

# SOME EXPERIMENTS IN NUMBER THEORY

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ABSTRACT. Following footsteps of Gauss, Euler, Riemann, Hurwitz, Smith, Hardy, Littlewood, Hedlund, Khinchin and Chebyshev, we visit some topics in elementary number theory. For matrices defined by Gaussian primes we observe a circular spectral law for the eigenvalues. We experiment then with various Goldbach conjectures for Gaussian primes, Eisenstein primes, Hurwitz primes or Octavian primes. These conjectures relate with Landau or Bunyakovsky or Andrica type conjectures for rational primes. The Landau problem asking whether infinitely many predecessors of primes are square is also related to a determinant problem for the prime matrices under consideration. Some of these matrices are adjacency matrices of bipartite graphs. Their Euler characteristics in turn is related to the prime counting function. When doing statistics of Gaussian primes on rows, we detect a sign of correlations: rows of even distance for example look asymptotically correlated. The expectation values of prime densities were conjectured to converge by Hardy-Littlewood almost 100 years ago. We probe the convergence to these constants, following early experimenters. After factoring out the dihedral symmetry of Gaussian primes, they are bijectively related to the standard primes but the sequence of angles appears random. A similar story happens for Eisenstein primes. Gaussian or Eisenstein primes have now a unique angle attached to them. We also look at the eigenvalue distribution of greatest common divisor matrices whose explicitly known determinants are given number theoretically by Jacobi totient functions and where unexplained spiral patterns can appear in the spectrum. Related are a class of graphs for which the vertex degree density is related to the Euler summatory totient function. We then apply cellular automata maps on prime configurations. Examples are Conway's life and moat-detecting cellular automata which we ran on Gaussian primes. Related to prime twin conjectures and more general pattern conjectures for Gaussian primes is the question whether "life" exists arbitrary far away from the origin, even if is primitive life in form of a blinker obtained from a prime twin. Most questions about Gaussian primes can be asked for Hurwitz primes inside the quaternions, for which the zeta function is just shifted. There is a Goldbach statement for quaternions: we see experimentally that every Lipschitz integer with entries larger than 1 is a sum of two Hurwitz primes with positive entries and every Hurwitz prime with entries larger than 3 is a sum of a Hurwitz and Lipschitz prime. For Eisenstein primes, we see that all but finitely many Eisenstein integers with coordinates larger than 2 can be written as a sum of two Eisenstein primes with positive coordinates. We also predict that every Eisenstein integer is the sum of two Eisenstein primes without any further assumption. For coordinates larger than 1, there are two curious ghost examples. For Octonions, we see that there are arbitrary large Gravesian integer with entries larger than 1 which are not the sum of two Kleinian primes with positive coordinates but we ask whether every Octavian integers larger than some constant  $K$  is a sum of two Octavian primes with positive coordinates. Finally we look at some spectra of almost periodic pseudo random matrices defined by Diophantine irrational rotations, where fractal spectral phenomena occur. The matrix is the real part of a van der Monde matrix whose determinant has relations with the curlicue problem in complex analysis or the theory of partitions of integers. Diophantine properties allow to estimate the growth rate of the determinants of these complex matrices if the rotation number is the golden mean.

## 1. INTRODUCTION

In this medley of experiments, we pick up some number theoretical themes. Our approach is mostly elementary and experimental and sometimes in linear algebra, graph theoretical or statistics context. The few mathematical remarks which appear in this text all have quick derivations. The topics have emerged in the last couple of years while teaching linear algebra courses or a course on "teaching math with a historical perspective" in the "math for teaching" program at the Harvard extension school. New phenomena came up especially while writing exam problems or computer algebra projects. Many of the featured experiments remain unexplained. We take the opportunity to illuminate the material also from a historical perspective focusing on mathematicians like Gauss, Euler, Goldbach, Riemann, Hurwitz, Smith, Hardy and Littlewood and Chebyshev, who were all interested in multiple fields of Mathematics like number theory and analysis.

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The history of mathematics illustrates that mathematical explorations initially are often experimental by nature: the Pythagorean theorem was first explored experimentally without proof, as writings on Clay tablets show [28]. Descartes discovered both the Euler polyhedral formula as well as the Goldbach conjecture by doing experiments, Fermat discovered the two square theorem experimentally. Conjectures by Fermat, Euler Gauss as well as Hardy and Littlewood came from experiments. The computations of Gauss counting primes led to the prime number theorem. Also his first work on quadratic reciprocity was experimental at first, before proofs got available, and Gauss himself got a few proofs. Many results in random matrix theory were exploratory at the beginning, including work of Wigner, Ginibre or Girko, who found the circular law mentioned below. Some open problems are constantly probed experimentally, like finding more structure in the roots of the zeta function by pushing the limits of Goldbach, investigating the statistics of frequencies of prime twins or searching for even perfect integers or perfect Euler bricks. The situation of Hardy and Littlewood is remarkable as it might have been the first time that “pure math” research mathematicians started to get assisted by collaborators who helped doing the computations, before electronic computers became available. Even today, in a time where “experimental mathematics” has become its own field [45, 19] and groups of research mathematicians like “polymath” collaborate, including experimental mathematicians who write and run code. Experiments feed the intuition and also help to fill computer assisted parts. The story of unsolved problems in number theory is closely linked to computations and experiments.

Primes in division algebras like Gaussian or Hurwitz or Octavian primes in Complex, Quaternionic or Octonion spaces give plenty of opportunity to make experiments. We got dragged into this also by reading [59] who proposed an exercise in Section 5, which totally mixed up our other summer plans. We certainly scratch only the surface while exploring this a bit. We take the opportunity to include seemingly remote topics like greatest common divisor matrices or matrices obtained from rotations using Diophantine angles. We will see for example, again following Hardy and Littlewood, that Goldbach problems in division algebras can be ported guided by calculus problems related to the circle method. There is plenty of opportunity for new questions. An example of a question which seems never have been asked is the existence of “Gaussian prime life” arbitrary far from the origin or whether a Prime twin theorem holds for Hurwitz or Octavian integers. Evidence that Goldbach holds for Hurwitz primes comes from relating the toughest boundary cases with Landau type problem which have quantitative generalizations given by Hardy-Liouville statistical laws which we test ourself up to  $2^{37}$ . In the Gaussian prime case already, we have full assurance that Gaussian Goldbach is hard, as it would imply an open Landau problem. Connections with other mathematical fields like topology comes in as prime numbers define classes of graphs. One can look for example at all the positive integers as the vertex set of a graph and connect two numbers  $a, b$  if  $a + ib$  is a Gaussian prime. Similarly, one can take the set of Gaussian integers  $a + ib, c + id$  in the first quadrant of the complex plane and connect two, if either  $a + ib + cj + dk$  is a Lipschitz prime or  $(a + ib + cj + dk)/2$  is a Hurwitz prime.

Primes in division algebras form an even grander arena for explorations than the traditional primes. It could even be an El Dorado for early explorations or in education. For Gaussian primes, there were attempts [58] in the 1960'ies to implement them into secondary education curricula. The question why quaternions have disappeared from mainstream calculus is interesting. [17] give as a reason the success of notations put forward by Gibbs [33] with precursor notes distributed as early as 1881. Coauthor Wilson in that Gibbs textbook acknowledges that quaternions were useful in helping the text. While quaternions are maybe a bit in the background in calculus frameworks, they are unbeaten in elegance in number theory: Gaussian primes produce the most natural frame work for proving some Diophantine problems and Hurwitz proof of the Lagrange four square theorem using quaternions can not be beaten in simplicity. This theorem implies that unlike for Gaussian primes, where half of the rational primes remain prime, all rational primes decay into Quaternionic primes.

The richness of the topic can be illustrated by the presence of open questions for Gaussian, Hurwitz and Octonion integers. Especially in the context of primes. One can ask questions about the **existence** of geometric patterns in the set of Hurwitz integers, like the prime twin problem for Hurwitz integers, additive number theoretical questions of Goldbach type in division algebras as done here, percolation problems of topological type like moat type problems in Hurwitz integers, as well as growth problems which deal with asymptotic behavior. We start also to look at correlation, determinant and trace type questions for Gaussian primes.

An other twist comes in when looking at primes as the input for dynamical systems. We can apply cellular automata to prime constellations for example. Cellular automata are dynamical systems, which emerged from topological dynamics are in some sense partial differential equations, where not only time and space is discrete, but where also the target space of the functions, here the alphabet  $A$  is discrete. They are useful in computer graphics, in algorithms for edge detection, for seeing morphological features, for smoothing or sharpening operations. Almost all image processing filters are cellular automata acting on the color frames of the lattice of pixels. One can apply such maps to configurations like Gaussian primes. Dynamical systems of a different kind enter when constructing matrices whose entries are  $A_{nm} = \cos(km\alpha + m\beta)$  where  $\alpha, \beta$  are Diophantine. These are real parts of van der Monde matrices  $B_{nm}$ . The spectra are unexplained. We are able to give a bound on the growth of the determinant of the  $B$  matrices if  $\alpha$  is the golden mean.

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## 2. GAUSSIAN PRIMES

**Gaussian primes** are the irreducible elements in the ring of **Gaussian integers**. The **arithmetic norm**  $N(z)$  of a Gaussian integer  $z = a + ib$  is defined as  $a^2 + b^2$ . If  $|z|$  denotes the **absolute value** of the complex number  $z$  then  $N(z) = |z|^2$  is the norm. Integers of norm 1 are called **units**. The four unit elements  $\{1, i, -1, -i\}$  form a cyclic multiplicative subgroup of the ring  $Z[i]$ . A number in  $Z[i]$  is called a **Gaussian prime** if it is not the product of two other numbers with smaller norm and norm larger than 1. Examples of Gaussian primes are  $2 + i$  or  $3$ . Examples of non-primes are  $1 + 3i$  or  $5$  as  $(1 + 3i) = (1 + i)(1 + 2i)$  and  $5 = (2 + i)(2 - i)$ . Also 0 and the units are not primes by definition. Since a unique factorization theorem holds in the ring  $Z[i]$  which is part of a **division algebra** satisfying  $|zw| = |z||w|$ , one can produce a list of Gaussian primes easily using the **baby prime test**: a Gaussian integer  $z$  is prime, if and only if for all  $w$  satisfying  $1 < |w| \leq \sqrt{|z|}$ , the Gaussian integer  $z$  is not a multiple of  $w$ . Fortunately, there is a much faster way to give a list of all Gaussian primes: one can just take the list of traditional primes  $\{2, 3, 5, 7, 11, \dots\}$  in  $\mathbb{N}$  called **rational primes** and associate to each a **Gaussian quadruple**  $z, iz, -z, -iz$  or an **octuple**  $z, iz, -z, iz, \bar{z}, i\bar{z}, -\bar{z}, i\bar{z}$  of Gaussian primes. The Gaussian primes can be seen as a “cover” of the rational primes: take a Gaussian prime  $z$ . Draw



the circle centered at 0 passing through a prime  $p$ . The radius  $|p|$  is either  $\sqrt{2}$  or a  $(4k + 3)$ -prime or then the square root of a  $(4k + 1)$ -prime. In the first two cases, there are 4 other Gaussian primes  $uz$  on the circle where  $u$  is one of the four **units**  $\{1, i, -1, -i\} = \{|z| = 1\} \cap Z[i]$ . In the last case, if  $z = a + ib$  with  $a \neq b$ , there are 8 primes  $uz$  and  $uz'$ , where  $u$  is a unit and  $z' = b + ia = i\bar{z}$ .

**Unique factorization** holds in  $Z[i]$  modulo units. The quotient  $X = \mathbb{C}/A$  of the complex plane by the dihedral group  $A$  generated by multiplication by units and the conjugation  $z \rightarrow \bar{z}$  is a cone on which the usual primes live. Due to the fact that for the primes on the axes have **prime length** and the primes off the axes have **prime norm**  $N(z) = |z|^2$ , the order is shuffled: the  $(4k + 1)$ -primes  $p$  are located at height  $\sqrt{p}$  while the  $(4k + 3)$ -primes  $p$  are located at height  $p$ . This will enrich slightly the Gauss zeta function by multiplying it with a beta

function as Dirichlet already knew. The cone  $X$  is an example of what topologists call an **orbifold**. This space  $\mathbb{C}/A$  is the quotient of the manifold  $\mathbb{C}$  by the equivalence relation defined by a finite group action. A geometer would say that the complex plane as a **ramified cover** over the space  $X$ . The rational primes  $2, 3, 11, \dots$  are special in that their ‘ramification index’ is different and 2 is very special as they are also invariant under conjugation. Topological jargon has entered number theory not accidentally: some number theorists were also geometers like Hurwitz, Gauss and Euler. There are plenty of accessible texts explaining the structure of Gaussian primes well [78, 15, 17, 74]. Gaussian integers are not without applications: for example in the study of eigenfunctions of the Laplacian on the torus [10] or to study discrete velocity models for the Boltzmann equation [25]. Engineers use it already for watermarks or codes [12], RSA [3] or zero-knowledge identifications [82]. Any question about primes in the integers  $\mathbb{Z}$  can also be asked in the integral domain  $Z[i]$  of Gaussian integers [31].

### Remarks.

1) According to Dickson [21], the Fermat two square theorem was noticed empirically first by Albert Girard, who was also the first to introduce the now common notations  $\sin, \cos, \tan$  for trigonometric functions as well

as the recurrence for the Fibonacci sequence [21] page 393. Since Fermat announced a proof of his theorem to Mersenne on December 25, that two-square theorem is also called the **Christmas theorem**.

2) There are various definitions for orbifold. We use the version telling that the quotient  $M/A$  of a finite group  $A$  acting on a smooth manifold  $M$  by smooth diffeomorphisms, such that for every  $a \in A$ , the set of fixed points is a smooth submanifold of  $M$ . In the Gaussian integer case, the group  $A$  is the dihedral group  $D_4$  on the complex plane generated by rotations  $z \rightarrow iz$  and reflections  $z \rightarrow \bar{z}$ . All elements clearly have fixed point sets which qualify. They are either the entire manifold, or a line or then a single point.

3) The covering story also applies to the ring  $\mathbb{Z}[i]$  of Gaussian integers. After dividing out the dihedral group  $A$  generated by the multiplication by units and the conjugation  $z \rightarrow \bar{z}$ , we obtain a structure in which the Gaussian primes correspond bijectively to the usual primes. But the larger dihedral symmetry also includes **complex conjugation**. The nature of an odd rational prime  $p$  can be read off from the **Jacobi symbol**  $(-1)^{(p-1)/2} = \left(\frac{-1}{p}\right) = (-1|p)$  which is in the case of primes  $p$  also called the **Legendre symbol**. Each prime of the form  $4k+1$  is by the **Fermat two square theorem** of the form  $p = a^2 + b^2$  and so the product of two Gaussian primes  $(a+ib), (a-ib)$ , the Gaussian primes on the axes are the  $4k+3$  rational primes. Away from the axes, we have also the Gaussian primes  $1+i$  and its conjugates.

4) A mathematical message of this section is that one can put a natural order structure the Gaussian primes after a suitable identification. It motivates questions like the following: define the sequence of numbers  $\theta(n) = \arg(p_n) - \pi/8 \in (-\pi/8, \pi/8)$ , where  $p_n$  is the  $n$ 'th prime of the form  $4k+1$ . As usual in topological dynamics, the half sequence  $\{\theta(n)\}_{n \in \mathbb{N}}$  defines a compact topological space  $X$ , the hull in the compact product space  $[-\pi/8, \pi/8]^{\mathbb{Z}}$  given by all the accumulation points. The shift  $T$  is now homeomorphism of this space and notions like topological entropy are defined. One can for example ask about the structure of the **invariant measures** of  $T$  etc. A natural conjecture is that the limiting system for these **prime angles** is a Bernoulli system, meaning that on the hull, the random variable  $X_k(\omega) = \theta_k(\omega)$  are all independent identically distributed, with uniform distribution on  $[-\pi/8, \pi/8]$ . Especially, **central limit theorems** and **laws of iterated logarithms** would hold. There are similar easy looking problems, like the system given by the integer digits of  $\pi$  or  $\sqrt{2}$  to some base, where a Bernoulli statement is believed to be true but not accessible to a proof yet.

5) One way to see that a number is composite if there are two fundamentally different ways to write it as a sum of two squares  $n = a^2 + b^2 = c^2 + d^2$  is due to Euler who pointed out  $n = ((a-c)^2 + (b-d)^2) * ((a+c)^2 + (b+d)^2) / (4(b-d)^2)$  which leads to a factorization. The relation with factorization could indicate that finding the angle  $\theta = \arg(a+ib)$  of a prime  $p = a^2 + b^2$  could be hard if only  $p = 4k+1$  is known. We are not aware of a faster determination of the angle  $\theta(p)$  than searching in  $O(\sqrt{p})$  time. One can ask for example whether it is possible to find the angle  $\theta$  in  $O(p^{1/4})$  computation steps from  $p$  or even in  $O(\log(p))$  time.

We can now look at the random walk when using the angles  $\theta(n)$ :

**Question:** How fast does  $S(n) = \sum_{k=1}^n \theta(p(n))$  grow, where  $p(n)$  is the  $n$ 'th prime of the form  $4k+1$ .

Our experiments indicate that this random walk does not behave differently than a usual random variable with uniform distribution in  $[-\pi/8, \pi/8]$ . It is known that the distribution of  $\theta$  is the uniform distribution. (References are given in [56]). We made experiments with correlations of neighboring  $\theta$ : the sequences  $Y(n) = X(2n)$  and  $Z(n) = X(2n+1)$  for example appear to be decorrelated, indicating that it is difficult to predict the angle of prime  $k$  if the angle of prime  $k-1$  is known.

Any way, we are not aware of a **fast way** to compute the unique  $(a, b)$  with  $0, b < a < \pi/4$  satisfying  $a^2 + b^2 = p$  if  $p$  is a  $4k+1$  prime.

**Remark:** On the quotient space  $\mathbb{Z}[i]/D_4$ , the Gaussian primes admit a natural total order and one-to-one bijection to the rational primes.

Of course, since both sets are rational, there existed also other bijections. The point is that we can relate them naturally. While this does not appear to be useful, it is a source for new questions like about the distribution of the angle.

**Remarks:**

1) Some results have been pushed over from rational primes to Gaussian primes. There is for example an analogue of Dirichlet's theorem on arithmetic progressions: for an arbitrary finite set in  $\mathbb{Z}[i]$ , there exist

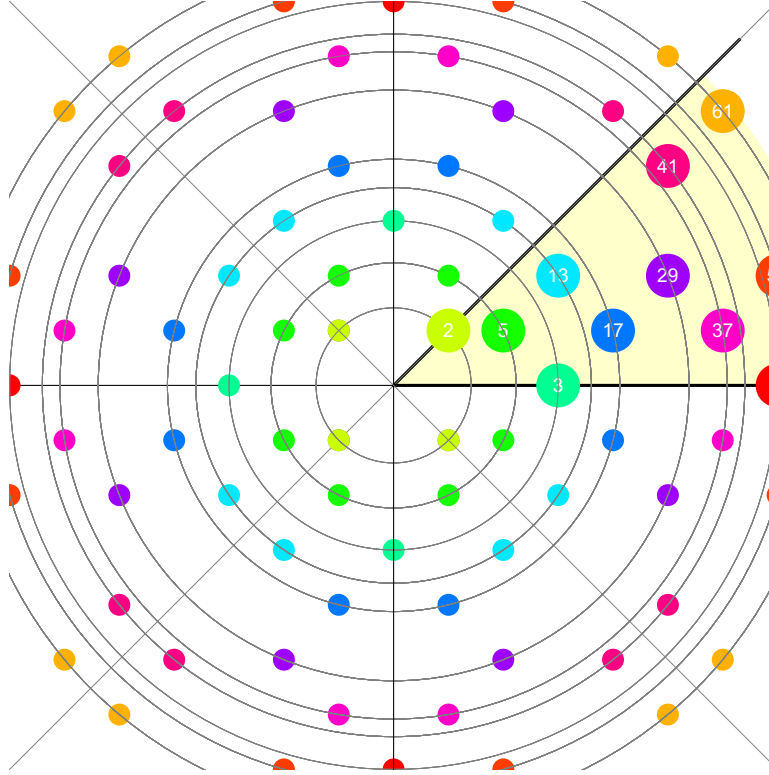


FIGURE 1. Gaussian primes are either located on the real or imaginary axes, given by primes of the form  $4k + 3$ , or then lattice points on circles of radius  $r$ , where  $p = r^2$  is prime. In other words, the circles on which Gaussian primes are located are in one to one correspondence to the standard rational primes. Only the order is shuffled. The second smallest circle corresponds to the primes  $\pm 2 \pm i$  and so to the prime 5. The third then to the prime 3. Each of the primes on the symmetry lines come with multiplicity 4, the rest with multiplicity 8. The picture shows the first octant sector, which is the fundamental domain for the action of the dihedral group  $A = D_4$  on the complex plane  $\mathbb{C}$ .

infinitely many  $a \in \mathbb{Z}[i]$  and  $r \in \mathbb{R}^+$  such that  $a + r \sum_{f \in F} v_f$  is a Gaussian prime [79]. An other example is that the density of the **prime quotients**  $p/q$  in  $\mathbb{R}^+$  generalizes to the proven statement that the Gaussian prime quotients  $p/q$  are dense in the complex plane [30]. Patterns are explored in [47, 46].

2) The connection between Gaussian primes and rational primes has been used by Gauss as a tool for **quadratic reciprocity**. Also Dirichlet realized the connection. The reason is that the multiplicative part of  $\mathbb{Z}[i]/p$  which has size  $N(p) - 1$  can be generated as  $a^k \bmod p$  as in the rational case. Now define the birational Jacobi symbol  $(a|p)$  as  $a^{(N(p)-1)/2} \bmod p$ . If this is  $-1$ , then there is a solution to  $x^2 = a \bmod p$  by Fermat's little theorem.

3) While the bijective map  $p \rightarrow a + ib$  with  $a > 0, b \geq 0, a < b$  from rational primes to equivalence classes of Gaussian primes  $\mathbb{C}/A$  has an inverse  $a + ib \rightarrow N(a + ib)$  which can be computed with no effort, the map itself needs some searching. Since most cases are where  $a$  is very small, this is usually found fast, but its not elegant.

4) While the identification of Gaussian primes with rational primes after taking equivalence classes is nice, there is still much more structure in the Gaussian primes. We will see that when formulating a Goldbach conjecture, when applying cellular automata map on them or by looking at the Gaussian Riemann zeta function, which is an example of a **Dedekind zeta function**. It leads to interesting structures like the **complex Möbius function** which is defined as in the rational case but for which the corresponding Mertens function fluctuates more.

5) There is a "one line" topological proof of the two square theorem [97]. But as it relies on a fixed point theorem, it is not constructive. Lets briefly sketch it: it uses the compact surface with boundary  $x^2 + 4yz = 13, x \geq 0, y \geq 0, z \geq 0$  on which there are two involutions  $T(x, y, z) = (x, z, y)$  and  $S(x, y, z)$  which is  $(x + 2z, z, y - x - z)$  if  $x < y - z$  and  $2y - x, y, x - y + z)$  if  $y - x < x < 2y$  and  $(x - 2y, x - y + z, y)$  if  $x > 2y$ .

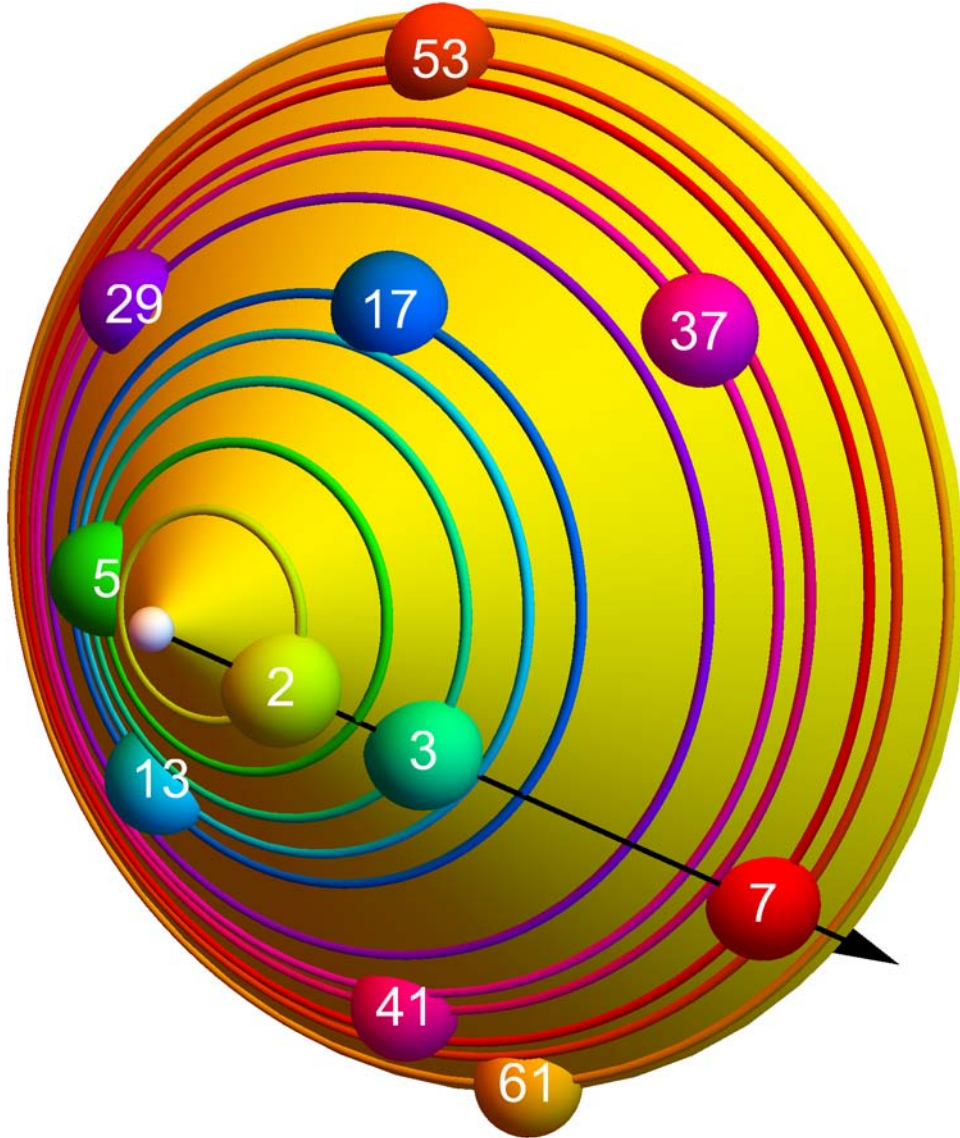


FIGURE 2. When drawn on the cone orbifold  $\mathbb{C}/A$ , where  $A$  is the dihedral symmetry group, every traditional rational prime can be matched in a one-to-one manner with an equivalence class of Gaussian primes. The rational primes of the form  $4k + 3$  are on the real axes where the cone is glued. The angles  $\theta(n)$  at which prime  $p(n)$  is located, appear to pretty random. All correlation tests done so far indicate this.

if  $2y < x$ . The map  $T$  has exactly one fixed point  $(1, 1, k)$  if  $p = 4k + 1$ . Since the fixed points  $T$  and the number of fixed points of  $S$  are the same modulo 2, also  $S$  must have a fixed point of the form  $(x, y, y)$ . This is topological as the argument about fixed points is essentially a Riemann-Hurwitz argument relating the difference  $\chi(X) - 2\chi(X/T)$  as a sum of ramification indices of fixed points.

**6)** Also the Minkowski proof using geometry of numbers or the Jacobi proof using theta functions  $\theta(x) = \sum_n x^{n^2}$  is not constructive. To the later: as  $\theta(x)^2$  is the generating function of the number  $2a(n)$  counting the way to write a number  $n$  as a sum of two squares and this is equal to  $1 + 4 \sum_{n \geq 0} x^{4n+1}/(1-x^{4n+1}) - x^{4n+3}/(1-x^{4n+3})$ , this is  $a(n) = 4(d_1(n) - d_3(n))$ , as  $2d_k(n)$  counting twice the number of divisors larger than 1 of  $n$  which are congruent to  $k$  modulo 4 has the generating function  $\sum_n x^{4n+k}/(1-x^{4n+k})$ . For a number  $n = p$  which is a  $4k + 1$  prime, this means  $a(n) = 4$ . Also this is not constructive. Also the proof using the solution  $x^2 = -1$  modulo  $p$  which is possible as  $-1$  is a quadratic residue using finding “square roots” of  $-1$  to reduce the problem.

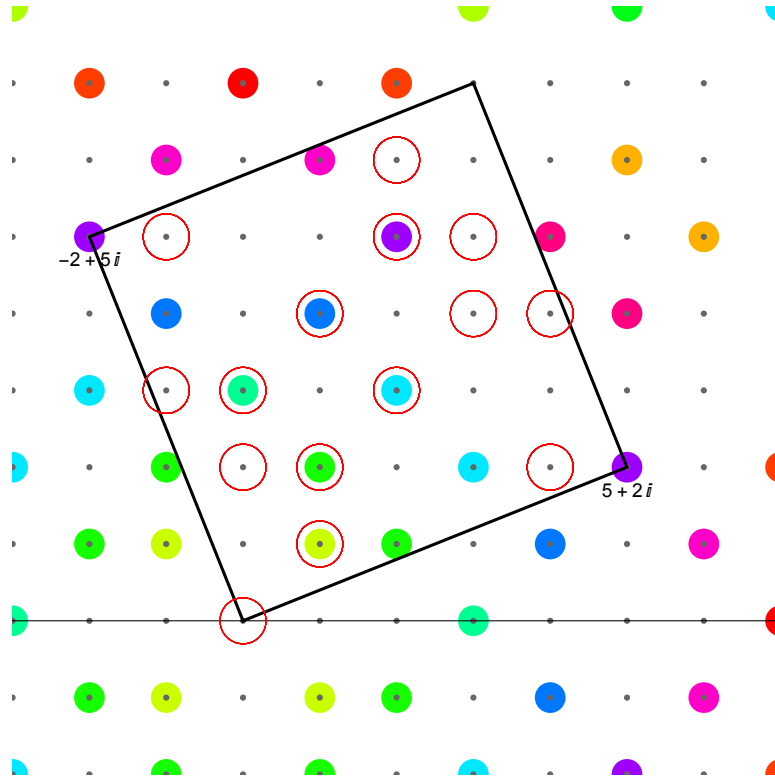


FIGURE 3. We see the region  $\mathbb{C}$  modulo a prime  $p = 5 + 2i$  and unit multiples. A number  $z \bmod p$  is in the square. There are  $N(p)$  equivalence classes. For odd  $N(p)$ , half of them are quadratic residues. The quadratic reciprocity result which Gauss proved already knew for Gaussian primes tells which primes are quadratic residues.

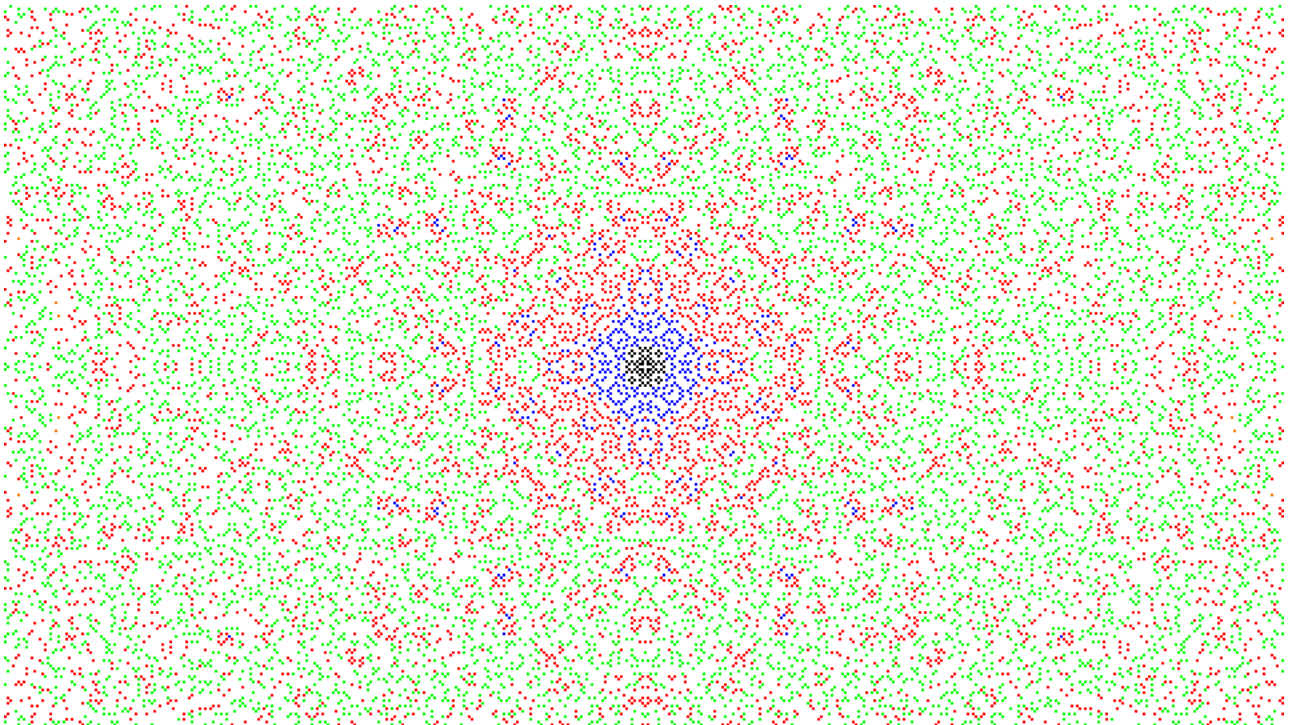


FIGURE 4. A larger part of the Gaussian primes.

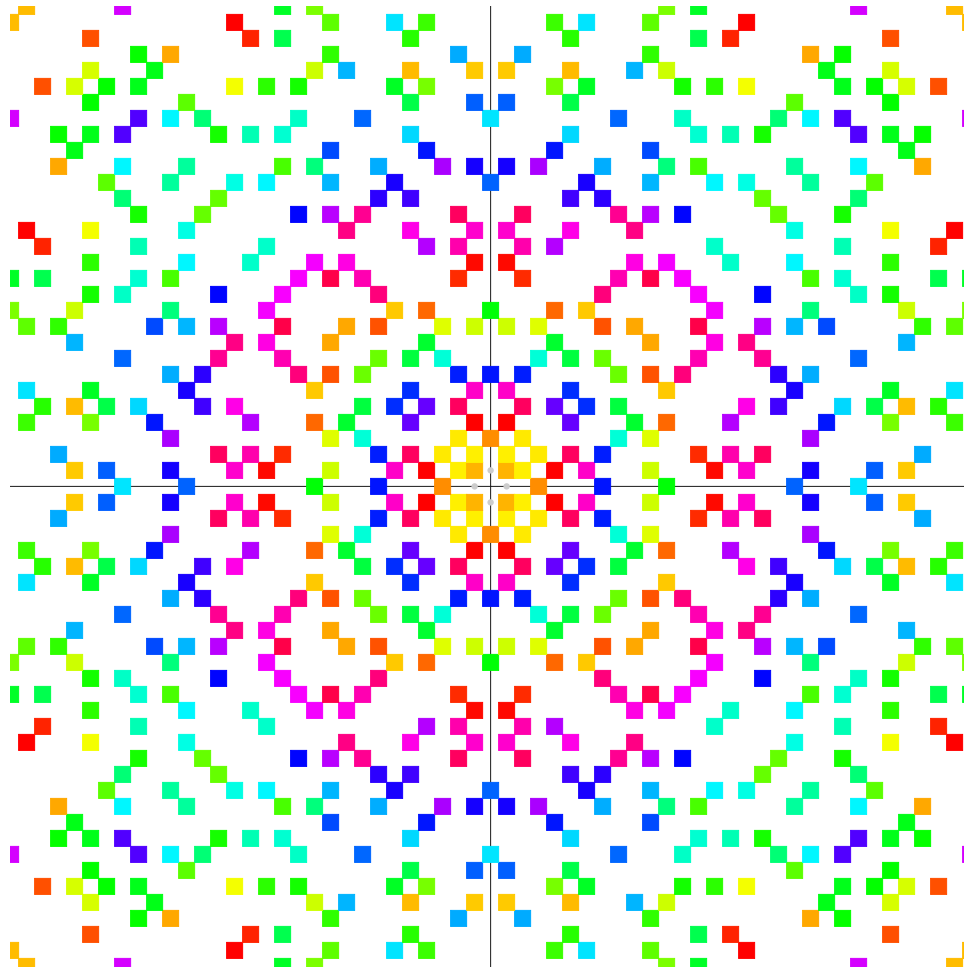


FIGURE 5. Gaussian primes have more structure near the origin which is amplified by the dihedral symmetry. According to a “Fortune” article of June 1958 (which we unfortunately can not get hold of), the Gaussian prime pattern has been used for a tea cloth.



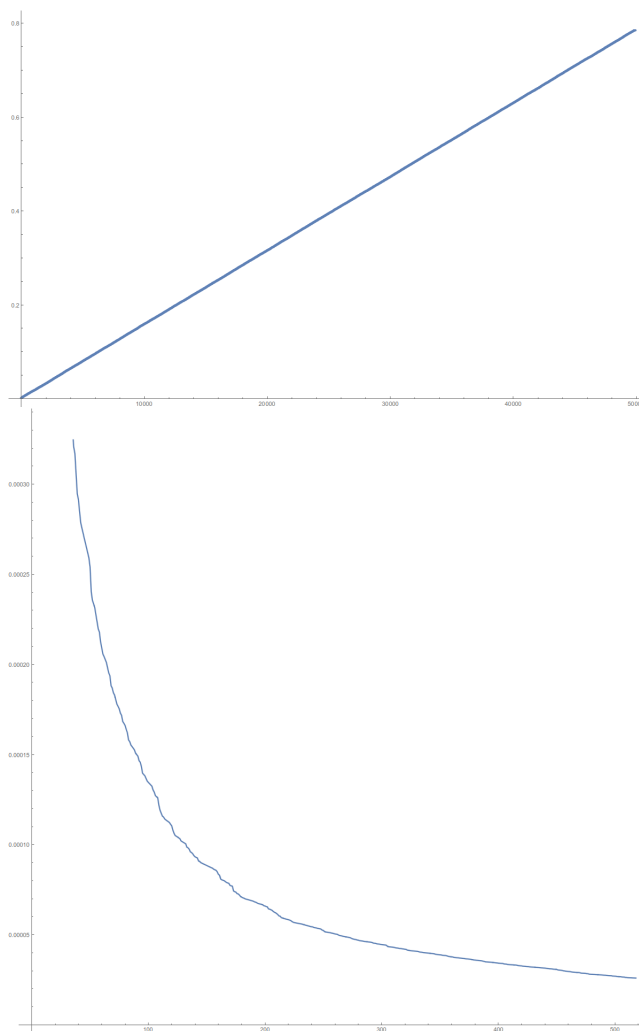


FIGURE 6. The argument  $\arg(z)$  over primes becomes uniform: no angular direction is favored. The function  $p \rightarrow \arg(P)$ , where  $P$  is the unique Gaussian prime with argument  $\theta(p) \in [0, \pi/4)$  belonging to  $p$  is a random variable which is pretty random for the primes  $p$  of the form  $4k + 1$ . The second figure shows the correlation between the vectors  $X(n) = (\theta(P(1)), \dots, \theta(P(2k+1)), \dots, \theta(P(2n+1)))$  and  $Y(n) = (\theta(P(2)), \dots, \theta(P(2k)), \dots, \theta(P(2n)))$  as a function of  $n$ .

## 3. LIPSCHITZ AND HURWITZ PRIMES

Next to the field of **complex numbers** comes the skew field of **quaternions**. Quaternions are numbers of the form  $a + bi + cj + dk = (a, b, c, d)$ , where  $i, j, k$  are symbols satisfying  $i^2 = j^2 = k^2 = ijk = -1$ . Discovered by Hamilton in 1843 they introduced simultaneously the **inner product** and **cross product** to vector calculus because  $(0, v_1, v_2, v_3) \cdot (0, w_1, w_2, w_3) = (-v \cdot w, v \times w)$ , even so the later structures were only promoted by Gibbs. Taking the product of three such quaternions produces the **triple product**  $(0, v_1, v_2, v_3) \cdot (0, v_1, v_2, v_3) \cdot (0, w_1, w_2, w_3) = (u \cdot (v \times w), u \times (v \times w))$ . We see that both the **triple scalar product** as well as the **triple cross product** are present. The textbook [33] is the key to see how experts in quaternions used their knowledge to introduce simpler and now popular notions and by doing so erase the quaternions from calculus textbooks.



1

Quaternions are useful in **computer vision** to realize a rotation in space around a unit vector  $u$  by an angle  $\theta$  by building the two quaternions  $s = v_1i + v_2j + v_3k$  and the  $r = \cos(\theta/2) + \sin(\theta/2)(u_1i + u_2j + u_3k)$ , then get the rotated vector  $Rv$  from the spacial components of the quaternion  $r \cdot s \cdot r^*$ .

Number theory in the ring of quaternions is a bit more strange, as quaternion multiplication is **no more commutative**. Furthermore, the ring of integer quaternions is no more a **unique factorization domain**. Even the Euclidean algorithm comes short. Hurwitz [43] found a way out of it and realized that

one can get a **Euclidean domain** when including half units. Quaternions with "integer" coordinates are called **Lipschitz integers**  $a + ib + jc + kd$ , where  $(a, b, c, d) \in \mathbb{Z}^4$ . The others which must be of the form  $(a, b, c, d) \in \mathbb{Z}^4 + 1/2(1, 1, 1, 1)$  to satisfy  $N(z) \in \mathbb{N}$  are called the **Hurwitz integers**. In this text, we want to use these two names to distinguish between the two distinct classes of integers and call their union simply **quaternion integers**. The quaternion integers form a non-commutative ring of the quaternion division ring.

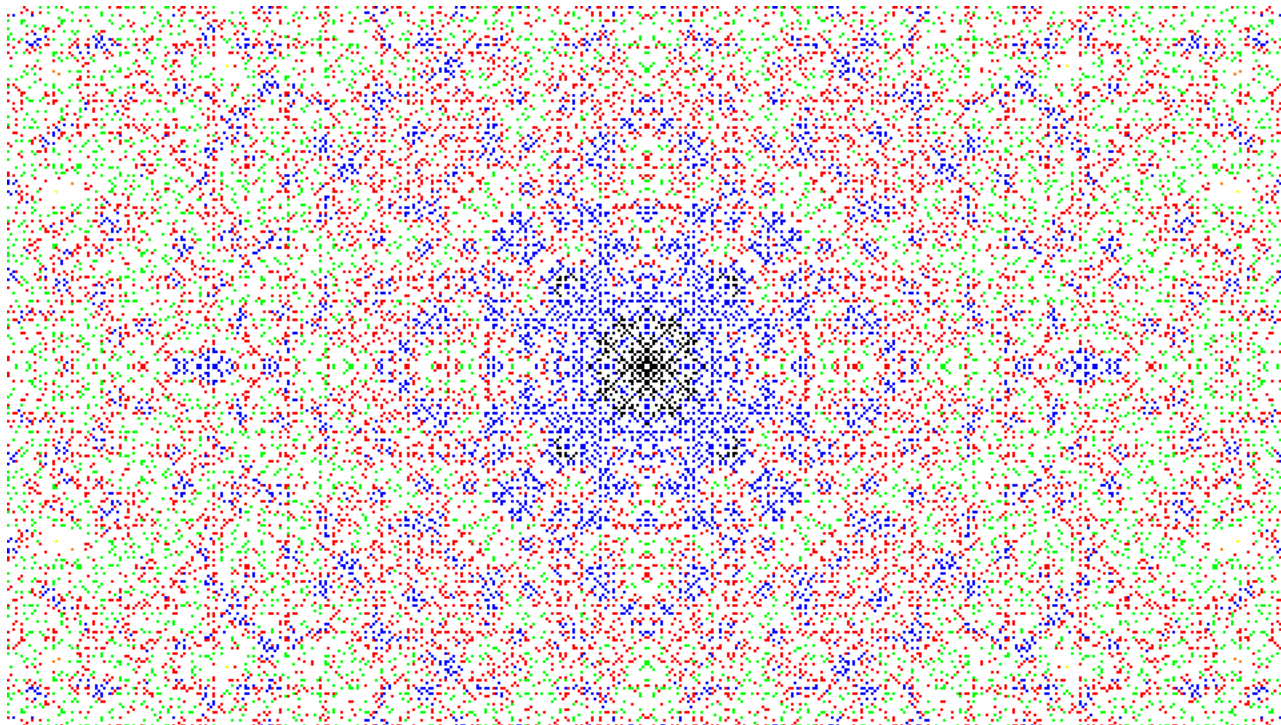
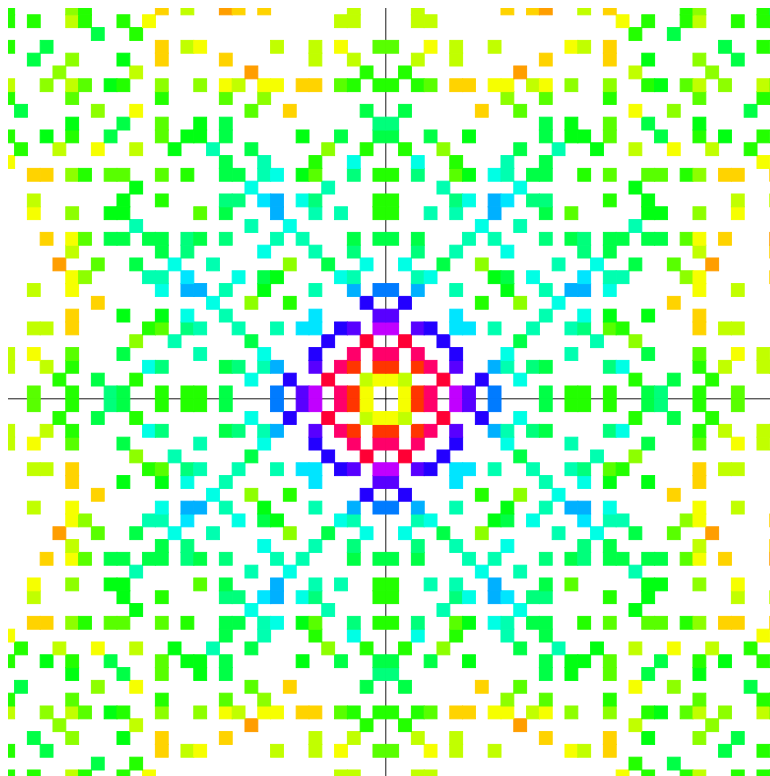


FIGURE 7. A slice through the Hurwitz primes: it is the two-dimensional plane  $(1, 1, 2x + 1, 2y + 1)/4$ . The pixel color  $(x, y)$  depends on the number of neighbors in that plane.

<sup>1</sup> The quaternions remind of the Middlesex canal which connected the Merrimack river with Boston: it was used to build the railroads, which in turn made the canal obsolete. Similarly, the success and applicability of the dot and cross product, which were historically built through quaternions led to their disappearance in vector calculus [17]



There are 24 **units** in the set of quaternion integers. Eight of them are Lipschitz integers given by permutations of  $(\pm 1, 0, 0, 0)$  and 16 are Hurwitz integers of the form  $(\pm 1, \pm 1, \pm 1, \pm 1)/2$ . The units build a subgroup of the quaternion group. It goes under the name **binary tetrahedral group**  $U = SL(2, 3)$ . As quaternions can be written as unitary  $2 \times 2$  matrices,  $U$  is a discrete finite subgroup of the unitary group  $SU(2)$ . It can be identified as the semi-direct product of the **quaternion sub group**  $Q$  built by the 8 Lipschitz units and the cyclic group  $Z_3$ , generated by conjugation  $i \rightarrow j \rightarrow k \rightarrow i$ . The group  $U$  is a finitely presented group with relations  $(ab)^2 = a^3 = b^3 = 1$  and generators  $a = (1 + i + j + k)/2$  and  $b = (1 + i + j - k)/2$ . All cyclic subgroups have order 2, 3 or 6. The element  $-a$  for example generates a cyclic group of order 6.

There is an other group  $V$  with 24 elements acting on the ring of Hurwitz integers. It is the group  $S_4$  of permutations of the 4 basis elements. Both  $U$  and  $V$  are non-abelian but solvable. Group theorists have counted that there are exactly 15 finite groups of order 24 up to isomorphisms and the two just mentioned groups belong to the interesting ones. One can see  $V$  also as a subgroup of the **group of automorphisms** of  $U$ . The set of **inner automorphisms**  $x \rightarrow e^{-1}xe$  of  $U$  has order 12 and is isomorphic to the **alternating group** of four elements. But it is not the group of even permutations of  $\{1, i, j, k\}$  which has the same order: while the rotation  $i \rightarrow j \rightarrow k \rightarrow i$  is inner, the rotation  $1 \rightarrow i \rightarrow j \rightarrow 1$  is of course not, because 1 is mapped into itself by any inner automorphism.

Before we move on, lets just remember that we had a similar situation for Gaussian integers. There was also the group of units  $U$  with four elements and the group of automorphisms  $V$ . commutativity has prevented the existence of non-trivial inner automorphisms and conjugation was the only outer automorphism, as the units formed the group  $U = Z_4$  which has  $V = Z_2$  as automorphism group. When identifying integers using both groups  $U, V$ , we factor out the dihedral group  $D_4$  leading to the bijection between rational primes and **Gaussian prime classes**.

Returning back to the quaternions, we have two groups  $U, V$  acting on the ring  $X$  of quaternion integers. It is so possible to do identifications with either of them and form  $X/U$  or  $X/V$ . We can also factor out both of them. Note that as  $U$  and  $V$  are both subgroups of the symmetry group  $S_{24}$  of all permutations on  $U$ , one can look at the subgroup of order 48 generated by both.

The **irreducible elements** in in the ring of Hurwitz integers are the **quaternion primes** or simply **primes** if the context is clear. They are known to be Lipschitz or Hurwitz integers  $z$  with prime norm  $p = N(z) = z\bar{z}$ .

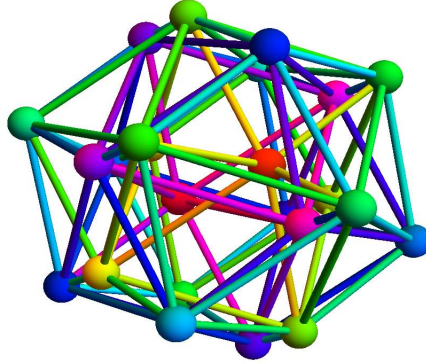


FIGURE 8. The units of the Gaussian integers is the **binary tetrahedral group**. It is a discrete subgroup of  $SU(2)$  with 24 elements. It contains the 8 Lipschitz units located on the 4 coordinate axes  $\pm 1, \pm i, \pm j, \pm k$  and additionally the 16 units of the form  $(a, b, c, d)/2$  with  $a, b, c, d \in \mathbb{Z}_2$ . The members of  $U$  can be arranged as the vertices of the **24 cell**, one of the six **Platonic solids** of dimension 3 realized as regular 4-polytop located on the 3-sphere in four dimensional space.

Geometrically, they are the integer or half integer lattice points on the sphere of radius  $\sqrt{p}$  in  $\mathbb{R}^4$ . To render terminology simpler, we call the integer coordinates ones the **Lipschitz primes** and the others called **Hurwitz primes**. Since by the **Lagrange's four square theorem**, every positive integer can be written as a sum of four squares, there are no quaternion primes on the four coordinate axes. We can factor an integer  $p + 0i + 0j + 0k$  as  $(a, b, c, d) \cdot (a, -b, -c, -d) = a^2 + b^2 + c^2 + d^2 = p$ . Actually, Hurwitz gave a proof of the Lagrange four square theorem using quaternions. Hurwitz proved also [43]:

**Remark:** (Hurwitz) If  $p$  is a rational prime, then the number of quaternion primes modulo  $U$  sitting above  $p$  is exactly  $p + 1$ .

The result holds only for odd  $p$  as for  $p = 2$ , there there is only the prime  $(1, 1, 0, 0)$ . Note that  $2 = (1 + i)(1 - i) = (1 + j)(1 - j) = (1 + k)(1 - k)$  are factorizations of 2. For  $p = 3$ , the  $p + 1 = 4$  prime classes are  $i + j + k, (1 + 3i + j + k)/2, (1 + 3j + i + k)/2, (1 + 3k + i + j)/2$ .

Let us look at the Hurwitz integers on the sphere of radius  $\sqrt{p}$  and factor out first only the automorphism group  $S_4$  which commutes the vector entries of a Hurwitz integer  $(a, b, c, d)$ . The equivalence classes are Hurwitz integers of the form  $(a, b, c, d)$  with  $a, b, c, d \geq 0$  and  $a \leq b \leq c \leq d$ . We call such a Hurwitz integer **positively ordered**.

In the case  $p = 2$ , the only representative is  $(0, 0, 1, 1)$ , for  $p = 3$ , the classes are represented by the primes  $(0, 1, 1, 1)$  and  $(1, 1, 1, 3)/2$ . As an other example, for  $p = 13$ , the positively ordered Hurwitz primes lying above  $p$  are  $(0, 0, 2, 3), (1, 1, 1, 7)/2, (1, 1, 5, 5)/2, (1, 2, 2, 2)$ , and  $(3, 3, 3, 5)/2$ . We see that unlike for Gaussian integers, there are now in general several equivalence classes of primes sitting above each prime.

Now, these classes can be divided into equivalence classes using the group  $U$ : two positively ordered primes  $p, q$  are **equivalent** if either  $p/q$  is unit of  $p/\bar{q}$  is a unit.

**Remark:** The orbits of the group  $U$  acting on these positively ordered primes belonging to an odd rational prime  $p$  all have length 2 or 3.

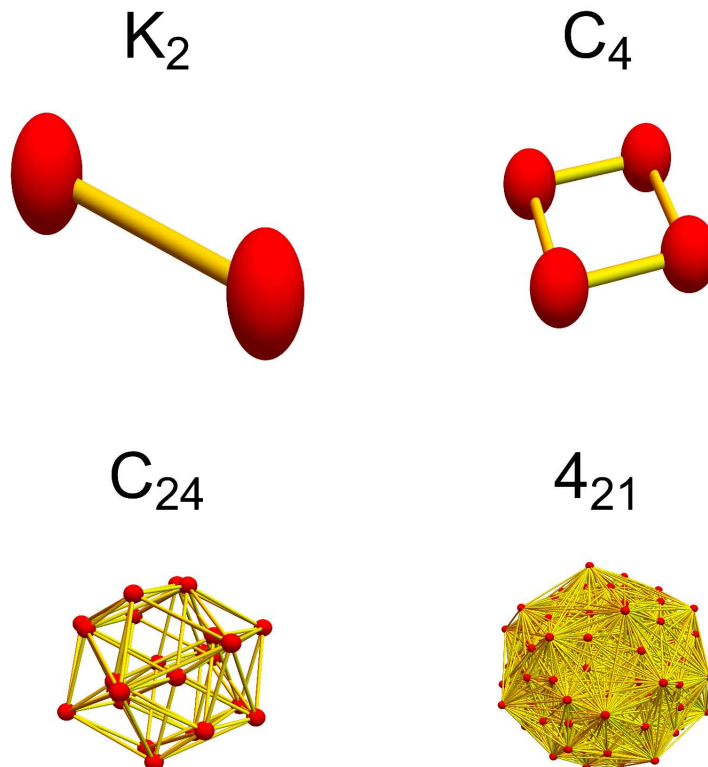


FIGURE 9. These are illustrations of the units in the four normed division algebras  $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ . We see the  $K_2$  for  $\{\pm 1\} \subset \mathbb{R}$ ,  $C_4 = \{1, i, -1, -i\} \subset \mathbb{C}$ , the 24 cell in  $\mathbb{H}$  and the Gosset polytope in  $\mathbb{Q}$  generating the  $E_8$  lattice. They all are known to generate the densest sphere packings, except for  $\mathbb{C}$ , where one has to look at the lattice generated by the Eisenstein integers instead.

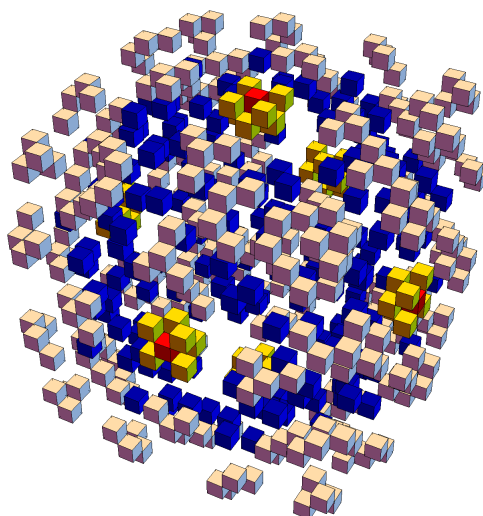


FIGURE 10. An illustration of a slice through all Lipschitz primes  $(a, x, y, z)$  for  $a = 701$ . They are given by the integer vectors  $(a, x, y, z)$  for which  $a^2 + x^2 + y^2 + z^2$  is a rational prime.

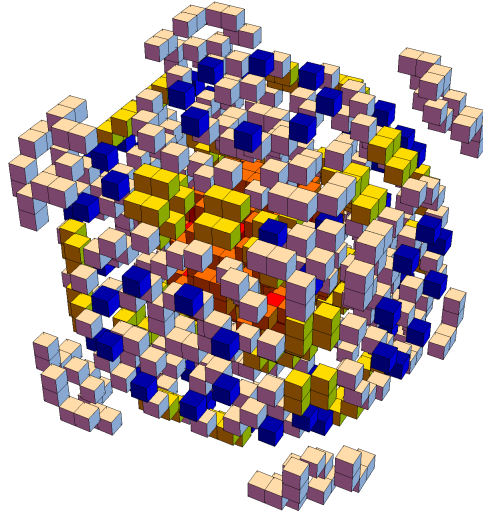


FIGURE 11. An illustration of a slice through all Hurwitz primes  $(1+2a, 1+2x, 1+2y, 1+2z)/2$  for  $a = 3001$ . They are given by the integer vectors  $(a, x, y, z) + (1, 1, 1, 1)/2$  for which the sum of the squares is a rational prime.

For example, there are exactly two positively ordered units. They are  $\{(1, 0, 0, 0), (1, 1, 1, 1)/2\}$ . The positively ordered Lipschitz primes with coefficients  $\leq 2$  are  $\{(0, 0, 1, 2), (0, 1, 1, 1), (1, 1, 1, 2), (1, 2, 2, 2)\}$  with norm 5, 3, 7, 13. The positively ordered Hurwitz primes with coefficients  $\leq 2$  are  $\{(1, 1, 1, 3)/2, (1, 1, 3, 3)/2, (1, 3, 3, 3)/2\}$  with norm 3, 5, 7.

One can visualize the primes by intersecting them with two dimensional planes while the plane moves further and further away from the origin.

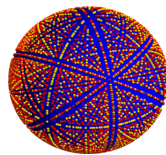


FIGURE 12. All the Hurwitz (red) and Lipschitz primes (yellow)  $(a, b, c, d)$  with  $|a|, |b|, |c|, |d| < M = 8$ , projected onto the first three coordinates and then projected onto the unit sphere.

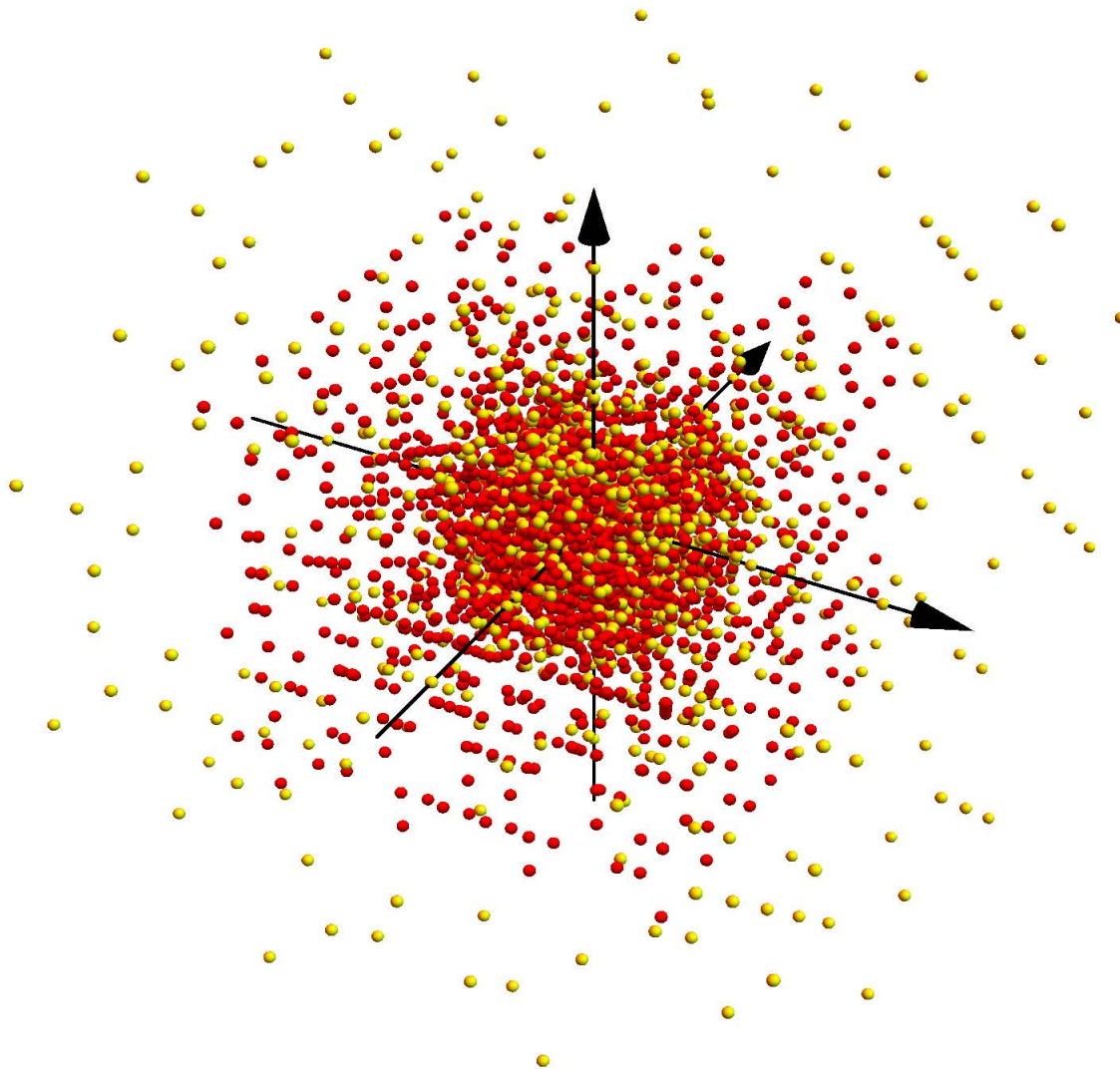


FIGURE 13. A stereographic projection of the Hurwitz prime sphere from  $\mathbb{H}$  to  $\mathbb{R}^3$  but for the cube radius  $M = 5$ .

#### 4. GOLDBACH FOR GAUSSIAN PRIMES

In this section, we formulate a Goldbach conjecture for Gaussian integers. It is not the first version of the kind: Mitsui got a version for integer rings in number fields, [62], a second one is given [41] for Gaussian primes. Holben and Jordan use an angle condition: their statement is that every even Gaussian integer of norm larger than 2 can be written as a sum of two Gaussian integers  $a, b$  such that  $\arg(a/z), \arg(b/z) < \pi/4$ . Mitsui assumes only a positivity condition for real conjugates and so no positivity condition in the Gaussian case. He just asks every even  $z$  to be representable as a sum of two primes.



As Goldbach in  $\mathbb{Z}$  deals with the sum of positive primes, we use primes in the **open first quadrant** of the complex plane:

$$Q = \{a + ib \mid a > 0, b > 0\} .$$

A Gaussian integer  $z$  is called **even** if it is of the form  $z(1+i)$ , where  $z$  is a Gaussian integer. The even Gaussian integers  $z$  are of the form  $z = a + ib$  with  $a + b$  even. Equivalently, they are even if  $N(z) = a^2 + b^2$  is even. The

**Goldbach conjecture for Gaussian primes** is:

**Conjecture:** Every **even** Gaussian integer  $z = a + ib$  with  $a > 1, b > 1$  is the sum of two Gaussian primes in  $\mathcal{Q}$ .

In other word, for every Gaussian integer  $z = (1+i)w$  in  $\mathcal{Q}$  we have a sum  $z = p + q$ , where  $p, q$  are Gaussian primes in  $\mathcal{Q}$ . It implies the following **ternary Gaussian prime conjecture**:

**Conjecture:** Every Gaussian integer  $a + ib$  in  $\mathcal{Q}$  satisfying  $a > 2, b > 2$  is the sum of three Gaussian primes in  $\mathcal{Q}$ .

Details are important. The formulation uses the **open** first quadrant, which does not include points on the real and positive axes. It uses a **definite number of summands** and not “at least” two or three summands as 0 is not counted as a prime. We also do not mean that the primes are required to be different. The even Gaussian integer  $2 + 2i$  for example can only be written as the sum of two identical primes  $p = 1 + i$  and  $q = 1 + i$ . It could well be that for large  $|z|$ , the additional requirement that the summands have to be different could be imposed additionally. One could weaken the conjecture also by allowing the summands to be in the **closed first quadrant**  $\overline{\mathcal{Q}} = \{a + ib \mid a \geq 0, b \geq 0\}$ , but there is strong enough evidence for the open version. The open version given here also has the advantage that it relates to the difficult Landau problem, so that it is not in danger of being “trivial”. As any conjecture it is in danger to be wrong however.

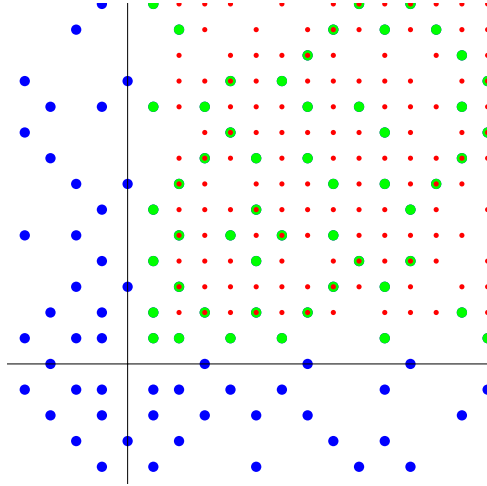


FIGURE 14. The Gaussian Goldbach conjecture.

The statement is natural because it can be reformulated for Taylor or Fourier coefficients of powers  $f^2$  and  $f^3$ , where a **Gaussian Goldbach function** given by

$$f = \sum_{a+ib \in P \cap \mathcal{Q}} x^a y^b / (a!b!)$$

sums over the set of Gaussian primes in  $\mathcal{Q}$ . For  $x = e^{i\phi}/2, y = e^{i\theta}/2$ , one gets smooth double periodic functions  $f(\phi, \theta)$ . These are Gaussian Goldbach functions on the two-dimensional torus. The factorial terms are an example but assure that the functions are smooth. As for rational primes, the Goldbach problem for Gaussian primes is now a **calculus problem** which lies at the basis of the **Hardy-Littlewood circle method**. A naive hope is to find coefficients  $g_{ab}$  with support on the Gaussian primes such that  $f = \sum g_{ab} x^a y^b$  is expressible as a finite sum of known functions, allowing to check the conjecture by computing Taylor or Fourier coefficients of  $f^2$ : in the Taylor picture, the Gaussian Goldbach conjecture.

**Remark:** The Goldbach conjecture follows from the statement that  $(f^2)_{kl} \neq 0$  for every pair  $k, l > 1$  for which  $k + l$  is even. Alternatively, it follows from  $\hat{f}^2_{k,l} \neq 0$  for every  $k, l > 1$  with  $k + l$  even.



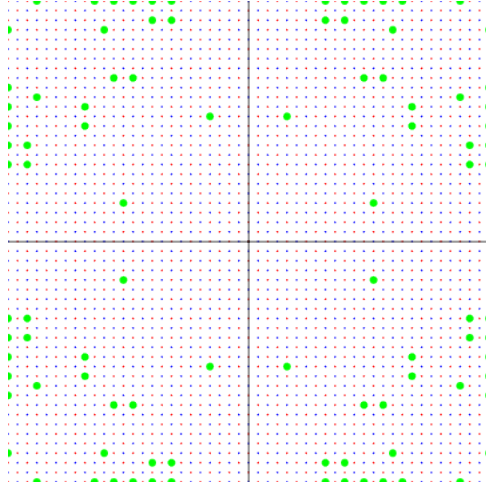


FIGURE 15. Which Gaussian integers are not the sum of two Gaussian primes? The smallest example is  $4 + 13i$ . A rather weak Gaussian Goldbach conjecture predicts that all such examples are odd.

There are interesting subproblems of the Goldbach problem for Gaussian integers. One can restrict for example to the **diagonal** and ask whether every Gaussian integer  $k + ik$  with  $k > 1$  can be written as a sum of two primes with positive entries. Here is the **diagonal Gaussian Goldbach** conjecture:

**Conjecture:** Every Gaussian diagonal integer  $k + ki$  with  $k > 1$  can be written as a sum of two Gaussian diagonal primes  $a + ib, c + id$  in  $\mathcal{Q}$ .

While Gaussian Goldbach would imply the diagonal version, the point of a diagonal version is that it appears to be much easier so that there is some chance that it could be solved with some already available techniques. As the map from a  $(4k + 1)$ -prime  $p = a + ib$  to  $a + b$  with  $b < a, a > 0, b > 0$  is defined, the question is related to the existence of **large gaps** on the class of  $4k + 1$  primes. If gaps near  $x$  are of size smaller than  $O(\sqrt{x})$  then the result follows. The classical **Andrica conjecture** asks whether the gap size is always smaller than  $2\sqrt{x} + 1$ . The diagonal Goldbach subquestion appears easier as there are in general many more ways to write  $k + ki$  as a sum of two Gaussian primes. Also, the general belief is that the prime gaps are much smaller: the **Cramer conjecture** for example predicts that gaps are not bigger than  $C \log(x)^2$ . And the same can be expected for gaps in the set of primes of the order  $4k + 1$ .

An other possibility is to weaken the conjecture and for a Schnirelman type result:

**Conjecture:** There exists a constant  $M$ , such that every Gaussian integer  $z = a + ib$  with  $a \geq M, b \geq M$  in  $\mathcal{Q}$  can be written as a sum of maximal  $M$  Gaussian primes in  $\mathcal{Q}$ .

This statement looks more approachable as the corresponding statement for rational integers has shown. Schnirelman proved the rational case using the concept of **Schnirelman density**. But it still would imply the Landau problem. And the boundary case is the Achilles heel as it could be that the Schnirelman density is zero at the boundary.

The ternary Goldbach problem is implied by the Gauss Goldbach problem and also weaker. Again, establishing the diagonal case appears to be the easiest: write every integer  $k \geq 3$  as a sum  $k = a + b + c = d + e + f$ , where  $a^2 + d^2, b^2 + e^2, c^2 + f^2$  are prime. Establishing the result for the third row  $k + 3i$  again would imply the open Landau problem about the infinitude of primes of the form  $n^2 + 1$ . So, also the ternary Goldbach problem appears to be hard.

What about the following statement?

**Conjecture:** Every even Gaussian integer is the sum of two Gaussian primes.

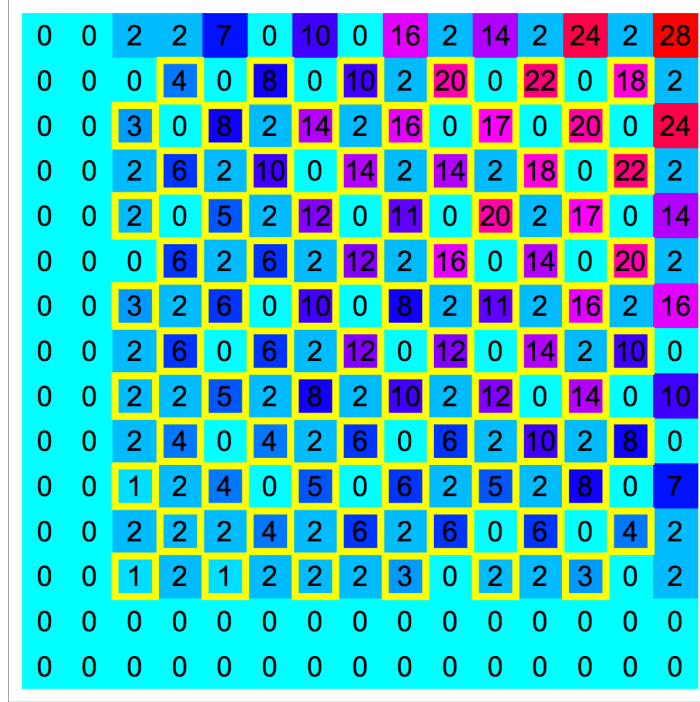


FIGURE 16. The **Gaussian Goldbach problem** for the open first quadrant  $Q = \mathbb{Z}^+[i]$  claims that every even  $a + ib \in \mathbb{Z}^+[i]$  with  $a, b \geq 2$  can be written as a sum of exactly two primes in  $\mathbb{Z}^+[i]$ . The matrix entries in the picture show how many times a Gaussian integer can be written as a sum of two primes. A first challenge is to establish the claim for the diagonal. The hardest appears the boundary row, where the result implies Landau's open problem about the infinitude of primes of the form  $n^2 + 1$  because the lowest row entries  $z = c + 2i$  must be the sum of two primes of the form  $a + i, b + i$  which both must have prime norm  $a^2 + 1, b^2 + 1$ . Algebraically, the Goldbach conjecture is a claim about the Goldbach function  $f = \sum_{a+ib \in P^+} x^a y^b$ . The claim is that the even derivatives  $(f^2)_{kl}$  are positive if  $k, l \geq 2$ .

One can not leave away the evenness condition. Already the real number 29 is not a sum of two Gaussian primes. It is actually the second largest integer which can not be written as a sum or difference of rational primes. This statement looks like the type of statement which Mitsui looked at in number fields.

#### Remarks.

1) The Goldbach statement also implies that there are infinitely many rational primes of the form  $n^2 + 4$ . Proof: assume there are only finitely many. If every even Gaussian integer of the form  $n + 3i$  could be written as a sum of two primes as such a summation would require one to be of the form  $p + i$ , the other  $q + 2i$ . Assume the set of primes of the form  $q + 2i$  is finite. It would imply that every even integer  $w$  of the form  $n + 3i$  has a prime  $w$  within some fixed distance  $R + 3$ . Define  $Z$  as the product of all Gaussian integers  $z = a + ib$  with  $|z| \leq R$ . This number has no prime in a neighborhood  $B(Z, R + 3)$  because any  $P = Z + A$  has the factor  $A$ . Therefore, also  $Z + 3i$  has no prime in a neighborhood of radius  $R$ .

2) Lets look again at [41] which they formulate as "conjecture E": if  $n$  is a Gaussian integer with  $n\bar{n} = N(n) > 2$ , then it can be written as a sum  $n = p + q$  of two primes for which the angles between  $n$  and  $p$  as well as  $n$  and  $q$  are both  $\leq \pi/4$ . [41] can strengthen their statement differently: their "conjecture F" claims that for  $N(n) > 10$ , one can write  $n$  as a sum of two primes  $p, q$  for which the angles between  $n$  and  $p$  and  $n$  and  $q$  are both  $\leq \pi/6$ . The angle could be reduced, but not arbitrarily since 4 is then not the sum of two Gaussian primes. Still, the **Gaussian Goldbach conjectures with angle conditions** is different from the open quadrant formulation given here. Our condition is easier to reformulate algebraically and also can be stated almost identically in all division algebras.

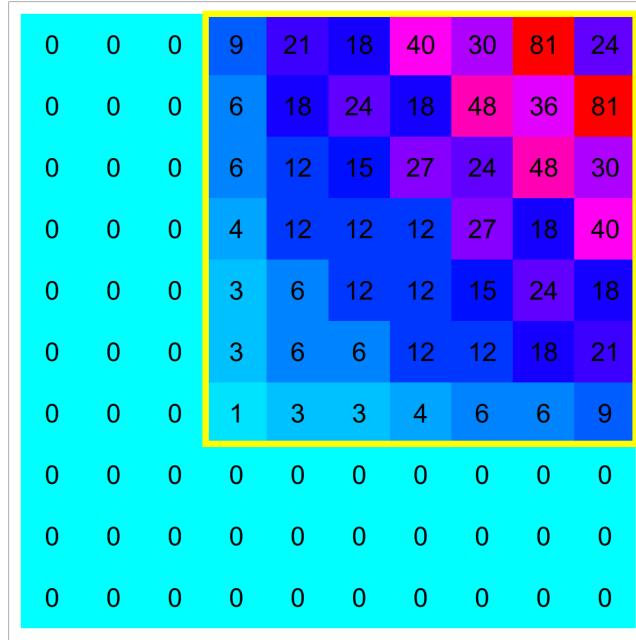


FIGURE 17. The **ternary Gauss Goldbach problem** claims that every Gaussian integer  $a + ib \in \mathbb{Z}^+[i]$  with  $a, b > 2$  is the sum of exactly three Gaussian primes in  $\mathbb{Z}^+[i]$ . Algebraically, it tells that coefficients of  $f^3$  are positive if  $a \geq 3, b \geq 3$ .

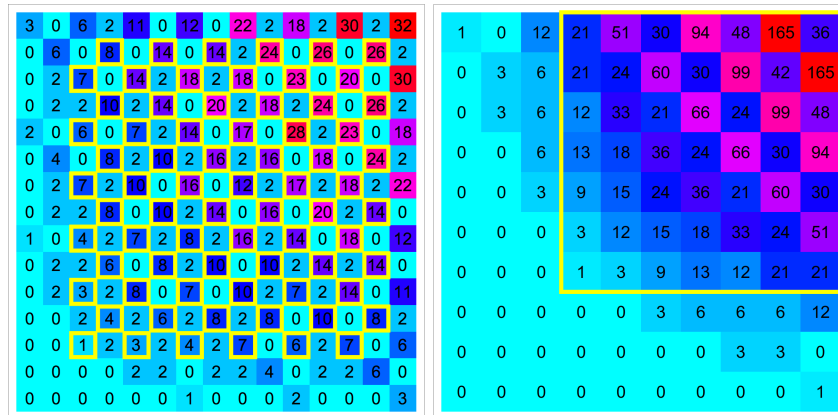


FIGURE 18. A weaker **Goldbach problem** for Gaussian integers is that for every even  $n + m$  and  $n, m \geq 2$  (now including the boundary of  $Q$ ), the prime  $z = n + im$  can be written as a sum of two Gaussian primes  $z = p + q$ .

3) [62] formulates a conjecture in general number fields. It does not use a cone restriction. Gaussian and Eisenstein primes are special cases. In the Gaussian case, it states that every even Gaussian integer is the sum of two Gaussian primes. The evenness condition is necessary. The smallest Gaussian integer which is not the sum of two Gaussian primes is  $4 + 13i$ . The Holben-Jordan conjecture implies the Mitsui statement. In the Eisenstein case, we see that every Eisenstein integer is the sum of two Eisenstein primes without evenness condition. The question makes sense also in the  $\mathbb{Z}$ : is every even integer the sum of two **signed primes**, where the set of **signed primes** is  $\{\dots, -7, -5, -3, -2, 2, 3, 5, 7, \dots\}$ . The smallest number which is not the sum of two signed primes is 23. All even numbers seem to be the sum of two signed primes! We are not aware of a proof of this **signed Goldbach statement**. It might not be of interest, since Goldbach implies it.

4) The Goldbach conjecture is not the only statement which involves the additive structure and primes (which inherently rely on the multiplicative structure of the ring): any additive function  $f(zw) = f(z) + f(w)$  which satisfies  $f(p + 1) = 0$  for all Gaussian primes is 0 [61].

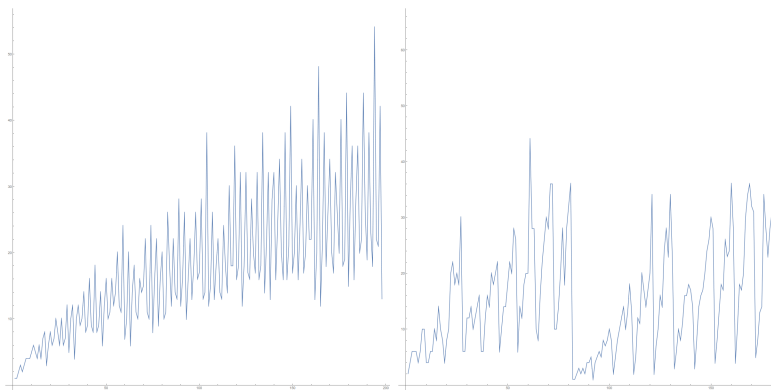


FIGURE 19. The coefficients of  $g(x) = f(x)^2$  with  $f(x) = \sum_p x^p$  form the **Goldbach comet** for primes. The coefficient  $x^n$  tells in how many ways one can write  $n$  as a sum of two primes. The picture to the right shows the comet in the case of Gaussian primes, where the coefficient of  $x^n y^m$  tells in how many times the Gaussian integer  $n + im$  can be written as a sum of two Gaussian primes. The Goldbach conjecture for Gaussian primes claims that every Gaussian integer  $2n + 2mi$  with  $n, m > 0$  is the sum of two Gaussian primes with nonnegative coordinates.

5) We do not know for sure whether the angle or open quadrant statement are not equivalent but believe the open quadrant statement harder (due to the Landau boundary problem). Both have their advantages: the angle version can be strengthened or weakened by changing the angle. The quadrant version of Goldbach is algebraically natural, especially when reformulating using Taylor series where no negative powers can occur.

6) The work [56] extends Rademacher, Hecke and Vinogradov to show that the number of primes in a sector  $\arg(z) \in \alpha, \beta, |z| \leq r$  is  $(\beta - \alpha)/(2\pi) \int_2^r dx/\log(x)$  with an error of the form  $O(re^{-c\sqrt{\log(r)}})$  meaning that in any radial sector one sees the distribution of the prime number theorem. The prime twin and Goldbach problem has been stated in [41] using some angle conditions. Holben-Jordan also define **Gaussian prime twins** as Gaussian prime pairs of distance  $\sqrt{2}$ . (We believe the prime twin problem for Gaussian primes must have been asked before but we don't find an earlier source than Holben-Jordan from 1968.) The twin prime conjecture was also featured in Hilbert's problem 8 and is part of the list of Landau's problems.

## 5. GOLDBACH FOR HURWITZ PRIMES

Most number theoretical question can also be asked for Hurwitz integers: one can in particular look at the **twin prime** or **Goldbach problems**. Also the **zeta function**  $\sum_z 1/N(z)$  can be considered for Hurwitz primes. It will turn out to be a scaled and shifted Riemann zeta function because it can be written as  $\sum_p (p+1)/p^s$  by Hurwitz's result. But lets first look at Goldbach for Hurwitz. It might more likely to be true as we are in higher dimensions but numerical experiments already become costly. It would not be a complete surprise if there was large counter example.

Lets look first a bit at the well known history of the one-dimensional problem [36]. The Goldbach conjecture for rational primes was proposed in 1742 by Christian Goldbach in a letter to Leonard Euler [55, 49]. Goldbach's original statement was that every



integer  $n \geq 5$  is a sum of three primes, which Euler reformulated as every even function  $\geq 4$  being a sum of 2 primes. Goldbach's footnote in the handwritten letter is well readable. Some transcribe it as  $> 2$  rather than  $\geq 5$  which would imply Goldbach considered 1 as a possible summand in "aggregatum trium numerorum primorum" as custom at the time [68]. The statement that every odd integer larger than 4 can be written as a sum of three primes has first been formulated by Waring in 1770 [68]. The book [44] transcribes on page 443 with  $\geq 5$ . More on the correspondence can be found in [55]. As Erdős liked to point out, Descartes already earlier voiced a similar conjecture [73, 85] so that the Goldbach formulation sometimes is also called **Descartes Conjecture** [68]. Dickson [21] mentions on page 421 that Descartes formulated the conjecture

in the form that every even number is the sum of 1,2 or 3 primes and that Waring conjectured in 1770 that every odd number is either a prime or the sum of three primes. Many have done experiments. Even Georg Cantor checked Goldbach up to 1000 [21] p. 422. Much progress has been done. Landmark results in theory were Hardy-Littlewood [38] with the circle method, the Lev Schnirelemans theorem [42] using density and Ivan Vinogradov's theorem [89] using trigonometric sums. The genesis paper for the circle method is in [86] attributed to the paper of Hardy with Srinivasa Ramanujan in 1918 [37]. Chen's theorem [13] tells that any sufficiently large even  $n$  is the sum of a prime and a semi prime. Taos theorem tells that any odd number larger than 1 is the sum of at most 5 primes. Helfgott announced the unconditional solution of the ternary Goldbach problem [39].

For now, lets look at the quaternion case, where we are not aware even of experiments about Goldbach have been done. Recall that there are two type of primes, the **Lipschitz primes**, which are of the form  $(a, b, c, d)$  with integers  $(a, b, c, d)$ . The rest of the **quaternion primes**. They are of the form  $(a + 1/2, b + 1/2, c + 1/2, d + 1/2)$  with integers  $(a, b, c, d)$ . To make the formulations easier, lets call the primes in set of Hurwitz integers just "primes" or "quaternion primes" and call the quaternion primes which are not Lipschitz primes the **Hurwitz primes**. Hurwitz himself called primes "Primquaternion" which is the German expression for prime quaternion.

Define  $Q$  as the set of Hurwitz integers for which all coordinates are positive.

**Conjecture:** Every Lipschitz integer quaternion with entries  $> 1$  is the sum of two Hurwitz primes in  $Q$ .

We believe that it will be difficult to prove if it holds up to be true. We also hope for

**Question:** Every Hurwitz integer quaternion with entries  $> 2$  is the sum of a Hurwitz and Lipschitz primes in  $Q$ .

The Hurwitz integer  $(3/2, 3/2, 3/2, 3/2)$  is not the sum of a Hurwitz and Lipschitz prime because the only decomposition would be  $(1, 1, 1, 1) + (1, 1, 1, 1)/2$  but both are not prime. Together:

**Question:** Every integer quaternion with entries larger than 2 is the sum of two quaternion primes in  $Q$ .

Here are some computations showing in how many ways a Lipschitz integer can be written as a sum of two Hurwitz primes. Since we are in a 4 dimensional lattice, we fix the first two coordinates  $a, b$ , then build the matrix  $G(a, b)$  for which  $G_{cd}(a, b)$  tells in how many ways one can write the Lipschitz prime  $(a, b, c, d)$  as a sum of two Hurwitz primes:

$$\begin{aligned}
 G(1, 1) &= \begin{bmatrix} 0 & 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 4 & 2 \\ 1 & 4 & 3 & 4 & 6 \\ 2 & 4 & 4 & 8 & 6 \\ 3 & 2 & 6 & 6 & 5 \end{bmatrix}, G(1, 2) = \begin{bmatrix} 0 & 2 & 4 & 4 & 2 \\ 2 & 6 & 6 & 6 & 10 \\ 4 & 6 & 8 & 10 & 10 \\ 4 & 6 & 10 & 10 & 10 \\ 2 & 10 & 10 & 10 & 10 \end{bmatrix} \\
 G(1, 3) &= \begin{bmatrix} 1 & 4 & 3 & 4 & 6 \\ 4 & 6 & 8 & 10 & 10 \\ 3 & 8 & 13 & 12 & 11 \\ 4 & 10 & 12 & 10 & 14 \\ 6 & 10 & 11 & 14 & 20 \end{bmatrix}, G(1, 4) = \begin{bmatrix} 2 & 4 & 4 & 8 & 6 \\ 4 & 6 & 10 & 10 & 10 \\ 4 & 10 & 12 & 10 & 14 \\ 8 & 10 & 10 & 14 & 20 \\ 6 & 10 & 14 & 20 & 14 \end{bmatrix}, G(1, 5) = \begin{bmatrix} 3 & 2 & 6 & 6 & 5 \\ 2 & 10 & 10 & 10 & 10 \\ 6 & 10 & 11 & 14 & 20 \\ 6 & 10 & 14 & 20 & 14 \\ 5 & 10 & 20 & 14 & 39 \end{bmatrix} \\
 G(2, 1) &= \begin{bmatrix} 2 & 6 & 6 & 6 & 10 \\ 6 & 14 & 14 & 16 & 16 \\ 6 & 14 & 20 & 24 & 24 \\ 6 & 16 & 24 & 22 & 24 \\ 10 & 16 & 24 & 24 & 34 \end{bmatrix}, G(2, 2) = \begin{bmatrix} 4 & 6 & 8 & 10 & 10 \\ 6 & 14 & 20 & 24 & 24 \\ 8 & 20 & 20 & 26 & 36 \\ 10 & 24 & 26 & 34 & 38 \\ 10 & 24 & 36 & 38 & 42 \end{bmatrix} \\
 G(2, 3) &= \begin{bmatrix} 4 & 6 & 10 & 10 & 10 \\ 6 & 16 & 24 & 22 & 24 \\ 10 & 24 & 26 & 34 & 38 \\ 10 & 22 & 34 & 38 & 36 \\ 10 & 24 & 38 & 36 & 56 \end{bmatrix}, G(2, 4) = \begin{bmatrix} 2 & 10 & 10 & 10 & 10 \\ 10 & 16 & 24 & 24 & 34 \\ 10 & 24 & 36 & 38 & 42 \\ 10 & 24 & 38 & 36 & 56 \\ 10 & 34 & 42 & 56 & 44 \end{bmatrix} \\
 G(3, 3) &= \begin{bmatrix} 3 & 8 & 13 & 12 & 11 \\ 8 & 20 & 20 & 26 & 36 \\ 13 & 20 & 46 & 34 & 33 \\ 12 & 26 & 34 & 40 & 46 \\ 11 & 36 & 33 & 46 & 61 \end{bmatrix}, G(3, 4) = \begin{bmatrix} 4 & 10 & 12 & 10 & 14 \\ 10 & 24 & 26 & 34 & 38 \\ 12 & 26 & 34 & 40 & 46 \\ 10 & 34 & 40 & 52 & 62 \\ 14 & 38 & 46 & 62 & 72 \end{bmatrix}, G(3, 5) = \begin{bmatrix} 6 & 10 & 11 & 14 & 20 \\ 10 & 24 & 36 & 38 & 42 \\ 11 & 36 & 33 & 46 & 61 \\ 14 & 38 & 46 & 62 & 72 \\ 20 & 42 & 61 & 72 & 96 \end{bmatrix}
 \end{aligned}$$

$$G(4, 4) = \begin{bmatrix} 8 & 10 & 10 & 14 & 20 \\ 10 & 22 & 34 & 38 & 36 \\ 10 & 34 & 40 & 52 & 62 \\ 14 & 38 & 52 & 62 & 74 \\ 20 & 36 & 62 & 74 & 86 \end{bmatrix}, G(4, 5) = \begin{bmatrix} 6 & 10 & 14 & 20 & 14 \\ 10 & 24 & 38 & 36 & 56 \\ 14 & 38 & 46 & 62 & 72 \\ 20 & 36 & 62 & 74 & 86 \\ 14 & 56 & 72 & 86 & 104 \end{bmatrix}, G(5, 5) = \begin{bmatrix} 6 & 10 & 14 & 20 & 14 \\ 10 & 24 & 38 & 36 & 56 \\ 14 & 38 & 46 & 62 & 72 \\ 20 & 36 & 62 & 74 & 86 \\ 14 & 56 & 72 & 86 & 104 \end{bmatrix}$$

For example, here are all the 14 summands of the Lipschitz integer  $z = (2, 3, 2, 2)$  as a sum of two Hurwitz integers  $z = p + q$ . The rational primes  $N(p), N(q)$  can vary.

2p	2q	N(p)	N(q)
(1, 1, 1, 3)	(3, 5, 3, 1)	3	11
(1, 1, 3, 1)	(3, 5, 1, 3)	3	11
(1, 3, 1, 3)	(3, 3, 3, 1)	5	7
(1, 3, 3, 1)	(3, 3, 1, 3)	5	7
(1, 3, 3, 3)	(3, 3, 1, 1)	7	5
(1, 5, 1, 1)	(3, 1, 3, 3)	7	7
(1, 5, 3, 3)	(3, 1, 1, 1)	11	3
(3, 1, 1, 1)	(1, 5, 3, 3)	3	11
(3, 1, 3, 3)	(1, 5, 1, 1)	7	7
(3, 3, 1, 1)	(1, 3, 3, 3)	5	7
(3, 3, 1, 3)	(1, 3, 3, 1)	7	5
(3, 3, 3, 1)	(1, 3, 1, 3)	7	5
(3, 5, 1, 3)	(1, 1, 3, 1)	11	3
(3, 5, 3, 1)	(1, 1, 1, 3)	11	3

Already for  $z = (2, 2, 2, 2)$ , the smallest allowable integer for the conjecture, there are 14 summands. They are of the form  $(3, 1, 1, 1)/2 + (1, 3, 3, 3)/2$  (8 cases) or  $(1, 1, 3, 3)/2 + (3, 3, 1, 1)/2$  (6 cases).

Let us look at the special case, when the Lipschitz integer is  $z = (2, 2, 2, n)$ . The two primes summing up to it must then have the form of one of the following cases modulo permutations:  $p = (1, 1, 1, x)/2, q = (3, 3, 3, n - x)/2$ , or then  $p = (1, 1, 3, x)/2, q = (3, 3, 1, 2n - x)/2$  for an unknown odd integer  $x$ . In the first case, we need simultaneously to have  $(3 + x^2)/4$  and  $(27 + (2n - x)^2)/4$  to be rational primes. In the second case, we need simultaneously to have  $(11 + x^2)/4$  and  $(19 + (2n - x)^2)/4$  to be prime. Since  $x$  needs to be odd for  $p$  to be a Hurwitz prime, we can write  $x = 2k + 1$ . Now  $(3 + x^2)/4 = 1 + k + k^2$  and  $(27 + (2n - x)^2)/4 = 7 - (n - k) + (n - k)^2$ . In the second case  $3 + k + k^2$  and  $5 - (n - k) + (2n - k)^2$ . We see: If the Hurwitz Goldbach conjecture holds, then for any  $n$ , there exists  $k < n$  for which both  $3 + k + k^2$  and  $7 + k + k^2 - n - 2kn + n^2$  are prime or for which both  $1 + k + k^2$  and  $5 + k + k^2 - n - 2kn + n^2$  are prime. Now, if Hurwitz-Goldbach is true and if there existed only finitely many primes of the form  $3 + k + k^2$  and  $1 + k + k^2$ , then for all  $m = n - k$  large enough,  $7 - m + m^2$  and  $5 - m + m^2$  would always have to be prime. This is obviously not true if  $m$  is a multiple of 7 or 5. We see that Goldbach implies a special case of the Bunyakovsky conjecture, which like Landau's problem is likely not so easy to prove:

**Remark:** If the Hurwitz Goldbach conjecture is true, then one of the sequences  $1 + k + k^2$  or  $3 + k + k^2$  contains infinitely many primes.

Let us quickly verify the **Bunyakovsky conditions** which must be checked for the **Bunyakovsky conjecture**. First,  $\phi_3(k) = 1 + k + k^2$  is already the cyclotomic polynomial and also  $k^2 + k + 3$  satisfies the conditions of the Bunyakovsky conjecture: the maximal coefficient of the polynomial  $f$  is positive, the coefficients have no common divisor and there is a pair of integers  $n, m$  such that  $f(n), f(m)$  have no common divisor. Hardy-Littlewood analogue density results have been formulated in [6]. The set of  $k$  for which  $k^2 + k + 1$  is prime is the sequence A002384 in [1].

Our experiments confirm that the sequences produce primes with the frequency given by analogous Hardy-Littlewood density constants.

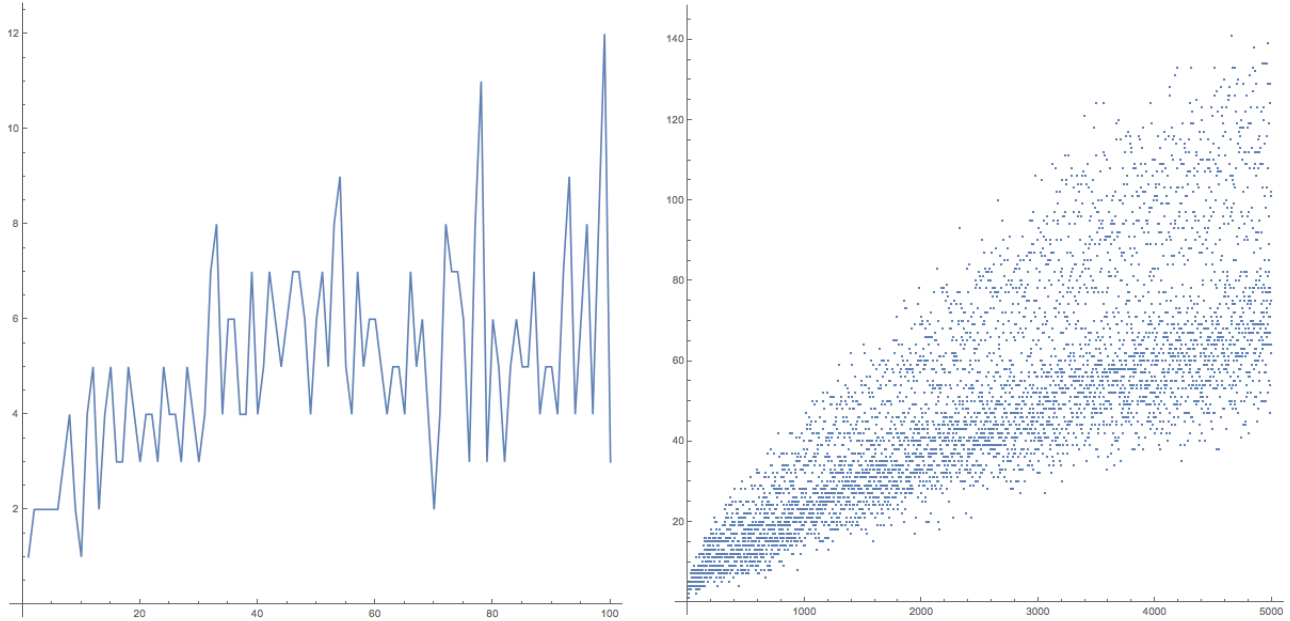


FIGURE 20. The **Hurwitz boundary comet** shows the function  $f(n)$  which gives the number of solutions  $p + q = (2, 2, 2, n)$  with ordered Hurwitz primes  $p, q$ .

## 6. GOLDBACH FOR OCTONIONS

Besides  $\mathbb{R}, \mathbb{C}$  and  $\mathbb{H}$ , there is a fourth division algebra  $\mathbb{O}$ . It is the space of **Cayley numbers**. Because it is eight dimensional, it has also been called the space of **Octonions**. It is this name which has stuck. The members of  $\mathbb{O}$  can either be written as a linear combination of a basis  $1, i, j, k, l, m, n, o$  or then, according to a suggestion of Cayley-Dixon, as pairs  $(z, w)$  of quaternions, defining  $(z, w) \cdot (u, v) = (zu - v^*w, vz + wu^*)$ . The algebra is no more associative.



In order to do number theory, one has to specify what the **integers** are in  $\mathbb{O}$ . It turns out that there are several classes of integers and even several maximal ones. First of all there are the **Gravesian integers**  $(a, b, c, d, e, f, g)$  which play the role of the Lipschitz primes in the Hurwitz case. Then there are the **Kleinian integers**  $(a, b, c, d, e, f, g) + (1, 1, 1, 1, 1, 1, 1)/2$  which play the role of the Hurwitz primes in  $\mathbb{H}$ . But then there are more, the **Kirmse integers** which includes elements for which 4 of the entries are half integers. There are 7 **maximal orders** which Kirmse classified [52]. (He counted 8 which was later corrected by Coxeter [18]). They turn out all to be equivalent and produce the ultimate class of integers. They are often called **Octonion** or **Cayley integers** or then more catchy, the **Octavian integers** [52]. In the mathematical genealogy database, there is no advisor listed, but since he acknowledges Herglotz in [51], this is our most likely guess. Johannes Kirmse wrote his pioneering Octonion paper in 1925. We could not get hold of the 1925 paper [52] but it is mentioned and discussed in [18] who's critics appears maybe a bit too harsh today when considering that Kirmse explored new ground where nobody has been before.

The condition  $N(zw) = N(z)N(w)$  which assures that the algebra is a normed **division algebra**, is also called the **Degen eight square identity**. Ferdinand Degen discovered it first around 1818 and John Thomas Graves in 1843. Also Arthur Cayley in 1845 rediscovered the identity.

The primes in the Octonions are either called **Gravesian primes** containing the Lipschitz primes, or **Kleinian primes** containing the Hurwitz primes or then the **Kirmse primes**. After choosing a maximal order, lets just call them **Octavian primes**. We have followed mostly the nomenclature of [17] but split the three type of primes up so that the Octavian primes are made up of three distinct type of primes, similarly as

the Quaternion primes were made up of two type of primes, the Lipschitz and Hurwitz primes. Already the Kirmse integers or the Octavian integers form a lattice called the  $E_8$  **lattice**. It is important as it produces the densest **sphere packing** in  $\mathbb{R}^8$ .

The units form what is now called a **Moufang loop** named after German mathematician **Ruth Moufang** (1905-1977) [64], who was a student of Max Dehn (1878-1952) and is considered one of the first PhD mathematicians working also in the industry. Dehn is also known for the Dehn-Sommerville relations. There is a smaller loop of 16 unit octonions containing Gravesian integers like  $(\pm 1, 0, 0, 0, 0, 0, 0, 0)$ . The units placed in the unit sphere of  $R^8$  form the **Gosset polytope**  $4_{21}$  which was discovered by Thorold Gosset (1869-1962) who as a lawyer without much clients amused himself as an amateur mathematician. The vertices of  $4_{21}$  are the roots of the exceptional Lie algebra  $E_8$  belonging to the 248 dimensional Lie group  $E_8$ . As the dimension of the maximal torus is 8, the root system lives in  $R^8$ . One can write points on the sphere of radius 2 taking vertices  $(a, b, 0, 0, 0, 0, 0, 0)$  with  $a, b \in \{-1, 1\}$  or  $(a, b, c, d, e, f, g, h)$  with with entries in  $\{-1/2, 1/2\}$  summing up to an even number. This lattice  $E_8$  has just recently been verified by Maryna Viazovska to be the densest sphere packing in  $R^8$  [87].

Let us fix now one of the 7 classes of Octonian integers. For the Goldbach statement, we are going to look up, it does not matter which. Proof. We use the Cayley-Dickson notation which writes an Octonion as a pair of quaternions. Assume we have integers of the form  $(w, z), (w/2, z/2), (w/2, z), (w, z/2)$  where  $w, z$  are Lipschitz quaternions and  $w/2, z/2$  are Hurwitz quaternions. Their units of the octonions do not form a multiplicative **group** any more, as multiplication is not associative. But it is a **loop** an algebraic structure more primitive than a group in which one does not insist on associativity.

When looking for **Octonion Goldbach conjectures** one has to check any of them. The most obvious case fails for  $K = 2$ :

One certainly has to distinguish cases. In the simplest case, within Gravesian integers if  $(z, w) = (z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4)$ , we want to find  $(x, y) = (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4)$  such that both  $p = \sum_i x_i^2 + y_i^2$  and  $q = \sum_i (z_i - x_i)^2 + (w_i - y_i)^2$  are prime. The integer  $(2, 2, 2, 2, 2, 2, 2, 2)$  can not be written as a sum of two Gravesian primes. Also Kleinian primes do not work: there are Gravesian integers with entries  $\geq 2$  which are not the sum of two Kleinian primes: Assume  $(2, 2, 2, 2, 2, 2, 2n)$  is the sum of two Kirmse primes. Here are the four cases:

$$\begin{aligned} & (1, 1, 1, 1, 1, 1, 2k+1)/2, (3, 3, 3, 3, 3, 3, 2n-2k-1)/2 \\ & (1, 1, 1, 1, 1, 1, 3, 2k+1)/2, (3, 3, 3, 3, 3, 3, 1, 2n-2k-1)/2 \\ & (1, 1, 1, 1, 1, 3, 3, 2k+1)/2, (3, 3, 3, 3, 3, 1, 1, 2n-2k-1)/2 \\ & (1, 1, 1, 1, 3, 3, 3, 2k+1)/2, (3, 3, 3, 3, 1, 1, 1, 2n-2k-1)/2 . \end{aligned}$$

Define  $m$  by  $2m+1 = 2n-2k-1$ . Now this either one of the four pairs are both prime:  $(7 + (2k+1)^2)/4 = 2 + k + k^2$  and  $(63 + (2m+1)^2)/4 = 16 + m + m^2$

$$(6 + 9 + (2k+1)^2)/4 = 4 + k + k^2 \text{ and } (1 + 54 + (2m+1)^2)/4 = 14 + m + m^2$$

$$(5 + 18 + (2k+1)^2)/4 = 6 + k + k^2 \text{ and } (2 + 45 + (2m+1)^2)/4 = 12 + m + m^2$$

$$(4 + 27 + (2k+1)^2)/4 = 8 + k + k^2 \text{ and } (3 + 36 + (2m+1)^2)/4 = 10 + m + m^2$$

with  $2m+1 = 2n-2k-1$ . But this is not true as in any case all these numbers are divisible by 2. The failure is related to the fact that the Kirmse integers are not yet a maximal order. But also a brute force search over Kirmse does not work for  $(2, 2, 2, 2, 2, 2, 2, 2)$ .

It appears that it is not obvious how to come up with a conjecture which is both convincing and also justifiably difficult. Brute force searches are difficult as the volume of a box of size  $r$  grows like  $r^8$ . Here is a first attempt of get to a conjecture. We formulate as a question since our experiments did not get far yet, nor do we have an idea how difficult the statement could be. Anyway, lets denote by  $Q$  again the set of all integer octonions, for which all coordinates are positive.

**Question:** There exists  $K$  such that every Octavian integer  $Z$  with coordinates  $\geq K$  is a sum of two Octavian primes  $P, Q$ .

#### Remarks.

1) In physics, one is interested in these structures as they have used them to build a  $10 = 8+1+1$ -dimensional space-time, where time and string parametrization add two more dimensions. Some theoreticians take the structures of division algebras serious in describing matter [35, 22]. Some history on their use in quantum mechanics is [4, 5].



## 7. LANDAU AND BUNYAKOVSKY PROBLEM

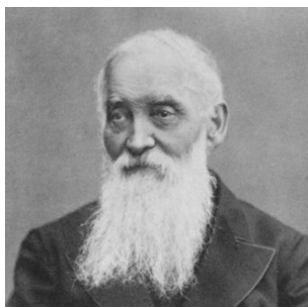
One of the four problems presented by **Edmund Landau** at the 1912 International congress of mathematicians claims that there are infinitely many primes of the form  $n^2 + 1$ . The problem is also known as the **fifth Hardy-Littlewood conjecture**, but the problem has been raised already by Euler [68]. It is now part of a more general **Bunyakovsky conjecture** or of the **Schinzel hypothesis  $H$**  or more generally the **Bateman-Horn conjecture** [6]. Early laboratory experiments have been done by Euler who computed all primes of the form  $n^2 + 1$  for  $n$  up to 1500 and saw that there are many primes in this list.

Despite the extreme simplicity of question: **”are there infinitely many primes  $p$  for which  $p - 1$  is a square?”**,



it is considered a “hard” problem. One indication is that despite of having been in the spotlight for more than over a hundred years, there is no solution in sight. The tools of analytic number theory look helpless. Just to illustrate some attacks, in [27] it is shown that there are infinitely many Gaussian primes  $a + ib$ , for which  $a$  is a given prime. While this existence result does not prove the Landau problem, it gives hope even so the result giving existence is far from proving that there are infinitely many.

A much more general conjecture was already formulated by Victor Bunyakovsky in 1857. Bunyakovsky was a student of Cauchy and a gifted educator. His conjecture is that if  $f(x)$  is an irreducible polynomial for which some obvious non-triviality conditions are satisfied, then the range of  $f$  has infinitely many primes.



It is therefore amazing that Hardy and Littlewood quantized the Landau problem in their landmark paper of 1923. They predicted an explicit limiting density  $C$  for the number of such primes and more generally predicted a precise density ration for primes of the form  $k + ia$  and  $k + ib$  as  $C_a/C_b$  with the product  $C_a = \prod_p (1 - \left(\frac{-a}{p}\right)/(p-1))$  over all odd primes. These are the **Hardy-Littlewood density conjectures**.

The constant  $C = C_1$  agrees with the ratio between real Gaussian primes and Gaussian primes of the form  $n + i$  in the first row. We focussed our own experiments primarily on this number.

The first **Hardy-Littlewood ratio**  $C_1$  can be rewritten as

$$C = \prod_{p \in P_1} \left[1 - \frac{1}{p-1}\right] \prod_{p \in P_3} \left[1 + \frac{1}{p-1}\right] = 1.37279\dots,$$

where  $P_k$  is the set of rational primes congruent to  $k$  modulo 4. Intuitively, one can understand this probabilistically. The probability to be in a multiplicative remainder class of  $p \in P_3$  is  $p/(p-1)$  and the probability to be not in a specific remainder class modulo  $p \in P_1$  is  $1 - 1/(p-1)$ .

More general **Hardy-Littlewood density estimates** predict a density relation between primes of the form  $f(x)$  or  $g(x)$  given two irreducible polynomials  $f, g$  of the same degree to be given as  $C_f/C_g$  with  $C_f = \prod_p \frac{(1 - \omega_f(p)/p)}{(1 - 1/p)}$ . Here,  $\omega_f(p)$  is the number of solutions  $f(x) = 0$  modulo  $p$ . In order for the estimate to work, one needs that the polynomials a positive leading coefficients and  $\omega_f(p) \neq p$  for all  $p$ . This especially implies  $\omega_f(2) = 0$  and even so not irreducible, one can include  $f(x) = x^n$ , where  $C_f = 1$ . (See [6, 14], where one can even look at the product of irreducible cases, combine several conjectures of Hardy and Littlewood). The constants  $C_k$  are then shorts cuts for  $C_{n^2+k}$ . In the prototype case  $f(x) = x^2 + 1$  one has  $\omega_f(p) = 1$  for  $p \in P_3$  and  $\omega_f(p) = 2$  for  $p \in P_1$  by quadratic reciprocity so that  $(1 - \omega_f(p)/p)/(1 - 1/p) = 1 - \frac{1}{p-1}$  for  $p \in P_1$  and  $1 + \frac{1}{p-1}$  for  $p \in P_3$ .

The intuition which led to the conjectures of Hardy and Littlewood are of a probabilistic nature: the key assumption is that solving equations like  $a^2 = -1$  modulo  $p$  or then solving modulo  $q$  is pretty much independent, if  $p, q$  are different odd primes. Of course this is not justified. But it is part of the magic of primes that it seems to work.

Lets try to explain the intuition behind the constant: every time a new prime is added, the size of the space changes by  $(p-1)/p = (1-1/p)$  because we can only take numbers which are not multiples of  $p$ . The product of these size changes is  $1/\zeta(1) = 0$  and reflects the infinitude of primes. But if we look at the **ratio** of solution sets for two polynomials, we don't have to do this **re-normalization** because it happens on both sides. Now, when looking at solutions of the form  $x^2 + 1$ , then whenever  $-1$  is a quadratic residue, the probability decreases by  $(1-1/(p-1))(1-1/p) = (p-2)/p$  but if  $-1$  is not a quadratic residue, then the probability increases stays the same. Including back the volume change gives  $(p-2)/p/(1-1/p) = 1-1/(p-1)$  in the residue case and  $1/(1-1/p) = p/(p-1) = 1+1/(p-1)$  in the non-residue case. This explains the formula for  $C$ . [6] explain this skillfully. Using this frame work, many of the formulas of [38] make sense, like density formulas for the estimated number of prime twins.

**Remark:** The Goldbach conjecture for Gaussian primes implies the existence of infinitely many Gaussian primes of the form  $a + i$  or  $a + 2i$  and especially implies Landau's first problem.

Landau's problem asks whether infinitely many primes exist on the first row of the complex plane. One can also ask about existence of primes on rows. This appears much easier but is also open. We wanted to call it the "highway crossing frog problem" but the popular US comedian Will Ferrell once said this is a "lame name" so that we call it the **frogger problem**. It could be a problem which will be solved first in the arena of Gaussian integers:

Assume the rows in the complex plane are the lanes of the "highway" and the primes are the "gaps between cars". A frog can walk freely horizontally between the highway lanes and hop through gaps. The question is whether it can hop arbitrarily far in the vertical direction.

Its obviously possible if and only if there exists at least one prime on each highway lane:

**Conjecture:** For any integer  $a > 0$  there exists a rational prime of the form  $x^2 + a^2$  with integer  $x$ .

The frogger problem would follow from Landau type conjectures the existence of infinitely many primes of the form  $x^2 + a^2$ . Work of Hecke shows that the frog can jump through lines for which  $a$  is prime. In general, it appears open.

The **Hurwitz frogger problem** asks whether for every integer  $a$ , there are integer vectors  $(x, y, z)$  for which  $a^2 + x^2 + y^2 + z^2$  is a rational prime and whether for every half integer  $a + 1/2$ , there exists a half integer  $(x + 1/2, y + 1/2, z + 1/2)$  for which  $(x + 1/2)^2 + (y + 1/2)^2 + (z + 1/2)^2 + (a + 1/2)^2$  is a rational prime. The first part can be solved:

**Remark:** For every integer  $a$  there are integer triples  $(x, y, z)$  such that  $a^2 + x^2 + y^2 + z^2$  is a rational prime.

Proof: Fix  $a$  and an arithmetic progression  $p = u + a^2 + kv$  such that  $u + a^2, v$  are coprime. By the **Dirichlet's theorem on arithmetic progressions**, there are infinitely primes  $p$  like that. Now  $a^2 + x^2 + y^2 + z^2 = p = u + a^2 + kv$  is  $x^2 + y^2 + z^2 = u + kv$ . The **Legendre three square theorem** tells that there is a solution if  $u + kv$  is not of the form  $4^r(8s + 7)$ . Now just chose  $u$  odd with remainder different from  $-1$  modulo 8 and  $v$  a multiple of 8.

We don't know yet about the second part which is a Bunyakovsky type problem but for a function of several variables:

**Conjecture:** For any integer  $a$  there are  $(x, y, z)$  such that  $a^2 + x^2 + y^2 + z^2 + x + y + z + a + 1$  is a rational prime.

We have measured numerically that the number of solutions of this problem grows like  $C_a(n \log(n))^2$  with Hardy-Littlewood type constants  $C_a$ .

**Conjecture:** For any hyper-plane  $z = a$  in the space of Hurwitz integers, the number of Hurwitz primes in a ball of radius  $r$  grows like  $C_a(n \log(n))^2$ .

Hardy-Littlewood type statements about the asymptotic density of the number of solutions of such Diophantine problems appear more approachable since we are in higher dimensions, where some Landau type problems are already answered.

**Remarks.** 1) Randomness considerations also add intuition to the believe that factoring large composite integers is hard because the holy grail of integer factorization is since Fermat the ability to solve quadratic equations modulo the product  $n = pq$ . Say, if  $x, y$  were both solutions to  $x^2 + 1 = 0$  modulo  $pq$ , then  $x^2 = y^2$  modulo  $pq$  and  $\gcd(x - y, n)$  is either  $p$  or  $q$ . In some sense, the quadratic map on finite fields shows similar randomness features like the quadratic maps on the field of complex numbers, where one has the Julia-Fatou story. So, once accepting this intuition that a quadratic maps shuffles the residue system pretty well, rendering them independent, the formulas of Hardy and Littlewood become transparent. The connection with chaotic maps is not an accident, there are basic algorithms like the Pollard rho method which use the **iteration of a quadratic** map to get to factors.

2) Probability argument using independent can work, Here is an example where one has a product space what is the probability that two numbers have no common denominator? The probability to have a common denominator  $p$  is then  $1 - 1/p^2$  in the product space  $[1, \dots, n]^2$ . The product of all these probabilities gives  $\prod_p (1 - 1/p^2) = 1/\zeta(2) = 6/\pi^2$ .

## 8. THE HARDY-LITTLEWOOD CONSTANTS

Hardy and Littlewood [38] conjectured that the density of Gaussian primes of the form  $a + i$  divided by the density of Gaussian primes of the form  $a + 0i$  approaches a constant. This statement has attracted the attention of early pioneers in computer experiments like [75] who factored numbers of the type  $n^2 + 1$  and was therefore was interested in the density of cases, where  $n^2 + 1$  is prime. If  $\pi_0(n)$  is the number of Gaussian real Gaussian primes in  $[0, n]$  and  $\pi_1(n)$  the number of Gaussian primes  $a + i$  with  $a \in [0, n]$ , then  $\pi_0 \sim (1/2)\text{Li}(x)$  and Hardy-Littlewood predicted  $\pi_1(x) \sim (C/2)\text{Li}(x)$ . It is interesting to see the early pioneers go through relatively heavy mathematical gymnastics to compute the constant  $C$  efficiently. Daniel Shanks (1917-1996) [75] reports in 1959 that A.E. Western rewrote the constant as

$$C = \frac{3}{4} \frac{\zeta(6)}{\beta(2)\zeta(3)} \prod_{p \in P_1} \left(1 + \frac{2}{p^3 - 1}\right) \left(1 - \frac{2}{p(p-1)^2}\right),$$

where  $G$  is the **Catalan constant**. This formula gives 5 decimal places already when summing over three primes  $p = 5, 13, 17$ . A bit earlier, in 1922, even before the Hardy-Littlewood article appeared (and probably while helping to work on the numerical verifications with Hardy and Littlewood), A.E. Western [92] computed the constant  $C$  using further sophisticated identities involving various zeta values so that one can use two primes 5, 13 only to get  $C$  to 5 decimal places! Western was a giant in computation [93]. What a culture had been developed only about a seemingly tangential constant!

But one has to remember that these mathematicians had no access to computers. The Zuse Z3 was built only in 1941, machines like Colossus and Mark I appeared only in 1944 but all of them were slow: Mark I for example needed 6 seconds to multiply two numbers [65]. By the way, Shanks in 1960 used an IBM 704 with a 32K high-speed memory. In contrary, we have today access to machines which give each user 500 GBybes of RAM (I have used such a machine, Odyssee, for some computations in this paper). Shanks needed 10 minutes to factor all  $n^2 + 1$  from  $n = 1$  to  $n = 180000$ . Today, the command `"Timing[Table[FactorInteger[n * n + 1], n, 180000]][[1]]"` on a tiny laptop reports it done in 4 seconds.

When looking at the Hardy-Littlewood paper, the expression  $C \prod_{p>2} \left(1 - \frac{1}{p-1} \left(\frac{-1}{p}\right)\right)$  involves the **Legendre symbol**  $\left(\frac{-1}{p}\right)$ . In this special case, it is  $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$  which is 1 for primes of the form  $4k + 1$  and  $-1$  for the others. Shanks article of 1960 reveals through how much pain Mathematicians have gone to compute things effectively before without computers.

Hardy and Littlewood were not without assistance: On page 62 of their article, they mention that "some of their conjectures have been tested numerically by Mrs. Streatfield, Dr. A.E. Western and Mr. O. Western".

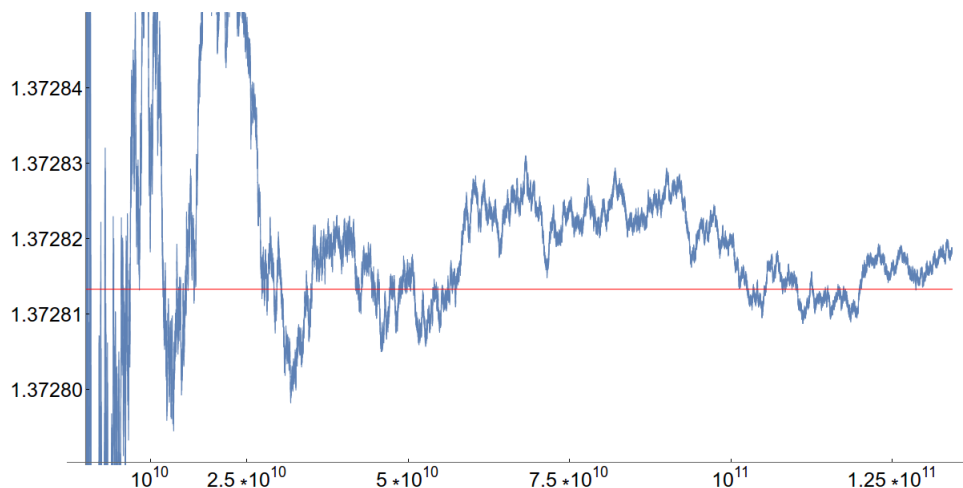


FIGURE 21. The convergence to the Hardy-Littlewood constant  $C$  giving the ratio of Gaussian prime density on  $\text{Im}(z) = 1$  and  $\text{Im}(z) = 0$ . Shanks [75] gave  $C = 1.37281346$ . This is conjecture E in [38]. The constant  $C$  is almost prophetic as by an open Landau's problem (which currently appears theoretically beyond reach), one does not even know whether  $C$  is positive!

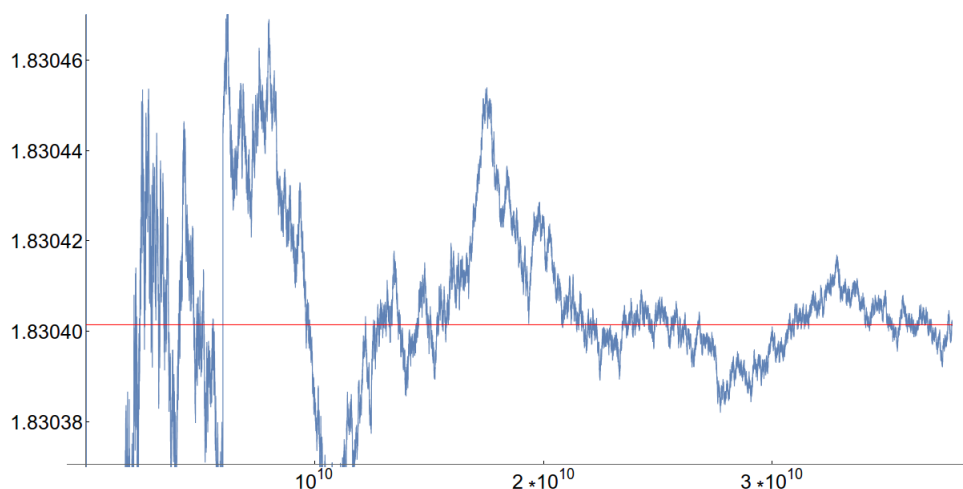


FIGURE 22. The convergence to the Hardy-Littlewood constant  $C_5$  giving the ratio of Gaussian prime density on  $\text{Im}(z) = 5$  and  $\text{Im}(z) = 0$ . In the 5th row of the Gaussian plane, there is a high density of primes.

Much has been written about the influence of computers in mathematical research [95], the story of the constant  $C$  illustrates that already early in the 20th century, when humans were doing the computations by hand, the experimental part has been important.

We report on some experiments on our own for computing the Hardy-Littlewood ratio  $C$ . Wunderlich [96] in the 70ies had computed up to  $n = 14'000'000$ . Our own runs go up to  $140'000'000'000$  which is  $10'000$  times more. We were surprised to see that while increasing  $n$  by several orders of magnitudes, we do not get closer to the actually predicted constant. Its not that we have



to compute more accurately the constant  $C$  which was obtained by Hardy and Littlewood from sieving considerations, but that the small fluctuations actually do only die out **very slowly**, the reason being that we look at the **absolute error** and not the **relative error**. As is evident from the papers of Shanks is that the computing assistants of Hardy and Littlewood got the computations refined so much to values of zeta functions that they had only to consider 2 primes in their version of the product to compute

the constant and already got to an accuracy of 5 digits. When checking how close we are to the constant for  $n = 2^{32}$ , it does not look much better. Note that the cryptologists Dan Shanks (1917-1996) and Marvin Wunderlich (1937-2013) primarily focused his computations on factorization and did measurements at a time, when the first magnetic card programmable computer HP-65 has hit the streets, and when Apple I did not even exist as a concept, and programming languages like Pascal (Niklaus Wirth 1970) and C (Dennis Richie, 1972) had just started to take off. Hardy and Littlewood themselves acknowledge the assisted by several human computational collaborators. Anyway, it is of course still possible that the Hardy-Littlewood claim was too strong and that the density ratios between different rows of the Gaussian integers remain fluctuating on a small order. Only the future will show whether Hardy-Littlewood were right.

On  $\{1, \dots, n = 2^{31} = 2'147'483'648\}$ , there are  $\pi_0(n) = 52549599$  Gaussian primes and on  $\{1 + i, \dots, 1 + 2^{31}\}$ , there are  $\pi_1(n) = 72139395$ . The Hardy-Littlewood ratio up to this  $n$  is  $\pi_1(n)/\pi_0(n) = 1.37279$ .

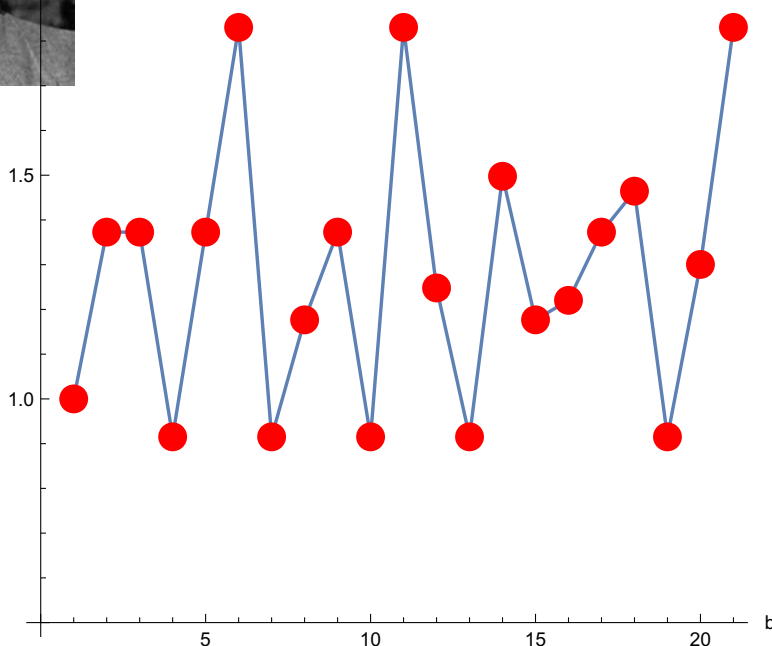
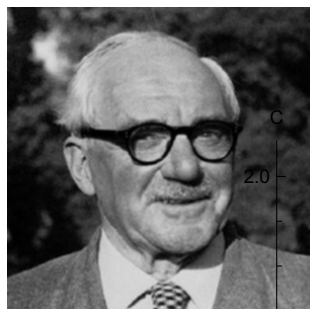


FIGURE 23. The Hardy-Littlewood constants  $C_k$  giving the density ratios between the first and  $k$ 'th row in the Gaussian primes.

To investigate the constant  $C_1$ , we repeated some measurements of Shanks in 1953 and Wunderlich 1973, We could push it further thanks to faster computers. Wunderlich stayed below  $10^{24}$  we went to  $2^{36}$  which illustrates Moore's law. We see that on the interval  $x \in \{1, \dots, 2^{36}\}$ , there are 1884245341 Gaussian primes  $x+i$  and 1372531868 real Gaussian primes. The fraction is 1.3728244749214085. As we don't even have 6 digits reliably while Shanks essentially confirmed 5 digits already. We initially started to doubt the Hardy-Littlewood conjecture until realizing that one should not expect a better convergence than in the case of the prime counting function itself. Why is the error going to zero so slowly?

Does it have to do the **Chebyshev bias** which is a strange phenomenon in the **prime race**? No, the reason is much simpler:

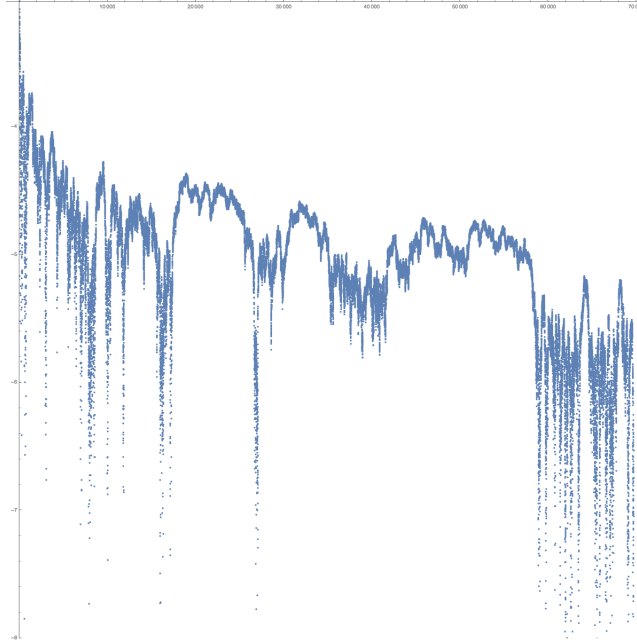


FIGURE 24. The deviation from the Hardy constant  $C$  when increasing the fraction of primes of the form  $a + i$  and real Gaussian primes. These are up to  $10^{11}$  (one unit is  $10^6$ ). Convergence is slow. If the Riemann hypothesis holds, we expect an error of the order  $(\log(x))^2/\sqrt{x}$ . For  $x = 2^{36}$  this is  $2 \cdot 10^{-3}$  only.

The Riemann hypothesis is known to be equivalent to  $|\pi(x) - \text{Li}(x)| \leq C\sqrt{x} \log(x)$ . Because  $\text{Li}(x) = x/\log(x)$  means  $|\pi(x)/\text{Li}(x) - 1| \leq C \log^2(x)/\sqrt{x}$ , we expect an error of the same size also for the convergence to the Hardy-Littlewood constant. In order to get 6 digits reliably, we expect having to go up to  $10^{18}$ .

## 9. GAUSSIAN PRIME MATRICES

In this section we introduce and investigate some **linear algebra problems** for matrices defined by Gaussian primes. The idea is to look at a square window in the Gaussian integer lattice and place a 1 in the corresponding matrix, where we have a Gaussian prime. Otherwise, we place a 0. This construction produces a square **Gauss prime matrix**  $A(z, n)$  which can be studied experimentally. We noticed empirically for example that two rows with odd distance are **negatively correlated** and rows of even distance are **positively correlated**. We follow here a bit the great Chebyshev, who has both relations to probability theory and number theory. Chebyshev also noticed a bias between the prime counting functions  $\pi_1$  and  $\pi_3$  for  $4k + 1$  and  $4k - 1$  primes. The Bertrand-Chebyshev theorem telling that there is always a prime between  $n$  and  $2n$  finally was a precursor of the prime number theorem.

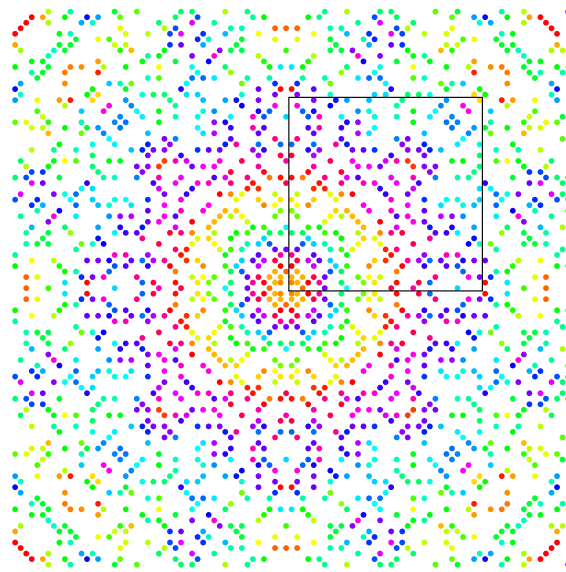


FIGURE 25. The picture shows Gaussian primes in  $|\operatorname{Re}(z)|, |\operatorname{Im}(z)| < 50$  color-coded according to how many primes there are in a neighborhood of radius 4. Outlined is the part produced to generate the matrix  $A(35)$ .

We start with a simple but curious **super symmetry** which has the effect that half of the determinants disappear. This symmetry is based on the fact that for a Gaussian prime  $a + ib$  with  $a^2 + b^2 > 2$  exactly one of the  $a$  or  $b$  is even. Since all except 4 Gaussian primes fit this "checkerboard" pattern, it gives some intuition why the correlation claim is reasonable: the claim would obviously hold if the Gaussian primes were generated by a random process. For matrices not affected by the symmetry, we see that they become invertible if they are sufficiently large and that the determinant grows with a definite super exponential growth rate. Also here, a disprove of Landau's problem on the infinitude of primes  $n^2 + 1$  would lead to arbitrary large matrices singular  $L(k + i \cdot 0, n)$  as the bottom row would then be zero. Since the problem of determinants is linked to a hard problem in number theory, the problem of invertibility of the matrices  $A(n)$  is least as hard.

Let us start to describe the set-up in more detail: for any positive integer  $n \in \mathbb{N}$  and any **Gaussian integer**  $z \in \mathbb{Z}[i]$ , define the  $n \times n$  matrix  $A_{kl}(z, n) = 1$  if  $z + k + il$  is a **Gaussian prime** and 0 else. We call it a **Gaussian prime matrix**. For non-zero and even  $z$ , that is if  $z$  is a non-zero multiple of  $1 + i$ , then  $A(z, n)$  is the adjacency matrix of a triangle free graph  $G(z, n)$  which for large enough  $n$  is a non-planar of chromatic number 2. When looking at spectra, the non-selfadjoint case is more interesting. The simplest non-self adjoint case  $z = 1$  defines the matrix  $A(n) = A(1, n)$ . See Figure (44). Let  $\mu(A) = \frac{1}{n} \sum_{\lambda} \delta_{\lambda}$  denote the **density of states** of a matrix  $A$ . It is the normalized uniform discrete Dirac point measure supported on the discrete

spectrum of  $A$  in the complex plane  $\mathbb{C}$ . In random matrix theory, one calls it the **empirical measure**. Denote by  $\rightarrow$  weak-\* convergence of measures meaning  $\mu_n \rightarrow \mu$  if  $\int_{\mathbb{C}} f(z) d\mu_n(z) \rightarrow \int_{\mathbb{C}} f(z) d\mu(z)$  for any continuous function  $f$  of compact support. Let  $P$  denote the diagonal matrix with entries  $P_{jj} = (-1)^j$ . Since for every  $A = A(z, n)$  with  $z = a + ib \neq 0, a \geq 0, b \geq 0$  and even  $a + b > 0$ , the **anti-commutation relation**  $\{P, A\} = 0$  holds, the spectrum of  $A(z, n)$  has then a **reflection symmetry**  $\sigma(A) = -\sigma(A)$  like Dirac matrices in physics. This symmetry assures that for odd  $n$ , the matrix  $A(z, n)$  always has a kernel, leading in that case to some unexpected linear relations between columns of the Gaussian prime matrices. In the other cases, for  $A(a + ib, n)$  with  $a, b \geq 0, a + b$  odd or  $(a, b) = (0, 0)$ , we detect a **threshold**  $n_0(z)$  so that the matrices are invertible for all  $n > n_0$ .

**Remark:** The matrix  $A = A(a + ib, n)$  with  $a \geq 0, b \geq 0$  and even  $a + b > 0$ , satisfies the anti-commutation relation  $\{P, A\} = PA - AP = 0$ . Its spectrum satisfies the symmetry  $\sigma(A) = -\sigma(A)$ .

Proof. The statement that  $A(z, n)$  anti-commutes with the matrix  $P = \text{Diag}(1, -1, 1, \dots)$  if  $z = k + il$  with  $k + l$  odd and  $k, l > 0$  follows from the fact that any Gaussian prime  $n + im$  has the property that exactly one of the two numbers  $n, m > 0$  is even if  $n, m$  are not both 1. The number  $n^2 + m^2 = p$  is then a prime congruent 1 modulo 4. Now  $PAP$  changes to its negative. We see that  $A$  is then conjugated to its negative implying in the odd  $n$  case that there is a zero eigenvalue.

The **characteristic polynomial**  $p(A, x) = \det(A - x) = p_0(-x)^n + p_1(-x)^{n-1} + \dots + p_n$  of  $A$  defines a real-valued piecewise constant function

$$f_n(x) = \frac{\log |p_{[nx]}(A(n))|}{\log |\text{Det}(A(n))|}$$

on the interval  $[0, 1]$ . We call it the **characteristic polynomial function** of  $A(n)$ . Here,  $\text{Det}(A(n))$  is the **pseudo determinant** [54], the product of non-zero eigenvalues, which is up to a sign the last nonzero coefficient  $p_k$  of the characteristic polynomial; the function  $t \rightarrow [t]$  is the floor function rendering the largest integer smaller or equal to  $t$ . In many statistical settings like for Erdős-Renyi graphs or random matrices, the coefficient functions  $f_n(x)$  is observed to converge uniformly to a limit. Let  $Q_n, R_n$  be the matrices in the **QR decomposition**  $A(n) = Q_n R_n$  and let  $q_n(x, y) = Q_n([nx] + 1, [ny] + 1)$  on  $[0, 1] \times [0, 1]$ . The row vectors  $X_k = (q_n(1, k), \dots, q_n(n, k))$  have constant  $L_2$  norm 1 and are pairwise perpendicular. As usual for data, the **expectation** of  $X$  is  $E[X] = \frac{1}{n} \sum_k X(k)$ , the **covariance** as  $\text{Cov}[X, Y] = E[XY] - E[X]E[Y]$ , the **standard deviation** as  $\sigma[X] = \sqrt{\text{Cov}[X, X]}$  and the **correlation** is  $\text{Cor}[X, Y] = \text{Cov}[X, Y]/(\sigma(X)\sigma(Y))$ . We can apply these notions especially to the row vectors  $R_k(n) = \{1 + ki, 2 + ki, \dots, n + ki\}$  of the Gaussian prime matrices. The function  $\text{Li}(x)$  is the **Eulerian logarithmic integral**  $\text{Li}(x) = \int_2^x dt/\log(t)$ . For the graphs  $G(n) = (\{2, \dots, n + 1\}, \{(a, b) \mid a + ib \text{ prime}\})$  the Euler characteristic decreases monotonically like  $-Cn^2/\log(n)$ . The graphs are bipartite and so triangle free. We have  $\chi(G(n)) = n - \pi(n)/2$ , where  $\pi(n)$  counts the number of primes in  $\{(a + ib) \mid 2 \leq a \leq n + 1, 2 \leq b \leq n + 1\}$ .

Our measurements indicate that there are real constants  $C_0, C_1, C_2, C_3$ , a rotationally symmetric measure  $\mu = \rho(r)drd\theta$  with smooth  $\rho$  in the complex plane and a measure  $\nu = \sigma(x)dx$  with smooth  $\sigma$  on the interval  $[0, 1]$  such that:

<b>A) Invertibility</b> $\det(A(n)) > 0$ for $n > 28 = C_0$ .	<b>B) Determinant</b> $\frac{1}{n \log(n) \log(\log(n))} \log  \det(A(n))  \rightarrow C_1$
<b>C) Trace</b> $\frac{1}{\text{Li}(n)} \text{tr}(A(n)) \rightarrow C_2$	<b>D) Eigenvalues</b> $\mu\left(\frac{C_3 \sqrt{\log(n)}}{\sqrt{n}} A(n)\right) \rightarrow \mu$
<b>E) Characteristic polynomial</b> $f_n(x)dx \rightarrow \sigma(x)$ .	<b>F) QR factorization</b> $E[X_y] \rightarrow 0$ for $y \in (0, 1)$ .
<b>G) Row correlation</b> $\text{Cor}[R_k(n), R_l(n)] \rightarrow 0$ for $k \neq l$ .	<b>H) Covariance sign</b> $\text{sign}(\text{Cov}[R_k(n), R_l(n)]) \rightarrow (-1)^{k-l}$

Computer code to investigate all these statements is included in the TeX source code of this file.



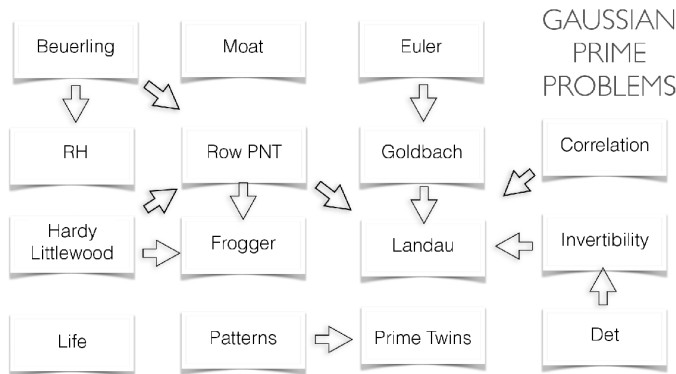


FIGURE 26. Some problems about Gaussian integers.

0	42	0	42	0	42	0	45	0	43	0
43	0	43	0	43	0	43	0	44	0	43
0	44	0	44	0	44	0	45	0	44	0
45	0	45	0	45	0	45	0	45	0	45
0	22	0	31	0	19	0	45	0	43	0
26	0	19	0	19	0	19	0	44	0	42
0	20	0	17	0	19	0	45	0	43	0
28	0	26	0	17	0	31	0	44	0	42
0	27	0	26	0	19	0	45	0	43	0
28	0	27	0	20	0	22	0	44	0	42
28	28	0	28	0	26	0	45	0	43	0

FIGURE 27. As an illustration to A) we compute the apparent non-invertibility thresholds of a matrix  $A(z, n)$  with  $z = a + ib$  and  $0 \leq a \leq 30, 0 \leq b \leq 30$ . An entry 0 means that the matrix  $A(a + ib, n)$  is singular for arbitrary large  $n$ . We know from the anti-commutative relation  $\{P, A\} = 0$  which gives a spectral symmetry  $\sigma(A) = -\sigma(A)$  in the complex plane forces in the odd  $n$  case to have a zero eigenvalue of  $A(z, n)$ .

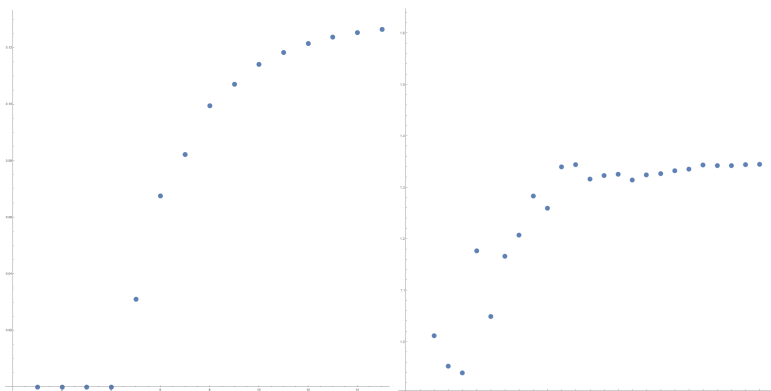


FIGURE 28. To the left we see the values of the determinant function in B) for  $n = 2^k$  with  $4 \leq k \leq 16$ . To the right we see the value of the trace function in C) for up to  $n = 2^{29}$ .

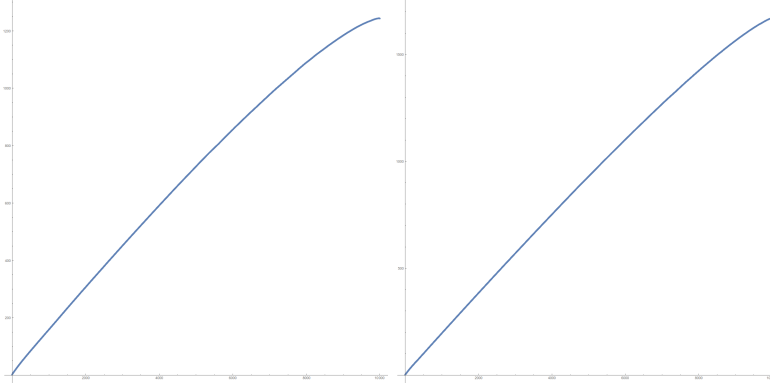


FIGURE 29. To the left, we see the characteristic polynomial function  $f_n(x)$  for  $A(n)$  with  $n = 10000$ . By B), the value  $p(0) = \det(A(n))$  to the left converges to a constant. Statement F) claims that all scaled coefficients converge. To the right we see the characteristic polynomial function for a random 0-1 matrix of the same size. As the coefficients of the characteristic polynomial of a matrix is  $p_k(A) = \text{tr}(\Lambda^k A) = \sum_{|P|=k} \det(A_P)$ , here the sum is over all  $k \times k$  sub matrices  $A_P$  of  $A$ , one knows in the random case that  $f_n(x) \rightarrow x$  for all  $x$ .

1) The numerical evidence has been accumulated only with relatively small  $n$ , namely  $n \leq 2^{15}$ , where we compute with  $32768 \times 32768$  matrices. We are confident about A) and predict it is the most reachable of all these statements. But it is certainly not easy as the generalized problem with matrices  $L(k + 0i, n)$  would imply the unresolved **Landau problem** about the existence of infinitely many rational primes of the form  $n^2 + 1$ . We contemplated about these problems while writing a linear algebra exam for the Math 21b service course at Harvard. The actual exam only featured the problem to compute  $\det(A(5))$  and mentioned A) as the "21b conjecture". The threshold A) will grow for matrices  $A(z, n)$  with larger  $|z|$ , where primes are more sparse as its easy to see that there are arbitrary large disks in  $\mathbb{Z}[i]$  without primes.

2) In B)-D), smaller order growth rates like  $\log(\log(\log(n)))$  might not be easy to detect in experiments. Anyhow, proving bounds like B) or C) appear theoretically out of sight. We do not even know whether the **logarithmic potential**  $\frac{1}{n} \log |\det(A(n))|$  diverges. This later divergence statement would definitely deserve the name conjecture (especially, when taking  $\limsup$ ) because we see it diverge logarithmically.

3) For E) the result holds for random matrices, as one can express  $p_k$  through minors. We also expect part F) to be true for random matrices. The characteristic polynomial function  $f_A(x)$  is interesting in more general setups. We see clear convergence in classes of matrices like if  $A(n)$  is the Kirchoff matrix of complete graphs  $K_n$ , circular graphs  $C_n$ , wheel graphs  $W_n$  or random graphs. To make the statement reasonable also in more graph theoretical setups, we divide by the Pseudo determinant. In our case, we could take the determinant.

4) Matrices like  $A(k(1+i), n)$  are symmetric. The symmetric case  $A(0, n)$  is the only where one still has a finite threshold 28 for zero determinant, all cases  $A(a+ai, n)$  have a kernel for odd  $n$  as the super-symmetry relation  $\{A(n), P\} = 0$  implies then a spectral symmetry  $\sigma(A(n)) = -\sigma(A(n))$  which for odd cases implies the existence of a zero eigenvalue. We see that  $A \rightarrow AP$  switches the sign of half of the eigenvalues and that  $A \rightarrow PA$  switches the other half. In the symmetric case  $B(n) = A(-n - ni, 2n)$ , we have  $\text{rank}(B(n)) = n$  for  $n > 28$ . As the spectrum is real, D) becomes a semi-circle law which should be compared in the random case with Wigner's theorem.

5) The divergence rate of  $\text{tr}(A(n))$  is a statement about arithmetic progressions of Gaussian primes on lines parallel to the coordinate axes. We measure what is expected from having the primes equally distributed on diagonal lines. Questions about arithmetic progressions is attributed to John Leech [36]. But density questions seem difficult as one does not even know whether for fixed  $a, b$ , infinitely many primes of the form  $k(a+ib)$  exist. For any prime  $a$ , work of Hecke shows that there are infinitely many primes of the form  $a+ib$ . The linear algebra questions posed here are related as if there was an  $a_0$  for which on the line  $a_0+ib$  only finitely many primes existed, then there exists  $b_0$  such that  $A(a+ib, n)$  is singular for  $a_0 - n < a < a_0, b > b_0$ . A statement like that for any  $a+ib$  in the first quadrant with  $a+b$  odd, there exists  $n_0$  such that  $A(a+ib, n)$

is nonsingular for all  $n > n_0$  would prove the infinitude of Gaussian primes on any axes parallel to the coordinate axes.

6) Observation D) is intuitively explained by **Girko's law** which holds for random  $\{0, 1\}$  matrices, where each entry takes value 1, 0 with some probability  $p$  going to zero for  $n \rightarrow \infty$ . Probabilistic thinking in that context are not new. For random percolation considerations, see [84]. More generally, one has looked at generalized primes and corresponding Beuerling integers [7, 40]. The random case in some sense investigates the statistics obtained from generalized Gaussian Beuerling primes.

7) Case E) is also observed for random matrices. One can write the coefficients of the characteristic polynomial in terms of minors [54] and get  $f_n(x) \rightarrow x$ . Special coefficients in the characteristic polynomial are the trace and the determinant, covered in more detail in statements B), C). One can get similar convergence statements for each of the coefficients  $p_k$  of the characteristic polynomial. If the picture in E) is correct, each of the  $p_k$  grows in the same way than the determinant, with the exception of the trace which grows at a slower pace.

8) Situation F) is very much expected as the QR decomposition "decorrelated" the columns of a matrix. It could be interesting but as the initial remark about spectral symmetry shows, there are unexpected linear relations between columns. The question about QR decomposition might be the least interesting, but we looked at QR decompositions of matrices  $A(n)$  when looking for problem sources for exam problems.

9) In the random matrix case, the expectation of having an invertible matrix goes to zero exponentially fast. Gaussian prime window matrices  $A(n)$  behave similarly as random matrices. The rate C) is what one expects from the prime number theorem, as there is no reason why on the diagonal of  $A(n)$ , the distribution of the Gaussian primes should be different than on any other side diagonal.

10) Since  $\frac{1}{n} \log(\det(A(n))) = \frac{1}{n} \sum_{\lambda} \log(\lambda)$  is the **Riesz potential**  $\Delta\mu$  of the measure  $\mu(A(n))$ , problems B) and D) are related and B) is the logarithmic potential at 0. Also the minimal absolute value of eigenvalues appears to be what one would expect from random matrices, where  $\log(\det(A(n)))/(n \log(n))$  converges. The claim that for large enough  $n$  the Riesz potential is bounded away from 0 by a definite constant is much weaker than observation B) but appears also out of reach.

11) The mathematics of estimating determinants of large matrices appears frequently in Hamiltonian dynamics or solid state physics. Here is an example of an open problem in ergodic theory which I had worked myself for a long time without success: if  $x(k+1) - 2x(k) + x(k-1) = cf(x(k))$  is the recursion of the symplectic map  $T(x, y) = (2x - y + cf(x), y)$  on the torus  $\mathbb{T}^2$  and  $L(n)u_k = (cf'(x_k) + 2)u_k + u_{k-1} + u_{k+1}$  linearizes a finite piece of orbit of length  $n$ , then the Green-McKay-Weiss formula  $\text{tr}(dT^n) - 2 = \det(L(n))$  for the Jacobian matrix  $dT^n$  of the iterate  $T^n$  leads to the Thouless formula assuring that  $\lambda(x_0, x_1) = \lim_{n \rightarrow \infty} (1/n) \log(\det(L(n)))$  is the **Lyapunov exponent** of the orbit starting at  $(x_0, x_1)$ , measuring sensitivity with respect to changes of initial conditions. By Oseledec's theorem, the limit exists for almost all  $(x_0, x_1)$ . Integrating  $\lambda(x_0, x_1)$  over the normalized Lebesgue measure on  $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$  is the Kolmogorov-Sinai entropy of  $T$ . Pesin theory shows that if it is positive, then  $T$  to a measure theoretical factor of a Bernoulli system on a set of positive Lebesgue measure. In the Gaussian prime case, where one deals not with band matrices, the scaling factor of the logarithmic potential is  $n \log(n)$  not  $n$ , but the difficulty of establishing a limit is similar than for Lyapunov exponents: the situation is deterministic and so of pseudo random nature. It is neither random nor integrable and in both cases, the difficulty is in the complement of integrable situations. Some progress about the Lyapunov exponent of the Standard map has been done in [9].

12) In random cases (probability theory) or integrable cases (harmonic analysis), there are usually attacks. For Jacobi matrices (like for the Anderson model) has been covered by 50 year old analysis Fürstenberg-Kesten. Also integrable almost periodic situations like almost Matthieu allow Fourier theory (Aubry duality) or subharmonic methods (Herman, Sorets-Spencer) are understood. There are more parallels: while almost periodic Jacobi matrices have in general Cantor spectrum, the eigenvalues of almost periodic matrices like  $A_n(i, j) = \cos(i\alpha + j\beta + \theta)$  with rationally independent  $\alpha, \beta, \theta$  appear to have a fractal limiting law in the complex plane. The Gaussian integers behave a bit like a mostly non-uniformly hyperbolic dynamical system, which both show order and randomness at the same time. Both, in the ergodic and number theoretical setups, the difficulty in the analysis is the complement of integrable structures: the complement of KAM islands playing the role of the Erastostenes sieve picture. This theme [80] also appears within dynamical system

theory [63]. The interplay between ergodic theory and number theory is hardly new [29] and has got through a renaissance in the last decade.

## 10. CIRCULAR LAWS

We also look at the eigenvalues of the Gaussian prime matrices and observe a **Wigner-Ginibre-Girko circular law** emerge, even so the density appears different than for **random matrices** with independent coefficients as random variables, where the law has been confirmed and a uniform density on the disc emerges. The idea is to take a large matrix defined by Gaussian primes and draw the eigenvalues in the complex planes. We see a uniform angular distribution to emerge but the radial distribution of the spectrum is different: the density inside is slightly larger than for random matrices. Furthermore, we look at the coefficients of the characteristic polynomial and observe that, like in many other random or pseudo random situations, that they converge to a concave limiting function if properly rescaled. The Gaussian primes appear to be random enough to smooth out the angular distribution. We see such phenomena in non hyperbolic situations which are located between random-Anosov-Anderson type or integrable almost periodic-KAM-Mathieu type situations. In any case, establishing a Girko type law for Gaussian prime matrices looks not easy. Even the random case is technically hard and it was only recently achieved by Tao and Vu [81] for matrices for which the entries can have discrete distribution. The circular law has been found by **Jean Ginibre** in 1965 and **Vyacheslav Girko** in 1984.

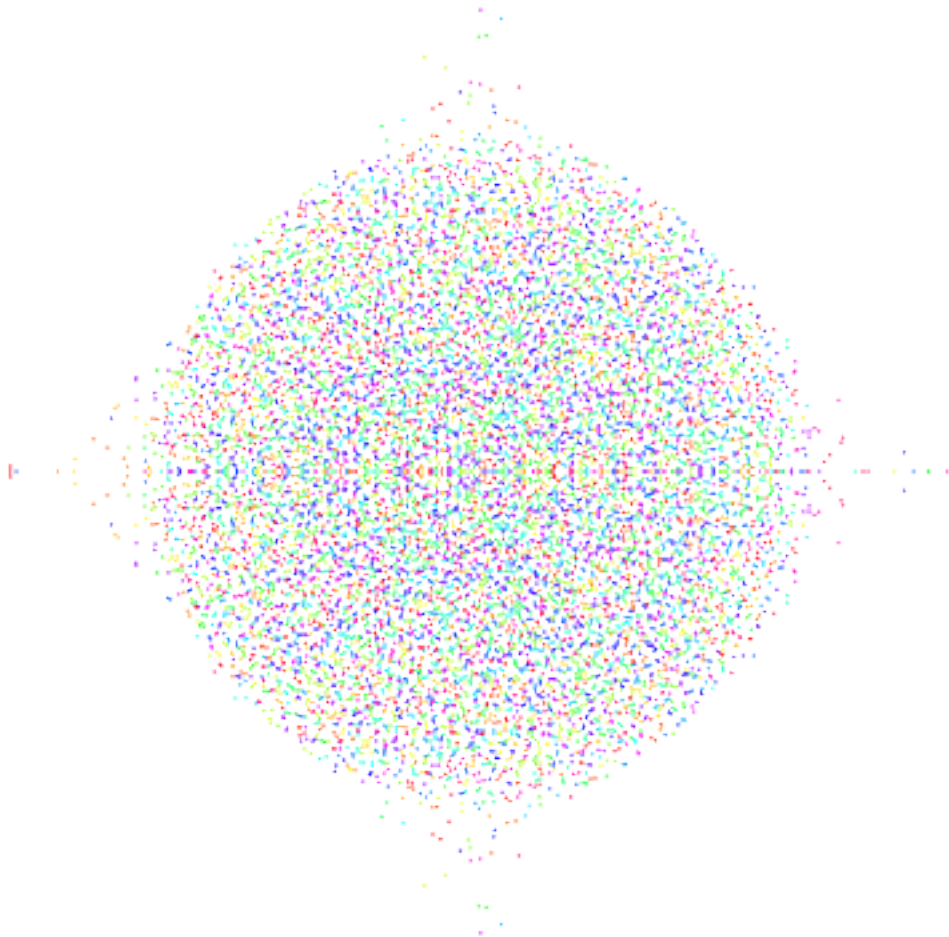
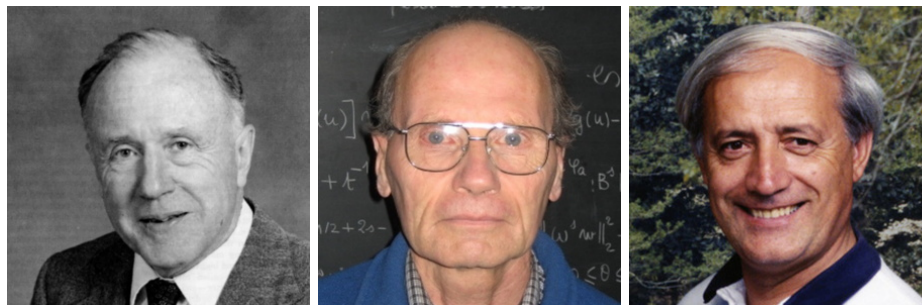


FIGURE 30. We see the eigenvalues of the matrix  $A(10000)$  in the complex plane. The colors encode the distance to the nearest neighbor, the idea being to detect any **clustering** patterns. Some **spectral leaking** near the coordinate axes can be observed but it appears to be too small to be statistically relevant leading to a violation of statement D). Still, there appears to be a larger inner density.



In the self-adjoint case, one has a semi-circle law.

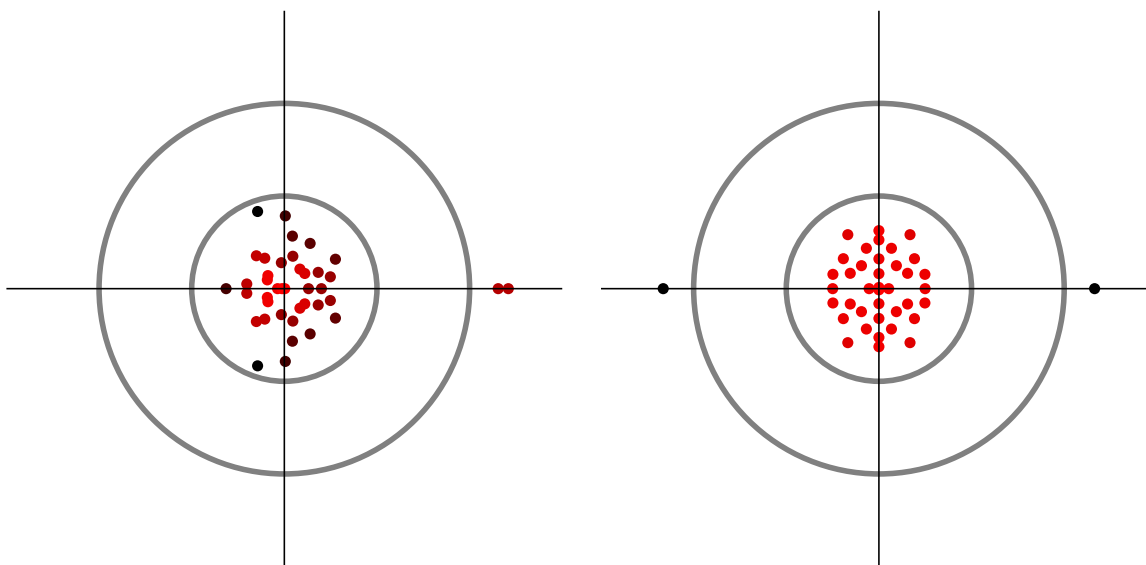


FIGURE 31. We see the eigenvalues of the matrix  $A(1, 201)$  in the complex plane to the left and the eigenvalues of the matrix  $A(2, 201)$  to the right. In the second case, there is a super symmetry present which is reflected in an additional symmetry  $\sigma(A) \rightarrow -\sigma(A)$ . The symmetry group of the left picture is  $Z_2$  for the right picture  $Z_2 \times Z_2$ .

**Remark.**

1) The laws for random matrices were first found by Wigner [94] in the selfadjoint case 1958, the **circular law** in the non-selfadjoint random was discovered by Ginibre in 1965 for the Gaussian case, by Girko in 1984 for more general case. Under rather minimal assumptions like matrices taking finitely many values is covered in [81]

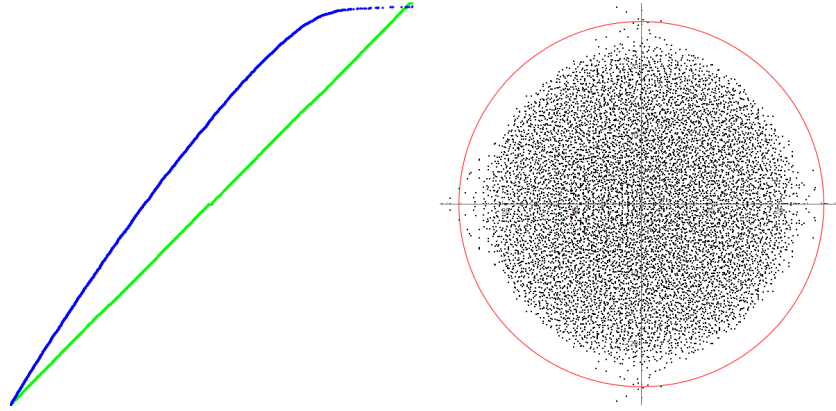


FIGURE 32. This figure shows the cumulative distribution functions of the argument and the absolute value of the eigenvalues. We see that the density decreases towards the boundary and that it is a bit larger inside than in the random uniform case. The actual eigenvalues are shown to the right together with a circle of radius  $\sqrt{n}/\log(n)$ . The **spectral radius** of  $A(n)$  grows like  $n/\log(n)$ , if the Hardy-Littlewood conjectures hold. In comparison, the spectral radius of random  $\{0, 1\}$  matrices grows linearly due to the presence of a single maximal real eigenvalue of size  $n/2$ .

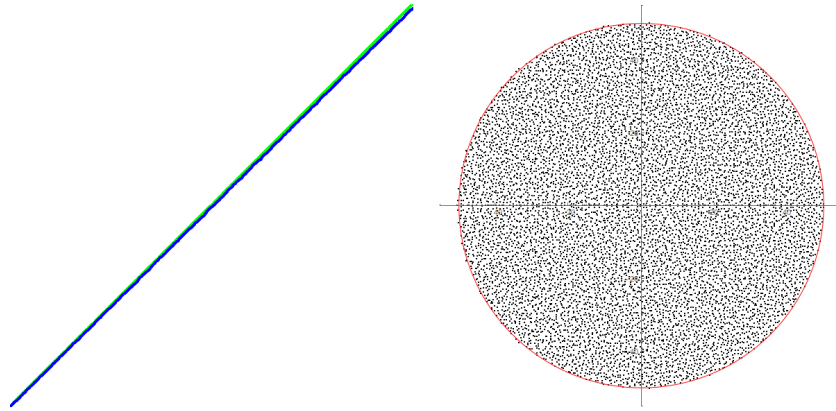


FIGURE 33. As a comparison, we see a random case, where the matrices have been chosen to take random  $\{0, 1\}$ -values with probability  $1/2$ . Also drawn is the circle of radius  $r = \sqrt{n}/2$ . Girko's law for such random matrices is known to hold [81].

## 11. GAUSSIAN PRIME GRAPHS

There is a topological connection between Gaussian primes and graphs. We will look in this section at graphs whose Euler characteristic is related to the **prime number theorem** for Gaussian primes.

We like to see the adjacency matrices of a class of bipartite graphs defined by Gaussian primes. Define  $G(n)$  as the finite simple graph which has as the vertex set the integers  $\{2, \dots, n+1\}$  and for which two vertices  $a, b$  are connected if  $a+ib$  is a Gaussian prime.

Since Gaussian primes  $k+il$  have the property that exactly one of the  $k, l$  are odd, the graphs  $G(n)$  are all **bipartite**. They consequently have no triangles and their Euler characteristic is  $|V| - |E|$  and by Euler-Poincaré given by  $b_0 - b_1$ , where  $b_0$  is the number of connectivity components and  $b_1$  is the **genus**. Since  $|V(n)| = n$  and  $|E(n)| = \pi(n)$  counts the number of Gaussian primes in  $[2 \times n + 1] \times [2 \times n + 1]$  we have a class of graphs for which we know the behavior of the Euler characteristic well.

What is the exact relation between the prime counting function for rational primes and Gaussian primes? Since every rational prime  $p$  in  $\{2\} \cup P_3$  corresponds to exactly 4 primes on the circle  $|z| = p$  and every

rational prime in  $P_1$  corresponds to exactly 8 primes on the circle  $|z| = \sqrt{p}$  the growth rates are related. Let  $\pi_G(r)$  is the **Gaussian prime counting function** giving the number of Gaussian primes in  $\{|z|^2 \leq r\}$ . We have now

**Remark:** The Gaussian prime counting function satisfies  $\pi_G(x) = 4 + 8\pi_1(x) + 4\pi_3(\sqrt{x})$ .

This implies that it behaves like  $8\pi_1(x)4\pi(x)$ , where  $\pi(x)$  is the prime counting function of the rational integers.

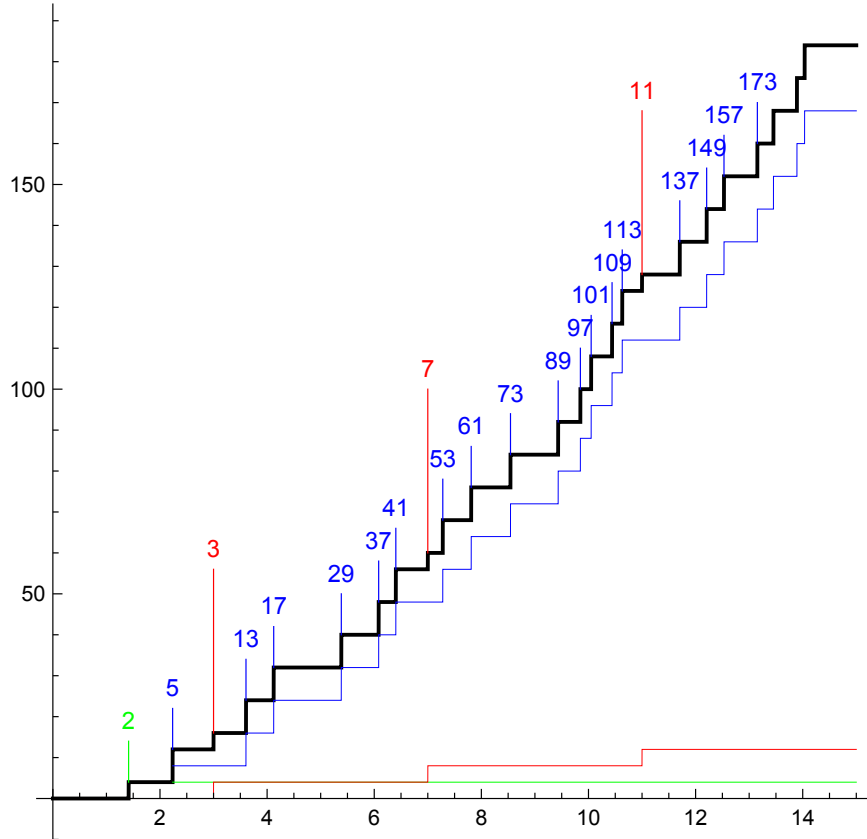


FIGURE 34. The prime counting function  $\pi_G(r)$  for Gaussian primes is the sum of three functions  $\pi_2(\sqrt{r})$  counting the even primes,  $8\pi_1(r^2)$  and  $4\pi_3(r)$ , where  $\pi_k(r)$  counts the number of primes giving remainder  $k$  when divided by 4. Since  $\pi_2$  is bounded by 1 and  $\pi_1(r)$  and  $\pi_3(r)$  grow in the same way, but  $\pi_1$  evaluates at  $r^2$ , the Gaussian prime counting function  $\pi_G$  is essentially governed by the growth of  $\pi_1(r^2)$ . In other words, one can pretty much neglect the even primes and the primes on the axes. The error  $\pi_G(r) - \text{Li}(r)/2$  is described by the Riemann hypothesis. Also the problem of estimating the lattice point error  $\pi_0(r) - \pi r^2$  is open but of a different kind: there are no primes involved and the number of solutions  $x^2 + y^2 = n$  for non-prime  $n$  can be much larger.

We know from the prime number theorem  $\pi(x) \sim \text{Li}(x)$  and  $\pi_1(x) \sim \pi(x)/2$  that  $\pi_G(x) \sim 4\text{Li}(x)$ .

**Remark:** The Euler characteristic of  $G(n)$  grows like  $\pi_G(x)$ .

Proof: The graphs are bipartite. This prevents any triangles to appear in the graph so that the Euler characteristic is  $\chi(G(n)) = |V| - |E| = n - \pi(n)$ , where  $\pi(n)$  is the number of Gaussian primes in the rectangle  $[2, n+1] \times [2, n+1]$  of the complex plane. The genus of these graphs is therefore  $b_1 = 1 - |V| + |E| = 1 - n + \pi_G(n)$ .

It follows that the Riemann hypothesis for Gaussian primes has a topological interpretation as the genus is directly linked to the prime counting function  $\pi$ .

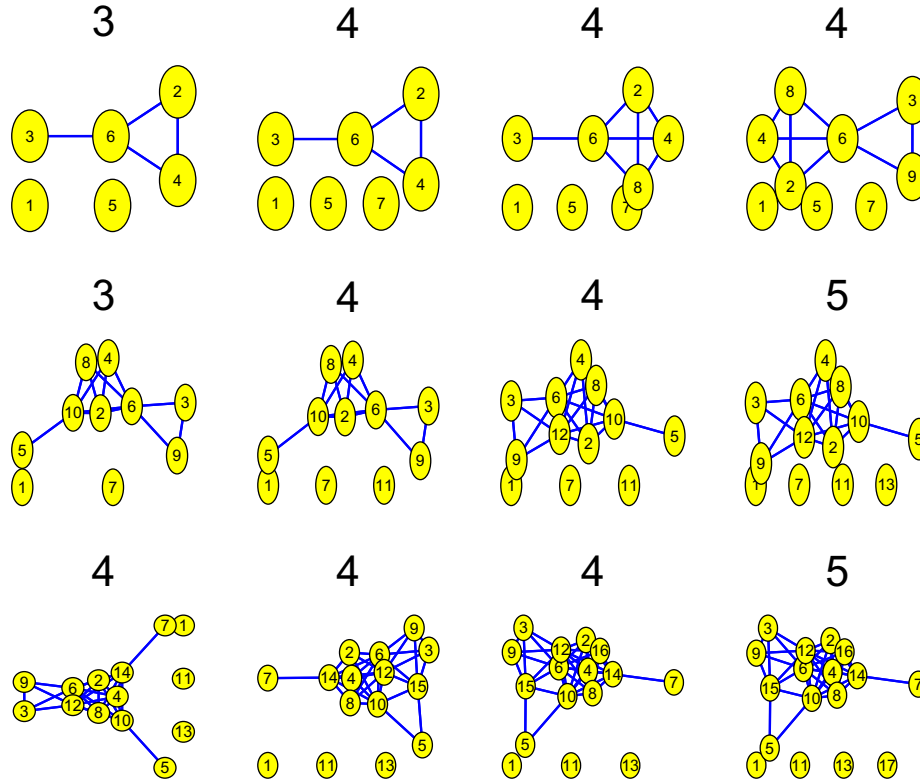
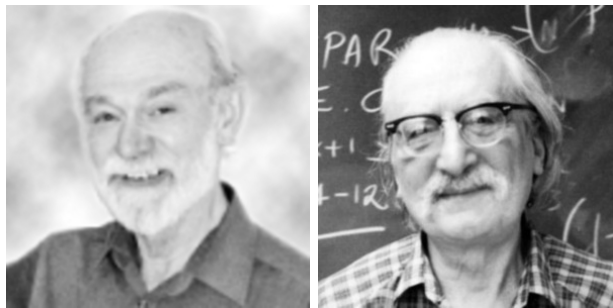


FIGURE 35. We see the bipartite graphs  $G(n)$  for  $n = 4, \dots, 19$ . The vertex set of  $G(n)$  is  $\{2, \dots, n+1\}$  and two vertices  $k, l$  in  $G(n)$  are connected if  $k + il$  is a Gaussian prime. The label above a graph gives its Euler characteristic  $b_0 - b_1 = |V| - |E|$ . The Euler characteristic  $\chi$  of  $G(n)$  is directly related to the Gaussian prime counting number:  $\chi(G(n)) = n - \pi_G(n)$ , where  $\pi_G(n)$  is the number of Gaussian primes in  $[2, n+1] \times [2, n+1]$ . Similarly as in a picture drawn by Knauf [53] but much more elementary, the Riemann hypothesis is here related to topological properties of graphs.



Similarly, we can look at graphs defined by Lipschitz or Hurwitz integers. Let  $H(n)$  denote the graph with vertex set  $a + ib$  with  $1 \leq a, b \leq n$  for which two vertices  $a + ib, c + id$  are connected, if  $a + ib + jc + kd$  is a Lipschitz prime, which means that  $a^2 + b^2 + c^2 + d^2$  is prime. Similarly, define  $L(n)$  denote the graph with the same vertex set for which two vertices  $a + ib, c + id$  are connected, if  $(a + ib + jc + kd)/2$  are Hurwitz primes (meaning  $(a^2 + b^2 + c^2 + d^2)/4$  being prime). Since even and odd can not be connected, the graph  $H(n)$  has at least two component. But we can conjecture to have enough Hurwitz primes so that

**Conjecture:** The graphs  $H(n)$  have exactly two components for  $n \geq 4$ .  
The graphs  $L(n)$  has exactly one component for  $n \geq 2$ .

Similarly as before for the Gaussian graphs, the Lipschitz graphs are bipartite so that the Euler characteristic is equal to  $|V| - |E|$ , where  $|V|$  are the number of vertices and  $|E|$  the number of edges.



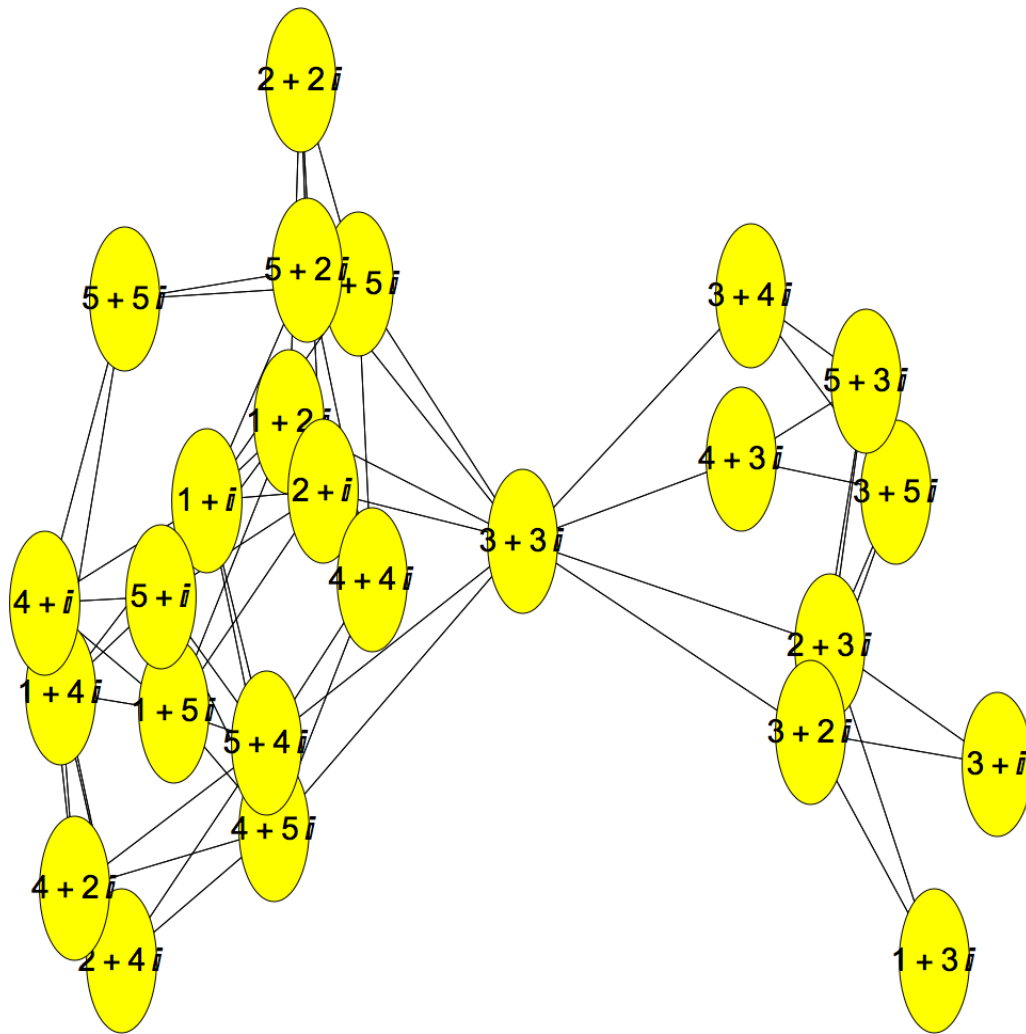


FIGURE 36. In this graph connects Gaussian integers  $a + ib, c + id$  with  $a > 0, b > 0$  for which  $a + ib + jc + kd$  is a Lipschitz prime.

**Remarks.** 1) Graphs can be introduced differently. For every integer  $n$  one can look at the countable simple graph  $G$  for which the Gaussian primes form the vertex set and where two primes are connected if their distance is  $\leq n$ . The **Gaussian moat problem** of Motzkin and Gordon [36] asks whether the connected component containing the prime  $1 + i$  can become infinite for some  $n$ , [24, 15]. The moat problem looks independent of the linear algebra problem posed here. But when seen like this, it is also an eigenvalue problem.

2) The prime counting function should be compared with the **Gaussian circle problem** which deals with the number of Gaussian integers within a disc of radius  $x$ . If  $\pi(x)$  counts the number of primes there, a result of Koch implies with the Riemann hypothesis that  $|\pi(x) - \text{Li}(x)/4| \leq C\sqrt{x} \log(x)$  for a constant  $C$  and sufficiently large  $x$ . The reason is that half of the primes remain on the axes and the other half gets their norm squeezed by the square root and multiply by 8. For the Gauss lattice problem on the other hand,

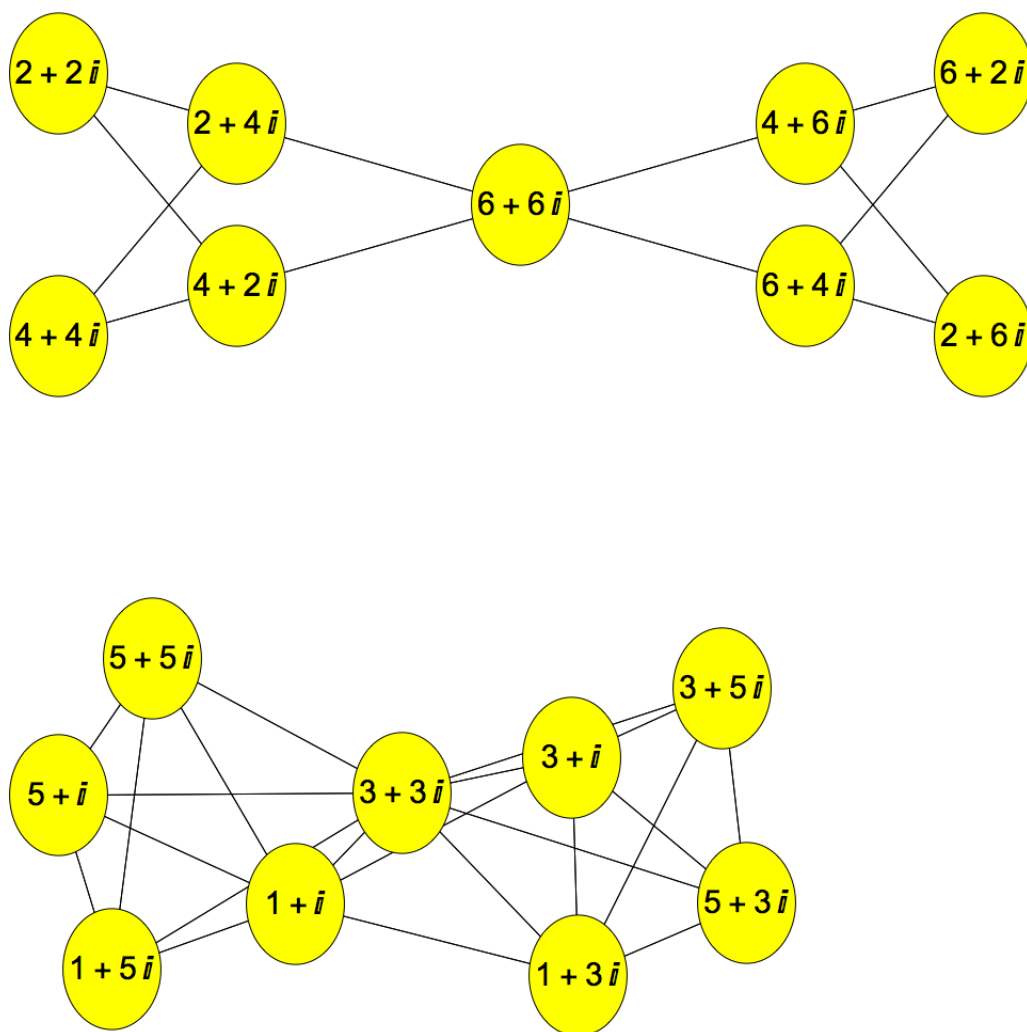


FIGURE 37. Here, we connect Gaussian integers  $a + ib, c + id$  with  $a > 0, b > 0$  if  $(a + ib + jc + kd)/2$  is a Hurwitz prime.

one believes that for every  $\epsilon > 0$ , one has  $|N(r) - \pi r^2| \leq r^{1/2+\epsilon}$  for large enough  $r$ .

**3)** The monotonicity question for the Euler characteristic of the graphs  $G(n)$  would follow from that fact that for any sufficiently large  $a$ , there exists  $2 \leq b \leq a$  for which  $a + ib$  is a Gaussian prime. A counter example would also lead to counter examples in A). Of course, if Landau's 4th problem about the existence of infinitely many primes  $p$  for which  $p - 1$  is a square is false, then there was a counter example. But having the density of primes on rows quantitatively so well nailed down by the Hardy-Littlewood constants, there is hardly anybody who would doubt that there are Gaussian primes on every row  $\{n + bi \mid n \in \mathbb{Z}\}$  of the complex integers.

## 12. GAUSSIAN ZETA FUNCTION

In the context of prime numbers, it is impossible to avoid mentioning zeta functions and especially the **Riemann hypothesis**, which is widely considered the most important open problem in mathematics. For any algebraic structure containing primes and multiplicative norm  $N$ , one has a zeta function  $\sum_p N(p)^{-s}$ . A bit more than a decade ago, the literature for the Riemann hypothesis aiming at the larger public has started to grow:



[20, 23, 71, 70, 91, 83, 59]. Popular expositions have appeared even before the Millennium problems were offered. Remarkable are the gorgeous paper [8] and [16], which is a paper which won a prize for expository writing. Some of the books mention also Gaussian primes. What about the zeta function for Gaussian integers or Hurwitz integers? As the zeta function is an example of a Dedekind zeta function for function fields, **Riemann hypothesis for Gaussian integers** is part of the **generalized Riemann hypothesis**. The story is quite similar than in the case of the standard zeta function.

The **Dedekind zeta function** of the Gaussian integers is

$$\zeta_G(s) = \sum_n N(n)^{-s},$$

where  $n$  ranges over all nonzero Gaussian integers  $n$  in the Euclidean domain  $\mathbb{Z}[i]$  and where  $N(a+ib) = a^2 + b^2$  is the usual arithmetic norm. As  $\mathbb{Z}[i]$  is a unique factorization domain, the **Euler product formula**  $\zeta_G(s) = 4 \prod_{p \in Q} (1 - N(p)^{-s})^{-1}$  still holds, where  $Q$  is set of Gaussian primes in a half open quadrant. It is the **Euler golden rule** for  $\zeta_G(s)$ . Note that the product is over all Gaussian primes in a quadrant only, similarly as for rational primes, where one only takes the product over positive primes. Let  $P_k$  denotes the set of rational primes which give remainder  $k$  modulo 4. The Gaussian zeta function relates with the **Dirichlet beta function**  $\beta(s) = \prod_{P_1} (1 - p^{-s}) \prod_{P_3} (1 + p^{-s})$  which is also called the **Dirichlet L-function**  $L(s, \chi_{4,3})$  for the character  $\chi_{4,3}$ .

While the Riemann hypothesis for Gaussian primes is part of the **generalized Riemann hypothesis**, the two statements for rational and Gaussian primes are known to be equivalent [60]. Already Dirichlet knew:

**Remark:** (Dirichlet) The Gauss Zeta function relates with the usual Zeta function by  $\zeta_G(s) = 4\zeta(s)\beta(s)$ .

Dirichlet knew such factorizations for quadratic number fields. We will look at it below also in the case of the Eisenstein zeta function which belongs to an other ring of integers.

The Riemann hypothesis appears first a bit stronger than the Riemann hypothesis because  $\zeta_G(s)$  involves also the **Dirichlet beta function** As the  $\zeta$  function, also the  $\beta$  function has both trivial as well as nontrivial roots. But it is known that the non-trivial roots of  $\beta(s)$  are on the critical line if the zeta function has the roots of the critical line [60]. The Gaussian Riemann hypothesis is therefore equivalent to the standard Riemann hypothesis.

The proof of the above formula for  $\zeta$  and  $\zeta_G$  appears in many places like [74] page 16, exercise 4.12.h in [88] or [46]).

There are many parallels between  $\zeta$  and  $\zeta_G$ : the **Euler golden keys** are

$$\beta(s) = \prod_{p \in P_1} (1 - p^{-s})^{-1} \prod_{p \in P_3} (1 + p^{-s})^{-1}, \zeta(s) = \prod_P (1 - p^{-s})^{-1},$$

where  $P$  is the set of rational primes. The functional equations of the two Dirichlet L-functions are very similar:

$$\begin{aligned} \beta(1-s) &= 2^s \pi^{-s} \sin(\pi s/2) \Gamma(s) \beta(s) \\ \zeta(1-s) &= 2\pi 2^{-s} \pi^{-s} \cos(\pi s/2) \Gamma(s) \zeta(s), \end{aligned}$$

the **functional equation for the Gaussian zeta function** is therefore even simpler and given by

$$\zeta_G(1-s) = \sin(\pi s) \Gamma(s)^2 \pi^{-2s} \zeta_G(s).$$

The **reduced Gaussian zeta function**  $\xi_G(s) = \zeta_G(s)\Gamma(s)/\pi^s$  is now invariant under the evolution  $s \rightarrow 1-s$  and  $s(1-s)\xi_G(s)$  has an **analytic continuation** to the entire plane.

The equivalence of the hypothesis for  $\mathbb{R}$  and  $\mathbf{C}$  implies statements about the growth rates of the corresponding **Mertens function**. This function is defined in the same way as for rational integers:

First of all, because  $\mathbf{C}$  is a division algebra implying  $N(n)$  to be multiplicative, the formula  $1/\zeta(s) = \sum_{n>0} \mu(n)/n^s$  for the usual Zeta function becomes now

$$\frac{1}{\zeta_G(s)} = \sum_{z, N(z)>0} \frac{\mu(N(z))}{N(z)^s}.$$

with the **Gaussian Möbius function**  $\mu_G(s)$  which is 0 if it contains a square prime factor and 1 if  $n$  has an even number of different prime factors and  $(-1)$  if there is an odd number of different prime factors. But now, for real Gaussian integers of the form  $z = 4k + 1$ , we have  $\mu(z) = 1$ . For  $z = 2$ , we have  $\mu(2) = 0$  as 2 is composed of two primes which are conjugated. In the same way than the **rational Mertens function**  $M(n) = \sum_{k=1}^n \mu(k)$ , define the **Gaussian Mertens function**  $M_G(n) = \sum_{N(k) \leq n} \mu_G(k)$ .

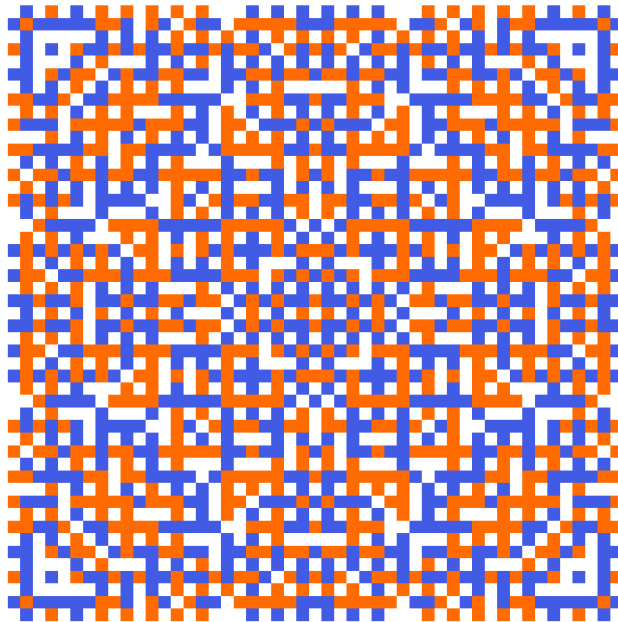


FIGURE 38. The Moebius function for Gaussian integers is defined in the same way as for rational integers. As it is zero on even Gaussian integers  $2a + 2bi$  because  $2 = (1+i)(1-i)$  and  $2a + 2bi = (1+i)^2(1-i)(a+ib)$  contains a square.

The equivalence of the Riemann hypothesis means that also than  $M_G(n) \leq n^{1/2+\epsilon}$  for every  $\epsilon > 0$  and large enough  $n$ .

A highlight of Riemann's theory is the fact that the **Chebyshev function**

$$\psi(x) = \sum_{p^k < x} \log(p) = \sum_{n \leq x} \Lambda(n)$$

which logarithmically counts primes, satisfies the **Riemann-Mangoldt formula**

$$\psi(x) = x - \sum_w \frac{x^w}{w} - \log(2\pi) - \frac{1}{2} \log(1-x^{-2}),$$

where  $w$  runs over the non-trivial roots of  $\zeta$ , and where  $\log(2\pi) = \zeta'(0)/\zeta(0)$  comes from the simple pole at 1 and  $\log(1-x^{-2})/2$  is the contribution of the trivial zeros  $-2, -4, -6, \dots$  and  $x^w/w = e^{\log(x)a + \log(x)ib}/w$  is the contribution from the nontrivial zeros. Pairing complex conjugated roots  $w_j = a_k + ib_j = |w_j|e^{i\alpha_j}, \bar{w}$  gives a sum of functions  $f_j(x) = e^{\log(x)a_j} 2 \cos(\log(x)b_j - \alpha_j)/|a_j + ib_j|$ . This is the major theme in books about the Riemann hypothesis for the more general audience like [83]. The functions  $f_j$  are the tunes of the

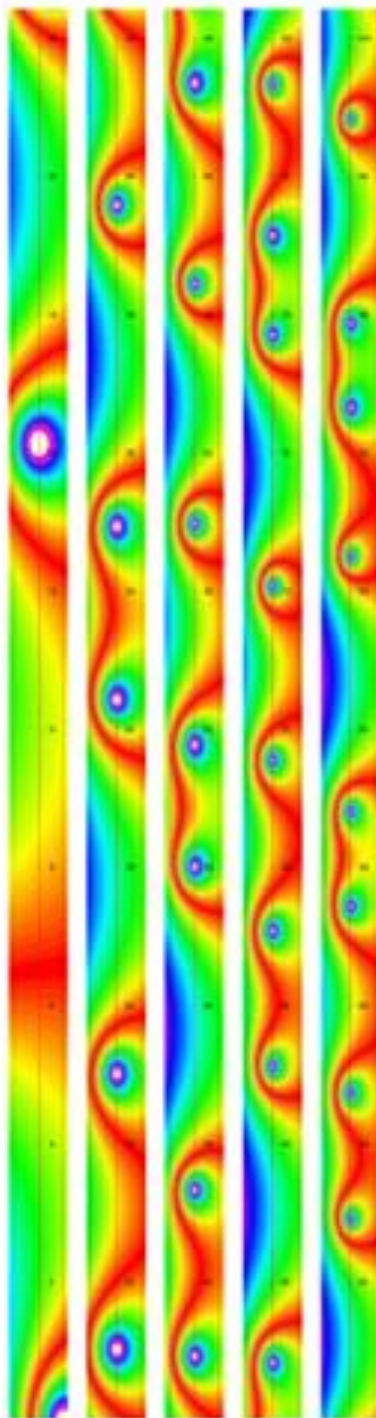


FIGURE 39. A contour plot of the Riemann zeta function in the **critical strip**  $z = a + ib$  with  $0 \leq a \leq 1, 0 \leq b \leq 100$ . The nontrivial roots are all located on the line  $\text{Re}(z) = 1/2$ .

**music of the primes**, as they guide the distribution of primes [8, 72, 67]. One can hardly be too much excited about this formula: [34] state "What an unexpected and delightful identity". Lets look at the picture of these functions:

For quaternions, for which unique prime factorization only holds modulo meta-commutation, permutation and recombination, one still has the golden key formula  $\zeta(s) = \prod_p (1 - 1/N(p)^s)^{-1}$  where the product is

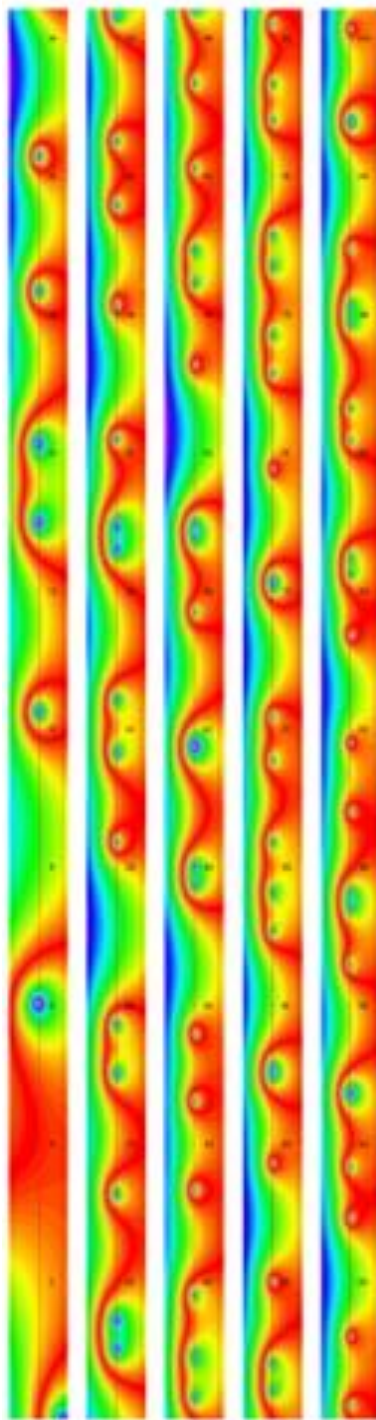


FIGURE 40. A contour plot of the **Dedekind Zeta function**  $\zeta_G(z)$  of the Gaussian primes. It is known that all its roots are on the critical line if the Riemann zeta function has this property. The zeta function is a product of the ordinary zeta function and the Dirichlet Beta function.

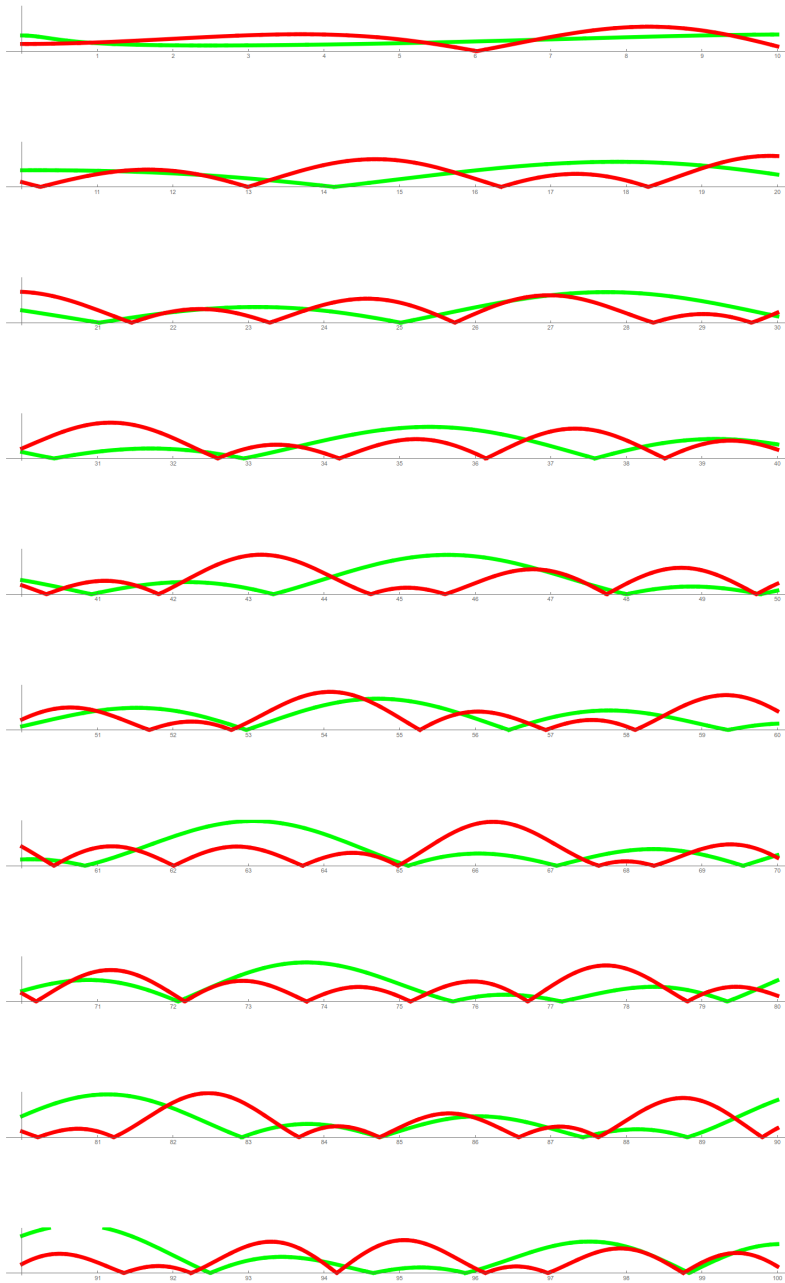


FIGURE 41. The absolute value of the  $\zeta$  and Dirichlet  $\beta$  function on the critical line  $\text{Re}(s) = 1/2$  for  $0 \leq \text{Im}(s) \leq 100$ . The roots have been computed for example in [66].

over all Hurwitz primes in the fundamental region of  $\mathbb{H}$ . This leads to relations between primes and the roots of zeta. Both in the Gaussian as well as the Hurwitz case, there are direct relations to the story in the rational prime case. In the Hurwitz case, the zeta function is just shifted by 1. The upshot is that for rational, Gaussian or Hurwitz arithmetic, the Riemann hypothesis are equivalent.

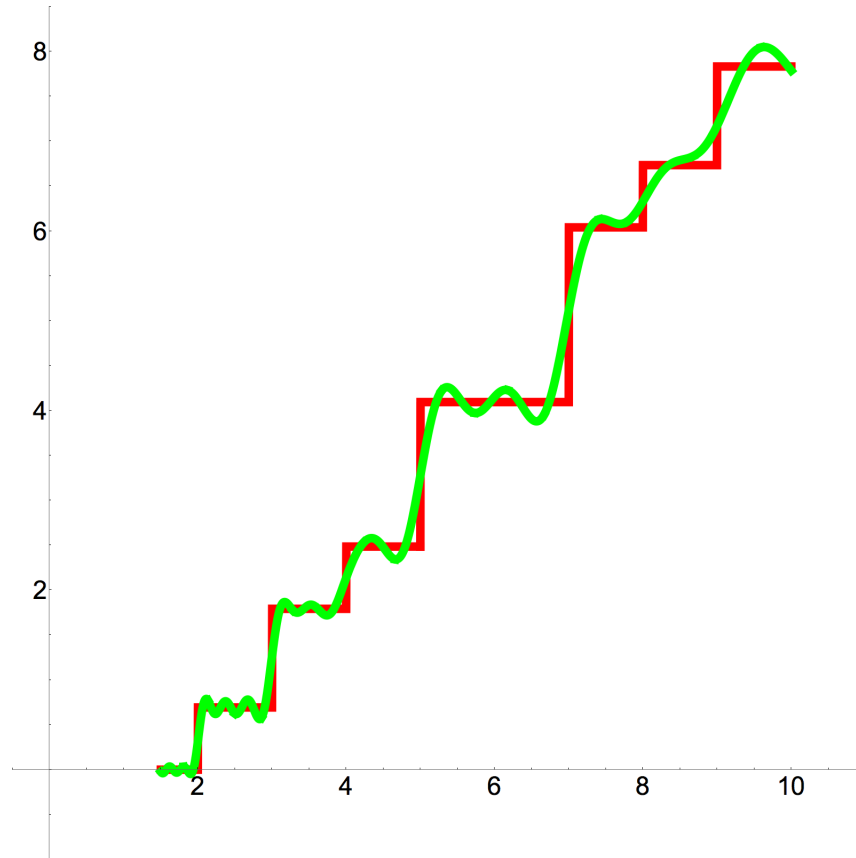


FIGURE 42. The graph of the Chebyshev function  $\psi$  with a finite Fourier approximation given by the Riemann-Mangoldt formula.

### 13. GREATEST COMMON DIVISOR MATRICES

A rather unexpected relation between number theory and matrices appears for the **greatest common divisor matrices**  $A_{ij}(n, s) = \gcd(i, j)^s$  introduced in 1876 by Henry John Smith [76] [76]. Smith was interested in various fields of mathematics. He also discovered the Cantor set well before Cantor [77].

The Smith matrices are finite matrices have entries  $A(n, s)_{ij} = \gcd(i, j)^s$ . For example,

$$S(7, s) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2^s & 1 & 2^s & 1 & 2^s & 1 \\ 1 & 1 & 3^s & 1 & 1 & 3^s & 1 \\ 1 & 2^s & 1 & 4^s & 1 & 2^s & 1 \\ 1 & 1 & 1 & 1 & 5^s & 1 & 1 \\ 1 & 2^s & 3^s & 2^s & 1 & 6^s & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 7^s \end{bmatrix}.$$

The determinants is explicitly known in terms of Jordan functions:

$$\det(A(n, s)) = \prod_{m=1}^n m^s \prod_{p|m} (1 - p^{-s}) = \prod_{k=1}^n \phi^s(k).$$

For  $s = 1$  already Smith has obtained  $\det(A(n, 1)) = \prod_{k=1}^n \phi(k)$ , where  $\phi$  is the Euler function  $\phi(n) = n \prod_{p|n} (1 - 1/p)$ .

More generally, Smith has looked at the matrices  $A_{ij} = \gcd(x_i, x_j)^s$  defined for any finite set  $S = \{x_1, \dots, x_n\}$  which is **factor closed**, meaning that every factor of an element in  $S$  must be in  $S$ . In that case, the determinant is





$$\prod_{k=1}^n \phi^s(x_k).$$

Note that the roots of the function  $h(s) = \det(A(n)^{-s})$  are all on the line  $\operatorname{Re}(s) = 0$  because it is a product of terms  $(1 - 1/p^s) = (1 - \exp(-s \log(p)))$  which are zero exactly for  $\operatorname{Re}(s) = 0$  and  $\operatorname{Im}(s) = 2\pi/\log(p)$ .

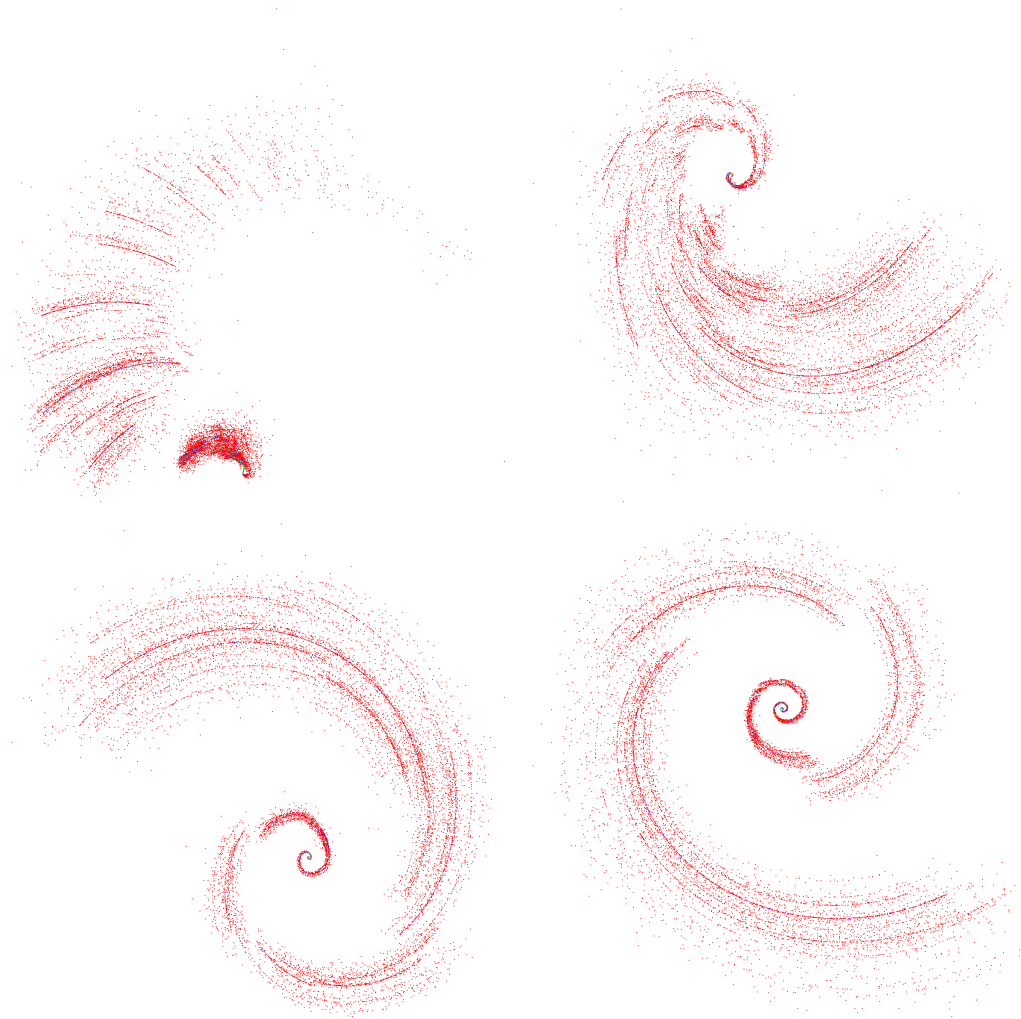


FIGURE 43. The spectrum of the GCD matrix  $A(n)^s$  with  $A_{ij}(n) = \gcd(i, j)$  for the values  $n = 10'000$  for the parameter  $s = 1 + ki$  for  $k = 1, 2, 3, 4$ .

We can also look at adjacency matrices of graphs  $G(n)$  with vertex set  $\{1, \dots, n\}$  for which two integers  $k, l$  are connected if  $\gcd(k, l) > 1$ . Let us call them **GCD graphs**. Since the rule making the connection is transitive, the graph is topologically quite trivial and its Euler characteristic is the number of connected components. Because these connected components must consist of primes larger than  $n/2$  and smaller or equal to  $n$ , we have:

**Remark:** The Euler characteristic of  $G(n)$  is 2 plus the number of primes in the half open interval  $(n/2, n]$ .

We look now at the vertex degrees. Look at a prime  $p \leq n$ . It is connected to  $[n/p] - 1$  other integers, where  $[t]$  is the floor function giving the largest integer smaller or equal to  $t$ . A product of two primes  $pq$  is connected to  $[n/p] + [n/q] - [n/(pq)] - 1$  other integers. For example, in  $G(30)$  the vertex 2 has degree  $[30/2] - 1 = 14$ , the vertex 3 has degree  $[30/3] - 1 = 9$  and the vertex 6 has degree  $19 = 14 + 9 - 4$ . Now, also

the vertex degree of a prime power  $p^k$  is  $[k/p]$  as it is connected to the same integers than 2. And the vertex degree of two prime powers  $p^k q^l$  are all equal to  $[n/p] + [n/q] - [n/(pq)] - 1$ . The vertex  $p^n q^m r^k$  with three primes  $p, q, r$  has degree  $[n/p] + [n/q] + [n/r] - f[n/(pq)] - f[n/(pr)] - f[n/(qr)] + f[n/(pqr)] - 1$ . By the **Euler handshake lemma** the sum of the vertex degrees is twice to the number of edges. This sequence is the sequence A185670 and explicitly given as  $n(n-1)/2 - \sum_{i=1}^n \phi(i) + 1$ .

**Remark:** The edge degree of  $G(n)$  is  $n(n-1)/2 - \sum_{k=2}^n \phi(k)$ .

This formula is attributed to Reinhard Zumkeller in [2] and as a formula counting the number of "non connections" then take this away from  $n(n-1)/2$ . It is amusing that for the determinants of the GCD matrices, we had the product of Euler totients  $\det(A(n)) = \prod_{k=1}^n \phi(k)$ . Now, the number of edges of of the GCD graphs  $G(n) = (V(n), E(n))$  is related to the sum of Euler totients  $\sum_{k=1}^n \phi(k)$ . This function is called the **totient summatory function**  $\Phi(n)$ , which grows like  $(3/\pi^2)n^2$ . Coming back to the GCD matrices, the Euler summatory totient function is just the sum of the entries of the last column of the matrix  $A(n)$ .

For the Kirchoff Laplacian of the graph  $G(n)$ , the trace is the sum of the vertex degrees. We see that the Euler summatory totient function has a spectral interpretation of the Laplacians of a sequence of graphs.

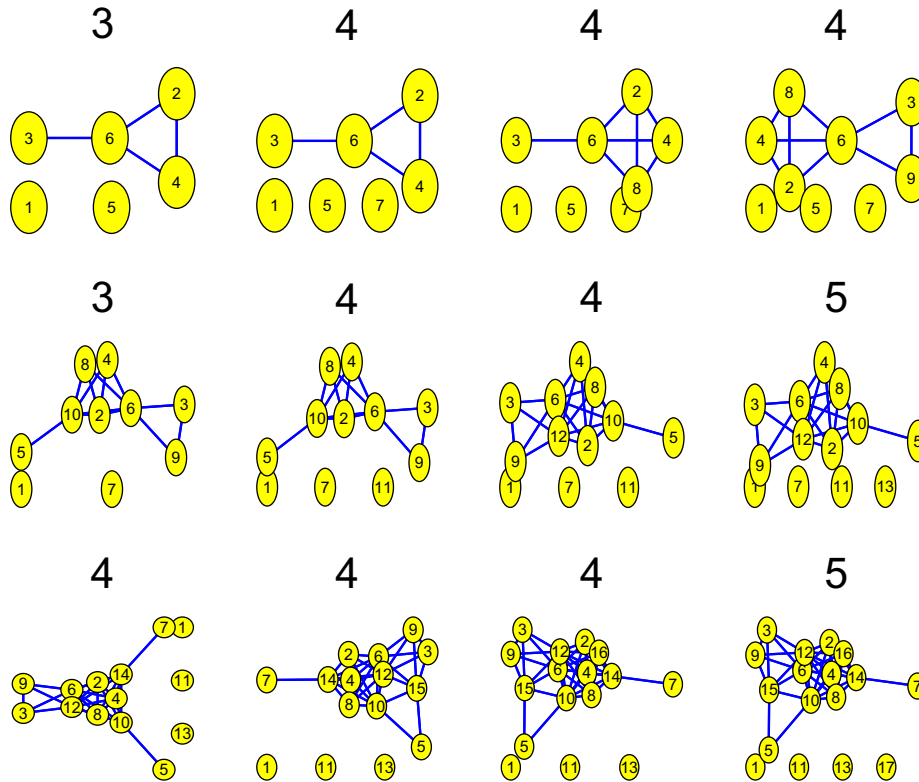


FIGURE 44. We see the  $G(n)$  for  $n = 4, \dots, 19$ . The vertex set of  $G(n)$  is  $\{1, \dots, n\}$  and two vertices  $k, l$  in  $G(n)$  are connected if  $\gcd(k, l) > 1$ . The label above a graph gives its Euler characteristic  $\sum_k v_k(G)$  where  $v_i(G)$  counts the number of complete subgraphs  $K_k$  of  $G$ .

**Remarks:**

1) It is fitting that similar formulas appear in the context of quaternions. [43] proves that if  $n$  is an integer, then the number of quaternions  $z$  congruent to 1 modulo  $n$  is

$$n^3 \prod_{p|n} \left(1 - \frac{1}{p^2}\right) = n\phi^2(n).$$

2) Much more is now known about GCD matrices. One has formulas for the inverse [11] which uses an explicit formula  $EL(n, s)E^T$ , with the unimodular lower triangular matrix  $E_{ij}$  which is 1 if  $i$  divides  $j$  and 0

otherwise. Since  $E_{ij}^{-1} = \mu(i/j)$  if  $i$  divides  $j$  and 0 else, the inverse is explicitly given as  $(E^{-1})^T L(n, s)^{-1} E^{-1}$ . Note that since  $E$  is not orthogonal, the explicit formula does not provide a diagonalization. The structure of the eigenvalues of  $S(n, s)$  is therefore not so clear.

3) Some properties of these determinants were rediscovered by Juan Jose Alba Gonzalez who communicated it to Omar Antolin, who showed it to me. The matrices became a fixture in our 21b linear algebra course of spring 2015, as they appeared in homework, project and exams. We have asked students to look at the eigenvalue structure of these matrices. While working on that Mathematica project, two of our students, Isabelle Steinhaus and Jerry Nelluvelili discovered the spiral for the parameter  $s = 1 + 4i$ . Similarly as Julia sets in complex dynamics, the spectral pictures are parameterized by a complex parameter. We would like to know for example whether for some  $s$  and  $n \rightarrow \infty$ , the spectrum converges to some fractal set. We know from the Smith formula that  $\det(A(n)^s/n^s)$  is a product of factors  $(1 - 1/p^s)$ , where  $p$  runs over all primes dividing  $n$ . In some sense, the determinants of the GCD matrices lurch up to the zeta function if  $n$  is the product of the first  $k$  primes and  $\zeta(s)$  is a limit of determinants.

#### 14. CELLULAR AUTOMATA

An other famous problem in additive number theory is the **twin prime conjecture**. It was first formulated by Polignac in 1849 [68] and mentioned by Kronecker in 1901 and Maillet in 1905.

Cellular automata were first introduced by Gustav Hedlund, who not only made early contributions to the calculus of variations and ergodic theory but also worked in symbolic and topological dynamics. The Hedlund-Curtis-Lyndon theorem assures that any continuous map  $T$  on a product space  $A^L$  of a compact topological space  $A$ , where  $L$  is a lattice such that  $T$  commutes with all translations, must be given by a **local rule**. In other words, such a topological dynamical system must be a cellular automaton.

One can see the set of Gaussian integers as an element in the linear space  $X$  of all functions from the two dimensional lattice  $Z^2$  to the field  $Z_2$ . When seen like this, the set  $X$  is a configuration over the alphabet

$A = \{0, 1\} = Z_2$  or as a compact metric space  $X = A^{Z^2}$  with the product topology. Gustav Hedlund was the first to consider continuous maps from  $X$  to  $X$  which commute with all translations. They are now called **cellular automata**. The most famous by far is the **game of life**. What happens if we apply this to the Gaussian prime initial condition  $x$ ? Let us call a region inside a configuration  $x$  "alive" for a cellular automaton  $T$ , if it is not a fixed point of  $T$ . It is alive, if it moves. One can conjecture that "the region which is alive is infinite". The prime twin conjecture would imply that.



Hardy and Littlewood have precise predictions about the number of classical prime twins. What about the complex cases? The **twin prime conjecture** for Gaussian primes asks for the existence of infinitely many **Gaussian prime twins**, pairs of primes for which the Euclidean distance is  $\sqrt{2}$ . [41]. While one does not know whether infinitely many Gaussian prime twins exist, one can estimate that there are asymptotically  $Cr/\log^2(r)$  of them in a ball of radius  $r$  [46]. Building the graph with the set  $P$  of Gaussian primes as vertex set, where two are connected if their distance is  $\sqrt{2}$ , the prime twin conjecture whether infinitely many components of length 2 exist. [41] also conjectured infinitely many connectivity components of size 3 and 4 and point out that there are only finitely many quintuplets. The **prime gap problem** is formulated analogue in the complex: define  $g_n$  as the radius of the largest punctured disc without primes in  $|z| \leq n$ . For rational primes, finite bounds for  $\liminf g_n$  are known and since Zhang's proof the bound has decreased to 246. This means that there are infinitely many prime pairs of distance 246. [69]. The twin gap problem could also be asked for Gaussian primes and smaller and smaller bounds searched until the twin prime problem is solved.

One can also use cellular automata to illustrate the moat problem. Just apply a map which gives 1 if there is a neighboring cell alive and 0 else. After applying this a couple of times, we can look for connected components. The computed moats of course depend on the cellular automaton used but its clear that if the moat conjecture is true, then also after applying such a CA a couple of times, the connected components

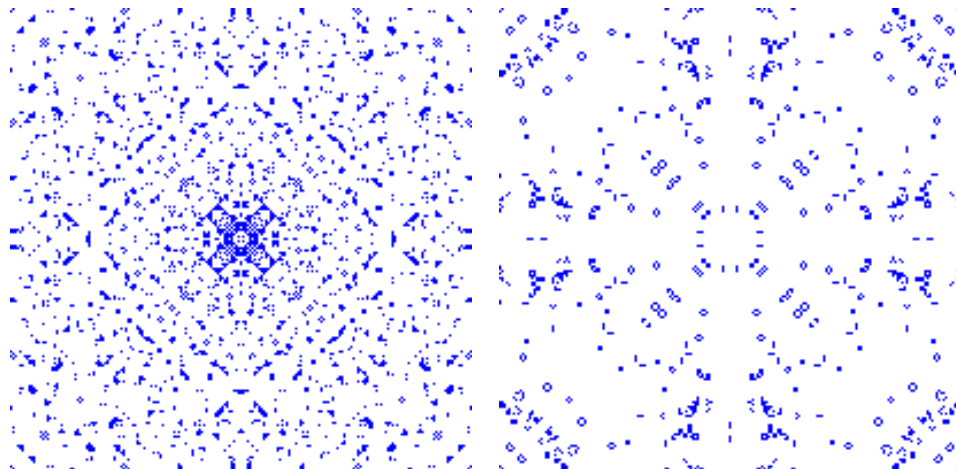


FIGURE 45. When seeing the Gaussian primes as a configuration in  $A^{\mathbb{Z}^2}$ , where  $A = \{0, 1\}$  is the alphabet, we can apply cellular automata on it. The first picture shows the prime configuration after applying the Game of life once, the second after three times. If the prime twin conjecture holds, then there is life arbitrary far away from the origin. “Life” in a region is a configuration which moves when applying the time map.

are still bounded. Indeed, for any cellular automaton, and every configuration  $x$ : if arbitrarily large moats exist for  $x$ , then for any finite  $m$ , arbitrarily large moats exist for  $T^m x$ . Now, like with additions of complex numbers, also the application of cellular automata maps is not compatible with the multiplicative structure of the complex numbers.

So, here is the conjecture about “Prime life arbitrary far, far away”:

**Conjecture:** When applying the game of life cellular automaton to the Gaussian integers, there is motion arbitrary far away from the origin.

If the conjecture is false, there are only finitely many living creatures in the Gaussian prime and that would be rather sad. The problem looks hard. It would follow from the prime twin conjecture for Gaussian primes as it would produce **blinkers** which are time periodic configurations.

**Remarks.**

1) The **Gaussian moat problem** is covered in [48, 32, 24, 90]. Possible Gaussian moat paths to infinity have some argument restrictions [57].

2) Percolation problems for Gaussian primes have been compared before with the case of random matrices [84], for which the distribution  $\pi(x) \sim \text{Li}(x)$  matches the Gaussian prime distribution. The spectral situation for the Gaussian pseudo random case appears similar as the random case. The random case can be seen as generalized primes leading to **Beurling zeta functions**.

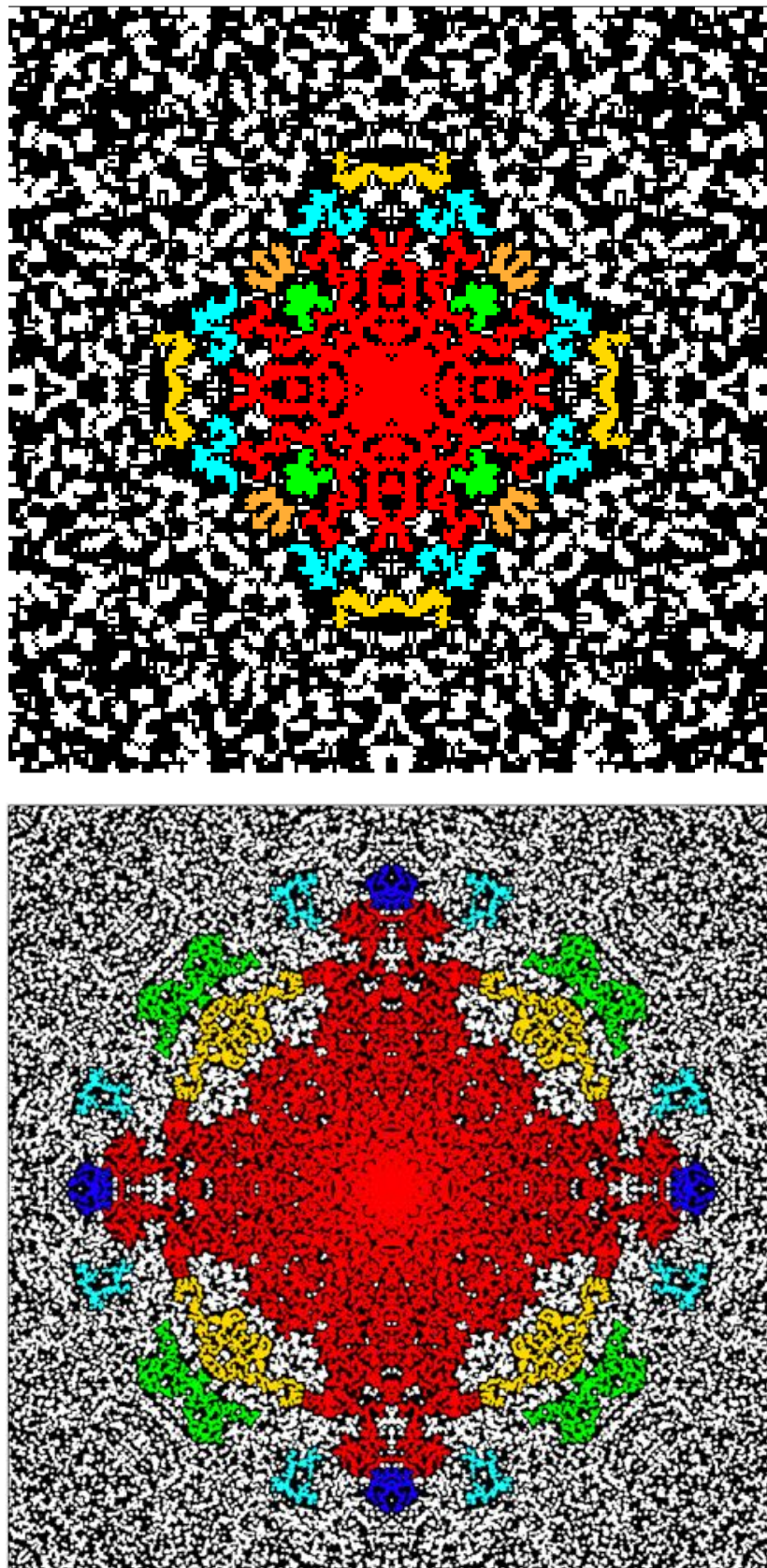


FIGURE 46. Illustration of the moat problem: we applied a cellular automaton map and looked at the central connected component. The first picture is after applying the map once, the second after applying the map two times. In principle, one can compute like that any moat. The problem is that it is inefficient. The best known moats have also been computed using computer assistance but are much more sophisticated.

## 15. ALMOST PERIODIC MATRICES

The last topic is related to an other passion of Hardy and Littlewood, as they studied almost periodic and especially quasi-periodic structures related to number theory.

These structures have been studied in solid state physics in the form of **quasi crystals**, which are special almost periodic structures.



An example of an extensively studied model is the **almost Mathieu operator**  $(Lu)_n = u_{n+1} + u_{n-1} + V_n u_n$  with  $V_n = c \cos(n\alpha)$ , where the almost periodicity is quasi-periodic. In its analysis number theoretical Diophantine properties relate to spectral properties of these Jacobi matrices. When plotting the picture of spectra for all  $\alpha$  in the interval 0 to  $\pi$  one sees the Hofstadter butterfly.

We look at almost periodic matrices defined by irrational rotations defined by

$$A_{km} = \cos(km\alpha + m\beta),$$

where  $\alpha, \beta$  are irrational numbers. We took  $\alpha = \sqrt{p}, \beta = \sqrt{q}$ , where  $p, q$  are primes. The spectrum of these non-selfadjoint  $n \times n$  matrices often feature an unusual spectral structure. We call them **snowflake spectra**.

The non-selfadjoint **van der Monde matrices**

$$B_{km} = \exp(i(km\alpha + m\beta)) = (z^k w)^m$$

with  $z = e^{i\alpha}, w = e^{i\beta}$  also produce interesting spectra in the complex plane but they look more like unions of curves. We have  $A = \operatorname{Re}(B)$  by Euler's formula.

Since the matrices  $A$  are real, the spectrum of  $A$  is symmetric with respect to complex conjugation. The spectrum of  $A(n)$  is however **not** symmetric with respect to  $\lambda \rightarrow -\lambda$  or  $\lambda \rightarrow i\lambda$  but in the large, this symmetry emerges. One can always see these model in a probabilistic setup and study the matrices

$$A_{km}(\theta) = \cos(\theta + km\alpha + m\beta),$$

experimentally. Here is a trivial upper bound:

**Remark:** The spectral norm of  $A$  is bounded above by  $n$ .

Proof; The spectral norm  $\|A\|_2$  is the square root of the maximal eigenvalue of  $A^T A$ . The norm  $\|A\|_1$  is the maximal  $l_1$  norm of its columns and  $\|A\|_\infty$  is the maximal  $l_1$  norm of the rows. One has  $\|A\|_2^2 \leq \|A\|_1 \|A\|_\infty \leq n^2$  so that  $\|A\|_2 \leq n$ .

Asymptotically, we expect the spectral radius of  $A$  to be bounded by  $\sqrt{n} + R(n)$  where  $R(n)$  goes to zero for  $n \rightarrow \infty$  at least if  $\alpha, \beta$  are Diophantine like square roots of primes. Since every column  $w_k$  has Euclidean length  $\leq \sqrt{n}$ , for every vector  $v$  of length 1, we have  $Av = \sum_k v_k \vec{w}_k$  with length  $\leq (\sum_k v_k^2 \sqrt{n}) + \sum_{l,k} v_k v_l \vec{w}_k \vec{w}_l \leq \sqrt{n} + R$  where  $R$  is expected to zero since the different columns become more and more orthogonal.

**Remark:** The matrix  $A$  is the real part of a Van der Monde matrix for the numbers  $a_k = z^k w$ .

We know therefore the determinant  $\prod_{k < l} (a_l - a_k) = \prod_{k < l} (w(z^l - z^k))$  which is  $w^{n(n-1)/2} \prod_{r=1}^{n-1} f_r(z)$  with  $f_r(z) = \prod_{m=1}^{r-1} (1 - z^m)$ . If we look at the absolute value and take the logarithm, this is a sum of Birkhoff sums which can be estimated from above by  $\log(n!)$ , as long as  $\alpha$  is strongly Diophantine like the golden mean.

In the case  $z = e^{i\alpha}$  and  $w = e^{i\beta} = 1$ , the complex matrices are

$$B_{km} = e^{ikm\alpha}$$

for which the determinant is  $\prod_{k < l} (z^l - z^k)$ . Taking absolute values  $\prod_{k,l,k < l} |1 - z^{k-l}|$  which is  $f_1(z) \dots f_{n-1}(z)$  if  $f_n(z) = \prod_{k=1}^n (1 - z^k)$ . We know that if  $\alpha$  is the **golden ratio**, then  $|f_n(z)| \leq \log(n)$ . The proof of the later uses that  $S_k(\alpha) = \sum_{j=1}^k g(j\alpha)$   $G(x) = \log(2 - 2 \cos(2\pi x)) = 2 \log |2 \sin(\pi x)| = 2 \log |1 - e^{2\pi i x}|$ . Now,

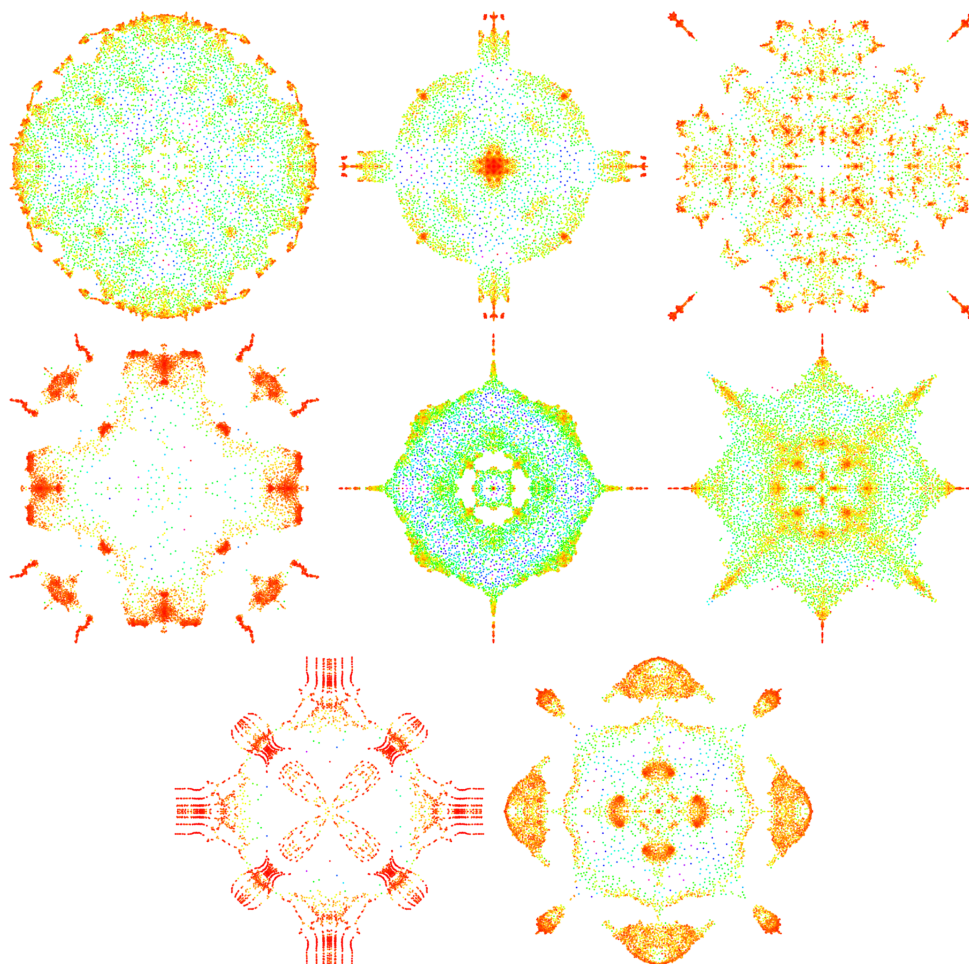


FIGURE 47. The spectrum of  $10000 \times 10000$  matrices  $A_{ij} = \cos(i^2\alpha + j\beta)$  with algebraic  $2\pi\alpha, 2\pi\beta$ . The color of an eigenvalue depends on the minimal distance of the eigenvalue to the rest of the spectrum.

the Birkhoff sum of its derivative  $G'(x) = \cot(\pi x)$  has a universal property.

So, we know at least how to handle the determinant in the complex case for special irrational rotations.

**Remark:** For  $\alpha$  the golden ratio and  $\beta = 0$ , the absolute value of the determinant of  $B$  is bounded above by  $\log(2^n n!)$ .

We see numerically a very similar behavior for the real case  $A$

**Conjecture:** For  $\alpha$  the golden ratio and  $\beta = 0$ , the absolute value of the determinant of  $A$  is bounded above by  $\log(2^n n!)$ .

If we animate the spectra of  $A$  by changing the rotation numbers  $\alpha, \beta$  we see the eigenvalues move around. One can observe **eigenvalue repulsion**. Let us illustrate this in the simplest case  $n = 2$ , where we have the  $2 \times 2$  matrix  $A$  given as

$$A = \begin{bmatrix} 1 & 1 \\ \cos(\beta) & \cos(\alpha + \beta) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ a & b \end{bmatrix}.$$

It has the characteristic polynomial  $x^2 - (1 + b)x + (b - a)$ . By changing  $\alpha$  and  $\beta$  we can achieve that the eigenvalues approach each other on the real axes, then bounce off and scatter away in the complex plane.

How come, almost periodic notions appear in number theory? One reason is the approximation of real numbers by fractions like the continued fraction expansion. How fast this can be done depends on Diophantine

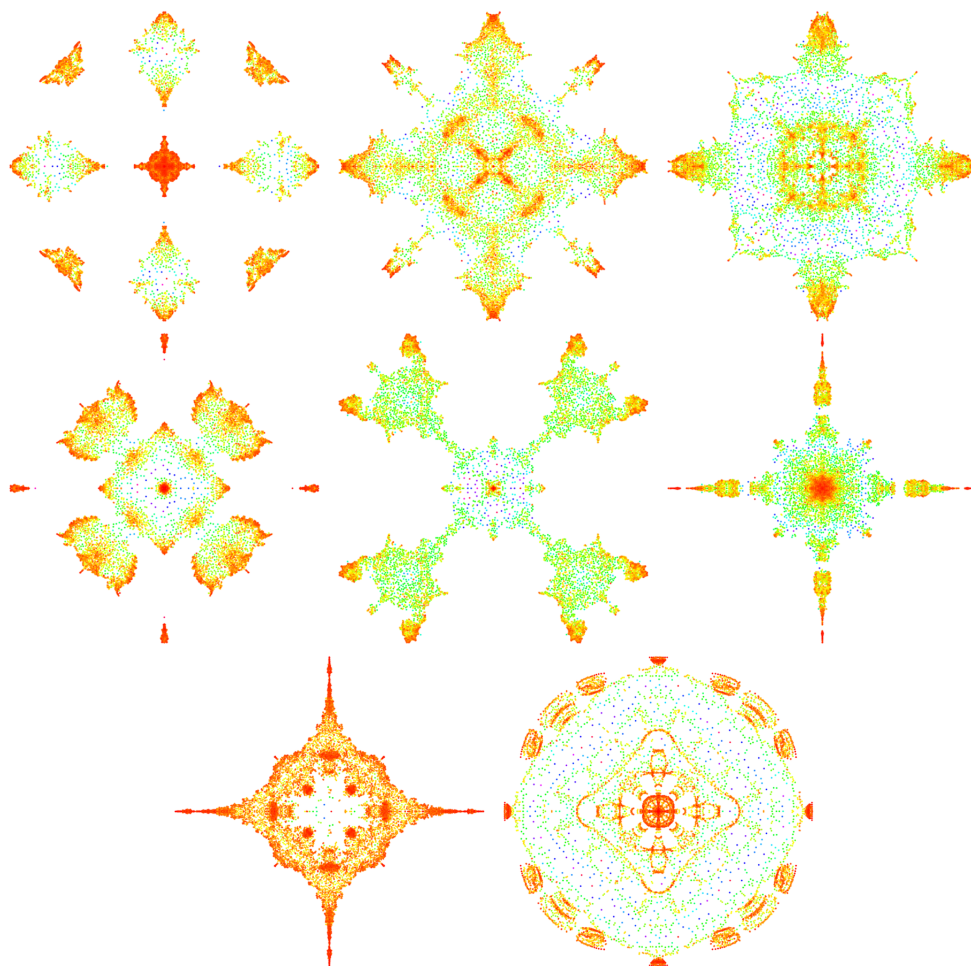


FIGURE 48. The snow flake spectra of these matrices were featured in a movie made for a talk in 2008 and were used in the spring 2010 for a Mathematica project.

properties. When looking at Diophantine equations or problems in the geometry of numbers, ergodic processes related to irrational translation appear naturally too. Here is where Aleksandr Yakovlevich Khinchin comes in who made profound contributions in this area [50] and also was a master of probability. Actually, his law of iterated logarithms has been put into connection with the Riemann hypothesis as popularized first in [26] and since been used in popular texts about the Zeta function like [20].

Unexplored is:

**Conjecture:** The empirical measure of the spectra of  $A(n)$  converges weakly for  $n \rightarrow \infty$ .

In any case, even a rough explanation for the strange spectral phenomena is missing.

## 16. EISENSTEIN PRIMES

In the ring  $R = \mathbb{Q}$  of rational numbers, the **integers** where the subring containing the multiples of 1. **Primes** were the integers different from 1 which can not be written as a product of smaller primes. With this definition, examples of primes are 2, 3, 101,  $-5$  or  $-11$ . When factoring out the **units**, the multiplicative subgroup  $U = \mathbb{Z}_2 = \{-1, 1\}$  of units one has a list of primes is 2, 3, 5, 7,  $\dots$  as  $-5$  and 5 are now identified. An different way to get rid of the “negative primes” is to look at **prime ideals** in the ring  $Z$ . The “primes” are now no more elements in the ring but they are **ideals**, additive subgroups  $I$  of the ring which absorb other integers in the sense that for all  $n$  in  $I$  and every integer  $m$ , also the product  $nm$  is in  $I$ . Each nonzero integer in  $Z$  generates the ideal of all multiples of the integer. An ideal  $P$  different from the ring is called



a **prime ideal** if for any pair  $n, m$  in the ring, the property that  $nm$  is in  $P$  implies that either  $n$  or  $m$  is in  $P$ . In the ring  $Z$  of ordinary integers the ideals are of the form  $(n) = \{\dots, -2n, -n, 0, n, 2n \dots\} = nZ$ , where  $n$  is an integer and the prime ideals are of the form  $(p)$  where  $p$  is a prime. The more abstract ideal setup has been developed by Ernst Kummer in 1844, while attempting to prove Fermat's last theorem. It was then generalized by Dedekind in 1876 to the notion we use today. The setup might be artificial at first, but the setup is unavoidable when generalizing primes to number fields.

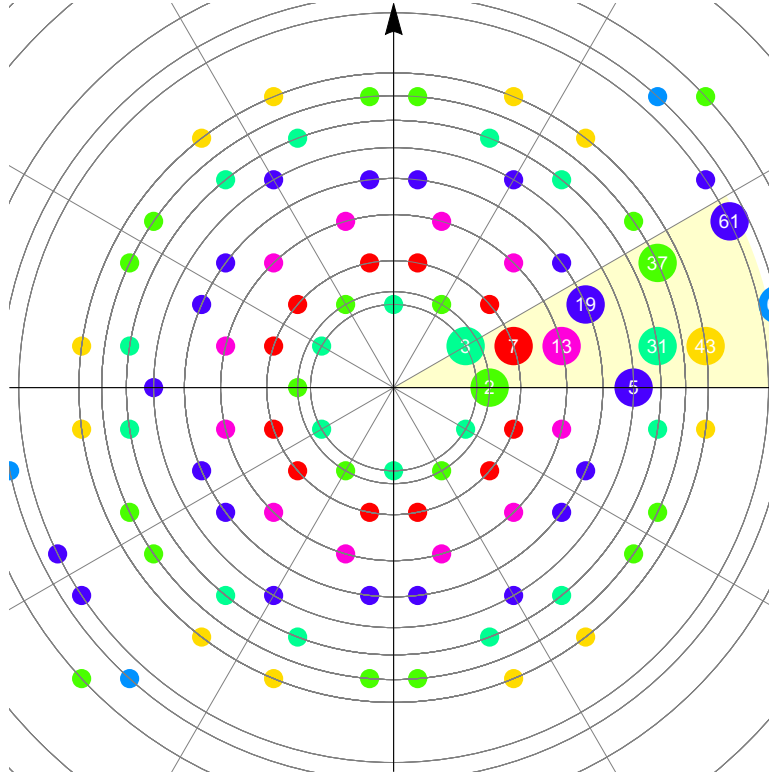


FIGURE 49. The Eisenstein primes in the fundamental region cover the rational primes. The ordering is different than for the Gaussian primes.

If  $R$  is an **algebraic number field**, an algebraic extension of  $\mathbb{Q}$ , the **ring of integers**  $O$  in  $R$  is the set of roots in  $R$  of a monic polynomial with coefficients in  $\mathbb{Z}$ . The ring of integers  $O$  is known to always be a **Dedekind ring**: the product of any two nonzero elements is always nonzero and every ideal different from  $R$  can be written as a product of prime ideals. While unique prime factorization can fail, like in the ring of integers  $R = \mathbb{Z}[\sqrt{-5}]$  in the number field  $\mathbb{Q}[i\sqrt{5}]$  obtained from  $\mathbb{Q}$  by adding the root of  $x^2 + 5$ . The integer 6 has there two different factorizations:  $6 = 2 \cdot 3 = (1 + i\sqrt{5})(1 - i\sqrt{5})$ . But when looking at ideals, the unique prime factorization is restored. If we write  $(a_1 \dots, a_k)$  for the ideal generated by  $a_1, \dots, a_k$ , then Now  $(2), (3), (1 + i\sqrt{5}), (1 - i\sqrt{5})$  are all ideals but they are not all prime ideals. There are "smaller primes" in the form of "larger ideals" like  $p = (2, 1 + i\sqrt{5})$  or  $q = (3, 1 + i\sqrt{5})$ . With these, the factorization  $6 = pq$  has become unique. The **principal ideals** generated by one element had to be enlarged and include **fractional ideals**. The **order** of a ring  $R$  is a sub-ring  $O$  which is an algebra over the field  $\mathbb{Q}$  which is the free Abelian group generated by a basis of  $O$ . The notion of "order" has been important already in non-commutative cases like the ring of quaternions, where the Lipschitz integers do produce an order, but not a maximal one forcing the Hurwitz integers. The example of the Octonion ring shows that the maximal orders are not necessarily unique. In an example like  $\mathbb{Q}[i]$ , the order  $\mathbb{Z}[2i]$  is not maximal neither. To summarize: one can define "primes", if there is a norm which is multiplicative  $N(x)N(y) = N(xy)$  implying that one has to be in a division algebra. This works for number fields in the complex plane. The **integers** of a number field are the maximal orders in that field. Sometimes these rings of integers have a unique factorization by itself, sometimes, it is necessary to use **fractional ideals** which are not necessarily principal. The **ideal class group** is the quotient of the set of fractional ideals divided by the set of principal ideals. Its cardinality is the **class number** of  $O$ .

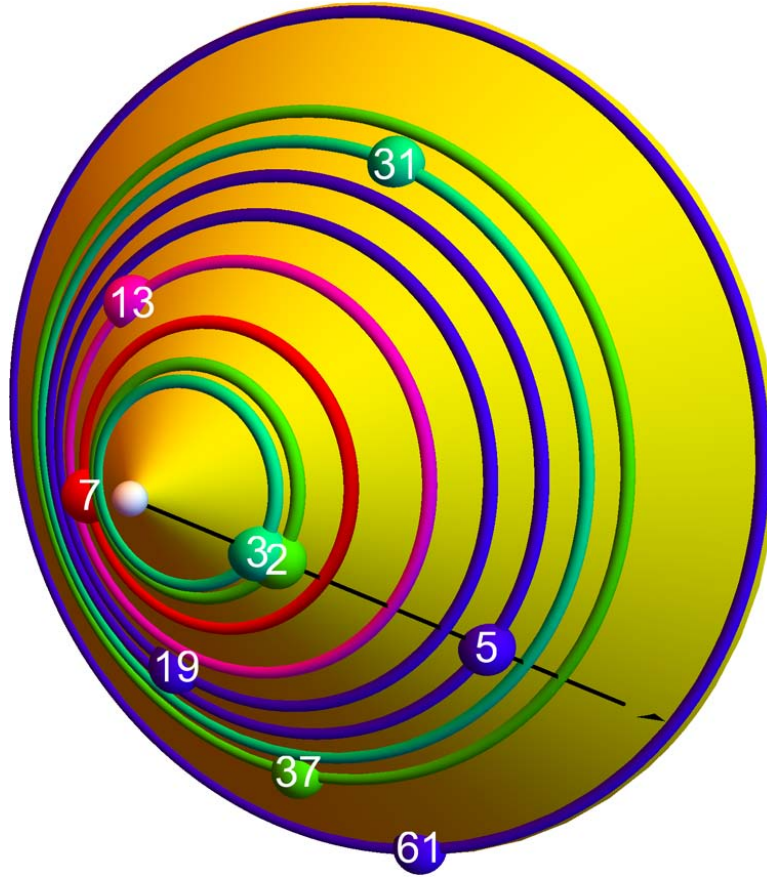


FIGURE 50. Eisenstein primes cover the rational primes.

But what about  $O = Z[\sqrt{-3}]$ ? One has the ideal  $p = (2, 1 + i\sqrt{3})$  but  $p^2 = (2)p$  and the fact that  $p$  is not equal to  $(2)$  shows that there is no unique factorization into ideals. The ring  $O$  appears not to be a Dedekind ring. Did we not just state that any ring of integers  $O$  of an algebraic number field like  $\mathbb{Q}[\sqrt{-3}]$  is a Dedekind ring? Yes, but the ring  $Z[\sqrt{-3}]$  is **not** a ring of integers. Indeed, there is a larger ring which needs to be considered. In other words,  $Z[\sqrt{-3}]$  is not a **maximal order**. There is a larger ring  $Z[(1 + \sqrt{-3})/2]$  which now contains the roots of the polynomial  $z^3 = 1$  and is called the ring of **Eisenstein integers**. This ring has already been constructed by Euler. The Eisenstein integers are now a **ring of integers** and unique factorization holds. The pictures of the primes in this ring are very beautiful as they feature a hexagonal symmetry. Why are the Eisenstein integers a ring of integers? We have to show that if  $\alpha = (1 + \sqrt{-3})/2$ , then every element  $a + b\alpha$  is a root of a monic polynomial. One can give it explicitly: as  $x^2 - (2a + b)x + (a^2 + ab + m^2)$ . (Note that we use  $\alpha$  for which both real and imaginary part are positive, one usually looks at  $\bar{\alpha}$  as a basis but for Goldbach, the positive sign is better). We have seen above that the class number of  $Z[i\sqrt{5}]$  is 1, but that the class number of  $Z[(1 + i\sqrt{3})/2]$  was 2. The class number quantifies, how far away the ring is from a unique factorization domain. While in general not understood yet, one knows the class numbers for quadratic number fields  $Z[\sqrt{d}]$  with negative  $d$ .

An Eisenstein integer  $z = a + b\alpha$  is prime if and only if  $p = N(z) = z\bar{z} = a^2 + b^2 + ab$  has one of the following properties: either  $p$  is prime and congruent to 0 or 1 to 3 or then  $\sqrt{p}$  is prime and congruent to  $-1$  modulo 3. The Eisenstein integers show a similar **dichotomy** like the Gaussian integers but with the 4 replaced by 3. Obviously, also 3 plays a special role now.

**Remark:** There is a natural bijection between Eisenstein primes and rational primes.

The picture is the same: identify  $\arg(z) = 0$  with  $\arg(z) = 2\pi/6$  to get a cone. Now the prime 2 represented by  $2 + 0w$  and 3 represented by  $1 - w$  as well as any prime of the form  $3k - 1$  are on the gluing line, while the

primes of the form  $3k + 1$  are in the interior. Again, we get for every of the later a unique angle  $0 < \alpha < \pi/6$  which is the argument of  $\arg(a + bw)$  with  $p = a^2 + b^2 + ab$ . Again we don't know of a fast way to compute that angle for a given prime of the form  $3k + 1$ .

Let us define  $Q$  to be the open **hexant**  $Q = \{x + yw \mid x > 0, y > 0\}$ , where  $\omega = (1 + \sqrt{-3})/2$ . Note that we use the cube root of  $-1$  and not the cube root  $(-1 + \sqrt{-3})/2$  as usual. The reason is Goldbach, were it is more convenient to work with  $w$ . When experimenting with these numbers, we notice that every Eisenstein integer in the open quadrant in distance 2 or larger from the boundary is a sum of two Eisenstein primes. Almost! There are two exceptions. Not on the boundary of integers of the form  $2 + kw$  which seems most vulnerable but on the row  $3 + kw$ !

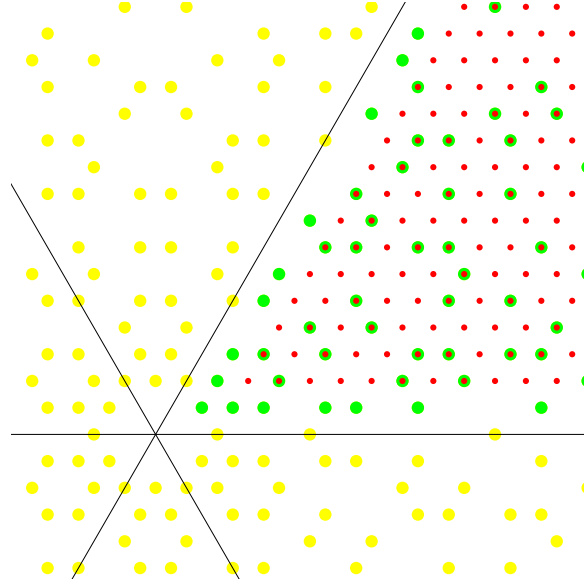


FIGURE 51. The Eisenstein Goldbach conjecture looks good at first for coefficients  $> 1$ . Here we see the initial part. But there are integers  $(109 + 3w)$  and  $(121 + 3w)$  (as well as their mirrors) which can not be written as the sum of two positive Eisenstein primes  $a + bw + (c + dw)$  with positive  $a, b, c, d$ . These are the bad **Eisenstein ghost twins**. They are not visible on this picture

We don't even need the condition that the integers are **even** except for the two ghost examples (we identify  $a + bw$  with  $b + aw$  when counting). The Eisenstein primes are so dense that their sum seems to cover the sector:

**Conjecture:** Every Eisenstein integer  $z = a + bw$  with  $a > 2, b > 2$  is a sum of two Eisenstein primes in  $Q$ .

and

**Conjecture:** Every Eisenstein integer  $z = a + bw$  with  $a > 1, b > 1$  is a sum of two Eisenstein primes in  $Q$  except for the two Eisenstein twin ghost examples  $3 + 109w, 3 + 121w$ .

**Conjecture:** Except for two counter examples, every rational integer  $n$  larger than 1 is a sum  $x + y$  where  $p = x^2 + x + 1$  and  $q = y^2 + 2y + 4$  are both rational primes. The two counter examples are  $n = 109$  and  $n = 121$ .

Again, we do not have to worry that a positive proof of the Goldbach question for Eisenstein integers is "easy" (except of course if there would exist an obvious counter example) To see this, look at the boundary case and consider the Eisenstein integers of the form  $n + 2w$ . If the Goldbach conjecture is true, then every  $n + 4w$  with odd  $n$  is a sum of two Eisenstein primes  $a + w, c + 3w$ . But since  $x^2 + x + 1$  is a cyclotomic

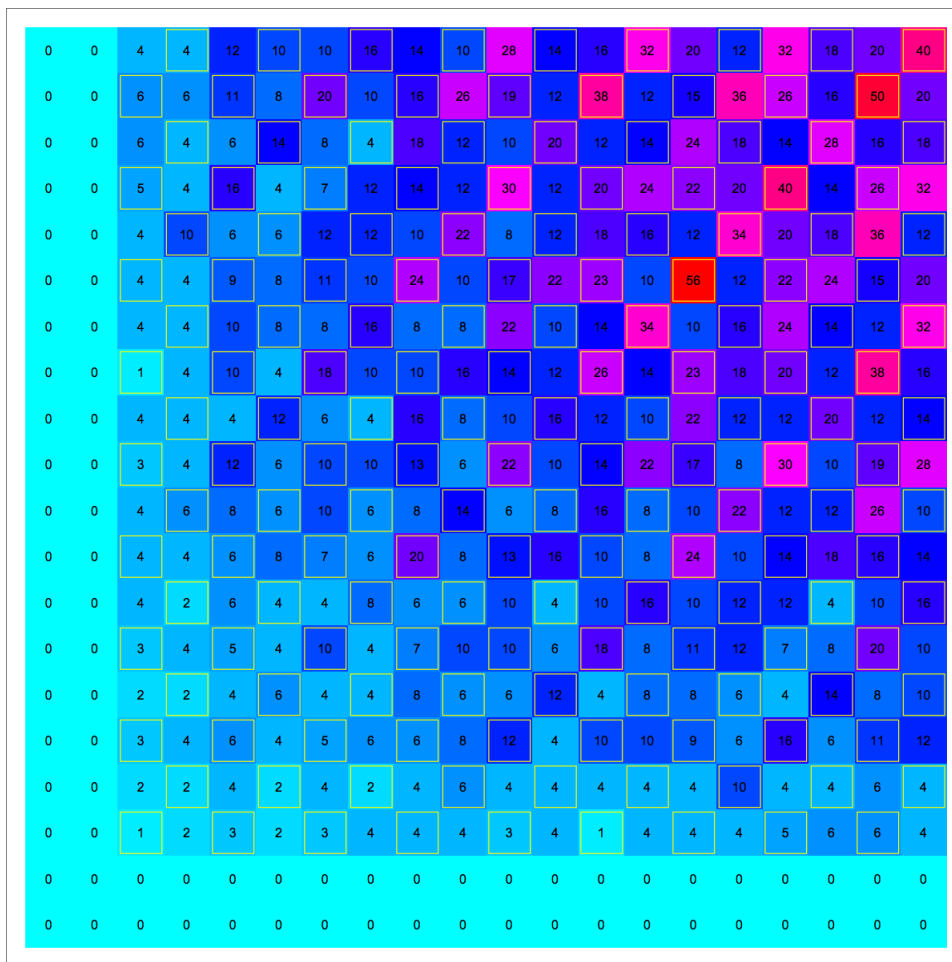


FIGURE 52. Here we see how many times an Eisenstein integer  $a + wb$  can be written as a sum of two Eisenstein primes.

polynomial, the Bunyakovsky condition is satisfied. In other words, we must have infinitely many primes of the form  $x^2 + x + 1$ , a problem which is considered similarly hard than the Landau problem:

**Remark:** If Goldbach conjecture holds for Eisenstein integers, then the Bunyakovsky conjecture holds for  $x^2 + x + 1$ .

Goldbach is even stronger at the boundary than Bunyakovsky: the density of the primes of the form  $n^2 + n + 1$  is so large that one can reach every rational integer larger than 2: one could call this the **boundary Eisenstein Goldbach conjecture**:

**Conjecture:** Every rational integer  $n > 1$  is a sum  $n = x + y$ , where  $x^2 + x + 1$  and  $y^2 + y + 1$  are both rational primes.

There is a surprise on the next column as there are two ghost counter examples. We call them the **Eisenstein ghost twins** (not to confuse with Eisenstein prime twins, which are neighboring Eisenstein primes).

When going further, it appears that every integer larger than 1 is the sum of  $n = x + y$  where  $p = x^2 + x + 1$  and  $q = y^2 + 3y + 9$  are rational primes.

On the next row  $5 + kw$  again, there are exceptional cases, of twins when writing  $n$  as a sum of primes of the form  $p = x^2 + x + 1$  and  $y^2 + 4y + 16$  where  $n = 21, 147$  are not possible or in the form  $p = x^2 + 2x + 4$  and  $q = y^2 + 3y + 9$  where  $n = 6, 27, 126$  are not possible. But now the two sets don't intersect and we appear always to be able to write the Eisenstein integer  $n = 5 + kw$  as a sum of two Eisenstein primes, either in the form  $n = (1 + xw) + (4 + yw)$  or then in the form  $n = (2 + xw) + (3 + yw)$ . As further away we are from the boundary, as more possibilities we have and the chance gets smaller and smaller.

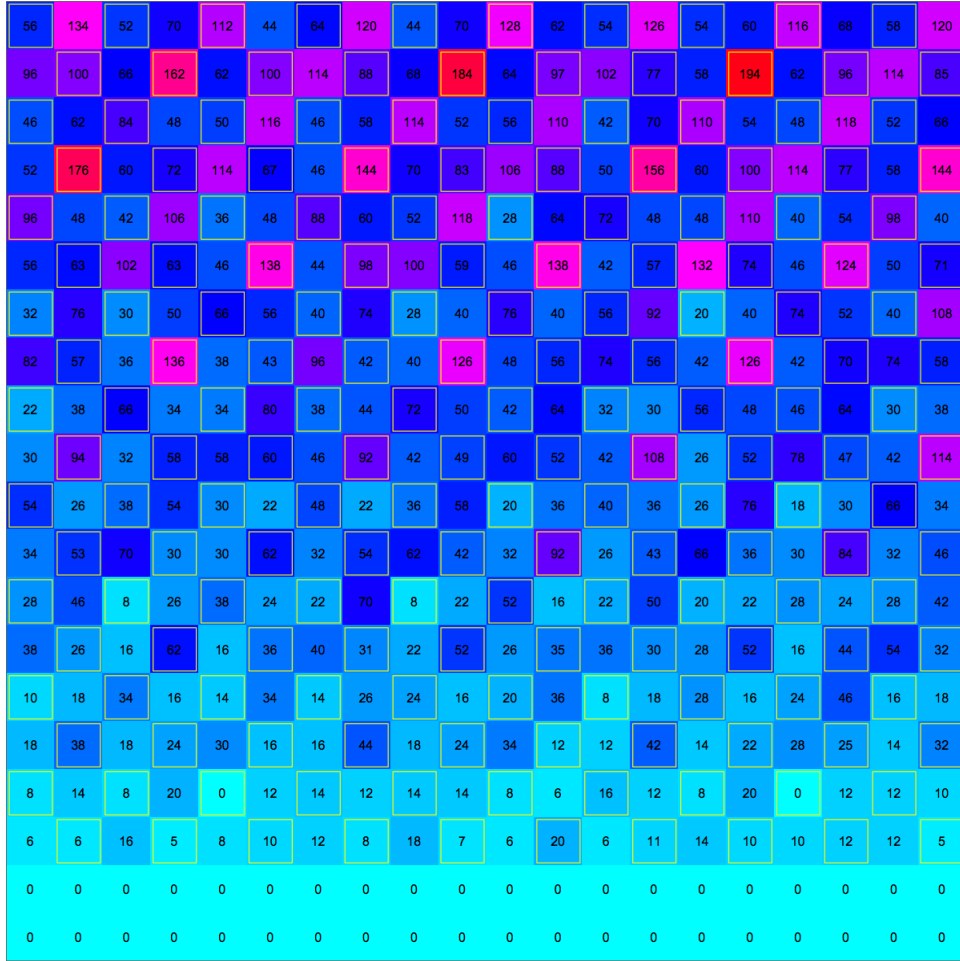


FIGURE 53. The Eisenstein ghosts  $(109 + 3w)$  and  $(121 + 3w)$  (as well as their mirrors) which can not be written as the sum of two positive Eisenstein primes  $a + bw + (c + dw)$  with positive  $a, b, c, d$ . These are the bad **Eisenstein ghost twins**. Can you find the ghosts in the matrix? They are caught in the interior of the matrix and not at the boundary.

**Remark: 1)** Having seen so many Goldbach versions, one can ask for more. Goldbach conjectures have been formulated in polynomial rings. One could also change the field and look for example at p-adic fields or then use an other division algebra. By a theorem of Frobenius, there are only the quaternions and octonions left. One could look at number fields and rings of integers inside the quaternions and octonions. Also to this there is literature but one has first to make sense of ideal theory in non-commutative and non-associative setups.

Lets at last look at the Zeta function in this case. The **Eisenstein zeta function** is an example of a Dedekind zeta function. Since the norm of  $z = n + mw$  is  $n^2 + nm + m^2$ , it is

$$\zeta_E(s) = \sum_{(n,m) \neq (0,0)} \frac{1}{(n^2 + nm + m^2)^s}.$$

As Eisenstein integers form a factorization domain, the **Euler product formula**

$$\zeta_E(s) = \prod_{p \in P} (1 - N(p)^{-s})^{-1}$$

still holds, where  $p$  runs over all Eisenstein primes as well as

$$\zeta_E(s)_{-1} = \sum_{(n,m) \neq (0,0)} \frac{\mu(n + mw)}{(n^2 + nm + m^2)^s}$$

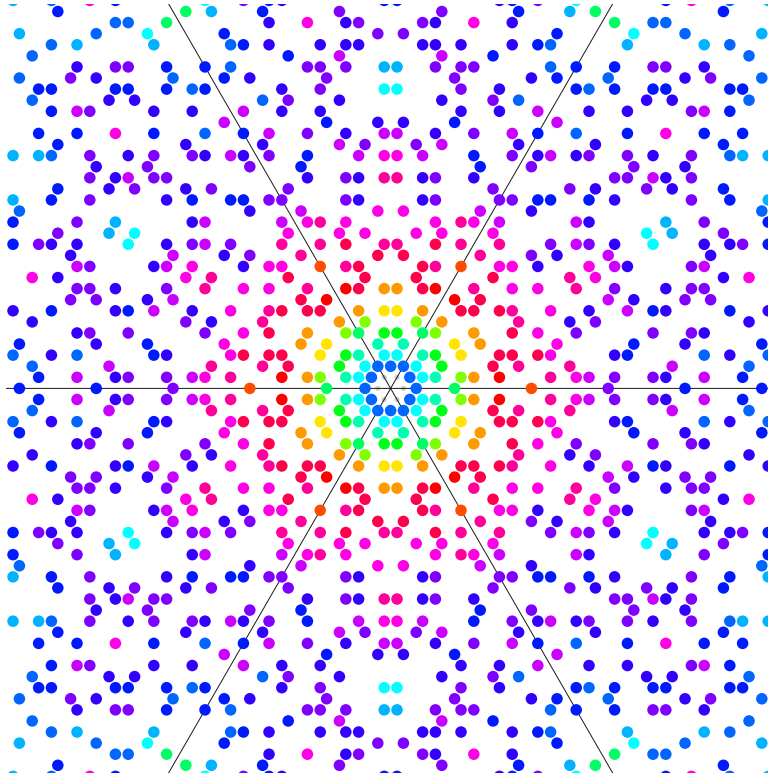


FIGURE 54. The Eisenstein primes.

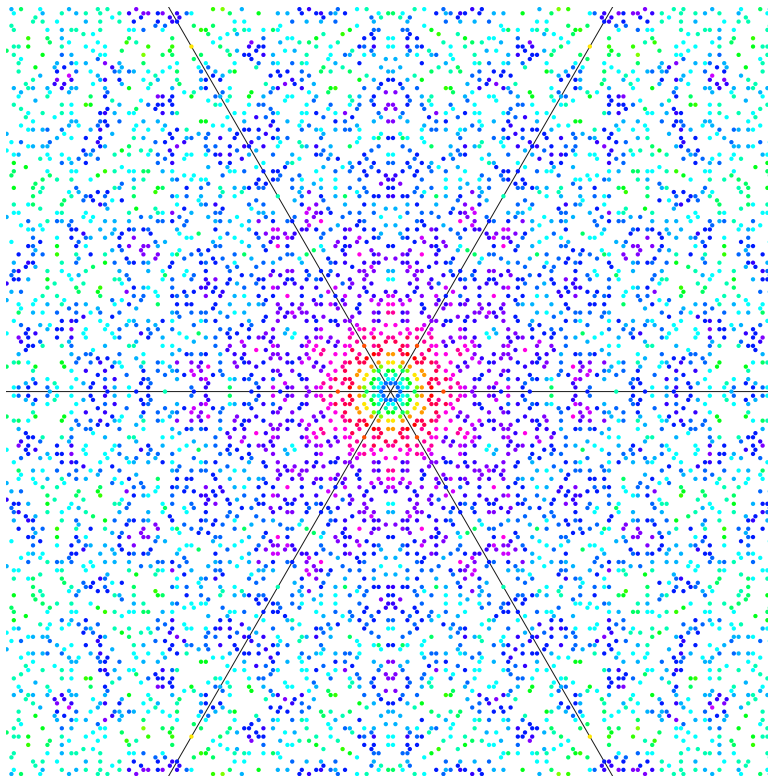


FIGURE 55. A larger view on the Eisenstein primes.

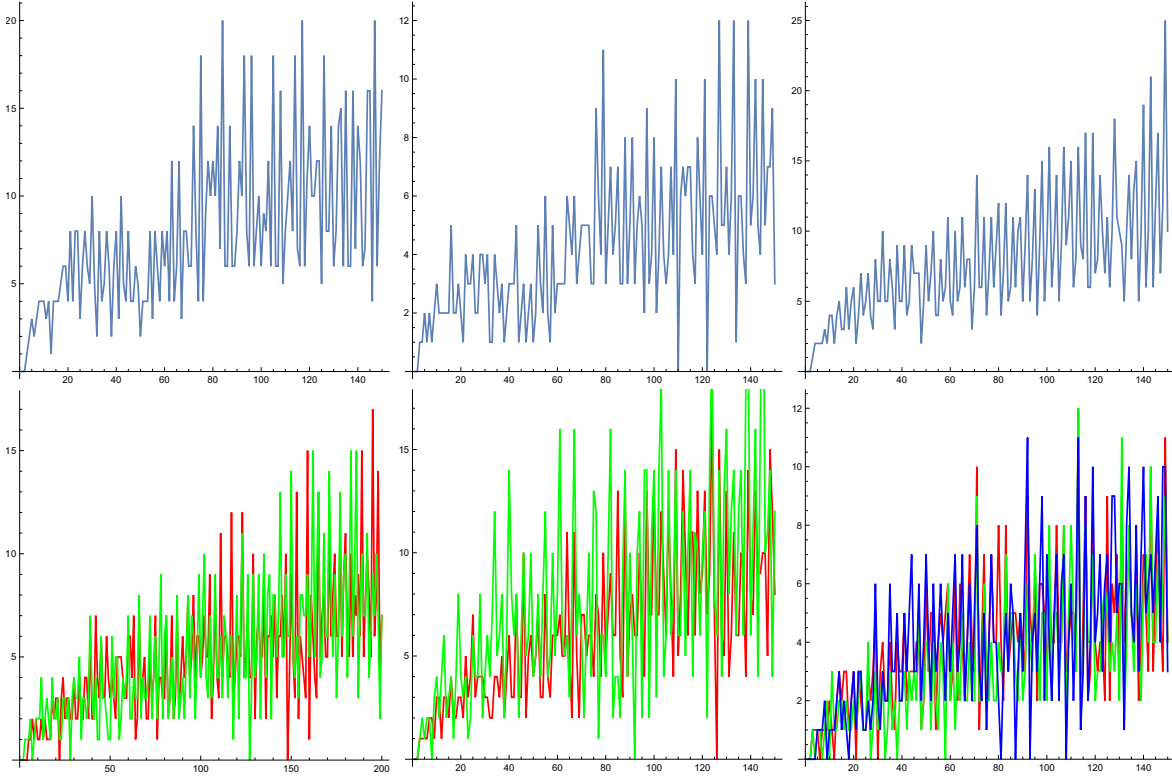


FIGURE 56. The number of times an Eisenstein integer  $a + bw$  can be written as a sum of Eisenstein primes. For  $b = 2$ , we measure that this is always possible: there are rational primes of the form  $p = x + w, q = y + w$  which add up to  $n = a + 2w$ . For  $b = 3$  we see two bad counter examples, the **Eisenstein twins** which are  $3 + 109w$  and  $3 + 121w$ . For  $b = 4$  we appear already fine by writing as a sum  $p = x + w$  and  $q = y + 3w$ . For  $b = 5$  there are gaps again for pairs  $p = x + w, q = y + 4w$  as well as pairs  $p = x + 2w, q = y + 3w$ , but they don't overlap. For  $b = 5$  the decomposition  $p = x + w$  and  $q = y + 4w$  is not possible for  $125 + 5w$  and  $p = x + 3w$  and  $q = y + 3w$  is not possible for  $91 + 5w$  but again they don't intersect.

where  $\mu(z)$  is the **Möbius function** on Eisenstein integers, encoding the parity of the prime factorization. Again, as realized by Dirichlet for quadratic number fields:

**Remark:** (Dirichlet) There is a factorization

$$\zeta_E(s) = 6\zeta(s)\beta_{3,2}(s) .$$

where  $\beta_3(s)$  is a  $L$ -function with character  $\chi_{3,2}(n) = 1$  if  $n \in P_1$  and  $\chi(n) = -1$  for  $n \in P_2$  and  $\chi(3k) = 0$ . The verification is the same.

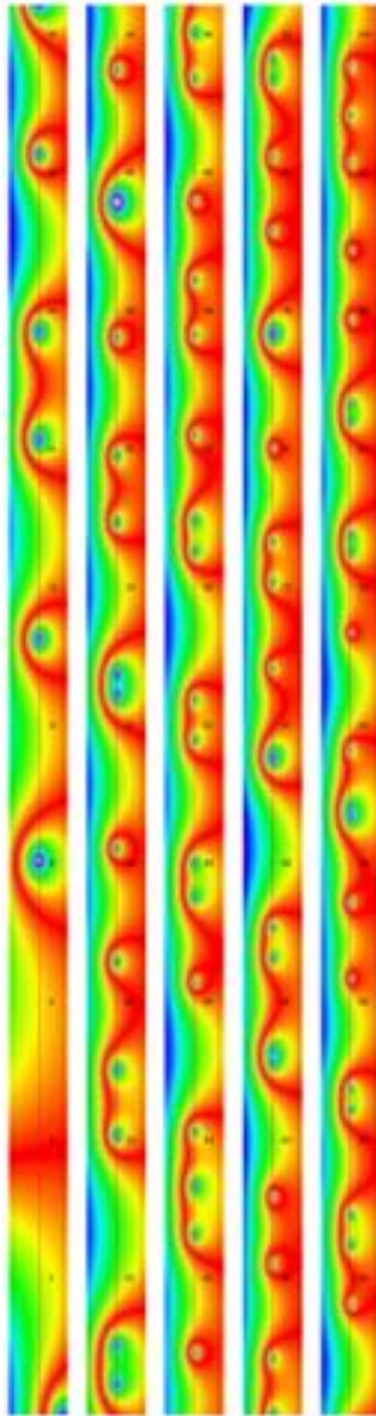


FIGURE 57. A contour plot of the **L-function**  $\zeta_E(z)$  of the Eisenstein primes. Part of the generalized Riemann hypothesis predicts that all roots are on the critical line. Due to the product property, it would imply the classical Riemann hypothesis.



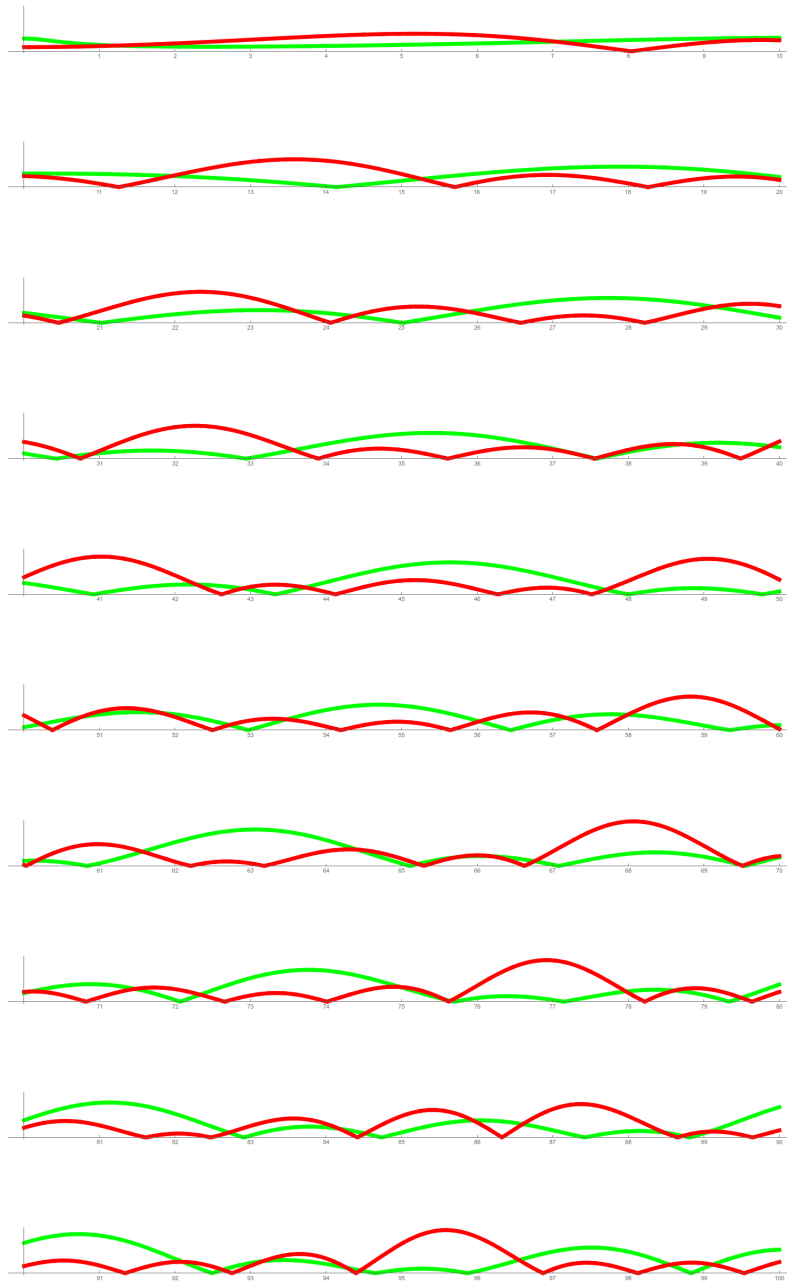


FIGURE 58. The absolute value of the  $\zeta$  function for the rational integers and the zeta function for the Eisenstein integers.

## 17. A PHYSICS ALLEGORY

Since the unit sphere in quaternions is  $SU(2)$  which relates to **Pauli matrices** in various ways, which plays a role in fundamental interactions, there has been early motivation to apply **particle phenomenology** in division algebras. But currently, the standard model is still king and no quaternion nor octonions are needed. In order not to be misunderstood, we don't think the following will change that. But we believe the following allegory illustrates that using an accelerator to smash **building blocks of matter** onto each other and experimenting with the **building blocks of the integers** is closely related. One should see it more as **entertainment**. We firmly believe that any physical theory of value must predict or explain something which can not be done otherwise. However the story which follows, renders some of the number theory in

the complex and in the quaternion case more pictorial:

**Leptons:** we think of Gaussian integers as a collection of **leptons**, where the individual Gaussian primes are indecomposable **Fermions**. A prime  $z$  of type  $4k + 1$  together with an opposite charged particle  $\bar{z}$  form an **electron-positron** pair. We gauge the integers with units  $U = \{1, i, -1, -i\}$  to be in the sector  $\pi/2 < \arg(z) \leq \pi/2$ . The **charge** of a lepton  $z$  is defined as the sign of the argument of  $z$  in the branch  $(-\pi, \pi]$ . Think of the logarithm of the norm  $N(z)$  as **mass**. A rational positive prime  $4k + 3$  is called a **neutrino** as it is lighter: its momentum  $|z|$  is prime while for  $4k + 1$  primes which are electrons or positrons, the energy  $N(z)$  is prime. A neutrino is **neutral** as it is located on the real axes. The largest known prime for example is a Mersenne prime and so a neutrino. An integer  $n = p_1 \dots p_k$  is a **Lepton configuration**. The fact that the Gaussian primes form a **unique factorization domain** translates into the statement that any lepton configuration can be decomposed uniquely into such leptons as well as a neutral mystery particle 2 which is its own anti-particle. The uniqueness holds only modulo **gauge transformations** which act here as multiplications by units. We will see that in the Quaternion case, this fact is no more the case, because that is, where the Hadrons will come in explaining why quarks form Baryons and Mesons. The electron-positron pair is **not bound** together: there is no unit which maps one into the other. Factoring out the symmetry of units renders the factorization unique. The product  $(-3)(-7)$  for example is gauge equivalent to the product  $3 \cdot 7$ . Let us now move from primes to rational integers and call a **rational integer**  $n \in \mathbb{Z}$  a **Boson configuration** if it contains an even number of Fermionic prime factors counted with multiplicity, otherwise it is a Fermion. Mathematically,  $n \in \mathbb{N}$  is a Fermion if and only if its **Jacobi symbol**  $(-1|n) = \left(\frac{-1}{n}\right)$  is  $-1$ . Otherwise, if  $(-1|p) = 1$ , it is a Boson. The Gauss law of **quadratic reciprocity result** tells now that two odd primes  $p, q$  satisfy the **commutation relations**  $(p|q) = (q|p)$  if at least one of them is a Boson and that the **anti-commutation relation**  $(p|q) = -(q|p)$  hold exactly if both primes  $p, q$  are Fermions. In other words, if we look at the Jacobi symbol as an operator  $p \cdot q$ , then Bosons commute with everything else, but the sign changes, if we switch two Fermions. The **two square theorem Fermat** telling that an integer  $n$  can be represented as  $a^2 + b^2$  if and only if  $n$  is a Bosonic integer can be interpreted as the fact that a Bosonic rational prime defines two leptons  $a + ib, a - ib$ , where  $a^2 + b^2 = p$ . The positron and electron are anti-particles of each other, but they are not equivalent since there is no gauge from one to the other. If we factor out the **gauge symmetries** given by the **units**, then the factorization aka particle decomposition of the Lepton set is unique. This is the **fundamental theorem of arithmetic** for Gaussian integers. It can be proven from the rational case using the 1 – 1 identifications on the orbifold  $\mathbb{C}/D_4$  so that we get the rational primes. The **Pauli exclusion principle** is encoded in the form of the **Moebius function**  $\mu_G(n)$  which is equal to 1 if a Gaussian integer  $n$  is the product of an even number of different Gaussian primes, and  $-1$  if it is the product of an odd number of different Gaussian primes and 0, if it contains to identical particles. Again this **particle allegory** is already useful as a **mnemonic** to remember theorems like the two square theorem, or the quadratic reciprocity theorem: "Quadratic Reciprocity means that only Fermion primes anti-commute  $(p|q) = -(q|p)$ . Fermat's two square theorem assures that Bosonic rational primes  $p = a^2 + b^2$  are composed of two Gaussian primes  $a \pm ib$ . The others are all real, light and neutral."

### Hadrons:

Hadrons are quaternion primes  $z$  with norm  $N(z)$  different from 2. The prime 2 is special also in quaternions. As Hurwitz already showed that despite non-commutativity, one can place them outside: every integer quaternion  $z$  is of the form  $z = (1 + i)^r b$  where  $w$  is an odd integer quaternion. There are two symmetry groups acting on hadrons. One is the group  $U$  of units, the other is the group  $V$  generated by coordinate permutations and conjugation. The group  $U$  has 24, the group  $V$  has 48 elements. The groups are no more contained into each other like in the complex case, where  $U = \mathbb{Z}_4$  was a subgroup of  $V = D_4$ . If we look at the  $U$  equivalence classes first and then look at the orbits of  $V$ , we see that some particles are fixed under  $V$  or then that they come in pairs. We will interpret this as **particle anti-particle pairs**. We can however also look at the  $V$  equivalence classes first and then look at the orbits of  $U$ , then we see 1 or 2 or 3 particles combined. This is remarkable. We call a quaternion prime equivalence class with is not a single element a **hadron**. Each hadron to an odd prime  $p$  consists either of two or three **quarks**. Quarks can be either Lipschitz or Hurwitz primes. Each hadron contains a single Lipschitz prime. The **Lagrange four square theorem** assures that there are no neutrino particles among Hadrons. So, we can think of Quaternion primes as **quarks** which form equivalence classes in the form of Baryons or Mesons. The conjugate quaternion is the **anti particle**. This can be seen from the Hurwitz factorization theorem which shows that in such a factorization, one never can have factors  $z\bar{z}$  near each other. The allegory is that they would "annihilate" into a shower of smaller particles. Going from one factorization to an other is a rather complex interaction

# 73

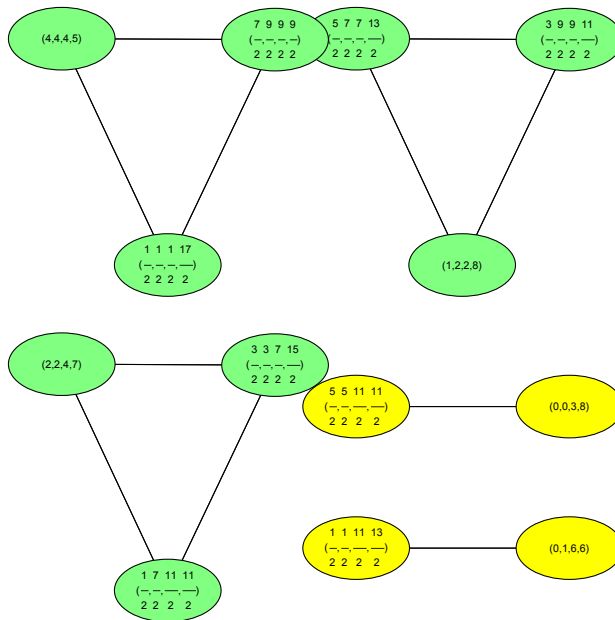


FIGURE 59. The hadrons belonging to the prime 73. There are three Baryons and two Mesons.

# 229

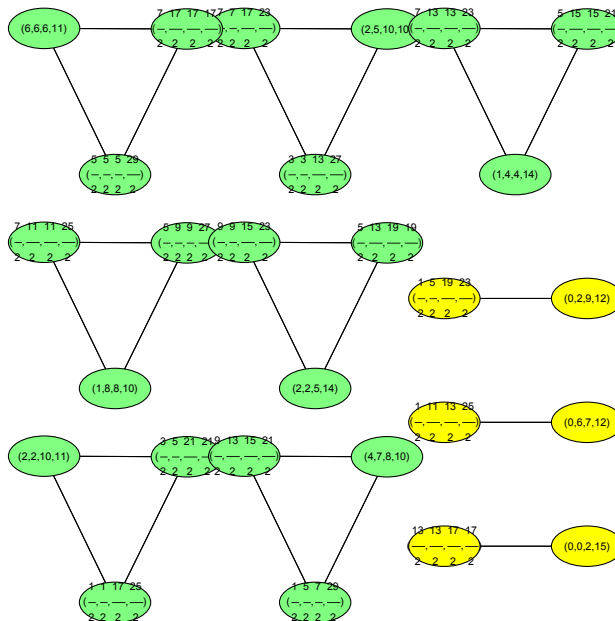


FIGURE 60. The hadrons belonging to the prime 229 features 7 Baryons and 3 mesons.

process changing the nature of some Baryons involving gauge bosons. Baryons are Fermions and Mesons are Bosons. Like in the complex case, we have a mystery  $p = 2$  case, which has only one equivalence class. Lets call it the 2 particle even so we would like to associate it with something real like Higgs because it is neutral, light, its own anti-particle and can give more mass to other particles by multiplying with it. The **group of units** contains 8 particles of the form  $(\pm 1, 0, 0, 0), (0, \pm 1, 0, 0), (0, 0, \pm 1, 0), (0, 0, 0, \pm 1)$  which modulo  $U$  are all equivalent to the neutral  $(1, 0, 0, 0)$ , the  $Z$ -Boson. There are 16 remaining units. Modulo  $V$  they are all equivalent to  $(1, 1, 1, 1)/2$ . This has a positive charge and is the  $W^+$  **boson**. Its conjugate is the  $W^-$  **boson**. Now lets look at a Meson  $(pq)$  containing a Lipschitz prime  $p$  and Hurwitz prime  $q$ . Since  $p$  and  $q$  are equivalent in  $U$ , there exists a permutation, possibly with a conjugation, such that  $\bar{p}$  is gauge equivalent to  $q$ . If a conjugation is involved, then  $p$  and  $q$  have the same sign of charge, otherwise opposite. Because there is a Lipschitz prime involved, it is not possible that all three quarks have the same charge. So, two must have one charge, and one the other. Lets postulate that the **charge** of a Lipschitz quark is  $\pm 2/3$ . Since we have identified modulo  $V$  we can assume that it is positive. In the Meson case, the charge of the other particle is  $1/3$  if it has the same charge and  $-2/3$  if it has opposite charge. In the Baryon case, if the two other quarks have the same charge sign, they have charge  $-1/3$ . If one has the same charge than the Lipschitz one, then both have charge  $2/3$  and the other  $-1/3$ . The charge of an equivalence class is the sum of the charges. We have now assigned a charge in a gauge invariant way: a Lipschitz quark  $(a, b, c, d)$  has charge  $+2/3$  if  $a \leq b \leq c \leq d$  and  $-2/3$  if it is obtained from that by switching two coordinates. The structure of the equivalence classes assures a compatible choice so that the total charge is an integer. In the Meson case, we observe that one of the Lipschitz primes is located on the three or two dimensional coordinate plane. If a coordinate is zero, then we can not perform flipping operations and we have to see whether we have to flip it in the **Gaussian subplane**. In this picture, all Hadrons have charge  $0, -1, 1$ . There are no Hadrons of charge 2. <sup>2</sup>

The non-uniqueness of prime factorization allows us to see going from one to the other factorization as a **particle process**. It involves the gauge bosons. It only becomes only unique modulo **unit migration**. (see [17]). This means that if  $x$  is a Hurwitz integer and  $N(x) = p_1 \dots p_n$  then  $x = (P_1 u_1)(u_1^{-1} P_2 u_2) \dots (u_{n-1}^{-1} P_n)$ , where the  $u_j$  are units and  $P_i$  are Hurwitz primes. In other words, the factorization becomes unique if we look at it on the Meson/Baryon scale but it depends on the order. It becomes unique when including meta-commutation: the prime factorization of a nonzero Hurwitz integer is unique up to meta-commutation, unit migration and recombination, the process of replacing  $P\bar{P}$  with  $Q\bar{Q}$  if  $P, Q$  have the same norm.

Let us look at some examples:

The  $p = 2$  is special  $(1, 1, 0, 0)$  and not included in the above remarks. It is neutral and equivalent to its anti-particle. It is "Higgs like", as in the lepton case and we can not place it yet. Also, we like to think about a situation, where space is actually a dyadic group of integers (a much more natural space as it is compact, features a smallest translation), here scaling by a factor 2 is a symmetry and multiplication by 2 makes the grid finer (making the 2-adic norm small). The mechanism of mass can anyway only be understood when looking at dynamical setups, where particles travel. When looking at wave equations in dyadic groups, the multiplication by 2 plays a special role and it can slow down particles, similar as mass does.

For  $p = 3$ ,  $(0, 1, 1, 1), (1, 1, 1, 3)/2$  form a Meson of charge  $1 = 2/3 + 1/3$ . For  $p = 5$ , we have a Meson  $(0, 0, 1, 2), (1, 1, 3, 3)/2$  of charge 1. For  $p = 13$ , we have a Baryon  $(1, 1, 1, 7)/2, (1, 2, 2, 2), (3, 3, 3, 5)/2$  of charge 0 and a Meson  $(0, 0, 2, 3), (1, 1, 5, 5)/2$  of charge 1. For  $p = 41$  we have a meson  $(0, 3, 4, 4), (3, 3, 5, 11)/2$  of charge  $\pm 1$ . and a meson  $(0, 1, 2, 6), (3, 5, 7, 9)/2$  for which all coordinates are different.

The integer units  $i, j, k$  are the gluons while the  $(1 + i + j + k)/2$  etc are vector bosons. In the complex the  $1, i$  generate photons. The  $(1 + i)$  is the Higgs in all cases. Can not have Lipschitz prime  $(a, a, b, b)$  as this is  $2a^2 + 2b^2 = 2(a^2 + b^2)$  which is  $0, 2$  modulo 4. You see in the figures some pictures. We are able to attach to any prime a collection of Baryons, for which the charges is determined. We believe it would be interesting to study the combinatorics of this setup more.

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<sup>2</sup>Particle physicists mention **beta uuu-hadrons** but seem not detected in experiments. Also strange quark matter consisting of more than 3 quarks have not been observed.

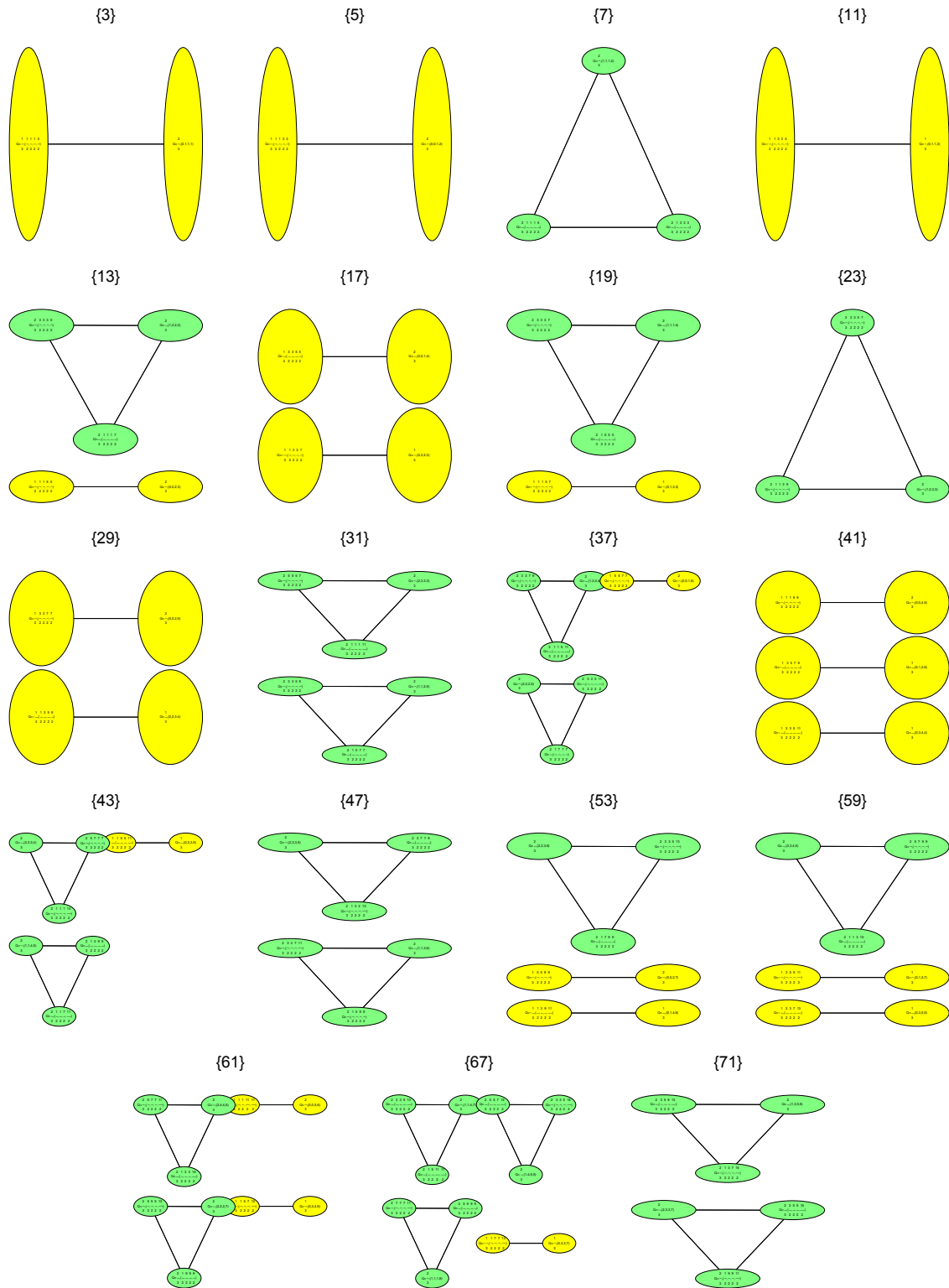


FIGURE 61. Hadrons with charges.

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