbf{y}}_1, \mathbf{z}_0 \rangle_{H^{-1}(G), H_0^1(G)} + \langle \mathbf{y}_1, \hat{\mathbf{z}}_0 \rangle_{L^2(G)} = -\langle \hat{\mathbf{y}}(0), \mathbf{z}_0 \rangle_{H^{-1}(G), H_0^1(G)} + \langle \mathbf{y}(0), \hat{\mathbf{z}}_0 \rangle_{L^2(G)}.$$

Noting that  $(\mathbf{z}_0, \hat{\mathbf{z}}_0)$  is an arbitrary element in  $H_0^1(G) \times L^2(G)$ , we obtain  $(\mathbf{y}(0), \hat{\mathbf{y}}(0)) = (\mathbf{y}_1, \hat{\mathbf{y}}_1)$ .

**The “only if” part.** We now prove (5.15) by a contradiction argument. Otherwise, one could find a sequence  $\{(\mathbf{z}_{0,k}, \hat{\mathbf{z}}_{0,k})\}_{k=1}^{\infty} \subset H_0^1(G) \times L^2(G)$  with  $(\mathbf{z}_{0,k}, \hat{\mathbf{z}}_{0,k}) \neq (0, 0)$  for all  $k \in \mathbb{N}$ , such that the corresponding solutions  $(\mathbf{z}_k, \hat{\mathbf{z}}_k)$  of (3.1) with the initial data  $(\mathbf{z}_{0,k}, \hat{\mathbf{z}}_{0,k})$  satisfying that

$$\left| \frac{\partial \mathbf{z}_k}{\partial \nu} \Big|_{L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0))} \right|^2 \leq \frac{1}{k^2} |(\mathbf{z}_{0,k}, \hat{\mathbf{z}}_{0,k})|_{H_0^1(G) \times L^2(G)}^2. \quad (5.18)$$

Write

$$\lambda_k = \frac{\sqrt{k}}{|(\mathbf{z}_{0,k}, \hat{\mathbf{z}}_{0,k})|_{H_0^1(G) \times L^2(G)}}, \quad \tilde{\mathbf{z}}_{0,k} = \lambda_k \mathbf{z}_{0,k}, \quad \tilde{\hat{\mathbf{z}}}_{0,k} = \lambda_k \hat{\mathbf{z}}_{0,k}$$

and denote by  $(\tilde{\mathbf{z}}_k, \tilde{\hat{\mathbf{z}}}_k)$  the solution of (3.1) (with  $(\mathbf{z}_0, \hat{\mathbf{z}}_0)$  replaced by  $(\tilde{\mathbf{z}}_{k,0}, \tilde{\hat{\mathbf{z}}}_{k,0})$ ). Then, it follows from (5.18) that, for each  $k \in \mathbb{N}$ ,

$$\left| \frac{\partial \tilde{\mathbf{z}}_k}{\partial \nu} \Big|_{L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0))} \right|^2 \leq \frac{1}{k} \quad (5.19)$$

and

$$|(\tilde{\mathbf{z}}_{k,0}, \tilde{\mathbf{z}}_{k,0})|_{H_0^1(G) \times L^2(G)} = \sqrt{k}. \quad (5.20)$$

Let us choose  $(\mathbf{y}^T, \hat{\mathbf{y}}^T) = (0, 0)$  in (5.1). Since the system (5.1) is exactly controllable, for any given  $(\mathbf{y}_1, \hat{\mathbf{y}}_1) \in L^2(G) \times H^{-1}(G)$ , there is a control  $\mathbf{h} \in L_{\mathbb{F}}^2(0, T; L^2(\Gamma_0))$  driving the corresponding solution of (5.1) to  $(\mathbf{y}_1, \hat{\mathbf{y}}_1)$ . It follows from the definition of the solution to (5.1) that

$$-\langle \hat{\mathbf{y}}_1, \mathbf{z}_0 \rangle_{H^{-1}(G), H_0^1(G)} + \langle \mathbf{y}_1, \hat{\mathbf{z}}_0 \rangle_{L^2(G)} = -\mathbb{E} \int_0^T \int_{\Gamma_0} \frac{\partial \mathbf{z}}{\partial \nu} \mathbf{h} d\Gamma ds.$$

Thus, for every  $k \in \mathbb{N}$ , we have that

$$-\langle \hat{\mathbf{y}}_1, \tilde{\mathbf{z}}_{k,0} \rangle_{H^{-1}(G), H_0^1(G)} + \langle \mathbf{y}_1, \tilde{\mathbf{z}}_{k,0} \rangle_{L^2(G)} = -\int_0^T \int_{\Gamma_0} \frac{\partial \tilde{\mathbf{z}}_k}{\partial \nu} \mathbf{h} d\Gamma ds. \quad (5.21)$$

This, together with (5.19) and the arbitrariness of  $(\mathbf{y}_1, \hat{\mathbf{y}}_1)$ , implies that

$$(\tilde{\mathbf{z}}_{k,0}, \tilde{\mathbf{z}}_{k,0}) \text{ tends to 0 weakly in } H_0^1(G) \times L^2(G) \text{ as } k \rightarrow +\infty.$$

Hence, by the Principle of Uniform Boundedness,  $\{(\tilde{\mathbf{z}}_{k,0}, \tilde{\mathbf{z}}_{k,0})\}_{k=1}^{\infty}$  is uniformly bounded in  $H_0^1(G) \times L^2(G)$ , which contradicts (5.20).  $\square$

## 6 A fundamental identity for stochastic hyperbolic-like operators

Throughout this section, we assume that  $b^{jk} \in C^3([0, T] \times \mathbb{R}^n)$  satisfies  $b^{jk} = b^{kj}$  for  $j, k = 1, 2, \dots, n$ , and  $\ell, \Psi \in C^2((0, T) \times \mathbb{R}^n)$ . Write

$$\begin{cases} \mathcal{C}^{jk} \triangleq (b^{jk} \ell_t)_t + \sum_{j', k'=1}^n [2b^{j'k'} (b^{j'k} \ell_{x_{j'}})_{x_{k'}} - (b^{jk} b^{j'k'} \ell_{x_{j'}})_{x_{k'}}] + \Psi b^{jk} \\ \mathcal{A} \triangleq (\ell_t^2 - \ell_{tt}) - \sum_{j,k=1}^n [b^{jk} \ell_{x_j} \ell_{x_k} - (b^{jk} \ell_{x_j})_{x_k}] - \Psi, \\ \mathcal{B} \triangleq \mathcal{A} \Psi + (\mathcal{A} \ell_t)_t - \sum_{j,k=1}^n (\mathcal{A} b^{jk} \ell_{x_j})_{x_k} + \frac{1}{2} \left[ \Psi_{tt} - \sum_{j,k=1}^n (b^{jk} \Psi_{x_j})_{x_k} \right]. \end{cases} \quad (6.1)$$

We shall derive a fundamental identity for the stochastic hyperbolic-like operator given in the following result.

**Lemma 6.1** *Let  $z$  be an  $H^2(\mathbb{R}^n)$ -valued semimartingale and  $\hat{z}$  be an  $L^2(\mathbb{R}^n)$ -valued semimartingale, and*

$$dz = \hat{z} dt + Z dW(t) \quad \text{in } (0, T) \times \mathbb{R}^n \quad (6.2)$$

for some  $Z \in L_{\mathbb{F}}^2(0, T; H^1(\mathbb{R}^n))$ . Set  $\theta = e^\ell$ ,  $v = \theta z$  and  $\hat{v} = \theta \hat{z} + \ell_t v$ . Then, for a.e.  $x \in \mathbb{R}^n$  and  $\mathbb{P}$ -a.s.  $\omega \in \Omega$ ,

$$\begin{aligned} & \theta \left( -2\ell_t \hat{v} + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} v_{x_k} + \Psi v \right) \left[ d\hat{z} - \sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} dt \right] \\ & + \sum_{j,k=1}^n \left[ \sum_{j',k'=1}^n \left( 2b^{j'k'} b^{j'k} \ell_{x_{j'}} v_{x_j} v_{x_{k'}} - b^{jk} b^{j'k'} \ell_{x_j} v_{x_{j'}} v_{x_{k'}} \right) \right] \end{aligned}$$

$$\begin{aligned}
& -2\ell_t b^{jk} v_{x_j} \hat{v} + b^{jk} \ell_{x_j} \hat{v}^2 + \Psi b^{jk} v_{x_j} v - \frac{\Psi_{x_j} b^{jk} v^2 - \mathcal{A} b^{jk} \ell_{x_j} v^2}{2} \Big]_{x_k} \\
& + d \left[ \ell_t \sum_{j,k=1}^n b^{jk} v_{x_j} v_{x_k} + \ell_t \hat{v}^2 - 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} v_{x_k} \hat{v} - \Psi v \hat{v} + \left( \mathcal{A} \ell_t + \frac{\Psi_t}{2} \right) v^2 \right] \\
& = \left\{ \left[ \ell_{tt} + \sum_{j,k=1}^n (b^{jk} \ell_{x_j})_{x_k} - \Psi \right] \hat{v}^2 + \sum_{j,k=1}^n c^{jk} v_{x_j} v_{x_k} - 2 \sum_{j,k=1}^n \left( (b^{jk} \ell_{x_k})_t + b^{jk} \ell_{t x_k} \right) v_{x_j} \hat{v} \right. \\
& \quad \left. + \mathcal{B} v^2 + \left( -2\ell_t \hat{v} + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} v_{x_k} + \Psi v \right)^2 \right\} dt + \ell_t (d\hat{v})^2 - 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} dv_{x_k} d\hat{v} - \Psi dv d\hat{v} \\
& \quad + \ell_t \sum_{j,k=1}^n b^{jk} (dv_{x_j})(dv_{x_k}) + \mathcal{A} \ell_t (dv)^2 - \left[ \theta \left( -2\ell_t \hat{v} + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} v_{x_k} + \Psi v \right) \ell_t Z \right. \\
& \quad \left. - \left( 2 \sum_{j,k=1}^n b^{jk} (\theta Z)_{x_k} \ell_{x_j} \hat{v} - \theta \Psi_t v Z + \theta \Psi \hat{v} Z \right) + 2 \left( \sum_{j,k=1}^n b^{jk} v_{x_j} (\theta Z)_{x_k} + \theta \mathcal{A} v Z \right) \ell_t \right] dW(t),
\end{aligned} \tag{6.3}$$

where  $(dv)^2$  and  $(d\hat{v})^2$  denote the quadratic variation processes of  $v$  and  $\hat{v}$ , respectively.

*Proof:* By (6.2), and recalling  $v = \theta z$  and  $\hat{v} = \theta \hat{z} + \ell_t v$ , we obtain that

$$dv = d(\theta z) = \theta_t z dt + \theta dz = \ell_t \theta z dt + \theta \hat{z} dt + \theta Z dW(t) = \hat{v} dt + \theta Z dW(t). \tag{6.4}$$

Hence,

$$\begin{aligned}
d\hat{z} &= d[\theta^{-1}(\hat{v} - \ell_t v)] = \theta^{-1} [d\hat{v} - \ell_{tt} v dt - \ell_t dv - \ell_t (\hat{v} - \ell_t v) dt] \\
&= \theta^{-1} \left[ d\hat{v} - (2\ell_t \hat{v} + \ell_{tt} v - \ell_t^2 v) dt - \theta \ell_t Z dW(t) \right].
\end{aligned} \tag{6.5}$$

Similarly, by  $b^{jk} = b^{kj}$  for  $j, k = 1, 2, \dots, n$ , we have

$$\sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} = \theta^{-1} \sum_{j,k=1}^n [(b^{jk} v_{x_j})_{x_k} - 2b^{jk} \ell_{x_j} v_{x_k} + (b^{jk} \ell_{x_j} \ell_{x_k} - b_{x_k}^{jk} \ell_{x_j} - b^{jk} \ell_{x_j x_k}) v]. \tag{6.6}$$

Therefore, from (6.5)–(6.6) and the definition of  $\mathcal{A}$  in (6.1), we get

$$\begin{aligned}
& \theta \left( -2\ell_t \hat{v} + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} v_{x_k} + \Psi v \right) \left[ d\hat{z} - \sum_{j,k=1}^n (b^{jk} z_{x_j})_{x_k} dt \right] \\
&= \left( -2\ell_t \hat{v} + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} v_{x_k} + \Psi v \right) \left[ d\hat{v} - \sum_{j,k=1}^n (b^{jk} v_{x_j})_{x_k} dt + \mathcal{A} v dt \right. \\
& \quad \left. + \left( -2\ell_t \hat{v} + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} v_{x_k} + \Psi v \right) dt - \theta \ell_t Z dW(t) \right] \\
&= \left( -2\ell_t \hat{v} + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} v_{x_k} + \Psi v \right) d\hat{v} + \left( -2\ell_t \hat{v} + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} v_{x_k} + \Psi v \right)^2 dt \\
& \quad + \left( -2\ell_t \hat{v} + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} v_{x_k} + \Psi v \right) \left[ - \sum_{j,k=1}^n (b^{jk} v_{x_j})_{x_k} + \mathcal{A} v \right] dt
\end{aligned} \tag{6.7}$$

$$-\theta \left( -2\ell_t \hat{v} + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} v_{x_k} + \Psi v \right) \ell_t Z dW(t).$$

We now analyze the first and third terms in the right-hand side of (6.7). Using Itô's formula and noting (6.4), we have

$$\begin{aligned} & \left( -2\ell_t \hat{v} + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} v_{x_k} + \Psi v \right) d\hat{v} \\ &= d \left( -\ell_t \hat{v}^2 + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} v_{x_k} \hat{v} + \Psi v \hat{v} \right) - 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} \hat{v} dv_{x_k} - \Psi \hat{v} dv \\ & \quad - \left[ -\ell_{tt} \hat{v}^2 + 2 \sum_{j,k=1}^n (b^{jk} \ell_{x_j})_t v_{x_k} \hat{v} + \Psi_t v \hat{v} \right] dt + \ell_t (d\hat{v})^2 - 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} dv_{x_k} d\hat{v} - \Psi dv d\hat{v} \\ &= d \left( -\ell_t \hat{v}^2 + 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} v_{x_k} \hat{v} + \Psi v \hat{v} - \frac{\Psi_t}{2} v^2 \right) - \sum_{j,k=1}^n (b^{jk} \ell_{x_j} \hat{v}^2)_{x_k} dt \\ & \quad + \left\{ \left[ \ell_{tt} + \sum_{j,k=1}^n (b^{jk} \ell_{x_j})_{x_k} - \Psi \right] \hat{v}^2 - 2 \sum_{j,k=1}^n (b^{jk} \ell_{x_k})_t v_{x_j} \hat{v} + \frac{\Psi_{tt}}{2} v^2 \right\} dt + \ell_t (d\hat{v})^2 \\ & \quad - 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} dv_{x_k} d\hat{v} - \Psi dv d\hat{v} - \left[ 2 \sum_{j,k=1}^n b^{jk} (\theta Z)_{x_k} \ell_{x_j} \hat{v} - \theta \Psi_t v Z + \theta \Psi \hat{v} Z \right] dW(t). \end{aligned} \quad (6.8)$$

Next,

$$\begin{aligned} & -2\ell_t \hat{v} \left( - \sum_{j,k=1}^n (b^{jk} v_{x_j})_{x_k} + \mathcal{A}v \right) dt \\ &= 2 \sum_{j,k=1}^n (\ell_t b^{jk} v_{x_j} \hat{v})_{x_k} dt - 2 \sum_{j,k=1}^n \ell_{tx_k} b^{jk} v_{x_j} \hat{v} dt - 2\ell_t \sum_{j,k=1}^n b^{jk} v_{x_j} \hat{v}_{x_k} dt - 2\mathcal{A}\ell_t v \hat{v} dt \\ &= 2 \sum_{j,k=1}^n (\ell_t b^{jk} v_{x_j} \hat{v})_{x_k} dt - 2 \sum_{j,k=1}^n \ell_{tx_k} b^{jk} v_{x_j} \hat{v} dt - 2\ell_t \sum_{j,k=1}^n b^{jk} v_{x_j} (dv - \theta Z dW(t))_{x_k} - 2\mathcal{A}\ell_t v (dv - \theta Z dW(t)) \\ &= 2 \sum_{j,k=1}^n (\ell_t b^{jk} v_{x_j} \hat{v})_{x_k} dt - 2 \sum_{j,k=1}^n \ell_{tx_k} b^{jk} v_{x_j} \hat{v} dt - d \left( \ell_t \sum_{j,k=1}^n b^{jk} v_{x_j} v_{x_k} + \mathcal{A}\ell_t v^2 \right) + \sum_{j,k=1}^n (\ell_t b^{jk})_t v_{x_j} v_{x_k} dt \\ & \quad + (\mathcal{A}\ell_t)_t v^2 dt + \ell_t \sum_{j,k=1}^n b^{jk} (dv_{x_j})(dv_{x_k}) + \mathcal{A}\ell_t (dv)^2 + 2 \left[ \sum_{j,k=1}^n b^{jk} v_{x_j} (\theta Z)_{x_k} + \theta \mathcal{A}v Z \right] \ell_t dW(t). \end{aligned} \quad (6.9)$$

Further, by direct computation, one may check that

$$\begin{aligned} & 2 \sum_{j,k=1}^n b^{jk} \ell_{x_j} v_{x_k} \left( - \sum_{j,k=1}^n (b^{jk} v_{x_j})_{x_k} + \mathcal{A}v \right) \\ &= - \sum_{j,k=1}^n \left[ \sum_{j',k'=1}^n \left( 2b^{jk} b^{j'k'} \ell_{x_{j'}} v_{x_j} v_{x_{k'}} - b^{jk} b^{j'k'} \ell_{x_j} v_{x_{j'}} v_{x_{k'}} \right) - \mathcal{A}b^{jk} \ell_{x_j} v^2 \right]_{x_k} \\ & \quad + \sum_{j,k,j',k'=1}^n \left[ 2b^{j'k'} (b^{jk} \ell_{x_{j'}})_{x_{k'}} - (b^{jk} b^{j'k'} \ell_{x_{j'}})_{x_{k'}} \right] v_{x_j} v_{x_k} - \sum_{j,k=1}^n (\mathcal{A}b^{jk} \ell_{x_j})_{x_k} v^2 \end{aligned} \quad (6.10)$$

and

$$\begin{aligned}
& \Psi v \left( - \sum_{j,k=1}^n (b^{jk} v_{x_j})_{x_k} + \mathcal{A}v \right) \\
&= - \sum_{j,k=1}^n \left( \Psi b^{jk} v_{x_j} v - \frac{\Psi_{x_j}}{2} b^{jk} v^2 \right)_{x_k} + \Psi \sum_{j,k=1}^n b^{jk} v_{x_j} v_{x_k} + \left[ - \frac{1}{2} \sum_{j,k=1}^n (b^{jk} \Psi_{x_j})_{x_k} + \mathcal{A}\Psi \right] v^2.
\end{aligned} \tag{6.11}$$

Finally, combining (6.7)–(6.11), we arrive at the desired equality (6.3).  $\square$

## 7 Observability estimate for the equation (5.2)

In this section, we establish the following observability estimate for the equation (5.2).

**Theorem 7.1** *Let Conditions 2.1 and 2.2 hold, and  $\Gamma_0$  be given by (2.5). Then, all solutions of the equation (5.2) with  $\mathbf{f} = 0$  and  $\hat{\mathbf{f}} = 0$  satisfy that*

$$|(\mathbf{z}_0, \hat{\mathbf{z}}_0)|_{H_0^1(G) \times L^2(G)} \leq C e^{Cr_4} \left| \frac{\partial \mathbf{z}}{\partial \nu} \right|_{L_{\mathbb{F}}^2(0, T; L^2(\Gamma_0))}. \tag{7.1}$$

*Proof:* We borrow some idea from [16, 20], and divide the proof into three steps.

**Step 1.** Let us choose

$$\ell(t, x) = \lambda \left[ \varphi(x) - c_1 \left( t - \frac{T}{2} \right)^2 \right]. \tag{7.2}$$

Put

$$\Lambda_i \triangleq \left\{ (t, x) \in Q \mid \varphi(x) - c_1 \left( t - \frac{T}{2} \right)^2 > \frac{R_0^2}{2(i+2)} \right\}, \quad \text{for } i = 0, 1, 2. \tag{7.3}$$

Let

$$\begin{cases} T_i \triangleq \frac{T}{2} - \varepsilon_i T, & T'_i \triangleq \frac{T}{2} + \varepsilon_i T, \\ Q_i \triangleq (T_i, T'_i) \times G, \end{cases} \quad \text{for } i = 0, 1, \tag{7.4}$$

where  $\varepsilon_0$  and  $\varepsilon_1$  are given below.

From Condition 2.2 and (7.2), we find that

$$\ell(0, x) = \ell(T, x) \leq \lambda \left( R_1 - \frac{c_1 T^2}{4} \right) < 0, \quad \forall x \in G. \tag{7.5}$$

Hence, there exists  $\varepsilon_1 \in (0, \frac{1}{2})$  such that  $\Lambda_2 \subset Q_1$  (7.6)

and that

$$\ell(t, x) < 0, \quad \forall (t, x) \in [(0, T_1) \cup (T'_1, T)] \times G. \tag{7.7}$$

Next, since  $\{T/2\} \times G \subset \Lambda_0$ , we know that there is an  $\varepsilon_0 > 0$  such that

$$Q_0 \triangleq \left( \frac{T}{2} - \varepsilon_0 T, \frac{T}{2} + \varepsilon_0 T \right) \times G \subset \Lambda_0. \tag{7.8}$$

**Step 2.** Apply Lemma 6.1 with  $(b^{jk})_{1 \leq j, k \leq n} = (a^{jk})_{1 \leq j, k \leq n}$  to the solution of the equation (5.2) with

$$\Psi = \ell_{tt} + \sum_{j,k=1}^n (a^{jk} \ell_{x_j})_{x_k} - c_0 \lambda, \quad (7.9)$$

and then estimate the resulting terms in (6.3) one by one.

Let us first analyze the terms which stand for the “energy” of the solution. To this end, we need to compute orders of  $\lambda$  in the coefficients of  $\hat{v}^2$ ,  $|\nabla v|^2$  and  $v^2$ .

Clearly, the term for  $\hat{v}^2$  reads

$$\left[ \ell_{tt} + \sum_{j,k=1}^n (a^{jk} \ell_{x_j})_{x_k} - \Psi \right] \hat{v}^2 = c_0 \lambda \hat{v}^2. \quad (7.10)$$

Noting that  $\ell_{tx_k} = \ell_{x_k t} = 0$ , we get

$$2 \sum_{j,k=1}^n \left[ (a^{jk} \ell_{x_k})_t + a^{jk} \ell_{tx_k} \right] v_{x_j} \hat{v} = 0. \quad (7.11)$$

From (7.9), we see that

$$\begin{aligned} & (a^{jk} \ell_t)_t + \sum_{j',k'=1}^n [2a^{jk'} (a^{j'k} \ell_{x_{j'}})_{x_{k'}} - (a^{jk} a^{j'k'} \ell_{x_{j'}})_{x_{k'}}] + \Psi a^{jk} \\ &= a^{jk} \ell_{tt} + \sum_{j',k'=1}^n [2a^{jk'} (a^{j'k} \ell_{x_{j'}})_{x_{k'}} - (a^{jk} a^{j'k'} \ell_{x_{j'}})_{x_{k'}}] + a^{jk} \left[ \ell_{tt} + \sum_{j',k'=1}^{\infty} (a^{j'k'} \ell_{x_{j'}})_{x_{k'}} - c_0 \lambda \right] \\ &= 2a^{jk} \ell_{tt} + \sum_{j',k'=1}^n [2a^{jk'} (a^{j'k} \ell_{x_{j'}})_{x_{k'}} - a_{x_{k'}}^{jk} a^{j'k'} \ell_{x_{j'}}] - a^{jk} c_0 \lambda \\ &= 2a^{jk} \ell_{tt} + \lambda \sum_{j',k'=1}^n [2a^{jk'} (a^{j'k} \psi_{x_{j'}})_{x_{k'}} - a_{x_{k'}}^{jk} a^{j'k'} \psi_{x_{j'}}] + \lambda \sum_{j',k'=1}^n 2a^{jk'} a^{j'k} \psi_{x_{j'}} \psi_{x_{k'}} - a^{jk} c_0 \lambda. \end{aligned}$$

This, together with Condition 2.1, implies that

$$\begin{aligned} & \sum_{j,k=1}^n \left\{ (a^{jk} \ell_t)_t + \sum_{j',k'=1}^n [2a^{jk'} (a^{j'k} \ell_{x_{j'}})_{x_{k'}} - (a^{jk} a^{j'k'} \ell_{x_{j'}})_{x_{k'}}] + \Psi a^{jk} \right\} v_{x_j} v_{x_k} \\ & \geq \lambda (\mu_0 - 4c_1 - c_0) \sum_{j,k=1}^n a^{jk} v_{x_j} v_{x_k}. \end{aligned} \quad (7.12)$$

Now we compute the coefficients of  $v^2$ . By (6.1), it is easy to obtain that

$$\begin{aligned} \mathcal{A} &= \ell_t^2 - \ell_{tt} - \sum_{j,k=1}^n [a^{jk} \ell_{x_j} \ell_{x_k} - (a^{jk} \ell_{x_j})_{x_k}] - \Psi \\ &= \lambda^2 \left[ c_1^2 (2t - T)^2 - \sum_{j,k=1}^n a^{jk} \varphi_{x_j} \varphi_{x_k} \right] + 4c_1 \lambda + c_0 \lambda. \end{aligned} \quad (7.13)$$



By the definition of  $\mathcal{B}$ , we see that

$$\begin{aligned}
\mathcal{B} &= \mathcal{A}\Psi + (\mathcal{A}\ell_t)_t - \sum_{j,k=1}^n (\mathcal{A}a^{jk}\ell_{x_j})_{x_k} + \frac{1}{2} \sum_{j,k=1}^n [\Psi_{tt} - (a^{jk}\Psi_{x_j})_{x_k}] \\
&= 2\mathcal{A}\ell_{tt} - \lambda c_0 \mathcal{A} - \sum_{j,k=1}^n a^{jk}\ell_j \mathcal{A}_k + \mathcal{A}_t \ell_t - \frac{1}{2} \sum_{j,k=1}^n \sum_{j',k'=1}^n [a^{jk}(a^{j'k'}\ell_{x_{j'}})_{x_{k'}}]_{x_k} \\
&= 2\lambda^3 \left[ -2c_1^3(2t-T)^2 + 2c_1 \sum_{j,k=1}^n a^{jk}\varphi_{x_j}\varphi_{x_k} \right] - \lambda^3 c_0 c_1^2 (2t-T)^2 + \lambda^3 c_0 \sum_{j,k=1}^n a^{jk}\varphi_{x_j}\varphi_{x_k} \\
&\quad + \lambda^3 \sum_{j,k=1}^n \sum_{j',k'=1}^n a^{jk}\varphi_{x_j}(a^{j'k'}\varphi_{x_{j'}}\varphi_{x_{k'}})_{x_k} - 4\lambda^3 c_1^3 (2t-T)^2 + O(\lambda^2) \\
&= (4c_1 + c_0)\lambda^3 \sum_{j,k=1}^n a^{jk}\varphi_{x_j}\varphi_{x_k} + \lambda^3 \sum_{j,k=1}^n \sum_{j',k'=1}^n a^{jk}\varphi_{x_j}(a^{j'k'}\varphi_{x_{j'}}\varphi_{x_{k'}})_{x_k} \\
&\quad - (8c_1^3 + c_0 c_1^2)\lambda^3 (2t-T)^2 + O(\lambda^2).
\end{aligned} \tag{7.14}$$

Similar to [16, (3.8)], we have

$$\sum_{j,k=1}^n \sum_{j',k'=1}^n a^{jk}\varphi_{x_j}(a^{j'k'}\varphi_{x_{j'}}\varphi_{x_{k'}})_{x_k} \geq \mu_0 \sum_{j,k=1}^n a^{jk}\varphi_{x_j}\varphi_{x_k}. \tag{7.15}$$

From (7.14), (7.15) and Condition 2.2, we obtain that

$$\begin{aligned}
\mathcal{B} &\geq \lambda^3 (4c_1 + c_0) \sum_{j,k=1}^n a^{jk}\varphi_{x_j}\varphi_{x_k} + \lambda^3 \mu_0 \sum_{j,k=1}^n a^{jk}\varphi_{x_j}\varphi_{x_k} - (8c_1^3 + 2c_0 c_1^2)\lambda^3 (2t-T)^2 + O(\lambda^2) \\
&\geq (\mu_0 + 4c_1 + c_0)\lambda^3 \sum_{j,k=1}^n a^{jk}\varphi_{x_j}\varphi_{x_k} - 8c_1^2(4c_1 + c_0)\left(t - \frac{T}{2}\right)^2 \lambda^3 + O(\lambda^2).
\end{aligned} \tag{7.16}$$

Since the diffusion term in the second equation of (5.2) is zero, we obtain that

$$\mathbb{E}\ell_t(d\hat{v})^2 = 0, \quad \mathbb{E} \sum_{j,k=1}^n a^{jk}\ell_{x_j} dv_{x_k} d\hat{v} = 0, \quad \mathbb{E}\Psi dv d\hat{v} = 0. \tag{7.17}$$

From (7.2) and noting that  $\mathbf{z}_{x_k} = (\theta^{-1}v)_{x_k} = \theta^{-1}v_{x_k} - \theta^{-1}\ell_{x_k}v$ , we see that

$$\begin{aligned}
&\ell_t \mathbb{E} \sum_{j,k=1}^n a^{jk}(dv_{x_j})(dv_{x_k}) \\
&= \mathbb{E} \left[ 2c_1 \lambda^3 \left(t - \frac{T}{2}\right) |a_5|^2 \theta^2 \sum_{j,k=1}^n a^{jk}\varphi_{x_j}\varphi_{x_k} |\mathbf{z}|^2 + 2c_1 \lambda \left(t - \frac{T}{2}\right) \theta^2 |a_5|^2 \sum_{j,k=1}^n a^{jk} \mathbf{z}_{x_j} \mathbf{z}_{x_k} \right. \\
&\quad + 2c_1 \lambda \left(t - \frac{T}{2}\right) \theta^2 |\mathbf{z}|^2 \sum_{j,k=1}^n a^{jk} a_{5,x_j} a_{5,x_k} + 4c_1 \lambda^2 \left(t - \frac{T}{2}\right) \theta^2 |a_5|^2 \mathbf{z} \sum_{j,k=1}^n a^{jk} \varphi_{x_j} \mathbf{z}_{x_k} \\
&\quad \left. + 4c_1 \lambda^2 \left(t - \frac{T}{2}\right) \theta^2 a_5 |\mathbf{z}|^2 \sum_{j,k=1}^n a^{jk} \varphi_{x_j} a_{5,x_k} \right]
\end{aligned} \tag{7.18}$$

$$\begin{aligned}
&= \mathbb{E} \left[ 4c_1 \lambda^3 \left( t - \frac{T}{2} \right) |a_5|^2 \sum_{j,k=1}^n a^{jk} \varphi_{x_j} \varphi_{x_k} |v|^2 + 2c_1 \lambda \left( t - \frac{T}{2} \right) |a_5|^2 \sum_{j,k=1}^n a^{jk} v_{x_j} v_{x_k} \right. \\
&\quad \left. + C(\lambda^2 |\nabla a_5| + \lambda |\nabla a_5|^2) |v|^2 \right].
\end{aligned}$$

Next, from (7.2) and (7.13), we find that

$$\begin{aligned}
\mathbb{E}[\mathcal{A}l_t(dv)^2] &= \lambda^3 c_1 (2t - T) \left[ c_1^2 (2t - T)^2 - \sum_{j,k=1}^n a^{jk} \varphi_{x_j} \varphi_{x_k} \right] \mathbb{E}(|a_5|^2 v^2) \\
&\quad + (4c_1 + c_0) c_1 (2t - T) \lambda^2 \mathbb{E}(|a_5|^2 v^2).
\end{aligned} \tag{7.19}$$

From (7.12), (7.16), (7.18) and (7.19), and noting the fourth inequality in Condition 2.2, we know that there is  $c_2 > 0$  such that for all  $(t, x) \in \Lambda_2$ , one has that

$$\begin{aligned}
&\mathbb{E} \left[ \sum_{j,k=1}^n c^{jk} v_{x_j} v_{x_k} + \mathcal{B}v^2 + \ell_t \sum_{j,k=1}^n a^{jk} (dv_{x_j})(dv_{x_k}) + \mathcal{A}l_t(dv)^2 \right] \\
&= \mathbb{E} \left\{ \lambda \left[ \mu_0 - 4c_1 - c_0 + 2c_1 \left( t - \frac{T}{2} \right) |a_5|^2 \right] \sum_{j,k=1}^n a^{jk} v_{x_j} v_{x_k} + (\mu_0 + 4c_1 + c_0) \lambda^3 \sum_{j,k=1}^n a^{jk} \varphi_{x_j} \varphi_{x_k} |v|^2 \right. \\
&\quad \left. - 8c_1^2 (4c_1 + c_0) \left( t - \frac{T}{2} \right)^2 \lambda^3 |v|^2 + 4c_1 \lambda^3 \left( t - \frac{T}{2} \right) |a_5|^2 \sum_{j,k=1}^n a^{jk} \varphi_{x_j} \varphi_{x_k} |v|^2 + O(\lambda^2) |v|^2 \right\} \\
&\geq \mathbb{E} [c_2 \lambda |\nabla v|^2 + c_2 \lambda^3 |v|^2 + O(\lambda^2) |v|^2].
\end{aligned}$$

Thus, there exist  $\lambda_1 > 0$  and  $c_3 > 0$  such that for all  $\lambda \geq \lambda_1$  and for every  $(t, x) \in \Lambda_2$ , one has that

$$\mathbb{E} \left[ \sum_{j,k=1}^n c^{jk} v_{x_j} v_{x_k} + \mathcal{B}v^2 + \ell_t \sum_{j,k=1}^n a^{jk} (dv_{x_j})(dv_{x_k}) + \mathcal{A}l_t(dv)^2 \right] \geq \mathbb{E} (c_3 \lambda |\nabla v|^2 + c_3 \lambda^3 |v|^2). \tag{7.20}$$

**Step 3.** For the boundary terms, by  $v|_{\Sigma} = 0$ , we have the following equality:

$$\begin{aligned}
&\mathbb{E} \int_{\Sigma} \sum_{j,k=1}^n \sum_{j',k'=1}^n \left( 2a^{jk} a^{j'k'} \ell_{x_j} v_{x_j} v_{x_{k'}} - a^{jk} a^{j'k'} \ell_{x_j} v_{x_j} v_{x_{k'}} \right) \nu^k d\Sigma \\
&= \lambda \mathbb{E} \int_{\Sigma} \sum_{j,k=1}^n \sum_{j',k'=1}^n \left( 2a^{jk} a^{j'k'} \varphi_{x_{j'}} v_{x_j} v_{x_{k'}} - a^{jk} a^{j'k'} \varphi_{x_j} v_{x_{j'}} v_{x_{k'}} \right) \nu^k d\Sigma \\
&= \lambda \mathbb{E} \int_{\Sigma} \sum_{j,k=1}^n \sum_{j',k'=1}^n \left( 2a^{jk} a^{j'k'} \varphi_{x_{j'}} \frac{\partial v}{\partial \nu} \nu^j \frac{\partial v}{\partial \nu} \nu^{k'} - a^{jk} a^{j'k'} \varphi_{x_j} \frac{\partial v}{\partial \nu} \nu^{j'} \frac{\partial v}{\partial \nu} \nu^{k'} \right) \nu^k d\Sigma \\
&= \lambda \mathbb{E} \int_{\Sigma} \left( \sum_{j,k=1}^n a^{jk} \nu^j \nu^k \right) \left( \sum_{j',k'=1}^n a^{j'k'} \varphi_{x_{j'}} \nu^{k'} \right) \left| \frac{\partial v}{\partial \nu} \right|^2 d\Sigma.
\end{aligned} \tag{7.21}$$

For any  $\tau \in (0, T_1)$  and  $\tau' \in (T_1, T)$ , put

$$Q_{\tau}^{\tau'} \triangleq (\tau, \tau') \times G. \tag{7.22}$$

Integrating (6.3) in  $Q_{\tau'}^{\tau'}$ , taking expectation and by (7.10), (7.11), (7.17) and (7.20), we obtain that

$$\begin{aligned}
& \mathbb{E} \int_{Q_{\tau'}^{\tau'}} \theta \left( -2\ell_t \hat{v} + 2 \sum_{i=1}^n a^{jk} \ell_{x_j} v_{x_k} + \Psi v \right) \left[ d\hat{\mathbf{z}} - \sum_{j,k=1}^n (a^{jk} \mathbf{z}_{x_j})_{x_k} dt \right] dx \\
& + \lambda \mathbb{E} \int_{\Sigma_0} \left( \sum_{j,k=1}^n a^{jk} \nu^j \nu^k \right) \left( \sum_{j',k'=1}^n a^{j'k'} \varphi_{x_{j'}} \nu^{k'} \right) \left| \frac{\partial v}{\partial \nu} \right|^2 d\Sigma \\
& + \mathbb{E} \int_{Q_{\tau'}^{\tau'}} d \left[ \ell_t \sum_{j,k=1}^n a^{jk} v_{x_j} v_{x_k} + \ell_t \hat{v}^2 - 2 \sum_{j,k=1}^n a^{jk} \ell_{x_j} v_{x_k} \hat{v} - \Psi v \hat{v} + \left( \mathcal{A} \ell_t + \frac{\Psi_t}{2} \right) v^2 \right] dx \quad (7.23) \\
& \geq c_0 \lambda \mathbb{E} \int_{Q_{\tau'}^{\tau'}} \hat{v}^2 dx dt + \mathbb{E} \int_{Q_{\tau'}^{\tau'}} \sum_{j,k=1}^n c^{jk} v_{x_j} v_{x_k} dx dt + \mathbb{E} \int_{Q_{\tau'}^{\tau'}} \mathcal{B} v^2 dx dt \\
& + \mathbb{E} \int_{Q_{\tau'}^{\tau'}} \left[ \ell_t \sum_{j,k=1}^n a^{jk} (dv_{x_j})(dv_{x_k}) + \mathcal{A} \ell_t (dv)^2 \right] dx + \mathbb{E} \int_Q \left( -2\ell_t \hat{v} + 2 \sum_{j,k=1}^n a^{jk} \ell_{x_j} v_{x_k} + \Psi v \right)^2 dx dt.
\end{aligned}$$

Clearly,

$$\begin{aligned}
& \mathbb{E} \int_{Q_{\tau'}^{\tau'}} d \left[ \ell_t \sum_{j,k=1}^n a^{jk} v_{x_j} v_{x_k} + \ell_t \hat{v}^2 - 2 \sum_{j,k=1}^n a^{jk} \ell_{x_j} v_{x_k} \hat{v} - \Psi v \hat{v} + \left( \mathcal{A} \ell_t + \frac{\Psi_t}{2} \right) v^2 \right] dx \\
& = \mathbb{E} \int_G \left[ \ell_t \sum_{j,k=1}^n a^{jk} v_{x_j}(\tau') v_{x_k}(\tau') + \ell_t \hat{v}(\tau')^2 - 2 \sum_{j,k=1}^n a^{jk} \ell_{x_j} v_{x_k}(\tau') \hat{v}(\tau') - \Psi v(\tau') \hat{v}(\tau') \right. \\
& \quad \left. + \left( \mathcal{A} \ell_t + \frac{\Psi_t}{2} \right) v(\tau')^2 \right] dx \quad (7.24) \\
& - \mathbb{E} \int_G \left[ \ell_t \sum_{j,k=1}^n a^{jk} v_{x_j}(\tau) v_{x_k}(\tau) + \ell_t \hat{v}(\tau)^2 - 2 \sum_{j,k=1}^n a^{jk} \ell_{x_j} v_{x_k}(\tau) \hat{v}(\tau) - \Psi v(\tau) \hat{v}(\tau) \right. \\
& \quad \left. + \left( \mathcal{A} \ell_t + \frac{\Psi_t}{2} \right) v(\tau)^2 \right] dx \\
& \leq C \lambda^3 \mathbb{E} \int_G \left\{ [\hat{v}(\tau)^2 + |\nabla v(\tau)|^2 + v(\tau)^2] + [\hat{v}(\tau')^2 + |\nabla v(\tau')|^2 + v(\tau')^2] \right\} dx.
\end{aligned}$$

From  $\theta = e^\ell$  and (7.7), we know that there is a  $\lambda_1 > 0$  such that for all  $\lambda > \lambda_1$ ,

$$\lambda^3 \theta(\tau) \leq 1, \quad \lambda^3 \theta(\tau') \leq 1. \quad (7.25)$$

Since  $v = \theta \mathbf{z}$  and  $\hat{v} = \theta \hat{\mathbf{z}}$ , it follows from (7.25) that

$$\begin{aligned}
& \lambda^3 \mathbb{E} \int_G \left\{ [\hat{v}(\tau)^2 + |\nabla v(\tau)|^2 + v(\tau)^2] + [\hat{v}(\tau')^2 + |\nabla v(\tau')|^2 + v(\tau')^2] \right\} dx \\
& \leq \mathbb{E} \int_G \left\{ [\hat{\mathbf{z}}(\tau)^2 + |\nabla \mathbf{z}(\tau)|^2 + \mathbf{z}(\tau)^2] + [\hat{\mathbf{z}}(\tau')^2 + |\nabla \mathbf{z}(\tau')|^2 + \mathbf{z}(\tau')^2] \right\} dx. \quad (7.26)
\end{aligned}$$

From (7.3), (7.6) and (7.22), we obtain that  $\Lambda_2 \subset Q_{\tau'}^{\tau'}$ . Thus,

$$\begin{aligned}
& \lambda \mathbb{E} \int_{Q_{\tau'}^{\tau'} \setminus \Lambda_2} \hat{v}^2 dx dt = \lambda \mathbb{E} \int_{Q_{\tau'}^{\tau'} \setminus \Lambda_2} (\theta \hat{\mathbf{z}} + \ell_t \theta \mathbf{z})^2 dx dt \\
& \leq 2\lambda \mathbb{E} \int_{Q_{\tau'}^{\tau'} \setminus \Lambda_2} \theta^2 \hat{\mathbf{z}}^2 dx dt + 2\lambda^3 \mathbb{E} \int_{Q_{\tau'}^{\tau'} \setminus \Lambda_2} (2t - T)^2 \theta^2 \mathbf{z}^2 dx dt \quad (7.27)
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E} \int_{Q_{\tau'} \setminus \Lambda_2} \sum_{j,k=1}^n c^{jk} v_{x_j} v_{x_k} dx dt = \mathbb{E} \int_{Q_{\tau'} \setminus \Lambda_2} \sum_{j,k=1}^n c^{jk} (\theta \mathbf{z}_{x_j}) (\theta \mathbf{z}_{x_k}) dx dt \\
& = \mathbb{E} \int_{Q_{\tau'} \setminus \Lambda_2} \sum_{j,k=1}^n c^{jk} \theta^2 (\lambda \varphi_{x_j} + \mathbf{z}_{x_j}) (\lambda \varphi_{x_k} + \mathbf{z}_{x_k}) dx dt \\
& \leq C \lambda \mathbb{E} \int_{Q_{\tau'} \setminus \Lambda_2} \theta^2 |\nabla \mathbf{z}|^2 dx dt + C \lambda^3 \mathbb{E} \int_{Q_{\tau'} \setminus \Lambda_2} \theta^2 |\mathbf{z}|^2 dx dt.
\end{aligned} \tag{7.28}$$

Furthermore, it follows from (7.14) that

$$\mathbb{E} \int_{Q_{\tau'} \setminus \Lambda_2} \mathcal{B} v^2 dx dt \leq C \lambda^3 \mathbb{E} \int_{Q_{\tau'} \setminus \Lambda_2} \theta^2 \mathbf{z}^2 dx dt. \tag{7.29}$$

Next, by (7.18) and (7.19), we get that

$$\begin{aligned}
& \mathbb{E} \int_{Q_{\tau'} \setminus \Lambda_2} \left[ \ell_t \sum_{j,k=1}^n a^{jk} (dv_{x_j}) (dv_{x_k}) + \mathcal{A} \ell_t (dv)^2 \right] dx \\
& \leq C \lambda \mathbb{E} \int_{Q_{\tau'} \setminus \Lambda_2} \theta^2 |\nabla \mathbf{z}|^2 dx dt + C \lambda^3 \mathbb{E} \int_{Q_{\tau'} \setminus \Lambda_2} \theta^2 |\mathbf{z}|^2 dx dt.
\end{aligned} \tag{7.30}$$

From (7.3), we know that  $\theta \leq \exp(\lambda e^{R_0^2 \mu/8})$  in  $Q_{\tau'} \setminus \Lambda_2$ . Consequently, there exists  $\lambda_2 \geq \max\{\lambda_0, \lambda_1\}$  such that for all  $\lambda \geq \lambda_2$ ,

$$C \lambda \max_{(x,t) \in Q_{\tau'} \setminus \Lambda_2} \theta^2 \leq e^{\lambda R_0^2/3}, \quad C \lambda^3 \max_{(x,t) \in Q_{\tau'} \setminus \Lambda_2} \theta^2 \leq e^{\lambda R_0^2/3}. \tag{7.31}$$

It follows from (7.20) and (7.27)–(7.31) that

$$\begin{aligned}
& \lambda \mathbb{E} \int_{Q_{\tau'}} \hat{v}^2 dx dt + \mathbb{E} \int_{Q_{\tau'}} \sum_{j,k=1}^n c^{jk} v_{x_j} v_{x_k} dx dt + \mathbb{E} \int_{Q_{\tau'}} \mathcal{B} v^2 dx dt \\
& + \mathbb{E} \int_{Q_{\tau'}} \left[ \ell_t \sum_{j,k=1}^n a^{jk} (dv_{x_j}) (dv_{x_k}) + \mathcal{A} \ell_t (dv)^2 \right] dx \\
& \geq \lambda \mathbb{E} \int_{\Lambda_2} \hat{v}^2 dx dt + c_2 \lambda \mathbb{E} \int_{\Lambda_2} |\nabla v|^2 dx dt + c_2 \lambda^3 \mathbb{E} \int_{\Lambda_2} |v|^2 dx dt - e^{\lambda R_0^2/3} \mathbb{E} \int_Q (|\hat{\mathbf{z}}|^2 + |\nabla \mathbf{z}|^2) dx dt.
\end{aligned} \tag{7.32}$$

Noting that  $(\mathbf{z}, \hat{\mathbf{z}})$  solves the equation (5.2), we deduce that

$$\begin{aligned}
& \mathbb{E} \int_{Q_{\tau'}} \theta \left( -2 \ell_t \hat{v} + 2 \sum_{i=1}^n a^{jk} \ell_{x_j} v_{x_k} + \Psi v \right) \left[ d\hat{\mathbf{z}} - \sum_{j,k=1}^n (a^{jk} \mathbf{z}_{x_j})_{x_k} dt \right] dx \\
& = \mathbb{E} \int_{Q_{\tau'}} \theta \left( -2 \ell_t \hat{v} + 2 \sum_{i=1}^n a^{jk} \ell_{x_j} v_{x_k} + \Psi v \right) \left[ -a_1 \cdot \nabla \mathbf{z} + (-\operatorname{div} a_1 + a_2 - a_3 a_5) \mathbf{z} \right] dx dt \\
& \leq \mathbb{E} \int_{Q_{\tau'}} \theta \left( -2 \ell_t \hat{v} + 2 \sum_{i=1}^n a^{jk} \ell_{x_j} v_{x_k} + \Psi v \right)^2 dx dt + r_2 \mathbb{E} \int_{Q_{\tau'}} (|\nabla \mathbf{z}|^2 + \mathbf{z}^2) dx dt.
\end{aligned} \tag{7.33}$$

Combing (7.23), (7.26), (7.32) and (7.33), we conclude that there is a  $\lambda_3 \geq \max\{\lambda_2, Cr_5 + 1\}$  such that for any  $\lambda \geq \lambda_3$ , one has that

$$\begin{aligned} & \mathbb{E} \int_{\Lambda_1} \theta^2 (|\hat{\mathbf{z}}|^2 + |\nabla \mathbf{z}|^2) dxdt + \mathbb{E} \int_{\Lambda_1} \theta^2 |\mathbf{z}|^2 dxdt \\ & \leq C \left[ e^{\lambda R_0^2/3} \mathbb{E} \int_Q (|\hat{\mathbf{z}}|^2 + |\nabla \mathbf{z}|^2) dxdt + e^{\lambda R_0^2/3} \mathbb{E} \int_Q |\mathbf{z}|^2 dxdt + e^{\lambda R_1^2} \mathbb{E} \int_{\Sigma_0} \left| \frac{\partial \mathbf{z}}{\partial \nu} \right|^2 d\Sigma \right. \\ & \quad \left. + \mathbb{E} \int_G (\hat{\mathbf{z}}(\tau)^2 + |\nabla \mathbf{z}(\tau)|^2 + \mathbf{z}(\tau)^2 + \hat{\mathbf{z}}(\tau')^2 + |\nabla \mathbf{z}(\tau')|^2 + \mathbf{z}(\tau')^2) dx \right]. \end{aligned} \quad (7.34)$$

Integrating (7.34) with respect to  $\tau$  and  $\tau'$  on  $[T_2, T_1]$  and  $[T'_1, T'_2]$ , respectively, we get that

$$\begin{aligned} & \mathbb{E} \int_{\Lambda_1} \theta^2 (|\hat{\mathbf{z}}|^2 + |\nabla \mathbf{z}|^2) dxdt + \mathbb{E} \int_{\Lambda_1} \theta^2 |\mathbf{z}|^2 dxdt \\ & \leq C e^{\lambda R_0^2/3} \mathbb{E} \int_Q (|\hat{\mathbf{z}}|^2 + |\nabla \mathbf{z}|^2) dxdt + C e^{\lambda R_0^2/3} \mathbb{E} \int_Q |\mathbf{z}|^2 dxdt \\ & \quad + C e^{\lambda R_1^2} \mathbb{E} \int_{\Sigma_0} \left| \frac{\partial \mathbf{z}}{\partial \nu} \right|^2 d\Sigma + C \mathbb{E} \int_Q (\hat{\mathbf{z}}^2 + |\nabla \mathbf{z}|^2 + \mathbf{z}^2) dxdt. \end{aligned} \quad (7.35)$$

From (7.3), we obtain that

$$\begin{aligned} \mathbb{E} \int_{\Lambda_1} \theta^2 (|\hat{\mathbf{z}}|^2 + |\nabla \mathbf{z}|^2 + |\mathbf{z}|^2) dxdt & \geq \mathbb{E} \int_{\Lambda_0} \theta^2 (|\hat{\mathbf{z}}|^2 + |\nabla \mathbf{z}|^2 + |\mathbf{z}|^2) dxdt \\ & \geq e^{\lambda R_0^2/2} \mathbb{E} \int_{Q_0} (|\hat{\mathbf{z}}|^2 + |\nabla \mathbf{z}|^2 + |\mathbf{z}|^2) dxdt. \end{aligned} \quad (7.36)$$

Combing (7.35) and (7.36), we arrive at

$$\begin{aligned} & \mathbb{E} \int_{Q_0} (|\hat{\mathbf{z}}|^2 + |\nabla \mathbf{z}|^2 + |\mathbf{z}|^2) dxdt \\ & \leq C \left[ e^{-\lambda R_0^2/6} \mathbb{E} \int_Q (|\hat{\mathbf{z}}|^2 + |\nabla \mathbf{z}|^2 + |\mathbf{z}|^2) dxdt + \lambda^2 e^{-\lambda R_0^2/6} \mathbb{E} \int_Q |\mathbf{z}|^2 dxdt \right. \\ & \quad \left. + e^{\lambda R_1^2} \mathbb{E} \int_{\Sigma_0} \left| \frac{\partial \mathbf{z}}{\partial \nu} \right|^2 d\Sigma + \mathbb{E} e^{-\lambda R_0^2/2} \int_Q (\hat{\mathbf{z}}^2 + |\nabla \mathbf{z}|^2 + \mathbf{z}^2) dxdt \right]. \end{aligned} \quad (7.37)$$

By standard energy estimate of the equation (5.2), we have that

$$|(\mathbf{z}_0, \hat{\mathbf{z}}_0)|_{H_0^1(G) \times L^2(G)}^2 \leq C e^{Cr_4} \mathbb{E} \int_{Q_0} (|\hat{\mathbf{z}}|^2 + |\nabla \mathbf{z}|^2 + |\mathbf{z}|^2) dxdt \quad (7.38)$$

and that

$$|(\mathbf{z}_0, \hat{\mathbf{z}}_0)|_{H_0^1(G) \times L^2(G)}^2 \geq C e^{-Cr_4} \mathbb{E} \int_{Q_0} (|\hat{\mathbf{z}}|^2 + |\nabla \mathbf{z}|^2 + |\mathbf{z}|^2) dxdt. \quad (7.39)$$

It follows from (7.37)–(7.39) that

$$|(\mathbf{z}_0, \hat{\mathbf{z}}_0)|_{H_0^1(G) \times L^2(G)}^2 \leq C e^{Cr_4} e^{-\lambda R_0^2/6} |(\mathbf{z}_0, \hat{\mathbf{z}}_0)|_{H_0^1(G) \times L^2(G)}^2 + C e^{\lambda R_1^2} \mathbb{E} \int_{\Sigma_0} \left| \frac{\partial \mathbf{z}}{\partial \nu} \right|^2 d\Sigma. \quad (7.40)$$

Let us choose  $\lambda_4 \geq \lambda_3$  such that  $C e^{Cr_4} e^{-\lambda_4 R_0^2/6} < 1$ . Then, for all  $\lambda \geq \lambda_4$ , we have that

$$|(\mathbf{z}_0, \hat{\mathbf{z}}_0)|_{H_0^1(G) \times L^2(G)}^2 \leq C e^{\lambda R_1^2} \mathbb{E} \int_{\Sigma_0} \left| \frac{\partial \mathbf{z}}{\partial \nu} \right|^2 d\Sigma. \quad (7.41)$$

This leads to the inequality (7.1) immediately.  $\square$

**Remark 7.1** It follows from Proposition 5.1 that

$$\left| \frac{\partial \mathbf{z}}{\partial \nu} \right|_{L^2_{\mathbb{F}}(0,T;L^2(\Gamma_0))} \leq C(|\mathbf{z}_0|_{H^1_0(G)} + |\hat{\mathbf{z}}_0|_{L^2(G)}).$$

This, together with Theorem 7.1, implies that

$$\frac{1}{C}(|\mathbf{z}_0|_{H^1_0(G)} + |\hat{\mathbf{z}}_0|_{L^2(G)}) \leq \left| \frac{\partial \mathbf{z}}{\partial \nu} \right|_{L^2_{\mathbb{F}}(0,T;L^2(\Gamma_0))} \leq C(|\mathbf{z}_0|_{H^1_0(G)} + |\hat{\mathbf{z}}_0|_{L^2(G)}).$$

Therefore, we can define a new norm  $\|\cdot\|$  on  $H^1_0(G) \times L^2(G)$ , which is equivalent to the norm  $|\cdot|_{H^1_0(G) \times L^2(G)}$  as follows:

$$\|(\xi, \eta)\| = \left| \frac{\partial \mathbf{z}}{\partial \nu} \right|_{L^2_{\mathbb{F}}(0,T;L^2(\Gamma_0))} \quad \text{for } (\xi, \eta) \in H^1_0(G) \times L^2(G),$$

where  $(\mathbf{z}, \hat{\mathbf{z}})$  is the solution to (5.2) and  $(\mathbf{z}_0, \hat{\mathbf{z}}_0) = (\xi, \eta)$ . This is the starting point of the duality argument to the proof of the controllability result via the observability estimate (See [12] for the details for wave equations).

## 8 Proofs of the main results

This section is addressed to proving our main results in this paper, i.e., Theorems 2.1–2.3.

*Proof of Theorem 2.2:* It follows from Propositions 5.5 and 5.6, and Theorem 7.1 immediately.  $\square$

Before proving Theorems 2.1 and 2.3, we recall the following known result ([24, Lemma 2.1]).

**Lemma 8.1** *There is a random variable  $\xi \in L^2_{\mathcal{F}_T}(\Omega)$  such that it is impossible to find  $(\varrho_1, \varrho_2) \in L^2_{\mathbb{F}}(0, T) \times C_{\mathbb{F}}([0, T]; L^2(\Omega))$  and  $\alpha \in \mathbb{R}$  satisfying*

$$\xi = \alpha + \int_0^T \varrho_1(t) dt + \int_0^T \varrho_2(t) dW(t).$$

We are now in a position to prove Theorems 2.1 and 2.3.

*Proof of Theorem 2.1:* We use the contradiction argument. Choose  $\psi \in H^1_0(G)$  satisfying  $|\psi|_{L^2(G)} = 1$  and let  $\tilde{y}_0 = \xi\psi$ , where  $\xi$  is given in Lemma 8.1. Assume that (1.1) was exactly controllable. Then, for any  $y_0 \in L^2(G)$ , we would find a triple of controls  $(g_1, g_2, h) \in L^2_{\mathbb{F}}(0, T; H^{-1}(G)) \times L^2_{\mathbb{F}}(0, T; H^{-1}(G)) \times L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0))$  such that the corresponding solution  $y \in C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G))) \cap C^1_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-1}(G)))$  to the equation (1.1) satisfies that  $y(T) = \tilde{y}_0$ . Clearly,

$$\int_G \tilde{y}_0 \psi dx - \int_G y_0 \psi dx = \int_0^T \langle y_t, \psi \rangle_{H^{-1}(G), H^1_0(G)} dt,$$

which leads to

$$\xi = \int_G y_0 \psi dx + \int_0^T \langle y_t, \psi \rangle_{H^{-1}(G), H^1_0(G)} dt.$$

This contradicts Lemma 8.1.  $\square$

*Proof of Theorem 2.3:* Let us employ the contradiction argument, and divide the proof into three cases.

**Case 1)**  $a_4 \in C_{\mathbb{F}}([0, T]; L^\infty(\Omega))$  and  $f$  is supported in  $G_0$ . Since  $G_0 \subset G$  is an open subset and  $G \setminus \overline{G_0} \neq \emptyset$ , we can find a  $\rho \in C_0^\infty(G \setminus G_0)$  satisfying  $|\rho|_{L^2(G)} = 1$ .

Assume that (1.10) was exactly controllable. Then, for  $(y_0, \hat{y}_0) = (0, 0)$ , one could find controls  $(f, g, h) \in L_{\mathbb{F}}^2(0, T; L^2(G)) \times L_{\mathbb{F}}^2(0, T; H^{-1}(G)) \times L_{\mathbb{F}}^2(0, T; L^2(\Gamma_0))$  with  $\text{supp } f \subset G_0$ , a.e.  $(t, \omega) \in (0, T) \times \Omega$  such that the corresponding solution to (1.10) fulfills  $(y(T), \hat{y}(T)) = (\rho\xi, 0)$ , where  $\xi$  is given in Lemma 8.1. Thus,

$$\rho\xi = \int_0^T \hat{y} dt + \int_0^T (a_4 y + f) dW(t). \quad (8.1)$$

Multiplying both sides of (8.1) by  $\rho$  and integrating it in  $G$ , we get that

$$\xi = \int_0^T \langle \hat{y}, \rho \rangle_{H^{-1}(G), H_0^1(G)} dt + \int_0^T \langle a_4 y, \rho \rangle_{L^2(G)} dW(t). \quad (8.2)$$

Since the pair  $(y, \hat{y}) \in C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G))) \times C_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-1}(G)))$  solves (1.10), then  $\langle \hat{y}, \rho \rangle_{H^{-1}(G), H_0^1(G)} \in C_{\mathbb{F}}([0, T]; L^2(\Omega))$  and  $\langle y, \rho \rangle_{L^2(G)} \in C_{\mathbb{F}}([0, T]; L^2(\Omega))$ , which, together with (8.2), contradicts Lemma 8.1.

**Case 2)**  $a_3 \in C_{\mathbb{F}}([0, T]; L^\infty(\Omega))$  and  $g$  is supported in  $G_0$ . Choose  $\rho$  as in Case 1).

If (1.10) was exactly controllable, then, for  $(y_0, \hat{y}_0) = (0, 0)$ , one can find controls  $(f, g, h) \in L_{\mathbb{F}}^2(0, T; L^2(G)) \times L_{\mathbb{F}}^2(0, T; H^{-1}(G)) \times L_{\mathbb{F}}^2(0, T; L^2(\Gamma_0))$  with  $\text{supp } g \subset G_0$ , a.e.  $(t, \omega) \in (0, T) \times \Omega$  such that the corresponding solution of (1.10) fulfills  $(y(T), \hat{y}(T)) = (0, \xi)$ .

It is clear that  $(\phi, \hat{\psi}) = (\rho y, \rho \hat{y})$  solves the following equation:

$$\left\{ \begin{array}{ll} d\phi = \hat{\phi} dt + (a_4 \phi + \rho f) dW(t) & \text{in } Q, \\ d\hat{\phi} - \sum_{j,k=1}^n (a^{jk} \phi_{x_j})_{x_k} dt = \zeta dt + a_3 \phi dW(t) & \text{in } Q, \\ \phi = 0 & \text{on } \Sigma, \\ \hat{\phi} = 0 & \text{on } \Sigma, \\ \phi(0) = 0, \quad \hat{\phi}(0) = 0 & \text{in } G, \end{array} \right. \quad (8.3)$$

where  $\zeta = \sum_{j,k=1}^n [(a^{jk} \rho_{x_j} y)_{x_k} + a^{jk} y_{x_j} \rho_{x_k}] + \rho a_1 \cdot \nabla y + \rho a_2 y$ . Further, we have  $\phi(T) = 0$  and  $\hat{\phi}(T) = \rho\xi$ .

Noting that  $(\phi, \hat{\phi})$  is the weak solution to (8.3), we see that

$$\begin{aligned} & \langle \rho\xi, \rho \rangle_{H^{-2}(G), H_0^2(G)} \\ &= \int_0^T \left[ \left\langle \sum_{j,k=1}^n (a^{jk} \phi_{x_j})_{x_k}, \rho \right\rangle_{H^{-2}(G), H_0^2(G)} + \langle \zeta, \rho \rangle_{H^{-1}(G), H_0^1(G)} \right] dt + \int_0^T \langle a_3 \phi, \rho \rangle_{L^2(G)} dW(t), \end{aligned}$$

which implies that

$$\xi = \int_0^T \left[ \left\langle \sum_{j,k=1}^n (a^{jk} \phi_{x_j})_{x_k}, \rho \right\rangle_{H^{-2}(G), H_0^2(G)} + \langle \zeta, \rho \rangle_{H^{-1}(G), H_0^1(G)} \right] dt + \int_0^T \langle a_3 \phi, \rho \rangle_{L^2(G)} dW(t). \quad (8.4)$$

Since  $(\phi, \hat{\phi}) \in C_{\mathbb{F}}([0, T]; L^2(\Omega; L^2(G))) \times C_{\mathbb{F}}([0, T]; L^2(\Omega; H^{-1}(G)))$ , then

$$\left\langle \sum_{j,k=1}^n (a^{jk} \phi_{x_j})_{x_k}, \rho \right\rangle_{H^{-2}(G), H_0^2(G)} + \langle \zeta, \rho \rangle_{H^{-1}(G), H_0^1(G)} \in L_{\mathbb{F}}^2(0, T)$$

and

$$\langle a_3\phi, \rho \rangle_{L^2(G)} \in C_{\mathbb{F}}([0, T]; L^2(\Omega)).$$

These, together with (8.4), contradict Lemma 8.1.

**Case 3)**  $h = 0$ . Assume that the system (1.10) was exactly controllable. Similar to the proof of Proposition 5.6, we could deduce that, for any  $(z^T, \hat{z}^T) \in L^2_{\mathcal{F}_T}(\Omega; H_0^1(G)) \times L^2_{\mathcal{F}_T}(\Omega; L^2(G))$ , the solution  $(z, Z, \hat{z}, \hat{Z})$  to (3.1) (with  $\tau = 0$  and  $(z(T), \hat{z}(T)) = (z^T, \hat{z}^T)$ ) satisfies

$$|(z^T, \hat{z}^T)|_{L^2_{\mathcal{F}_T}(\Omega; H_0^1(G)) \times L^2_{\mathcal{F}_T}(\Omega; L^2(G))} \leq C \left( |Z|_{L^2_{\mathbb{F}}(0, T; H_0^1(G))} + |\hat{Z}|_{L^2_{\mathbb{F}}(0, T; L^2(G))} \right). \quad (8.5)$$

For any nonzero  $(\eta_0, \eta_1) \in H_0^1(G) \times L^2(G)$ , let us consider the following random wave equation:

$$\begin{cases} \eta_{tt} - \sum_{j,k=1}^n (a^{jk} \eta_{x_j})_{x_k} dt = b_1 \cdot \nabla \eta + b_2 \eta & \text{in } (0, T) \times G, \\ z = 0 & \text{on } (0, \tau) \times \Gamma, \\ \eta(0) = \eta_0, \quad \eta_t(0) = \eta_1 & \text{in } G. \end{cases} \quad (8.6)$$

Clearly,  $(\eta, 0, \eta_t, 0)$  solves (3.1) with the final datum  $(z^T, \hat{z}^T) = (\eta(T), \hat{\eta}(T))$ , a contradiction to the inequality (8.5).  $\square$

## 9 Further comments and open problems

In this paper, we obtain the exact controllability of the system (1.10) with one boundary control and two internal controls. It is natural to consider the exact controllability problem for stochastic wave-like equations with three internal controls:

$$\begin{cases} dy = \hat{y} dt + (a_4 y + f) dW(t) & \text{in } Q, \\ d\hat{y} - \sum_{j,k=1}^n (a^{jk} y_{x_j})_{x_k} dt = (a_1 \cdot \nabla y + a_2 y + a_5 g + \chi_{G_0} h) dt + (a_3 y + g) dW(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0, \quad \hat{y}(0) = \hat{y}_0 & \text{in } G. \end{cases}$$

Here

$$G_0 \triangleq \{x \in G \mid \text{dist}(x, \Gamma_0) \leq \delta\}$$

for a  $\delta > 0$ ,  $(y_0, \hat{y}_0) \in H_0^1(G) \times L^2(G)$ ,  $f \in L^2_{\mathbb{F}}(0, T; H_0^1(G))$ ,  $g \in L^2_{\mathbb{F}}(0, T; L^2(G))$  and  $h \in L^2_{\mathbb{F}}(0, T; L^2(G_0))$  are the controls.

By a duality argument, one only need to show the following observability estimate:

$$\begin{aligned} & |(z^T, \hat{z}^T)|_{L^2_{\mathcal{F}_T}(\Omega; L^2(G)) \times L^2_{\mathcal{F}_T}(\Omega; H^{-1}(G))} \\ & \leq C \left( |\chi_{G_0} z|_{L^2_{\mathbb{F}}(0, T; L^2(\Gamma_0))} + |Z|_{L^2_{\mathbb{F}}(0, T; L^2(G))} + |\hat{Z}|_{L^2_{\mathbb{F}}(0, T; H^{-1}(G))} \right), \end{aligned} \quad (9.1)$$

where  $(z, Z, \hat{z}, \hat{Z})$  solves (3.1) with  $\tau = T$  and the final datum  $(z^T, \hat{z}^T)$ . Following [9], we can prove that (9.1) holds. Details are too lengthy to be presented here.

There are many open problems related to the topic of this paper. We shall list below some of them which, in our opinion, are particularly interesting:



- **Null controllability for stochastic wave-like equations**

In this paper, the exact controllability for stochastic wave-like equations are presented. As immediate consequences, we can obtain the null and approximate controllability for the same system. However, in order to show these two results, there seems no reason to use three controls. By Theorem 2.3, it is shown that only one control applied in the diffusion term is not enough. However, inspired by the result in [15], we believe that one boundary control is enough for the null and approximate controllability of (1.1) and (1.10). Unfortunately, some essential difficulties appear when we try to prove it, following the method in the present paper. For example, for the null controllability, we should prove the following inequality for the solution to (3.1):

$$|z(0)|_{H_0^1(G)}^2 + |\hat{z}(0)|_{L^2(G)}^2 \leq C \int_0^T \int_{\Gamma_0} \left| \frac{\partial z}{\partial \nu} \right|^2 d\Gamma dt.$$

However, if we utilize the method in this paper, we only get

$$|z(0)|_{H_0^1(G)}^2 + |\hat{z}(0)|_{L^2(G)}^2 \leq C \left( \int_0^T \int_{\Gamma_0} \left| \frac{\partial z}{\partial \nu} \right|^2 d\Gamma dt + \int_0^T |Z|_{H_0^1(G)}^2 + \int_0^T |\hat{Z}|_{L^2(G)}^2 dt \right).$$

There are two additional terms containing  $Z$  and  $\hat{Z}$  in the right hand side of the above inequality. These terms come from the fact that, in the Carleman estimate, we regard  $Z$  and  $\hat{Z}$  simply as nonhomogeneous terms rather than part of the solution. Therefore, we believe that one should introduce some new technique, for example, a Carleman estimate in which the fact that  $Z$  and  $\hat{Z}$  are part of the solution is essentially used, to get rid of the additional terms containing  $Z$  and  $\hat{Z}$ . However, we do not know how to achieve this goal at this moment.

- **Exact controllability for stochastic wave-like equations with less restrictive condition**

In this paper, we get the exact controllability of the system (1.10) for  $\Gamma_0$  given by (2.5). It is well known that a sharp sufficient condition for exact controllability of deterministic wave equations with time invariant coefficients is that the triple  $(G, \Gamma_0, T)$  satisfies the Geometric Control Condition introduced in [4]. It would be quite interesting and challenging to extend this result to the stochastic setting, but it seems that there are lots of things to be done before solving this problem. Indeed, the propagation of singularities for stochastic partial differential equations, at least, for stochastic hyperbolic equations, should be established. However, as far as we know, this topic is completely open.

- **Exact controllability for stochastic wave-like equations with more regular controls**

In this paper, we get the exact controllability of the system (1.10) a triple  $(f, g, h)$ , where  $g \in L_{\mathbb{F}}^2(0, T; H^{-1}(G))$ , which is very irregular. It is very interesting to see whether (1.10) is exactly controllable when  $g \in L_{\mathbb{F}}^2(0, T; L^2(G))$ . By duality argument, one can show that this is equivalent to the following observability estimate:

$$\begin{aligned} & |(z^T, \hat{z}^T)|_{L_{\mathcal{F}_T}^2(\Omega; H_0^1(G)) \times L_{\mathcal{F}_T}^2(\Omega; L^2(G))} \\ & \leq C \left( \left| \frac{\partial z}{\partial \nu} \right|_{L_{\mathbb{F}}^2(0, T; L^2(\Gamma_0))} + |Z|_{L_{\mathbb{F}}^2(0, T; L^2(G))} + |\hat{Z}|_{L_{\mathbb{F}}^2(0, T; L^2(G))} \right), \end{aligned} \tag{9.2}$$

where  $(z, Z, \hat{z}, \hat{Z})$  is the solution to (3.1) with  $\tau = T$  and final datum  $(z^T, \hat{z}^T)$ . By Lemma 6.1, we can prove that the inequality (9.2) holds if the term  $|Z|_{L_{\mathbb{F}}^2(0, T; L^2(G))}$  is replaced by  $|Z|_{L_{\mathbb{F}}^2(0, T; H_0^1(G))}$ . However, we do not know whether (9.2) is true or not.

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