

# A Lower Bound for the Fisher Information Measure

Manuel Stein, Amine Mezghani, and Josef A. Nossek  
 Institute for Circuit Theory and Signal Processing  
 Technische Universität München, Germany  
 E-Mail: {manuel.stein, josef.a.nossek}@tum.de

**Abstract**—The problem of approximating the absolute value of the Fisher information measure based on the output of a general parametric system is considered. Having available the first and second moment of the system output in a parametric form, it is shown that the information measure can be bounded from below through replacement of the original system by a Gaussian system with equivalent moments. Derivations are given for uni- and multivariate parametric systems. Finally, the result is applied to a system of practical importance and the potential quality of the presented bound is demonstrated.

**Index Terms**—estimation theory, minimum Fisher information, non-linear systems

## I. INTRODUCTION

Assessing the quality of a given probabilistic system with respect to the inference of unknown parameters plays a key roll in the design of signal processing systems. Through the Fisher information measure [1] [2], estimation theory provides the theoretical grounds to characterize the estimation capability of a system in a compact analytical way. As this measure is convex [3] [4] and inversely proportional to the minimum achievable variance of an unbiased estimator, it can be used for analytical performance characterization or as a figure-of-merit when heading at the optimum system design. However, the application of such analytical tools always requires a probabilistic system model which represents all underlying effects within the system of interest. As superposition shows to characterize in an appropriate way how noise enters into practical systems, additive noise models have become a popular assumption among engineers. If the noise is caused by a large number of independent sources, the central limit theorem can be applied in order to justify an additive noise model with Gaussian distribution. Using that, among all possible additive noise distributions with *fixed* variance, the Gaussian distribution minimizes Fisher information [5] [6] [7], allows to access the quality of any system model with independent additive noise in a conservative way. Note, that a pessimistic performance evaluation is crucial for the robust design of estimation systems and algorithms.

### A. Motivation

While mathematical tractability makes models with independent and additive noise attractive, this assumption must be questioned if one takes into account that in general probabilistic systems follow non-additive laws. This becomes relevant in practice when considering that a variety of technical systems exhibit severe non-linear characteristics behind their

dominant noise sources. Therefore, additive independent noise models do in general not reflect an accurate and general mathematical characterization for the output of a technical system. In order to contribute to the understanding of this general class of systems, it is shown here how to validate the Fisher information measure approximately for an arbitrary system model in a simple analytical way.

### B. Related Works

An early discussion on the least favorable distributions with respect to the location and the scale estimation problem is found in [8]. For a generalized form of the Fisher information measure (Fisher information measure of order  $s$ ), [5, pp. 73 ff.] shows that with a location parameter the exponential power distribution with *fixed*  $s$ -th moment attains minimum Fisher information and cites [3] for a proof on the standard case  $s = 2$ . The recent work [9] focuses on minimizing the Fisher information measure under constraints on moments of higher order. The works [10] and [11] analyze the problem under a restriction on the support of the system output. A lecture note on minimum Fisher information and additive univariate noise models is provided by [6], while [7] generalizes to additive multi-variate noise models with correlation.

### C. Contribution

To the best of our knowledge, none of the previous works provides a bounding approach for the Fisher information measure of a generic system model with *parametric* moments. Therefore, the strength of the presented method lies in its generality, providing the possibility to bound the Fisher information measure for any probabilistic model based on the dependency between the first two central output moments and the system parameters. The presented bounding technique can be interpreted as a replacement of the original system by an appropriate additive Gaussian model with equivalent moments.

## II. SYSTEM MODEL

For the discussion, we assume access to the output  $y \in \mathbb{R}$  of a general parametric system  $p_y(y; \theta)$ , where  $\theta \in \mathbb{R}$  is an unknown but deterministic parameter. Further, the first and second moment are available in the form

$$\mu(\theta) = \int y p_y(y; \theta) dy \quad (1)$$

$$\sigma^2(\theta) = \int (y - \mu(\theta))^2 p_y(y; \theta) dy. \quad (2)$$

### III. FISHER INFORMATION BOUND THE UNIVARIATE CASE

Given the possibility to observe the system output  $y$ , following a parameterized probability density function  $p_y(y; \theta)$ , the Fisher information measure  $F(y; \theta)$ , associated with the average information about the unknown parameter  $\theta$  carried by realizations of the system output  $y$ , is defined [?]

$$F(y; \theta) = \int p_y(y; \theta) \left( \frac{\partial \log p_y(y; \theta)}{\partial \theta} \right)^2 dy. \quad (3)$$

The goal of our discussion is to show, that if the derivative

$$\frac{\partial \mu(\theta)}{\partial \theta} \quad (4)$$

exists, the Fisher information measure can be bounded by

$$F(y; \theta) \geq \frac{1}{\sigma^2(\theta)} \left( \frac{\partial \mu(\theta)}{\partial \theta} \right)^2. \quad (5)$$

#### A. Proof

Starting from the definition of the information measure (3) and using the inequality (43) from Appendix A,

$$\begin{aligned} F(y; \theta) &= \int p_y(y; \theta) \left( \frac{\partial \log p_y(y; \theta)}{\partial \theta} \right)^2 dy \\ &\geq \frac{\left( \int (y - \mu(\theta)) \frac{\partial \log p_y(y; \theta)}{\partial \theta} p_y(y; \theta) dy \right)^2}{\int (y - \mu(\theta))^2 p_y(y; \theta) dy} \\ &= \frac{\left( \int (y - \mu(\theta)) \frac{\partial \log p_y(y; \theta)}{\partial \theta} p_y(y; \theta) dy \right)^2}{\sigma^2(\theta)} \\ &= \frac{\left( \int (y - \mu(\theta)) \frac{\partial p_y(y; \theta)}{\partial \theta} dy \right)^2}{\sigma^2(\theta)} \\ &= \frac{\left( \int y \frac{\partial p_y(y; \theta)}{\partial \theta} dy - \int \mu(\theta) \frac{\partial p_y(y; \theta)}{\partial \theta} dy \right)^2}{\sigma^2(\theta)} \\ &= \frac{\left( \frac{\partial}{\partial \theta} \int y p_y(y; \theta) dy - \mu(\theta) \frac{\partial}{\partial \theta} \int p_y(y; \theta) dy \right)^2}{\sigma^2(\theta)} \quad (6) \\ &= \frac{1}{\sigma^2(\theta)} \left( \frac{\partial \mu(\theta)}{\partial \theta} \right)^2, \quad (7) \end{aligned}$$

allows to bound (3) in general<sup>1</sup> from below.

### IV. FISHER INFORMATION BOUND THE MULTIVARIATE CASE

For the general case of a multivariate system output  $\mathbf{y} \in \mathbb{R}^N$  and a multidimensional parameter  $\boldsymbol{\theta} \in \mathbb{R}^K$ , the probabilistic system is written  $p_y(\mathbf{y}; \boldsymbol{\theta})$ . First and second moment are given

$$\boldsymbol{\mu}(\boldsymbol{\theta}) = \int \mathbf{y} p_y(\mathbf{y}; \boldsymbol{\theta}) d\mathbf{y} \quad (8)$$

$$\mathbf{R}(\boldsymbol{\theta}) = \int (\mathbf{y} - \boldsymbol{\mu}(\boldsymbol{\theta})) (\mathbf{y} - \boldsymbol{\mu}(\boldsymbol{\theta}))^T p_y(\mathbf{y}; \boldsymbol{\theta}) d\mathbf{y}, \quad (9)$$

<sup>1</sup>Note that  $p_y(y; \theta)$  has to fulfill certain regularity conditions for step (6). This is the case for most systems of technical relevance.

and the Fisher information measure is defined by

$$\mathbf{F}(\mathbf{y}; \boldsymbol{\theta}) = \int p_y(\mathbf{y}; \boldsymbol{\theta}) \left( \frac{\partial \log p_y(\mathbf{y}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)^2 d\mathbf{y}. \quad (10)$$

The goal is to show that if

$$\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \quad (11)$$

exists, the matrix inequality

$$\mathbf{F}(\mathbf{y}; \boldsymbol{\theta}) \succeq \left( \frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)^T \mathbf{R}(\boldsymbol{\theta})^{-1} \left( \frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right) \quad (12)$$

holds, where for two matrices  $\mathbf{A} \in \mathbb{R}^{K \times K}$  and  $\mathbf{B} \in \mathbb{R}^{K \times K}$

$$\mathbf{A} \succeq \mathbf{B}, \quad (13)$$

stands for the fact that

$$\mathbf{x}^T (\mathbf{A} - \mathbf{B}) \mathbf{x} \geq 0 \quad \forall \mathbf{x} \in \mathbb{R}^K. \quad (14)$$

#### A. Proof

Inequality (39) from Appendix A is used such that

$$\begin{aligned} \mathbf{F}(\mathbf{y}; \boldsymbol{\theta}) &= \int \frac{\partial \log p_y(\mathbf{y}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \left( \frac{\partial \log p_y(\mathbf{y}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)^T p_y(\mathbf{y}; \boldsymbol{\theta}) d\mathbf{y} \\ &\succeq \int \frac{\partial \log p_y(\mathbf{y}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} (\mathbf{y} - \boldsymbol{\mu}(\boldsymbol{\theta}))^T p_y(\mathbf{y}; \boldsymbol{\theta}) d\mathbf{y} \mathbf{R}(\boldsymbol{\theta})^{-1}. \\ &\quad \cdot \int (\mathbf{y} - \boldsymbol{\mu}(\boldsymbol{\theta})) \left( \frac{\partial \log p_y(\mathbf{y}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)^T p_y(\mathbf{y}; \boldsymbol{\theta}) d\mathbf{y} \\ &= \int \frac{\partial p_y(\mathbf{y}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} (\mathbf{y} - \boldsymbol{\mu}(\boldsymbol{\theta}))^T d\mathbf{y} \mathbf{R}(\boldsymbol{\theta})^{-1}. \\ &\quad \cdot \int (\mathbf{y} - \boldsymbol{\mu}(\boldsymbol{\theta})) \left( \frac{\partial p_y(\mathbf{y}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)^T d\mathbf{y} \\ &= \left( \frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)^T \mathbf{R}(\boldsymbol{\theta})^{-1} \left( \frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right). \quad (15) \end{aligned}$$

### V. EQUIVALENT PESSIMISTIC SYSTEM AN INTERPRETATION

For an interpretation of the bounding technique, consider instead of  $p_y(y; \theta)$ , the Gaussian system

$$q_y(y; \theta) = \frac{1}{\sqrt{2\pi\sigma^2(\theta)}} e^{-\frac{(y-\mu(\theta))^2}{2\sigma^2(\theta)}}, \quad (16)$$

with equivalent first and second order moments. If the dependency of  $\sigma^2(\theta)$  on  $\theta$  is ignored and

$$\frac{\partial \sigma^2(\theta)}{\partial \theta} = 0 \quad (17)$$

is postulated, the Fisher information measure

$$\tilde{F}(y; \theta) = \int q_y(y; \theta) \left( \frac{\partial \log q_y(y; \theta)}{\partial \theta} \right)^2 dy \quad (18)$$

attains the absolute value

$$\tilde{F}(y; \theta) = \frac{1}{\sigma^2(\theta)} \left( \frac{\partial \mu(\theta)}{\partial \theta} \right)^2. \quad (19)$$

For the multivariate case, the Gaussian system

$$q_y(\mathbf{y}; \theta) = \rho \cdot e^{-\frac{1}{2}(\mathbf{y} - \boldsymbol{\mu}(\theta))^T \mathbf{R}(\theta)^{-1}(\mathbf{y} - \boldsymbol{\mu}(\theta))}, \quad (20)$$

with

$$\rho = \frac{1}{(2\pi)^{\frac{N}{2}} (\det \mathbf{R}(\theta))^{\frac{1}{2}}}, \quad (21)$$

exhibits the Fisher information value

$$\begin{aligned} \tilde{F}(\mathbf{y}; \theta) &= \int q_y(\mathbf{y}; \theta) \left( \frac{\partial \log q_y(\mathbf{y}; \theta)}{\partial \theta} \right)^2 d\mathbf{y} \\ &= \left( \frac{\partial \boldsymbol{\mu}(\theta)}{\partial \theta} \right)^T \mathbf{R}(\theta)^{-1} \left( \frac{\partial \boldsymbol{\mu}(\theta)}{\partial \theta} \right), \end{aligned} \quad (22)$$

under the claim that

$$\frac{\partial \mathbf{R}(\theta)}{\partial \theta} = \mathbf{0}. \quad (23)$$

This shows, that the bounding approach can be interpreted as a replacement of the original system  $p_y(\mathbf{y}; \theta)$  by its equivalent pessimistic counterpart  $q_y(\mathbf{y}; \theta)$ , such that the inequality

$$\mathbf{F}(\mathbf{y}; \theta) \succeq \tilde{\mathbf{F}}(\mathbf{y}; \theta) \quad (24)$$

holds. So, the Fisher information of the original system  $p_y(\mathbf{y}; \theta)$  always dominates the information measure calculated for the equivalent pessimistic system  $q_y(\mathbf{y}; \theta)$ .

## VI. MINIMUM FISHER INFORMATION A SPECIAL CASE

A question which has received attention in the field of statistical signal processing is to specify the worst-case additive noise model under a *fixed* variance [6] [7]. In order to show that our approach contains this important class of systems as a special case, consider the additive system model

$$y = s(\theta) + \eta, \quad (25)$$

where the location parameter  $\theta$  modulates the signal  $s(\theta)$  and  $\eta$  is a zero-mean random process with constant variance  $\beta$ . As the first two moments are

$$\int y p_y(y; \theta) dy = s(\theta) \quad (26)$$

$$\int y^2 p_y(y; \theta) dy = \beta, \quad (27)$$

it is directly clear with (5), that assuming  $\eta$  to be normally distributed with zero-mean and variance  $\beta$  is the worst-case assumption under an estimation theoretic perspective. Note, that the approach presented here also allows a pessimistic statement for the general case with *parametric* variance  $\beta(\theta)$ . Therefore, it provides a tool for the analysis of a brighter class of systems than those considered in [6] and [7].

## VII. FISHER INFORMATION BOUND PREDICTIVE QUALITY

In order to show that the presented approach has the potential to bound the Fisher information measure in an accurate way, we consider the example of an additive Gaussian noise model which is processed by a hard-limiting device with threshold  $\alpha$

$$y = \text{sign}_\alpha(\theta + \eta), \quad (28)$$

where

$$\text{sign}_\alpha(x) = \begin{cases} +1 & \text{if } x \geq \alpha \\ -1 & \text{if } x < \alpha \end{cases} \quad (29)$$

and

$$p_\eta(\eta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\eta^2}{2}}. \quad (30)$$

In this case

$$\begin{aligned} p(y = +1; \theta) &= \int_{-\theta+\alpha}^{\infty} p_\eta(\eta) d\eta \\ &= \frac{1}{2} \left( 1 + \text{erf} \left( \frac{\theta - \alpha}{\sqrt{2}} \right) \right) \\ p(y = -1; \theta) &= \int_{-\infty}^{-\theta+\alpha} p_\eta(\eta) d\eta \\ &= \frac{1}{2} \left( 1 - \text{erf} \left( \frac{\theta - \alpha}{\sqrt{2}} \right) \right) \end{aligned} \quad (31)$$

and

$$\frac{\partial p(y = +1; \theta)}{\partial \theta} = \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{\theta - \alpha}{\sqrt{2}}\right)^2} \quad (32)$$

$$\frac{\partial p(y = -1; \theta)}{\partial \theta} = -\frac{1}{\sqrt{2\pi}} e^{-\left(\frac{\theta - \alpha}{\sqrt{2}}\right)^2}. \quad (33)$$

The first moment of the output (28) is given by

$$\begin{aligned} \mu(\theta) &= \int y p_y(y; \theta) dy \\ &= \text{erf} \left( \frac{\theta - \alpha}{\sqrt{2}} \right), \end{aligned} \quad (34)$$

while the second central moment is

$$\begin{aligned} \sigma^2(\theta) &= \int \left( y - \text{erf} \left( \frac{\theta - \alpha}{\sqrt{2}} \right) \right)^2 p_y(y; \theta) dy \\ &= 1 - \text{erf}^2 \left( \frac{\theta - \alpha}{\sqrt{2}} \right). \end{aligned} \quad (35)$$

Therefore, the exact Fisher information of system (28) is

$$\begin{aligned}
F(y; \theta) &= \int p_y(y; \theta) \left( \frac{\partial \log p_y(y; \theta)}{\partial \theta} \right)^2 dy \\
&= \int \frac{1}{p_y(y; \theta)} \left( \frac{\partial p_y(y; \theta)}{\partial \theta} \right)^2 dy \\
&= \left( \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{\theta-\alpha}{\sqrt{2}}\right)^2} \right)^2 \\
&\quad \cdot \left( \frac{2}{1 + \operatorname{erf}\left(\frac{\theta-\alpha}{\sqrt{2}}\right)} + \frac{2}{1 - \operatorname{erf}\left(\frac{\theta-\alpha}{\sqrt{2}}\right)} \right) \\
&= \frac{2}{\pi} \frac{1}{1 - \operatorname{erf}^2\left(\frac{\theta-\alpha}{\sqrt{2}}\right)} e^{-2\left(\frac{\theta-\alpha}{\sqrt{2}}\right)^2}, \tag{36}
\end{aligned}$$

while with the derivative of the first moment

$$\frac{\partial \mu(\theta)}{\partial \theta} = \sqrt{\frac{2}{\pi}} e^{-\left(\frac{\theta-\alpha}{\sqrt{2}}\right)^2}, \tag{37}$$

the pessimistic description for the Fisher information

$$\begin{aligned}
\tilde{F}(y; \theta) &= \frac{1}{\sigma^2(\theta)} \left( \frac{\partial \mu(\theta)}{\partial \theta} \right)^2 \\
&= \frac{2}{\pi} \frac{1}{1 - \operatorname{erf}^2\left(\frac{\theta-\alpha}{\sqrt{2}}\right)} e^{-2\left(\frac{\theta-\alpha}{\sqrt{2}}\right)^2} \tag{38}
\end{aligned}$$

is found to match (36) exactly.

### VIII. CONCLUSION

We have established a lower bound for the Fisher information measure of an arbitrary parametric and probabilistic system. Being able to characterize the first and second central moment of the system output as functions of the system parameters, consultation of an equivalent Gaussian system leads to a pessimistic approximation of the information measure. Using the approach for a non-linear system of practical relevance we have demonstrated that the presented bound has the potential to accurately describe the true information measure in the original system. In particular in situations where the analytic characterization of the information measure on the system of interest is difficult but the first two central moments are available, the presented technique allows to analyze the system with respect to estimation performance.

### APPENDIX A

**Proposition** Given two multivariate random variables  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$  following the joint probability distribution  $p_{xy}(\mathbf{x}, \mathbf{y})$

$$\mathbb{E}_y [\mathbf{y}\mathbf{y}^T] \succeq \mathbb{E}_{xy} [\mathbf{y}\mathbf{x}^T] \mathbb{E}_x [\mathbf{x}\mathbf{x}^T]^{-1} \mathbb{E}_{xy} [\mathbf{x}\mathbf{y}^T]. \tag{39}$$

*Proof:* Given  $\mathbf{x}$  construct the auxiliary random variable  $\hat{\mathbf{y}}(\mathbf{x})$  by

$$\hat{\mathbf{y}}(\mathbf{x}) = \mathbb{E}_{xy} [\mathbf{y}\mathbf{x}^T] \mathbb{E}_x [\mathbf{x}\mathbf{x}^T]^{-1} \mathbf{x}. \tag{40}$$

Observing that by construction

$$\mathbb{E}_{xy} [(\mathbf{y} - \hat{\mathbf{y}}(\mathbf{x}))(\mathbf{y} - \hat{\mathbf{y}}(\mathbf{x}))^T] \succeq \mathbf{0}, \tag{41}$$

proofs that

$$\mathbb{E}_y [\mathbf{y}\mathbf{y}^T] - \mathbb{E}_{xy} [\mathbf{y}\mathbf{x}^T] \mathbb{E}_x [\mathbf{x}\mathbf{x}^T]^{-1} \mathbb{E}_{xy} [\mathbf{x}\mathbf{y}^T] \succeq \mathbf{0}. \tag{42}$$

**Corollary** Given two scalar random variables  $x, y \in \mathbb{R}$  following the joint probability distribution  $p_{xy}(x, y)$  it holds that

$$\int y^2 p_y(y) dy \int x^2 p_x(x) dx \geq \left( \int \int xy p_{xy}(x, y) dx dy \right)^2. \tag{43}$$

*Proof:* With (39) and  $N = 1$

$$\int y^2 p_y(y) dy \geq \frac{\left( \int \int xy p_{xy}(x, y) dx dy \right)^2}{\int x^2 p_x(x) dx} \tag{44}$$

inequality (43) follows directly. ■

### REFERENCES

- [1] R. A. Fisher, "On the Mathematical Foundations of Theoretical Statistics," Philosophical Transactions of the Royal Society of London, vol. 222, pp. 309–368, Jan. 1922
- [2] R. A. Fisher, "Theory of Statistical Estimation," Proceedings of the Cambridge Philosophical Society, vol. 22, pp. 700–725, 1925
- [3] A. J. Stam, "Some mathematical properties of quantities of information," Ph. D. thesis, Delft University of Technology, Netherlands, Apr. 1959
- [4] M. Cohen, "The Fisher information and convexity," IEEE Transactions on Information Theory, vol. 14, no. 4, pp. 591–592, Jul. 1968
- [5] D. E. Boeke, "A generalization of the Fisher information measure," Ph. D. thesis, Delft University Press, Sept. 1977
- [6] P. Stoica, P. Babu, "The Gaussian data assumption leads to the largest Cramér-Rao Bound," IEEE Signal Processing Magazine, vol. 28, no. 3, pp. 132–133, 2011
- [7] S. Park, E. Serpedin, K. Qaraqe, "Gaussian assumption: The least favorable but the most useful," IEEE Signal Processing Magazine, vol. 30, no. 3, pp. 183–186, 2013
- [8] P. J. Huber, "Robust statistics: A review," Ann. Math. Statist., vol. 43, no. 4, pp. 1041–1067, 1972
- [9] V. Živojnović, "Minimum Fisher information of moment-constrained distributions with application to robust blind identification," Signal Processing, Elsevier, vol. 65, no. 2, pp. 297 – 313, Oct. 1998
- [10] E. Uhrmann-Klingen, "Minimal Fisher information distributions with compact-supports," Sankhyā: The Indian Journal of Statistics, Series A (1961-2002), vol. 57, no. 3, pp. 360-374, Oct. 1995
- [11] J. F. Bercher, C. Vignat, "On minimum Fisher information distributions with restricted support and fixed variance," Information Sciences, vol. 179, no. 22, Nov. 2009