

Decycling Number of Linear Graphs of Trees

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Abstract. The decycling number of a graph G is the minimum number of vertices whose removal from G results in an acyclic subgraph. It is known that determining the decycling number of a graph G is equivalent to finding the maximum induced forests of G . The line graphs of trees are the claw-free block graphs. These graphs have been used by Erdős, Saks and Sós to construct graphs with a given number of edges and vertices whose maximum induced tree is very small. In this paper, we give bounds on the decycling number of line graphs of trees and construct extremal trees to show that these bounds are the best possible. We also give bounds on the decycling number of line graph of k -ary trees and determine the exact the decycling number of line graphs of perfect k -ary trees.

Keywords: maximum induced forests, maximum linear forests, line graphs of trees, decycling number.

1 Introduction

Let $G = (V, E)$ be a simple graph, with vertex set V and edge set E . A subset $F \subset V(G)$ is called a *decycling set* if the subgraph $G - F$ is acyclic. The minimum cardinality of a decycling set is called *the decycling number* (or *feedback number*) of G proposed first by Beineke and Vandell [2]. We use the notation $\nabla(G)$ to denote the decycling number of G .

In fact, the problem of determining the decycling number of a graph is *NP*-complete by Karp [10] (also see [6]). The best known approximation algorithm for this problem has approximation ratio 2 [1]. Determining the decycling number is difficult even for some elementary graphs. We refer the reader to an original research paper [2] for some results. Bounds on the decycling numbers have been established for some well-known graphs, such as hypercubes [4], star graphs [12], generalized Petersen graphs [7], distance graphs and circulant graphs [11].

For a graph G , let $f(G)$ be the maximum number of vertices in an induced subgraph of G that is a forest. An induced forest with maximum number of vertices is called a maximum

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induced forest of G . Determining the decycling number of a graph G is equivalent to finding the maximum induced forest of G , since the sum of the two numbers equals the order of G .

One can also study induced trees rather than forests in graphs. Let $t(G)$ be the size of maximum induced trees in G . The problem of bounding $t(G)$ in the connected graph G was first studied by Erdős, Saks and Sós [3] thirty years ago. In their paper, Erdős, Saks and Sós studied the relationship between $t(G)$ and several natural parameters of the graph G . They were able to obtain asymptotically tight bounds on $t(G)$ when either the number of edges or the independent number of G were known. Their result showed that $t(G)$ can be very small over graphs with n vertices and m edges. Given a graph G , its line graph $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G . Erdős, Saks and Sós use line graphs of trees to construct graphs for which $t(G)$ is surprisingly small. Besides, Erdős, Saks and Sós also considered the problem of estimating the size of maximum induced tree in K_r -free graphs. They use line graphs of regular trees to construct K_r -free graphs for which $t(G)$ is small. Recently, Jacob Fox, Po-Shen Loh and Benny Sudakov improved the results on lower bounds of maximum induced trees in K_r -free graphs [5].

A *linear forest* in a graph G is a vertex disjoint union of simple paths of G . A *maximum linear forest* in G is a linear forest in G with maximum number of edges. The number of edges in maximum linear forests of graph G is denoted by $l(G)$. Define the hamiltonian completion number of graph G , denoted by $hc(G)$, to be the minimum number of edges that need to be added to make G hamiltonian. The hamiltonian completion problem was introduced in 1970s by Goodman and Hedetniemi [8,9]. Goodman and Hedetniemi [8] prove the following relation between $l(G)$ and $hc(G)$. For any graph G with n vertices, if $hc(G) > 0$, then $l(G) + hc(G) = n$; if $hc(G) = 0$, then $l(G) = n - 1$.

In this paper, we study the decycling number of line graphs of trees. We show that finding maximum induced forests in line graphs is equivalent to finding maximum linear forests in original graphs. Let T be a tree on n vertices with diameter $d \geq 4$. We give lower and upper bounds on $\nabla(L(T))$ as follows. If d is even, then

$$\left\lceil \frac{n-d-1}{d-1} \right\rceil \leq \nabla(L(T)) \leq n-d-1.$$

If d is odd, then

$$\left\lceil \frac{n-d-2}{d-2} \right\rceil \leq \nabla(L(T)) \leq n-d-1.$$

The extremal line graphs that achieve these bounds are also constructed.

A *k-ary tree* is a rooted tree where within each level every node has either 0 or k children. A *perfect k-ary tree* is a k -ary tree in which all leaf nodes are at the same depth. In this paper, we give bounds on decycling number of line graphs of k -ary trees as follows. Let T be a k -ary tree on n vertices. Then

$$\frac{(k-2)n-k+2}{k} \leq \nabla(L(T)) \leq \frac{(k-1)n-2k+1}{k}.$$

Moreover, we prove that if T is a perfect k -ary tree on n vertices with height h , then

$$\nabla(L(T)) = \frac{(k-1)n-k-(-1)^h}{k+1}.$$

The rest of this paper is organized as follows. In Section 2, we show that finding maximum induced forests in line graphs is equivalent to finding maximum linear forests in original graphs. In Section 3, we give lower and upper bounds on the decycling numbers of line graphs of trees with given diameter. In Section 4, we give lower and upper bounds on the decycling number of line graphs of k -ary trees.

2 Maximum Induced Forests in Line Graphs

In this section, we prove that the maximum induced forests in line graphs correspond to maximum linear forests in original graphs. Denoted $p(G)$ by the length of the longest paths in G .

Lemma 2.1. *A vertex-disjoint path P in G is longest if and only if $L(P)$ is a maximum induced tree in line graph $L(G)$. A linear forest F in G is maximum if and only if $L(F)$ is a maximum induced forest in line graph $L(G)$. Thus, $p(G) = t(L(G))$ and $l(G) = f(L(G))$.*

Proof. It is known that if line graphs are claw-free, then they contain no induced $K_{1,3}$. So do their induced trees and induced forests. It follows that every induced tree of a line graph is an induced path and every induced forest of a line graph is an induced linear forest.

Moreover, we shall show that the line graph of a vertex-disjoint path in G is an induced path in $L(G)$ and the induced path in $L(G)$ is also a line graph of a vertex-disjoint path in G . If $P = v_1e_1v_2e_2 \dots v_l e_l v_{l+1}$ is a vertex-disjoint path in G , in which v_i 's are vertices and e_j 's are edges of G . Then, we shall show $L(P) = (e_1, e_2, \dots, e_l)$ is an induced path in $L(G)$. Otherwise, assume that (e_j, e_k) forms an edge in $L(G)$ and $k > j + 1$. Then e_j and e_k share a common ending point in G . We have $\{v_j, v_{j+1}\} \cap \{v_k, v_{k+1}\} \neq \emptyset$, which contradicts with path P is vertex-disjoint. Conversely, if H is an induced path in line graph $L(G)$. Let e_1, e_2, \dots, e_l are l consecutive vertices in H . Clearly, $P = (e_1, e_2, \dots, e_l)$ is a path in G and $H = L(P)$.

Thus, a vertex-disjoint path P in G is longest if and only if $L(P)$ is a maximum induced tree in $L(G)$ and a linear forest F in G is maximum if and only if $L(F)$ is a maximum induced forest in $L(G)$. ■

Clearly, linear forests of G have at most $n - 1$ edges. It implies that $f(L(G)) \leq n - 1$. Therefore, we have the following corollary.

Corollary 2.1. *For any graph G with n vertices and m edges, $\nabla(L(G)) \geq m - n + 1$.*

3 The Decycling Number of Line Graphs of Trees

Let T be a tree with n vertices. An *inner vertex* is a vertex of degree at least two. Similarly, an *outer vertex* (or a *leaf*) is a vertex of degree one. Then the vertices of T can be partitioned into the set of leaves V_{out} and the set of inner vertices V_{in} . The cardinality of V_{out} is denoted by $out(T)$. For any inner vertex v , let $ex(v)$ be zero if v has at most two neighbors of degree less than three; let $ex(v)$ be $k - 2$ if v has k neighbors of degree less than three. Then we have the following lemma.

Lemma 3.1. For any tree T with n vertices,

$$\left\lceil \frac{out(T) + \sum_{v \in V_{in}} ex(v)}{2} \right\rceil \leq hc(T) \leq out(T) - 1.$$

Proof. Since T is a tree. By adding $hc(T)$ edges on T , we get a hamiltonian graph G . Since G has a hamiltonian cycle C , each leaf of T is incident to a new edge in C . For any inner vertex v , if $ex(v)$ is greater than zero, then at most two edges that incident to v are in C . It means that at least $ex(v)$ neighbors of v have degree less than or equal to two and do not adjacent to v in C . Each of these neighbors has to be incidence to a new edge in C . Thus, at least $out(T) + \sum_{v \in V_{in}} ex(v)$ vertices are incidence to new edges. Therefore, we have

$$hc(T) \geq \left\lceil \frac{out(T) + \sum_{v \in V_{in}} ex(v)}{2} \right\rceil.$$

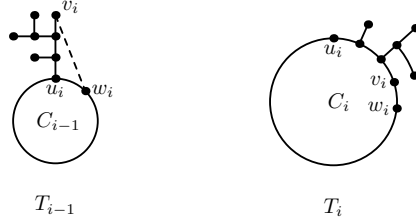


Figure 1: An example from tree T_{i-1} to T_i .

On the other hand, we can get a hamiltonian cycle by adding edges to T according to the following procedure. Firstly, we choose two leaves u, v of T and add an edge between them. Let G_0 be the graph $T + uv$. Then G_0 contains an unique cycle C_0 formed by new edge uv and the unique path $P_{u,v}$ in T . Let T_0 be the graph obtained from G_0 by contracting cycle C_0 , or $T_0 = G_0 \cdot C_0$. Denoted by C_0 the contracted vertex. It is easy to see that T_0 is a tree with $out(T) - 1$. Now choose a leaf v_1 outside C_0 in T_0 . Let $P_{C_0 v_1}$ be the unique path between C_0 and v_1 . Let u_1 be the vertex in the cycle C_0 that has the smallest distance to v_1 . Let w_1 be a neighbor of u_1 in C_0 . Then by adding edges $v_1 w_1$, we get a larger cycle $C_1 = C_0 - u_1 w_1 + P_{u_1 v_1} + v_1 w_1$. Now let T_1 be the graph obtained from T by contracting C_1 . Then T_1 is a tree with $out(T) - 2$ leaves. Now choose a leaf v_2 outside C_1 from T_1 . Let $P_{C_1 v_2}$ be the unique path between C_1 and v_2 . Let u_2 be the vertex in the cycle C_1 that has the smallest distance to v_2 . Let w_2 be a neighbor of u_2 in C_1 . Then by adding edge $v_2 w_2$, we get a larger cycle $C_2 = C_1 - u_2 w_2 + P_{u_2 v_2} + v_2 w_2$ in T . Do this procedure repeatedly, through each step we can get a tree T_i from tree T_{i-1} with leaves less than 1 (see Fig.1), the procedure has to be stopped when the contracted tree has only one vertex. Then we get a hamiltonian cycle by adding $out(T) - 1$ edges in T . Thus, $hc(T) \leq out(T) - 1$. ■

Since any tree on n vertices have $n - 1$ edges. Then $f(L(T)) + \nabla(L(T)) = n - 1$. By Lemma 2.1, we know that $l(T) = f(L(T))$. Moreover, it is true that $l(T) + hc(T) = n$. Therefore, we have the following corollary.

Corollary 3.1. For any tree T on n vertices,

$$\left\lceil \frac{out(T) + \sum_{v \in V_{in}} ex(v)}{2} \right\rceil - 1 \leq \nabla(L(T)) \leq out(T) - 2.$$

Now we introduce an operation on leaves of trees that does not decrease $l(T)$. For any two leaves u_i, u_j of T , suppose their neighbors are w_i, w_j . We define Leaf-Exchange operation on T as removing edge $w_i u_i$ from T and adding edge $u_j u_i$, the obtained tree is denoted by $T[u_i \rightarrow u_j]$.

Lemma 3.2. *For any two leaves u_i, u_j of T , $l(T[u_i \rightarrow u_j]) \geq l(T)$.*

Proof. Suppose F is a maximum linear forest in T . Then $F - w_i u_i + u_j u_i$ is a linear forest in $T[u_i \rightarrow u_j]$. Thus, we have $l(T) \leq l(T[u_i \rightarrow u_j])$. \blacksquare

Let T be a tree on n vertices. The center of a tree is the set of vertices, from which the greatest distance equals to its radius. Let v^* be one of the center of tree T . Then T can be viewed as a rooted tree with root v^* . Moreover, we can partition $V(T)$ into sets $V_0(T), V_1(T), \dots, V_r(T)$, where $V_i(T) = \{w | d(v^*, w) = i\}$ and r is the radius of T . In case of no confusion, $V_i(T)$ is often abbreviated as V_i . The vertex in V_i is called the vertex at depth i . Let $V_{\geq 2} = V_2 \cup V_3 \dots \cup V_r$. Let $d(T)$ be the diameter of T and $r(T)$ be the radius of T . Let $s(T)$ be the number of degree-two vertices in $V_1(T)$. Then, we define three family of rooted trees on n vertices with diameter at most d as follows.

$$\left\{ \begin{array}{l} \mathcal{T}_1(n, d) = \{T: |V(T)| = n, d(T) \leq d, r(T) \leq \lceil \frac{d}{2} \rceil \\ \quad \text{and } deg(v) \leq 2 \text{ for } v \in V_{\geq 2}, deg(v) \leq 3 \text{ for } v \in V_1\}, \\ \mathcal{T}_2(n, d) = \{T: s(T) \leq 3, T \in \mathcal{T}_1(n, r)\}, \\ \mathcal{T}_3(n, d) = \{T: 2 \leq s(T) \leq 3, T \in \mathcal{T}_1(n, r)\}. \end{array} \right.$$

By the following three lemmas, we shall show that finding the upper bounds for $l(T)$ on all trees is equivalence to finding that on $\mathcal{T}_3(n, r)$.

Lemma 3.3. *For any tree T on n vertices with diameter d , there exists a tree T' in $\mathcal{T}_1(n, d)$ such that $l(T) \leq l(T')$.*

Proof. Any tree can be viewed as a rooted tree with its center as the root. Suppose to the contrary, there exist trees that we cannot find trees with larger maximum linear forest in $\mathcal{T}_1(n, d)$. Let T be a counterexample with $|V_1|$ maximum. Clearly, T is not in $\mathcal{T}_1(n, d)$. Then, T has a vertex v in V_1 such that $deg(v) \geq 4$ or T has a vertex v in $V_{\geq 2}$ such that $deg(v) \geq 3$. We split the proof into two cases as follows.

Case 1. T has a vertex v in V_1 such that $deg(v) \geq 4$. Assume that $deg(v) = t + 1$ and $t \geq 3$. Then v has one neighbor v^* and t neighbors in V_2 . Let v_1, v_2, \dots, v_t be these t neighbors in V_2 and T_1, T_2, \dots, T_t be subtrees of v with root v_1, v_2, \dots, v_t . Let F be a maximum linear forest of T . Then at most two edges of $v^*v, vv_1, vv_2, \dots, vv_t$ are in F . Since $t \geq 3$, there exists one of vv_1, vv_2, \dots, vv_t that is not in F . Without loss of generality, we assume vv_t is not in F . Then by removing edge vv_t from T and adding edge v^*v_t , we get a new tree \bar{T} with $d(\bar{T}) \leq d(T)$. Clearly, we have $l(T) \leq l(\bar{T})$ since F is also a linear forest of \bar{T} . Moreover, $V_1(\bar{T})$ has more vertices than $V_1(T)$. Since T is the counterexample with $|V_1|$ maximum, we know that \bar{T} is no longer a counterexample. Therefore, there exists a tree T' in $\mathcal{T}_1(n, d)$ such that $l(\bar{T}) \leq l(T')$. Then $l(T) \leq l(\bar{T}) \leq l(T')$, which contradicts with that T is a counterexample.

Case 2. T has a vertex v in $V_k (k \geq 2)$ such that $\deg(v) \geq 3$. If $\deg(v) \geq 4$, then we can get a contradiction by the same argument as in Case 1. Thus, we only need to consider the case $\deg(v) = 3$. Then v has one neighbor w in V_{k-1} and has two neighbors v_1 and v_2 in V_{k+1} . T_1, T_2 be subtrees of v with root v_1, v_2 . Let F be a maximum linear forest of T . Then at most two edges of wv, vv_1, vv_2 are in F . If wv is not in F , by removing edge wv from T and adding edge v^*v , we get a new tree \bar{T} with $d(\bar{T}) \leq d(T)$. We have $l(T) \leq l(\bar{T})$ since F is also a linear forest in \bar{T} . Since $V_1(\bar{T})$ is increased by one, \bar{T} is no longer a counterexample. Therefore, there exists a tree T' in $\mathcal{T}_1(n, d)$ such that $l(\bar{T}) \leq l(T')$. We get a contradiction. If one of vv_1 and vv_2 is not in F , without loss of generality, we assume vv_2 is not in F . Then by removing edge vv_2 from T and adding edge v^*v_2 , we get a new tree \bar{T} , which also leads to a contradiction.

Therefore, the claim holds. ■

Lemma 3.4. *For any tree T in $\mathcal{T}_1(n, d)$, there exists a tree T' in $\mathcal{T}_2(n, d)$ such that $l(T) \leq l(T')$.*

Proof. Suppose to the contrary, there exist counterexamples. Let T be the one in $\mathcal{T}_1(n, d)$ with $s(T)$ minimum. Then v^* has at least four degree-two neighbors. Assume they are v_1, v_2, \dots, v_s . Then by definition of $\mathcal{T}_1(n, d)$, it is easy to see that subtrees with roots v_1, v_2, \dots, v_s are all paths. Let P_1, P_2, \dots, P_s be these paths. Clearly, v_i is one endpoint of P_i . Let F be a maximum linear forest in T . Then at least two of edges $v^*v_1, v^*v_2, \dots, v^*v_s$ are not in F . Without loss of generality, we suppose v^*v_1, v^*v_2 are not in F . Then all edges in P_1, P_2 are in F .

If one of paths P_1, P_2 has length at least two. Without loss of generality, we suppose P_1 has length at least 2. Clearly, v_1 is one endpoint of P_1 . Let u_1 be the other endpoint of P_1 and w_1 be the parent of u_1 in the rooted tree T . Then by removing edge w_1u_1 and adding edge v_1u_1 , we get a new tree \bar{T} . We have $l(T) \leq l(\bar{T})$ since $F - w_1u_1 + v_1u_1$ is a linear forest of \bar{T} . Since $s(\bar{T}) = s(T) - 1$, there exists a tree T' in $\mathcal{T}_2(n, d)$ such that $l(\bar{T}) \leq l(T')$. It leads to a contradiction. If P_1, P_2 all have length one. Assume that P_1 is an edge v_1u_1 and P_2 is an edge v_2u_2 . Then by removing edge v_1u_1 and adding edge v_2u_1 , we get a new tree \bar{T} with $l(T) \leq l(\bar{T})$, which also leads to a contradiction. Thus, the claim holds. ■

Lemma 3.5. *For any tree T in $\mathcal{T}_2(n, d)$ with $n \geq d$ and $d \geq 4$, there exists a tree T' in $\mathcal{T}_3(n, d)$ such that $l(T) \leq l(T')$.*

Proof. Suppose T is a counterexample in $\mathcal{T}_2(n, d)$ with $s(T)$ maximum. Clearly, $s(T) \leq 1$. Let F be a maximum linear forest in T .

Firstly, we claim T has no leaves in V_1 . Otherwise, assume v_0 is a leaf in V_1 . If there is a degree-3 vertex in V_1 , say w_0 . Let u_0 be a leaf of the subtree with root w_0 . Then by Leaf-Exchange operation on T , we get a new tree $T[u_0 \rightarrow v_0]$ with $s(T[u_0 \rightarrow v_0]) \geq s(T) + 1$ and $l(T[u_0 \rightarrow v_0]) \geq l(T)$. Since T is a counterexample with $s(T)$ maximum. Then there exists a tree T' in $\mathcal{T}_3(n, d)$ such that $l(T[u_0 \rightarrow v_0]) \leq l(T')$. It follows that $l(T) \leq l(T')$, a contradiction. If there is no degree-3 vertex in V_1 . Then all vertices in V_1 have degree at most two. Since $s(T) \leq 1$ and $n \geq d$, there are at least two leaves in V_1 , say v_1 and v_2 . Then by Leaf-Exchange operation on T , we get a new tree $T[v_1 \rightarrow v_2]$ with $s(T[v_1 \rightarrow v_2]) = s(T) + 1$. Since $d \geq 4$, the Leaf-Exchange operation cannot decrease the diameter. Then there exists a tree

T' in $\mathcal{T}_3(n, d)$ such that $l(T[v_1 \rightarrow v_2]) \leq l(T')$. It follows that $l(T) \leq l(T[v_1 \rightarrow v_2]) \leq l(T')$, a contradiction. Thus, T has no leaves in V_1 .

Now, we know that there is at most one degree-2 vertex and no leaf in V_1 . Since $n \geq d \geq 2r(T)$, the number of degree-3 vertices in V_1 has to be greater than one. Let v_1, v_2, \dots, v_s be degree-3 vertices in V_1 . If at least one of edges $v^*v_1, v^*v_2, \dots, v^*v_s$ is in F . Without loss of generality, we assume that v^*v_1 is in F . Let u_1, u_2 be two neighbors of v_1 in V_2 . Then at least one of edges v_1u_1 and v_1u_2 is not in F . Suppose v_1u_1 is not in F . Then by removing edge v_1u_1 and adding edge v^*u_1 , we get a new tree \bar{T} with $l(T) \leq l(\bar{T})$. Clearly, $s(\bar{T}) \geq s(T) + 1$ and \bar{T} is in $\mathcal{T}_2(n, r)$. Then there exists a tree T' in $\mathcal{T}_3(n, d)$ such that $l(\bar{T}) \leq l(T')$. It follows that $l(T) \leq l(\bar{T}) \leq l(T')$, a contradiction. If none of $v^*v_1, v^*v_2, \dots, v^*v_s$ is in F . Then at most one edge incident to v^* is in F , and edges in each subtree with root v_i are all in F . Suppose v_1 has two neighbors u_1, u_2 in V_2 . Then by removing edge v_1u_1 and adding edge v^*u_1 , we get a new tree \bar{T} . Then $F - v_1u_1 + v^*u_1$ is a linear forest in \bar{T} . It follows that $l(T) \leq l(\bar{T})$ and $s(\bar{T}) \geq s(T) + 1$. Then there exists a tree T' in $\mathcal{T}_3(n, d)$ such that $l(\bar{T}) \leq l(T')$. It follows that $l(T) \leq l(\bar{T}) \leq l(T')$, a contradiction.

Combining all the cases, we complete the proof. ■

Theorem 3.1. *For any tree T on n vertices with diameter $d \geq 4$, we have*

$$\begin{cases} d \leq l(T) \leq \left\lfloor \frac{(d-2)n+2}{d-1} \right\rfloor, & \text{for } d \text{ is even;} \\ d \leq l(T) \leq \left\lfloor \frac{(d-3)n+4}{d-2} \right\rfloor, & \text{for } d \text{ is odd.} \end{cases}$$

Proof. For the lower bounds, clearly we have $l(T) = f(L(T)) \geq t(L(T)) = d(H)$. Moreover, the extremal trees that achieve these lower bounds are shown in Fig.2.

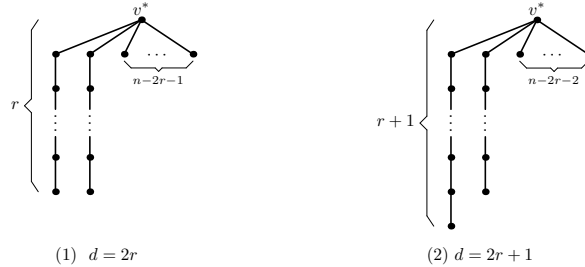


Figure 2: Extremal trees that achieve lower bounds.

Let v^* be the center of tree T . Then T can be viewed as a rooted tree with root v^* and radius r . Clearly, $n \geq d$. Then by Lemma 3.3, 3.4 and 3.5, we know that there exists a tree $T' \in \mathcal{T}_3(n, d)$ such that $l(T) \leq l(T')$. Therefore, we only need to consider the upper bounds on $l(T')$ for $T' \in \mathcal{T}_3(n, d)$. Now we split the proof into two cases by the parity of d .

Case 1. $d = 2r$. Let T be a tree in $\mathcal{T}_3(n, d)$. It is easy to see that T has radius at most r . Since $s(T) \geq 2$. Then, at least two vertices in $V_1(T)$ have degree two, say u and v . Then the subtree with root u and the subtree with root v are two paths. We call two leaves in these two subtrees critical leaves of T and call all the other leaves non-critical leaves. Let $T^*(n)$ be the tree in $\mathcal{T}_3(n, d)$ satisfying the following two properties as shown in Figure 3:

- (1) Two critical leaves of $T^*(n)$ are all at depth r ;

(2) All but at most one of its leaves are at depth r . If the only leaf with depth less than r lie in a subtree, whose root is a degree-2 vertices in $V_1(T^*(n))$, then $s(T^*(n)) = 3$. If the only leaf with depth less than r lie in a subtree, whose root is a degree-3 vertices in $V_1(T^*(n))$, then $s(T^*(n)) = 2$.

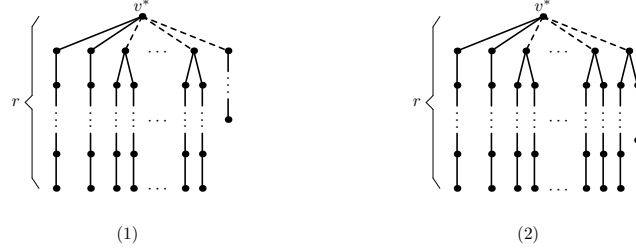


Figure 3: Extremal graphs $T^*(n)$ that achieve the upper bounds.

We claim for any tree T in $\mathcal{T}_3(n, d)$, $l(T) \leq l(T^*(n))$. Let v_1, v_2 be degree-2 vertices in V_1 and v_3, \dots, v_k be degree-3 vertices in V_1 . If $s(T) = 3$, let v_{k+1} be the third degree-2 vertex in V_1 . We arrange the subtrees of T with roots $v_1, v_2, v_3, \dots, v_k, v_{k+1}$ from left to right in the plane. Then do Leaf-Exchange operation from a rightmost leaf to a leftmost leaf with depth less than r convectively. Finally, we shall arrive the tree $T^*(n)$. Since Leaf-Exchange operation from leaf to leaf can never decrease the value of $l(T)$. It follows that $l(T) \leq l(T^*(n))$.

Let m be the remainder of dividing $n - 2r - 1$ by $2r - 1$. We splits the proof into two parts by the value of m .

Case 1.1. If $1 \leq m \leq r$, then $s(T^*(n)) = 3$ as shown in Fig.3 (1). It is easy to see that there are three vertices in $V_1(T^*(b))$. It follows that $ex(v^*) = 1$. Moreover, the number of leaves in $T^*(n)$ can be computed as follows.

$$\begin{aligned} out(T^*(n)) &= 2 \left\lfloor \frac{n - 2r - 1}{2r - 1} \right\rfloor + 3 \\ &= 2 \left\lfloor \frac{n - 2}{2r - 1} \right\rfloor + 1 \end{aligned}$$

By Lemma 3.1, we have

$$\begin{aligned} l(T^*(n)) &\leq n - hc(T^*(n)) \\ &\leq n - \left\lfloor \frac{out(T^*) + \sum_{v \in V_{in}} ex(v)}{2} \right\rfloor \\ &= n - \left\lfloor \frac{n - 2}{2r - 1} \right\rfloor - 1 \\ &= \left\lfloor \frac{(2r - 2)n + 2}{2r - 1} \right\rfloor - 1 = \left\lfloor \frac{(2r - 2)n + 2}{2r - 1} \right\rfloor. \end{aligned}$$

However, by removing the dashed edges as shown in Fig.3 (1), we get a linear forest with $n - 1 - \left\lfloor \frac{n - 2r - 1}{2r - 1} \right\rfloor = \left\lfloor \frac{(2r - 2)n + 2}{2r - 1} \right\rfloor$ edges. Thus, $l(T^*(n)) = \left\lfloor \frac{(2r - 2)n + 2}{2r - 1} \right\rfloor$.

Case 1.2. If m is 0 or between $d + 1$ and $2d - 2$, then $T^*(n)$ is as shown in Fig.3 (2). For the second case, we have $ex(v^*) = 0$ and $out(T^*(n)) = 2 \left\lceil \frac{n-2r-1}{2r-1} \right\rceil + 2 = 2 \left\lceil \frac{n-2}{2r-1} \right\rceil$. By Lemma 3.1, we have

$$\begin{aligned} l(T^*(n)) &\leq n - hc(T^*(n)) \\ &\leq n - \left\lceil \frac{out(T^*(n)) + \sum_{v \in V_{in}} ex(v)}{2} \right\rceil \\ &= \left\lfloor \frac{(2r-2)n+2}{2r-1} \right\rfloor. \end{aligned}$$

However, by removing the dashed edge as shown in Fig.3 (2), we get a linear forest with $n - 1 - \left\lceil \frac{n-2r-1}{2r-1} \right\rceil = \left\lfloor \frac{(2r-2)n+2}{2r-1} \right\rfloor$ edges. Thus, $l(T^*(n)) = \left\lfloor \frac{(2r-2)n+2}{2r-1} \right\rfloor$.

Combining the two subcases, we prove that for any tree T with n vertices and diameter $2r$ ($r \geq 2$),

$$l(T) \leq l(T^*) \leq \left\lfloor \frac{(2r-2)n+2}{2r-1} \right\rfloor.$$

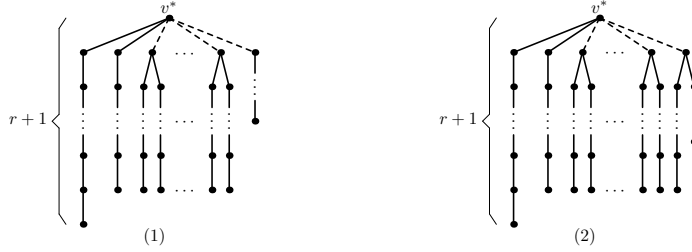


Figure 4: Extremal trees $T_1^*(n)$ that achieve the upper bounds.

Case 2. $d = 2r + 1$. Let T be a tree in $\mathcal{T}_3(n, d)$. Since $d(T) \leq d$ and $r(T) \leq r + 1$. Then if there are leaves in V_{r+1} , these leaves are all in the same subtree of v^* . Thus, we have $|V_{r+1}| = 1$ or $|V_{r+1}| = 2$.

Case 2.1. $|V_{r+1}| = 1$. Let u be the only leaf in V_{r+1} and w be its neighbor. By remove this vertex, we get a tree T' with diameter $2r$. Suppose F is a maximum linear forest in T . Then $F - uw$ is a linear forest in T' . Conversely, if F is a maximum linear forest in T' then $F + uw$ is a linear forest in T . It follows that $l(T) = l(T') + 1$. Therefore, by adding one leaf to $T^*(n-1)$, we obtain a new tree $T_1^*(n)$ with n vertices and diameter $2r + 1$ as shown in Fig.4. And it is easy to see that for any T in $\mathcal{T}_3(n, d)$, $l(T) \leq l(T_1^*(n))$. Thus,

$$\begin{aligned} l(T) &\leq l(T_1^*(n)) \\ &= l(T^*(n-1)) + 1 \\ &= \left\lfloor \frac{(2r-2)n+3}{2r-1} \right\rfloor. \end{aligned}$$

Case 2.2. $|V_{r+1}| = 2$. Let u_1 and u_2 be these two vertices and w_1 and w_2 be their neighbors, respectively. Clearly, $n \geq 4r + 2$. By remove these two vertices, we get a tree T'

with diameter $2r$. Clearly, we have $l(T) = l(T') + 2$. Thus, by adding two leaves to $T^*(n-1)$, we obtain a tree $T_2^*(n)$ with n vertices and diameter $2r+1$ as shown in Fig.5. And any tree T in $\mathcal{T}_3(n, d)$ with $|V_{r+1}| = 2$ has $l(T) \leq l(T_2^*(n))$. Thus,

$$\begin{aligned} l(T) &\leq l(T_2^*(n)) \\ &= l(T^*(n-2)) + 2 \\ &= \left\lfloor \frac{(2r-2)n+4}{2r-1} \right\rfloor. \end{aligned}$$

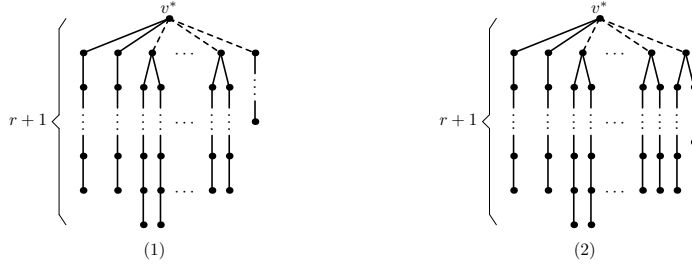


Figure 5: Extremal trees $T_2^*(n)$ that achieve the upper bounds.

Since $l(T_1^*) \leq l(T_2^*)$, then we have $l(T) \leq l(T_2^*(n)) = n \left\lfloor \frac{(2r-2)n+4}{2r-1} \right\rfloor$ if $n \geq 4r+2$ and $T_2^*(n)$ is the extremal tree that achieves the upper bound. $l(T) \leq l(T_1^*(n)) = \left\lfloor \frac{(2r-2)n+3}{2r-1} \right\rfloor$ if $n \leq 4r+1$ and $T_1^*(n)$ is the extremal graph that achieves the upper bound. ■

Let T be a tree on n vertices. Then $L(T)$ has $n-1$ edges. Since $f(L(T)) = l(T)$ and $f(L(T)) + \nabla(L(T)) = n-1$. Then, we have the following corollary.

Corollary 3.2. *Let T be a tree on n vertices with diameter $d \geq 4$, then*

$$\begin{cases} \left\lfloor \frac{n-d-1}{d-1} \right\rfloor \leq \nabla(L(T)) \leq n-d-1, & \text{for } d \text{ is even;} \\ \left\lfloor \frac{n-d-2}{d-2} \right\rfloor - 1 \leq \nabla(L(T)) \leq n-d-1, & \text{for } d \text{ is odd.} \end{cases}$$

Theorem 3.2. *For any connected graph G with n vertices and m edges, if the length of the longest path in G is p and $p \geq 4$, then we have*

$$\begin{cases} m - \left\lfloor \frac{(p-2)n+2}{p-1} \right\rfloor \leq \nabla(L(G)) \leq m-p, & \text{if } p \text{ is even;} \\ m - \left\lfloor \frac{(p-3)n+4}{p-2} \right\rfloor \leq \nabla(L(G)) \leq m-p, & \text{if } p \text{ is odd.} \end{cases}$$

Proof. According to Lemma 2.1, we have that $t(L(G)) = p(G) = p$ and $f(L(G)) = l(G)$. Then it is clear that $f(L(G)) \geq t(L(G)) = p$. Since any linear forest in G can be extended to a spanning tree of G , then there exists a spanning tree T of G such that $l(G) = l(T)$. Moreover, the maximum linear forest in any spanning tree of G is also a linear forest of G . It implies that for any spanning tree T of G , $l(T) \leq l(G)$. Consequently, let \mathcal{T} be the set of all spanning trees of G , then $l(G) = \max_{T \in \mathcal{T}} l(T)$. It is easy to see that the diameter of each

spanning tree of G is less than or equal to s . Moreover, upper bounds on $l(T)$ in Theorem 3.1 are all increasing functions of diameters. Thus,

$$\begin{cases} p \leq l(G) \leq \left\lfloor \frac{(p-2)n+2}{s-1} \right\rfloor, & \text{for } p \text{ is even;} \\ p \leq l(G) \leq \left\lfloor \frac{(p-3)n+4}{s-2} \right\rfloor, & \text{for } p \text{ is odd.} \end{cases}$$

Thus, the theorem follows. ■

4 The Decycling Number of Line Graphs of k -ary Trees

A k -ary tree is a rooted tree where within each level every node has either 0 or k children. The maximum degree of a k -ary tree is $k + 1$. It follows that line graphs of k -ary trees are K_{k+2} -free graphs. Since it is often interesting to consider the decycling number of K_{k+2} -free graphs. Thus, we consider the decycling number of line graphs of k -ary trees in this section.

Before that, we give a dynamic programming algorithm to find a maximum linear forest in rooted tree. Let T be a rooted tree with root v^* . Let T_1, T_2, \dots, T_t be subtrees of v^* with root v_1, v_2, \dots, v_t , respectively. Let $F(T)$ be the edge set of a maximum linear forest in T . Let $F'(T)$ be the edge set of a largest linear forest of T such that the degree of v^* is at most one.

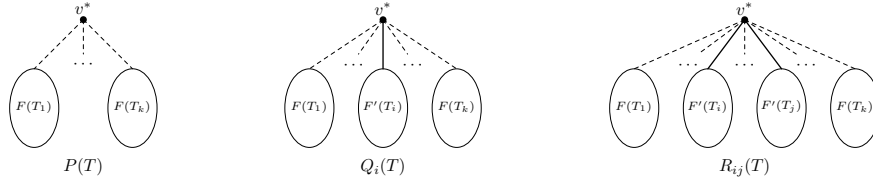


Figure 6: The structures of linear forests $P(T)$, $Q_i(T)$ and $R_{ij}(T)$.

Then, we define three kinds of linear forests $P(T)$, $Q_i(T)$ and $R_{ij}(T)$ as follows.

$$\begin{cases} P(T) = \cup_{i=1}^k F(T_i), \\ Q_i(T) = (\cup_{k \neq i} F(T_k)) \cup F'(T_i) \cup \{(v^*, v_i)\}, \\ R_{ij}(T) = (\cup_{k \neq i, j} F(T_k)) \cup F'(T_i) \cup F'(T_j) \cup \{(v^*, v_i), (v^*, v_j)\}. \end{cases}$$

As shown in Fig.6, $P(T)$ is a linear forest in T such that v^* has degree zero; $Q_i(T)$ is a linear forest such that v^* has degree one and (v^*, v_i) is an edge in the linear forest; $R_{ij}(T)$ is a linear forest such that v^* has degree two and (v^*, v_i) , (v^*, v_j) are edges in the linear forest. Let $S(T)$ be the largest linear forest among all $P(T)$, $Q_i(T)$ and $R_{ij}(T)$. Let $S'(T)$ be the largest linear forest among all $P(T)$, $Q_i(T)$.

Lemma 4.1. *For any tree T , $S(T)$ is a maximum linear forest in T , $S'(T)$ is a largest linear forest in T such that root v^* has degree at most one.*

Proof. For any tree T , if $S(T)$ is not a maximum linear forest in T . Let $F(T)$ be a maximum linear forest in T . Let $F[T_k] = F(T) \cap E(T_k)$, for $k = 1, 2, \dots, t$. We can divide the proof into three cases according to the degree of v^* in $F(T)$.

Case 1. The degree of v^* in $F(T)$ is zero. Clearly, $F[T_k]$ is a linear forest in subtree T_k . Then we have $|F[T_k]| \leq |F(T_k)|$. Therefore, $|F(T)| \leq |P(T)|$.

Case 2. The degree of v^* in $F(T)$ is one. Suppose v^*v_i is in $F(T)$. Then let $F[T_k] = F(T) \cap E(T_k)$, for $r = 1, 2, \dots, t$. Clearly, for $k \neq i$, $F[T_k]$ is a linear forest in subtree T_k and $F[T_k]$ is a linear forest in subtree T_k with degree of v^* at most one. Thus, for each $k \neq i$ we have $|F[T_k]| \leq |F(T_k)|$ and $|F[T_i]| \leq |F'(T_i)|$. Therefore, $|F(T)| \leq |Q_i(T)|$.

Case 3. The degree of v^* in $F(T)$ is two. Suppose v^*v_i and v^*v_j are in $F(T)$. Then let $F[T_k] = F(T) \cap E(T_k)$. Clearly, for $r \neq i, j$, $F[T_k]$ is a linear forest in subtree T_k . $F[T_i]$ is a linear forest in subtree T_i with degree of v^* at most one and $F[T_j]$ is a linear forest in subtree T_j with degree of v^* at most one. Thus, for each $k \neq i, j$ we have $|F[T_r]| \leq |F(T_r)|$. For each $r = i, j$, we have $|F[T_i]| \leq |F'(T_i)|$ and $|F[T_j]| \leq |F'(T_j)|$. Therefore, $|F(T)| \leq |R_{ij}(T)|$.

Combining these cases, we get the conclusion that $F(T) \leq S(T)$, which implies that $S(T)$ is a maximum linear forest in T . Similarly, we can prove $S'(T)$ is a largest linear forest in T such that root v^* has degree at most one. ■

Theorem 4.1. For any k -ary tree T with n vertices, we have

$$\frac{n+k-1}{k} \leq l(T) \leq \frac{2n-2}{k}.$$

Proof. For the lower bound, suppose T has x internal vertices and y leaves, then we have $kx + 1 = x + y = n$. It follows that $x = \frac{n-1}{k}$ and $y = \frac{(k-1)n+1}{k}$. By Lemma 3.1, we have $h(T) \leq \frac{(k-1)n+1}{k} - 1$. Thus, $l(T) \geq n - h(T) = \frac{n+k-1}{k}$.

For the upper bound, we can divide $n - 1$ edges of T into $\frac{n-1}{k}$ groups such that each k edges with the same parent are in the same group. Since the degree of vertices in the linear forest is at most two. Thus, at most two edges in each group are in the linear forest. Therefore, we get $l(T) \leq \frac{2n-2}{k}$. ■

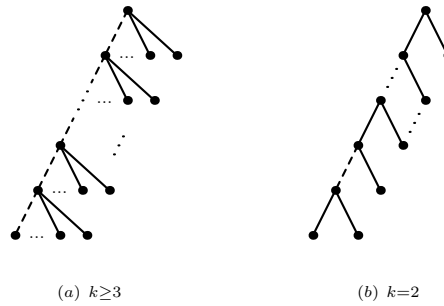


Figure 7: Special k -ary trees and their maximum linear forests.

Let T be a k -ary tree on n vertices such that in each layer there is only one node with k children, as shown in Fig. 7. Let v_i be the internal vertex at depth i and root v_0 is at depth 0. For $k \geq 3$, it is clear that $l(T) = \frac{2n-2}{k}$ as shown in Fig. 7 (a). For $k = 2$, by Theorem 4.1 it is easy to check that the linear forest shown in Fig. 7 (b) is maximum. Thus, $l(T) = \frac{3(n-1)}{4}$ for $\frac{n-1}{2}$ is even; $l(T) = \frac{3n-1}{4}$ for $\frac{n-1}{2}$ is odd.

Corollary 4.1. *Let T be a k -ary tree on n vertices, then*

$$\frac{(k-2)n - k + 2}{k} \leq \nabla(L(T)) \leq \frac{(k-1)n - 2k + 1}{k}.$$

A perfect k -ary tree is a k -ary tree in which all leaves are at the same depth. At last, we obtain the maximum linear forests in perfect k -ary trees as follows.

Theorem 4.2. *For any perfect k -ary tree T with n vertices, we have*

$$l(T) = \frac{2n - 1 + (-1)^h}{k + 1},$$

where $h = \log_k(n(k-1) + 1)$ is the height of T and leaves are at height 1.

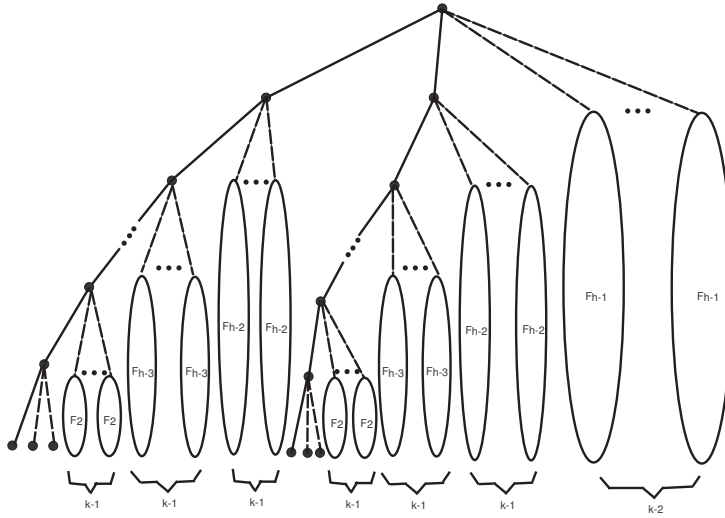


Figure 8: A linear forest in perfect k -ary trees.

Proof. Let $h = \log_k(n(k-1) + 1)$ be the height of T . We construct a linear forest of T as follows. Firstly, we choose two vertex-disjoint paths of length h that go from root to two leaves. Then tree T is decomposed into $k-2$ subtrees of height $h-1$, $2(k-1)$ subtrees of height $h-2$, $2(k-1)$ subtrees of height $h-3$, ..., $2(k-1)$ subtrees of height 3 and $2(k-1)$ subtrees of height 2 as shown in Fig. 8. Then for each subtree, we choose two vertex-disjoint paths that go from the root to the two leaves again. Do it recursively, then a linear forest of T is created.

Let F_h be the edge set of the obtained linear forest in T with height h and let f_h be the cardinality of F_h . Then it is easy to see that $f_1 = 0$ and $f_2 = 2$. According to the recursive construction of the obtained linear forest, we have

$$f_h = (k-2)f_{h-1} + 2(k-1) \sum_{i=1}^{h-2} f_i + 2(h-1),$$

and

$$f_{h-1} = (k-2)f_{h-2} + 2(k-1) \sum_{i=1}^{h-3} f_i + 2(h-2).$$

Combining the two equations, we get a recursive relation as follows.

$$f_h = (k-1)f_{h-1} + kf_{h-2} + 2.$$

By the technique of generating functions, we can derive a formula for f_h as follows.

$$f_h = \frac{2n-1+(-1)^h}{k+1}.$$

Thus, $l(T) \geq f_h = \frac{2n-1+(-1)^h}{k+1}$.

Now we prove that F_h is a maximum linear forest in T . Let v^* be the root of T and T_1, T_2, \dots, T_t be the t subtrees of v^* with root v_1, v_2, \dots, v_t . Since T is a perfect k -ary tree, each subtree T_k is identical to a perfect k -ary tree of height $h-1$. Suppose v^*v_1 and v^*v_2 are in F_h . Let F'_{h-1} be the subset of F_h in subtree T_1 and let f'_{h-1} be the cardinality of F'_{h-1} . Then F'_{h-1} is a linear forest in T_1 such that $\deg(v_1) = 1$.

We claim that in a perfect k -ary tree of height h , F_h is a maximum linear forest and F'_h is a largest linear forest such that the degree of the root is at most one. We prove the claim by induction on h . For $h=1$ and $h=2$, it is easy to check F_1, F_2 are maximum linear forests and F'_1, F'_2 are largest linear forests with degree of root at most one, where F_1, F'_1 are empty sets. Suppose the claim is true for perfect k -ary tree with height $h-1$. Let T be a perfect k -ary tree with height h . Define

$$\begin{cases} P(T) &= \cup_{i=1}^k F(T_i), \\ Q_i(T) &= (\cup_{k \neq i} F(T_k)) \cup F'(T_i) \cup \{(v^*, v_i)\}, \\ R_{ij}(T) &= (\cup_{k \neq i, j} F(T_k)) \cup F'(T_i) \cup F'(T_j) \cup \{(v^*, v_i), (v^*, v_j)\}. \end{cases}$$

By induction hypothesis, each $F(T_i)$ is identical to F_{h-1} and each $F'(T_i)$ is identical to F'_{h-1} . Then

$$\begin{cases} |P(T)| &= kf_{h-1}, \\ |Q_i(T)| &= (k-1)f_{h-1} + f'_{h-1} + 1, \\ |R_{ij}(T)| &= (k-2)f_{h-1} + 2f'_{h-1} + 2. \end{cases}$$

It is easy to see that each T_i has $\frac{n-1}{k}$ vertices. Since F_h is also consist of $k-2$ F_{h-1} 's, two F'_{h-1} 's and two extra edges, then we have $|R_{ij}(T)| = f_h$. It follows that $f'_{h-1} = \frac{1}{2}(f_h - (k-2)f_{h-1} - 2)$. Similarity, we have $|Q_i(T)| = f'_h$. Then,

$$\begin{aligned} f'_{h-1} + 1 - f_{h-1} &= \frac{1}{2}(f_h - (k-2)f_{h-1} - 2) + 1 - f_{h-1} \\ &= \frac{1}{2}(f_h - kf_{h-1}) \\ &= \frac{1}{2} \left(\frac{2n-1+(-1)^h}{k+1} - k \frac{2\frac{n-1}{k} - 1 + (-1)^{h-1}}{k+1} \right) \\ &= 1 - (-1)^{h-1} \geq 0. \end{aligned}$$

It follows that $|P(T)| \leq |Q_i(T)| \leq |R_{ij}(T)|$. By Theorem 4.1, we see that $R_{ij}(T)$ is a maximum linear forest in T and $Q_i(T)$ is a largest linear forest such that degree of the root is at most one, which are exactly F_h and F'_h . Therefore, we prove the claim and F_h is a maximum linear forest of T . Thus, we conclude that $l(T) = f_h = \frac{2n-1+(-1)^h}{k+1}$. ■

Corollary 4.2. *For any perfect k -ary tree T with n vertices, the decycling number of $L(T)$ is*

$$\nabla(L(T)) = \frac{2n - 1 + (-1)^h}{k + 1},$$

where $h = \log_k(n(k-1) + 1)$ is the height of T .

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