Cooperative Source Seeking via Networked Multi-vehicle Systems *

Zhuo Li, Keyou You^{*}, Shiji Song

Department of Automation, and BNRist, Tsinghua University, Beijing, 100084, China.

arXiv:1901.07716v2 [eess.SP] 10 Jan 2020

Abstract

This paper studies the cooperative source seeking problem via a networked multi-vehicle system. In contrast to existing literature, the multi-vehicle system is controlled to the source position that maximizes aggregated *multiple* unknown scalar fields and each sensor-enabled vehicle only samples measurements of *one* scalar field. Thus, a single vehicle is unable to localize the source and has to cooperate with its neighboring vehicles. By jointly exploiting the ideas of the consensus algorithm and the stochastic extremum seeking (ES), this paper proposes novel distributed stochastic ES controllers, which are gradient-free and do not need any absolute information, such that the multi-vehicle system simultaneously approaches the source position. The effectiveness of the proposed controllers is proved for quadratic scalar fields. Finally, illustrative examples are included to validate the theoretical results.

Key words: Cooperative source seeking, scalar field, multi-vehicle systems, consensus algorithms, stochastic ES.

1 Introduction

This paper is concerned with the design of distributed controllers to drive a multi-vehicle system to approach a source position of interest, which has great significance in various applications, such as environmental monitoring (Dhariwal et al. 2004), odor source detection (Gao, Acar & Sarangapani 2016), acoustic source localization (Zhao 2016) and pollution sensing (Gao, Li, Li & Sun 2016). Consider the problem of seeking an indoor fire source, where there are multiple indoor positions having either the highest temperature or the highest toxic gas concentration but only the position of the fire source attains the highest values of the both fields. Consequently, it is unable to localize the fire source by sensing only one of the two scalar fields. There are also examples that the position of interest may not be a maximum of any sensed scalar field. Based on these observations, we are interested in the complex environment where the source position maximizes the aggregated multiple unknown scalar *fields*, which is different from existing works exploring only one scalar field (Zhang et al. 2016, Dürr et al. 2017, Lin et al. 2017).

Our first challenge lies in the unknown distribution of any scalar field, i.e., any sensor cannot measure a continuum of the scalar field. Hence the distributed optimization algorithms explicitly using gradients cannot be directly applied here (Wang & Elia 2010, Gharesifard & Cortés 2014, You et al. 2019). To solve it, we adopt the stochastic extremum seeking (ES) (Manzie & Krstic 2009) to estimate local gradients by using samples of the sensed scalar fields, which is completed in the associated vehicle by superimposing a stochastic excitation signal. Moreover, the ES method does not require absolute position information.

The second is that each vehicle of this work is only able to sense one scalar field. In this case, seeking the source position for multiple scalar fields needs multiple vehicles and their cooperation. Although the networked multi-vehicle system has been employed in Frihauf et al. (2014), Khong et al. (2014), Brinon-Arranz et al. (2016), Turgeman & Werner (2018), the seeking position therein is the source of only one scalar field and all the vehicles take samples from the same field. Thus, the cooperation in their works is not indispensable.

By jointly using stochastic ES and the cooperation

^{*} This research was supported in part by the National Key Research and Development Program of China under Grant No.2016YFC0300801, and National Natural Science Foundation of China under Grants No.41576101 and No.41427806. * Corresponding author

^{*} Corresponding author

Email addresses: lizhuo16@mails.tsinghua.edu.cn (Zhuo Li), youky@tsinghua.edu.cn (Keyou You), shijis@tsinghua.edu.cn (Shiji Song).

among vehicles, this work proposes distributed stochastic ES (DSES) controllers to drive all vehicles to simultaneously approach the source position. In the literature, cooperative ES has been used for social games in Menon & Baras (2014), Dougherty & Guay (2017), Vandermeulen et al. (2018), Guay et al. (2018) and the resource allocation problem in Poveda & Quijano (2013). In their works, the objective of each agent is to reach the social equilibrium or compute its optimal resource via cooperation. Clearly, they cannot apply to our problem, since we require each vehicle to reach consensus at the same source position.

Our problem setting is closely related to Ye & Hu (2016), Kvaternik & Pavel (2012), Michalowsky et al. (2017). In Ye & Hu (2016), a consensus-based ES algorithm is developed to solve a saddle point problem. Since their algorithm does not involve the agent dynamics, it is unclear how to extend it to dynamical vehicle models, e.g., the unicycle model in Li et al. (2015). Moreover, the excitation of ES therein is based on deterministic signals, which should be orthogonal among agents. This renders it not as simple as using the stochastic signals for implementation in the multi-vehicle network. In Kvaternik & Pavel (2012), Michalowsky et al. (2017), authors show the existence of a distributed ES controller to find the position of interest in the deterministic regime, but do not provide an explicit ES controller.

We prove the effectiveness of DSES controllers for the quadratic scalar fields by the stochastic averaging theory, and conclude that the DSES controllers might also work for the non-quadratic case by simulations. A conference version of this work has been presented in Li et al. (2018) where the DSES controller is given for a special case that the position of interest simultaneously maximizes all the local scalar fields.

The rest of this paper is organized as follows. In Section 2, we describe the cooperative seeking problem by using a group of networked vehicles. In Section 3, we propose DSES controllers in both undirected and directed interaction graphs and prove their effectiveness by the stochastic averaging theory. Illustrative examples are provided in Section 4 and some remarks are drawn in Section 5.

Notation: Throughout this paper, any notation with a subscript *i* represents that of vehicle *i*, e.g., $x_i \in \mathbb{R}^m$, and $x = [x'_1, x'_2, \ldots, x'_n]' \in \mathbb{R}^{mn}$ for the networked multivehicle system. $O(\alpha)$ denotes the infinitesimal of the same order as a scalar α , i.e., $\lim_{\alpha \to 0} O(\alpha)/\alpha = c < \infty$ with $c \neq 0$. $\|\cdot\|$ denote the Euclidean norm for a vector or a matrix and \otimes denote the Kronecker product.

2 Problem Formulation

In this section, we explicitly describe our cooperative source seeking problem by using the networked multivehicle system, where each vehicle is embedded with only one sensor to measure the strength of one scalar field and has to cooperate with its neighboring vehicles.

2.1 The cooperative source seeking problem

There are *n* networked autonomous vehicles, each of which has only one sensor to measure the signal strength a scalar field $f_i(z) : \mathbb{R}^m \to \mathbb{R}, i = 1, 2, \dots n$ at the position $z \in \mathbb{R}^m$. The task of the multi-vehicle system is to autonomously approach the source position z^* that maximizes the sum of $f_i(z)$, i.e.,

$$z^* \in \arg\max_{z} F(z) := \sum_{i=1}^{n} f_i(z),$$
 (1)

where $\arg \max_{z} F(z)$ denotes the set of optimal points that maximize the aggregated multiple unknown scalar fields.

Since the value of $f_i(\cdot)$ is the *only* accessible information from the sensed scalar field for the *i*-th vehicle, each vehicle has to cooperate with others to complete the seeking task in (1). This problem setup is essentially motivated by two notable examples.

Example 1 Consider an indoor fire source seeking problem. The fire source z^* is the position of our interest, and is the unique point that simultaneously attains the highest temperature and toxic gas concentration, i.e.,

$$z^* \in \bigcap_{i=1}^2 \arg\max_z f_i(z), \tag{2}$$

where $f_1(z)$, $f_2(z)$ denote the temperature and the toxic gas concentration at the position $z \in \mathbb{R}^m$, respectively.

To approach the fire source z^* , there are two autonomous vehicles embedded with a temperature sensor and a gas sensor, respectively. Due to the complex sensing environment, it is possible that each $f_i(z)$ contains multiple or an infinite number of maximum points. Then, any vehicle cannot guarantee to exactly find the fire source z^* and has to cooperate with others. One can easily show that the multi-objective problem (2) is a special case of the cooperative seeking problem (1).

Example 2 Consider the following dynamical process

$$z(k+1) = Az(k)$$

where $A \in \mathbb{R}^{m \times m}$ and $z(k) \in \mathbb{R}^m$ denote the transition matrix and the state at time step k, respectively. Our objective is to recover the initial state $z(t_0)$ by using measurements from multiple sensors.

At each time step k, the *i*-th vehicle is able to take the measurement

$$m_i(k) = C_i z(k),$$

where the system (C_i, A) is not observable for any *i*, *i.e.*, the observability Gramian $\Phi_i := \sum_{k=0}^{m-1} (A')^k C'_i C_i A^k$ is rank deficient (Chen 1998). Therefore, it is impossible to recover the initial state $z(t_0)$ by only using measurements of any single vehicle. Now, suppose that the multi-vehicle system is jointly observable, *i.e.* the system (C, A) is observable with $C = (C'_1, \ldots, C'_n)'$. Then, the multi-vehicle system is able to cooperatively complete the recovering task.

To elaborate it, define the local objective function as

$$f_i(z) = -\sum_{k=0}^{m-1} \|m_i(k) - C_i A^k z\|^2.$$
(3)

It is easy to show that $\arg \max_z f_i(z) = z(t_0) + Null(\Phi_i)$ where $Null(\Phi_i)$ denotes the null space of Φ_i . Since (C_i, A) is not observable, then $Null(\Phi_i)$ is a non-trivial subspace of \mathbb{R}^m . Therefore, $z(t_0)$ is not the unique element of $\arg \max_z f_i(z)$. That is, $z(t_0)$ is unable to be recovered by only using the *i*-th vehicle's measurements. Similarly, we can show that $\arg \max_z F(z) = z(t_0) + Null(\sum_{i=1}^n \Phi_i)$. Since (C, A) is observable, then $\sum_{i=1}^n \Phi_i$ is non-singular, which in turn implies that $\arg \max_z F(z) = z(t_0)$. That is, $z(t_0)$ is the unique element of $\arg \max_z F(z) = z(t_0)$. That is able to be recovered by solving the cooperative seeking problem (1) with $f_i(z)$ given in (3).

In the above examples, each set of *local* optimal points arg max_z $f_i(z)$ may contain multiple elements, and we are only interested in the one lying in their intersection, which clearly maximizes F(z) the sum of all the local objective functions. It should be noted that the cooperative seeking problem (1) also includes the case where the maximum point of F(z), i.e., the source position z^* , may not maximize any $f_i(z)$. In both cases, a local objective function $f_i(z)$ can only offer limited information on the source position z^* , the localization of which obviously requires the cooperation among vehicles.

2.2 Networked multi-vehicle systems

The interactions (cooperations) between vehicles are modeled by a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the index set of nodes (vehicles) and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of the interaction edges between vehicles. Node i can measure its relative position to that of node j if and only if $(i, j) \in \mathcal{E}$. The set of neighbors of node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. A path from node j_0 to node j_k is a set of distinct nodes $\{j_0, j_1, \ldots, j_k\}$ such that $(j_{i-1}, j_i) \in \mathcal{E}$ for all $i \in \{1, 2, \ldots, k\}$. If any two nodes can be connected via a path, then \mathcal{G} is strongly connected. Let the adjacency matrix $[a_{ij}]_{n \times n}$ be defined such that $a_{ij} > 0$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The associated Laplacian matrix is $L = [l_{ij}]_{n \times n}$ where

 $l_{ii} = \sum_{j=1}^{n} a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$, and 1_n is the unique solution (within a multiplier) of Lx = 0 if \mathcal{G} is strongly connected (Ren & Beard 2008). If $L = L^T$, then \mathcal{G} is an undirected graph. Clearly, the connectivity of \mathcal{G} is necessary for our problem.

Assumption 3 G is strongly connected.

The networked n autonomous vehicles are modeled by single integrators

$$\dot{z}_i = u_i, i \in \mathcal{V},\tag{4}$$

where $z_i(t) \in \mathbb{R}^m$ and $u_i(t) \in \mathbb{R}^m$ represent the position and the control input of the *i*-th vehicle at time *t*, respectively. When it is clear from the context, we drop the dependence of the time index *t* for ease of notations.

2.3 The objective of this work

The objective of this work is to design distributed controllers for the networked multi-vehicle system to *simultaneously* approach the source position z^* of (1) under the following constraints:

- (a) Each vehicle *i* is only able to obtain the numerical value of $f_i(z_i)$ at its current position z_i .
- (b) Each vehicle *i* can only measure its relative positions to its neighbors and has no access to its absolute position in the global position system (GPS).

Under the first constraint, gradient-based methods cannot be directly applied to solve the cooperative source seeking problem (1). We adopt the stochastic ES method (Liu & Krstic 2010) to design a gradient-free controller for each vehicle *i*, which is rigorously proved for the quadratic $f_i(\cdot)$. Although the ES method has been widely applied to solve source seeking problems, the number of scalar fields is mostly restricted to one, i.e. n = 1 in (1). In these cases, a single vehicle is sufficient to complete the seeking task, and there is no need of vehicles' cooperation. In contrast, the cooperation among vehicles is indispensable in this work, and is achieved by using consensus algorithms (Ren & Beard 2008).

Under the second constraint, distributed controllers are designed by only using the relative positions to its neighbors, which is particularly useful in the GPS-denied environment, e.g., the indoor fire source seeking. This is essentially motivated by the observation that many sensors, e.g., the acoustic sensor and the vision sensor, can easily measure the relative positions between two vehicles, while it is difficult to obtain the vehicle's GPS information. From this point of view, our controller preserves the advantage of the ES method without using any absolute position information.

It is worth mentioning that if the vehicle can only take a noisy measurement $f_i(z) + \varepsilon_i(z)$ at the position z, where



Fig. 1. The DSES controller (5) for undirected graphs.

 $\varepsilon_i(z)$ is an additive white noise and is spatially independent, our major results still hold.

3 Distributed Stochastic ES Controller Design

In this section, we design distributed stochastic ES (DSES) controllers for the networked vehicles and prove that the multi-vehicle system converges to the source position z^* in (1). We first consider undirected graphs and then extend to directed graphs.

3.1 The DSES controller for undirected graphs

If \mathcal{G} is undirected, the DSES controller for the *i*-th autonomous vehicle is devised as

$$\begin{cases} u_i = \alpha \sum_{j \in \mathcal{N}_i} a_{ij}(z_j - z_i + v_j - v_i) \\ + \beta \sin(\eta_i) \Delta_i(z_i) + \gamma \operatorname{dsin}(\eta_i) / \operatorname{dt}, \quad (5) \\ \dot{v}_i = \sum_{j \in \mathcal{N}_i} a_{ij}(z_i - z_j), \end{cases}$$

where $\alpha, \beta, \gamma \in \mathbb{R}_+$ are positive control parameters, $v_i \in \mathbb{R}^m$, $\Delta_i(z_i)$ is the output of a washout filter $\frac{s}{s+h}$ under the input signal $f_i(z_i)$, and $\sin(\eta_i)$ is the sinusoid of a stochastic excitation signal $\eta_i \in \mathbb{R}^m$, which is the state output of the following diffusion process

$$\mathrm{d}\eta_i = -\frac{1}{\epsilon}\eta_i \mathrm{d}t + \frac{g}{\sqrt{\epsilon}}\mathrm{d}w_i,\tag{6}$$

where $\epsilon \in (0, \epsilon_0]$ is a given parameter for some fixed $\epsilon_0 > 0, g$ is any positive parameter, and w_i is a standard m-dimensional Brownian motion (Øksendal 2003). Note that w_i is generated independently of w_j for any $j \neq i$. The last term $d \sin(\eta_i)/dt$ is the Ito derivative of $\sin(\eta_i)$ to persistently excite the system. In Manzie & Krstic (2009), Frihauf et al. (2014), the selection of γ, h, ϵ_0, g is discussed for the standard ES method and is not repeated here. See Fig. 1 for the DSES controller of each vehicle.

The cooperation among vehicles is exploited in the first term of u_i in (5), where $\sum_{j \in \mathcal{N}_i} a_{ij}(z_i - z_j)$ is also known as the consensus term (Ren & Beard 2008) and v_i is its

integration. This term only relies on the relative positions of the *i*-th vehicle to its neighbors and is to coordinate vehicles. Thus, each vehicle utilizes not only its own local measurements, but also its neighbors' trajectories, which is exactly the advantage of cooperation. This idea is significantly different from Vandermeulen et al. (2018), Dougherty & Guay (2017), Guay et al. (2018) where a consensus algorithm is used to estimate the sum of local objective functions $f_i(\cdot)$.

To implicitly estimate the gradient of $f_i(\cdot)$, we adopt the stochastic ES technique in Liu & Krstic (2010). The last two terms of u_i in (5) are the approximated gradient and the stochastic excitation signal, respectively. Note that it is always difficult for the deterministic ES to satisfy orthogonality requirements for a large number of networked vehicles. Excitation signals using the Brownian motion w_i for the stochastic ES are much easier to implement as we only require their independence.

Overall, the DSES controller (5) jointly exploits the ideas from the consensus algorithm and the stochastic ES technique. From this perspective, the strictly positive parameters α and β balance the importance of the consensus and the stochastic ES terms. Specifically, if α is relatively large, the vehicles tend to reach consensus faster, otherwise they tend to be attracted to their own individual sets of optimal points, i.e., $\arg \max_z f_i(z)$. This has been validated in the simulations. Roughly speaking, α can be interpreted as the rate of learning neighboring vehicles' behaviors and β is the rate of learning its local objective function. To localize the source position z^* , both rates are essential and cannot be neglected.

3.2 Convergence analysis

We establish the convergence of the networked n autonomous vehicles (4) with the DSES controller (5) under a similar assumption as Liu & Krstic (2010).

Assumption 4 The objective function $f_i(z)$ in (1) is quadratic¹, i.e.

$$f_i(z) = -\frac{1}{2}z'H_i z + b'_i z + c_i, \forall i \in \mathcal{V},$$
(7)

where $H_i \in \mathbb{R}^{m \times m}$ is positive semi-definite. Moreover, $\sum_{i=1}^{n} H_i$ is strictly positive definite.

Though our theoretical result is established for the quadratic case, simulations in Section 4 indicate the applicability of the proposed DSES controller to non-quadratic cases.

 $^{^1\,}$ For the non-quadratic case, it serves as a local quadratic approximation and the convergence results hold in the local sense.

Let $e_i = f_i(z_i) - \Delta_i(z_i) - f_i(z^*)$. Inserting the DSES controller (5) to the vehicle's dynamical equation (4) leads to the closed-loop system

$$\begin{cases} dz_i = \alpha \sum_{j \in \mathcal{N}_i} a_{ij}(z_j - z_i + v_j - v_i) dt \\ + \beta \sin(\eta_i) \Delta_i dt + \gamma d\sin(\eta_i), \\ dv_i = \sum_{j \in \mathcal{N}_i} a_{ij}(z_i - z_j) dt, \\ de_i = h \Delta_i dt, \end{cases}$$
(8)

where we adopt Δ_i to denote $\Delta_i(z_i)$ for notational simplicity.

Proposition 5 Given an undirected graph \mathcal{G} , consider the networked n autonomous vehicles (4) under the DSES controller (5). Let $\tilde{z}_i(t) = z_i(t) - \gamma \sin(\eta_i) - z^*$ where z^* is the source position defined in (1) and suppose that Assumptions 3 and 4 hold. Then, there exists a positive constant ρ_1 and a function $T_1(\epsilon) : (0, \epsilon_0) \to \mathbb{N}$ such that for any $\delta > 0$ and bounded initial condition (i.e. $\|z_i(t_0)\| < +\infty$, $\|v_i(t_0)\| < +\infty$ for all $i \in \mathcal{V}$), it holds that $\forall i \in \mathcal{V}$,

 $\lim_{\epsilon \to 0} \mathcal{P}\left\{\|\tilde{z}_i(t)\| \le \rho_1 \exp(-\lambda_1 t) + \delta, \forall t \in [t_0, T_1(\epsilon)]\right\} = 1,$ (9)

where $\lim_{\epsilon \to 0} T_1(\epsilon) = +\infty$ and $\lambda_1 > 0$ is the smallest eigenvalue of the following positive definite matrix

$$M := lpha(L \otimes I_m) + \kappa eta H^d$$

with $\kappa = rac{\gamma}{2}(1 - \exp(-g^2))$ and $H^d = diag(H_1, \dots, H_n)$

By (9), the DSES controller is able to drive the multivehicle system to the neighborhood of the source position z^* with an error size $O(\gamma) + O(\delta)$ in probability. Since δ is an arbitrarily small constant, it follows that the parameter γ controls the distance of the vehicle's final position to the source position z^* .

The exponential convergence in Liu & Krstic (2010) for a single field still holds in the present problem. The major difference is that the rate here also depends on the interaction graph among vehicles. In view of the exponent λ_1 , we conclude that the larger the control parameters α, β, γ and g, the faster the convergence rate of the position error $\tilde{z}_i(t)$ in the continuous-time regime. However, this does not apply to the discretized system in application. Particularly, if the control parameter α, β, γ or g is too large, it might lead to divergence.

The argument ϵ in $T_1(\epsilon)$ is exactly the parameter ϵ in the stochastic excitation signal η_i . That is, the convergence depends on the stochastic excitation signal. A sufficiently small ϵ is required to guarantee the reliability of the convergence.

Proof of Proposition 5: By the dynamical equation in (8), we obtain the following dynamics²

$$\begin{cases} \dot{\tilde{z}}_{i} = \alpha \sum_{j \in \mathcal{N}_{i}} a_{ij} (\tilde{z}_{j} + \gamma \sin(\eta_{j}) - \tilde{z}_{i} - \gamma \sin(\eta_{i}) \\ + v_{j} - v_{i}) + \beta \sin(\eta_{i}) \Delta_{i}, \\ \dot{v}_{i} = \sum_{j \in \mathcal{N}_{i}} a_{ij} (\tilde{z}_{i} + \gamma \sin(\eta_{i}) - \tilde{z}_{j} - \gamma \sin(\eta_{j})), \\ \dot{e}_{i} = h \Delta_{i}. \end{cases}$$
(10)

The proof is completed via three steps.

Step 1: Stochastic average system of (10).

Let the excitation signal $\eta_i(t) = \chi_i(t/\epsilon)$ and substitute it into (10). The average of $\dot{\tilde{z}}_i$ is defined as

$$\dot{\tilde{z}}_{i}^{a} = \lim_{T \to +\infty} \frac{1}{T} \int_{t_{0}}^{T+t_{0}} \alpha \sum_{j \in \mathcal{N}_{i}} a_{ij} (\tilde{z}_{j} + \gamma \sin(\chi_{j}) - \tilde{z}_{i} - \gamma \sin(\chi_{i}) + v_{j} - v_{i}) + \beta \sin(\chi_{i}) \Delta_{i} dt,$$
(11)

where $t_0 \geq 0$ and χ_i denotes $\chi_i(t/\epsilon)$ for ease of notations. To compute the average in (11), we adopt $\varpi_i(t) = 1/\sqrt{\epsilon} dw_i(\epsilon t)$ to denote a standard Brownian motion and obtain that $d\chi_i(t) = -\chi_i(t)dt + gd\varpi_i(t)$ from the diffusion process (6). Clearly, $\{\chi_i(t)\}_{t\geq t_0}$ is an ergodic Ornstein-Uhlenbeck process and has an invariant distribution $\mu(ds) = 1/(\sqrt{\pi}g)\exp(-s^2/g^2)ds$. It follows from the ergodic theorem (Ash & Doléans-Dade 2000) that

$$\lim_{T \to +\infty} \frac{1}{T} \int_{t_0}^{T+t_0} \sin(\chi_i(t)) \mathrm{d}t = \int_{\mathbb{R}} \sin(s)\mu(\mathrm{d}s) \quad (12)$$

almost surely, which is uniform in $\forall t_0 \geq 0$. Moreover, it holds that

$$\int_{\mathbb{R}} \sin^{k}(s)\mu(\mathrm{d}s) = \begin{cases} 0, & k = 1, 3, \\ \frac{1}{2}(1 - \exp(-g^{2})), & k = 2, \end{cases}$$
(13)

by Liu & Krstic (2010, Section III), and

$$\int_{\mathbb{R}^2} \sin^k(s) \sin^l(t) \mu(\mathrm{d}s) \mu(\mathrm{d}t) = \frac{1}{\pi g^2}$$
$$\int_{\mathbb{R}} \sin^k(s) \exp(-\frac{s^2}{g^2}) (\mathrm{d}s) \int_{\mathbb{R}} \sin^l(t) \exp(-\frac{t^2}{g^2}) (\mathrm{d}t).$$
(14)

Moreover, Δ_i in (11) can be expressed by the definition of e_i , i.e.,

$$\Delta_{i} = -\frac{1}{2}\tilde{z}_{i}'H_{i}\tilde{z}_{i} - \frac{\gamma^{2}}{2}\sin(\chi_{i})'H_{i}\sin(\chi_{i}) - \gamma\sin(\chi_{i})'H_{i}\tilde{z}_{i} -\gamma\sin(\chi_{i})'H_{i}z^{*} - \tilde{z}_{i}'H_{i}z^{*} + b_{i}'(\tilde{z}_{i} + \gamma\sin(\chi_{i})) - e_{i}.$$

 $^{^2\;}$ The derivative is interpreted as the Ito derivative, which is clear from the context.

Thus, together with the ergodicity of χ_i in (12) and the relations in (13)-(14), we compute the average in (11) and obtain that $\dot{\tilde{z}}_i^a = \alpha \sum_{j \in \mathcal{N}_i} a_{ij} (\tilde{z}_j^a - \tilde{z}_i^a + v_j^a - v_i^a) - \kappa \beta (H_i(\tilde{z}_i^a + z^*) - b_i)$, which is uniform in t_0 and independent of e_i . Further, applying the similar technique to \dot{v}_i leads to the following stochastic average system

$$\begin{cases} \dot{\tilde{z}}_i^a = \alpha \sum_{j \in \mathcal{N}_i} a_{ij} (\tilde{z}_j^a - \tilde{z}_i^a + v_j^a - v_i^a) \\ -\kappa \beta (H_i (\tilde{z}_i^a + z^*) - b_i), \\ \dot{v}_i^a = \sum_{j \in \mathcal{N}_i} a_{ij} (\tilde{z}_i^a - \tilde{z}_j^a). \end{cases}$$
(15)

Step 2: Stability of the stochastic average system (15).

The stochastic average system of the networked \boldsymbol{n} vehicles can be compactly expressed as

$$\begin{cases} \dot{\tilde{z}}^a = -\alpha(L \otimes I_m)(\tilde{z}^a + v^a) \\ -\kappa\beta(H^d(\tilde{z}^a + 1_n \otimes z^*) - b), \qquad (16) \\ \dot{v}^a = (L \otimes I_m)\tilde{z}^a. \end{cases}$$

Denote an equilibrium point of (16) by $((\tilde{z}^a_{eq})', (v^a_{eq})')'$. It follows that

$$-\kappa\beta(H^d(\tilde{z}_{eq}^a+1_n\otimes z^*)-b) = \alpha(L\otimes I_m)v_{eq}^a, \qquad (17)$$
$$(L\otimes I_m)\tilde{z}_{eq}^a = 0_{mn}.$$

Pre-multiplying both sides of the first equality of (17) by $(1'_n \otimes I_m)$ and noting $1'_n L = 0_n$, we obtain that

$$-\kappa\beta(1'_n\otimes I_m)(H^d(\tilde{z}^a_{eq}+1_n\otimes z^*)-b)=0_{mn}.$$
 (18)

By the second equality of (17) and Assumption 3, there exists a vector $z_1 \in \mathbb{R}^m$ such that $\tilde{z}_{eq}^a = 1_n \otimes z_1$. Inserting it into (18) yields that $\left(\sum_{i=1}^n H_i\right)(z_1 + z^*) = \sum_{i=1}^n b_i$. Since the source position z^* in (1) satisfies that $z^* = \left(\sum_{i=1}^n H_i\right)^{-1} \left(\sum_{i=1}^n b_i\right)$ by Assumption 4, we obtain that $z_1 = 0_m$, i.e., $\tilde{z}_{eq}^a = 0_{mn}$. That is, an equilibrium point of (16) is $(0'_{mn}, (v_{eq}^a)')'$.

Now, we study the convergence of the networked average system (16). Defining $V = \frac{1}{2} \left(\|\tilde{z}^a\|^2 + \alpha \cdot \|v^a - v^a_{eq}\|^2 \right)$ and taking its derivative along (16), we obtain that

$$\dot{V} = -\alpha(\tilde{z}^a)'(L \otimes I_m)(\tilde{z}^a + v^a) - \kappa\beta(\tilde{z}^a)'(H^d\tilde{z}^a + H^d(1_n \otimes z^*) - b) + \alpha(v^a - v^a_{eq})'(L \otimes I_m)\tilde{z}^a.$$
(19)

By the first equality of (17) and $\tilde{z}_{eq}^a = 0$, \dot{V} in (19) can be further simplified as

$$\dot{V} = (\tilde{z}^a)'(-\alpha(L \otimes I_m) - \kappa\beta H^d)\tilde{z}^a = -(\tilde{z}^a)'M\tilde{z}^a.$$
 (20)

Clearly, the matrix M is positive semi-definite, since $\alpha(L \otimes I_m)$ and $\kappa \beta H^d$ are positive semi-definite under

Assumptions 3 and 4 (Ren & Beard 2008). Suppose that there exists a non-zero vector $\zeta \in \mathbb{R}^{mn}$ such that $\zeta' M \zeta = 0$. That is, $\zeta' (L \otimes I_m) \zeta = 0$ and $\zeta' H^d \zeta = 0$. In light of Assumption 3, ζ must be of the form that $\zeta = 1_n \otimes \zeta_1$ for some non-zero vector $\zeta_1 \in \mathbb{R}^m$. Then, we can obtain that $\zeta'_1 (\sum_{i=1}^n H_i) \zeta_1 = 0$, which contradicts Assumption 4 that $\sum_{i=1}^n H_i$ is strictly positive definite. Thus, the matrix M is strictly positive definite.

Let λ_1 denote the smallest eigenvalue of M. It follows from (20) that $\dot{V}(t) \leq -\lambda_1 \|\tilde{z}^a(t)\|^2$. Since $\frac{1}{2} \|\tilde{z}^a(t)\|^2 \leq V(t) = V(t_0) + \int_{t_0}^t \dot{V}(s) ds \leq V(t_0) + \int_{t_0}^t -\lambda_1 \|\tilde{z}^a(s)\|^2 ds$, there exists a finite constant $\rho_1 > 0$ such that

$$\|\tilde{z}_i^a(t)\| \le \|\tilde{z}^a(t)\| \le \rho_1 \exp(-\lambda_1 t).$$

Note that ρ_1 depends on the initial condition.

Step 3: Stability of the dynamics (10).

By Øksendal (2003, Proposition 5.5) and Assumption 4, the dynamics (10) admits a unique (almost surely) continuous solution on $[t_0, +\infty)$. Together with Liu & Krstic (2010, Proposition 2), the rest of proof is completed.

In addition, since $F(z_i) = F(z^* + \tilde{z}_i + \gamma \sin(\eta_i))$, it follows from (9) that for any $\delta > 0$, $\lim_{\epsilon \to 0} \mathcal{P}\{|F(z_i) - F(z^*)| \le O(\gamma) + O(\delta) + O(\exp(-\lambda_1 t)), \forall t \in [t_0, T_1(\epsilon)]\} = 1.$

3.3 Extension to directed graphs

For the case of directed graphs, the essential difference is that the Laplacian matrix L is asymmetric and \mathcal{G} is unbalanced, i.e., $\sum_{j} a_{ij} \neq \sum_{j} a_{ji}$ for some $i \in \mathcal{V}$. To solve it, the DSES controller in (5) is modified as

$$\begin{cases} u_i = \alpha \sum_{j \in \mathcal{N}_i} a_{ij} (\varphi(z_j - z_i) + v_j - v_i) \\ + \beta / r_{ii} \cdot \sin(\eta_i) \Delta_i + \gamma \operatorname{dsin}(\eta_i) / \operatorname{dt}, \\ \dot{v}_i = \sum_{j \in \mathcal{N}_i} a_{ij} (z_i - z_j), \\ \dot{r}_i = \sum_{j \in \mathcal{N}_i} a_{ij} (r_j - r_i). \end{cases}$$
(21)

where $\alpha, \beta, \gamma, \varphi \in \mathbb{R}_+$ are positive control parameters, $\varphi = \varrho + (1 + \alpha)/(\alpha \varrho)$ with a constant $\varrho > 0$ and $r_i = (r_{1i}, r_{2i}, \ldots, r_{ni})' \in \mathbb{R}^n$ with $r_{ii}(t_0) = 1$, $r_{ji}(t_0) = 0$ for $j \neq i$ is an estimate of a left eigenvector of L associated with zero eigenvalue and can be computed in a distributed manner.

Proposition 6 Given a directed graph \mathcal{G} , consider the networked n autonomous vehicles (4) under the DSES controller (21). Let $\tilde{z}_i(t) = z_i(t) - \gamma \sin(\eta_i) - z^*$ where z^* is the source position defined in (1) and suppose that Assumptions 3 and 4 hold. Then, there exists a positive constant ρ_2 and a function $T_2(\epsilon)$: $(0, \epsilon_0) \rightarrow \mathbb{N}$ such



Fig. 2. The undirected (left) and directed (right) interaction graphs of the multi-vehicle system.

that for any $\delta > 0$ and bounded initial condition (i.e. $||z_i(t_0)|| < +\infty$, $||v_i(t_0)|| < +\infty$ for all $i \in \mathcal{V}$), it holds that $\forall i \in \mathcal{V}$,

$$\lim_{\epsilon \to 0} \mathcal{P}\{\|\tilde{z}_i(t)\| \le \rho_2 \exp(-\lambda_2 t) + \delta, \forall t \in [t_0, T_2(\epsilon)]\} = 1,$$

where $\lim_{\epsilon \to 0} T_2(\epsilon) = +\infty$ and $\lambda_2 > 0$ is given in (A.13).

The proof and the explicit expression of λ_2 are given in Appendix A. Note that it usually holds that $\lambda_2 < \lambda_1$, i.e., the convergence rate here is slower than that in undirected graphs due to the directed interaction between vehicles.

4 Illustrative Examples

In this section, examples are given to illustrate the effectiveness of the proposed DSES controllers. Let n = 4 in (1) and

$$H_1 = \begin{bmatrix} -2 & 1 \\ 1 & -0.5 \end{bmatrix}, \quad H_2 = \begin{bmatrix} -0.25 & 0.5 \\ 0.5 & -1 \end{bmatrix},$$
$$H_3 = \begin{bmatrix} -0.5 & 1.5 \\ 1.5 & -4.5 \end{bmatrix}, \quad H_4 = \begin{bmatrix} -3 & 1 \\ 1 & -0.33 \end{bmatrix},$$

 $b_1 = [1.5, -0.75]', b_2 = [-0.5, 1]', b_3 = [-2, 6]', b_4 = [2.5, -0.83]', c_1 = 0.44, c_2 = 0.5, c_3 = -3, c_4 = -0.042.$ Note that H_i is positive semi-definite, which clearly implies that $\arg \max_z f_i(z)$ contains an infinite number of elements. Thus, a single vehicle is unable to guarantee to approach the source position $z^* = [1.53, 1.82]'$.

The interaction graphs among the vehicles are depicted in Fig. 2, which satisfy Assumption 3. We select $\epsilon = 0.05$, g = 0.6, h = 1 for the excitation signals. Except Section 4.3, the vehicles are initially placed at $z_1 = [0,0]', z_2 = [0.9,0]', z_3 = [0.9,0.9]'$, and $z_4 = [0,0.9]'$.

4.1 Simulations of undirected graphs

Consider the multi-vehicle system (4) under the DSES controller (5) with $\alpha = 0.01$, $\beta = 2.5$, and $\gamma = 0.01$. The results are shown in Fig. 3, where the solid lines denote the trajectories of the vehicles, and the dashed lines represent the optimal points of $f_i(z)$. Clearly, each



Fig. 3. Trajectories of vehicles under the DSES controller (5) in the undirected graph.



Fig. 4. Measurement processes of vehicles with the DSES controller (5) in the undirected graph.

vehicle is attracted to its own optimal points while tending to achieve consensus. Finally, all the vehicles revolve around the source position z^* , which is consistent with Proposition 5, and the corresponding measurement processes are presented in Fig. 4.

We also test our DSES controller (5) for non-quadratic $f_i(\cdot)$, where f_1, f_2 remain quadratic with H_1, H_2 unchanged, $b_1 = [2, -1]', b_2 = [-0.5, 1]', c_1 = -1, c_2 = 0$, and f_3, f_4 are redefined as non-quadratic

$$f_3 = 0.083(x - 2.44)^3 - 0.25(x - 2.44) + 0.83,$$

$$f_4 = -e^{-x^2 - (y-1)^2} + 2x^4 e^{-x^2 - (y-2)^2} - 0.037.$$

Thus, the optimal points of f_3 are two lines, i.e., x = 1.44and x = 3.44 (we only draw the first one in Fig. 5), f_4 have two isolated optimal points denoted by small magenta circles, and the optimal point of the aggregated objective function is $z^* = [1.443, 2.041]'$. Fig. 5 illustrates that the multi-vehicle system simultaneously approaches the source position, indicating that the cooperative source seeking method also works even for the non-quadratic case.

4.2 Simulations of directed graphs

Consider the vehicles in directed interaction graphs. The control parameters of the DSES controller (21) are set



Fig. 5. Trajectories of vehicles under the DSES controller (5) for non-quadratic $f_i(\cdot)$.



Fig. 6. Trajectories of vehicles under the DSES controller (21) in the directed graph.



Fig. 7. Measurement processes of vehicles with the DSES controller (21) in the directed graph.

as $\alpha = 0.002$, $\beta = 1.2$, $\gamma = 0.0125$ and $\varphi = 50$. Fig. 6 shows that all vehicles converge to the neighborhood of the source position z^* , and Fig. 7 shows the measurement processes. Comparing Fig. 7 with Fig. 4, we can see the convergence rate of the directed graph is slower than that of the undirected graph. This is reasonable since there are fewer interaction links in the directed graph than in the undirected graph.



Fig. 8. Trajectories of vehicles starting from a consensus position (left) and optimal points of $f_i(z)$ (right).



Fig. 9. Measurement processes of vehicles starting from a consensus position (top) and optimal points of $f_i(z)$ (bottom).

4.3 Effects of control parameters

In this subsection, we illustrate the effects of the two parameters α and β by the DSES controller (5) for the networked vehicles in the undirected graph. First, let the initial positions $z_1 = z_2 = z_3 = z_4 = [0.5, 0.5]'$ and $\alpha = 0.005$. This indicates that all vehicles start from a consensus position, and α in Section 4.1 is reduced by one half. Then, let the initial positions $z_1 = [0.75, 0]', z_2 =$ $[0.5, 1.25]', z_3 = [0.75, 1.58]', z_4 = [1, 0.5]'$ and $\beta = 1.25$. One can verify that each vehicle starts from an optimal point of its local objective function, and β in Section 4.1 is reduced by one half.

The comparisons of the two cases are shown in Fig. 8 and 9. In the left subfigure of Fig. 8, each vehicle tends to approach a consensus state, while it moves to its local optimal point in the right subfigure. That is, the consensus term forces the vehicles to reach consensus and the stochastic ES term for gradient estimation drives the vehicles to their own optimal points. Both objectives are essential to approach the source position.

5 Conclusion

We have proposed the DSES controllers of the multivehicle system for the cooperative source seeking problem. In our scheme, each vehicle is only required to obtain the values of its local objective function and the relative positions to its neighbors. Via the stochastic averaging theory, we establish the convergence of the networked vehicles under our DSES controllers in probability, in both undirected and directed interaction graphs. Finally, simulations are included to verify our theoretical results.

A Proof of Proposition 6

In the appendix, the concatenated vector of two vectors x and y is simply written as [x; y] = (x', y')'.

Consider the autonomous vehicle (4) under the DSES controller (21), it holds that

$$\begin{cases} \dot{\tilde{z}}_i = \alpha \sum_{j \in \mathcal{N}_i} a_{ij} (\varphi(\tilde{z}_j + \gamma \sin(\eta_j) - \tilde{z}_i - \gamma \sin(\eta_i)) \\ + v_j - v_i) + \beta / r_{ii} \cdot \sin(\eta_i) \Delta_i, \\ \dot{v}_i = \sum_{j \in \mathcal{N}_i} a_{ij} (\tilde{z}_i + \gamma \sin(\eta_i) - \tilde{z}_j - \gamma \sin(\eta_j)), \\ \dot{r}_i = \sum_{j \in \mathcal{N}_i} a_{ij} (r_j - r_i). \end{cases}$$
(A.1)

Similar to the proof of Proposition 5, we define the stochastic average system of the networked multi-vehicle system, apply the same technique as in (12)-(14), and obtain that

$$\begin{cases} \dot{\tilde{z}}^a = -\alpha(L \otimes I_m)(\varphi \tilde{z}^a + v^a) - \kappa \beta(R^{-1} \otimes I_m) \\ (H^d(\tilde{z}^a + 1_n \otimes z^*) - b), \\ \dot{v}^a = (L \otimes I_m) \tilde{z}^a, \\ \dot{r} = -(L \otimes I_n)r, \end{cases}$$
(A.2)

where $R = \text{diag}(r_{11}, ..., r_{nn}).$

To study the convergence of (A.2), we first consider its approximated system below

$$\begin{cases} \dot{\tilde{z}}^a = -\alpha(L \otimes I_m)(\varphi \tilde{z}^a + v^a) - \kappa \beta(\Xi^{-1} \otimes I_m) \\ (H^d(\tilde{z}^a + 1_n \otimes z^*) - b), \\ \dot{v}^a = (L \otimes I_m) \tilde{z}^a, \end{cases}$$
(A.3)

where R in (A.2) is replaced by $\Xi = \text{diag}(\xi)$ with $\xi \in \mathbb{R}^{1 \times n}_+$ being a positive left eigenvector of L associated with zero eigenvalue (Zhu et al. 2019, Lemma 1). Let $[\tilde{z}^a_{eq}; v^a_{eq}]$ be an equilibrium point of (A.3). Then we have that

$$-\kappa\beta(\Xi^{-1}\otimes I_m)(H^d(\tilde{z}_{eq}^a+1_n\otimes z^*)-b) = \alpha(L\otimes I_m)v_{eq}^a.$$
(A.4)

Since $1'_n \Xi = \xi$, we pre-multiply both sides by $1'_n \Xi \otimes I_m$ and obtain the same relation as that in (18). Then, we can obtain $\tilde{z}^a_{eq} = 0_{mn}$ as in the proof of Proposition 5. Consider the following Lyapunov functional candidate for the system (A.3)

$$V(x) = \frac{1}{2}x' \left(\begin{bmatrix} \varrho^2 + 1 \ \varrho \\ \varrho & 1 \end{bmatrix} \otimes (\Xi \otimes I_m) \right) x, \qquad (A.5)$$

where $x = [\tilde{z}^a; v^a - v^a_{eq}]$. Clearly, it holds that

$$a_1 ||x||^2 \le V(x) \le a_2 ||x||^2, ||\nabla V(x)|| \le 2a_2 ||x||,$$
 (A.6)

where $\nabla V(x)$ is the gradient of V(x), and the positive constant $a_i, i \in \{1, 2\}$ is not explicitly given for brevity. We take derivative of (A.5) along (A.3) and obtain that $\dot{V} = -x'(P \otimes (\Xi L + L'\Xi) \otimes I_m + Q \otimes H^d)x$ with

$$P = \frac{\alpha}{2} \begin{bmatrix} \varrho^3 + 2\varrho + \frac{\alpha+1}{\alpha\varrho} & 1+\varrho^2 \\ 1+\varrho^2 & \varrho \end{bmatrix} \text{ and } Q = \kappa\beta \begin{bmatrix} 1+\varrho^2 & \frac{\varrho}{2} \\ \frac{\varrho}{2} & 0 \end{bmatrix}$$

by the selection of $\varphi = \varrho + (1 + \alpha)/(\alpha \varrho)$ and the relation in (A.4). Since $\xi L = 0'_n$, pre-multiplying both sides of the second equality in (A.3) by $\xi \otimes I_m$ yields that $(\xi \otimes I_m)\dot{v}^a = 0_m$, which implies that $(\xi \otimes I_m)(v^a - v^a_{eq}) = 0_m$. Hence we only need to investigate the convergence of (A.3) on the subspace $\mathcal{X} = \{[x_1; x_2] \in \mathbb{R}^{2mn} | (\xi \otimes I_m)x_2 = 0_m\}.$

Let $\mathcal{X}_1 = \{ [x_1; x_2] \in \mathbb{R}^{2mn} | x_1 = 1_n \otimes \zeta, x_2 = 0_{mn}, \zeta \in \mathbb{R}^m \}$ denote the null space of $P \otimes (\Xi L + L' \Xi) \otimes I_m$ on \mathcal{X} . Then, for any $x = [1_n \otimes \zeta; 0_{mn}] \in \mathcal{X}_1$, it holds that

$$\dot{V} = -[1_n \otimes \zeta; 0]' (Q \otimes H^d) [1_n \otimes \zeta; 0]$$

= $-\kappa\beta(1+\varrho^2)\zeta' \Big(\sum_{i=1}^n H_i\Big)\zeta \le -c_0 \|x\|^2$ (A.7)

where c_0 is positive by Assumption 4. For any $x \in \mathcal{X} - \mathcal{X}_1$, applying Weyl's inequality (Horn & Johnson 2012, Theorem 4.3.1) yields that

$$\dot{V} = -x'(P \otimes (\Xi L + L'\Xi) \otimes I_m + Q \otimes H^d)x$$

$$\leq -(\lambda_P \lambda_L - \lambda_Q \lambda_H) \|x\|^2, \qquad (A.8)$$

where $\lambda_P = O(\varrho)$ is the smallest eigenvalue of P, λ_L is the smallest nonzero eigenvalue of $\Xi L + L'\Xi$, $\lambda_Q = O(\varrho^2)$ is the largest eigenvalue of -Q, and λ_H is the largest eigenvalue of H^d . Since $\lim_{\varrho \to 0} \lambda_P / \lambda_Q = +\infty$, there exists a positive ϱ_0 such that $\lambda_P \lambda_L - \lambda_Q \lambda_H > 0$ for any $\varrho \in (0, \varrho_0)$. Jointly with (A.6), (A.7) and (A.8), it holds that $\forall x \in \mathcal{X}$ and $\varrho \in (0, \varrho_0)$, there exists a positive $a_3 > 0$ such that

$$\dot{V} \le -a_3 V, \tag{A.9}$$

Therefore, $[0_{mn}; v_{eq}^a]$ is an exponentially stable equilibrium point of (A.3).

Now, we consider the dynamics of \tilde{z}^a in (A.2) and rewrite it as

$$\dot{\tilde{z}}^a = -\alpha(L \otimes I_m)(\varphi \tilde{z}^a + v^a) - \kappa \beta(\Xi^{-1} \otimes I_m) \\
(H^d(\tilde{z}^a + 1_n \otimes z^*) - b) + d,$$
(A.10)

where $d = -\kappa\beta(R^{-1} - \Xi^{-1}) \otimes I_m(H^d(\tilde{z}^a + 1_n \otimes z^*) - b)$ is treated as a perturbation term of (A.3) and satisfies that

$$\begin{aligned} \|d\| &\leq \|\kappa\beta(R^{-1} - \Xi^{-1}) \otimes I_m H^d \tilde{z}^a\| \\ &+ \|\kappa\beta(R^{-1} - \Xi^{-1}) \otimes I_m(H^d(1_n \otimes z^*) - b)(A.11) \\ &\leq \kappa\beta\varsigma(t) \|H^d \tilde{z}^a\| + \kappa\beta\varsigma(t)\|(H^d(1_n \otimes z^*) - b)\|, \end{aligned}$$

where $\varsigma(t) = ||R^{-1} - \Xi^{-1}|| \otimes I_m$. In view of Lemma 1 in Zhu et al. (2019), $||r(t) - 1_n \otimes \xi'||$ converges to zero less slowly than the rate of $\exp(-\ell t)$, where ℓ is the real part of the eigenvalue of L that is closest to the left half plane and is positive under Assumption 3. This implies that $\varsigma(t)$ tends to zero at the same rate,

$$\sup_{t \ge t_0} \varsigma(t) < +\infty \text{ and } \int_{t_0}^{+\infty} \varsigma(s) \mathrm{d}s < a_4 \qquad (A.12)$$

for some $a_4 > 0$. Note that (A.3) has an exponentially stable equilibrium point $[0; v_{eq}^a]$, and its Lyapunov function (A.5) satisfies (A.6) and (A.9) in \mathcal{X} . Provided the perturbation term d(t) satisfies (A.11) and (A.12), for any bounded initial condition of (A.10), there exists a positive ρ_2 such that $\|\tilde{z}^a\| \leq \frac{\rho_2}{2} (\exp(-a_3 t) + \exp(-\ell t))$ (Khalil 2002, Lemma 9.4), which further implies that

$$\|\tilde{z}_i^a\| \le \rho_2 \exp(-\min\{a_3, \ell\}t) := \rho_2 \exp(-\lambda_2 t).$$
 (A.13)

The rest of the proof is similar to Step 3 in Proposition 5 and is omitted.

References

- Ash, R. B. & Doléans-Dade, C. A. (2000), *Probability* and measure theory, Academic Press.
- Brinon-Arranz, L., Schenato, L. & Seuret, A. (2016), 'Distributed source seeking via a circular formation of agents under communication constraints', *IEEE Transactions on Control of Network Systems* 3(2), 104–115.
- Chen, C. T. (1998), *Linear system theory and design*, Oxford University Press, Inc.
- Dhariwal, A., Sukhatme, G. S. & Requicha, A. A. (2004), Bacterium-inspired robots for environmental monitoring, *in* 'International Conference on Robotics and Automation (ICRA)', IEEE, pp. 1436–1443.
- Dougherty, S. & Guay, M. (2017), 'An extremum-seeking controller for distributed optimization over sensor networks', *IEEE Transactions on Automatic Control* 62(2), 928–933.

- Dürr, H. B., Krstić, M., Scheinker, A. & Ebenbauer, C. (2017), 'Extremum seeking for dynamic maps using lie brackets and singular perturbations', *Automatica* 83, 91–99.
- Frihauf, P., Liu, S. & Krstic, M. (2014), 'A single forward-velocity control signal for stochastic source seeking with multiple nonholonomic vehicles', *Jour*nal of Dynamic Systems, Measurement, and Control 136(5), 051024.
- Gao, B., Li, H., Li, W. & Sun, F. (2016), '3D mothinspired chemical plume tracking and adaptive step control strategy', *Adaptive Behavior* 24(1), 52–65.
- Gao, X., Acar, L. & Sarangapani, J. (2016), 'Detection and tracking of odor source in sensor networks using reasoning system', *Journal of Automation and Control Engineering* 4(6), 418–423.
- Gharesifard, B. & Cortés, J. (2014), 'Distributed continuous-time convex optimization on weightbalanced digraphs', *IEEE Transactions on Automatic Control* 59(3), 781–786.
- Guay, M., Vandermeulen, I., Dougherty, S. & McLellan, P. J. (2018), 'Distributed extremum-seeking control over networks of dynamically coupled unstable dynamic agents', *Automatica* **93**, 498–509.
- Horn, R. A. & Johnson, C. R. (2012), Matrix analysis, Cambridge university press.
- Khalil, H. K. (2002), Nonlinear systems, Prentice-Hall.
- Khong, S. Z., Tan, Y. & Manzie, C. (2014), 'Multiagent source seeking via discrete-time extremum seeking control', *Automatica* 50(9), 2312–2320.
- Kvaternik, K. & Pavel, L. (2012), An analytic framework for decentralized extremum seeking control, *in* 'American Control Conference (ACC)', IEEE, pp. 3371– 3376.
- Li, C., Qu, Z. & Weitnauer, M. A. (2015), 'Distributed extremum seeking and formation control for nonholonomic mobile network', Systems & Control Letters 75, 27–34.
- Li, Z., You, K., Song, S. & Dong, S. (2018), Distributed extremum seeking with stochastic perturbations, *in* 'International Conference on Control and Automation (ICCA)', IEEE, pp. 367–372.
- Lin, J., Song, S., You, K. & Krstic, M. (2017), 'Stochastic source seeking with forward and angular velocity regulation', *Automatica* 83, 378–386.
- Liu, S. & Krstic, M. (2010), 'Stochastic averaging in continuous time and its applications to extremum seeking', *IEEE Transactions on Automatic Control* 55(10), 2235–2250.
- Manzie, C. & Krstic, M. (2009), 'Extremum seeking with stochastic perturbations', *IEEE Transactions on Au*tomatic Control 54(3), 580–585.
- Menon, A. & Baras, J. S. (2014), Collaborative extremum seeking for welfare optimization, in 'Conference on Decision and Control (CDC)', IEEE, pp. 346– 351.
- Michalowsky, S., Gharesifard, B. & Ebenbauer, C. (2017), Distributed extremum seeking over directed graphs, in 'Conference on Decision and Control

(CDC)', IEEE, pp. 2095–2101.

- Øksendal, B. (2003), Stochastic differential equations, Springer.
- Poveda, J. & Quijano, N. (2013), Distributed extremum seeking for real-time resource allocation, *in* 'American Control Conference (ACC)', IEEE, pp. 2772–2777.
- Ren, W. & Beard, R. W. (2008), *Distributed consensus* in multi-vehicle cooperative control, Springer.
- Turgeman, A. & Werner, H. (2018), Multiple source seeking using glowworm swarm optimization and distributed gradient estimation, in 'American Control Conference (ACC)', IEEE, pp. 3558–3563.
- Vandermeulen, I., Guay, M. & McLellan, P. J. (2018), 'Discrete-time distributed extremum-seeking control over networks with unstable dynamics', *IEEE Transactions on Control of Network Systems* 5(3), 1182– 1192.
- Wang, J. & Elia, N. (2010), Control approach to distributed optimization, in 'Allerton Conference on Communication, Control, and Computing (Allerton)', IEEE, pp. 557–561.
- Ye, M. & Hu, G. (2016), 'Distributed extremum seeking for constrained networked optimization and its application to energy consumption control in smart grid', *IEEE Transactions on Control Systems Technology* 24(6), 2048–2058.
- You, K., Tempo, R. & Xie, P. (2019), 'Distributed algorithms for robust convex optimization via the scenario approach', *IEEE Transactions on Automatic Control* 64(3), 880–895.
- Zhang, Y., Makarenkov, O. & Gans, N. (2016), 'Extremum seeking control of a nonholonomic system with sensor constraints', *Automatica* **70**, 86–93.
- Zhao, N. (2016), Acoustic source localization, PhD thesis, Massachusetts Institute of Technology.
- Zhu, Y., Yu, W., Wen, G. & Ren, W. (2019), 'Continuous-time coordination algorithm for distributed convex optimization over weight-unbalanced directed networks', *IEEE Transactions on Circuits* and Systems II: Express Briefs 66(7), 1202–1206.