

# EXACT REGULARITY AND THE COHOMOLOGY OF TILING SPACES

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ABSTRACT. Exact regularity was introduced recently as a property of homological Pisot substitutions in one dimension. In this paper, we consider the analog of exact regularity for arbitrary tiling spaces. Let  $\mathbf{T}$  be a  $d$  dimensional repetitive tiling, and let  $\Omega$  be its hull. If  $\check{H}^d(\Omega, \mathbb{Q}) = \mathbb{Q}^k$ , then there exist  $k$  patches whose appearance govern the number of appearances of every other patch. This gives uniform estimates on the convergence of all patch frequencies to the ergodic limit. If the tiling  $\mathbf{T}$  comes from a substitution, then we can quantify that convergence rate. If  $\mathbf{T}$  is also one-dimensional, we put constraints on the measure of any cylinder set in  $\Omega$ .

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Ever since the seminal paper of Anderson and Putnam [AP], there has been a small industry devoted to computing topological invariants of tiling spaces. A key question throughout this effort has been “what do these invariants actually mean?” Put another way, if we determine (say) that the first cohomology of a 1-dimensional tiling space is  $\mathbb{Z}[1/2] \oplus \mathbb{Z}$ , what does that tell us about the properties of tilings in that space? Progress has been made, relating tiling cohomology to gap labeling [BBG], to deformations of tilings [CS], and to measures and patch frequencies [CGU]. In this paper we continue this last direction of inquiry and show how the top-dimensional cohomology governs not only patch frequencies over an entire tiling, but the number of appearances of a patch in any finite region.

Exact regularity was introduced in [BBJS]. In that paper, we considered one-dimensional substitution tilings with tile lengths chosen according to the left eigenvector of the substitution matrix, where the stretching factor  $\lambda$  is a Pisot number of algebraic degree  $k$ . If the rank of the first rational (Čech) cohomology of the tiling space is also  $k$ , then the number of appearances of a patch  $P$  in a return word is determined *exactly* by the Euclidean length (in  $\mathbb{R}^1$ ) of the return word.

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The proof never used the condition that  $\lambda$  was a Pisot number. It did rely on the tiling being 1-dimensional, and on the dimension of  $\check{H}^1$  equaling the algebraic degree of the stretching factor. The point is that, under these conditions, the integer span of the tile lengths is a rank- $k$  free Abelian group, so that specifying the length (in  $\mathbb{R}$ ) of a return word is equivalent to specifying  $k$  integers. These  $k$  integers then determine how many times  $P$  appears, up to a boundary term. For appropriately chosen return words, the contributions of the two boundaries cancel and we are left with an exact formula. Tilings for which this exact formula works are said to have the *Exact Regularity Property*, or ERP.

In this paper we study the situation where the dimension of  $\check{H}^1$  is unrelated to the algebraic degree of the stretching factor, or where the tiling does not come from a substitution at all, or when the tiling is of a higher-dimensional Euclidean space. In each of these cases the rank of the top cohomology determines how many integers are needed to keep track of an arbitrary patch  $P$  in an arbitrary region  $R$  (Theorem 1). The formulas are generally *not* exact, but involve correction terms that depend on the boundary of  $R$ . However, this is sufficient to prove estimates on the rate at which the frequency of  $P$  approaches its ergodic average (Theorems 2 and 3).

In some cases (e.g., substitution tilings in one dimension, and some special 2-dimensional examples) it is possible to choose a region where the boundary terms vanish. This allows us to set constraints on the possible measures of cylinder sets in 1-dimensional substitution tilings (Theorem 4).

Let  $\mathbf{T}_0$  be a tiling of  $\mathbb{R}^d$  that is translationally finite and repetitive. Being translationally finite (also called having finite local complexity) means that there are only a finite number of tile types, up to translation and there are only a finite number of ways that two tiles can meet. Being repetitive means that, for every patch  $P$ , there is an  $r_P$  such that in every ball of radius  $r_P$  there is at least one copy of  $P$ . Note that these assumptions exclude the pinwheel tiling, where tiles point in an infinite number of directions, and all tilings where patterns appear with frequency zero (e.g., a one dimensional tiling with one black tile and infinitely many white tiles). It is possible to address the pinwheel tiling, but the techniques are somewhat different. See the end of Section 4.

The *hull*  $\Omega$  of  $\mathbf{T}_0$ , also called the tiling space associated with  $\mathbf{T}_0$ , is the set of tilings  $\mathbf{T}$  with the property that every patch in  $\mathbf{T}$  is a translate of a patch in  $\mathbf{T}_0$ . Under the above assumptions,  $\Omega$  is a minimal dynamical system with respect to translations and is a compact space, with a metric where two tilings are close if they agree on a big ball centered at the origin, up to a small translation. If  $\mathbf{T}$  is in the hull of  $\mathbf{T}_0$ , then the hull of  $\mathbf{T}$  is the same as the hull of  $\mathbf{T}_0$ . For this reason, we usually speak of a minimal tiling space  $\Omega$ , rather than the hull of any one particular tiling.

**Theorem 1.** *Let  $\Omega$  be a compact and minimal space of translationally finite tilings of  $\mathbb{R}^d$ . Suppose that  $\check{H}^d(\Omega, \mathbb{Q}) = \mathbb{Q}^k$  for some integer  $k$ . Then there exist patches  $P_1, \dots, P_k$  with the following property: For any other patch  $P$  there exist rational numbers  $c_1(P), \dots, c_k(P)$  such that, for any region  $R$  in any tiling  $\mathbf{T} \in \Omega$ , the number of appearances of  $P$  in  $R$  equals  $\sum_{i=1}^k c_i(P)n_i + e(P, R)$ , where  $n_i$  is the number of appearances of  $P_i$  in  $R$ , and  $e(P, R)$  is an error term computable from the patterns that appear on the boundary of  $R$ . In particular, the magnitude of  $e(P, R)$  is bounded by a constant times the measure of the boundary of  $R$ .*

Note that Theorem 1 is not limited to substitution tilings, nor even to non-periodic tilings, but applies to any compact and minimal tiling space. For instance, if  $\mathbf{T}_0$  is a periodic tiling, then  $\Omega$  is a  $d$ -dimensional torus and  $\check{H}^d(\Omega, \mathbb{Q}) = \mathbb{Q}$ , so we only need to count one patch (say, a fundamental domain) to determine the number of all other patches. The Penrose tiling has  $\check{H}^2(\Omega, \mathbb{Q}) = \mathbb{Q}^8$ , so there are eight patches whose appearance controls the appearance of all other patches. Further examples are given in Section 4.

This theorem has immediate implications for the existence of ergodic limits, and for the rate of convergence to those limits.

**Theorem 2.** *Suppose that a tiling space  $\Omega$  satisfies the conditions of Theorem 1, and suppose that the patches  $P_1, \dots, P_k$  occur with well-defined frequencies  $f_1, \dots, f_k$ . That is, the number of occurrences of  $P_i$  in any ball of volume  $V$ , divided by  $V$ , approaches  $f_i$  as  $V \rightarrow \infty$ . Then  $\Omega$  is uniquely ergodic, and the frequency of any patch  $P$  approaches  $\sum c_i(P)f_i$  at least as quickly as the slowest of the  $P_i$ 's, or as  $V^{-1/d}$ , whichever is slower.*

In particular, a uniquely ergodic tiling space whose patch frequencies do not converge uniformly must have infinitely generated rational top cohomology.

This theorem is in some sense dual to the results of [CGU], who study possible measures on tilings spaces by considering frequencies of (possibly collared) tiles and requiring that they satisfy a set of homological ‘‘Kirchoff’s Rules’’. Knowing the (co)homology of the tiling space then puts constraints on the possible invariant measures.

When the tiling comes from a substitution, we can further sharpen our convergence estimates. Let  $\mathbf{T}$  be a repetitive and non-periodic tiling derived from a primitive self-similar substitution. Let  $M$  be the matrix of the substitution. That is,  $M_{ij}$  is the number of times that the  $i$ -th tile type appears in the substitution of the  $j$ -th tile type. Arrange the eigenvalues  $\lambda_i$  of  $M$  in decreasing order of size. Note that  $\lambda_1$  is real and positive and is strictly larger than  $|\lambda_2|$ .

**Theorem 3.** *If  $\mathbf{T}$  is a self-similar tiling as above, and if  $P$  is any patch in  $\mathbf{T}$ , then there exist constants  $K$  and  $\nu$  such that, for any ball  $R$  of sufficiently large volume  $V$ ,*

$$(1) \quad \left| \frac{\text{number of } P \text{'s in } R}{V} - \sum c_i(P) f_i \right| < KV^{-\gamma} (\log V)^\nu,$$

where  $\gamma = \min(d^{-1}, 1 - \frac{\log |\lambda_2|}{\log |\lambda_1|})$ .

We obtain our strongest results when the tiling is 1-dimensional and comes from a substitution. Let  $\Omega$  be a 1-dimensional tiling space coming from a primitive substitution  $\phi$ . The top cohomology  $\check{H}^1(\Omega, \mathbb{Q}) = \mathbb{Q}^k$  is finitely generated [BD]. The substitution  $\phi$  maps  $\Omega$  to itself, and therefore maps  $\check{H}^1(\Omega, \mathbb{Q})$  to itself. This last action can be expressed by a non-singular  $k \times k$  integer matrix  $A$ . (Note:  $A$  is typically different from the matrix  $M$  of the substitution, but both matrices have the same leading eigenvalue [BD], namely the stretching factor  $\lambda$ .)

The *minimal polynomial* of a matrix  $A$  is the lowest order monic polynomial  $p(x)$  for which  $p(A) = 0$ . Likewise, the minimal polynomial of the leading eigenvalue  $\lambda$  is the lowest order monic polynomial  $q(x)$  for which  $q(\lambda) = 0$ . Since  $p(\lambda) = 0$ ,  $q(x)$  divides  $p(x)$  and we can write  $p(x) = q(x)r(x)$  for some integer polynomial  $r(x)$ . If  $A$  is primitive, then the polynomials  $q(x)$  and  $r(x)$  have no roots in common, so we can find a nonzero integer  $D$  and polynomials  $Q(x)$  and  $R(x)$  with integer coefficients such that  $Q(x)q(x) + R(x)r(x) = D$ . The smallest such integer  $D$  is called the *congruence number* [WT] or *reduced resultant* [Po] of  $q(x)$  and  $r(x)$ .

Connected patches in a 1-dimensional tiling correspond to sequences of letters, also called words, where each letter designates a tile type. If the word  $\ell_1 \dots \ell_n \ell_{n+1}$  occurs in the tiling, and if  $\ell_1 = \ell_{n+1}$ , then  $\ell_1 \dots \ell_n$  is called a *return word* and the total length of the tiles corresponding to the letters  $\ell_1 \dots \ell_n$  is called a *return length*.

**Theorem 4.** *Let  $\Omega_\phi$  be a tiling space obtained from a primitive one-dimensional substitution  $\phi$ , let  $\lambda$  be the stretching factor of  $\phi$ , let  $L > 0$  be a return length, let  $\mathbf{T} \in \Omega_\phi$ , and let  $P$  be any patch in  $\mathbf{T}$ . Then the frequency of  $P$  in  $\mathbf{T}$  takes the form*

$$(2) \quad f(P) = \frac{u_P(\lambda)}{LDq'(\lambda)q_0^n},$$

where  $u_P(x)$  is a polynomial with integer coefficients,  $q(x)$  is the minimal polynomial of  $\lambda$ ,  $q'(x)$  is its derivative,  $q_0$  is the constant coefficient of  $q(x)$ , and  $D$  is the reduced resultant of  $q(x)$  and  $r(x)$ .

This theorem has a very different flavor from Theorems 1–3. The previous theorems are ergodic in nature and describe convergence to an ergodic limit, while Theorem 4 is algebraic and puts constraints on what that limit can be.

In Section 2 we provide necessary background for understanding the proofs of these theorems. In Section 3 we prove the four theorems, and in Section 4 we provide examples that illustrate how exact regularity works in practice.

## 2. BACKGROUND

A tile is a topological disk that is the closure of its interior, together with a label. A tiling of  $\mathbb{R}^d$  is a collection of tiles that overlap only on their boundaries, whose union is all of  $\mathbb{R}^d$ . A *patch* is a finite collection of tiles that intersect only on their boundaries, and the support of a patch is the union of these tiles.  $[B_r]^{\mathbf{T}}$  denotes the patch consisting of all tiles that intersect a closed ball of radius  $r$  around the origin in the tiling  $\mathbf{T}$ . The translation group acts naturally on tilings by moving each tile simultaneously. If  $\mathbf{T}$  is a tiling, then  $\mathbf{T} - x$  is the same tiling translated by the vector  $x$ , so that a neighborhood of the origin in  $\mathbf{T} - x$  looks like a neighborhood of  $x$  in  $\mathbf{T}$ .

A tiling *space*  $\Omega$  is a topological space whose elements are tilings, with a metric topology in which two tilings are close if they agree on a big ball around the origin, up to a small translation. In this paper we assume throughout that  $\Omega$  is compact and is a minimal dynamical system with respect to translations. This is equivalent to  $\Omega$  being the hull of a tiling that is translationally finite (for compactness) and repetitive (for minimality).

Every compact tiling space is topologically conjugate to another tiling space whose tiles are polytopes. Without loss of generality, we henceforth assume that all of our tiles are polytopes.

A self-similar substitution is a map that replaces each tile with a patch whose support is the original tile, scaled by a *linear stretching factor*  $\lambda_0$ . This action extends by concatenation to patches, and indeed to tilings. A *supertile of order  $n$*  is a patch obtained by applying the substitution  $n$  times to a tile.

The *matrix of the substitution* (also called the *abelianization* when the tilings in question are 1-dimensional) is the matrix  $M$  whose entries  $M_{ij}$  count the number of times that the  $i$ -th tile type appears in a substituted  $j$  tile. A square matrix  $A$  with non-negative entries is *primitive* if all the entries of some power  $A^n$  are positive. Every primitive matrix has a largest positive eigenvalue  $\lambda_{PF}$ , called the Perron-Frobenius eigenvalue. This eigenvalue has (algebraic and geometric) multiplicity one, and the corresponding left- and right-eigenvectors have strictly positive entries. All other eigenvalues are strictly smaller than  $\lambda_{PF}$  in magnitude.

If  $\phi$  is a tiling substitution with matrix  $M$ , and if  $M$  is primitive, then the relative frequencies of all tile types is given by the right Perron-Frobenius eigenvector of  $M$ .

The relative volumes of the tile types in a self-similar tiling are given by the left Perron-Frobenius eigenvector of  $M$ . The Perron-Frobenius eigenvalue is  $\lambda = \lambda_0^d$ , where  $\lambda_0$  is the linear stretching factor.

A substitution  $\phi$  *forces the border* [Kel1] if there is an integer  $n$  with the following property. If  $t_1$  and  $t_2$  are tiles in  $\mathbf{T}$  of the same type, then the patches  $\phi^n(t_1)$  and  $\phi^n(t_2)$  not only agree in  $\phi^n(\mathbf{T})$ , up to translation, but their nearest neighbors also agree. If a substitution forces the border, then the tiling space is the inverse limit of the Anderson-Putnam complex [AP], and all cohomology classes are generated by the duals to the vertices, edges, faces, etc., of this complex. If a substitution does not force the border, then we can define a new tile set using *collared tiles*. A collared tile is a tile together with a label describing the immediate neighbors of that tile. For instance, in the tiling  $\dots baab \dots$ , the two  $a$ 's are different as collared tiles, in that the first is preceded by a  $b$  and followed by an  $a$ , while the second is preceded by an  $a$  and followed by a  $b$ .

Collaring greatly increases the number of tile types, leading to a more complicated substitution matrix. The new substitution forces the border while describing the same tiling space as the original [AP].

Let  $M$  be the matrix of a self-similar substitution and let  $M'$  be the matrix of the same substitution rewritten in terms of collared tiles. If  $t_1$  and  $t_2$  are tiles of the same type, but are different as collared tiles, then the supertiles  $\phi^n(t_1)$  and  $\phi^n(t_2)$  are the same (up to translation) as collections of regular tiles. Viewed as collections of collared tiles, they can disagree only near the boundary. Thus the differences between substituted collared tiles of the same uncollared type can grow at most as  $\lambda_0^{n(d-1)}$ . This implies that, if an eigenvalue  $\lambda'$  of  $M'$  is not an eigenvalue of  $M$ , then  $|\lambda'| \leq \lambda_0^{d-1}$ .

A 1-dimensional substitution is *proper* if every substituted letter begins with the same letter, and every substituted letter ends with the same letter. For instance, the substitution  $a \rightarrow abbabb$ ,  $b \rightarrow aabab$  is proper, in that all substituted letters start with  $a$  and end with  $b$ . In a proper substitution, every substituted letter is a return word. It is always possible to rewrite a substitution to make it proper, without changing the underlying tiling space. This rewriting will change the substitution matrix, but the only eigenvalues that can appear or disappear from the rewriting process are zero and roots of unity. In particular, the Perron-Frobenius eigenvalue is unchanged. If a substitution is proper, then the first Čech cohomology is the direct limit of the transpose of the substitution matrix [BD].

Pattern-equivariant cohomology was first defined by Kellendonk and Putnam [Kel2, KP] using differential forms, and then recast in [Sa2] in terms of cochains. Here we consider rational pattern-equivariant cohomology using cochains.

A  $d$ -dimensional tiling  $\mathbf{T}$  gives  $\mathbb{R}^d$  the structure of a CW complex, with the vertices serving as 0-cells, the edges serving as 1-cells, the 2-dimensional faces as 2-cells, and so on. We consider rational cellular cochains on this CW complex, with a  $n$ -cochain assigning a rational number to each  $n$ -cell.

**Definition 5.** A rational 0-cochain is said to be pattern-equivariant with radius  $r$  if, whenever  $x$  and  $y$  are vertices of  $\mathbf{T}$  and  $[B_r]^{\mathbf{T}-x} = [B_r]^{\mathbf{T}-y}$ , the cochain takes the same values at  $x$  and  $y$ . A 0-cochain is pattern-equivariant if it is pattern-equivariant with radius  $r$  for some finite  $r$ . Pattern-equivariant  $n$ -cochains for  $n > 0$  are defined similarly – their values on a  $n$ -cell depend only on the pattern of the tiling out to a fixed finite distance around that  $n$ -cell.

If  $\beta$  is a rational pattern-equivariant  $n$ -cochain, its coboundary,  $\delta_n(\beta)$ , is a rational pattern-equivariant  $(n + 1)$ -cochain.

**Definition 6.** The rational  $n$ -th pattern-equivariant cohomology of  $\mathbf{T}$  is  $H_{PE}^n(\mathbf{T}, \mathbb{Q}) = \text{Ker}(\delta_n) / \text{Im}(\delta_{n-1})$ .

A priori this would seem to depend on  $\mathbf{T}$ , but if the tiling space  $\Omega$  is minimal, then  $H^n(\mathbf{T}, \mathbb{Q})$  is the same for all  $\mathbf{T} \in \Omega$  and is isomorphic to  $\check{H}^n(\Omega, \mathbb{Q})$  [Kel2, KP, Sa2]. (Even if  $\Omega$  is not minimal,  $H_{PE}^n(\mathbf{T}, \mathbb{Q})$  is isomorphic to  $\check{H}^n(\Omega_{\mathbf{T}}, \mathbb{Q})$ , where  $\Omega_{\mathbf{T}}$  is the hull of  $\mathbf{T}$ .)

An *indicator cochain* for a patch  $P$  is a  $d$ -cochain that evaluates to 1 on a particular tile of  $P$ , and evaluates to 0 on all other tiles, whether in  $P$  or not. In other words, it counts the occurrences of  $P$ . It's easy to see that all indicator cochains for a specific  $P$  are cohomologous, and that every pattern-equivariant  $d$ -cochain is a linear combination of indicator cochains. In particular, every cohomology class in  $H_{PE}^d(\mathbf{T}, \mathbb{Q})$  can be represented by a linear combination of indicator cochains.

If two monic polynomials  $q(x)$  and  $r(x)$  with integer coefficients have no roots in common, then the *resultant* of  $q$  and  $r$  is  $\text{Res}(q, r) = \prod_{i,j} (\lambda_i - \mu_j) = \prod_i r(\lambda_i) = \pm \prod_j q(\mu_j)$ , where  $q(x) = \prod (x - \lambda_i)$  and  $r(x) = \prod (x - \mu_j)$ . This quantity is easily computed and is closely related to the reduced resultant  $D$  discussed earlier.  $D$  and  $\text{Res}(q, r)$  have the same prime factors and  $D$  always divides  $\text{Res}(q, r)$ , but the two numbers are not always equal. Computing  $D$  is usually more difficult; see [WT] for an algorithm.

### 3. PROOFS

*Proof of Theorem 1.* Every pattern-equivariant cochain is a linear combination of indicator cochains. Furthermore, each  $d$ -dimensional cochain is closed, and hence defines a cohomology class. If  $\check{H}^d(\Omega, \mathbb{Q})$  is  $k$ -dimensional, we can find  $k$  patches  $P_1, \dots, P_k$ ,

whose indicator cochains  $\chi_1, \dots, \chi_k$  represent linearly independent classes in  $\check{H}^d(\Omega, \mathbb{Q})$ . Let  $[\chi_i]$  be the cohomology class of  $\chi_i$ . Likewise, let  $\chi_P$  be an indicator cochain of the patch  $P$ , and let  $[\chi_P]$  be its cohomology class. Since  $\{[\chi_1], \dots, [\chi_k]\}$  is a basis for  $\check{H}^d(\Omega, \mathbb{Q})$ , there exist rational numbers  $c_1(P), \dots, c_k(P)$  such that  $[\chi_P] = \sum_i c_i(P)[\chi_i]$ , hence

$$(3) \quad \chi_P = \sum c_i(P)\chi_i + \delta\alpha,$$

for some pattern-equivariant  $(d-1)$ -cochain  $\alpha$ . Now apply both sides of equation (3) to a region  $R$ . The left hand side gives the number of  $P$ 's in  $R$ , while the right-hand side gives  $\alpha(\partial R) + \sum_{i=1}^k c_i(P)n_i$ , where  $\partial R$  is the boundary of  $R$ , viewed as a chain. The term  $\alpha(\partial R)$  is our error term  $e(P, R)$ , and is bounded in magnitude by a constant times the size of  $\partial R$ .  $\square$

Note that we have actually proved something stronger than Theorem 1, since we have obtained a formula for the error. This will become important when we consider 1-dimensional tilings, and some special 2-dimensional tilings, where for appropriate regions we can get the error term to vanish.

*Proof of Theorem 2.* This is an immediate corollary of Theorem 1. For any region  $R$  of volume  $V$ , let  $n$  be the number of times  $P$  appears in  $R$ , and let  $n_i$  be the number of times that  $P_i$  appears.

$$\begin{aligned} \frac{n}{V} - \sum f_i c_i(P) &= \frac{e(P, R) + \sum n_i c_i(P)}{V} - \sum f_i c_i(P) \\ &= \frac{e(P, R)}{V} + \sum c_i(P) \left( \frac{n_i}{V} - f_i \right). \end{aligned}$$

The first term goes to zero as  $V^{-1/d}$ , while the others converge at worst at the rate of the slowest  $P_i$ .  $\square$

Before proving Theorem 3, which concerns the convergence of the frequencies of arbitrary patches, we consider the convergence of the frequencies of the basic tile types.

**Lemma 7.** *There exist constants  $c_0$  and  $\nu_0$  such that, if tile type  $i$  occurs with frequency  $f_i$ , and if  $\phi^n(t_j)$  is an  $n$ -th order supertile of volume  $V$ , then*

$$(4) \quad |\text{Number of tiles of type } i \text{ in } \phi^n(t_j) - (f_i \times \text{Volume of } \phi^n(t_j))| \leq c_0 |\lambda_2|^n n^{\nu_0},$$

where  $\lambda_2$  is the second-largest eigenvalue of  $M$ .

*Proof.* This is straightforward linear algebra. If  $v_n$  is a column vector whose  $i$ -th entry is the number of tiles of type  $i$  in  $\phi^n(t_j)$  minus  $f_i$  times the volume of  $\phi^n(t_j)$ , then  $v_{n+1} = Mv_n$  and  $v_n$  has no component in the Perron-Frobenius eigenspace of  $M$ . The



vector  $v_n$  thus grows at most as  $|\lambda_2|^n$  if  $M$  is diagonalizable, and at most as a polynomial in  $n$  times  $|\lambda_2|^n$  if  $M$  is not diagonalizable.  $\square$

**Lemma 8.** *There exist constants  $c$  and  $\nu$  such that, for any ball  $R$  of volume  $V$ ,*

$$(5) \quad |\text{Number of tiles of type } i \text{ in } R - f_i V| < c(\log V)^\nu V^{\max\left(\frac{d-1}{d}, \frac{\log|\lambda_2|}{\log\lambda_1}\right)}.$$

*Proof.* Let  $n$  be the smallest integer such that  $V$  is less than the volume of the smallest  $n + 1$ -st order supertile. This means that  $V$  is bounded both above and below by a constant times  $\lambda_0^{nd}$ . We write  $R$  as the union of supertiles of order  $m$ , where  $m$  ranges from 0 to  $n$ , plus a number of partial tiles at the boundary of  $R$ . First we identify any complete  $n$ -th order supertiles inside  $R$ , then identify the complete  $(n - 1)$ -st order supertiles in the remainder of  $R$ , then identify the complete  $(n - 2)$ -nd order supertiles in what is left, and so on. The number of partial tiles is bounded by a constant times the surface area of  $R$ , which goes as  $V^{\frac{d-1}{d}} \sim \lambda_0^{n(d-1)}$ . The number of supertiles of order  $m$  is bounded by a constant times the surface area of  $R$  scaled down by  $\lambda_0^m$ , hence  $\lambda_0^{(n-m)(d-1)}$ . The contributions of each supertile are governed by equation (4), so the total contribution of all the supertiles of level  $m$  is of order  $\lambda_0^{n(d-1)} \left(\frac{|\lambda_2|}{\lambda_0^{\frac{d-1}{d}}}\right)^m m^{\nu_0}$ .

The sum  $\sum_{m=0}^n \lambda_0^{n(d-1)} \left(\frac{|\lambda_2|}{\lambda_0^{\frac{d-1}{d}}}\right)^m m^{\nu_0}$  is bounded by the number of terms times the largest term, hence by a constant times  $n^{\nu_0+1}$  times either  $|\lambda_2|^n$  or  $\lambda_0^{n(d-1)}$ , whichever is larger. Since  $V$  is of order  $\lambda_0^{nd}$ ,  $|\lambda_2|^n$  is of order  $V^{\frac{\log|\lambda_2|}{\log(\lambda_1)}}$ , while  $\lambda_0^{n(d-1)}$  is of order  $V^{\frac{d-1}{d}}$ .  $\square$

*Proof of Theorem 3.* First suppose that the substitution forces the border, so that  $\check{H}^d(\Omega, \mathbb{Q})$  is generated by the duals to the various (uncollared) tile types  $[AP]$ . This implies that the top pattern-equivariant cohomology is generated by the indicator cochains of these tile types, and that we can take our patches  $P_1, \dots, P_k$  to simply be different types of uncollared tiles. By Theorem 2, the number of appearances of  $P$  converges no slower than the slowest of the  $P_i$ 's. However, the number of each tile type in the region  $R$  is governed by equation (5), which is tantamount to Theorem 3.

If the substitution does not force the border, then we rewrite it using collared tiles. This changes the substitution matrix, but equation (5) still applies, albeit with  $\lambda_2$  being the second-largest eigenvalue of the new matrix. If this is the same as the second-largest eigenvalue of the old matrix, then we are done. If not, then  $|\lambda_2| \leq \lambda_0^{d-1} = \lambda_1^{\frac{d-1}{d}}$ , so  $\max\left(\frac{d-1}{d}, \frac{\log|\lambda_2|}{\log(\lambda_1)}\right) = \frac{d-1}{d}$ . Dividing by  $V$ , we again get the estimate of Theorem 3.  $\square$

*Proof of Theorem 4.* Without loss of generality, we assume that the substitution  $\phi$  is proper. Imagine applying the indicator cochain  $\chi_P$  to  $\phi^n(\ell_2)$  for some letter  $\ell_2$  that sits

in the 3-letter word  $\ell_1\ell_2\ell_3$ . Since the beginning of  $\phi(\ell_2)$  and  $\phi(\ell_3)$  are the same, and since the end of  $\phi(\ell_1)$  and  $\phi(\ell_2)$  are the same, the exact piece  $\delta\alpha$  in the decomposition (3) of  $\chi_P$  evaluates to zero on  $\phi^n(\ell_2)$  if  $n$  is sufficiently large. This means that counting  $P$ 's in  $\phi^n(\ell_2)$ , for  $n$  large, is purely a cohomological calculation.

Now let  $\beta$  be a rational pattern-equivariant cochain. We say that  $\beta$  is *regular* if, for any letter  $\ell$  and for any sufficiently large  $n$ ,  $\beta(\phi^n(\ell))$  is an integer. Clearly, every indicator cochain is regular, but a rational linear combination of indicator cochains may not be.

Let  $\gamma$  be a 1-cochain, which we can write as a linear combination  $\sum_P c_P \chi_P$  of indicator cochains with rational coefficients. Define the *trace of  $\gamma$*  to be

$$(6) \quad Tr(\gamma) = \sum_P c_P f_P,$$

where  $f_P$  is the frequency of the patch  $P$ . This trace is the same as the Ruelle-Sullivan map in the context of [KP], and is closely related to the trace operation on  $K^0$  of the  $C^*$ -algebra defined by the action of translations on  $\Omega_T$ .

**Lemma 9.** *Let  $\mathbf{T}$  be a 1-dimensional tiling obtained from a primitive substitution  $\phi$ . Let  $p(x)$  be the minimal polynomial of the operator  $A$  that represents the action of substitution on  $H_{PE}^1(\mathbf{T}, \mathbb{Q})$ , let  $q(x)$  be the minimal polynomial of the Perron-Frobenius eigenvalue of the substitution matrix, and let  $p(x) = q(x)r(x)$ . Let  $L$  be a return length. If  $\beta$  is a regular 1-cochain and  $q(A)[\beta] = 0$ , then*

$$(7) \quad Tr(\beta) = \frac{u_\beta(\lambda)}{Lq'(\lambda)q_0^n}$$

for some polynomial  $u_\beta$  with integer coefficients.

*Proof.* The following proof is a small modification of the proof of Theorem 9 of [BBJS]. Let  $s$  be the algebraic degree of  $\lambda$  and let  $w$  be a return word of length  $L$ . Since  $\phi(w), \phi^2(w), \dots$  are return words,  $\lambda L, \lambda^2 L, \dots$  are return lengths, and  $L\mathbb{Z}[\lambda]$  is a rank- $s$  Abelian group, hence a finite-index subgroup of the span of all the tile lengths. In particular, the length of any tile  $t$  can be written uniquely in the form  $|t| = L(c_0(t) + c_1(t)\lambda + \dots + c_{s-1}(t)\lambda^{s-1})$ , where each  $c_i(t)$  is rational. Define pattern-equivariant 1-cochains  $\xi_0, \dots, \xi_{s-1}$  by  $\xi_i(t) = c_i(t)$ . It is not hard to see that the cohomology classes  $[\xi_i]$  are linearly independent (see Lemma 8 of [BBJS]). Under substitution, the cochains

$\xi_0, \dots, \xi_{s-1}$  transform via the matrix

$$(8) \quad C = \begin{pmatrix} 0 & 0 & \cdots & 0 & -q_0 \\ 1 & 0 & \cdots & 0 & -q_1 \\ 0 & 1 & \cdots & 0 & -q_2 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -q_{s-1} \end{pmatrix},$$

where  $q(x) = x^s + q_{s-1}x^{s-1} + \cdots + q_0$  is the minimal polynomial of  $\lambda$ . Note that the characteristic polynomial of  $C$  is precisely  $q(x)$ , and hence that the classes  $[\xi_i]$  span the kernel of  $q(A)$ . In particular, the cohomology class  $[\beta]$  is a rational linear combination of the  $[\xi_i]$ 's. We can therefore write  $\beta = \sum \beta_i \xi_i + \delta\alpha$ , where  $\alpha$  is a pattern-equivariant 0-cochain and the coefficients  $\beta_i$  are rational. Applied to any word of the form  $\phi^n(w)$ , for  $n$  large enough,  $\delta\alpha$  yields zero, while  $\beta(\phi^n(w))$  yields an integer.

Note that, for any integers  $n \geq i \geq 0$ ,  $q_0^{n-i}\lambda^i$  is a linear combination of  $\lambda^n, \lambda^{n+1}, \dots, \lambda^{n+s-1}$ . To see this, divide the equation  $q(\lambda) = 0$  by  $\lambda$  to get  $q_0\lambda^{-1} = -(\lambda^{s-1} + q_{s-1}\lambda^{s-2} + \cdots + q_1)$ . Taking the  $n-i$ -th power, applying the equation  $q(\lambda) = 0$  to eliminate large powers of  $\lambda$ , and finally multiplying by  $\lambda^n$ , gives the result. This implies that  $q_0^{n-i}(\beta - \delta\alpha)$ , applied to  $\phi^i(w)$ , yields an integer, which in turn implies that each coefficient  $\beta_i$  is an integer divided by  $q_0^i$ .

The trace of  $\xi_i$  is easily computed. We just apply  $\xi$  to  $\phi^n(w)$ , divide by the length of  $\phi^n(w)$ , and take the limit as  $n \rightarrow \infty$ . This is equivalent to writing  $\lambda^n = \sum_{i=0}^{s-1} c_{n,i}\lambda^i$  and taking the limit of  $c_{n,i}/\lambda^n$ , and is precisely the  $i$ -th entry of the right eigenvector  $\vec{v}_r$  of  $C$ , with eigenvalue  $\lambda$ , normalized so that  $(1, \lambda, \dots, \lambda^{s-1})\vec{v}_r = 1$ . This eigenvector is:

$$(9) \quad \vec{v}_r = \frac{1}{q'(\lambda)} \begin{pmatrix} \lambda^{s-1} + q_{s-1}\lambda^{s-2} + \cdots + q_1 \\ \lambda^{s-2} + q_{s-1}\lambda^{s-3} + \cdots + q_2 \\ \lambda^{s-3} + q_{s-1}\lambda^{s-4} + \cdots + q_3 \\ \vdots \\ \lambda + q_{s-1} \\ 1 \end{pmatrix}.$$

Since each entry of  $\vec{v}_r$  is a polynomial in  $\lambda$  divided by  $q'(\lambda)$ , and since each  $\beta_i$  is an integer divided by a power of  $q_0$ , the trace of  $\beta$  is of the desired form.  $\square$

**Lemma 10.** *Let  $\beta$  be any pattern-equivariant 1-cochain. If  $r(A)[\beta] = 0$ , then  $Tr(\beta) = 0$ .*

*Proof.* If  $r(A)[\beta] = 0$ , then the cohomology class  $[\beta]$  has no component that scales under  $n$ -fold substitution as  $\lambda^n$ . Thus  $\lim \beta(\phi^n(\ell))/\lambda^n = \lim (A^n(\beta))(\ell)/\lambda^n = 0$ . Since, in the limit,  $\beta$  averages to zero on patches of the form  $\phi^n(\ell)$ , the trace of  $\beta$  is zero.  $\square$

Finally, let  $\chi_P$  be an indicator cochain. Since  $\chi_P$  is regular,  $A(\chi_P) = \chi_P \circ \phi$  is regular, and so is any polynomial in  $A$  applied to  $\chi_P$ . In particular,  $Q(A)q(A)\chi_P$  and  $R(A)r(A)\chi_P$  are regular. Since  $[r(A)Q(A)q(A)\chi_P] = p(A)[Q(A)\chi_P] = 0$ , the trace of  $R(A)r(A)\chi_P$  is of the form indicated in Lemma 9. Likewise, the trace of  $Q(A)q(A)\chi_P$  is zero by Lemma 10. Since  $D\chi_P = Q(A)q(A)\chi_P + R(A)r(A)\chi_P$ , the trace of  $\chi_P$  is  $D^{-1}$  times something of the form indicated in Lemma 9, which completes the proof of Theorem 4.  $\square$

#### 4. EXAMPLES

**4.1. The Thue-Morse Tiling.** The Thue-Morse tiling is a 1-dimensional tiling given by the substitution  $\phi_1(a) = ab$ ,  $\phi_1(b) = ba$ . This substitution is not proper, but being proper is not a requirement for Theorems 1 and 4. In the case of Thue-Morse, it is not difficult to compute the cohomology directly by a variety of methods. The first rational PE-cohomology is  $\mathbb{Q}^2$ , and is generated by the indicators of the patches  $P_1 = ab$  and  $P_2 = aa$ . The matrix  $A$  (in an appropriate basis) is  $\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$  with eigenvalues  $\lambda_{PF} = 2$  and  $\lambda_2 = -1$ . Note that  $q(x) = x - 2$ ,  $r(x) = x + 1$ , and  $D = 3$ , insofar as  $3 = (x + 1) - (x - 2)$ . We will show directly that the appearance of every patch in a return word is governed, up to coboundaries, by the appearance of  $P_1$  and  $P_2$ . Note that the frequencies of  $P_1$  and  $P_2$  are  $1/3$  and  $1/6$ , respectively, which are *not* in  $Z[1/2]$ . The factor of  $D^{-1}$  in Theorem 4 is indeed necessary.

Let  $P_3 = aababb$ . We will show how the appearance of  $P_3$  is controlled by the appearance of  $P_1$  and  $P_2$ . For definiteness, pick indicator cochains  $\chi_i$  ( $i = 1, 2, 3$ ) that equal one on the first letter of  $P_i$  and are zero elsewhere. We will show that, for any region  $R$ ,  $\chi_3(R) = \frac{7}{8}\chi_2(R) - \frac{1}{8}\chi_1(R) +$  boundary terms.

We begin by evaluating each  $\chi_i$  on  $\phi_1^3(a) = abbabaab$  and  $\phi_1^3(b) = baababba$ . The results are:

$$(10) \quad \begin{aligned} \chi_1(\phi_1^3(a)) &= 3 & \chi_1(\phi_1^3(b)) &= \begin{cases} 2 & \text{if } \phi_1^3(b) \text{ is followed by } \phi_1^3(a) \\ 3 & \text{if } \phi_1^3(b) \text{ is followed by } \phi_1^3(b) \end{cases} \\ \chi_2(\phi_1^3(a)) &= 1 & \chi_2(\phi_1^3(b)) &= \begin{cases} 2 & \text{if } \phi_1^3(b) \text{ is followed by } \phi_1^3(a) \\ 1 & \text{if } \phi_1^3(b) \text{ is followed by } \phi_1^3(b) \end{cases} \\ \chi_3(\phi_1^3(a)) &= \begin{cases} 0 & \text{if } \phi_1^3(a) \text{ is followed by } \phi_1^3(a) \\ 1 & \text{if } \phi_1^3(a) \text{ is followed by } \phi_1^3(b) \end{cases} & \chi_3(\phi_1^3(b)) &= 1 \end{aligned}$$

Next, suppose that  $w$  is a return word. The number of times that the patch  $ab$  appears in  $\phi_1(w)$  equals the number of times that  $ba$  appears. Also,  $\phi_1(w)$  has as many appearances of  $a$  as of  $b$ . Thus, if we treat  $\phi_1^4(w)$  as the concatenation of 3rd order

“supertiles” of the form  $\phi_1^3(a)$  and  $\phi_1^3(b)$ , then there are equal numbers of  $a$  and  $b$  type supertiles, and the number of  $a$ -supertiles that are followed by  $b$ -supertiles equals the number of  $b$ -supertiles that are followed by  $a$ -supertiles. If there are  $k_1$   $a$ -supertiles,  $k_2$  of which are followed by  $b$ -supertiles, then

$$(11) \quad \begin{aligned} \chi_1(\phi_1^4(w)) &= 6k_1 - k_2; \\ \chi_2(\phi_1^4(w)) &= 2k_1 + k_2 \\ \chi_3(\phi_1^4(w)) &= k_1 + k_2 = -\frac{1}{8}\chi_1(\phi_1^4(w)) + \frac{7}{8}\chi_2(\phi_1^4(w)). \end{aligned}$$

Finally, let  $R$  be any region. We can always write  $R = p\phi_1^4(w)s$ , where  $w$  is a return word and the prefix  $p$  and the suffix  $s$  each have length at most 48. The number  $n_i$  of occurrences of  $P_i$  in the prefix and suffix do *not* have to satisfy  $n_3 = \frac{-n_1}{8} + \frac{7n_2}{8}$ , but the deviation from this rule is computable from the local patterns  $p$  and  $s$ . We therefore have a pattern-equivariant 0-cochain  $\alpha$ , with radius at most 48, such that

$$(12) \quad \chi_3 = \frac{-\chi_1}{8} + \frac{7\chi_2}{8} + \delta\alpha.$$

The exact same argument would work for any patch  $P_4$  of length at most 8. We just have to evaluate  $\chi_4$  on  $\phi_1^3(a)$  and  $\phi_1^3(b)$ , and use this information to evaluate  $\chi_4$  on  $\phi_1^4(w)$  for any return word  $w$ , with the answer being a linear function of  $k_1$  and  $k_2$ . Since  $\chi_1(\phi_1^4(w))$  and  $\chi_2(\phi_1^4(w))$  are linearly independent functions of  $k_1$  and  $k_2$ , we can always express  $\chi_4(\phi_1^4(w))$  as a linear combination of  $\chi_1(\phi_1^4(w))$  and  $\chi_2(\phi_1^4(w))$ .

If we have a patch  $P_5$  that is longer than 8 letters, we just have to work with higher-order supertiles. If  $2^{n-1} < |P_5| \leq 2^n$ , we count the appearances of  $P_1$ ,  $P_2$  and  $P_5$  on  $\phi_1^n(a)$  and  $\phi_1^n(b)$ , and write an arbitrary word as  $p\phi_1^{n+1}(w)s$ , where  $w$  is a return word and  $p$  and  $s$  are words of length at most  $6 \cdot 2^n$ .

**4.2. Thue-Morse variants.** A couple of variants on the Thue-Morse substitution help illustrate the extent to which the bounds of Theorem 4 are sharp. In both cases, as with the original Thue-Morse substitution, the stretching factor is a power of 2, both tiles can be given length 1, and there are return words of length 1, so Theorem 4 essentially says that all patch frequencies live in  $\frac{1}{D}\mathbb{Z}[1/2]$ .

The first variant is the substitution  $\phi_2 = \phi_1^4$ , or explicitly  $\phi_2(a) = abbabaabbaabba$ ,  $\phi_2(b) = baababbaabbabaab$ . The tilings for  $\phi_2$  are exactly the same as those for  $\phi_1$ . In particular, the appearance of the patches  $ab$  and  $aa$  govern the appearance of all patches, and all patch frequencies live in  $\frac{1}{3}\mathbb{Z}[1/2]$ .

However, the substitution, acting on cohomology, has eigenvalues 16 and 1 rather than 2 and  $-1$ , and the number theoretic constant  $D$  is now 15 rather than 3. A naive application of Theorem 4 says that all patch frequencies live in  $\frac{1}{15}\mathbb{Z}[1/2]$ , which is true, but this is not sharp; the factor of 5 in the denominator is spurious.

Now consider the substitution  $\phi_3(a) = aaaaaaaabbbbbbb = a^8b^8$ ,  $\phi_3(b) = a^7b^9$ . This tiling space also has  $H^1 = \mathbb{Q}^2$ , and the substitution, acting on cohomology, has eigenvalues 16 and 1. However, in this example there *are* patches whose frequencies are not in  $\frac{1}{3}\mathbb{Z}[1/2]$ . In particular, the letter  $a$  occurs with frequency  $\frac{7}{15}$ , while the letter  $b$  occurs with frequency  $\frac{8}{15}$ .

Theorem 4 appears to be the strongest estimate that can be made using only the minimal polynomial of  $A$ , or equivalently using the eigenvalues of  $A$  (and the size of any Jordan blocks). However, as  $\phi_2$  shows, we can sometimes obtain stronger estimates by studying the *eigenvectors* of  $A$ . The key is decomposing the cohomology class of an arbitrary indicator cochain into the sum of two pieces, one that is annihilated by  $q(A)$  and one that is annihilated by  $r(A)$ . Although we can always do this using rational coefficients with denominator  $D$ , some matrices  $A$  allow us to do better.

**4.3. A Fibonacci variant.** Theorems 1–4 were stated in terms of the rational cohomology of the tiling space. This avoids complications relating to torsion and divisibility. However, there are times when integer-valued cohomology can be used more effectively.

Consider the 1-dimensional substitution on two letters  $\phi(a) = baaab$ ,  $\phi(b) = aba$ . This is an irreducible Pisot substitution, with substitution matrix  $A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$  and stretching factor  $\lambda = 2 + \sqrt{5}$ , which is the cube of the golden mean. The first cohomology is  $\check{H}^1(\Omega_\phi) = \mathbb{Z}^3$ , with generators corresponding to the indicator cochains of  $P_1 = a$ ,  $P_2 = b$  and  $P_3 = ab$ .

Since every indicator cochain is cohomologous to an integer linear combination of  $\chi_1$ ,  $\chi_2$  and  $\chi_3$ , every patch frequency is an integer linear combination of  $f_1 = \frac{1}{2\sqrt{5}}$ ,  $f_2 = \frac{\sqrt{5}-1}{4\sqrt{5}}$ , and  $f_3 = \frac{1}{4\sqrt{5}}$ , where we have chosen the tiles to have length  $|a| = \sqrt{5} + 1 = \lambda - 1$  and  $|b| = 2$ . In other words, all patch frequencies are of the form  $\frac{(m+n\sqrt{5})}{4\sqrt{5}}$ , where  $m$  and  $n$  are integers.

This is stronger than applying Theorem 4 with the return length  $L = |b| = 2$ , which only says that patch frequencies must be of the form  $\frac{(m+n\sqrt{5})}{16\sqrt{5}}$ .

**4.4. A random tiling.** Next consider a random 1-dimensional tiling  $\mathbf{T}$ , with two tile types, each of length 1. We assume that the label of the tiles are chosen independently, with each tile having a probability  $p$  of being type  $a$  and a  $1 - p$  probability of being type  $b$ . With probability one, every finite word in  $a$  and  $b$  appears in  $\mathbf{T}$ , with a well-defined overall frequency given by the Bernoulli measure. If  $p$  is transcendental, then the frequencies of  $a$ ,  $aa$ ,  $aaa$ , etc. are all linearly independent over the rationals. This

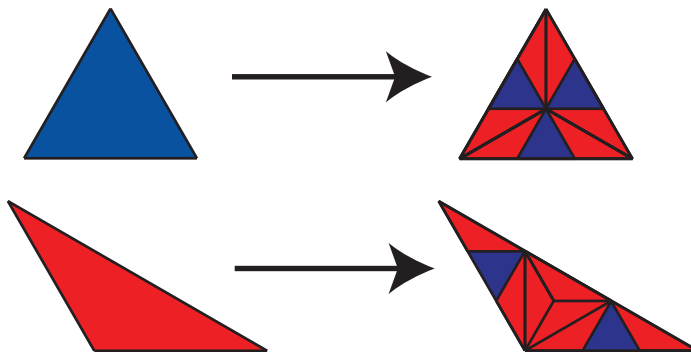


FIGURE 1. The Equithirds Substitution

implies that the pattern-equivariant cohomology of  $\mathbf{T}$  is infinitely generated.<sup>1</sup> If  $p$  is algebraic, or even rational, then the PE cohomology is *still* infinitely generated, since the set of possible patches is independent of  $p$  as long as  $0 < p < 1$ .

**4.5. The equithirds tiling.** We have limited ourselves to one-dimensional examples so far because, in one dimension, it is possible to eliminate the error term in Theorem 1 by choosing an appropriate return word. In two dimensions, that is usually much more difficult. However, some two-dimensional tilings, like the half-hex, admit regions whose boundaries are homologically trivial. Another such example is the equithirds tiling, discovered independently by Ludwig Danzer (unpublished) and Bill Kalahurka [Kal].

The equithirds tiling is a two-dimensional substitution tiling based on the substitution of Figure 1. Each tile is either an equilateral triangle of side length 1, or a 30-30-120 triangle with sides of length 1, 1, and  $\sqrt{3}$ . The equilateral triangle appears in two orientations, while the isosceles triangle appears in six orientations. All triangles have area  $\sqrt{3}/4$ . The isosceles triangles come in pairs, forming rhombi, and the equilateral triangles also come in pairs, also forming rhombi.

The set of vertices of an equithirds tiling is a translate of the triangular lattice  $L$  generated by  $(1, 0)$  and  $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ . The unit cell of this lattice has area  $V_0 = \frac{\sqrt{3}}{2}$ , or twice the area of a triangle. The vertices of first-order supertiles comprise a translate of  $3L$ , and vertices of  $n$ -th order supertiles comprise a translate of  $3^n L$ . The locations of  $n$ -th order vertices mod  $3^n L$  gives a map from the tiling space to the 2-torus, and the collection of all such locations gives a map from the tiling space to  $\varprojlim(L, \times 3)$ , which is

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<sup>1</sup>Strictly speaking this is not a consequence of Theorems 1 and 2, since  $\mathbf{T}$  is not repetitive. However,  $\mathbf{T}$  being in the support of the Bernoulli measure is an adequate substitute for repetitivity and unique ergodicity.

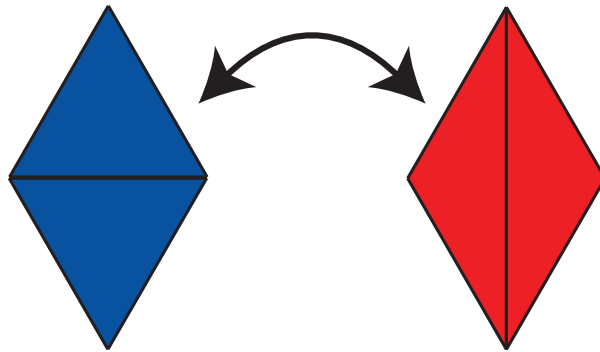


FIGURE 2. Triangles Assemble into Rhombi of Two Types

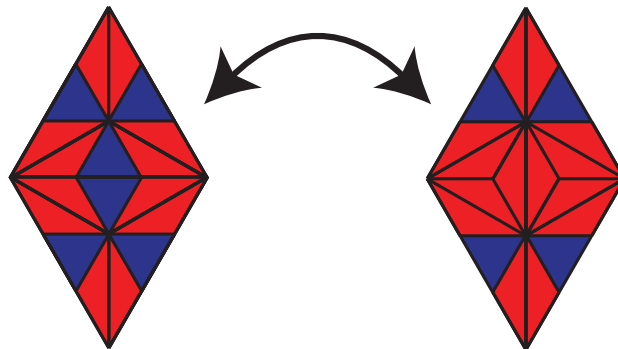


FIGURE 3. Substituted Rhombi

topologically the product of two 3-adic solenoids. This map is a measurable conjugacy, and one might expect all patch frequencies to live in  $\frac{1}{v_0}\mathbb{Z}[1/9]$ .<sup>2</sup>

This is not the case. Each orientation of equilateral triangle, and each orientation of isosceles triangle, actually appears with frequency  $\frac{1}{4v_0}$ . Overall,  $3/4$  of the triangles are isosceles, while  $1/4$  are equilateral. The appearance of a factor of 4 is analogous to the appearance of  $1/3$  in the patch frequencies of the Thue-Morse tiling.

Moreover, the patches shown in Figure 2, and these patches rotated by multiples of 120 degrees, play a role analogous to return words. When substituted one or more times, the patterns on opposite legs of the rhombus match perfectly, as seen in Figure 3. The term  $\delta\alpha$  in equation (3) vanishes when applied to a sufficiently substituted rhombus. This means that the number of appearance of any patch  $P$  in a sufficiently

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<sup>2</sup>We write  $\mathbb{Z}[1/9]$  rather than  $\mathbb{Z}[1/3]$  to emphasize that substitution corresponds to multiplication by 9, but of course the set of 9-adic rational numbers is the same as the set of 3-adic rationals.



substituted rhombus is determined exactly by the number of appearances of the control patches  $P_1, \dots, P_k$ .

The equithirds tiling forces the border, which makes it easy to compute the cohomology [Kal] using the methods of Anderson and Putnam [AP]. The answers are that  $\check{H}^1(\Omega) = \mathbb{Z}[1/3]^2$  and  $\check{H}^2(\Omega) = \mathbb{Z}[1/9] \oplus \mathbb{Z}^3$ , hence  $\check{H}^2(\Omega, \mathbb{Q}) = \mathbb{Q}^4$ , or  $k = 4$ . We can choose our control patches as follows. Let  $P_1$  be an equilateral triangle with horizontal base, let  $P_2$  be the second rhombus shown in Figure 2, and let  $P_3$  and  $P_4$  be rotated versions of  $P_2$ . The frequency of each  $P_i$  is  $\frac{1}{4V_0}$ , and every patch frequency lives in  $\frac{1}{4V_0}\mathbb{Z}[1/9]$ .

Finally, it is possible to directly understand the various terms in  $\check{H}^2(\Omega)$ . The patches shown in Figure 3, obtained by substituting the rhombi of Figure 3, are identical except in the very middle. Substituting again yields even bigger patches that are identical except for one rhombus in the middle. Taking a limit we obtain two tilings that agree completely except for that one rhombus. One of the classes in  $\check{H}^2(\Omega)$ , represented by  $\chi_1 - \chi_2$ , measures the difference between these two tilings, and is invariant under substitution. Likewise,  $\chi_1 - \chi_3$  and  $\chi_1 - \chi_4$  measure the same thing rotated by 120 and 240 degrees. Each of these classes has trace zero. A fourth class,  $\chi_1 + \chi_2 + \chi_3 + \chi_4$ , counts area in units of  $V_0$ , scales by 9 under substitution, and has trace  $\frac{1}{V_0}$ . This corresponds to a generator of  $\mathbb{Z}[1/9]$ . These four classes span  $\check{H}^2(\Omega, \mathbb{Q})$ , but some indicator cochains cannot be written as integer linear combinations. For instance,  $\chi_1 = \frac{1}{4}[(\chi_1 + \chi_2 + \chi_3 + \chi_4) + (\chi_1 - \chi_2) + (\chi_1 - \chi_3) + (\chi_1 - \chi_4)]$ , so the trace of  $\chi_1$  (and likewise  $\chi_{2,3,4}$ ) is  $\frac{1}{4V_0}$ .

**4.6. The pinwheel tiling.** As noted earlier, the pinwheel tiling [Rad] falls outside the assumptions of this paper. It is not translationally finite, and each patch of a tiling appears with frequency zero. To make sense of patch frequencies, we must count how many times a patch *or a rotated version of that patch* appears per unit area. This involves looking at rotationally invariant indicator cochains. (These are a special case of the pattern-equivariant cochains with a representation developed in [Ran].) Since the coboundary of a rotationally invariant cochain is rotationally invariant, we can define a rotationally invariant pattern-equivariant cohomology  $H_{PE,rot}^n(\mathbf{T}, \mathbb{Q})$  to be the closed rotationally invariant cochains modulo coboundaries of rotationally invariant cochains.

The correspondence between  $H_{PE,rot}^n(\mathbf{T}, \mathbb{Q})$  and the Čech cohomology of the pinwheel tiling space is subtle.  $H_{PE,rot}^2(\mathbf{T}, \mathbb{Q})$  actually corresponds to  $\check{H}^3(\Omega, \mathbb{Q})$ , where  $\Omega$  is a compactification of the hull of a pinwheel tiling, using a metric where two tilings are close if they agree on a big ball up to a small rigid motion (which may include a small rotation). See [BDHS] for this correspondence and for a computation of  $\check{H}^*(\Omega, \mathbb{Q})$ , with

the result that  $\check{H}^3(\Omega, \mathbb{Q}) = \mathbb{Q}^8$ , hence that  $H_{PE,rot}^2(\mathbf{T}, \mathbb{Q}) = \mathbb{Q}^8$ , hence that there are eight patches that control the appearance of all other patches, up to boundary terms.

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