Quantum enigma machines

Seth Lloyd

Department of Mechanical Engineering Massachusetts Institute of Technology MIT 3-160, Cambridge MA, 02139, USA

Abstract: Enigma machines are devices that perform cryptography using pseudo-random numbers. The original enigma machine code was broken by detecting hidden patterns in these pseudo-random numbers. This paper proposes a model for a quantum optical enigma machine and shows that the phenomenon of quantum data locking makes such quantum enigma machines provably secure even in the presence of noise and loss.

The enigma machine used for cryptography during the second world war was a device which, given a short keyword, produced a pseudorandom output [1-3] which could be decoded by a second machine using the same keyword. The original enigma machine consisted of a series of rotors through which electrical current could pass in a way that depended on the relative orientation of the rotors. The path taken by the current connected an input symbol to an output system. After each key press the rotors went through a stepping motion that changed the functional relationship betweein input and output for the next key press. A sender and receiver who prepared their machines using the same initial setting, determined by the keyword, could then exchange encrypted messages. While the enigma machine did a pretty good job of scrambling the input, the outputs deviated sufficiently from pseudorandom sequences that the enigma code could be broken. In general, classical codes based on pseudo-random numbers are secure only if $P \neq NP$

[1-3]: proving the security of such codes is accordingly difficult. This paper proposes a quantum optical version of the enigma machine and shows that it is secure in principle, in the sense that amount of information that Eve can access about the message can be made arbitrarily small, even in the presence of arbitrary amounts of loss.

The security of quantum enigma machines relies on the phenomenon of quantum data locking [4-10]. Suppose that Alice possesses an n-bit message j that she wishes to send to Bob. Alice and Bob initially possess a secret, fully random m-bit string k (the 'seed'), where m << n. They publicly agree upon a set of 2^m unitary operations U_k , randomly selected according to the Haar measure. Alice first maps the message j to a quantum state $|j\rangle$. She then applies the transformation U_k corresponding to the shared seed k and sends the resulting state $|j\rangle_k = U_k|j\rangle$ to Bob. Bob decodes the message by applying the inverse transformation $U_k^{\dagger}|j\rangle_k = |j\rangle$. The devices that perform Alice's and Bob's encoding and decoding operations can be termed quantum enigma machines, in analogue to classical enigma machines that encode and decode via classical invertible transformations.

The method of quantum data locking [4-10] can now be used to show that quantum enigma machines are secure against eavesdropping. Suppose that Eve intercepts the full state sent by Alice. Assume that all 2^n messages that Alice could send are equally likely (as is the case, for example, if Alice has performed good classical data compression on her original message). Since Eve does not know the seed k, the state that she receives is

$$2^{-m-n} \sum_{jk} U_k |j\rangle \langle j| U_k^{\dagger} = 2^{-m-n} \sum_{jk} |j\rangle_k \langle j|. \tag{1}$$

The maximum amount of information that Eve can obtain about Alice's message is then limited by the accessible information I_c , equal to the maximum mutual information between the inputs j and the outcome of a measurement made by Eve on the encoded state. In [4], it is shown by a simple convexity argument that the accessible information in turn is limited by

$$I_c \le n + 2^{-m} \max_{|\phi\rangle} \sum_{jk} |\langle \phi | j \rangle_k|^2 \log |\langle \phi | j \rangle_k|^2.$$
 (2)

Hayden et al. [5] have shown that if the U_k are fully random, i.e., selected from the set of all n-qubit unitaries according to the Haar measure, then Eve's accessible information

can be made arbitrarily small using a key of size $m = O(\log n)$. A number of papers have extended this result [6-10]. In particular, as shown in [10], if $m = O(4\log(1/\epsilon))$, then for any $\epsilon > 0$ there exists an n sufficiently large that Eve's accessible information is lower than ϵn . Bob can decode the full n-bit message, while in the absence of those m bits, Eve can obtain only an arbitrarily small amount of information. For example, if $\epsilon = n/2^{10}$, then a key of length $m = O(4\log n + 40)$ allows Eve access to less than a thousandth of a bit. Quantum enigma machines are secure in principle by the accessible information criterion.

There are several hurdles to overcome in order to make quantum enigma machines secure in practice. Quantum data locking hides large amounts of data with a small key. Consequently, Alice and Bob must be very careful to ensure the security of that key. Data locking is susceptible to plain-text attack: if Eve knows part of the message, she can use that to determine the key and unlock the rest [7,9]. Alice and Bob can evade the plaintext attack by using data locking to distribute a random secret key [11]: Alice simply sends Bob a random number, known only to her. Performing a fully random unitary is computationally inefficient: in [10], however, it is shown how locking can be performed using non-random unitaries that can be constructed in time almost linear in n. That is, unlike the classical pseudo-random transformations of the original enigma machine, quantum pseudo-random transformations perform a sufficiently good job of scrambling the message that Eve cannot decipher it. Perhaps the most pressing issue, however, is the ability of quantum enigma machines to function effectively in the presence of noise and loss.

Noise and Loss

Bob's ability to decode the message is sensitive to noise and loss on the channel. As stated, the protocol requires a communications channel capable of sending quantum information, which suggests that more than 50% loss would render the protocol ineffective. I now show that, contrary to that intuition, quantum enigma machines can function with arbitrarily high levels of loss.

First consider the depolarizing channel. Alice sends qudits $\in C^d$ to Bob. With probability η the qudit is transmitted faithfully, and with probability $1 - \eta$ it is replaced

with a fully mixed state. For $\eta \leq 1/2$, this channel is anti-degradable and has neither quantum capacity nor private capacity [12-13]. The enigmatic version of the depolarizing channel is straightforward: Alice applies a random unitary to the qudit before sending, and Bob applies the inverse transformation on the other side. To Alice and Bob, then, the channel behaves like the ordinary depolarizing channel, which has classical capacity for all $\eta > 0$. By contrast, even if Eve intercepts the locked qudit in a noiseless state, she can extract only a vanishingly small amount of accessible information.

To send information securely by the accessible information criterion, Alice and Bob agree on a particular error correcting code for sending classical information down the depolarizing channel. The code has block length b appropriate for the depolarizing rate η and the degree of accuracy that they wish to attain. As shown in [10], if a single use of the locked channel bounds Eve's accessible information by ϵ , then the composition of b uses of a locked channel bounds her accessible information by $b\epsilon$. For sufficiently large d, then, Alice and Bob can pick a key of length $O(b \log(b/\epsilon))$ to lock the b uses of the channel, allowing Eve access to at most a fraction ϵ of the transmitted information.

The example of the qudit depolarizing channel shows that quantum data locking combined with suitable error correction can still send information that is secure by the accessible information criterion down a quantum channel whose private capacity is zero. Now turn to an example of how quantum data-locking can be used in an experimentally achievable setting, even in the presence of large amounts of loss. Consider the lossy bosonic channel [14-16]. Alice's and Bob's quantum enigma machines can consist of passive linear elements such as beam splitters and phase shifters. Alice's quantum enigma machine maps the annihilation operators for N input modes $\vec{\alpha} \to U_k \vec{\alpha}$, where U_k is an $N \times N$ random unitary transformation acting on the mode labels, selected from a set of M such transformations. Bob's enigma machine performs the inverse transformation, $\vec{\alpha} \to U_k^{\dagger} \vec{\alpha}$. Assume for the moment that there is no loss in the encoding and decoding procedures – all the loss is in the communication channel.

To see that Alice's and Bob's enigma machines can still perform effectively in the presence of high levels of loss, consider a 'unary' encoding, in which $n = \log_2 N$ message bits are encoded on a single photon spread amongst N modes, so that the j'th message

is encoded in the state $|j\rangle$ with the single photon in mode j. The security analysis for this unary encoding is mathematically the same as the qubit case above. Alice's quantum enigma machine uses linear optics to apply a random unitary transformation U_k to the mode labels, yielding the state $|j\rangle_k = U_k|j\rangle$. In the absence of knowledge of the secret key k, Eve's accessible information about the message can be made less than ϵ bits using a key of size $O(\log n \log(n/\epsilon))$.

A simple strategy now allows Alice and Bob to communicate even in the presence of high amounts of loss. Since Bob's quantum enigma machine undoes Alice's linear transformation, if the photon that Alice sent does shows up, Bob's enigma machine maps it into the j'th mode. If no photon shows up, Bob simply asks Alice to resend with new key. If ϵ is the fraction of the information that Eve has access to for one transmission, then $T\epsilon$ is the fraction that she can obtain after T transmissions. For a fixed level of loss per mode, the probability of losing the photon is independent of the number of modes N. Consequently, by making ϵ small and N large, Alice and Bob can cope with arbitrily high levels of loss while guaranteeing that Eve gains an arbitrarily small fraction of the transmitted information.

The ability of Bob to unlock information even in the presence of large amounts of loss comes about because the lossy bosonic channel retains quantum capacity with feedback up to arbitrarily high loss levels. The non-trivial feature of this quantum enigma machine is that the locking/unlocking protocol survives the transmission/feedback process. Since unary coding over blocks of N modes attains the channel capacity for a lossy channel in the limit of low photon number per mode [14-16], the capacity of a secure quantum enigma machine channel is essentially the same as that of an insecure bosonic channel in this regime.

In the presence of thermal noise and loss, the capacity of the bosonic quantum enigma machine is bounded below by the forward and reverse coherent information [17-18]. At optical frequencies, the background thermal noise is neglible. Moreover, the conventional way for Alice to generate her single photons is by parametric downconversion of a pair of photons: detection of one photon of the pair then heralds the arrival of the other. To jam the channel, Eve must introduce enough photons to insure that the exact time bin of the her-

alded photon contains excess photons. As long as the average noise photon number within the time bin is below the threshold for non-zero forward or reverse coherent information, Alice and Bob retain secret capacity via quantum locking, both for direct transmission (with error-correcting codes) and for key generation (with privacy amplification).

Coherent states

An important open question is whether it is possible to construct a provably secure quantum enigma machine using linear optics and coherent states. An example of a high-power coherent-state quantum enigma machine is Yuen's α - η model [19-23]. In the quantum enigma model discussed here, the transformations U_k map the message to sets of overlapping coherent states is selected at random, whereas in the α - η model these transformations are selected in a systematic fashion using binary phase shift keying [19]. The α - η model has been shown to yield error probabilities approaching 1 for common types of eavesdropper [19-23], although its security based on accessible information or on Holevo information [24-25] has not been shown.

Conclusions:

This paper showed that quantum enigma machines, unlike their classical counterparts, are provably secure against eavesdropping. The security of the quantum enigma machine is guaranteed by its ability to spread encoded states over Hilbert space via quantum data locking, thereby limiting the ability of an eavesdropper to obtain information about the encoded message. A quantum enigma machine based on single photons and unary encoding was exhibited and shown to retain its security in the presence of noise and high amounts of loss in the communication channel. The general question of how to define the capacity of quantum enigma machines that render a specified channel secure by the accessible information remains open [26]. Here we exhibited enigmatic quantum coding schemes for the depolarizing and lossy channels: it would be useful to have codes for quantum data locking on general quantum channels. As with any cryptographic system, the security of practically realizable quantum enigma machines must be investigated on a case-by-case basis to probe for possible attacks.

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