

# A Survey on Nonconvex Regularization Based Sparse and Low-Rank Recovery in Signal Processing, Statistics, and Machine Learning

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**Abstract**—In the past decade, sparse and low-rank recovery have drawn much attention in many areas such as signal/image processing, statistics, bioinformatics and machine learning. To achieve sparsity and/or low-rankness inducing, the  $\ell_1$  norm and nuclear norm are of the most popular regularization penalties due to their convexity. While the  $\ell_1$  and nuclear norm are convenient as the related convex optimization problems are usually tractable, it has been shown in many applications that a nonconvex penalty can yield significantly better performance. In recent, nonconvex regularization based sparse and low-rank recovery is of considerable interest and it in fact is a main driver of the recent progress in nonconvex and nonsmooth optimization. This paper gives an overview of this topic in various fields in signal processing, statistics and machine learning, including compressive sensing (CS), sparse regression and variable selection, sparse signals separation, sparse principal component analysis (PCA), large covariance and inverse covariance matrices estimation, matrix completion, and robust PCA. We present recent developments of nonconvex regularization based sparse and low-rank recovery in these fields, addressing the issues of penalty selection, applications and the convergence of nonconvex algorithms. Code is available at <https://github.com/FWen/ncreg.git>.

**Index Terms**—Sparse, low-rank, nonconvex, compressive sensing, regression, covariance matrix estimation, matrix completion, principal component analysis.

## I. INTRODUCTION

In the past decade, sparse and low-rank recovery have attracted much study attention in many areas, such as signal processing, image processing, statistics, bioinformatics and machine learning. To achieve sparsity and low-rankness promotion, sparsity and low-rankness constraints or penalties are commonly employed. Among the sparsity and low-rankness inducing penalties, the  $\ell_1$ -norm and nuclear norm penalties are of the most popular. This is mainly due to their convexity as it makes the involved optimization problems tractable in that, existing convex optimization techniques with well-established convergence properties can be directly used or can be applied after some extension.

Generally, under certain conditions, the  $\ell_1$  and nuclear norm penalties can reliably recover the underlying true sparse signal and low-rank matrix with high probability. However, both of them have a bias problem, which would result in significantly biased estimates, and cannot achieve reliable recovery with the least observations [1], [2], [3]. In comparison, a nonconvex

penalty, such as the  $\ell_0$ ,  $\ell_q$  ( $0 < q < 1$ ), smoothly clipped absolute deviation (SCAD) or minimax concave penalty (MCP), is superior in that it can ameliorate the bias problem of the  $\ell_1$ -one. In recent, nonconvex regularization based sparse and low-rank recovery have drawn considerable interest and achieved significant performance improvement in many applications over convex regularization. This progress benefited a lot from the recent developments in nonconvex and nonsmooth optimization, and, meanwhile, promoted the developments of the latter.

The goal of this article is to give an overview of the recent developments in nonconvex regularization based sparse and low-rank recovery in signal/image processing, statistics and machine learning. In a field as wide as this, we mainly focus on the following eight important topics.

1) *Compressive sensing (CS)*: CS aims to acquire sparse signals (or signals can be sparsely represented in some basis) at a significantly lower rate than the classical Nyquist sampling [14]–[16]. In CS, exploiting the sparsity (or sparse representation) of the desired signals is the key for their reconstruction. There exist a number of recent works addressing nonconvex regularized sparse reconstruction, e.g., using the  $\ell_0$  and  $\ell_q$  regularization [39]–[66]. It has been demonstrated that, nonconvex regularization not only ameliorates the bias problem of the  $\ell_1$ -one but also requires fewer measurements for reliable recovery.

2) *Sparse regression and variable selection*: Sparse regression aims to simultaneously select variables and estimate coefficients of variables, which is a fundamental problem in high-dimensional statistical analysis. Nonconvex penalties, such as the SCAD,  $\ell_0$ ,  $\ell_q$ , MCP penalties, have been widely employed to attain more accurate estimation over the  $\ell_1$ -one [11], [12], [83]–[96].

3) *Sparse signals separation and image inpainting*: Sparse signals separation problems arise in many important applications, such as source separation, super-resolution, inpainting, interference cancellation, saturation and clipping restoration, and robust sparse recovery in impulsive (sparse) noise. In these applications, the  $\ell_q$  penalty can attain considerable restoration performance improvement over the  $\ell_1$  penalty [120].

4) *Sparse principal component analysis (PCA)*: PCA is a useful statistical tool for data analysis and dimensionality reducing, which is widely used in virtually all areas of science and engineering. Sparse PCA aims to obtain sparse loading vectors to enhance the interpretability of the princi-

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ple components (PCs). Nonconvex regularization, such as  $\ell_0$  regularization, has been widely used for sparsity promotion [128]–[141].

5) *Large sparse covariance matrix estimation*: Large covariance matrix estimation is a fundamental problem in modern high-dimensional statistical analysis, which has found wide applications such as in economics, finance, bioinformatics, social networks, climate studies, and health sciences. In the high-dimensional setting where the dimensionality is often comparable to (or even larger than) the sample size, sparsity regularized estimation is especially effective. Some works addressing nonconvex regularized covariance matrix estimation include [148], [152].

6) *Large sparse inverse covariance matrix estimation*: Large inverse covariance matrix estimation is also fundamental to high-dimensional statistical analysis, which is closely related to undirected graphs under a Gaussian model. Some papers addressing nonconvex regularized inverse covariance matrix estimation include [160]–[166].

7) *Matrix completion*: Matrix completion deals with the recovery of a low-rank matrix from its partially observed entries. Such problems arise in many applications such as in recommender systems, computer vision and system identification. Various algorithms have been developed using the nonconvex Schatten- $q$  ( $0 \leq q < 1$ ) norm, truncated nuclear norm, and MCP penalties for low-rankness inducing, e.g., [6], [48], [175]–[182], to achieve better recovery performance over the nuclear norm.

8) *Robust PCA*: Robust PCA aims to enhance the robustness of PCA against outliers or sparse corruption which is ubiquitous in modern applications such as image processing, web data analysis, and bioinformatics. Basically, robust PCA is a joint sparse and low-rank recovery problem, which seeks to identify a low-dimensional structure from grossly corrupted observations. Existing works using different combinations of nonconvex sparse and low-rank penalties for robust PCA include [9], [193]–[197].

Among these topics, CS, sparse regression, sparse signals separation, and sparse PCA are sparse vector recovery problems, large sparse covariance and inverse covariance matrices estimation are sparse matrix recovery problems, whilst matrix completion and robust PCA are low-rank recovery problems. To be more precise, sparse PCA is not a vector recovery problem, but in many popular greedy methods, the PCs are estimated in a vector-by-vector manner. Meanwhile, robust PCA is a joint sparse and low-rank recovery problem.

In this paper, we also provide some critical perspectives. As it is often the case that, when new techniques are introduced, there may also be some overexcitement and abuse. We comment on the following points: There exist certain instances in both sparse and low-rank recovery, where the use of nonconvex regularization is simply unnecessary and will not significantly improve performance [95]. The use of nonconvex regularization does not always guarantee distinct performance improvement over convex regularization. Moreover, employing nonconvex regularization models can even be disadvantageous because the related nonconvex and nonsmooth optimization problems are less tractable than convex problems. Perfor-

mance should be defined in a broader sense that includes not only recovery accuracy, but also other properties of the algorithm, such as convergence speed. Generally, for a nonconvex regularized algorithm, the performance is usually closely related to the initialization and the convergence rate is usually slower than that of a convex regularized algorithm. In comparison, a convex algorithm has better stability and convergence properties, and is insensitive to initialization since converging to a global minimal is usually easy guaranteed [202]. Meanwhile, for first-order convex algorithms, well-developed acceleration techniques can be used with guaranteed convergence [26]. However, when such acceleration techniques are applied to nonconvex algorithms, there is no guarantee of the convergence and stability.

Therefore, the question of whether to use convex or nonconvex regularization models requires careful deliberation. We show that convex models may be preferable when the signal is not strictly sparse (or the matrix is not strictly low-rank) or the signal-to-noise ratio (SNR) is low, since in these cases the performance improvement of nonconvex models are often not distinct and may not deserve the price of more slowly converging nonconvex algorithms. We provide a number of concrete examples that clarify these points.

We also address the issues of penalty selection and the convergence of related nonconvex algorithms. We hope that our paper will illuminate the role nonconvex regularization plays in sparse and low-rank recovery in signal processing, statistics and machine learning, and demonstrate when and how it should be used.

There also exist some recent overview articles related to the topics of this work, e.g., on low-rank matrix recovery [203], [204], and nonconvex optimization in machine learning [205]. While the works [203], [204] mainly focus on matrix factorization based (nonconvex) low-rank recovery problems, the low-rank recovery problems investigated in this work are more general. Meanwhile, while the article [205] mainly focuses on the optimization aspect of general nonconvex problems in machine learning, we focus on nonconvex regularized problems characterized with the non-convexity and non-smoothness of the involved problems. Moreover, we provide a more wide scope on the applications in various fields.

*Outline*: The rest of this paper is organized as follows. In section II, we review the proximity operator for nonconvex penalties, and present extended vector proximity operator (for joint sparse recovery) and singular value shrinkage operator (for low-rank matrix recovery) for generalized nonconvex penalties. Section III discusses nonconvex regularized sparse vector recovery problems, including CS, sparse regression, sparse signals separation and sparse PCA. Section IV reviews nonconvex regularized sparse matrix recovery problems, including large sparse covariance and inverse covariance matrices estimation. Section V discusses nonconvex regularized low-rank recovery problems, including matrix completion and robust PCA. Section VI further discusses other applications involving nonconvex regularized sparse and low-rank recovery. Section VII concludes the overview.

*Notations*: For a matrix  $\mathbf{M}$ ,  $\text{rank}(\mathbf{M})$ ,  $\text{tr}(\mathbf{M})$ ,  $|\mathbf{M}|$ ,  $\|\mathbf{M}\|_2$  and  $\|\mathbf{M}\|_{\mathbb{F}}$  are the rank, trace, determinant, spectral norm and

Frobenius norm, respectively, whilst  $\text{eig}_{\max}(\mathbf{M})$ ,  $\text{eig}_{\min}(\mathbf{M})$  and  $\sigma_i(\mathbf{M})$  denote the maximal eigenvalue, minimal eigenvalue and the  $i$ -th largest singular value of  $\mathbf{M}$ . For a matrix  $\mathbf{M}$ ,  $\text{diag}(\mathbf{M})$  is a diagonal matrix which has the same diagonal elements as that of  $\mathbf{M}$ , whilst for a vector  $\mathbf{v}$ ,  $\text{diag}(\mathbf{v})$  is a diagonal matrix with diagonal elements be  $\mathbf{v}$ .  $\mathbf{M} \geq \mathbf{0}$  mean that  $\mathbf{M}$  is positive-semidefinite.  $\langle \cdot, \cdot \rangle$  and  $(\cdot)^T$  stand for the inner product and transpose, respectively.  $\nabla f$  and  $\partial f$  stand for the gradient and subdifferential of the function  $f$ , respectively.  $\text{sign}(\cdot)$  denotes the sign of a quantity with  $\text{sign}(0)=0$ .  $\mathbf{I}_m$  stands for an  $m \times m$  identity matrix.  $\|\cdot\|_q$  with  $q \geq 0$  denotes the  $\ell_q$ -norm defined as  $\|\mathbf{x}\|_q = (\sum_i |x_i|^q)^{1/q}$ .  $\delta_{i,j}$  is the Kronecker delta function.  $E\{\cdot\}$  denotes the expectation.  $I(\cdot)$  denotes the indicator function.

## II. PROXIMITY OPERATOR FOR NONCONVEX REGULARIZATION PENALTIES

Proximity operator plays a central role in developing efficient proximal splitting algorithms for many optimization problems, especially for nonconvex and nonsmooth inverse problems encountered in the applications addressed in this paper. As will be shown later, for convex or nonconvex penalized minimization problems, proximity operator is the core of most highly-efficient first-order algorithms which scale well for high-dimensional problems. This section reviews nonconvex regularization penalties and their corresponding proximity operator, including the hard-thresholding,  $\ell_q$ -norm, an explicit  $q$ -shrinkage, SCAD, MCP, and firm thresholding.

### A. Scalar Proximity Operator

For a proper and lower semi-continuous penalty function  $P_\lambda(\cdot)$  where  $\lambda > 0$  is a threshold parameter, consider the following scalar proximal mapping

$$\text{prox}_{P_\lambda}(t) = \arg \min_x \left\{ P_\lambda(x) + \frac{1}{2}(x-t)^2 \right\}. \quad (1)$$

As  $P_\lambda(\cdot)$  is separable, the proximity operator of a vector  $\mathbf{t} = [t_1, \dots, t_n]^T \in \mathbb{R}^n$ , denoted by  $\text{prox}_{P_\lambda}(\mathbf{t})$ , can be computed in an element-wise manner as

$$\text{prox}_{P_\lambda}(\mathbf{t}) = [\text{prox}_{P_\lambda}(t_1), \dots, \text{prox}_{P_\lambda}(t_n)]^T. \quad (2)$$

Table 1 shows the penalties and the corresponding proximal mapping operator. Among the presented penalties, only the soft-thresholding penalty is convex, while the other penalties are (symmetric) folded concave functions, as shown in Fig. 1.

The hard-thresholding was first introduced in [36] and then applied for wavelet applications in statistics [4], which is a natural selection for sparse inducing [77]–[81]. The well-known soft-thresholding rule was first observed by Donoho, Johnstone, Hoch and Stem [35] and then used in wavelet applications [36], which forms the core of the LASSO introduced by Tibshirani [37]. The  $\ell_1$  penalty is the most popular one as its convexity makes the related optimization problems more tractable than that using a nonconvex one.

However, the  $\ell_1$  penalty has a bias problem. More specifically, when the true parameter has a relatively large magnitude, the soft-thresholding estimator is biased since it imposes a

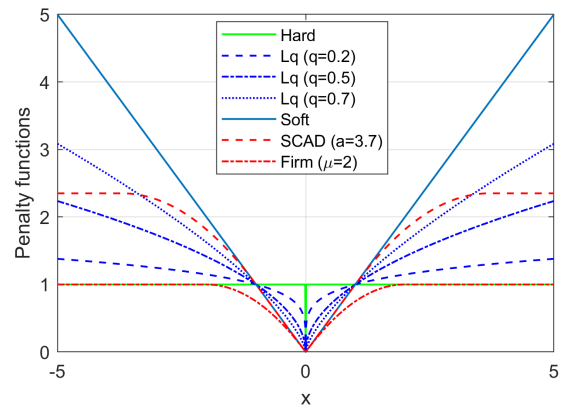


Fig. 1: Plot of penalty functions for  $\lambda = 1$ .

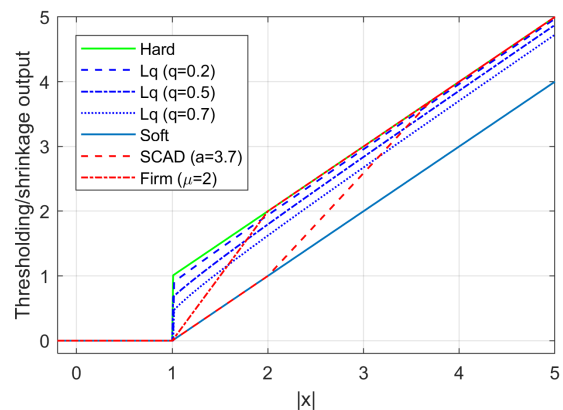


Fig. 2: Plot of thesholding/shrinkage functions (with a same threshold).

constant shrinkage on the parameter, as shown in Fig. 2. Fig. 2 plots the thresholding/shrinkage functions for the hard-, soft-,  $\ell_q$ - and SCAD penalties with a same threshold. In contrast, the hard-thresholding and SCAD estimators are unbiased for large parameter. Meanwhile, the thresholding rule corresponding to SCAD,  $\ell_q$  and  $q$ -shrinkage fall in (sandwiched) between hard- and soft-thresholding.

The  $\ell_q$  penalty with  $0 < q < 1$  bridges the gap between the  $\ell_0$  and  $\ell_1$  penalties, and intuitively its shrinkage function is less biased than soft-thresholding. The proximity operator of the  $\ell_q$ -norm penalty does not have a closed-form expression except for the two special cases of  $q = 1/2$  and  $q = 2/3$  [5] (in these two cases the proximal mapping can be explicitly expressed as the solution of a cubic or quartic equation), but it can be efficiently solved, e.g., by a Newton's method. Moreover, explicit  $q$ -shrinkage mapping has been proposed in [8]–[10], which has some qualitative resemblance to the  $\ell_q$  proximal mapping while being continuous and explicit. The  $q$ -shrinkage reduces to the soft-thresholding when  $q = 1$ , while it tends pointwise to the hard-thresholding in the limit as  $q \rightarrow -\infty$ . As an acceptable price to pay for having an explicit proximal mapping, its penalty does not have a closed-form expression.

The SCAD penalty has been widely used in variable selection problems, and it has shown favorable effectiveness compared with other penalties in high-dimensional variable

TABLE I: Regularization penalties and the corresponding proximity operator ( $\lambda > 0$  is a thresholding parameter).

Penalty name	Penalty formulation	Proximity operator
(i) Soft thresholding [35]	$P_\lambda(x) = \lambda x $	$\text{prox}_{P_\lambda}(t) = \text{sign}(t) \max\{ t  - \lambda, 0\}$
(ii) Hard thresholding [36]	$P_\lambda(x) = \lambda[2 - ( x  - \sqrt{2})^2 I( x  < \sqrt{2})]$ or $P_\lambda(x) = \lambda x _0$	$\text{prox}_{P_\lambda}(t) = \begin{cases} 0, &  t  < \sqrt{2}\lambda \\ \{0, t\}, &  t  = \sqrt{2}\lambda \\ t, &  t  > \sqrt{2}\lambda \end{cases}$
(iii) $\ell_q$ -norm [6], [7]	$P_\lambda(x) = \lambda x ^q, 0 < q < 1$	$\text{prox}_{P_\lambda}(t) = \begin{cases} 0, &  t  < \tau \\ \{0, \text{sign}(t)\beta\}, &  t  = \tau \\ \text{sign}(t)y, &  t  > \tau \end{cases}$ where $\beta = [2\lambda(1-q)]^{1/(2-q)}$ , $\tau = \beta + \lambda q \beta^{q-1}$ , $h(y) = \lambda q y^{q-1} + y -  t  = 0$ and $y \in [\beta,  t ]$
(iv) $q$ -shrinkage [10]	N/A ( $q < 1$ )	$\text{prox}_{P_\lambda}(t) = \text{sign}(t) \max\{ t  - \lambda^{2-q} t^{q-1}, 0\}$
(v) SCAD [11]	$P_\lambda(x) = \begin{cases} \lambda x , &  x  < \lambda \\ \frac{2a\lambda x  - x^2 - \lambda^2}{2(a-1)}, & \lambda \leq  x  < a\lambda \\ (a+1)\lambda^2/2, &  x  \geq a\lambda \end{cases}$	$\text{prox}_{P_\lambda}(t) = \begin{cases} \text{sign}(t) \max\{ t  - \lambda, 0\}, &  t  \leq 2\lambda \\ \frac{(a-1)t - \text{sign}(t)a\lambda}{a-2}, & 2\lambda <  t  \leq a\lambda \\ t, &  t  > a\lambda \end{cases}$
(vi) MCP [12]	$P_{\lambda,\gamma}(x) = \lambda \int_0^{ x } \max(1 - t/(\gamma\lambda), 0) dt$ , where $\gamma > 0$	$\text{prox}_{P_{\lambda,\gamma}}(t) = \begin{cases} 0, &  t  \leq \lambda \\ \frac{\text{sign}(t)( t  - \lambda)}{1 - 1/\gamma}, & \lambda <  t  \leq \gamma\lambda \\ t, &  t  > \gamma\lambda \end{cases}$
(vii) Firm thresholding [13]	$P_{\lambda,\mu}(x) = \begin{cases} \lambda[ x  - x^2/(2\mu)], &  t  \leq \mu \\ \lambda\mu/2, &  t  \geq \mu \end{cases}$ where $\mu > \lambda$	$\text{prox}_{P_{\lambda,\mu}}(t) = \begin{cases} 0, &  t  \leq \lambda \\ \frac{\text{sign}(t)( t  - \lambda)\mu}{\mu - \lambda}, & \lambda \leq  t  \leq \mu \\ t, &  t  \geq \mu \end{cases}$

selection problems [11]. As well as the  $\ell_0$ ,  $\ell_q$ , and SCAD penalties, the MCP penalty can also ameliorate the bias problem of the  $\ell_1$  penalty [12], and it has been widely used for penalized variable selection in high-dimensional linear regression. For the MCP penalty with each  $\lambda > 0$ , we can obtain a continuum of penalties and threshold operators by varying  $\gamma$  in the range  $(0, +\infty)$ . Moreover, the firm thresholding is a continuous and piecewise-linear approximation of the hard-thresholding [13]. In addition, a class of nonconvex penalties which can maintain the convexity of the global cost function have been designed in [209]–[211], whilst log penalties have been considered in [212], [213].

Among the presented shrinkage functions, the soft-, SCAD,  $q$ -shrinkage and firm thresholding are continuous while the hard- and  $\ell_q$ -thresholding are discontinuous. For each of the presented penalties, the corresponding thresholding/shrinkage operator  $\text{prox}_{P_\lambda}(t)$  satisfies

$$\begin{aligned} \text{i)} \quad & \text{prox}_{P_\lambda}(t) = \text{sign}(x)\text{prox}_{P_\lambda}(|t|) \\ \text{ii)} \quad & |\text{prox}_{P_\lambda}(t)| \leq |t| \\ \text{iii)} \quad & \text{prox}_{P_\lambda}(t) = 0 \quad \text{for some threshold } |t| \leq T_\lambda \\ \text{iv)} \quad & |\text{prox}_{P_\lambda}(t) - t| \leq \lambda \end{aligned} \quad (3)$$

where  $T_\lambda > 0$  is a threshold dependent on  $\lambda$ . These conditions establish the sign consistency, shrinkage, thresholding, and limited shrinkage properties for the thresholding/shrinkage operator corresponding to a generalized penalty.

### B. Vector Proximity Operator (for Multitask Joint Sparse Recovery)

In many applications, as will be shown in section III, it is desirable to jointly recover multichannel signals to exploit

the multichannel joint sparsity pattern. In this case, it involves solving the following vector optimization problem

$$\mathbf{prox}_{P_\lambda}(\mathbf{t}) = \arg \min_{\mathbf{x}} \left\{ P_\lambda(\|\mathbf{x}\|_2) + \frac{1}{2} \|\mathbf{x} - \mathbf{t}\|_2^2 \right\} \quad (4)$$

where  $\mathbf{x} \in \mathbb{R}^L$  and  $\mathbf{t} \in \mathbb{R}^L$  with  $L$  be the channel number. The case of  $P_\lambda$  be the  $\ell_q$  penalty has been considered in [120], we extend the result to generalized penalties in the following.

**Theorem 1.** Suppose that  $P_\lambda(x)$  is a nondecreasing function for  $x \in [0, \infty)$ , for any  $\mathbf{x} \in \mathbb{R}^L$ , the solution to (4) is given by

$$\mathbf{prox}_{P_\lambda}(\mathbf{t}) = \begin{cases} \mathbf{0}, & \mathbf{t} = \mathbf{0} \\ \frac{\text{prox}_{P_\lambda}(\|\mathbf{t}\|_2)}{\|\mathbf{t}\|_2} \mathbf{t}, & \text{otherwise} \end{cases} \quad (5)$$

*Proof:* See Appendix A.

### C. Singular Value Shrinkage Operator (for Low-Rank Matrix Recovery)

For a matrix  $\mathbf{M} \in \mathbb{R}^{m \times n}$ , consider the singular value decomposition (SVD) of rank  $r$ ,  $\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$ , where  $\mathbf{\Sigma} = \text{diag}\{\sigma_1, \dots, \sigma_r\}$  contains the singular values,  $\mathbf{U} \in \mathbb{R}^{m \times r}$  and  $\mathbf{V} \in \mathbb{R}^{r \times n}$  contain the orthonormal singular vectors. In low-rank matrix recovery, the low-rankness promotion on a matrix is usually achieved by sparsity promotion on the singular values of the matrix. We denote a generalized penalty for low-rankness promotion by  $\bar{P}_\lambda$ , which is defined as

$$\bar{P}_\lambda(\mathbf{M}) = \sum_i P_\lambda(\sigma_i) \quad (6)$$

where  $P_\lambda$  is a generalized penalty for sparsity inducing as introduced in Table 1. In the two cases of  $P_\lambda(\cdot) = \|\cdot\|_0$  and  $P_\lambda(\cdot) = \|\cdot\|_1$ ,  $\bar{P}_\lambda(\mathbf{M})$  becomes the rank  $\text{rank}(\mathbf{M})$  and nuclear

norm  $\|\mathbf{M}\|_*$  of  $\mathbf{M}$ , respectively. When  $P_\lambda$  is the  $\ell_q$  penalty,  $\bar{P}_\lambda$  becomes the Schatten- $q$  quasi-norm of matrix.

In the following, we provide the generalized singular-value shrinkage operator for a generalized penalty  $\bar{P}_\lambda$ .

**Theorem 2.** For a rank  $r$  matrix  $\mathbf{M} \in \mathbb{R}^{m \times n}$ , suppose that it has an SVD  $\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$ , where  $\mathbf{\Sigma} = \text{diag}\{\sigma_1, \dots, \sigma_r\}$  and  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$ ,  $\mathbf{U}$  and  $\mathbf{V}$  contain the left and right singular vectors. Then, for any  $\bar{P}_\lambda$  defined as (6) with  $P_\lambda$  satisfying (3), the solution to the optimization problem

$$\text{prox}_{\bar{P}_\lambda}(\mathbf{M}) = \arg \min_{\mathbf{X}} \left\{ \bar{P}_\lambda(\mathbf{X}) + \frac{1}{2} \|\mathbf{X} - \mathbf{M}\|_F^2 \right\} \quad (7)$$

is given by

$$\text{prox}_{\bar{P}_\lambda}(\mathbf{M}) = \mathbf{U} \cdot \text{diag} \{ \text{prox}_{P_\lambda}(\sigma_1, \dots, \sigma_r) \} \cdot \mathbf{V}^T \quad (8)$$

where  $\text{prox}_{P_\lambda}$  is the proximity operator defined in (2).

*Proof:* See Appendix B.

### III. SPARSE VECTOR RECOVERY

This section reviews nonconvex regularization based sparse vector signals recovery, mainly on the following four topics, CS, sparse regression and variable selection, sparse signals separation with application to image inpainting and super-resolution, and sparse PCA. Strictly speaking, sparse PCA is not a vector recovery problem, but in many popular greedy approaches, the principle components are estimated in a one-by-one (vector-by-vector) manner.

#### A. Compressive Sensing

In the past decade, compressive sensing has attracted extensive studies [14]–[17] and has found wide applications in radar [18], [19], communications [20], medical imaging [21], image processing [22], and speech signal processing [23]. In the CS framework, sparse signals (or signals can be sparsely represented in some basis) can be acquired at a significantly lower rate than the classical Nyquist sampling, and signals only need to be sampled at a rate proportional to their information content.

In CS, the objective is to reconstruct a sparse signal  $\mathbf{x} \in \mathbb{R}^n$  from its compressed measurement

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n} \quad (9)$$

where  $\mathbf{A} \in \mathbb{R}^{m \times n}$  with  $m < n$  is the sensing matrix (also called measurement matrix),  $\mathbf{n} \in \mathbb{R}^m$  is additive measurement noise. Since  $m < n$ , the recovery of  $\mathbf{x}$  from the compressed measurement is generally ill-posed. However, provided that  $\mathbf{x}$  is sparse and the sensing matrix  $\mathbf{A}$  satisfies some stable embedding conditions [17],  $\mathbf{x}$  can be reliably recovered with an error upper bounded by the noise strength. This can be achieved in the noiseless case by the formulation

$$\begin{aligned} & \min_{\mathbf{x}} P(\mathbf{x}) \\ & \text{subject to } \mathbf{A}\mathbf{x} = \mathbf{y} \end{aligned} \quad (10)$$

or in the noisy case by the formulation

$$\begin{aligned} & \min_{\mathbf{x}} P(\mathbf{x}) \\ & \text{subject to } \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2 \leq \sigma \end{aligned} \quad (11)$$

where  $P$  is a penalty for sparsity inducing (a special case of  $P_\lambda$  with  $\lambda = 1$ ), and  $\sigma > 0$  bounds the  $\ell_2$ -norm of the residual error. This constrained formulation (11) can be converted into an unconstrained formulation as

$$\min_{\mathbf{x}} \frac{1}{2} \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2^2 + P_\lambda(\mathbf{x}). \quad (12)$$

Naturally, using the  $\ell_0$ -norm penalty, i.e.,  $P(\mathbf{x}) = \|\mathbf{x}\|_0$  and  $P_\lambda(\mathbf{x}) = \lambda \|\mathbf{x}\|_0$ , which counts the number of nonzero components in the vector  $\mathbf{x}$ , (10), (11) and (12) are the exact formulations of finding a sparse vector to fulfill the linear constraint  $\mathbf{A}\mathbf{x} = \mathbf{y}$ , satisfy the residual constraint  $\|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2 \leq \sigma$ , and minimize the quadratic loss function in (12), respectively. However, with the  $\ell_0$  penalty the problems (10)–(12) are nonconvex and NP-hard, thus, convex relaxation methods are often considered, e.g., replace the  $\ell_0$  penalty by the  $\ell_1$  one. With  $P(\mathbf{x}) = \|\mathbf{x}\|_1$  and  $P_\lambda(\mathbf{x}) = \lambda \|\mathbf{x}\|_1$ , the formulations (10), (11) and (12) are the well-known basis-pursuit (BP) [24], basis-pursuit denoising (BPDN) and LASSO [13], respectively. In this case, the formulations are convex and hence tractable. A large number of algorithms have been developed in the past decade for these  $\ell_1$  minimization problems (see [25]–[29] and the reference therein).

The CS theory has established that if the sensing matrix satisfies some conditions, such as the restricted isometry property (RIP) [17], [30]–[32], the null space property [33], and the incoherence condition [34], the sparse signal can be reconstructed by  $\ell_1$  regularization reliably. However, due to the relaxation, the recovery accuracy is often degraded, e.g., it often introduces extra bias [1], [2] and cannot reconstruct a signal with the least observations [3]. Furthermore, for some applications, the result of the  $\ell_1$ -minimization is not sparse enough and the original signals cannot be recovered. A simple example of such a case has been given in [38] with intuitive explanation.

To address this problem, a number of improved algorithms have been developed via employing the nonconvex  $\ell_q$ -norm penalty instead of the  $\ell_1$  one, i.e.,  $P(\mathbf{x}) = \|\mathbf{x}\|_q$  and  $P_\lambda(\mathbf{x}) = \lambda \|\mathbf{x}\|_q^q$  with  $0 < q < 1$ . For  $0 < q < 1$ ,  $\|\mathbf{x}\|_q^q$  is the  $\ell_q$  quasi-norm defined as  $\|\mathbf{x}\|_q^q = \sum_i |x_i|^q$ . Compared with the  $\ell_1$ -norm, the  $\ell_q$ -norm is a closer approximation of the  $\ell_0$ -norm. It has been shown in [37] that under certain RIP conditions of the sensing matrix,  $\ell_q$ -regularized algorithms require fewer measurements to achieve reliable recovery than  $\ell_1$ -regularized algorithms. Moreover, the sufficient conditions in terms of RIP for  $\ell_q$  regularization are weaker than those for  $\ell_1$  regularization [39], [40]. Meanwhile, it has been shown in [41] that for any given measurement matrix with restricted isometry constant  $\delta_{2k} < 1$ , there exists some  $q \in (0, 1)$  that guarantees exact recovery of signals with support smaller than  $k < m/2$  by  $\ell_q$ -minimization.

Recently,  $\ell_q$ -regularized sparse reconstruction has been extensively studied for CS, e.g., [5], [8], [10], [39]–[66], [200], and extended to structured sparse recovery [201]. As the  $\ell_q$  penalty is nonsmooth and nonconvex, many of these algorithms solve a smoothed (approximated)  $\ell_q$ -minimization problem, e.g., the works [47]–[49] use an approximation of

$\|\mathbf{x}\|_q^q$  as

$$\|\mathbf{x}\|_{q,\varepsilon}^q = \sum_{i=1}^n (x_i^2 + \varepsilon^2)^{\frac{q}{2}} \quad (13)$$

where  $\varepsilon > 0$  is a smoothing parameter. Furthermore, the iteratively reweighted algorithms [38], [44], [46] use the following two penalties (at the  $k + 1$ -th iteration)

$$\begin{aligned} \|\mathbf{x}\|_{q,\varepsilon}^q &= \sum_{i=1}^n (|x_i^k| + \varepsilon)^{q-1} |x_i| \\ \|\mathbf{x}\|_{q,\varepsilon}^q &= \sum_{i=1}^n (|x_i^k|^2 + \varepsilon^2)^{\frac{q}{2}-1} |x_i|^2 \end{aligned}$$

which explicitly relate to the  $\ell_q$ -norm approximation.

These algorithms have been shown to achieve better recovery performance relative to the  $\ell_1$ -regularized algorithms. However, due to the non-convexity of  $\ell_q$ -minimization, many of these algorithms are generally inefficient and impractical for large-scale problems. For example, the StSALq method in [47] requires repetitive computation of matrix inversion of dimension  $n \times n$ , whilst the IRucLq method in [48] solve a set of linear equations using matrix factorization (with matrix dimension of  $n \times n$ ). In comparison, the Lp-RLS method in [49], which uses an efficient conjugate gradient (CG) method in solving a sequence of smoothed subproblems, has better scalable capability. Moreover, the iteratively reweighted method in [50] also involves solving a sequence of weighted  $\ell_1 - \ell_2$  mixed subproblems. Although efficient first-order algorithms can be used to solve the subproblems involved in the methods in [49], [50], both the methods have a double loop which hinders the overall efficiency. While subsequence convergence is guaranteed for the iteratively reweighted methods [48] and [50], there is no such guarantee for StSALq [47] and Lp-RLS [49].

In comparison, the proximal gradient descent (PGD) and alternative direction method of multipliers (ADMM) algorithms for the problem (12) are globally convergent under some mild conditions, while being much more efficient. Specifically, let  $f(\mathbf{x}) = 1/2 \|\mathbf{Ax} - \mathbf{y}\|_2^2$ , consider the following quadratic approximation of the objective in the formulation (12) at iteration  $k + 1$  and at a given point  $\mathbf{x}^k$  as

$$\begin{aligned} Q_{L_f}(\mathbf{x}; \mathbf{x}^k) \\ = f(\mathbf{x}^k) + (\mathbf{x} - \mathbf{x}^k)^T \nabla f(\mathbf{x}^k) + \frac{L_f}{2} \|\mathbf{x} - \mathbf{x}^k\|_2^2 + P_\lambda(\mathbf{x}) \end{aligned} \quad (14)$$

where  $L_f > 0$  is a proximal parameter. Then, minimizing  $Q_{L_f}(\mathbf{x}; \mathbf{x}^k)$  reduces to the proximity operator introduced in section 2 as

$$\mathbf{x}^{k+1} = \text{prox}_{(1/L_f)P_\lambda} \left( \mathbf{x}^k - \frac{1}{L_f} \nabla f(\mathbf{x}^k) \right) \quad (15)$$

which can be computed element-wise via the shrinkage/thresholding function in Table I.

This PGD algorithm fits the frameworks of the forward-backward splitting method [67] and the generalized gradient projection method [68]. Very recently, the convergence properties of this kind of algorithms have been established via

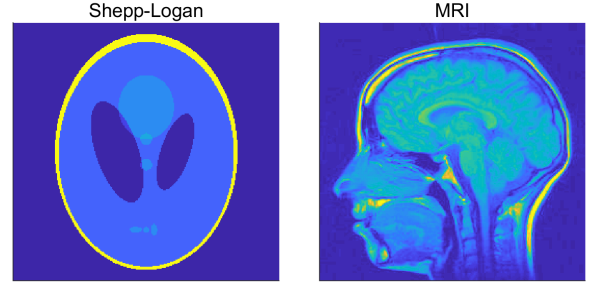


Fig. 3: The two considered  $256 \times 256$  images.

exploiting the Kurdyka-Lojasiewicz (KL) property of the objective function [69]–[71]. Suppose that  $P_\lambda$  is a closed, proper, lower semi-continuous, KL function, if  $L_f > \lambda_{\max}(\mathbf{A}^T \mathbf{A})$ , then, the sequence  $\{\mathbf{x}^k\}$  generated by PGD (15) converges to a stationary point of the problem (12). Further, under some more conditions, the convergence of PGD to a local minimizer can be guaranteed [72], [73].

For ADMM algorithm, using an auxiliary variable

$$\mathbf{z} = \mathbf{Ax} - \mathbf{y}$$

the problem (12) can be equivalently reformulated as

$$\min_{\mathbf{x}} \frac{1}{2} \|\mathbf{z}\|_2^2 + P_\lambda(\mathbf{x}) \quad (16)$$

$$\text{subject to } \mathbf{Ax} - \mathbf{y} - \mathbf{z} = \mathbf{0}.$$

Then, the ADMM algorithm consists of the following steps

$$\begin{aligned} \mathbf{x}^{k+1} &= \arg \min_{\mathbf{x}} \left( P_\lambda(\mathbf{x}) + \frac{\rho}{2} \left\| \mathbf{Ax} - \mathbf{y} - \mathbf{z}^k + \frac{\mathbf{w}^k}{\rho} \right\|_2^2 \right) \\ \mathbf{z}^{k+1} &= \arg \min_{\mathbf{z}} \left( \frac{1}{2} \|\mathbf{z}\|_2^2 + \frac{\rho}{2} \left\| \mathbf{Ax}^{k+1} - \mathbf{y} - \mathbf{z} + \frac{\mathbf{w}^k}{\rho} \right\|_2^2 \right) \\ \mathbf{w}^{k+1} &= \mathbf{w}^k + \rho(\mathbf{Ax}^{k+1} - \mathbf{y} - \mathbf{z}^{k+1}) \end{aligned}$$

where  $\mathbf{w}$  is the dual variable,  $\rho > 0$  is a penalty parameter. As a standard trick, the  $\mathbf{x}$ -subproblem can be solved approximately via linearizing the quadratic term. For a closed, proper, lower semi-continuous and KL  $P_\lambda$ , under some condition of the proximal parameter and  $\rho$  (should be chosen sufficiently large), this proximal ADMM algorithm globally converges to a stationary point of the problem (16) [74]–[75].

For PGD and ADMM, the dominant computational load in each iteration is matrix-vector multiplication with complexity  $O(mn)$ , thus, scale well for high-dimension problems. These two algorithms may be further accelerated by the schemes in [26], [76], however, for a nonconvex  $P_\lambda$ , there is no guarantee of convergence when using such acceleration schemes.

*Example 1 (Image reconstruction).* We evaluate the PGD (15) with hard-,  $\ell_q$ - and soft-thresholding on image reconstruction. The used images are of size  $256 \times 256$  ( $n = 65536$ ), include a synthetic image, ‘‘Shepp-Logan’’ and an MRI image, as shown in Fig. 3. We use the Haar wavelets as the basis for sparse representation of the images, and  $\mathbf{A}$  is a partial DCT matrix with  $m = \text{round}(0.4n)$ . The  $\ell_q$ -thresholding is initialized by the solution of the soft-thresholding. The regularization parameter  $\lambda$  is selected by providing the best performance.

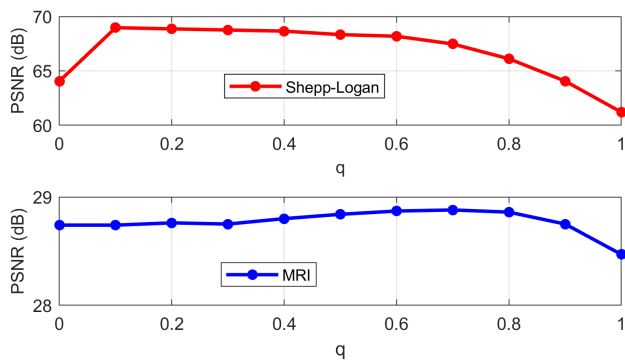


Fig. 4: Recovery performance of  $\ell_q$ -thresholding versus  $q$  in reconstructing two  $256 \times 256$  images.

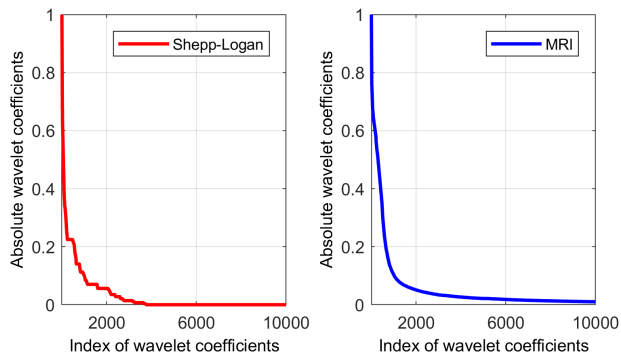


Fig. 5: Sorted and normalized absolute values of the wavelet coefficients of the two images (the first 10000 large values).

Fig. 4 shows the peak-signal noise ratio (PSNR) of recovery of  $\ell_q$ -thresholding for different value of  $q$  with SNR = 50 dB, including the hard- and soft-thresholding as special cases. It can be seen that each method is able to achieve a high PSNR greater than 60 dB in recovering the synthetic image, but degrades significantly in recovering the MRI image (less than 30 dB). This is due the nature that, as shown in Fig. 5, the wavelet coefficients of the synthetic image are truly sparse (approximately 5.7% nonzeros), while that of a real-life image are not strictly sparse but rather approximately follow an exponential decay, which is referred to as compressible. Also due to this, for the two images, the best performance of  $\ell_q$ -thresholding are given by  $q = 0.1$  and  $q = 0.7$ , respectively, as shown in Fig. 4. That is, a relatively small value of  $q$  should be used for strictly sparse signals, while a relatively large value of  $q$  should be used for compressible (non-strictly sparse) signals. Moreover, the hard- and  $\ell_q$ -thresholding can achieve significant performance improvement over the soft-thresholding only for strictly sparse signals.

### B. Sparse Regression and Variable Selection

Nowadays, the analysis of data sets with the number of variables comparable to or even much larger than the sample size arises in many areas, such as genomics, health sciences, economics and machine learning [126]. In this context with high-dimensional data, most traditional variable selection procedures, such as AIC and BIC, becomes infeasible and

impractical due to too expensive computational cost. In this scenario, sparse regression, which can simultaneously select variables and estimate coefficients of variables, has become a very popular topic in the last decade due to its effectiveness in the high-dimensional case [37], [82].

Let  $\mathbf{X} \in \mathbb{R}^{m \times n}$  denote the (deterministic) design matrix,  $\beta \in \mathbb{R}^n$  contains the unknown regression coefficients, and  $\mathbf{y} \in \mathbb{R}^m$  is the response vector. Further, assume that  $\mathbf{y}$  depends on  $\beta$  through a linear combination  $\mathbf{X}\beta$  and the conditional log-likelihood given  $\mathbf{X}$  is  $\mathcal{L}(\beta)$ . In the variable selection problem, the assumption is that majority of the true regression coefficients are zero, and the goal is to identify and estimate the subset model. Under the sparse assumption of the true regression coefficients, a natural method for simultaneously locating and estimating those nonzero coefficients in  $\beta$  is to maximize the following penalized likelihood of the form

$$\max_{\beta} \mathcal{L}(\beta) - P_{\lambda}(\beta) \quad (17)$$

where  $P_{\lambda}$  is a generalized sparsity promotion penalty as defined in section 2. Moreover, there exists a popular alternative which uses the following penalized least-square (LS) formulation

$$\min_{\beta} \frac{1}{2} \|\mathbf{y} - \mathbf{X}\beta\|_2^2 + P_{\lambda}(\beta). \quad (18)$$

The well-known LASSO method is first proposed in [37] for the linear regression problem (18). In the same spirit of LASSO, nonconcave penalty functions, such as SCAD [11] and MCP [12], have been proposed to select significant variables for various parametric models, including linear regression, generalized linear regression and robust linear regression models [83]. Extension to some semiparametric models, such as the Cox model and partially linear models have been considered in [84]–[86]. It has been shown in these works that, with appropriately selected regularization parameters, nonconvex penalized estimators can perform as well as the oracle procedure in selecting the correct subset model and estimating the true nonzero coefficients. Further, even for super-polynomial of sample size, nonconvex penalized likelihood methods possess model selection consistency with oracle properties [87]. In addition, adaptive LASSO has been proposed in [88], which uses an adaptively weighted  $\ell_1$  penalty. While LASSO variable selection can be inconsistent in some scenarios, adaptive LASSO enjoys the oracle properties and also leads to a near-minimax optimal estimator. The oracle property of adaptive LASSO has also been demonstrated in the high-dimensional case [89].

To solve a nonconvex penalized sparse regression problem, locally approximation of the penalty function can be utilized, such as local quadratic approximation (LQA) [11], [90] and local linear approximation (LLA) [91]. While the LQA based methods uses a backward stepwise variable selection procedure, the LLA based method [91] is one-step and hence more favorable.

More recently, benefit from the progress in nonconvex and nonsmooth optimization, direct methods have been widely designed. Specifically, in [92], a coordinate-descent algorithm has been proposed for the linear regression model (18).

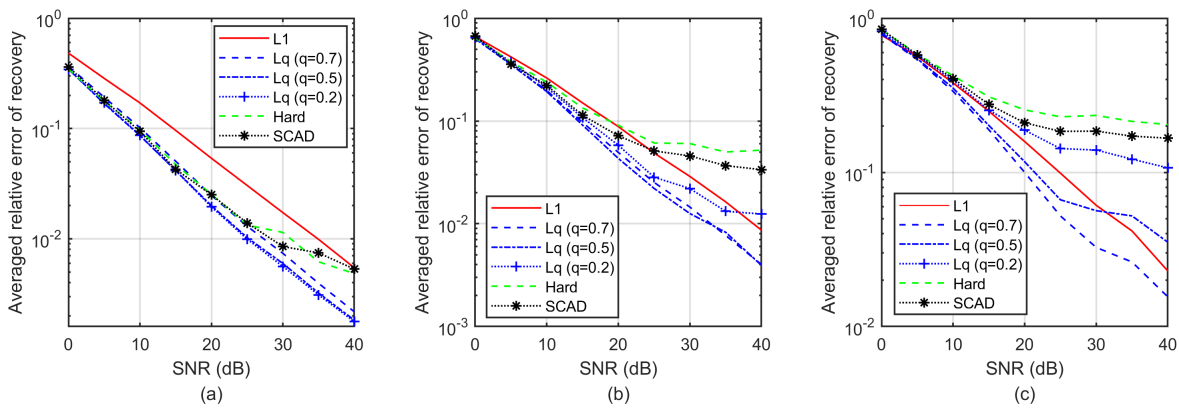


Fig. 6: Recovery performance of the hard-, SCAD-,  $\ell_q$ - and soft-thresholding versus SNR with zero initialization. (a) 2% active coefficients in  $\beta$ , (b) 5% active coefficients in  $\beta$ , (c) 10% active coefficients in  $\beta$ .

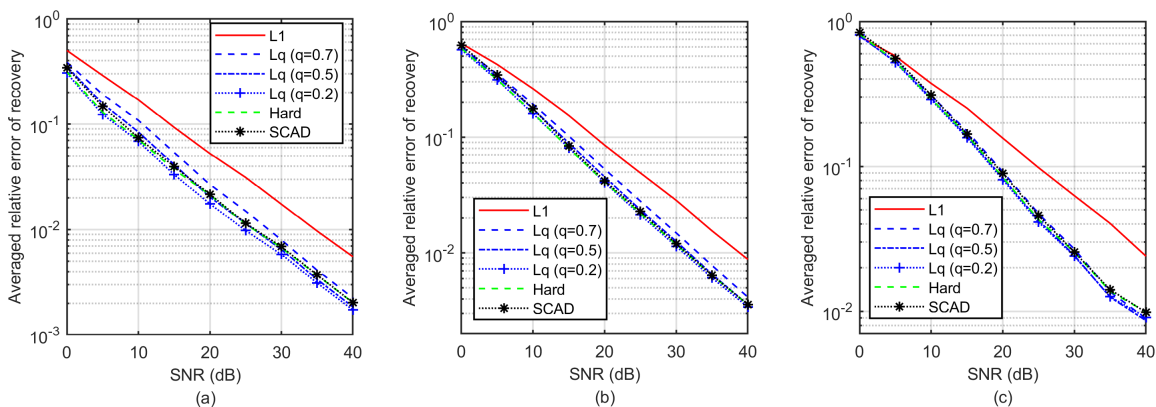


Fig. 7: Recovery performance of the hard-, SCAD-,  $\ell_q$ - and soft-thresholding versus SNR, initialized by the solution of the convex  $\ell_1$  penalty. (a) 2% active coefficients in  $\beta$ , (b) 5% active coefficients in  $\beta$ , (c) 10% active coefficients in  $\beta$ .

By using a continuation process of the parameters of the SCAD or MCP penalty, its convergence to a minimum is guaranteed under certain conditions. Meanwhile, an alternative coordinate-descent algorithm has been presented in [93] with guaranteed convergence to a local minimum. Then, a cyclic descent algorithm employing the  $\ell_0$  penalty for multivariate linear regression has been introduced in [94]. Subsequently, a cyclic descent algorithm for the  $\ell_q$  penalized LS problem has been proposed in [7], [95], whilst a majorization-minimization (MM) algorithm with momentum acceleration for the  $\ell_0$  penalized LS formulation has been developed in [96]. For both the methods in [95] and [96], convergence to a local minimizer is guaranteed under certain conditions. Moreover, as introduced in the last subsection, there exist numerous algorithms, e.g., the PGD and ADMM algorithms, can be applied to the penalized LS formulation (18).

In addition, a global optimization approach has been recently proposed in [206] for concave penalties (e.g., SCAD and MCP) based nonconvex learning via reformulating the nonconvex problems as general quadratic programs. Meanwhile, theoretical analysis on the statistical performance of local minimizer methods using folded concave penalties has been provided in [207]. Moreover, a class of nonconvex penalties termed *trimmed Lasso* has been considered in [208],

which enables exact control of the desired sparsity level.

*Example 2 (Estimation accuracy in different SNR and different sparsity level).* We evaluate the performance of different penalties for the linear sparse regression problem (18) using the PGD (15), with  $\beta \in \mathbb{R}^{256}$  and  $\mathbf{X} \in \mathbb{R}^{100 \times 256}$ . Three sparsity levels, 2%, 5%, and 10% active coefficients of  $\beta$ , are considered. Fig. 6 shows the performance of different penalties versus SNR when the PGD is initialized by zero, while Fig. 7 shows the results when the PGD is initialized by the solution of the convex  $\ell_1$  penalty. It can be seen that the performance of the nonconvex penalties is heavily dependent on the initialization. The advantage of a nonconvex penalty over the  $\ell_1$  penalty is significant when the sparsity level of the coefficient vector is relatively low and/or the SNR is relatively high.

### C. Sparse Signals Separation and Image Inpainting

Sparse signals separation has wide applications, such as source separation, super-resolution and inpainting, interference cancellation, saturation and clipping restoration, and robust sparse recovery in impulsive (sparse) noise. The objective of the problem is to demix the two sparse vectors  $\mathbf{x}_k \in \mathbb{R}^{n_k}$ ,  $k = 1, 2$ , from their mixed linear measurements  $\mathbf{y} \in \mathbb{R}^m$  as

$$\mathbf{y} = \mathbf{A}_1 \mathbf{x}_1 + \mathbf{A}_2 \mathbf{x}_2 \quad (19)$$



where  $\mathbf{A}_k \in \mathbb{R}^{m \times n_k}$  are known deterministic dictionaries. More specifics on the applications involving the model (19) are as follows.

1) *Source separation*: such as the separation of texture in images [97], [98] and the separation of neuronal calcium transients in calcium imaging [99],  $\mathbf{A}_1$  and  $\mathbf{A}_2$  are two dictionaries allowing for sparse representation of the two distinct features,  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are the (sparse or approximately sparse) coefficients describing these features [100]–[102]. 2) *Super-resolution and inpainting*: in the super-resolution and inpainting problem for images, audio, and video signals [103]–[105], only a subset of the desired signal  $\mathbf{y}_0 = \mathbf{A}_1 \mathbf{x}_1$  is available. Given  $\mathbf{y}$ , the objective is to fill in the missing parts in  $\mathbf{y}_0$ , in which case  $\mathbf{A}_2 = \mathbf{I}_m$  and  $\mathbf{x}_2$  stands for the missing parts. 3) *Interference cancellation*: in some audio, video, or communication applications, it is desired to restore a signal corrupted by narrowband interference, such as electric hum [101]. As narrowband interference can be sparsely represented in the frequency domain,  $\mathbf{A}_2$  can be an inverse discrete Fourier transform matrix. 4) *Saturation and clipping restoration*: in many practical systems where the measurements are quantized, nonlinearities in amplifiers may result in signal saturation and causes significant nonlinearity and potentially unbounded errors [101], [106], [107]. In this case,  $\mathbf{y}_0 = \mathbf{A}_1 \mathbf{x}_1$  is the desired signal,  $\mathbf{y}$  is the situated measurement with  $\mathbf{x}_2$  stands for the saturation errors. 5) *Robust sparse recovery in impulsive noise*: impulsive noise is usually approximately sparse and has a distribution with heavy tail. In practical image and video processing applications [108]–[110], impulsive noise may come from the measurement process, or caused by transmission problems, faulty memory locations, buffer overflow and unreliable memory [111]–[114]. In these cases,  $\mathbf{x}_2$  represents the (sparsely) impulsive noise and  $\mathbf{A}_2 = \mathbf{I}_m$ .

Exploiting the sparsity,  $\mathbf{x}_1$  and  $\mathbf{x}_2$  can be reconstructed via the following formulation

$$\begin{aligned} & \min_{\mathbf{x}_1, \mathbf{x}_2} \mu g_1(\mathbf{x}_1) + g_2(\mathbf{x}_2) \\ & \text{subject to } \mathbf{A}_1 \mathbf{x}_1 + \mathbf{A}_2 \mathbf{x}_2 = \mathbf{y} \end{aligned} \quad (20)$$

where  $g_1$  and  $g_2$  are penalties for sparsity promotion,  $\mu > 0$  is a parameter takes the statistic difference between the two components into consideration. As will be shown later, when  $\mathbf{x}_1$  and  $\mathbf{x}_2$  have different sparsity properties, using two different penalties for  $\mathbf{x}_1$  and  $\mathbf{x}_2$  can obtain performance gain over using a same one.

When both  $g_1$  and  $g_2$  are the  $\ell_1$  penalty, i.e.,  $g_1 = g_2 = \|\cdot\|_1$ , (20) reduces to the sparse separation formulation in [102]. Further, when  $g_1 = g_2 = \|\cdot\|_1$  and  $\mu = 1$ , the formulation (20) degenerates to the BP form considered in [100]. Moreover, when  $\mathbf{A}_2 = \mathbf{I}_m$  and  $g_1 = g_2 = \|\cdot\|_1$ , (20) reduces to the  $\ell_1$ -regularized least-absolute problem for robust sparse recovery [115], which has outstanding robustness in the presence of impulsive noise. In addition, the M-estimation method in [218] can also be considered as a special case of (20) with a selected robust loss term.

Naturally, nonconvex penalties can be expected to yield better reconstruction performance over the above convex methods. For example, the  $\ell_0$  penalty has been used in [116]–

[119] to obtain more robust restoration of images corrupted by salt-and-pepper impulsive noise. Very recently, a generalized formulation using  $\ell_q$  penalty has been proposed in [120] with  $g_1 = \|\cdot\|_{q_1}^{q_1}$  and  $g_2 = \|\cdot\|_{q_2}^{q_2}$ ,  $0 \leq q_1, q_2 < 1$ . The formulation (20) can be directly solved by a standard two-block ADMM procedure [29], but it often fails to converge in the nonconvex case [120]. To develop convergent algorithms, a recent work [120] proposed to solve a quadratic approximation of (20) and developed two first-order algorithms based on the proximal block coordinate descent (BCD) and ADMM frameworks. The proximal BCD method consists of the following two update steps

$$\begin{aligned} \mathbf{x}_1^{k+1} &= \text{prox}_{(\beta\mu/\eta_1)g_1} \left\{ \mathbf{x}_1^k - \frac{2}{\eta_1} \mathbf{A}_1^T (\mathbf{A}_1 \mathbf{x}_1^k + \mathbf{A}_2 \mathbf{x}_2^k - \mathbf{y}) \right\} \\ \mathbf{x}_2^{k+1} &= \text{prox}_{(\beta/\eta_2)g_2} \left\{ \mathbf{x}_2^k - \frac{2}{\eta_2} \mathbf{A}_2^T (\mathbf{A}_1 \mathbf{x}_1^{k+1} + \mathbf{A}_2 \mathbf{x}_2^k - \mathbf{y}) \right\} \end{aligned} \quad (21)$$

where  $\eta_1 > 0$  and  $\eta_2 > 0$  are proximal parameters,  $\beta > 0$  is quadratic approximation parameter. Suppose that  $g_1$  and  $g_2$  are closed, proper, lower semi-continuous, KL functions, if  $\eta_1 > 2\text{eig}_{\max}(\mathbf{A}_1^T \mathbf{A}_1)$  and  $\eta_2 > 2\text{eig}_{\max}(\mathbf{A}_2^T \mathbf{A}_2)$ , the algorithm updated via (21) and (22) is a descent algorithm and the generated sequence  $\{(\mathbf{x}_1^k, \mathbf{x}_2^k)\}$  converges to a stationary point of the approximated problem.

In many applications such as super-resolution and inpainting for color images with 3 (RGB) channels, multichannel joint recovery is more favorable than independently recovery of each channel, as the former can exploit the feature correlation among different channels. In the multitask case, the linear measurements  $\mathbf{Y} \in \mathbb{R}^{m \times L}$  of  $L$  channels can be expressed as

$$\mathbf{Y} = \mathbf{A}_1 \mathbf{X}_1 + \mathbf{A}_2 \mathbf{X}_2 \quad (23)$$

where  $\mathbf{X}_k \in \mathbb{R}^{n_k \times L}$ ,  $k = 1, 2$ , are the sparse features in the two components. To exploit the joint sparsity among the  $L$  channels, joint sparsity penalties can be used, which is defined as

$$\begin{aligned} \tilde{P}_\lambda(\mathbf{X}) &= \sum_i P_\lambda(\|\mathbf{X}[i, :]\|_2) \\ &= \sum_i P_\lambda \left( \left( \sum_j \mathbf{X}^2[i, j] \right)^{1/2} \right). \end{aligned}$$

Using such a penalty, e.g.,  $G_1$  for  $\mathbf{X}_1$  and  $G_2$  for  $\mathbf{X}_2$ , the proximal BCD algorithm for the multitask problem consists of the following two steps [120]

$$\begin{aligned} \mathbf{X}_1^{k+1} &= \arg \min_{\mathbf{X}_1} \mu G_1(\mathbf{X}_1) + \\ & \frac{\eta_3}{2\beta} \left\| \mathbf{X}_1 - \mathbf{X}_1^k + \frac{2}{\eta_3} \mathbf{A}_1^T (\mathbf{A}_1 \mathbf{X}_1^k + \mathbf{A}_2 \mathbf{X}_2^k - \mathbf{Y}) \right\|_{\text{F}}^2 \\ \mathbf{X}_2^{k+1} &= \arg \min_{\mathbf{X}_2} G_2(\mathbf{X}_2) + \\ & \frac{\eta_4}{2\beta} \left\| \mathbf{X}_2 - \mathbf{X}_2^k + \frac{2}{\eta_4} \mathbf{A}_2^T (\mathbf{A}_1 \mathbf{X}_1^{k+1} + \mathbf{A}_2 \mathbf{X}_2^k - \mathbf{Y}) \right\|_{\text{F}}^2 \end{aligned} \quad (25)$$

where  $\eta_3 > 0$  and  $\eta_4 > 0$  are proximal parameters. These two subproblems can be solved row-wise as (5). Suppose that  $G_1$

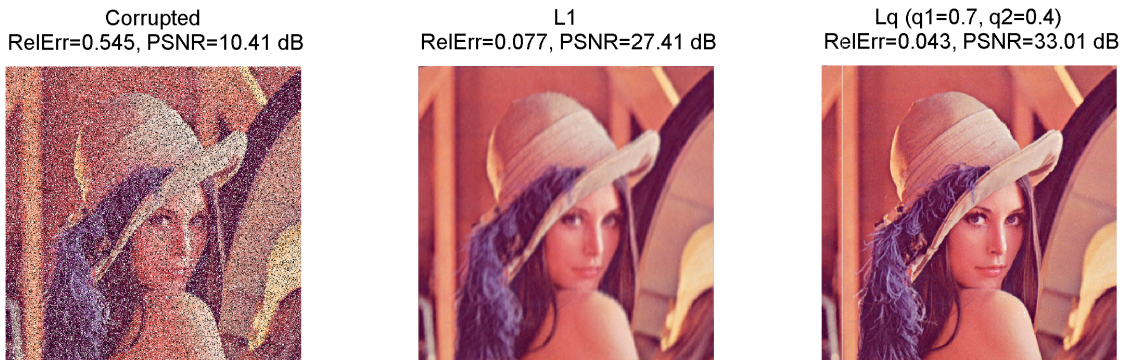
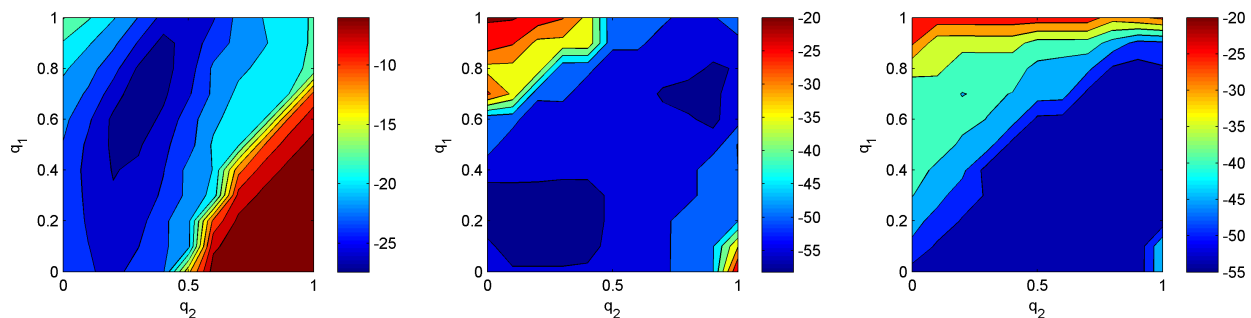
(a) Inpainting of a  $318 \times 500$  image corrupted by overwritten text(b) Inpainting of a  $512 \times 512$  image corrupted by salt-and-pepper noise (30% of the pixels are corrupted)Fig. 8: Restoration performance of the  $\ell_1$  and  $\ell_q$  regularization in color images inpainting.

Fig. 9: Restoration performance versus  $q_1$  and  $q_2$  in terms of RelErr in dB defined as  $20\log_{10}(\|\hat{\mathbf{x}}_1 - \mathbf{x}_1\|_2 / \|\mathbf{x}_1\|_2)$ . *Left*: Case 1: restoration of the image in Fig. 8 (a),  $\mathbf{X}_1$  contains the DCT coefficients of the image and  $\mathbf{X}_2$  represents the overwritten text. *Middle*: Case 2: both  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are strictly sparse. *Right*: Case 3:  $\mathbf{x}_1$  is strictly sparse and  $\mathbf{x}_2$  is  $\alpha$ -stable (S $\alpha$ S) noise (non-strictly sparse).

and  $G_2$  are closed, proper, lower semi-continuous, KL functions, if  $\eta_3 > 2\text{eig}_{\max}(\mathbf{A}_1^T \mathbf{A}_1)$ , and  $\eta_4 > 2\text{eig}_{\max}(\mathbf{A}_2^T \mathbf{A}_2)$ , the algorithm updated via (24) and (25) is a descent algorithm and the generated sequence  $\{(\mathbf{X}_1^k, \mathbf{X}_2^k)\}$  converges to a stationary point of the approximated problem.

*Example 3 (Color image inpainting)*. We compare the performance of the  $\ell_q$  and  $\ell_1$  regularization on color images inpainting using the BCD method (24) and (25). The task is to restore the original image from text overwriting or salt-and-pepper noise corruption. In this case,  $\mathbf{A}_2 = \mathbf{I}$  and  $\mathbf{X}_2$  represents the sparse corruption in the three (RGB) channels. We select  $\mathbf{A}_1$  as an inverse discrete cosine transformation (IDCT) matrix, and, accordingly,  $\mathbf{X}_1$  contains the DCT coefficients of the image. As shown in Fig. 8,  $\ell_q$  regularization significantly outperforms the  $\ell_1$  one, e.g., the improvement is more than 9 dB in the overwritten text case and more than 5 dB in the

salt-and-pepper noise case.

*Example 4 (Evaluation on different penalties)*. An important problem in practice is how to select a nonconvex penalty. This example sheds some light on this by three application cases. Case 1: the color image inpainting experiment in Fig. 8 (a). Case 2:  $L = 1$ ,  $\mathbf{A}_1 \in \mathbb{R}^{128 \times 128}$  and  $\mathbf{A}_2 \in \mathbb{R}^{128 \times 128}$  are respectively DCT and Gaussian matrices,  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are strictly sparse vectors with sparsity  $K = 25$ . Case 3 (robust sparse recovery in impulsive noise):  $L = 1$ ,  $\mathbf{A}_1 \in \mathbb{R}^{100 \times 256}$  is a Gaussian matrix,  $\mathbf{A}_2 = \mathbf{I}_{100}$ ,  $\mathbf{x}_1$  is a strictly sparse vector with  $K = 20$ , and  $\mathbf{x}_2$  is symmetric  $\alpha$ -stable (S $\alpha$ S) noise with characteristic exponent  $\alpha = 1$  and dispersion  $\gamma = 10^{-3}$ .

As the  $\ell_q$  penalty has a flexible parametric form that adapts to different thresholding functions, we evaluate the effect of the values of  $q_1$  and  $q_2$  on the recovery performance in the three cases. Fig. 9 indicates that for strictly sparse signals, a

relatively small value of  $q$  is favorable, while for non-strictly sparse signals, a relatively large value of  $q$  is favorable. For example, in case 1,  $\mathbf{X}_1$  (DCT coefficients of the image) is non-strictly sparse whilst  $\mathbf{X}_2$  is strictly sparse, thus, a relatively large value of  $q_1$  and a relatively small value of  $q_2$  should be used. In case 3,  $\mathbf{x}_1$  is strictly sparse whilst  $\mathbf{x}_2$  (S $\alpha$ S noise) is non-strictly sparse, thus, a relatively small value of  $q_1$  and a relatively large value of  $q_2$  would result in good performance.

#### D. Sparse PCA

PCA is a useful tool for dimensionality reduction and feature extraction, which has been applied in virtually all areas of science and engineering, such as signal processing, machine learning, statistics, biology, medicine, finance, neurocomputing, and computer networks, to name just a few. In many real applications, sparse loading vectors are desired in PCA to enhance the interpretability of the principle components (PCs). For instance, in gene analysis, the sparsity of PCs can facilitate the understanding of the relation between the whole gene microarrays and certain genes; in financial analysis, the sparsity of PCs implies fewer assets in a portfolio thus is helpful to reducing the trading costs. In these scenarios, it is desirable not only to achieve the dimensionality reduction but also to reduce the number of explicitly used variables.

To achieve sparse PCA, ad hoc methods were firstly designed via thresholding the PC loadings [124], [125]. Then, more popular methods incorporating a sparsity inducing mechanism into the traditional PCA formulation have been developed [126]. Let  $\mathbf{X} \in \mathbb{R}^{d \times n}$  be a centered data matrix with  $d$  and  $n$  respectively be the dimensionality and the size of the data. Employing a sparsity constraint in the traditional PCA model, the sparse PCA problem to find an  $m$ -dimensional subspace can be formulated as

$$\mathbf{w}_i^* = \arg \max_{\mathbf{w}_i \in \mathbb{R}^d} \|\mathbf{w}_i^T \mathbf{X}\|_2^2 \quad (26)$$

$$\text{subject to } \mathbf{w}_i^T \mathbf{w}_j = \delta_{i,j}, P(\mathbf{w}_i) \leq k$$

for  $i, j = 1, \dots, m$ .  $P$  is a sparsity inducing penalty. Instead of solving (26), a widely used greedy strategy is to find the approximate solution of (26) by sequentially solving the following single PC problem

$$\mathbf{w}^* = \arg \max_{\mathbf{w} \in \mathbb{R}^d} \|\mathbf{w}^T \mathbf{X}\|_2^2 \quad (27)$$

$$\text{subject to } \|\mathbf{w}\|_2 \leq 1, P(\mathbf{w}) \leq k.$$

An alternative of the sparsity constrained formulation (27) is the sparsity penalized formulation as follows

$$\mathbf{w}^* = \arg \max_{\mathbf{w} \in \mathbb{R}^d} \|\mathbf{w}^T \mathbf{X}\|_2^2 - P_\lambda(\mathbf{w}) \quad (28)$$

$$\text{subject to } \|\mathbf{w}\|_2 \leq 1$$

where  $\lambda > 0$  is a regularization parameter.

There exist numerous algorithms employing different penalty functions for the formulations (26)–(28) and their variants. In [126], the  $\ell_1$  constraint is used for the formulation (26). In [127], an elastic-net regression based algorithm has been proposed. Then, a semidefinite relaxation method has been developed in [128], [129] for the formulation (27) with  $\ell_0$

constraint. Meanwhile, in [130], a regularized low rank matrix approximation method has been designed with the consideration of different penalties, i.e.,  $\ell_1$ ,  $\ell_0$  and SCAD. In [131], with  $\ell_1$  and  $\ell_0$  penalties, reformulations of (28) and its block variant have been solved via gradient algorithms. Generalized version of (28) with  $\ell_0$  penalty has been considered in [132]. Moreover, an alternative discrete spectral formulation of  $\ell_0$  constrained (27) and an effective greedy approach have been presented in [133]. In [134], unified alternating maximization framework for  $\ell_0$  and  $\ell_1$  constrained or penalized sparse PCA problems (using  $\ell_1$  or  $\ell_2$  loss) has been proposed.

More recently, robust sparse PCA using  $\ell_1$  loss and  $\ell_q$  penalty has been considered in [135]. Meanwhile, an ADMM based distributed sparse PCA algorithm has been proposed in [136] which covers the  $\ell_1$ , log sum and MCP penalties. Moreover, Shatten- $q$  penalty has been used for structured sparse PCA in [137]. In addition, there also exist several other methods for  $\ell_0$  constrained or penalized sparse PCA problems, e.g., [138]–[141].

## IV. SPARSE MATRIX RECOVERY

This section reviews the nonconvex regularization based sparse matrix recovery problems, mainly on large covariance matrix and inverse covariance matrix estimation, which are two fundamental problems in modern multivariate analysis [142]. Nowadays, the advance of information technology makes massive high-dimensional data widely available for scientific discovery. In this context, effective statistical analysis for high-dimensional data is becoming increasingly important. In many applications involving statistical analysis of high-dimensional data, estimating large covariance or inverse covariance matrices is necessary for effective dimensionality reduction or discriminant analysis. Such applications arise in economics and finance, bioinformatics, social networks, smart grid, climate studies, and health sciences [142]–[144]. In the high-dimensional setting, the dimensionality is often comparable to (or even larger than) the sample size. In these cases, the sample covariance matrix estimator has a poor performance [145], and intrinsic structures such as sparsity can be exploited to improve the estimation of covariance and inverse covariance matrices [142], [146]–[148], [155].

### A. Large Sparse Covariance Matrix Estimation

Consider a vector  $\mathbf{x} \in \mathbb{R}^d$  with covariance  $\Sigma = E\{\mathbf{x}\mathbf{x}^T\}$ , the objective is to estimate its covariance from  $n$  observations  $\mathbf{x}_1, \dots, \mathbf{x}_n$ . Usually, compared to estimate  $\Sigma$  directly, it is more favorable to estimate the correlation matrix first,  $\mathbf{R} = \text{diag}(\Sigma)^{-1/2} \Sigma \text{diag}(\Sigma)^{-1/2}$ . Then, given the estimated correlation matrix  $\hat{\mathbf{R}}$ , the corresponding estimation of the covariance matrix is  $\hat{\Sigma} = \text{diag}(\mathbf{S})^{1/2} \hat{\mathbf{R}} \text{diag}(\mathbf{S})^{1/2}$ , where  $\mathbf{S}$  is the sample covariance matrix. That is because the correlation matrix has the same sparsity pattern of the covariance matrix but with all the diagonal elements known to be one, thus, it can be estimated more accurately than the covariance matrix [149]–[151].

Given the sample correlation matrix  $\mathbf{S}$ , the generalized thresholding estimator [148] solves the following problem

$$\min_{\mathbf{R}} \frac{1}{2} \|\mathbf{R} - \mathbf{S}\|_F^2 + \sum_{i \neq j} P_\lambda(\mathbf{R}_{ij}) \quad (29)$$

with  $P_\lambda$  be generalized penalty function for sparsity promotion as introduced in section 2. Note that, the diagonal elements are not penalized since the diagonal elements of a correlation (also covariance) matrix are always positive. The solution to (29) is a thresholding of the sample correlation matrix  $\mathbf{S}$ , which can be efficiently computed as shown in section 2.

The thresholding estimator (29) have good theoretical properties. It is consistent over a large class of (approximately) sparse covariance matrices [148]. However, in practical finite sample applications, such an estimator is not always positive-definite although it converges to a positive-definite limit in the asymptotic setting [149], [151]. To simultaneously achieve sparsity and positive-definiteness, positive-definite constraint can be added into (29) as [152]

$$\min_{\mathbf{R}} \frac{1}{2} \|\mathbf{R} - \mathbf{S}\|_F^2 + \sum_{i \neq j} P_\lambda(\mathbf{R}_{ij}) \quad (30)$$

subject to  $\text{diag}(\mathbf{R}) = \mathbf{I}_d$  and  $\mathbf{R} \geq \varepsilon \mathbf{I}_d$

where  $\varepsilon > 0$  is the lower bound for the minimum eigenvalue. An alternating minimization algorithm has been proposed for (30) in [152], which is guaranteed to be globally convergent for a generalized nonconvex penalty  $P_\lambda$  (when it is a closed, proper, lower semi-continuous, KL function).

Suppose the ‘‘approximately sparse’’ covariance matrix satisfies

$$\mathcal{U}(\kappa, p, M_d, \xi) := \left\{ \Sigma : \max_i \Sigma_{ii} \leq \kappa, \tilde{\Sigma}^{-1} \Sigma \tilde{\Sigma}^{-1} \in \mathcal{M}(p, M_d, \xi) \right\}$$

where  $\tilde{\Sigma} = \text{diag}(\sqrt{\Sigma_{11}}, \dots, \sqrt{\Sigma_{dd}})$  and

$$\mathcal{M}(p, M_d, \xi) := \left\{ \mathbf{R} : \max_i \sum_{j \neq i} |\mathbf{R}_{ij}|^p \leq M_d, \mathbf{R}_{ii} = 1, \text{eig}_{\min}(\mathbf{R}) = \xi \right\}$$

with  $0 \leq p < 1$ . It has been shown in [152] that, for a generalized nonconvex penalty which satisfies (3) and for large enough  $n$ , the positive-definite estimator (30) satisfies

$$\|\hat{\Sigma} - \Sigma\|_2 = O_P \left( M_d \left( \frac{\log d}{n} \right)^{\frac{1-p}{2}} \right)$$

which achieves the minimax lower bound over the class  $\mathcal{U}(\kappa, p, M_d, \xi)$  under the Gaussian model [154], as the estimator (29). The estimators (29) and (30) give the same estimation with overwhelming probability in the asymptotic case. Thus, for covariance matrices in  $\mathcal{U}(\kappa, p, M_d, \xi)$ , both the estimators (29) and (30) are asymptotically rate optimal under the flexible elliptical model.

*Example 5 (Estimation on simulated datasets).* We evaluate the alternating minimization algorithm [152] solving (30) ( $\varepsilon = 10^{-3}$ ) with different penalties in terms of relative error of estimation under spectral norm. Each provided result is an

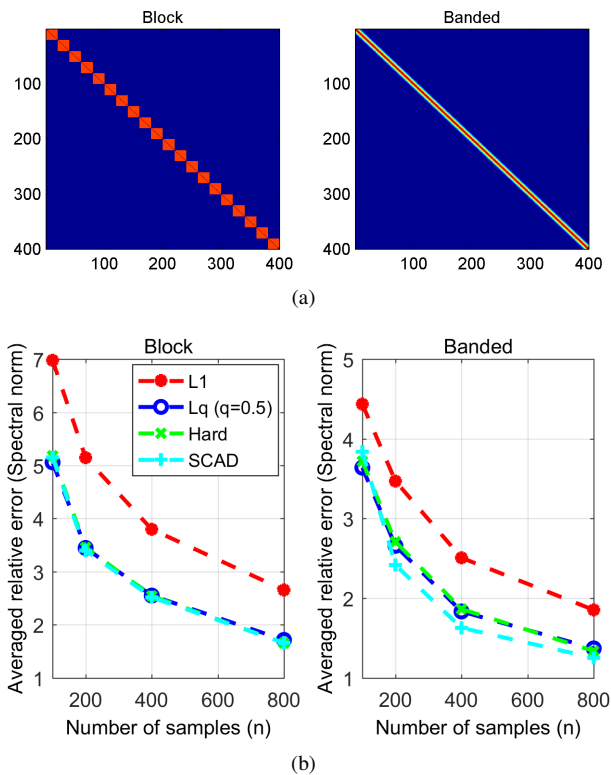


Fig. 10: (a) Heat maps of the two simulated covariance matrices. (b) Performance of different penalties in estimating the two simulated covariance matrices.

average over 100 independent runs. Two typical sparse covariance matrix models, block and banded of size  $d = 400$ , are considered. Fig. 10 demonstrates that, the nonconvex SCAD, hard- and  $\ell_q$ -penalties can yield considerable performance gain over the  $\ell_1$  penalty.

*Example 6 (Gene clustering).* We further consider a gene clustering example using a gene expression dataset from a small round blue-cell tumors (SRBCTs) microarray experiment [153]. This dataset contains 88 SRBCT tissue samples, and 2308 gene expression values are recorded for each sample. We use the 63 labeled calibration samples and pick up the top 40 and bottom 160 genes based on their F-statistic. Accordingly, the top 40 genes are informative while the bottom 160 genes are non-informative. Fig. 11 shows the heat maps of the absolute values of estimated correlations by the compared penalties for the selected 200 genes. Each heat map is ordered by group-average-agglomerative clustering based on the estimated correlation. It can be seen that, compared with the  $\ell_1$  penalty, each nonconvex penalty can give cleaner and more informative estimates of the sparsity pattern.

## B. Large Sparse Inverse Covariance Matrix Estimation

Large inverse covariance matrix estimation is another fundamental problem in modern multivariate analysis. While the covariance matrix  $\Sigma = E\{\mathbf{x}\mathbf{x}^T\} \in \mathbb{R}^{d \times d}$  only captures the marginal correlations among the variables in  $\mathbf{x}$ , the inverse covariance matrix  $\Theta = \Sigma^{-1}$  captures the conditional correlations among these variables and is closely related to undirected

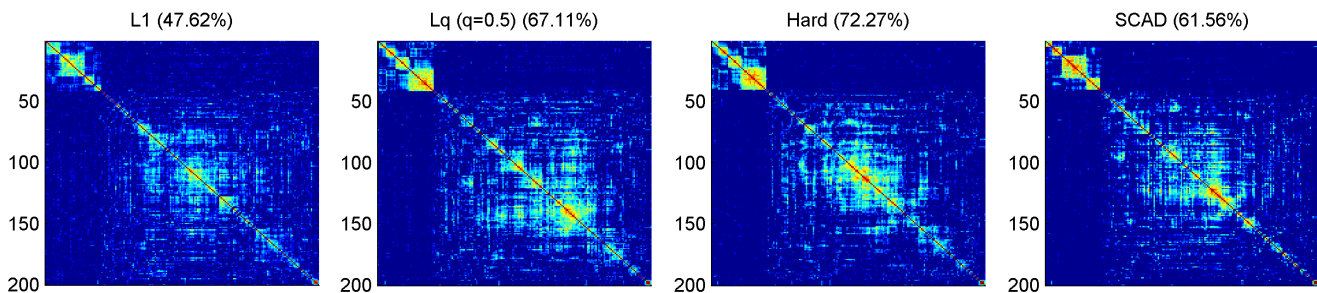


Fig. 11: Heat maps of the absolute values of estimated correlations for the selected 200 genes (the percentage of the entries with absolute values less than  $10^{-5}$  is given in parentheses).

graphs under a Gaussian model. Following the parsimony principle, it is desirable to choose the simplest model (i.e., the sparsest graph) that adequately explains the data. To achieve this, sparsity promotion can be used to improve the interpretability of the model and prevent overfitting. In addition to the graphical model problem, the interest in sparse inverse covariance estimation also arises in many other areas such as high dimensional discriminant analysis, portfolio allocation, and principal component analysis [142], [156]–[158].

One of the most popular approaches of estimating sparse inverse covariance matrices is through the penalized maximum likelihood. Specifically, assume that  $\mathbf{x}_1, \dots, \mathbf{x}_n$  are independently and identically Gaussian distributed with zero-mean and covariance  $\Sigma$ , the negative log-likelihood function is  $\mathcal{L}(\Theta) = \text{tr}(\mathbf{S}\Theta) - \log|\Theta|$ , where  $\mathbf{S}$  is the sample covariance. Then, the sparsity penalized likelihood estimator is given by

$$\min_{\Theta} \text{tr}(\mathbf{S}\Theta) - \log|\Theta| + \sum_{i \neq j} P_{\lambda}(\Theta_{ij}). \quad (31)$$

The positive-definiteness of this estimator is naturally ensured thanks to the logarithmic barrier term.

As the loss term (the log-likelihood  $\mathcal{L}(\Theta)$ ) in (31) is convex, when  $P_{\lambda}$  is the  $\ell_1$  penalty, the formulation (31) is convex and efficient ADMM algorithm can be applied with guaranteed convergence [29], [159]. However, with a nonconvex  $P_{\lambda}$ , there is no guarantee of the convergence for such an algorithm, since the gradient of the loss is not Lipschitz continuous.

For a folded concave penalty, such as the SCAD,  $\ell_q$ , or MCP penalty, a strategy is to use local quadratic approximation (LQA) or local linear approximation (LLA) of the penalty [160]–[162]. In such a manner, a concave penalized problem is converted into a sequence of reweighted  $\ell_1$  penalized problems. Very recently, direct methods have been developed. Specifically, a block cyclic descent (CD) algorithm has been proposed for the formulation (31) with  $\ell_q$  penalty in [163], [164]. Subsequently, a coordinate-by-coordinate CD algorithm with guaranteed convergence (to a local minimizer) has been proposed in [165] for the  $\ell_0$  penalized log-likelihood formulation

$$\min_{\Theta} \text{tr}(\mathbf{S}\Theta) - \log|\Theta| + \lambda \|\Theta\|_0. \quad (32)$$

Meanwhile, a highly efficient block iterative method for (32) has been developed in [166], which is capable of handling large-scale problems (e.g.,  $d = 10^4$ ). Moreover, extension of the  $\ell_0$  penalized formulation (32) for time-varying sparse

inverse covariance estimation for the application of tracking dynamic functional magnetic resonance imaging (fMRI) brain networks has been considered in [167].

With a general nonconvex penalty function, the rate of convergence for estimating sparse inverse covariance matrices based on penalized likelihood has been established in [162], which under the Frobenius norm is of order  $\sqrt{s \cdot \log d/n}$ , where  $s$  and  $d$  are respectively the number of nonzero elements and size of the matrix, and  $n$  is the sample size. Further, while the  $\ell_1$ -penalty can achieve simultaneously the optimal rate of convergence and sparsistency when the number of nonzero off-diagonal entries is no larger than  $O(d)$ , there is no need of such a restriction for an unbiased nonconvex penalty, e.g., the SCAD or hard-thresholding penalty.

## V. LOW-RANK MATRIX RECOVERY

This section reviews nonconvex regularization based low-rank recovery problems, mainly on matrix completion and robust PCA, which are two hot topics in the past few years. Matrix completion aims to recover a low-rank matrix from partially observed entries, while robust PCA aims to decompose a low-rank matrix from sparse corruption.

### A. Matrix Completion

Matrix completion problems deal with the recovery of a matrix from its partially observed (may be noisy) entries [168]–[171], which has found applications in various fields such as recommender systems [172], computer vision [173] and system identification [174], to name a few. The goal of matrix completion is to recover a matrix  $\mathbf{X} \in \mathbb{R}^{m \times n}$  from its partially known entries  $\{\mathbf{M}_{i,j} = \mathbf{X}_{i,j} : (i,j) \in \Omega\}$ , where  $\Omega \subset [1, \dots, m] \times [1, \dots, n]$  is a random subset. This can be achieved via exploiting the low-rankness of  $\mathbf{X}$  by the following formulation

$$\begin{aligned} & \min_{\mathbf{X}} \bar{P}(\mathbf{X}) \\ & \text{subject to } \mathcal{P}_{\Omega}(\mathbf{X}) = \mathcal{P}_{\Omega}(\mathbf{M}) \end{aligned} \quad (33)$$

where  $\bar{P}$  is a penalty for low-rank promotion as introduced in section 2.3,  $\mathcal{P}_{\Omega} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$  denotes projection onto  $\Omega$ . In the case of  $\bar{P}(\mathbf{X}) = \|\mathbf{X}\|_0 = \text{rank}(\mathbf{X})$ , (33) is a nonconvex rank minimization problem. A popular convex relaxation method is to approximate the rank function using the nuclear norm, i.e.,  $\bar{P}(\mathbf{X}) = \|\mathbf{X}\|_*$ . It has been shown in [169], [171]

that under certain conditions, a matrix  $\mathbf{M} \in \mathbb{R}^{m \times n}$  of rank  $r \leq \min\{m, n\}$  can be exactly recovered from a small of its entries by using the nuclear norm.

In the noisy case,  $\{\mathbf{M}_{i,j} = \mathbf{X}_{i,j} + \epsilon_{i,j} : (i, j) \in \Omega\}$  where  $\epsilon_{i,j}$  is i.i.d. Gaussian noise, a robust variant of (33) is more favorable as

$$\begin{aligned} & \min_{\mathbf{X}} \bar{P}(\mathbf{X}) \\ & \text{subject to } \|\mathcal{P}_{\Omega}(\mathbf{X}) - \mathcal{P}_{\Omega}(\mathbf{M})\|_{\text{F}} \leq \sigma \end{aligned} \quad (34)$$

where  $\sigma > 0$  is the noise tolerance. This constrained formulation (34) can be converted into an unconstrained formulation as

$$\min_{\mathbf{X}} \frac{1}{2} \|\mathcal{P}_{\Omega}(\mathbf{X}) - \mathcal{P}_{\Omega}(\mathbf{M})\|_{\text{F}}^2 + \bar{P}_{\lambda}(\mathbf{X}) \quad (35)$$

where  $\lambda > 0$  is the a regularization parameter.

The superiority of a nonconvex regularization (e.g., the Schatten- $q$  norm) over the nuclear norm has been widely demonstrated in [6], [48], [175]–[179]. In [6], [175], a proximal descent (PD) algorithm has been proposed, which can be viewed as a special case of the PD algorithm (37) with  $\bar{P}_{\lambda}$  be the Schatten- $q$  norm. In [48], an iteratively reweighted algorithm for the unconstrained formulation (35) has been designed via smoothing the Schatten- $q$  norm, which involves solving a sequence of linear equations. It has been shown in [177] that, for reliable recovery of low-rank matrices from compressed measurements, the sufficient condition of Schatten- $q$  norm regularization is weaker than that of nuclear norm regularization. Moreover, robust matrix completion using Schatten- $q$  regularization has been considered in [179]. Meanwhile, in [180], a truncated nuclear norm has been designed to gain performance improvement over the nuclear norm. More recently, the MCP penalty has been used and an ADMM algorithm has been developed in [181], whilst an approximated  $\ell_0$  penalty has been considered in [182].

Among the above nonconvex methods, subsequence convergence is proved for the methods [6], [48], [175], [176], [181]. In fact, based on the recent convergence results for nonconvex and nonsmooth optimization [69]–[71], global convergence of the methods [6], [175], [181] can be guaranteed under some mild conditions.

Following the recent results in [69]–[71], efficient and globally convergent first-order algorithms for the unconstrained problem (35) can be developed for a generalized nonconvex  $\bar{P}_{\lambda}$ . Using PD method for example, let  $F(\mathbf{X}) = 1/2 \|\mathcal{P}_{\Omega}(\mathbf{X}) - \mathcal{P}_{\Omega}(\mathbf{M})\|_{\text{F}}^2$ , consider the following approximation of the objective in (35) at iteration  $k + 1$  and at a given point  $\mathbf{X}^k$  as

$$\begin{aligned} Q_{L_F}(\mathbf{X}; \mathbf{X}^k) &= F(\mathbf{X}^k) + \langle \mathbf{X} - \mathbf{X}^k, \nabla F(\mathbf{X}^k) \rangle \\ &+ \frac{L_F}{2} \|\mathbf{X} - \mathbf{X}^k\|_{\text{F}}^2 + \bar{P}_{\lambda}(\mathbf{X}) \end{aligned} \quad (36)$$

where  $\nabla F(\mathbf{X}^k) = \mathcal{P}_{\Omega}(\mathbf{X}^k) - \mathcal{P}_{\Omega}(\mathbf{M})$  and  $L_F > 0$  is a proximal parameter. Then, with the definition of  $\bar{P}_{\lambda}$  in (6), minimizing  $Q_{L_F}(\mathbf{X}; \mathbf{X}^k)$  reduces to the generalized proximity operator in Theorem 2 (section 2.3) as

$$\mathbf{X}^{k+1} = \text{prox}_{(1/L_F)\bar{P}_{\lambda}} \left( \mathbf{X}^k - \frac{1}{L_F} \nabla F(\mathbf{X}^k) \right) \quad (37)$$

which can be computed as (8).

This iterative singular-value thresholding algorithm also fits into the framework of majorization-minimization (MM) method. From the results in [69]–[71], a sufficient condition of convergence for this algorithm is given as follows. Suppose that  $\bar{P}_{\lambda}$  is a closed, proper, lower semi-continuous, KL function, if  $L_F > 1$ , then, the sequence  $\{\mathbf{X}^k\}$  generated by (37) converges to a stationary point of the problem (35). Meanwhile, the efficient ADMM algorithm also can be applied to the problem (35) with convergence guarantee under some mild conditions [71].

For large-scale matrix completion problems, matrix factorization is a popular approach since a matrix factorization based optimization formulation, even with huge size, can be solved very efficiently by standard optimization algorithms [183]. Very recently, scalable equivalent formulations of Schatten- $q$  quasi-norm have been proposed in [184]–[186], which facilitate the design of highly efficient algorithms that only need to update two much smaller factor matrices.

*Example 7 (Low-rank image recovery).* We evaluate the algorithm (37) with hard-,  $\ell_q$ - and soft-thresholding on grayscale image recovery on the “Boat” image of size  $512 \times 512$ , where 50% of pixels are observed in the presence of Gaussian noise with SNR = 40 dB, as shown in Fig. 12. Two cases are considered. 1) *Non-strictly low-rank*: the original image is used, which is not strictly low-rank with the singular values approximately following an exponential decay, as shown in Fig. 13. 2) *Strictly low-rank*: the singular values of the original image are truncated and only the 50 largest values are retained, which results in a new image which is strictly low-rank used for performance evaluation. The regularization parameter  $\lambda$  is selected by providing the best performance. In implementing the algorithm with hard- and  $\ell_q$ -thresholding, we first run it with soft-thresholding to obtain an initialization.

Fig. 12 shows the recovered images along with the relative error of recovery (RelErr). Fig. 14 shows the performance of the  $\ell_q$ -thresholding versus  $q$  in the two cases. For the two cases, the best performance of  $\ell_q$ -thresholding are given by  $q = 0.8$  and  $q = 0.2$ , respectively. The results indicate that a relatively small value of  $q$  should be used in the strictly low-rank case, while a relatively large value of  $q$  should be used in the non-strictly low-rank case. The  $\ell_0$  and  $\ell_q$  penalties can achieve significant performance improvement over the  $\ell_1$  one only in the strictly low-rank case.

## B. Robust PCA

The objective of robust PCA is to enhance the robustness of PCA against outliers or corrupted observations [187]–[189]. As these traditional methods cannot yield a solution in polynomial-time with performance guarantees under mild conditions, an improved version of robust PCA has been proposed in [190], [191]. In this new version, the robust PCA problem is treated as a low-rank matrix recovery problem in the presence of sparse corruption, which in fact is a joint sparse and low-rank recovery problem. Specifically, the goal is to

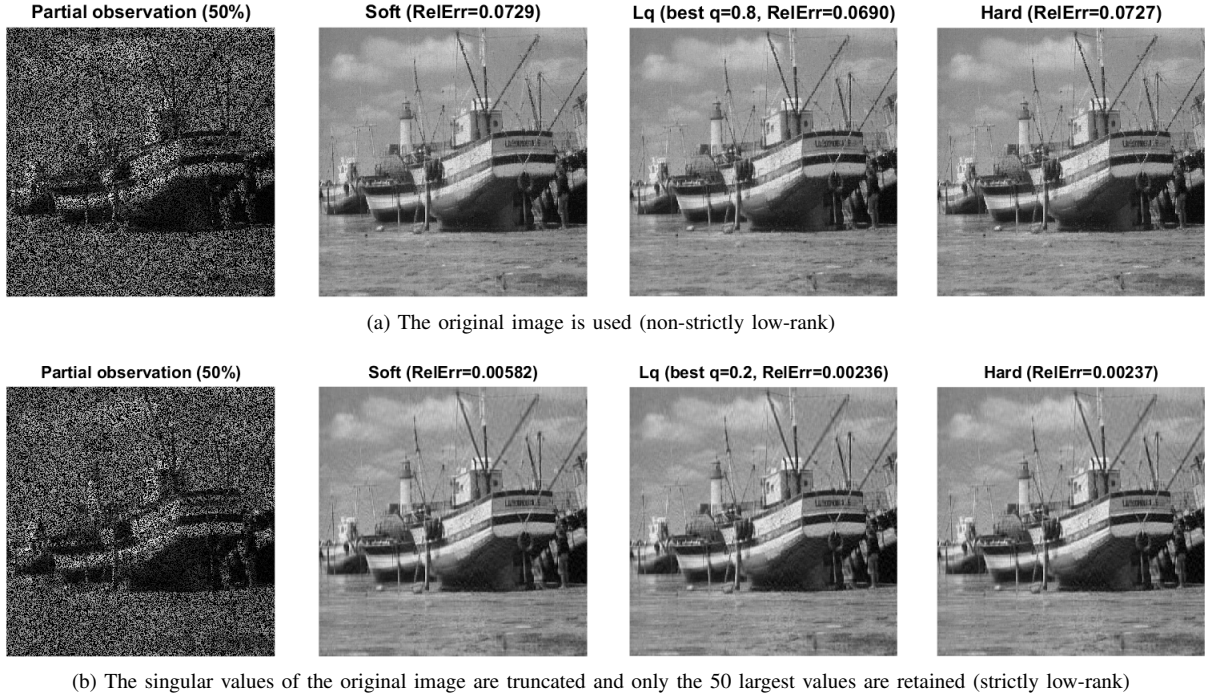


Fig. 12: Recovery performance of the hard-,  $\ell_q$ - and soft-thresholding in reconstructing an  $512 \times 512$  image.

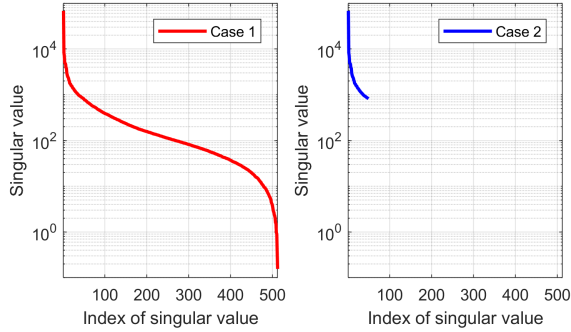


Fig. 13: Sorted singular values in the two cases.

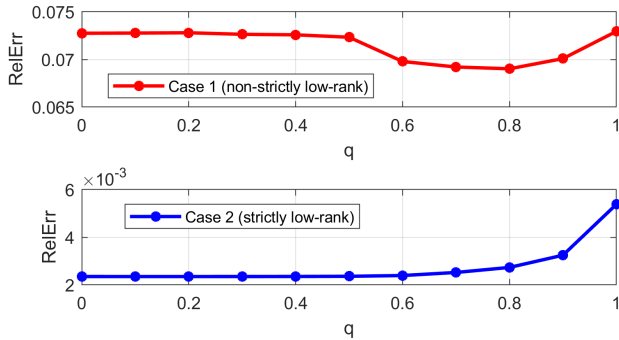


Fig. 14: Recovery performance of  $\ell_q$ -thresholding versus  $q$ .

recover a low-rank matrix  $\mathbf{L} \in \mathbb{R}^{m \times n}$  from highly (sparsely) corrupted observation

$$\mathbf{M} = \mathbf{L} + \mathbf{S}$$

where  $\mathbf{S} \in \mathbb{R}^{m \times n}$  represents the sparse corruption, in which the entries can have arbitrarily large magnitude and their support can be assumed to be sparse but unknown. There exist many important applications involving such a low-rank and sparse decomposition problem, such as video surveillance, face recognition, latent semantic indexing and ranking and collaborative filtering [191], to name a few. This can be achieved via exploiting the low-rankness of  $\mathbf{M}$  by the following formulation

$$\begin{aligned} \min_{\mathbf{L}, \mathbf{S}} G_1(\mathbf{L}) + \lambda G_2(\mathbf{S}) \\ \text{subject to } \mathbf{M} = \mathbf{L} + \mathbf{S} \end{aligned} \quad (38)$$

where  $G_1$  and  $G_2$  are penalties for low-rank and sparsity promotion, respectively. With  $G_1(\mathbf{L}) = \|\mathbf{L}\|_*$  and  $G_2(\mathbf{S}) = \|\mathbf{S}\|_1$ , (38) becomes the principal component pursuit (PCP) formulation [191]. It has been shown in [191] that under rather weak conditions, exactly recovery of the low-rank  $\mathbf{L}$  and the sparse  $\mathbf{S}$  can be achieved by the convex PCP formulation.

Consider both small entry-wise noise and gross sparse errors, which is a more practical case in many realistic applications, the formulation (38) can be extended to

$$\begin{aligned} \min_{\mathbf{L}, \mathbf{S}} G_1(\mathbf{L}) + \lambda G_2(\mathbf{S}) \\ \text{subject to } \|\mathbf{M} - \mathbf{L} - \mathbf{S}\|_F \leq \sigma \end{aligned} \quad (39)$$

where  $\sigma > 0$  is the noise tolerance. A more tractable alternative formulation of (39) is the following unconstrained formulation

$$\min_{\mathbf{L}, \mathbf{S}} G_1(\mathbf{L}) + \lambda G_2(\mathbf{S}) + \frac{1}{2\mu} \|\mathbf{M} - \mathbf{L} - \mathbf{S}\|_F^2 \quad (40)$$

where  $\mu > 0$  is the a penalty parameter. With  $G_1(\mathbf{L}) = \|\mathbf{L}\|_*$  and  $G_2(\mathbf{S}) = \|\mathbf{S}\|_1$ , (39) reduces to the stable principal

component pursuit in [192], and its variant (40) is solved by accelerated proximal gradient algorithm [192].

Recently, to attain performance improvement over using the convex nuclear norm and  $\ell_1$  penalties, nonconvex penalties have been considered in [9], [193]–[196]. Specifically, in [9], the  $q$ -shrinkage regularization has been used for both  $\mathbf{L}$  and  $\mathbf{S}$  in (40) and an ADMM algorithm has been proposed. In [193], a nonconvex method for (38) has been developed via alternating projection of the residuals onto the set of low-rank matrices and the set of sparse matrices. In [194], the capped norm penalty has been considered in the formulation (39) and an ADMM algorithm has been proposed. More recently, based on the formulation (40) and using the  $\ell_0$  penalty for  $\mathbf{S}$  and a low-rank factorization for  $\mathbf{L}$ , an cyclic descent algorithm has been designed in [195]. Meanwhile, using the formulation (38), an ADMM algorithm employing a rank approximation penalty has been proposed in [196].

Among all these methods, except for the convex methods [191], [192], there is no global convergence guarantee for the nonconvex algorithms. Following the recent results in nonconvex and nonsmooth optimization [69], [70], we can develop an efficient and globally convergent algorithm for generalized nonconvex penalties  $G_1$  and  $G_2$ , e.g., for the formulation (40), based on the block coordinate descent (BCD) method (also known as alternating minimization) as

$$\begin{aligned} \mathbf{L}^{k+1} = \arg \min_{\mathbf{L}} G_1(\mathbf{L}) \\ + \frac{1}{2\mu} \|\mathbf{M} - \mathbf{L} - \mathbf{S}^k\|_{\text{F}}^2 + \frac{c_k}{2} \|\mathbf{L} - \mathbf{L}^k\|_{\text{F}}^2 \end{aligned} \quad (41)$$

$$\begin{aligned} \mathbf{S}^{k+1} = \arg \min_{\mathbf{S}} \lambda G_2(\mathbf{S}) \\ + \frac{1}{2\mu} \|\mathbf{M} - \mathbf{L}^{k+1} - \mathbf{S}\|_{\text{F}}^2 + \frac{d_k}{2} \|\mathbf{S} - \mathbf{S}^k\|_{\text{F}}^2 \end{aligned} \quad (42)$$

where  $c_k > 0$  and  $d_k > 0$ . These two subproblems can be efficiently solved by the proximity operator introduced in section 2.

This BCD algorithm considers the proximal regularization of the Gauss-Seidel scheme by coupling the Gauss-Seidel iteration scheme with a proximal term. Using this proximal regularization strategy, it can be derived following the results in [69], [70] that, when  $G_1$  and  $G_2$  are closed, proper, lower semi-continuous and KL functions, the sequence  $\{\mathbf{L}^k, \mathbf{S}^k\}$  generated via (41) and (42) converges to a stationary point of the problem (40). Meanwhile, the ADMM method can also be applied to solve (40) with guaranteed convergence under some mild conditions [197].

## VI. OTHER APPLICATIONS INVOLVING NONCONVEX SPARSE AND LOW-RANK REGULARIZATION

The goal of this article is to provide a comprehensive overview on nonconvex regularization based sparse and low-rank recovery. In a field as wide as this, except for the applications introduced above, in this section we further briefly review some other applications where nonconvex regularized sparse and low-rank recovery has been applied.

*Subspace Learning and Tracking:* In addition to the matrix completion and robust PCA problems introduced above, nonconvex regularization based low-rank recovery has also been

used in subspace learning and tracking, which has applications in pattern recognition (e.g., object tracking, activity recognition and video surveillance) and image classification. For such applications, the Schatten- $q$  norm has been used to achieved better subspace learning and tracking in [198], [199].

*Dictionary Learning for Sparse Coding:* Dictionary learning for sparse coding aims to learn an overcomplete dictionary on which the input data can be succinctly represented, i.e., a linear combination of only a few atoms of the learned dictionary. It has wide applications in signal/image processing, computer vision, and machine learning [228]. The  $\ell_0$  norm penalty has been widely used in dictionary learning for sparse coding [229]. For general non-convex sparse coding problems, alternating algorithms with established convergence guarantee have been proposed in [214]. More recently, the  $\ell_{1/2}$  penalty and the log penalty have been employed for dictionary learning in [213], [215].

*Nonconvex Regularizers with Redistributing Nonconvexity:* To facilitate the efficient solving of nonconvex regularized problems, a nonconvexity redistributing method has been proposed recently in [216]. The core idea is to move the nonconvexity associated with a nonconvex penalty to the loss term. The nonconvex regularization term is convexified to be a convex one, e.g., the  $\ell_1$  norm, whilst the augmented loss term maintains the Lipschitz smooth property. In such a manner, the transformed problem can be efficiently solved by well-developed existing algorithms designed for convex regularized problems.

*Maximum Consensus Robust Fitting in Computer Vision:* Robust model fitting is a fundamental problem in many computer vision applications, where it is needed to deal with real-life raw data, e.g., in multi-view geometry and vision-based localization in robotics navigation. For robust model fitting, the maximum consensus criterion is of the most popular and useful, which is in fact a linearly constrained  $\ell_0$  minimization problem. Deterministic algorithms for the maximum consensus  $\ell_0$  minimization have been proposed recently in [217], [230], which showed superior performance in both solution quality and efficiency.

*Image Deconvolution and Restoration:* For the image deconvolution and restoration application,  $\ell_q$  regularization has been used in [121]–[123] to attain improved performance over the  $\ell_1$  regularization. Meanwhile,  $\ell_0$  norm combined with total variation has been considered in [119] to achieve robust restoration in the presence of impulsive noise.

*Least-Mean-Square (LMS) Filter:* For sparse system identification, regularized least-mean-square (LMS) algorithms have shown advantage over traditional LMS algorithms, e.g., be more accurate, more efficient and more robust to additive noise.  $\ell_0$  constrained LMS algorithms have been designed and analyzed in [224], [225]. Then, a weighted  $\ell_2$  regularization based normalized LMS filter has been proposed in [219], with application to acoustic system identification and active noise control. Moreover,  $\ell_p$  regularization with  $0 < p \leq 1$  has been considered in [226], [227].

*Simultaneously Sparse and Low-Rank Matrix Recovery:* While robust PCA introduced in the last section aims to decompose a low-rank component from sparsely corrupted



observation, the works [220]–[223], [231] consider the recovery of matrices which are simultaneously sparse and low-rank. For simultaneously sparse and low-rank recovery, it has been shown in [222] that nonconvex formulations can achieve reliable performance with less measurements than convex formulations. In [220], an ADMM algorithm using an iteratively reweighted  $\ell_1$  scheme has been proposed and applied to hyperspectral image unmixing. More recently, a nonconvex and nonseparable regularization method derived based on the sparse Bayesian learning framework has been presented in [223].

*Matrix Factorization Based Low-Rank Recovery:* In addition to the general low-rank recovery models introduced in the last section, there also exist a class of low-rank models based on low-rank matrix factorization [203], [204]. A significant feature (advantage) of such matrix factorization based methods is that, matrix factorization enables the algorithms to scale well to large-scale problems. Although matrix factorization makes the related formulation nonconvex (more precisely biconvex), it has been proven that such formulation for matrix completion has no spurious local minima, i.e., all local minima are also global [183], [232].

## VII. CONCLUSION AND DISCUSSION

In this overview paper, we have presented recent developments of nonconvex regularization based sparse and low-rank recovery in various fields in signal/image processing, statistics and machine learning, and addressed the issues of penalty selection, applications and the convergence of nonconvex algorithms. In recent, nonconvex regularization has attracted much study interest and promoted the progress in nonconvex and nonsmooth optimization. As a result, for many applications, convergent and efficient first-order algorithms have been developed for nonconvex regularized problems.

As shown in many applications, a nonconvex penalty can achieve significant performance improvement over the  $\ell_1$  norm penalty. However, there exist certain instances where the use of nonconvex regularization will not significantly improve performance, e.g., when the signal is not strictly sparse (or the matrix is not strictly low-rank) and/or the SNR is low. In such a case, the use of nonconvex regularization may be unnecessary, considering that the related nonconvex optimization problems are less tractable than convex problems. Specifically, for a nonconvex regularized algorithm, the performance is closely related to the initialization and the convergence rate is usually slower than that of a convex regularized algorithm.

Although it is difficult to determine the best selection of the penalty for a special instance, it can be selected in an application dependent manner. Specifically, from the results in the experimental examples, when the intrinsic component is strictly sparse (or has relatively high sparsity) and the noise is relatively low, a penalty with aggressive thresholding function (e.g., the  $\ell_q$  norm with a relatively small value of  $q$ ) should be used. Whereas, when the intrinsic component is non-strictly sparse (or has relatively low sparsity) and/or the noise is relatively high, a penalty with less aggressive thresholding function (e.g., the  $\ell_q$  norm with a relatively large value of  $q$ )

tend to yield better performance. The same philosophy applies to the low-rank recovery problems, depends on whether the intrinsic component is strictly low-rank or not.

For the nonconvex and nonsmooth problems reviewed in this paper, first-order algorithms are usually of the most efficient, such as the proximal gradient descent, block coordinate descent, and ADMM algorithms. The dominant computational complexity of such algorithms in each iteration is matrix-vector multiplication for the sparse recovery problems, e.g., the problems in section III. Meanwhile, the dominant computational complexity of these first-order algorithms in each iteration is SVD calculation for the low-rank recovery problems, e.g., the matrix completion and robust PCA problems in section V. However, the theoretical convergence rate of a nonconvex algorithm is generally difficult to derive. To the best of our knowledge, local (eventually) linear convergence rate has been established only for the proximal gradient descent algorithm for some special penalties with discontinuous thresholding functions, such as the hard and  $\ell_q$  thresholding [73]. For other algorithms with a general nonconvex penalty, the theoretical convergence rate is still an open problem.

## APPENDIX A PROOF OF THEOREM 1

Denote

$$f(\mathbf{x}) = P_\lambda(\|\mathbf{x}\|_2) + \frac{1}{2}\|\mathbf{x} - \mathbf{t}\|_2^2.$$

By simple geometrical arguments, we first show that a minimizer  $\mathbf{x}^*$  of  $f$  satisfies that  $\mathbf{x}^* = \alpha\mathbf{t}$  for some  $\alpha \geq 0$ . Specifically, assume that  $\|\mathbf{x}^* - \mathbf{t}\|_2 = r$  and consider the set  $\Omega = \{\mathbf{x} : \|\mathbf{x} - \mathbf{t}\|_2 = r\}$ , the points in the set  $\Omega$  are lying on the ball with center at  $\mathbf{t}$  and radius  $r$ . Since  $P_\lambda$  is a non-decreasing function in  $[0, \infty)$ , in the set  $\Omega$ , a minimal value of  $P_\lambda(\|\cdot\|_2)$  is given by the point which is the intersection of the ball and the vector  $\mathbf{t}$ . Thus,  $\mathbf{x}^* = \alpha\mathbf{t}$  with some  $\alpha \geq 0$ . Using this property, we have

$$f(\mathbf{x}^*) = P_\lambda(\|\mathbf{t}\|_2\alpha) + \frac{1}{2}\|\mathbf{t}\|_2^2(\alpha - 1)^2.$$

Further,  $\alpha$  should be the minimizer of the function

$$h(\alpha) = P_\lambda(\|\mathbf{t}\|_2\alpha) + \frac{1}{2}\|\mathbf{t}\|_2^2(\alpha - 1)^2,$$

which is given by 0 when  $\|\mathbf{t}\|_2 = 0$  and otherwise given by  $\text{prox}_{P_\lambda}(\|\mathbf{t}\|_2)/\|\mathbf{t}\|_2$ , which results in (5).

It is easy to see that, when  $\mathbf{t}$  is a scalar, i.e.,  $L = 1$ , (5) reduces to the proximity operator (1) for a scalar. For the  $\ell_1$  penalty, i.e.,  $P_\lambda(\cdot) = \lambda\|\cdot\|_1$ , we have

$$\text{prox}_{P_\lambda}(\mathbf{t}) = \mathbf{t} \cdot \max\left(1 - \frac{\lambda}{\|\mathbf{t}\|_2}, 0\right). \quad (43)$$

Meanwhile, for the  $\ell_0$  penalty, i.e.,  $P_\lambda(\cdot) = \lambda\|\cdot\|_0$ , the proximity operator becomes

$$\text{prox}_{P_\lambda}(\mathbf{t}) = \begin{cases} \mathbf{0}, & \|\mathbf{t}\|_2 < \sqrt{2\lambda} \\ \{\mathbf{0}, \mathbf{t}\}, & \|\mathbf{t}\|_2 = \sqrt{2\lambda} \\ \mathbf{t}, & \|\mathbf{t}\|_2 > \sqrt{2\lambda} \end{cases}. \quad (44)$$

APPENDIX B  
PROOF OF THEOREM 2

Without loss of generality, we assume that  $m \geq n$ . Let  $\mathbf{X} = \mathbf{SDE}^T$  be an SVD of  $\mathbf{X}$ , where  $\mathbf{D} = \text{diag}(d_1, \dots, d_n) \in \mathbb{R}^{n \times n}$  contains the singular values,  $\mathbf{S} = [\mathbf{s}_1, \dots, \mathbf{s}_n] \in \mathbb{R}^{m \times n}$  and  $\mathbf{E} = [\mathbf{e}_1, \dots, \mathbf{e}_n] \in \mathbb{R}^{n \times n}$  contain the left and right singular vectors, respectively. The objective function in (7) can be expressed as

$$\begin{aligned} \bar{P}_\lambda(\mathbf{X}) + \frac{1}{2} \|\mathbf{X} - \mathbf{M}\|_{\text{F}}^2 \\ = \sum_{i=1}^n P_\lambda(d_i) + \frac{1}{2} \left( \sum_{i=1}^n d_i^2 - 2 \sum_{i=1}^n d_i \mathbf{s}_i^T \mathbf{M} \mathbf{e}_i + \|\mathbf{M}\|_{\text{F}}^2 \right). \end{aligned}$$

Then, the formulation (7) can be equivalently expressed as

$$\begin{aligned} \text{prox}_{\bar{P}_\lambda}(\mathbf{M}) = \arg \min_{\mathbf{S}, \mathbf{D}, \mathbf{E}} \sum_{i=1}^n \frac{1}{2} d_i^2 - d_i \mathbf{s}_i^T \mathbf{M} \mathbf{e}_i + P_\lambda(d_i) \\ \text{subject to } \mathbf{S}^T \mathbf{S} = \mathbf{I}_n, \mathbf{E}^T \mathbf{E} = \mathbf{I}_n \\ \text{and } d_i \geq 0 \text{ for } i = 1, \dots, n. \end{aligned} \quad (45)$$

Since the  $i$ -th summand in the objective in (45) is only dependent on the  $\mathbf{s}_i$ ,  $\mathbf{e}_i$  and  $d_i$ , the minimization in (45) can be equivalently reformulated as

$$\begin{aligned} \min_{\mathbf{s}_i, \mathbf{e}_i, d_i \geq 0} \frac{1}{2} d_i^2 - d_i \mathbf{s}_i^T \mathbf{M} \mathbf{e}_i + P_\lambda(d_i) \\ \text{subject to } \mathbf{s}_i^T \mathbf{s}_j = \delta_{i,j}, \mathbf{e}_i^T \mathbf{e}_j = \delta_{i,j}, i, j \in \{1, \dots, m\}. \end{aligned} \quad (46)$$

For fixed  $d_i \geq 0$ , minimizing the objective in (46) is equivalent to maximizing  $\mathbf{s}_i^T \mathbf{M} \mathbf{e}_i$  with respect to  $\mathbf{s}_i$  and  $\mathbf{e}_i$ .

Let  $\tilde{\mathbf{s}}_1, \dots, \tilde{\mathbf{s}}_{i-1}$  and  $\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_{i-1}$  be any left and right singular vectors of  $\mathbf{M}$  respectively corresponding to the singular values  $\sigma_1(\mathbf{M}) \geq \sigma_2(\mathbf{M}) \geq \dots \geq \sigma_{i-1}(\mathbf{M})$ , consider the following fact that, for any  $i \geq 1$  the problem

$$\begin{aligned} \max_{\mathbf{s}_i, \mathbf{e}_i} \mathbf{s}_i^T \mathbf{M} \mathbf{e}_i \\ \text{subject to } \mathbf{s}_i \perp \{\tilde{\mathbf{s}}_1, \dots, \tilde{\mathbf{s}}_{i-1}\}, \|\mathbf{s}_i\|_2 \leq 1 \\ \text{and } \mathbf{e}_i \perp \{\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_{i-1}\}, \|\mathbf{e}_i\|_2 \leq 1 \end{aligned}$$

is solved by  $\tilde{\mathbf{s}}_i$  and  $\tilde{\mathbf{e}}_i$ , the left and right singular vectors of  $\mathbf{M}$  corresponding to the  $i$ -th largest singular value  $\sigma_i(\mathbf{M})$  (this result is well established in PCA). With such a solution, the formulation (46) becomes

$$\min_{d_i \geq 0} \frac{1}{2} d_i^2 - \sigma_i(\mathbf{M}) d_i + P_\lambda(d_i). \quad (47)$$

This problem is a form of the proximity operator (1), and the solution is given by  $d_i^* = \text{prox}_{P_\lambda}(\sigma_i(\mathbf{M}))$ , which consequently results in (8).

REFERENCES

- [1] N. Meinshausen and B. Yu, "Lasso-type recovery of sparse representations for high-dimensional data," *Annals of Statistics*, vol. 37, no. 1, pp. 246–270, 2009.
- [2] T. Hastie, R. Tibshirani, M. Wainwright. *Statistical learning with sparsity: the lasso and generalizations*. CRC Press, 2016.
- [3] R. Chartrand and V. Staneva, "Restricted isometry properties and non-convex compressive sensing," *Inverse Problems*, vol. 24, no. 3, 2008.
- [4] A. Antoniadis, "Wavelets in Statistics: A Review" (with discussion), *Journal of the Italian Statistical Association*, vol. 6, pp. 97–144, 1997.
- [5] Z. Xu, X. Chang, F. Xu, and H. Zhang, "L1/2 regularization: a thresholding representation theory and a fast solver," *IEEE Trans. Neural Networks Learning Systems*, vol. 23, no. 7, pp. 1013–1027, 2012.
- [6] G. Marjanovic and V. Solo, "On  $\ell_q$  optimization and matrix completion," *IEEE Trans. Signal Process.*, vol. 60, no. 11, pp. 5714–5724, 2012.
- [7] G. Marjanovic and V. Solo, "On exact  $\ell_q$  denoising," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, 2013, pp. 6068–6072.
- [8] R. Chartrand, "Fast algorithms for nonconvex compressive sensing: MRI reconstruction from very few data," in *Proc. IEEE Int. Symp. Biomed. Imag.*, 2009, pp. 262–265.
- [9] R. Chartrand, "Nonconvex splitting for regularized low-rank+sparse decomposition," *IEEE Trans. Signal Process.*, 2012, 60(11): 5810–5819.
- [10] J. Woodworth, R. Chartrand, "Compressed sensing recovery via non-convex shrinkage penalties," *Inverse Problems*, vol. 32, no. 7, pp. 1–25, 2016.
- [11] J. Fan and R. Li, "Variable selection via nonconcave penalized likelihood and its oracle properties," *Journal of the American Statistical Association*, vol. 96, no. 456, pp. 1348–1360, 2001.
- [12] C. Zhang, "Nearly unbiased variable selection under minimax concave penalty," *The Annals of Statistics*, vol. 38, no. 2, pp. 894–942, 2010.
- [13] H.-Y. Gao and A. G. Bruce, "Wave shrink with firm shrinkage," *Statistica Sinica*, vol. 7, no. 4, pp. 855–874, 1997.
- [14] D. L. Donoho, "Compressed sensing," *IEEE Trans. Inf. Theory*, vol. 52, no. 4, pp. 1289–1306, 2006.
- [15] E. J. Candès and M. Wakin, "An introduction to compressive sampling," *IEEE Signal Process. Mag.*, vol. 25, no. 2, pp. 21–30, Mar. 2008.
- [16] R. G. Baraniuk, "Compressive sensing," *IEEE Signal Process. Magazine*, vol. 24, no. 4, pp. 118–121, 2007.
- [17] E. J. Candès, J. Romberg, and T. Tao, "Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information," *IEEE Trans. Inf. Theory*, vol. 52, no. 2, pp. 489–509, Feb. 2006.
- [18] M. A. Herman and T. Strohmer, "High-resolution radar via compressed sensing," *IEEE Trans. Signal Process.*, vol. 57, no. 6, pp. 2275–2284, 2009.
- [19] L. C. Potter, E. Ertin, J. T. Parker, and M. Cetin, "Sparsity and compressed sensing in radar imaging," *IEEE Proceedings*, vol. 98, no. 6, pp. 1006–1020, 2010.
- [20] C. R. Berger, Z. Wang, J. Huang, and S. Zhou, "Application of compressive sensing to sparse channel estimation," *IEEE Commun. Mag.*, vol. 48, no. 11, pp. 164–174, 2010.
- [21] M. Lustig, D. Donoho, and J. M. Pauly, "Sparse MRI: The application of compressed sensing for rapid MR imaging," *Magnetic Resonance Medicine*, vol. 58, no. 6, pp. 1182–1195, 2007.
- [22] J. Yang, J. Wright, T. S. Huang, and Y. Ma, "Image super-resolution via sparse representation," *IEEE Trans. Image Process.*, vol. 19, no. 11, pp. 2861–2873, 2010.
- [23] X. Jiang, R. Ying, F. Wen, et al., "An improved sparse reconstruction algorithm for speech compressive sensing using structured priors," in *Proc. IEEE Int. Conf. on Multimedia and Expo (ICME 2016)*, 2016.
- [24] S. S. Chen, D. L. Donoho, and M. A. Saunders, "Atomic decomposition by basis pursuit," *SIAM J. Sci. Comput.*, vol. 20, no. 1, pp. 33–61, 1998.
- [25] Y. Nesterov, "A method of solving a convex programming problem with convergence rate  $O(1/k^2)$ ," *Soviet Math. Doklady*, vol. 27, no. 2, pp. 372–376, 1983.
- [26] A. Beck and M. Teboulle, "A fast iterative shrinkage-thresholding algorithm for linear inverse problems," *SIAM J. Imag. Sci.*, vol. 2, no. 1, pp. 183–202, 2009.
- [27] S. Becker, J. Bobin, and E. Candes, *NESTA: A fast and accurate first-order method for sparse recovery*. Berkeley, CA, USA: Univ. of California Press, Apr. 2009.
- [28] D. Donoho, A. Maleki, and A. Montanari, "Message-passing algorithms for compressed sensing," *PNAS*, vol. 106, no. 45, pp. 18914–18919, Jul. 2009.
- [29] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, "Distributed optimization and statistical learning via the alternating direction method of multipliers," *Found. Trends Mach. Learn.*, vol. 3, no. 1, pp. 1–122, 2011.
- [30] E. J. Candès and T. Tao, "Decoding by linear programming," *IEEE Trans. Inf. Theory*, vol. 51, no. 12, pp. 4203–4215, Dec. 2005.
- [31] E. J. Candès, "The restricted isometry property and its implications for compressed sensing," *Comptes. Rendus. Mathématique*, vol. 346, no. 9, pp. 589–592, 2008.
- [32] R. Baraniuk, M. Davenport, R. DeVore, and M. Wakin, "A simple proof of the restricted isometry property for random matrices," *Construct. Approx.*, vol. 28, no. 3, pp. 253–263, 2008.

- [33] A. Cohen, W. Dahmen, R. DeVore, "Compressed sensing and best k-term approximation," *J. Am. Math. Soc.*, vol. 22, no. 1, pp. 211–231, 2009.
- [34] J. A. Tropp, "Greed is good: algorithmic results for sparse approximation," *IEEE Trans. Inf. Theory*, vol. 50, no. 10, pp. 2231–2242, 2004.
- [35] D. L. Donoho, I. M. Johnstone, J. C. Hoch, and A. S. Stern, "Maximum entropy and the nearly black object," *J. Roy. Statist. Soc. B (Methodological)*, vol. 54, no. 1, pp. 41–81, 1992.
- [36] D. L. Donoho, I. M. Johnstone, "Ideal spatial adaptation by wavelet shrinkage," *biometrika*, vol. 81, no. 3, pp. 425–455, 1994.
- [37] R. Tibshirani, "Regression shrinkage and selection via the lasso," *J. Roy. Statist. Soc. B (Methodological)*, vol. 58, no. 1, pp. 267–288, 1996.
- [38] E. J. Candès, M. B. Wakin, and S. P. Boyd, "Enhancing sparsity by reweighted  $\ell_1$  minimization," *J. Fourier Anal. Appl.*, vol. 14, pp. 877–905, 2008.
- [39] S. Foucart and M.-J. Lai, "Sparsest solutions of underdetermined linear systems via  $\ell_q$ -minimization for  $0 < q \leq 1$ ," *Appl. Comput. Harmon. Anal.*, vol. 26, no. 3, pp. 395–407, May 2009.
- [40] Q. Sun, "Recovery of sparsest signals via  $\ell_q$ -minimization," *Appl. Comput. Harmon. Anal.*, vol. 32, no. 3, pp. 329–341, 2012.
- [41] R. Wu and D.-R. Chen, "The improved bounds of restricted isometry constant for recovery via lp-minimization," *IEEE Trans. Inf. Theory*, vol. 59, no. 9, pp. 6142–6147, Sep. 2013.
- [42] L. Zheng, A. Maleki, Q. Liu, et al., "An  $\ell_p$ -based reconstruction algorithm for compressed sensing radar imaging," *IEEE Radar Conference*, 2016.
- [43] H. Mohimani, M. Babie-Zadeh, and C. Jutten, "A fast approach for overcomplete sparse decomposition based on smoothed  $\ell_0$ -norm," *IEEE Trans. Signal Process.*, vol. 57, no. 1, pp. 289–301, Jan. 2009.
- [44] I. Daubechies, R. DeVore, M. Fornasier, et al. "Iteratively reweighted least squares minimization for sparse recovery," *Communications on Pure and Applied Mathematics*, vol. 63, no. 1, pp. 1–38, 2010.
- [45] R. Chartrand, "Exact reconstruction of sparse signals via nonconvex minimization," *IEEE Signal Process. Lett.*, vol. 14, no. 10, pp. 707–710, 2007.
- [46] R. Chartrand and W. Yin, "Iteratively reweighted algorithms for compressive sensing," in *Proc. IEEE Int. Conf. Acoust. Speech, Signal Process.*, 2008, pp. 3869–3872.
- [47] R. Saab, R. Chartrand, and O. Yilmaz, "Stable sparse approximations via nonconvex optimization," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, 2008, pp. 3885–3888.
- [48] M. J. Lai, Y. Xu, and W. Yin, "Improved iteratively reweighted least squares for unconstrained smoothed  $\ell_q$  minimization," *SIAM J. Numer. Anal.*, vol. 51, no. 2, pp. 927–957, 2013.
- [49] J. K. Pant, W. Lu, and A. Antoniou. "New Improved Algorithms for Compressive Sensing Based on  $\ell_p$ -Norm," *IEEE Trans. Circuits and Systems II: Express Briefs*, vol. 61, no. 3, pp. 198–202, 2014.
- [50] Z. Lu, "Iterative reweighted minimization methods for lp regularized unconstrained nonlinear programming," *Mathematical Programming*, vol. 147, pp. 277–307, 2014.
- [51] W. Bian and X. Chen, "Smoothing SQP algorithm for non-Lipschitz optimization with complexity analysis," *Preprint*, February, 2012.
- [52] X. Chen, D. Ge, Z. Wang, et al., "Complexity of unconstrained L2-Lp minimization," *Mathematical Programming*, vol. 143, pp. 371–383, 2014.
- [53] X. Chen, L. Niu, and Y. Yuan, "Optimality conditions and smoothing trust region Newton method for non-Lipschitz optimization," *Preprint*, March 2012.
- [54] X. Chen, F. Xu, and Y. Ye, "Lower bound theory of nonzero entries in solutions of l2-lp minimization," *SIAM J. Sci. Comput.*, vol. 32, no. 5, pp. 2832–2852, 2010.
- [55] X. Chen and W. Zhou, "Convergence of reweighted l1 minimization algorithms and unique solution of truncated lp minimization," *Preprint*, April 2010.
- [56] D. Ge, X. Jiang, and Y. Ye, "A note on the complexity of Lp minimization," *Math. Program.*, vol. 129, no. 2, pp. 285–299, 2011.
- [57] M. Lai and J. Wang, "An unconstrained lq minimization with  $0 < q < 1$  for sparse solution of underdetermined linear systems," *SIAM J. Optim.*, vol. 21, no. 1, pp. 82–101, 2011.
- [58] N. Mourad and J. P. Reilly, "Minimizing nonconvex functions for sparse vector reconstruction," *IEEE Trans. Signal Process.*, vol. 58, no. 7, pp. 3485–3496, 2010.
- [59] Z. Xu, H. Zhang, Y. Wang, et al., "L1/2 regularization," *Science China Information Sciences*, vol. 53, no. 6, pp. 1159–1169, 2010.
- [60] F. Wen, L. Pei, Y. Yang, W. Yu, and P. Liu, "Efficient and robust recovery of sparse signal and image using generalized nonconvex regularization," *IEEE Transactions on Computational Imaging*, vol. 3, no. 4, pp. 566–579, Dec. 2017.
- [61] L. Chen and Y. Gu, "On the null space constant for Lp minimization," *IEEE Signal Processing Letters*, vol. 22, no. 10, pp. 1600–1603, 2015.
- [62] L. Zheng, A. Maleki, H. Weng, et al., "Does Lp-minimization outperform L1-minimization," *IEEE Transactions Information Theory*, vol. 63, no. 11, pp. 6896–6935, 2017.
- [63] F. Wen, P. Liu, Y. Liu, R. C. Qiu, and W. Yu, "Robust sparse recovery for compressive sensing in impulsive noise using Lp-norm model fitting," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, 2016, pp. 3885–3888.
- [64] F. Wen, P. Liu, Y. Liu, R. C. Qiu, and W. Yu, "Robust sparse recovery in impulsive noise via Lp-L1 optimization," *IEEE Trans. Signal Process.*, vol. 65, no. 1, pp. 105–118, Jan. 2017.
- [65] W. Bian and X. Chen, "Linearly constrained non-Lipschitz optimization for image restoration," *SIAM Journal on Imaging Sciences*, vol. 8, no. 4, pp. 2294–2322, 2015.
- [66] H. Weng, L. Zheng, A. Maleki, et al., "Phase transition and noise sensitivity of  $\ell_p$ -minimization for  $0 \leq p \leq 1$ ," in *Proc. IEEE International Symposium Information Theory (ISIT)*, 2016, pp. 675–679.
- [67] H. Attouch, J. Bolte, and B. Svaiter, "Convergence of descent methods for semi-algebraic and tame problems: Proximal algorithms, forward-backward splitting, and regularized Gauss-Seidel methods," *Math. Program. A*, vol. 137, no. 1, pp. 91–129, 2013.
- [68] K. Bredies, D. Lorenz, and S. Reiterer, "Minimization of non-smooth, non-convex functionals by iterative thresholding," *J. Optim. Theory Appl.*, vol. 165, no. 1, pp. 78–122, 2015.
- [69] H. Attouch, J. Bolte, P. Redont, and A. Soubeyran, "Proximal alternating minimization and projection methods for nonconvex problems: an approach based on the Kurdyka-Lojasiewicz inequality," *Mathematics of Operations Research*, vol. 35, no. 2, pp. 438–457, 2010.
- [70] J. Bolte, S. Sabach, and M. Teboulle, "Proximal alternating linearized minimization for nonconvex and nonsmooth problems," *Mathematical Programming*, vol. 146, pp. 459–494, 2014.
- [71] G. Li and T. K. Pong, "Global convergence of splitting methods for nonconvex composite optimization," *SIAM J. Optimization*, vol. 25, no. 4, pp. 2434–2460, Jul. 2015.
- [72] J. Zeng, S. Lin, Y. Wang, and Z. Xu, "L1/2 regularization: Convergence of iterative half thresholding algorithm," *IEEE Trans. Signal Process.*, vol. 62, no. 9, pp. 2317–2329, Jul. 2014.
- [73] J. Zeng, S. Lin, and Z. Xu, "Sparse regularization: Convergence of iterative jumping thresholding algorithm," *IEEE Trans. Signal Process.*, vol. 64, no. 19, pp. 5106–5118, Oct. 2016.
- [74] M. Hong, Z. Q. Luo, and M. Razaviyayn, "Convergence analysis of alternating direction method of multipliers for a family of nonconvex problems," *SIAM J. Optimization*, vol. 26, no. 1, pp. 337–364, 2016.
- [75] F. Wang, Z. Xu, and H.-K. Xu, "Convergence of bregman alternating direction method with multipliers for nonconvex composite problems," *arXiv preprint*, arXiv:1410.8625, Dec. 2014.
- [76] T. Goldstein, B. O'Donoghue, S. Setzer, and R. Baraniuk, "Fast alternating direction optimization methods," *SIAM J. Imag. Sci.*, vol. 7, no. 3, pp. 1588–1623, 2014.
- [77] T. Blumensath, M. Yaghoobi, and M. E. Davies, "Iterative hard thresholding and l0 regularisation," *IEEE ICASSP*, 2007, pp. 877–880.
- [78] N. Zhang, Q. Li, "On optimal solutions of the constrained  $\ell_0$  regularization and its penalty problem," *Inverse Problems*, vol. 33, no. 2, 2017.
- [79] Y. Jiao, B. Jin, X. Lu, "A primal dual active set with continuation algorithm for the  $\ell_0$ -regularized optimization problem," *Applied and Computational Harmonic Analysis*, vol. 39, no. 3, pp. 400–426, 2015.
- [80] C. Bao, B. Dong, L. Hou, et al., "Image restoration by minimizing zero norm of wavelet frame coefficients," *Inverse Problems*, vol. 32, no. 11, 2016.
- [81] M. Nikolova, "Relationship between the optimal solutions of least squares regularized with  $\ell_0$ -norm and constrained by k-sparsity," *Applied and Computational Harmonic Analysis*, vol. 41, no. 1, pp. 237–265, 2016.
- [82] T. Hastie, R. Tibshirani, and J. Friedman, *Unsupervised learning: The elements of statistical learning*. Springer New York, 2009: pp. 485–585.
- [83] J. Fan and H. Peng, "On non-concave penalized likelihood with diverging number of parameters," *Annals of Statistics*, vol. 32, no. 3, pp. 928–961, 2004.
- [84] J. Fan and R. Li, "Variable selection for Cox's proportional hazards model and frailty model," *Annals of Statistics*, vol. 30, no 1, pp. 74–99, 2002.
- [85] J. Fan and R. Li, "New estimation and model selection procedures for semiparametric modeling in longitudinal data analysis," *Journal American Statistical Association*, vol. 99, no. 467, pp. 710–723, 2004.
- [86] J. Fan, H. Lin, and Y. Zhou, "Local partial likelihood estimation for life time data," *Annals of Statistics*, vol. 34, no. 1, pp. 290–325, 2006.

- [87] J. Fan and J. Lv, "Nonconcave penalized likelihood with NP-dimensionality," *IEEE Trans. Inf. Theory*, vol. 57, no. 8, pp. 5467–5484, 2011.
- [88] H. Zou, "The adaptive lasso and its oracle properties," *Journal American Statistical Association*, vol. 101, no. 476, pp. 141–1429, 2006.
- [89] J. Huang, S. Ma, and C. H. Zhang, "Adaptive Lasso for sparse high-dimensional regression models," *Statistica Sinica*, vol. 18, no. 4, pp. 1603–1618, 2008.
- [90] D. Hunter and R. Li, "Variable selection using mm algorithms," *Annals of Statistics*, vol. 33, no. 4, pp. 1617–1642, 2005.
- [91] H. Zou and R. Li, "One-step sparse estimates in nonconcave penalized likelihood models," *Annals of Statistics*, vol. 36, no. 4, pp. 1509–1533, 2008.
- [92] R. Mazumder, J. H. Friedman, and T. Hastie, "Sparsenet: Coordinate descent with nonconvex penalties," *Journal American Statistical Association*, vol. 106, no. 495, pp. 1125–1138, 2011.
- [93] P. Breheny and J. Huang, "Coordinate descent algorithms for nonconvex penalized regression, with applications to biological feature selection," *Annals of Applied Statistics*, vol. 5, no. 1, pp. 232–253, 2011.
- [94] A. Seneviratne and V. Solo, "On vector l0 penalized multivariate regression," *IEEE ICASSP*, 2012, pp. 3613–3616.
- [95] G. Marjanovic and V. Solo, "lq sparsity penalized linear regression with cyclic descent," *IEEE Trans. Signal Process.*, vol. 62, no. 6, pp. 1464–1475, 2014.
- [96] G. Marjanovic, M. O. Ulfarsson, and A. O. Hero III, "MIST: l0 Sparse linear regression with momentum," *IEEE ICASSP*, 2015, pp. 3551–3555.
- [97] M. Elad, J. L. Starck, P. Querre, and D. L. Donoho, "Simultaneous cartoon and texture image inpainting using morphological component analysis (MCA)," *Appl. Comput. Harmon. Anal.*, vol. 19, no. 3, pp. 340–358, 2005.
- [98] J.-F. Cai, S. Osher, and Z. Shen, "Split Bregman methods and frame based image restoration," *Multiscale Model. Simul.*, vol. 8, no. 2, pp. 337–369, 2009.
- [99] W. Göbel and F. Helmchen, "In vivo calcium imaging of neural network function," *Physiology*, vol. 22, no. 6, pp. 358–365, 2007.
- [100] C. Studer, P. Kuppinger, G. Pope, and H. Bölcskei, "Recovery of sparsely corrupted signals," *IEEE Trans. Inf. Theory*, vol. 58, no. 5, pp. 3115–3130, 2012.
- [101] C. Studer and R. G. Baraniuk, "Stable restoration and separation of approximately sparse signals," *Applied and Computational Harmonic Analysis*, vol. 37, no. 1, pp. 12–35, 2014.
- [102] M. B. McCoy, V. Cevher, Q. T. Dinh, A. Asaci, and L. Baldassarre, "Convexity in source separation: Models, geometry, and algorithms," *IEEE Signal Processing Magazine*, vol. 31, no. 3, pp. 87–95, 2014.
- [103] S. G. Mallat and G. Yu, "Super-resolution with sparse mixing estimators," *IEEE Trans. Image Process.*, vol. 19, no. 11, pp. 2889–2900, Nov. 2010.
- [104] M. Elad and Y. Hel-Or, "Fast super-resolution reconstruction algorithm for pure translational motion and common space-invariant blur," *IEEE Trans. Image Process.*, vol. 10, no. 8, pp. 1187–1193, Aug. 2001.
- [105] M. Bertalmio, G. Sapiro, V. Caselles, and C. Ballester, "Image inpainting," in *Proc. 27th Annu. Conf. Comp. Graph. Int. Technol.*, 2000, pp. 417–424.
- [106] A. Adler, V. Emiya, M. G. Jafari, M. Elad, R. Gribonval, and M. D. Plumbley, "Audio inpainting," *IEEE Trans. Audio Speech Lang. Process.*, vol. 20, no. 3, pp. 922–932, 2012.
- [107] J. N. Laska, P. T. Boufounos, M. A. Davenport, and R. G. Baraniuk, "Democracy in action: Quantization, saturation, and compressive sensing," *Appl. Comput. Harmon. Anal.*, vol. 31, no. 3, pp. 429–443, 2011.
- [108] E. J. Candès and P. A. Randall, "Highly robust error correction by convex programming," *IEEE Trans. Inf. Theory*, vol. 54, no. 7, pp. 2829–2840, 2008.
- [109] B. Popilka, S. Setzer, and G. Steidl, "Signal recovery from incomplete measurements in the presence of outliers," *Inverse Problems Imag.*, vol. 1, no. 4, pp. 661–672, Nov. 2007.
- [110] R. Chan, C.-W. Ho, and M. Nikolova, "Salt-and-pepper noise removal by median-type noise detectors and de tail-preserving regularization," *IEEE Trans. Image Process.*, vol. 14, no. 10, pp. 1479–1485, Oct. 2005.
- [111] T. Hashimoto, "Bounds on a probability for the heavy tailed distribution and the probability of deficient decoding in sequential decoding," *IEEE Trans. Inf. Theory*, vol. 51, no. 3, pp. 990–1002, Mar. 2005.
- [112] L. Bar, A. Brook, N. Sochen, and N. Kiryati, "Deblurring of color images corrupted by impulsive noise," *IEEE Trans. Image Process.*, vol. 16, no. 4, pp. 1101–1111, Apr. 2007.
- [113] P. Civioglu, "Using uncorrupted neighborhoods of the pixels for impulsive noise suppression with ANFIS," *IEEE Trans. Image Process.*, vol. 16, no. 3, pp. 759–773, Mar. 2007.
- [114] P. Windyga, "Fast impulsive noise removal," *IEEE Trans. Image Process.*, vol. 10, no. 1, pp. 173–179, Jan. 2001.
- [115] J. F. Yang and Y. Zhang, "Alternating direction algorithms for 11-problems in compressive sensing," *SIAM J. Sci. Comput.*, vol. 33, no. 1, pp. 250–278, 2011.
- [116] Y. Xiao, T. Zeng, J. Yu, and M. K. Ng, "Restoration of images corrupted by mixed Gaussian-impulse noise via L1-L0 minimization," *Pattern Recognition*, vol. 44, no. 8, pp. 1708–1720, 2011.
- [117] M. Yan, "Restoration of images corrupted by impulse noise and mixed Gaussian impulse noise using blind inpainting," *SIAM J. Imag. Sci.*, vol. 6, no. 3, pp. 1227–1245, 2013.
- [118] K. Hohm, M. Storath, and A. Weinmann, "An algorithmic framework for Mumford-Shah regularization of inverse problems in imaging," *Inverse Probl.*, vol. 31, no. 11, 2015.
- [119] G. Yuan and B. Ghanem, "L0 TV: A new method for image restoration in the presence of impulse noise," in *Proc. IEEE Conf. Comput. Vis. Pattern Recognit.*, 2015, pp. 5369–5377.
- [120] F. Wen, L. Adhikari, L. Pei, et al., "Nonconvex regularization based sparse recovery and demixing with application to color image inpainting," *IEEE Access*, vol. 5, pp. 11513–11527, May 2017.
- [121] X. Zhou, F. Zhou, X. Bai, "Parameter estimation for Lp regularized image deconvolution," *IEEE Int. Conf. Image Process. (ICIP)*, 2015, pp. 4892–4896.
- [122] C. Mia and H. Yu, "Alternating iteration for Lp regularized CT reconstruction," *IEEE Access*, vol. 4, pp. 4355–4363, 2016.
- [123] X. Zhou, R. Molina, F. Zhou, et al., "Fast iteratively reweighted least squares for lp regularized image deconvolution and reconstruction," *IEEE Int. Conf. Image Process. (ICIP)*, 2014, pp. 1783–1787.
- [124] I. T. Jolliffe, "Rotation of principal components: choice of normalization constraints," *J. Appl. Stat.*, vol. 22, no. 1, pp. 29–35, 1995.
- [125] J. Cadima and I. T. Jolliffe, "Loadings and correlations in the interpretation of principal components," *J. Appl. Stat.*, vol. 22, no. 2, pp. 203–214, 1995.
- [126] I. T. Jolliffe, N. T. Trendafilov, and M. Uddin, "A modified principal component technique based on the LASSO," *Journal Computational Graphical Statistics*, vol. 12, no. 3, pp. 531–547, 2003.
- [127] H. Zou, T. Hastie, and R. Tibshirani, "Sparse principal component analysis," *J. Comput. Graph. Stat.*, vol. 15, no. 2, pp. 26–286, 2006.
- [128] A. d'Aspremont, L. El Ghaoui, M. I. Jordan, et al., "A direct formulation for sparse PCA using semidefinite programming," *SIAM Rev.*, vol. 49, no. 3, pp. 434–448, 2007.
- [129] A. d'Aspremont, F. R. Bach, and L. E. Ghaoui, "Optimal solutions for sparse principal component analysis," *J. Mach. Learn. Res.*, vol. 9, pp. 1269–1294, 2008.
- [130] H. P. Shen and J. Z. Huang, "Sparse principal component analysis via regularized low rank matrix approximation," *J. Multivariate Anal.*, vol. 99, pp. 1015–1034, 2008.
- [131] M. Journée, Y. Nesterov, P. Richtarik, et al., "Generalized power method for sparse principal component analysis," *J. Mach. Learn. Res.*, vol. 11, pp. 517–553, 2010.
- [132] B. K. Sriperumbudur, D. A. Torres, G. R. Lanckriet, "Sparse eigen methods by D.C. programming," in *Proc. 24th Int. Conf. Machine Learning*, pp. 831–838, Corvallis, 2007.
- [133] B. Moghaddam, Y. Weiss, and S. Avidan, "Spectral bounds for sparse PCA: Exact and greedy algorithms," *Advances in Neural Information Processing Systems*, 2006.
- [134] P. Richtarik, M. Takac, and S. D. Ahipasaoglu, "Alternating maximization: Unifying framework for 8 sparse pca formulations and efficient parallel codes," *arXiv Preprint*, arXiv:1212.4137, 2012.
- [135] Q. Zhao, D. Y. Meng, and Z. B. Xu, "Robust sparse principal component analysis," *Science China Information Sciences*, vol. 57, no. 9, pp. 1–14, 2014.
- [136] D. Hajimezhad and M. Hong, "Nonconvex alternating direction method of multipliers for distributed sparse principal component analysis," *GlobalSIP*, pp. 255–259, 2015.
- [137] H. Chang, L. Luo, J. Yang, et al., "Schatten p-norm based principal component analysis," *Neurocomputing*, vol. 207, pp. 754–762, 2016.
- [138] Y. Wang and Q. Wu, "Sparse PCA by iterative elimination algorithm," *Advances Computational Mathematics*, vol. 36, no. 1, pp. 137–151, 2012.
- [139] M. O. Ulfarsson and V. Solo, "Vector l0 sparse variable PCA," *IEEE Trans. Signal Process.*, vol. 59, no. 5, pp. 1949–1958, 2011.
- [140] M. O. Ulfarsson and V. Solo, "Sparse loading noisy PCA using an l0 penalty," *ICASSP*, 2012, pp. 3597–3600.
- [141] M. Luessi, M. S. Hamalainen, and V. Solo, "Vector  $\ell_0$  latent-space principal component analysis," *ICASSP*, 2014, pp. 4229–4233.

- [142] J. Fan, Y. Liao, and H. Liu, "An overview on the estimation of large covariance and precision matrices," *The Econometrics Journal*, vol. 19, no. 1, pp. 1–32, 2016.
- [143] J. Fan, F. Han, and H. Liu, "Challenges of big data analysis," *National science review*, vol. 1, pp. 293–314, 2014.
- [144] R. C. Qiu and P. Antonik, *Smart Grid and Big Data: Theory and Practice*. John Wiley Sons, 2015.
- [145] I. Johnstone, "On the distribution of the largest eigenvalue in principal components analysis," *The Annals of Statistics*, vol. 29, pp. 295–327, 2001.
- [146] P. Bickel and E. Levina, "Covariance Regularization by Thresholding," *The Annals of Statistics*, vol. 36, pp. 2577–2604, 2008.
- [147] N. E. Karoui, "Operator Norm Consistent Estimation of Large Dimensional Sparse Covariance Matrices," *The Annals of Statistics*, vol. 36, pp. 2717–2756, 2008.
- [148] A. Rothman, E. Levina, and J. Zhu, "Generalized Thresholding of Large Covariance Matrices," *Journal of the American Statistical Association*, vol. 104, pp. 177–186, 2009.
- [149] A. J. Rothman, "Positive definite estimators of large covariance matrices," *Biometrika*, vol. 99, no. 3, pp. 733–740, 2012.
- [150] L. Xue, S. Ma, and H. Zou, "Positive Definite L1 Penalized Estimation of Large Covariance Matrices," *Journal of the American Statistical Association*, vol. 107, pp. 1480–1491, 2012.
- [151] H. Liu, L. Wang, and T. Zhao, "Sparse covariance matrix estimation with eigenvalue constraints," *Journal of Computational and Graphical Statistics*, vol. 23, no. 2, pp. 439–459, 2014.
- [152] F. Wen, Y. Yang, P. Liu, and R. C. Qiu, "Positive definite estimation of large covariance matrix using generalized nonconvex penalties," *IEEE Access*, vol. 4, pp. 4168–4182, 2016.
- [153] J. Khan, J. Wei, M. Ringner, et al., "Classification and Diagnostic Prediction of Cancers Using Gene Expression Profiling and Artificial Neural Networks," *Nature Medicine*, vol. 7, pp. 673–679, 2001.
- [154] T. Cai, C. Zhang, and H. Zhou, "Optimal rates of convergence for covariance matrix estimation," *The Annals of Statistics*, vol. 38, pp. 2118–2144, 2010.
- [155] J. Bien and R. J. Tibshirani, "Sparse estimation of a covariance matrix," *Biometrika*, vol. 98, no. 4, pp. 807–820, 2011.
- [156] J. Fan, Y. Fan, and J. Lv, "High dimensional covariance matrix estimation using a factor model," *Journal of Econometrics*, vol. 147, no. 1, pp. 186–197, 2008.
- [157] T. Cai and W. Liu, "Adaptive thresholding for sparse covariance matrix estimation," *Journal of the American Statistical Association*, vol. 106, no. 494, pp. 672–684, 2011.
- [158] J. Fan, J. Zhang, and K. Yu, "Vast portfolio selection with gross-exposure constraints," *Journal of the American Statistical Association*, vol. 107, no. 498, pp. 592–606, 2012.
- [159] K. Scheinberg, S. Ma, and D. Goldfarb, "Sparse inverse covariance selection via alternating linearization methods," in *Advances in Neural Information Processing Systems*, 2010.
- [160] H. Zou and R. Li, "One-step sparse estimates in nonconcave penalized likelihood models (with discussion)," *The Annals of Statistics*, vol. 36, no. 4, pp. 1509–1533, 2008.
- [161] J. Fan, Y. Feng, and Y. Wu, "Network exploration via the adaptive LASSO and SCAD penalties," *Annals of Applied Statistics*, vol. 3, no. 2, pp. 521–541, 2009.
- [162] C. Lam and J. Fan, "Sparsistency and rates of convergence in large covariance matrix estimation," *The Annals of Statistics*, vol. 37, pp. 42–54, 2009.
- [163] G. Marjanovic and A. O. Hero III, "On lq estimation of sparse inverse covariance," *IEEE ICASSP*, 2014.
- [164] G. Marjanovic and V. Solo, "On lq optimization and sparse inverse covariance selection," *IEEE Trans. Signal Proces.*, vol. 62, no. 7, pp. 1644–1654, 2014.
- [165] G. Marjanovic and A. O. Hero III, "L0 Sparse Inverse Covariance Estimation," *IEEE Trans. Signal Proces.*, vol. 63, no. 12, pp. 3218–3231, 2015.
- [166] G. Marjanovic, M. Ulfarsson, and V. Solo, "Large-scale l0 sparse inverse covariance estimation," *IEEE ICASSP*, 2016, pp. 4767–4771.
- [167] Z. Fu, S. Han, A. Tan, et al., "L0-regularized time-varying sparse inverse covariance estimation for tracking dynamic fMRI brain networks," *37th Annual Int. Conf. IEEE Engin. Medicine and Biology Society*, 2015, pp. 1496–1499.
- [168] E. J. Candès and Y. Plan, "Matrix completion with noise," *Proceedings of the IEEE*, vol. 98, no. 6, pp. 925–936, 2010.
- [169] E. Candès and B. Recht, "Exact matrix completion via convex optimization," *Communications of the ACM*, vol. 55, no. 6, pp. 111–119, 2012.
- [170] J. F. Cai, E. J. Candès, and Z. Shen, "A singular value thresholding algorithm for matrix completion," *SIAM Journal on Optimization*, vol. 20, no. 4, pp. 1956–1982, 2010.
- [171] E. J. Candès, T. Tao, "The power of convex relaxation: Near-optimal matrix completion," *IEEE Trans. Inf. Theory*, vol. 56, no. 5, pp. 2053–2080, 2010.
- [172] Y. Koren, R. Bell, and C. Volinsky, "Matrix factorization techniques for recommender systems," *IEEE Comput.*, vol. 42, no. 8, pp. 30–37, Aug. 2009.
- [173] P. Chen and D. Suter, "Recovering the missing components in a large noisy low-rank matrix: Application to SFM," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 26, no. 8, pp. 1051–1063, Aug. 2004.
- [174] Z. Liu and L. Vandenberghe, "Interior-point method for nuclear norm approximation with application to system identification," *SIAM J. Matrix Anal. Appl.*, vol. 31, no. 3, pp. 1235–1256, 2009.
- [175] G. Marjanovic and V. Solo, "Lq matrix completion," *ICASSP*, 2012, pp. 3885–3888.
- [176] Z. Lu and Y. Zhang, "Schatten-p quasi-norm regularized matrix optimization via iterative reweighted singular value minimization," *arXiv Preprint*, arXiv:1401.0869v2, 2015.
- [177] M. Malek-Mohammadi, M. Babaie-Zadeh, and M. Skoglund, "Performance guarantees for Schatten-p quasi-norm minimization in recovery of low-rank matrices," *Signal Processing*, vol. 114, pp. 225–230, 2015.
- [178] F. Nie, H. Huang, and C. H. Q. Ding, "Low-rank matrix recovery via efficient Schatten p-norm minimization," *AAAI*, 2012.
- [179] F. Nie, H. Wang, X. Cai, et al., "Robust matrix completion via joint Schatten p-norm and lp-norm minimization," *ICDM*, 2012, pp. 566–574.
- [180] Y. Hu, D. Zhang, J. Ye, et al., "Fast and accurate matrix completion via truncated nuclear norm regularization," *IEEE Trans. Pattern Analysis Machine Intelligence*, vol. 35, no. 9, pp. 2117–2130, 2013.
- [181] Z. F. Jin, Z. Wan, Y. Jiao, et al., "An alternating direction method with continuation for nonconvex low rank minimization," *Journal of Scientific Computing*, vol. 66, no. 2, pp. 849–869, 2016.
- [182] Z. X. Cui and Q. Fan, "A nonconvex nonsmooth regularization method for compressed sensing and low rank matrix completion," *Digital Signal Processing*, vol. 62, pp. 101–111, 2017.
- [183] R. Sun and Z. Q. Luo, "Guaranteed matrix completion via non-convex factorization," *IEEE Trans. Inf. Theory*, vol. 62, no. 11, pp. 6535–6579, 2016.
- [184] F. Shang, Y. Liu, and J. Cheng, "Scalable algorithms for tractable Schatten quasi-norm minimization," in *AAAI*, 2016, pp. 2016–2022.
- [185] F. Shang, Y. Liu, and J. Cheng, "Tractable and scalable Schatten quasi-norm approximations for rank minimization," in *Proc. 19th Int. Conf. Artif. Intell. Statist.*, 2016, pp. 620–629.
- [186] F. Shang, Y. Liu, and J. Cheng, "Unified scalable equivalent formulations for Schatten quasi-norms," *arXiv preprint*, arXiv:1606.00668, 2016.
- [187] C. Croux and G. Haesbroeck, "Principal component analysis based on robust estimators of the covariance or correlation matrix: influence functions and efficiencies," *Biometrika*, vol. 87, no. 3, pp. 603–618, 2000.
- [188] F. De la Torre and M. J. Black, "Robust principal component analysis for computer vision," *ICCV*, 2001, pp. 362–369.
- [189] F. De La Torre and M. J. Black, "A framework for robust subspace learning," *International Journal of Computer Vision*, vol. 54, no. 1-3, pp. 117–142, 2003.
- [190] J. Wright, A. Ganesh, S. Rao, Y. Peng, and Y. Ma, "Robust principal component analysis: Exact recovery of corrupted low-rank matrices via convex optimization," in *Advances in Neural Information Processing Systems*, 2009, pp. 2080–2088.
- [191] E. J. Candès, X. Li, Y. Ma, and J. Wright, "Robust principal component analysis?" *Journal of the ACM (JACM)*, vol. 58, no. 3, p. 11, 2011.
- [192] Z. Zhou, X. Li, J. Wright, et al., "Stable principal component pursuit," in *Proc. IEEE Int. Symposium Information Theory (ISIT)*, 2010, pp. 1518–1522.
- [193] P. Netrapalli, U. N. Niranjan, S. Sanghavi, et al., "Non-convex robust pca," in *Advances in Neural Information Processing Systems*, 2014, pp. 1107–1115.
- [194] Q. Sun, S. Xiang, and J. Ye, "Robust principal component analysis via capped norms," in *Proc. 19th ACM SIGKDD*, 2013, pp. 311–319.
- [195] M. O. Ulfarsson, V. Solo, and G. Marjanovic, "Sparse and low rank decomposition using l0 penalty," *IEEE ICASSP*, 2015.
- [196] Z. Kang, C. Peng, and Q. Cheng, "Robust pca via nonconvex rank approximation," *IEEE Int. Conf. Data Mining (ICDM)*, 2015, pp. 211–220.
- [197] F. Wang, W. Cao, and Z. Xu, "Convergence of multi-block Bregman ADMM for nonconvex composite problems," *arXiv preprint*, arXiv:1505.03063, 2015.

- [198] F. Seidel, C. Hage, and M. Kleinsteuber, "pROST: a smoothed  $\ell_p$ -norm robust online subspace tracking method for background subtraction in video," *Machine Vision and Applications*, vol. 25, no. 5, pp. 1227–1240, 2014.
- [199] Q. Wang, F. Chen, Q. Gao, et al., "On the Schatten norm for matrix based subspace learning and classification," *Neurocomputing*, vol. 216, pp. 192–199, 2016.
- [200] J. K. Pant and S. Krishnan, "Two-pass  $\ell_p$ -regularized least-squares algorithm for compressive sensing," *IEEE Int. Symp. Circuits and Systems (ISCAS)*, 2017, pp. 1–4.
- [201] Y. Gao, J. Peng, and S. Yue, "Stability and robustness of the  $l_2/l_q$ -minimization for block sparse recovery," *Signal Processing*, vol. 137, pp. 287–297, 2017.
- [202] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge university press, 2004.
- [203] Y. Chen and Y. Chi, "Harnessing structures in big data via guaranteed low-rank matrix estimation," *IEEE Signal Processing Magazine*, vol. 35, no. 4, pp. 14–31, 2018.
- [204] Y. Chi, Y. M. Lu, and Y. Chen, "Nonconvex optimization meets low-rank matrix factorization: An overview," *arXiv preprint, arXiv:1809.09573*, 2018.
- [205] P. Jain and P. Kar, "Non-convex optimization for machine learning," *Foundations and Trends in Machine Learning*, vol. 10, no. 3-4, pp. 142–336, 2017.
- [206] H. Liu, T. Yao and R. Li, "Global solutions to folded concave penalized nonconvex learning," *Annals of statistics*, vol. 44, no. 2, pp. 629–659, 2016.
- [207] H. Liu, T. Yao, R. Li, and Y. Ye, "Folded concave penalized sparse linear regression: sparsity, statistical performance, and algorithmic theory for local solutions," *Mathematical programming*, vol. 166, no. 1-2, pp. 207–240, 2017.
- [208] D. Bertsimas, S. C. Martin, and R. Mazumder, "The trimmed Lasso: sparsity and robustness," *arXiv preprint, arXiv:1708.04527*, 2017.
- [209] I. Selesnick, "Sparse regularization via convex analysis," *IEEE Transactions Signal Processing*, vol. 65, no. 17, pp. 4481–4494, 2017.
- [210] A. Parekh and I. Selesnick, "Enhanced low-rank matrix approximation," *IEEE Signal Processing Letters*, vol. 23, no. 4, pp. 493–497, 2016.
- [211] I. Selesnick, "Total variation denoising via the Moreau envelope," *IEEE Signal Processing Letters*, vol. 24, no. 2, pp. 216–220, 2017.
- [212] D. Malioutov and A. Aravkin, "Iterative log thresholding," in *IEEE Int. Conf. Acoustics, Speech and Signal Processing (ICASSP)*, 2014.
- [213] Z. Li, S. Ding, T. Hayashi and Y. Li, "Incoherent dictionary learning with log-regularizer based on proximal operators," *Digital Signal Processing*, vol. 63, pp. 86–99, 2017.
- [214] C. Bao, H. Ji, Y. Quan, and Z. Shen, "Dictionary learning for sparse coding: Algorithms and convergence analysis," *IEEE Transactions Pattern Analysis and Machine Intelligence*, vol. 38, no. 7, pp. 1356–1369, 2016.
- [215] Z. Li, S. Ding, W. Chen, Z. Yang, and S. Xie, "Proximal alternating minimization for analysis dictionary learning and convergence analysis," *IEEE Transactions Emerging Topics in Computational Intelligence*, 2018.
- [216] Q. Yao, and J. T. Kwok, "Efficient learning with a family of nonconvex regularizers by redistributing nonconvexity," *The Journal of Machine Learning Research*, vol. 18, no. 1, pp. 1–52, 2018.
- [217] F. Wen, D. Zou, R. Ying, and P. Liu, "Efficient outlier removal for large scale global structure-from-motion," *arXiv preprint, arXiv:1808.03041*, 2018.
- [218] Z. G. Zhang, S. C. Chan, Y. Zhou, and Y. Hu, "Robust linear estimation using M-estimation and weighted L1 regularization: Model selection and recursive implementation," in *IEEE Int. Symp. Circuits and Systems*, 2009, pp. 1193–1196.
- [219] S. C. Chan, Y. J. Chu, and Z. G. Zhang, "A new variable regularized transform domain NLMS adaptive filtering algorithm—acoustic applications and performance analysis," *IEEE Transactions Audio Speech and Language Processing*, vol. 21, no. 4, pp. 868–878, 2013.
- [220] C. G. Tsinos, A. A. Rontogiannis, and K. Berberidis, "Distributed blind hyperspectral unmixing via joint sparsity and low-rank constrained non-negative matrix factorization," *IEEE Transactions Computational Imaging*, vol. 3, no. 2, pp. 160–174, June 2017.
- [221] S. Wu, X. Zhang, N. Guan, D. Tao, X. Huang, and Z. Luo, "Non-negative low-rank and group-sparse matrix factorization," in *Int. Conf. Multimedia Modeling*, 2015, pp. 536–547.
- [222] S. Oymak, A. Jalali, M. Fazel, Y. C. Eldar, and B. Hassibi, "Simultaneously structured models with application to sparse and low-rank matrices," *IEEE Transactions Information Theory*, vol. 61, no. 5, pp. 2886–2908, May 2015.
- [223] W. Chen, "Simultaneously sparse and low-Rank matrix reconstruction via nonconvex and nonseparable regularization," *IEEE Transactions Signal Processing*, vol. 66, no. 20, pp. 5313–5323, 2018.
- [224] Y. Gu, J. Jin, and S. Mei, " $\ell_0$  norm constraint LMS algorithm for sparse system identification," *IEEE Signal Processing Letters*, vol. 16, no. 9, pp. 774–777, Sep. 2009.
- [225] G. Su, J. Jin, Y. Gu, and J. Wang, "Performance analysis of  $\ell_0$  norm constraint least mean square algorithm," *IEEE Trans. Signal Process.*, vol. 60, no. 5, pp. 2223–2235, May 2012.
- [226] F. Y. Wu and F. Tong, "Gradient optimization  $p$ -norm-like constraint LMS algorithm for sparse system estimation," *Signal Processing*, vol. 93, no. 4, pp. 967–971, Apr. 2013.
- [227] L. Weruaga, and S. Jimaa, "Exact NLMS algorithm with  $\ell_p$ -norm constraint," *IEEE Signal Processing Letters*, vol.22, no.3, pp. 366–370, 2015.
- [228] I. Tomic and P. Frossard, "Dictionary learning," *IEEE Signal Process. Mag.*, vol. 28, no. 2, pp. 27–38, Mar. 2011.
- [229] M. Aharon, M. Elad, and A. Bruckstein, "K-SVD: An algorithm for designing overcomplete dictionaries for sparse representation," *IEEE Trans. Signal Process.*, vol. 54, no. 11, pp. 4311–4322, Nov. 2006.
- [230] H. Le, T. J. Chin, A. Eriksson, and D. Suter, "Deterministic approximate methods for maximum consensus robust fitting," *arXiv preprint, arXiv:1710.10003*, 2017.
- [231] P. V. Giampouras, K. E. Themelis, A. A. Rontogiannis, and K. D. Koutroumbas, "Simultaneously sparse and low-rank abundance matrix estimation for hyperspectral image unmixing," *IEEE Transactions Geoscience and Remote Sensing*, vol. 54, no. 8, pp. 4775–4789, 2016.
- [232] R. Ge, J. D. Lee, and T. Ma, "Matrix completion has no spurious local minimum," in *Advances in Neural Information Processing Systems*, 2016, pp. 2973–2981.