Eigenvalues of Cayley graphs

Xiaogang Liu

Department of Applied Mathematics

Northwestern Polytechnical University

Xi'an, Shaanxi 710072, P. R. China

`xiaogliu@nwpu.edu.cn`

Sanming Zhou

School of Mathematics and Statistics

The University of Melbourne

Parkville, VIC 3010, Australia

`sanming@unimelb.edu.au`

January 22, 2019

Abstract

We survey some of the known results on eigenvalues of Cayley graphs and their applications, together with related results on eigenvalues of Cayley digraphs and generalizations of Cayley graphs.

Keywords: Spectrum of a graph; Eigenvalues of a graph; Cayley graphs; Integral graphs; Energy of a graph; Ramanujan graph; Second largest eigenvalue of a graph; Distance-regular graphs; Strongly regular graphs; Perfect state transfer

AMS Subject Classification (2010): 05C50, 05C25

Contents

1 Introduction

The study of eigenvalues of graphs is an important part of modern graph theory. In particular, eigenvalues of Cayley graphs have attracted increasing attention due to their prominent roles in algebraic graph theory and applications in many areas such as expanders, chemical graph theory, quantum computing, etc. A large number of results on spectra of Cayley graphs have been produced over the last more than four decades. This paper is a survey of the literature on eigenvalues of Cayley graphs and their applications.

All definitions below are standard and can be found in, for example, [59, 92, 133]. A finite undirected *graph* consists of a finite set whose elements are called the *vertices* and a collection of unordered pairs of (not necessarily distinct) vertices each called an *edge*. As usual, for a graph G, we use $V(G)$ and $E(G)$ to denote its vertex and edge sets, respectively, and we call the size of $V(G)$ the *order* of G. An edge $\{u, v\}$ of G is usually denoted by uv or vu. If uv is an edge of G, we say that u and v are joined by this edge, u and v are *adjacent* in G, and both u and v are *incident* to the edge uv. An edge joining a vertex to itself is called a *loop*, and two or more edges joining the same pair of distinct vertices are called *parallel edges*. The *degree* of u in G, denoted by $d_G(u)$ or simply $d(u)$ if there is no risk of confusion, is the number of edges of G incident to u, each loop counting twice. A graph without loops is called a *multigraph*, and a graph is *simple* if it has no loops or parallel edges.

Let G be a finite undirected graph of order n. The *adjacency matrix* of G, denoted by $A(G)$, is the $n \times n$ matrix with rows and columns indexed by the vertices of G such that the (u, v)-entry is equal to the number of edges joining u and v, with each loop counting as two edges. The eigenvalues of $A(G)$ are called the *eigenvalues* of G, and the collection of eigenvalues of G with multiplicities is called the *spectrum* of G. Since G is undirected, $A(G)$ is a real symmetric matrix and hence the eigenvalues of G are all real numbers. If $\lambda_1, \lambda_2, \ldots, \lambda_r$ are distinct eigenvalues of G and $m_1, m_2, \ldots, m_r$ the corresponding multiplicities, then the spectrum of G is denoted by

$$\mathrm{Spec}(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_r \\ m_1 & m_2 & \ldots & m_r \end{pmatrix}$$

or

$$(\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_r^{m_r}),$$

4

where in the latter notation we usually omit m_i if $m_i = 1$ for some i. An eigenvalue with multiplicity 1 is called a *simple* eigenvalue. The *spectral radius* of a graph is the maximum modulus of its eigenvalues. Two graphs are said to be *cospectral* if they have the same spectrum.

Let $D(G)$ be the $n \times n$ diagonal matrix whose (u, u)-entry is equal to the degree $d(u)$ of u in G, for each $u \in V(G)$. The matrices $D(G) - A(G)$ and $D(G) + A(G)$ are called the *Laplacian matrix* and *signless Laplacian matrix* of G, respectively, and their eigenvalues are called the *Laplacian eigenvalues* and *signless Laplacian eigenvalues* of G, respectively. In the case when G is *k-regular* for some integer $k \geq 0$, that is, each vertex has degree k, a real number x is an eigenvalue of G if and only if $k - x$ is a Laplacian eigenvalue of G. Since all graphs considered in this paper are regular, this implies that all results to be reviewed in the paper can be presented in terms of Laplacian eigenvalues. Henceforth we will mostly talk about eigenvalues of graphs.

Similar to undirected graphs, a finite *directed graph* (or *digraph*) can be defined by specifying a finite set of vertices and a collection of ordered pairs of (not necessarily distinct) vertices; each ordered pair (u, v) in the collection is called an *arc* from u to v. The *underlying graph* of a digraph is the undirected graph obtained by replacing each arc by an edge with the same end-vertices. The *adjacency matrix* of a finite digraph G is the matrix whose (u, v)-entry is equal to the number of arcs from u to v, and the eigenvalues of this matrix are called the *eigenvalues* of G. Note that, unlike the undirected case, a digraph may have complex eigenvalues as its adjacency matrix is not necessarily symmetric.

A digraph without loops is called a *multidigraph*. A digraph is *symmetric* if whenever (u, v) is an arc, (v, u) is also an arc. We often identify a loopless symmetric digraph with its underlying simple graph, which is obtained by replacing each pair of arcs $(u, v), (v, u)$ by a single edge joining u and v. Note that a loopless symmetric digraph and its underlying simple graph have the same adjacency matrix and hence the same spectrum.

Let Γ be a finite group with identity element 1, and let S be a subset of Γ. The *Cayley (di)graph on* Γ with *connection set* S, denoted by $\mathrm{Cay}(\Gamma, S)$, is defined to be the digraph with vertex set Γ and arcs (x, y) for all pairs $x, y \in \Gamma$ such that $xy^{-1} \in S$. Clearly, this digraph has $|\Gamma|$ vertices, with each vertex having in-degree and out-degree $|S|$. Moreover, $\mathrm{Cay}(\Gamma, S)$ is connected if and only if $\langle S \rangle = \Gamma$, where $\langle S \rangle$ is the subgroup of Γ generated by S. In the case when $1 \notin S$ and S is *inverse-closed* (that is, $S = S^{-1} := \left\{ s^{-1} : s \in S \right\}$), the digraph $\mathrm{Cay}(\Gamma, S)$ is symmetric with no loops and hence may be identified with its underlying simple graph. In other words, if $S \subseteq \Gamma \setminus \{1\}$ is inverse-closed, then $\mathrm{Cay}(\Gamma, S)$ is understood as a k-regular simple graph, where $k = |S|$.

Throughout this paper, unless stated otherwise when we say a graph we mean a finite undirected simple graph. In particular, unless stated otherwise Cayley graphs considered in this paper are finite, undirected and simple.

1.1 Outline of the paper

It is well known that eigenvalues of Cayley (di)graphs can be expressed in terms of the irreducible characters of the underlying groups. In section 2 we will review these fundamental results together with their counterparts for vertex-transitive graphs.

An important problem in spectral graph theory is to understand when all eigenvalues of a graph are integers. A graph with this property is called integral, or more specifically, integral

over the field $\mathbb{Q}$. Similarly, one can consider graphs which are integral over some other algebraic number fields such as the Gaussian field $\mathbb{Q}(i)$. A number of results on integral Cayley graphs have been produced in the past more than ten years. In section 3 we will review these results, along with a few other results on Cayley graphs which are integral over $\mathbb{Q}(i)$ or a general algebraic number field. In section 4 we will survey results on cospectral Cayley graphs and Cayley graphs which are determined by their spectra. In section 5 we will discuss several families of Cayley graphs on finite commutative rings, including unitary Cayley graphs and quadratic unitary Cayley graphs of finite commutative rings, as well as some Cayley graphs on finite chain rings.

The energy of a graph is a concept that arises from chemical graph theory. This notion has been studied extensively over the past more than four decades. In particular, a number of results on energies of Cayley graphs have been produced in recent years. We will give an account of such results in section 6.

It is well known [150] that the expansion of a regular graph is determined by its second largest eigenvalue and that Cayley graphs play an important role in constructing expander graphs. Roughly, regular graphs achieving best possible expansion (in terms of the well-known Alon-Boppana bound) are called Ramanujan graphs. In sections 7 and 8 we will review some results on Ramanujan Cayley graphs and the second largest eigenvalue of Cayley graphs, respectively.

Perfect state transfer in graphs is an important concept that arises from quantum computing. In general, it is challenging to construct graphs admitting perfect state transfer or prove the existence of perfect state transfer in a graph. In section 9 we will survey recent results on perfect state transfer in several families of Cayley graphs along with related results on the periodicity of such graphs.

A number of distance-regular graphs (and in particular strongly regular graphs) are known to be Cayley graphs. In section 10 we will review some results on distance-regular Cayley graphs. This discussion will be continued in section 11.2, where we will review several results on strongly regular n-Cayley graphs, where $n \geq 2$. In section 11 we will discuss a few notions of generalized Cayley graphs, including n-Cayley graphs, Cayley sum graphs and group-subgroup pair graphs, with a focus on their eigenvalues.

In section 12 we will give a brief account of results on eigenvalues of directed Cayley graphs. We will conclude this paper in section 13 with some miscellaneous results, including results on eigenvalues of random Cayley graphs, distance eigenvalues of Cayley graphs, eigenvalues of mixed Cayley graphs, etc.

Despite its length this survey is far from being a comprehensive treatise on eigenvalues of Cayley graphs. Some important omissions are as follows.

- Only finite Cayley graphs are considered in our paper. Some results on eigenvalues of infinite Cayley graphs can be found in [246, Section 5].

- We do not survey results on families of expanders with a fixed degree but increasing orders (see, for example, [60]). We believe that this important area involving deep group theory and other mathematical tools deserves a separate treatment by experts in the area. The reader is referred to [221] for a survey on eigenvalues of Cayley graphs and their connections with expanders and random walks. See also [204] for a survey on Ramanujan graphs and [287] for a book on expansion in Cayley graphs and finite simple groups of Lie type. A survey on analytic, geometric and combinatorial properties of finitely generated groups

that are related to expansion properties of Cayley graphs as measured by the isoperimetric number (Cheeger constant) can be found in [310].

- Strongly regular Cayley graphs are essentially partial difference sets in groups. A large number of results on partial difference sets exist in the literature (see [226] for a not-so-recent survey on this topic), but we believe that a survey of them should be a subject of its own. So this area is mostly omitted in our paper, with the exception of a few results whose proofs use methods which seem to be nontypical in the area of partial difference sets. The reader is referred to [65] for a classic treatment and [296] for a recent survey on distance-regular graphs. Most results on distance-regular Cayley graphs reviewed in our paper are not covered in [296].

- It is well known [56] that the number of spanning trees in a regular graph is determined by its eigenvalues. Nevertheless, results on the number of spanning trees in Cayley graphs will not be surveyed in our paper.

We conclude this section by mentioning a few references which complement the present paper. In [76], Chan and Godsil surveyed some of what can be deduced about automorphisms of a graph from information on its eigenvalues and eigenvectors. A survey on Laplacian eigenvalues of graphs can be found in [245], and a survey on spectra of digraphs is given in [67]. Basic results on eigenvalues of Cayley graphs can be found in [56] and [133], which are widely used textbooks on algebraic graph theory.

1.2 Terminology and notation

The reader is referred to [59], [160], [145] and [34, 57] for terminology and notation on graph theory, group theory, number theory and the theory of finite commutative rings, respectively.

The *complement* $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in G. The *line graph* $L(G)$ of G is the graph with vertices the edges of G such that two vertices are adjacent if and only if the corresponding edges have a common end-vertex. The *distance* between two vertices u and v in a graph G, denoted by $d_G(u,v)$ or simply $d(u,v)$, is the length of a shortest path between them in G; if there is no path between u and v in G, then we set $d(u,v) = \infty$. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum distance between two vertices of G.

The *Cartesian product* $G \square H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ in which (u,v) and (x,y) are adjacent if and only if either $u = x$ and $vy \in E(H)$, or $v = y$ and $ux \in E(G)$. The *tensor product* (also known as direct product, Kronecker product and categorical product in the literature) of G and H, denoted by $G \otimes H$, is the graph with vertex set $V(G) \times V(H)$ in which (u,v) is adjacent to (x,y) if and only if u is adjacent to x in G and v is adjacent to y in H. These two operations are associative and so the Cartesian (tensor) product of any finite number of graphs is well defined.

A group Γ is said to *act* on a set Ω if each pair $(\alpha, g) \in \Omega \times \Gamma$ corresponds to some $\alpha^g \in \Omega$ such that $\alpha^1 = \alpha$ and $(\alpha^g)^h = \alpha^{gh}$ for $g, h \in \Gamma$, where 1 is the identity element of Γ. The *stabilizer* of $\alpha \in \Omega$ in Γ is the subgroup $\Gamma_\alpha := \{g \in \Gamma : \alpha^g = \alpha\}$ of Γ. The Γ-*orbit* containing α is defined to be $\alpha^\Gamma := \{\alpha^g : g \in \Gamma\}$. If $\Gamma_\alpha = \{1\}$ for all $\alpha \in \Omega$, then Γ is called *semiregular* on Ω. If $\alpha^\Gamma = \Omega$ for some (and hence all) $\alpha \in \Omega$, then Γ is *transitive* on Ω. If Γ is both transitive and semiregular on Ω, then it is said to be *regular* on Ω.

An *automorphism* of a graph G is a permutation of $V(G)$ which maps adjacent vertices to adjacent vertices and nonadjacent vertices to nonadjacent vertices. The *automorphism group* of G, denoted by $\mathrm{Aut}(G)$, is the group of automorphisms of G under the usual composition of permutations. Of course $\mathrm{Aut}(G)$ acts on $V(G)$ in a natural way, and this action induces natural actions on the set of edges, the set of arcs and the set of 2-arcs of G, where an *arc* is an ordered pair of adjacent vertices, and a *2-arc* is an ordered triple (u, v, w) of distinct vertices such that v is adjacent to both u and w. If a subgroup Γ of $\mathrm{Aut}(G)$ is transitive on $V(G)$, then G is said to be Γ-*vertex-transitive*. Similarly, if Γ is transitive on the set of edges of G, then G is Γ-*edge-transitive*; if Γ is transitive on the set of arcs of G, then G is Γ-*arc-transitive*; and if Γ is transitive on the set of 2-arcs of G, then G is $(\Gamma, 2)$-*arc-transitive*. A graph G is called *vertex-transitive*, *edge-transitive*, *arc-transitive* or *2-arc-transitive* if it is $\mathrm{Aut}(G)$-vertex-transitive, $\mathrm{Aut}(G)$-edge-transitive, $\mathrm{Aut}(G)$-arc-transitive or $(\mathrm{Aut}(G), 2)$-arc-transitive, respectively. Clearly, any vertex-transitive graph must be regular and any arc-transitive graph without isolated vertices must be vertex-transitive. It is known that any edge-transitive but not vertex-transitive graph must be bipartite (see, for example, [56, Proposition 15.1]). It is well known that all Cayley graphs are vertex-transitive, but the converse is not true, the Petersen graph being a counterexample. It is also well known that a graph is isomorphic to a Cayley graph if and only if its automorphism group contains a subgroup which is regular on the vertex set (see, for example, [56, Lemma 16.3]). A *circulant graph*, or simply a *circulant*, is a Cayley graph on a cyclic group.

Throughout the paper we use the following notation:

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{P}$: Sets of positive integers, integers, primes, respectively

$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$: Fields of rational numbers, real numbers, complex numbers, respectively

$|V|$: Cardinality of a set V (in particular, if Γ is a group, then $|\Gamma|$ is the order of Γ)

n, d, r: Positive integers

$\omega_n = \exp(2\pi i/n)$: An nth primitive root of unity, where $i^2 = -1$

$\phi(n)$: Euler's totient function, which gives the number of positive integers up to n that are coprime to n

$e_p(n)$: Exponent of prime p in n

$\mathrm{Cay}(\Gamma, S)$: Cayley (di)graph of a group Γ with respect to connection set $S \subseteq \Gamma$

$\mathbb{Z}_n$: Ring of integers modulo n

$\mathbb{Z}_n^\times$: Set of units of ring $\mathbb{Z}_n$

C_n, K_n: Cycle and complete graph of order n, respectively

$K_{m,n}$: Complete bipartite graph with m and n vertices in the bi-parts of the bipartition, respectively

$H(d, q)$: Hamming graph, namely the Cartesian product $K_q \square \cdots \square K_q$ (d factors)

$H(d, 2)$: Hypercube of dimension d

$\mathrm{NEPS}(G_1, \ldots, G_d; \mathcal{B})$: NEPS of graphs $G_1, \ldots, G_d$ with respect to basis $\mathcal{B} \subseteq \mathbb{Z}_2^d \setminus \{\mathbf{0}\}$

$L(G)$: Line graph of a graph G

$\overline{G}$: Complement of a graph G

$\mathrm{Spec}(G)$: Spectrum of a graph G

$\mathcal{E}(G)$: Energy of a graph G

$D(n)$: Set of positive divisors of n

ICG(n, D): gcd graph of the cyclic group $\mathbb{Z}_n$ with respect to $D \subseteq D(n) \setminus \{n\}$

ICG$(R/(c), D)$: gcd graph of a UFD R with respect to a set D of proper divisors of $c \in R \setminus \{0, 1\}$

$G_R = \mathrm{Cay}(R, R^\times)$: Unitary Cayley graph of a finite commutative ring R, where $R^\times$ is the set of units of R

$G_{\mathbb{Z}_n} = \mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^\times) = \mathrm{ICG}(n, \{1\})$: Unitary Cayley graph of the cyclic group $\mathbb{Z}_n$

$\mathcal{G}_R$: Quadratic unitary Cayley graph of a finite commutative ring R

$\mathbb{F}_q$: Finite field of q elements, q being a prime power

$w(\mathbf{a})$: Hamming weight, namely the number of nonzero coordinates of $\mathbf{a}$

S_n: Symmetric group of degree n

D_{2n}: Dihedral group $\langle a, b \,|\, a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ of order $2n$

$\langle S \rangle$: Subgroup of a group Γ generated by a subset $S \subseteq \Gamma$ (in particular, $\langle x \rangle = \langle \{x\} \rangle$, $\langle x, y \rangle = \langle \{x, y\} \rangle$, and so on)

XY: The subset $\{xy : x \in X, y \in Y\}$ of a group Γ, for given $X, Y \subseteq \Gamma$

$\Gamma \,\mathrm{wr}\, \Sigma$: Wreath product of a group Γ by a group Σ

$\mathrm{Aut}(G)$: Automorphism group of a graph G

$\mathrm{Aut}(\Gamma)$: Automorphism group of a group Γ

2 Eigenvalues of Cayley graphs

2.1 Characters

The reader is referred to [159, 162] for representation theory of finite groups and properties of characters. Here we recall only a few basic definitions. The *general linear group* $\mathrm{GL}(n, \mathbb{C})$ is the group of all invertible $n \times n$ matrices over $\mathbb{C}$, with operation the product of matrices, where $n \geq 1$ is an integer. A *representation* of a finite group Γ is a group homomorphism $\pi : \Gamma \to \mathrm{GL}(n, \mathbb{C})$; we call n the *degree* or *dimension* of π. The *character* of π is the mapping $\chi_\pi : \Gamma \to \mathbb{C}$ defined by

$$\chi_\pi(g) = \mathrm{Tr}(\pi(g)), \; g \in \Gamma,$$

where Tr denotes the trace of a matrix. It is readily seen that $\chi_\pi(1) = \mathrm{Tr}(\pi(1)) = \mathrm{Tr}(I_n) = n$, which is also called the *degree* of χ_π, where 1 is the identity element of Γ and I_n the identity matrix of size $n \times n$. The representation $\pi : \Gamma \to \mathrm{GL}(1, \mathbb{C})$ defined by $\pi(g) = (1)$, $g \in \Gamma$ is called the *trivial representation* of Γ and the corresponding character is called the *trivial character* of Γ.

Let X and Y be matrices. Define $X \oplus Y = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$, where the two zero-blocks are all-0 matrices of appropriate sizes. Let π_1 and π_2 be representations of Γ. The *direct sum* of π_1 and π_2, denoted by $\pi_1 \oplus \pi_2$, is defined to be the representation of Γ given by $(\pi_1 \oplus \pi_2)(g) = \pi_1(g) \oplus \pi_2(g)$ for $g \in \Gamma$. A representation π of Γ is *reducible* if there exist representations π_1, π_2 of Γ such that $\pi \cong \pi_1 \oplus \pi_2$, and *irreducible* otherwise. In the latter case we say that χ_π is an *irreducible character*.

It is well known that every irreducible character of any finite abelian group is one-dimensional. In other words, the characters of any finite abelian group Γ are precisely homomorphisms $\chi : \Gamma \to \mathbb{C}^*$ (that is, $\chi(gh) = \chi(g)\chi(h)$ for $g, h \in \Gamma$). Note that the modulus $|\chi(g)| = 1$ for all $g \in \Gamma$. It is also well known that Γ has $|\Gamma|$ distinct characters and the dual group $\hat{\Gamma}$ of Γ is isomorphic to Γ, where $\hat{\Gamma}$ is the group of all characters of Γ with operation defined by $(\chi\psi)(g) = \chi(g)\psi(g)$ for $\chi, \psi \in \hat{\Gamma}, g \in \Gamma$. In particular, the characters of the cyclic group $\mathbb{Z}_n$ are given by $\chi_k(g) = \omega_n^{kg}$, $0 \le k \le n - 1$, where $\omega_n = \exp(2\pi i/n)$, $i^2 = -1$ is an nth primitive root of unity.

2.2 Eigenvalues of Cayley graphs

In this section we survey a few basic results on eigenvalues of Cayley (di)graphs and Cayley colour (di)graphs. Many results on eigenvalues of Cayley graphs to be surveyed in later sections are based on these fundamental results.

Let Γ be a finite group of order n and $\alpha : \Gamma \to \mathbb{C}$ a function. The *Cayley colour digraph* of Γ with *connection function* α (see, for example, [36]), denoted by $\mathrm{Cay}(\Gamma, \alpha)$, is defined to be the directed graph with vertex set Γ and arc set $\{(x, y) : x, y \in \Gamma\}$ such that each arc (x, y) is coloured by $\alpha(xy^{-1})$. The *adjacency matrix* of $\mathrm{Cay}(\Gamma, \alpha)$ is defined as the matrix with rows and columns indexed by the elements of Γ such that the (x, y)-entry is equal to $\alpha(xy^{-1})$. The *eigenvalues* of $\mathrm{Cay}(\Gamma, \alpha)$ are simply the eigenvalues of its adjacency matrix. In the special case when $\alpha : \Gamma \to \{0, 1\}$ and the set $S = \{g : \alpha(g) = 1\}$ satisfies $1 \notin S$ and $S = S^{-1}$, the Cayley colour digraph $\mathrm{Cay}(\Gamma, \alpha)$ can be identified with the Cayley graph $\mathrm{Cay}(\Gamma, S)$, and the adjacency matrix of $\mathrm{Cay}(\Gamma, \alpha)$ agrees with that of $\mathrm{Cay}(\Gamma, S)$.

In 1975, Lovász [217] proved the following result.

Theorem 2.1. ([217]; also [36, Corollary 3.2]) *Let Γ be an abelian group of order n with irreducible characters $\chi_1, \chi_2, \ldots, \chi_n$. Let $\alpha : \Gamma \to \mathbb{C}$ be a function. Then the eigenvalues of the Cayley colour digraph $\mathrm{Cay}(\Gamma, \alpha)$ of Γ are given by*

$$\lambda_i = \sum_{g \in \Gamma} \alpha(g)\chi_i(g), \ i = 1, 2, \ldots, n.$$

Inspired by the work in [217], Babai obtained the following more general result.

Theorem 2.2. ([36, Theorem 3.1]) *Let Γ be a finite group. Let $\chi_1, \chi_2, \ldots, \chi_h$ be the irreducible characters of Γ and $n_1, n_2, \ldots, n_h$ be their degrees, respectively. Let $\alpha : \Gamma \to \mathbb{C}$ be a function. Then the eigenvalues of the Cayley colour digraph $\mathrm{Cay}(\Gamma, \alpha)$ of Γ can be arranged as*

$$\Lambda = \{\lambda_{ijk} : i = 1, 2, \ldots, h; j, k = 1, 2, \ldots, n_i\}$$

such that $\lambda_{ij1} = \lambda_{ij2} = \cdots = \lambda_{ijn_i}$ (this common value is denoted by $\lambda_{i,j}$), and for any positive integer t,

$$\lambda_{i,1}^t + \cdots + \lambda_{i,n_i}^t = \sum_{g_1, \ldots, g_t \in \Gamma} \left(\prod_{s=1}^{t} \alpha(g_s) \right) \chi_i \left(\prod_{s=1}^{t} g_s \right).$$

In the case when $\alpha : \Gamma \to \{0, 1\}$, Theorems 2.1 and 2.2 yield the following corollaries, respectively.

Corollary 2.3. *Let Γ be an abelian group of order n with irreducible characters $\chi_1, \chi_2, \ldots, \chi_n$. Then the eigenvalues of any Cayley (di)graph $\mathrm{Cay}(\Gamma, S)$ of Γ are given by*

$$\lambda_i = \sum_{g \in S} \chi_i(g), \; i = 1, 2, \ldots, n.$$

Corollary 2.4. *Let Γ be a finite group. Let $\chi_1, \chi_2, \ldots, \chi_h$ be the irreducible characters of Γ and $n_1, n_2, \ldots, n_h$ be their degrees, respectively. Then the eigenvalues $\lambda_{i,j}$ $(1 \le i \le h, \; 1 \le j \le n_i)$ of any Cayley (di)graph $\mathrm{Cay}(\Gamma, S)$ of Γ satisfy*

$$\lambda_{i,1}^t + \cdots + \lambda_{i,n_i}^t = \sum_{A \subseteq S, \; |A| = t} \chi_i \left(\prod_{g \in A} g \right)$$

for any positive integer t.

As an example we obtain the eigenvalues of any circulant graph from Corollary 2.3 immediately (see also the treatment in [56]). Recall that a circulant graph is a Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, S)$ on a cyclic group $\mathbb{Z}_n$, where $n \ge 3$. Since the characters of $\mathbb{Z}_n$ are given by $\chi_k(g) = \omega_n^{kg}$, $0 \le k \le n-1$, from Corollary 2.3 it follows that the eigenvalues of $\mathrm{Cay}(\mathbb{Z}_n, S)$ are given by

$$\lambda_k = \sum_{g \in S} \omega_n^{kg}, \; k = 0, 1, \ldots, n-1. \tag{2.1}$$

In particular, the eigenvalues of the cycle C_n of length n are $2\cos(2\pi k/n)$, $0 \le k \le n-1$. Eigenvalues of circulants were reviewed in [251] as part of a survey of some aspects of quadratic Gauss sums over finite rings and fields with applications. In the case when S is the set of nonzero quadratic residues, formula (2.1) together with the explicit formula for the Gauss sum gives a complete determination of the spectrum of the corresponding circulant $\mathrm{Cay}(\mathbb{Z}_n, S)$.

As another example, let us look at the d-dimensional hypercube $H(d, 2)$, $d \ge 1$, which can be defined as the Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^d, S)$, where $S = \{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ is the standard basis of $\mathbb{Z}_2^d$. The characters of $\mathbb{Z}_2^d$ are given by $\chi_{\mathbf{a}}(x) = (-1)^{\langle \mathbf{a}, \mathbf{x} \rangle}$, $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Z}_2^d$. Hence $\chi_{\mathbf{a}}(\mathbf{e}_i) = -1$ if $a_i = 1$ and $\chi_{\mathbf{a}}(\mathbf{e}_i) = 1$ if $a_i = 0$. The eigenvalue of $H(d, 2)$ associated with $\mathbf{a}$ is $\sum_i \chi_{\mathbf{a}}(\mathbf{e}_i) = (d - w(\mathbf{a})) - w(\mathbf{a}) = d - 2w(\mathbf{a})$, where $w(\mathbf{a})$ is the Hamming weight of $\mathbf{a}$. Therefore, the eigenvalues of $H(d, 2)$ are $d - 2i$ with multiplicity $\binom{d}{i}$, $0 \le i \le d$.

A Cayley graph $\mathrm{Cay}(\Gamma, S)$ is called *normal*[1] if S is closed under conjugation (that is, S the union of some conjugacy classes of the group Γ). The following result enables us to compute explicitly the eigenvalues of any normal Cayley graph using character values of the underlying group.

Theorem 2.5. ([308, Theorem 1]) *Let Γ be a finite group and let $\{\chi_1, \chi_2, \ldots, \chi_h\}$ be the set of all irreducible characters of Γ. Then the eigenvalues of any normal Cayley graph $\mathrm{Cay}(\Gamma, S)$ of Γ are given by*

$$\lambda_j = \frac{1}{\chi_j(1)} \sum_{g \in S} \chi_j(g), \; j = 1, 2, \ldots, h. \tag{2.2}$$

Moreover, the multiplicity of λ_i is equal to $\sum_{1 \le k \le h, \, \lambda_k = \lambda_j} \chi_k(1)^2$.

---

[1]Note that this definition is different from the following notion which is also widely used in the literature: A Cayley graph $G = \mathrm{Cay}(\Gamma, S)$ is called a normal Cayley graph if the right regular representation of Γ is normal in $\mathrm{Aut}(G)$.

A consequence of this result is that the number of distinct eigenvalues of a normal Cayley graph on a finite group Γ does not exceed the number of irreducible complex representations of Γ. Another application of Theorem 2.5 is a formula (see [308, Theorem 2]) for the number of walks of length n, $n \in \mathbb{N}$, between any two vertices in a normal Cayley graph. In the special case when Γ contains a cyclic self-normalized subgroup W of order pq for distinct primes p, q, and $S = \cup_{g \in \Gamma} W_0^g$, with W_0 the set of elements of W with order pq, $\mathrm{Cay}(\Gamma, S)$ has 4 or 5 distinct eigenvalues and for any $n \in \mathbb{N}$ the number of walks of length n between two adjacent vertices in $\mathrm{Cay}(\Gamma, S)$ does not depend on the choice of the two adjacent vertices (see [308, Theorem 9]).

Theorem 2.5 can be used to compute explicitly the eigenvalues of specific normal Cayley graphs in the case when the irreducible characters of the underlying groups are known. For example, all groups of order $p^2 q$ together with their character tables are known, where p and q are odd primes with $p < q$. Using this and Theorem 2.5, the eigenvalues of each normal Cayley graph of order $p^2 q$ were determined in [128]. Similarly, the spectra of all normal Cayley graphs on groups of order $2pq$ and $3pq$ in terms of the character tables of such groups were determined in [129], where p and q are distinct odd primes.

Theorem 2.6. ([116, Theorem 4.3 and Corollary 4.5]) *Let Γ be a finite group and let $f : \Gamma \to \mathbb{C}$ be a class function. Then the eigenvalues of the matrix $M_f = (f(xy^{-1}))_{x,y \in \Gamma}$ are*

$$\theta_\chi = \frac{1}{\chi(1)} \sum_{x \in \Gamma} f(x)\chi(x), \quad \text{with multiplicity } \chi(1)^2,$$

where χ ranges over the irreducible characters of Γ.

Moreover, the spectral radius of M_f is equal to $\sum_{x \in \Gamma} |f(x)|$. In particular, if $f : \Gamma \to \mathbb{R}^+$ takes nonnegative real numbers, then the spectral radius of M_f is the eigenvalue θ_1 corresponding to the trivial character, and in addition θ_1 is the largest eigenvalue of M_f provided that all eigenvalues θ_χ are real.

2.3 Eigenvalues of vertex-transitive graphs

Vertex-transitive graphs can be considered as generalizations of Cayley graphs, and the computation of their eigenvalues can be reduced to that of Cayley graphs (see [217]). As such we review a few results about eigenvalues of vertex-transitive graphs in this section.

In [217], Lovász proved that the determination of the spectrum of any vertex-transitive graph trivially reduces to that of a Cayley graph, and moreover he gave a formula for the spectrum in terms of irreducible characters of the underlying group. Using this method, he obtained the eigenvalues of circulant graphs and cubelike graphs, the latter being Cayley graphs on elementary abelian 2-groups.

In 1969, Petersdorf and Sachs [257] proved that for any vertex-transitive graph G with degree k and order n, if n is odd then k is the only simple eigenvalue of G, and if n is even then any simple eigenvalue of G is contained in the set $\{2i - k : 0 \leq i \leq k\}$. This result was strengthened by Sachs and Stiebitz [268] in the follow way.

Theorem 2.7. ([268, Theorem 13]) *Let G be a vertex-transitive graph with degree k and order $n = 2^e m$, m being odd. If $e = 0$, then k is the only simple eigenvalue of G; if $e = 1$, then G has at most one simple eigenvalue other than k, and moreover it is of the form $4i - k$ for some $i \in \{0, 1, \ldots, (k-1)/2\}$ if it exists; if $e \geq 2$, then G has at most 2^e simple eigenvalues including k, and each of them is contained in the set $\{2i - k : 0 \leq i \leq k\}$.*

Several other interesting results on simple eigenvalues of (directed or undirected) vertex-transitive graphs have also been proved in [268]. We present a few of them for undirected graphs in the following. Let Γ be a permutation group of degree n. Then each element $g \in \Gamma$ corresponds to a permutation matrix P_g. Denote by $z(\Gamma)$ the number of vectors $\mathbf{x} = (x_1, \ldots, x_n)^T$ with $x_1 = 1$ such that for every $g \in \Gamma$ there is a number $a_g(\mathbf{x})$ with $P_g \mathbf{x} = a_g(\mathbf{x})\mathbf{x}$.

Theorem 2.8. ([268, Theorem 5]) *Let G be a vertex-transitive graph. Then G has at most $z(\mathrm{Aut}(G))$ simple eigenvalues.*

Theorem 2.9. ([268, Proposition 4 and Theorem 14]) *Let G be a vertex-transitive graph of order n and degree k. Then the following hold:*

(a) *G has at most $k + 1$ simple eigenvalues;*

(b) *if G has more than $n/2$ simple eigenvalues, then $\mathrm{Aut}(G)$ is abelian.*

Theorem 2.10. ([268, Theorem 16]) *Let G be a connected vertex-transitive graph of order n and degree k. Suppose that $\mathrm{Aut}(G)$ is primitive on $V(G)$. Then the following hold:*

(a) *if n is not a prime, then G has exactly one simple eigenvalue, which is k;*

(b) *if $n = p$ is a prime, then either G has exactly one simple eigenvalue or it has exactly p simple eigenvalues, and the latter occurs if and only if $\mathrm{Aut}(G)$ is a cyclic group of order p.*

Theorem 2.11. ([268, Corollary]) *Let G be a Γ-vertex-transitive graph, where $\Gamma \leq \mathrm{Aut}(G)$. Then G has at most $t + 1$ simple eigenvalues, where t is the number of involutions of Γ.*

3 Integral Cayley graphs

A graph is called *integral* if its eigenvalues are all integers. This is equivalent to saying that all eigenvalues are rational numbers, because one can easily show that if all eigenvalues of a graph are rational then they must be integers. Which graphs are integral? This question was first asked by Harary and Schwenk [144] in 1973, with an immediate remark that the general problem appears to be challenging and intractable. It is known that there exist infinitely many integral graphs. However, in general it is nontrivial to construct integral graphs or determine all integral graphs in a given class of graphs. See [40, 300] for two surveys on integral graphs. In this section we give a survey of integral Cayley graphs, and in section 5 we will focus on two families of integral Cayley graphs on finite commutative rings. Our treatment here has almost no overlap with the survey papers [40, 300].

3.1 Characterizations of integral Cayley graphs

3.1.1 Integral circulant graphs, unitary Cayley graphs, and gcd graphs

In view of (2.1) one can see easily that not every circulant graph is integral. So it is natural to ask when a circulant graph is integral. This question was answered completely by So in [283]. Given an integer $n \geq 1$ and a positive divisor d of n, define

$$S_n(d) = \{a : 1 \leq a \leq n, \ \gcd(a, n) = d\}. \tag{3.1}$$

Theorem 3.1. ([283, Theorem 7.1]) *Let $n \geq 3$ be an integer. A circulant graph $\mathrm{Cay}(\mathbb{Z}_n, S)$ is integral if and only if S is the union of $S_n(d)$ for some proper divisors d of n.*

As a corollary it was noted [283, Corollary 7.2] that there are at most $2^{\tau(n)-1}$ integral circulants on n vertices, where $\tau(n)$ is the number of divisors of n. It was further conjectured that there are exactly $2^{\tau(n)-1}$ integral circulants on n vertices (see [283, Conjecture 7.3]).

The sufficiency in Theorem 3.1 was obtained independently by Klotz and Sander in [174] using a slightly different language. Denote by $\mathbb{Z}_n^{\times}$ the set of units (multiplicatively invertible elements) of ring $\mathbb{Z}_n$. The *unitary Cayley graph* of $\mathbb{Z}_n$ is defined as the Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^{\times})$. This graph was first introduced in [53, 54] in the study of induced cycles of a given length and related chromatic uniqueness problem for the graph, and in [120, 121] longest induced cycles in $\mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^{\times})$ were further studied. In [174], Klotz and Sander studied several combinatorial properties of unitary Cayley graphs such as connectivity, diameter, chromatic number, clique number, independence number and perfectness. In the same paper they introduced the concept of gcd graphs as a generalization of unitary Cayley graphs. Define

$$D(n) = \{a : 1 \leq a \leq n, \ a \text{ is a divisor of } n\} \tag{3.2}$$

to be the set of positive divisors of n. So $D(n) \setminus \{n\}$ is the set of positive proper divisors of n. Given $D \subseteq D(n) \setminus \{n\}$, where $n \geq 2$, the *gcd graph* of the cyclic group $\mathbb{Z}_n$ with respect to D, denoted by $\mathrm{ICG}(n, D)$, is defined by

$$V(\mathrm{ICG}(n, D)) = \mathbb{Z}_n, \ \ E(\mathrm{ICG}(n, D)) = \{\{x, y\} : x, y \in \mathbb{Z}_n, \gcd(x - y, n) \in D\}. \tag{3.3}$$

In other words,

$$\mathrm{ICG}(n, D) = \mathrm{Cay}(\mathbb{Z}_n, S_n(D)),$$

where

$$S_n(D) = \{a \in \mathbb{Z}_n : \gcd(a, n) \in D\}. \tag{3.4}$$

In particular,

$$\mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^{\times}) \cong \mathrm{ICG}(n, \{1\}).$$

It is evident that the graphs in Theorem 3.1 are precisely the gcd graphs of cyclic groups. Thus the following result obtained by Klotz and Sander in [174] gives the sufficiency in Theorem 3.1.

Theorem 3.2. ([174, Theorem 16]) *The gcd graphs of cyclic groups are all integral. In particular, unitary Cayley graphs of cyclic groups are integral circulant graphs.*

Moreover, in [174] the eigenvalues of any gcd graph $\mathrm{ICG}(n, D)$ were given in terms of Euler's totient function $\varphi(n)$ and the Möbius function

$$\mu(n) = \sum_{1 \leq k \leq n, \gcd(k, n) = 1} \omega_n^k.$$

The well-known *Ramanujan sum* [145] is defined as

$$c(k, n) = \sum_{1 \leq j \leq n, \gcd(j, n) = 1} \omega_n^{kj} = \mu\left(\frac{n}{\gcd(k, n)}\right) \cdot \frac{\varphi(n)}{\varphi\left(\frac{n}{\gcd(k, n)}\right)}, \ k = 0, 1, \dots, n - 1. \tag{3.5}$$

We know from number theory that $\mu(n)$ takes value from $\{-1, 0, 1\}$ and $c(k, n)$ takes only integral values. In [174, Theorem 13], Klotz and Sander proved that the eigenvalues of $\mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^\times)$ are $c(k, n)$, $0 \le k \le n - 1$. In general, they proved [174, Theorem 16] further that the eigenvalues of $\mathrm{ICG}(n, D)$ are

$$\lambda_l(n, D) = \sum_{d \in D} c(l, n/d), \ l = 0, 1, \ldots, n - 1. \tag{3.6}$$

Therefore, $\mathrm{ICG}(n, D)$ is integral as claimed in Theorem 3.2.

Using the language of gcd graphs, Theorem 3.1 can be restated as follows.

Corollary 3.3. *A circulant graph is integral if and only if it is a gcd graph of a cyclic group.*

3.1.2 Integral Cayley graphs on abelian groups

Now that integral circulant graphs have been characterized, it is natural to move on to Cayley graphs on abelian groups. Let A be a set and $\mathcal{F}$ a family of subsets of A. The *Boolean algebra* $B(\mathcal{F})$ generated by $\mathcal{F}$ is the lattice of those subsets of A each obtained by taking unions, intersections and complements of members of $\mathcal{F}$ in an arbitrary way but a finite number of times. The following result, due to Klotz and Sander [175], gives a sufficient condition for a Cayley graph on an abelian group to be integral.

Theorem 3.4. ([175, Theorem 8]) *A Cayley graph $\mathrm{Cay}(\Gamma, S)$ on a finite abelian group Γ is integral provided that S belongs to the Boolean algebra generated by the family of subgroups of Γ.*

With the help of Theorem 3.1 Klotz and Sander also proved that this sufficient condition is necessary in the special case when Γ is a cyclic group.

Theorem 3.5. ([175, Theorem 10]) *A circulant graph $\mathrm{Cay}(\mathbb{Z}_n, S)$ with order $n \ge 2$ is integral if and only if S belongs to the Boolean algebra generated by the family of subgroups of $\mathbb{Z}_n$.*

In the same paper, Klotz and Sander further conjectured that the same should be true for any finite abelian group. One year later they confirmed their own conjecture in the following special case.

Theorem 3.6. ([177, Theorems 2 and 4]) *Let $\Gamma = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ be an abelian group such that $\gcd(n_i, n_j) \le 2$ for $i \ne j$, where each $n_i \ge 2$. Then a Cayley graph $\mathrm{Cay}(\Gamma, S)$ on Γ is integral if and only if S belongs to the Boolean algebra generated by the family of subgroups of Γ.*

The above-mentioned conjecture in [175] was finally confirmed by Alperin and Peterson in [22] in its general form. To present this result we need a few definitions first. Let Γ be a finite group. Let $\widehat{\Gamma}$ be the set of characters of representations of Γ over $\mathbb{C}$. A subset $A \subseteq \Gamma$ is called *integral* if $\chi(A) \in \mathbb{Z}$ for every $\chi \in \widehat{\Gamma}$, where $\chi(A) = \sum_{a \in A} \chi(a)$. The group Γ is called *cyclotomic* [22] if the Boolean algebra generated by the family $\mathcal{I}(\Gamma)$ of integral sets of Γ equals the Boolean algebra generated by the family $\mathcal{F}(\Gamma)$ of subgroups of Γ. Alperin and Peterson proved that any finite abelian group is cyclotomic [22, Theorem 5.1], and that the converse is also true [22, Theorem 8.3]. Using the former statement, they obtained the following result, which confirms the above-mentioned conjecture of Klotz and Sander [175].

Theorem 3.7. ([22, Corollary 7.2]) *A Cayley graph* $\mathrm{Cay}(\Gamma, S)$ *on a finite abelian group* Γ *is integral if and only if* S *is an integral set. Equivalently, a Cayley graph* $\mathrm{Cay}(\Gamma, S)$ *on a finite abelian group* Γ *is integral if and only if* S *belongs to the Boolean algebra generated by the family of subgroups of* Γ.

Alperin and Peterson also proved that for any group Γ (not necessarily abelian), all atoms of the Boolean algebra $B(\mathcal{F}(\Gamma))$ are integral and so $B(\mathcal{F}(\Gamma)) \subseteq B(\mathcal{I}(\Gamma))$ (see [22, Section 4]), and for the dihedral group D_{2n} of order $2n$, $B(\mathcal{I}(D_{2n}))$ equals the power set of D_{2n} (see [22, Theorem 6.1]).

Let G be a graph and $\emptyset \neq N \subseteq \mathbb{N} \cup \{\infty\}$. The *distance power* of G with respect to N, denoted by G^N, is the graph with vertex set $V(G)$ such that u and v are adjacent if and only if the distance between them in G belongs to N. In particular, for any positive integer k, $G^{\{1,2,...,k\}}$ is called the *kth power* of G and is usually denoted by G^k. The second power G^2 is usually called the *square* of G.

Theorem 3.8. ([178, Theorem 1]) *Let* $G = \mathrm{Cay}(\Gamma, S)$ *be a Cayley graph on a finite abelian group* Γ. *If* G *is integral, then for any set* N *of non-negative integers (possibly including* ∞), *the distance power* G^N *of* G *is also an integral Cayley graph on* Γ.

Recently, Liu and Li [212] studied distance powers of the unitary Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^\times)$. Since $\mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^\times)$ is integral, by Theorem 3.8, such distance powers are all integral circulant graphs.

Cayley graphs on abelian groups Γ and their eigenvalues have been studied in [62] via the algebra $\mathbb{C}[\Gamma]$ over $\mathbb{C}$ generated by the permutation matrices representing the elements of Γ and the algebra over $\mathbb{Q}$ of those elements of $\mathbb{C}[\Gamma]$ with rational entries and rational eigenvalues. The objects of study in [62] are graphs admitting an abelian group Γ as an automorphism group regular on the vertex set. Note that such graphs are precisely Cayley graphs on Γ.

3.1.3 Integral normal Cayley graphs

Recall that a Cayley graph $\mathrm{Cay}(\Gamma, S)$ is normal if S is the union of some conjugacy classes of the group Γ. A subset S of Γ is called *power-closed* if, for every $x \in S$ and $y \in \langle x \rangle$ with $\langle y \rangle = \langle x \rangle$, we have $y \in S$. The next result follows from a more general result [134, Theorem 1.3] proved by Godsil and Spiga.

Theorem 3.9. ([134, Theorem 1.1]) *A finite normal Cayley graph* $\mathrm{Cay}(\Gamma, S)$ *is integral if and only if* S *is power-closed.*

3.1.4 Integral Cayley multigraphs

A *multiset* is a set S together with a multiplicity function $\mu_S : S \to \mathbb{N}$, where for each $x \in S$ the positive integer $\mu_S(x)$ indicates the number of times that x occurs in the multiset. As a convention we set $\mu_S(x) = 0$ for $x \notin S$. Let Γ be a finite group. A multiset S of some elements of Γ is called *inverse-closed* if $\mu_S(x) = \mu_S(x^{-1})$ for every $x \in S$. In this case the *Cayley multigraph* on Γ with connection set S, $\mathrm{Cay}(\Gamma, S)$, is defined to be the multigraph with vertex set Γ such that the number of edges joining $x, y \in \Gamma$ is equal to $\mu_S(xy^{-1})$. In other words, $\mathrm{Cay}(\Gamma, S)$ is precisely the Cayley colour graph $\mathrm{Cay}(\Gamma, \mu_S)$. The *adjacency matrix* of $\mathrm{Cay}(\Gamma, S)$ is the matrix

with (x, y)-entry $\mu_S(xy^{-1})$, and its eigenvalues are called the *eigenvalues* of $\mathrm{Cay}(\Gamma, S)$. A Cayley multigraph is callled *integral* if all its eigenvalues are integers. In the special case when $\mu_S(x) = 1$ for every $x \in S$, $\mathrm{Cay}(\Gamma, S)$ is a Cayley graph in the usual sense. Hence we use the same notation for both Cayley graphs and Cayley multigraphs.

Denote by $\mathcal{N}(\Gamma)$ the set of normal subgroups of Γ. Define $B(\mathcal{N}(\Gamma))$ to be the Boolean algebra generated by $\mathcal{N}(\Gamma)$. This notion can be extended to multisets by including multiset operations. Formally, we take all atoms of the Boolean algebra $B(\mathcal{N}(\Gamma))$ and take all multisets that can be expressed as non-negative integer combinations of these atoms. This defines the collection $\mathcal{C}(\Gamma)$ of multisets, which is called the *integral cone* over $B(\mathcal{N}(\Gamma))$.

In 1982, Bridges and Mena [63] completely characterized integral Cayley multigraphs on abelian groups.

Theorem 3.10. ([63, Theorem 2.4]); also [103, Theorem 1] *Let Γ be an abelian group and S an inverse-closed multiset of Γ. Then the Cayley multigraph $\mathrm{Cay}(\Gamma, S)$ is integral if and only if $S \in \mathcal{C}(\Gamma)$.*

In [103], DeVos et al. generalized the sufficiency part of this result to Cayley multigraphs on any finite group.

Theorem 3.11. ([103, Theorem 2]) *Let Γ be a finite group. Then for any $S \in \mathcal{C}(\Gamma)$ the Cayley multigraph $\mathrm{Cay}(G, S)$ is integral.*

Of course, by Theorem 3.10, the converse statement in Theorem 3.11 holds for abelian groups. In [103], DeVos et al. also investigated to what extent the converse would hold for some other groups. They provided necessary and sufficient conditions for integrality of Cayley multigraphs over Hamiltonian groups (see [103, Theorem 11]), where a *Hamiltonian group* is a non-abelian Dedekind group, whilst a *Dedekind group* is a group in which every subgroup is normal. It is known that every finite Hamiltonian group is the direct product $Q_8 \times A$ of the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ and a finite abelian group A. Given a Hamiltonian group $\Gamma = Q_8 \times A$ and an inverse-closed multiset S of elements of Γ, for every $q \in Q_8$, define $B_q = \{a \in A : (q, a) \in S\}$ to be the multiset in which the multiplicity of $a \in B_q$ is equal to the multiplicity of (q, a) in S. Then $B_1 = B_1^{-1}$, $B_{-1} = B_{-1}^{-1}$ and $B_{-q} = B_q^{-1}$ for $q \in Q_8 \setminus \{-1, 1\}$. For an irreducible character λ of A, define $\hat{\lambda}(B_q) = \lambda(B_q) - \lambda(B_q^{-1}) = \lambda(B_q) - \lambda(B_{-q})$ for $q \in Q_8$.

Theorem 3.12. ([103, Theorem 11]) *Let $\Gamma = Q_8 \times A$ be a Hamiltonian group, where A is an abelian group, and let S be an inverse-closed multiset of elements of Γ. Then the Cayley multigraph $\mathrm{Cay}(\Gamma, S)$ is integral if and only if the following hold:*

(a) $B_1, B_{-1} \in \mathcal{C}(A)$;

(b) *the multiset union $B_q \cup B_{-q} \in \mathcal{C}(A)$, for every $q \in Q_8 \setminus \{-1, 1\}$;*

(c) $\hat{\lambda}(B_i)^2 + \hat{\lambda}(B_j)^2 + \hat{\lambda}(B_k)^2$ *is a negative square of an integer, for every irreducible character λ of A.*

Using this, DeVos et al. also obtained a few results on the integrality of Cayley graphs and Cayley multigraphs on some special families of Hamiltonian groups, including $Q_8 \times \mathbb{Z}_p$ and $Q_8 \times \mathbb{Z}_p^d$, where p is a prime and $d \geq 2$. See [103, Section 6] for details.

3.2 A few families of integral Cayley graphs on abelian groups

3.2.1 Unitary finite Euclidean graphs

In [203], Li and Vinh defined the *unitary finite Euclidean graphs* $T_n^{(d)}$ by

$$V(T_n^{(d)}) = \mathbb{Z}_n^d, \ \ E(T_n^{(d)}) = \left\{ (\mathbf{a}, \mathbf{b}) : \mathbf{a}, \mathbf{b} \in \mathbb{Z}_n^d, \ \sum_{i=1}^d (a_i - b_i)^2 \in \mathbb{Z}_n^\times \right\},$$

where $n \geq 2$, $d \geq 1$ and $\mathbb{Z}_n^\times$ is the set of units of $\mathbb{Z}_n$. Clearly,

$$T_n^{(d)} = \mathrm{Cay}(\mathbb{Z}_n^d, S_n^{(d)}),$$

where

$$S_n^{(d)} = \left\{ \mathbf{x} \in \mathbb{Z}_n^d : \sum_{i=1}^d x_i^2 \in \mathbb{Z}_n^\times \right\}.$$

The following result was proved by Li and Vinh in [203].

Theorem 3.13. ([203, Theorem 3.6]) *Let $n \geq 2$ and $d \geq 1$ be integers. Then the unitary finite Euclidean graph $T_n^{(d)}$ is integral when n is odd or d is even.*

In the same paper Li and Vinh also conjectured that $T_n^{(d)}$ is integral for any $n \geq 2$ and $d \geq 1$ (see [203, Conjecture 3.7]). This conjecture was proved in [93] as a special case of a more general result. Given $U \subseteq \mathbb{Z}_n$, the *distance graph* over $\mathbb{Z}_n^d$ generated by U, denoted by $T_n^{(d)}(U)$, was defined [93] in the same way as $T_n^{(d)}$ except that $\mathbb{Z}_n^\times$ is replaced by U. In particular, $T_n^{(d)} = T_n^{(d)}(\mathbb{Z}_n^\times)$. Obviously,

$$T_n^{(d)}(U) = \mathrm{Cay}(\mathbb{Z}_n^d, S_n^{(d)}(U)),$$

where $S_n^{(d)}(U)$ is defined in the same way as $S_n^{(d)}$ with $\mathbb{Z}_n^\times$ replaced by U. Since this is a Cayley graph on an abelian group, by Corollary 2.3 its eigenvalues can be easily found to be

$$\lambda_{\mathbf{b}}(U) = \sum_{\mathbf{x} \in S_n^{(d)}(U)} \exp\{2\pi i (\mathbf{b}^T \mathbf{x})/n\}, \ \mathbf{b} \in \mathbb{Z}_n^d,$$

where $\mathbf{b}^T$ is the transpose of $\mathbf{b}$ and $\mathbf{b}^T \mathbf{x}$ is the dot product of $\mathbf{b}$ and $\mathbf{x}$.

Recall from (3.1) that $S_n(d) = \{1 \leq a \leq n : \gcd(a, n) = d\}$ for any divisor d of n.

Theorem 3.14. ([93, Theorem 3]) *Let $n \geq 2$ and $d \geq 1$ be integers. Let D be a set of divisors of n and $U = \cup_{d \in D} S_n(d)$. Then $T_n^{(d)}(U)$ is integral.*

In the special case when $D = \{1\}$, this implies that the above-mentioned conjecture in [203, Conjecture 3.7] is true.

Theorem 3.15. ([93]) *The unitary finite Euclidean graph $T_n^{(d)}$ is integral for any integers $n \geq 2$ and $d \geq 1$.*

3.2.2 NEPS of complete graphs, gcd graphs of abelian groups, and generalized Hamming graphs

Let $G_1, \ldots, G_d$ be graphs, and let $\emptyset \neq \mathcal{B} \subseteq \mathbb{Z}_2^d \setminus \{\mathbf{0}\}$, where $d \geq 1$ and $\mathbf{0} = (0, \ldots, 0) \in \mathbb{Z}_2^d$. The *non-complete extended p-sum* (NEPS) of $G_1, \ldots, G_d$ with *basis* $\mathcal{B}$ (see [92, Definition 2.5.1]), denoted by $\mathrm{NEPS}(G_1, \ldots, G_d; \mathcal{B})$, is the graph with vertex set $V(G_1) \times \cdots \times V(G_d)$ in which two vertices $(x_1, \ldots, x_d)$ and $(y_1, \ldots, y_d)$ are adjacent if and only if there exists $\beta = (\beta_1, \ldots, \beta_d) \in \mathcal{B}$ such that $x_i = y_i$ whenever $\beta_i = 0$ and x_i is adjacent to y_i in G_i whenever $\beta_i = 1$. This notion is a generalization of several graph operations such as tensor product, Cartesian product, and strong product. In fact, $\mathrm{NEPS}(G_1, \ldots, G_d; \{(1, \ldots, 1)\})$ is simply the *tensor product* $G_1 \otimes \cdots \otimes G_d$; if $\mathcal{B}_0 = \{(1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1)\}$ is the standard basis of $\mathbb{Z}_2^d$, then $\mathrm{NEPS}(G_1, \ldots, G_d; \mathcal{B}_0)$ is the *Cartesian product* $G_1 \square \cdots \square G_d$. Thus $\mathrm{NEPS}(K_{n_1}, \ldots, K_{n_d}; \mathcal{B}_0)$ is the *Hamming graph* $H(n_1, \ldots, n_d)$, where $n_1, \ldots, n_d \geq 2$. In particular, $\mathrm{NEPS}(K_2, \ldots, K_2; \mathcal{B}_0)$ is the hypercube $H(d, 2)$ of dimension d. In general, for any $\emptyset \neq \mathcal{B} \subseteq \mathbb{Z}_2^d \setminus \{\mathbf{0}\}$, $\mathrm{NEPS}(K_2, \ldots, K_2; \mathcal{B})$ is called a *cubelike graph* [217]. In other words, a cubelike graph is precisely a Cayley graph on an elementary abelian 2-group. A fundamental result (see [92, Theorem 2.5.4]) on NEPS asserts that, if $\lambda_{i1}, \ldots, \lambda_{in_i}$ are eigenvalues of G_i for $1 \leq i \leq d$, where $n_i = |V(G_i)|$, then $\mathrm{NEPS}(G_1, \ldots, G_d; \mathcal{B})$ has eigenvalues

$$\mu_{i_1, \ldots, i_d} = \sum_{(\beta_1, \ldots, \beta_d) \in \mathcal{B}} \lambda_{1i_1}^{\beta_1} \cdots \lambda_{di_d}^{\beta_d}, \ i_t = 1, 2, \ldots, n_t, \ t = 1, 2, \ldots, d. \tag{3.7}$$

As an immediate consequence, we have the following result.

Corollary 3.16. *Every NEPS of integral graphs is integral.*

This result is relevant since NEPS graphs of complete graphs form an interesting family of integral Cayley graphs on abelian groups, as we now explain. Let $\mathbf{m} = (m_1, \ldots, m_d)$, $\mathbf{n} = (n_1, \ldots, n_d)$ and $\mathbf{k} = (k_1, \ldots, k_d)$ be d-tuples of positive integers. If $k_i = \gcd(m_i, n_i)$ for $i = 1, \ldots, d$, then $\mathbf{k}$ is called the *greatest common divisor* of $\mathbf{m}$ and $\mathbf{n}$, written $\mathbf{k} = \gcd(\mathbf{m}, \mathbf{n})$. Define

$$\mathbf{D}(\mathbf{n}) = \{(a_1, \ldots, a_d) : 1 \leq a_i \leq n_i \text{ and } a_i \text{ is a divisor of } n_i \text{ for } 1 \leq i \leq d\}. \tag{3.8}$$

Let $\Gamma = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_d}$ be an abelian group, where $d \geq 1$ and each $n_i \geq 2$. Let $\mathbf{n} = (n_1, \ldots, n_d)$ and $\mathbf{D} \subseteq \mathbf{D}(\mathbf{n}) \setminus \{\mathbf{n}\}$. Denote by $S_\Gamma(\mathbf{D})$ the set of elements $\mathbf{x} = (x_1, \ldots, x_d)$ of Γ such that $0 \leq x_i \leq n_i - 1$ for $1 \leq i \leq d$ and $\gcd(\mathbf{x}, \mathbf{n}) \in \mathbf{D}$. (If $\mathbf{D} = \{\mathbf{k}\}$, then we write $S_\Gamma(\mathbf{k})$ in place of $S_\Gamma(\{\mathbf{k}\})$.) The *gcd graph* of the abelian group Γ with respect to $\mathbf{D}$, denoted[2] by $\mathrm{ICG}(\mathbf{n}, \mathbf{D})$, is defined [177, 179] to be the Cayley graph $\mathrm{Cay}(\Gamma, S_\Gamma(\mathbf{D}))$. This definition is a generalization of the gcd graphs of cyclic groups, because in the special case when $\Gamma = \mathbb{Z}_n$ (so $\mathbf{n} = (n)$) and $\mathbf{D} = D \subseteq D(n) \setminus \{n\}$ is a set of positive proper divisors of n, $\mathrm{ICG}(\mathbf{n}, \mathbf{D})$ is exactly the gcd graph $\mathrm{ICG}(n, D)$ of the cyclic group $\mathbb{Z}_n$. Note that $\mathrm{ICG}(\mathbf{n}, \mathbf{D})$ relies on the specific expression of the abelian group Γ as the direct sum $\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_d}$.

In [179, Theorems 2.5 and 2.6], it was shown that a graph isomorphic to a gcd graph of an abelian group if and only if it is isomorphic to the NEPS of some complete graphs. In fact, the proof of [179, Theorem 2.6] implies that

$$\mathrm{NEPS}(K_{n_1}, \ldots, K_{n_d}, \mathcal{B}) \cong \mathrm{ICG}(\mathbf{n}, \mathbf{D}),$$

---

[2]As seen in section 3.1.1, the acronym ICG arose from Integral Circulant Graphs. We use the same acronym for gcd graphs of abelian groups here and gcd graphs of unique factorization domains later (section 6.5) to indicate that all these graphs are defined in the same fashion, though they are not necessarily circulants in general.

where $\mathbf{D} = \cup_{\beta \in \mathcal{B}} \mathbf{D}(\beta)$, where for each $\beta = (\beta_1, \ldots, \beta_d) \in \mathcal{B}$, $\mathbf{D}(\beta)$ consists of those $(a_1, \ldots, a_d)$ such that $a_i = n_i$ if $\beta_i = 0$ and $a_i \geq 1$ is a proper divisor of n_i if $\beta_i = 1$. Since complete graphs are integral, this together with Corollary 3.16 implies the following result.

Corollary 3.17. ([179, Proposition 3.2]; see also [177]) *NEPS graphs of complete graphs are integral. In other words, every gcd graph of any abelian group is integral.*

In [179], Klotz and Sander proved further that gcd graphs of abelian groups have a particularly simple eigenspace structure, namely every eigenspace of the adjacency matrix of a gcd graph has a basis with entries $-1, 0, 1$ only. Corollary 3.17 implies the following result due to Lovász.

Corollary 3.18. ([217]) *Every cubelike graph is integral.*

Let $\Gamma = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_d}$ be an abelian group, where $d \geq 1$ and $n_i \geq 2$ for $1 \leq i \leq d$. Let $T = \{t_1, \ldots, t_k\}$ be a set of integers such that $1 \leq t_i \leq d$ for each i. The *generalized Hamming graph* $H(n_1, \ldots, n_d; T)$ is defined to have vertex set Γ such that $\mathbf{x} = (x_1, \ldots, x_d) \in \Gamma$ and $\mathbf{y} = (y_1, \ldots, y_d) \in \Gamma$ are adjacent if and only if the *Hamming distance* $w(\mathbf{x} - \mathbf{y})$ between them belongs to T. As observed in [179, Example 2.8], $H(n_1, \ldots, n_d; T)$ is an NEPS graph of $K_{n_1}, \ldots, K_{n_d}$. Thus it is integral by Corollary 3.18 and its eigenvalues can be computed using formula (3.7).

Theorem 3.19. ([175, Proposition 14]) *Every generalized Hamming graph is an integral Cayley graph.*

In the special case when $T = \{1\}$, we have $H(n_1, \ldots, n_d; \{1\}) \cong K_{n_1} \square \cdots \square K_{n_d} = H(n_1, \ldots, n_d)$. In particular, $H(q, \ldots, q; \{1\})$ is the classical Hamming graph $H(d, q)$. It is well known [92] that the spectra of the Cartesian product of graphs can be expressed in terms of that of the factor graphs. Since the eigenvalues of any complete graph are known, this implies [65] that the eigenvalues of $H(d, q)$ are $q(d-j) - d$ with multiplicities $\binom{d}{j}(q-1)^j$ for $0 \leq j \leq d$. In [276], T. Sander proved that $H(n_1, \ldots, n_d; \{d\})$ is isomorphic to the complement graph of $H(n_1, \ldots, n_d; \{1, \ldots, d-1\})$, and for distinct primes $p_1, \ldots, p_d$, $H(p_1, \ldots, p_d; \{d\})$ is isomorphic to the unitary Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^\times)$, where $n = \prod_{i=1}^d p_i$.

In [207], Li et al. characterized isomorphisms between the squares of generalized Hamming graphs and the NEPS of some complete graphs. As an application they determined the eigenvalues of the squares of generalized Hamming graphs. In view of Theorem 3.8 and Corollary 3.16, such graphs are all integral Cayley graphs.

3.2.3 Sudoku graphs and positional Sudoku graphs

Let $n \geq 2$ be an integer. An *n-Sudoku* is an arrangement of $n \times n$ square blocks each consisting of $n \times n$ cells, with each cell filled with a number (colour) from $\{1, 2, \ldots, n^2\}$ such that every block, row or column contains all of the colours $1, 2, \ldots, n^2$. The *Sudoku graph* $\mathrm{Sud}(n)$ is defined [275, 175] to have vertices the n^4 cells of an n-Sudoku such that distinct vertices (cells) are adjacent if and only if they are in the same block, row or column. The *positional Sudoku graph* $\mathrm{SudP}(n)$ is defined [175] to have vertices the n^4 cells of an n-Sudoku such that distinct vertices (cells) are adjacent if and only if they are in the same block, row, column or in the same

position of their respective blocks. It can be shown that both $\mathrm{Sud}(n)$ and $\mathrm{SudP}(n)$ are Cayley graphs on $\mathbb{Z}_n^4$; see [175, Section 4.2] for an explicit expression of the corresponding connection sets. In [275, Lemma 3.1], it was shown that $\mathrm{Sud}(n)$ is the NEPS of four copies of K_n with basis $\{(0, 1, 0, 1), (1, 1, 0, 0), (0, 0, 1, 1), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$. Using this it was proved in [275, Theorem 3.2] that $\mathrm{Sud}(n)$ is integral and each of its eigenspaces admits a basis whose entries are all from the set $\{-1, 0, 1\}$.

Theorem 3.20. ([275, 175]) *Let $n \geq 2$ be an integer. Then both $\mathrm{Sud}(n)$ and $\mathrm{SudP}(n)$ are integral Cayley graphs. Moreover, the following hold:*

(a) *the spectrum of the Sudoku graph $\mathrm{Sud}(n)$ is*

$$
\begin{pmatrix}
3n^2 - 2n - 1 & 2n^2 - 2n - 1 & n^2 - n - 1 & n^2 - 2n - 1 & -1 & -1 - n \\
1 & 2(n-1) & 2n(n-1) & (n-1)^2 & n^2(n-1)^2 & 2n(n-1)^2
\end{pmatrix};
$$

(b) *the spectrum of the positional Sudoku graph $\mathrm{SudP}(n)$ is*

$$
\begin{pmatrix}
4n(n-1) & 2n^2 - 3n & n(n-2) & 0 & -n & -2n \\
1 & 4(n-1) & 4(n-1)^2 & (n-1)^4 & 4(n-1)^3 & 2(n-1)^2
\end{pmatrix}.
$$

In the expository paper [176], Klotz and Sander used symmetry properties of Sudoku graphs to determine their eigenvalues. In particular, they recovered the above-mentioned result that all eigenvalues of Sudoku graphs are integral.

3.2.4 Pandiagonal Latin square graphs

Let $n \geq 2$ be an integer. The *pandiagonal Latin square graph* $\mathrm{PLSG}(n)$ is defined [175] to have vertices the n^2 positions of an $n \times n$ matrix such that distinct vertices (positions) are adjacent if and only if they are in the same row, the same column, the same (broken) parallel to the main diagonal, or the same (broken) parallel to the secondary diagonal. This is a Cayley graph on $\mathbb{Z}_n^2$ whose connection set can be found in [175, Section 4.3].

Theorem 3.21. ([175, Proposition 16]) *Let $n \geq 2$ be an integer. The pandiagonal Latin square graph $\mathrm{PLSG}(n)$ is an integral Cayley graph. Moreover, the following hold:*

(a) *if n is odd, then the spectrum of $\mathrm{PLSG}(n)$ is*

$$
\begin{pmatrix}
4n - 4 & n - 4 & -4 \\
1 & 4(n-1) & n^2 - 4n + 3
\end{pmatrix};
$$

(b) *if n is even, then the spectrum of $\mathrm{PLSG}(n)$ is*

$$
\begin{pmatrix}
4n - 5 & 2n - 5 & n - 3 & n - 5 & -3 & -5 \\
1 & 1 & n & 3n - 6 & n^2/2 - n & n^2/2 - 3n + 4
\end{pmatrix}.
$$

3.3 Integral Cayley graphs on non-abelian groups

3.3.1 Cayley graphs on dihedral groups

In this subsection $D_{2n} = \langle a, b \,|\, a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ is the dihedral group of order $2n \geq 4$ and S is a subset of $D_{2n} \setminus \{1\}$ with $S^{-1} = S$. The Cayley graph $\mathrm{Cay}(D_{2n}, S)$ is called a

dihedrant in the literature, and its eigenvalues can be computed using Theorem 2.2 and the character table of D_{2n} (see [184, Section 7.5] and [154, Corollary 2.7]). Recently, Gao and Luo [123] gave a simpler way to compute the spectra of $\mathrm{Cay}(D_{2n}, S)$ using a more general result (see Theorem 11.1) on eigenvalues of semi-Cayley (bi-Cayley) graphs on abelian groups. We present their result in the theorem and defer our discussion on bi-Cayley graphs to section 11.1. Set $\langle a \rangle b = \{a^i b : 0 \le i \le n-1\}$.

Theorem 3.22. ([123, Theorem 4.3]) *The following hold:*

(a) *if $S \cap \langle a \rangle b = \emptyset$, then the eigenvalues of $\mathrm{Cay}(D_{2n}, S)$ are*

$$\sum\nolimits_{a^i \in S} \omega_n^{ir}, \ r = 0, 1, \ldots, n-1,$$

each with multiplicity 2;

(b) *if $S \cap \langle a \rangle b \ne \emptyset$, say $a^{i_0} b \in S$, then the eigenvalues of $\mathrm{Cay}(D_{2n}, S)$ are*

$$\sum\nolimits_{a^i \in S} \omega_n^{ir} \pm \left| \sum\nolimits_{a^i b \in S} \omega_n^{(i_0 - i)r} \right|, \ r = 0, 1, \ldots, n-1.$$

Since $\left| \sum_{a^i b \in S} \omega_n^{(i_0-i)r} \right| = \left| \omega_n^{i_0 r} \right| \left| \sum_{a^i b \in S} \omega_n^{-ir} \right| = \left| \sum_{a^i b \in S} \omega_n^{-ir} \right|$, Theorem 3.22 implies the following characterization of integral dihedrants.

Theorem 3.23. ([13, Theorem 3.5.1]) *A dihedrant $\mathrm{Cay}(D_{2n}, S)$ is integral if and only if $\sum_{a^i \in S} \omega_n^{ir}$ and $\left| \sum_{a^i b \in S} \omega_n^{-ir} \right|$ are all integers for $0 \le r \le n-1$.*

We have $\sum_{j \in S_n(d)} \omega_n^{jr} \in \mathbb{Z}$ for $0 \le r \le n-1$ (see [283, Theorem 4.4]). It can be easily shown that, for $A \subseteq \mathbb{Z}_n$ and $0 \le r \le n-1$, if $\sum_{j \in A} \omega_n^{jr} \in \mathbb{Z}$, then $\sum_{j \in A} \omega_n^{-jr} = \sum_{j \in A} \omega_n^{jr}$. This together with Theorem 3.23 implies the following result, where $S_n(d)$ is as in (3.1).

Theorem 3.24. ([213]) *The following hold:*

(a) *$\mathrm{Cay}(D_{2n}, S)$ is integral provided that each of $\{i : a^i \in S\}$ and $\{i : a^i b \in S\}$ is a union of $S_n(d)$ for some divisors d with $d < n$;*

(b) *$\mathrm{Cay}(D_{2n}, S)$ is integral provided that $\{i : a^i \in S\}$ is a union of $S_n(d)$ for some divisors d with $d < n$ and $|\{i : a^i b \in S\}| = 1$.*

Since $S_n(d) = dS_{n/d}(1)$ for any divisor d of n, in the special case when $\{i : a^i \in S\} = \{i : a^i b \in S\} = S_n(1) \cup S_n(d)$, part (a) of Theorem 3.24 implies the following result.

Theorem 3.25. ([7, Theorem 1.2]) *A dihedrant $\mathrm{Cay}(D_{2n}, S)$ is integral if $n \ge 3$ is odd and*

$$S = \{a^i : i \in S_n(1)\} \cup \{a^{di} : i \in S_{n/d}(1)\} \cup \{ba^i : i \in S_n(1)\} \cup \{ba^{di} : i \in S_{n/d}(1)\}$$

for some proper divisor d of n.

Recently, Lu et al. [219] obtained several results about integral dihedrants. By the character table of dihedral groups, for each h, $0 \le h \le n-1$, the mapping $\chi_h : D_{2n} \to \mathbb{C}$ defined by $\chi_h(a^i) = 2\cos(2hi\pi/n)$ and $\chi_h(ba^i) = 0$ for $0 \le i \le n-1$ is a character of D_{2n}. Recall from section 3.1.2 that $B(\mathcal{F}(\Gamma))$ is the Boolean algebra generated by the family $\mathcal{F}(\Gamma)$ of subgroups of a group Γ.

Theorem 3.26. ([219]) *Let $S = S_1 \cup S_2$ be a subset of $D_{2n} \setminus \{1\}$ with $S^{-1} = S$, where $S_1 \subseteq \langle a \rangle$ and $S_2 \subseteq b\langle a \rangle$. Then the following hold:*

(a) *$\mathrm{Cay}(D_{2n}, S)$ is integral if and only if both $\chi_h(S_1)$ and $\chi_h(S_1^2) + \chi_h(S_2^2)$ are integers, and $2(\chi_h(S_1^2) + \chi_h(S_2^2)) - (\chi_h(S_1^2))^2$ is a square, for $1 \le h \le \lfloor (n-1)/2 \rfloor$;*

(b) *$\mathrm{Cay}(D_{2n}, S)$ is integral if and only if $S_1 \in B(\mathcal{F}(\langle a \rangle))$ and $2\chi_h(S_2^2)$ is a square, for $1 \le h \le \lfloor (n-1)/2 \rfloor$;*

(c) *if $S_1 \in B(\mathcal{F}(\langle a \rangle))$ and $bS_2 \in B(\mathcal{F}(\langle a \rangle))$, then $\mathrm{Cay}(D_{2n}, S)$ is integral;*

(d) *if $n = p$ is an odd prime, then $\mathrm{Cay}(D_{2p}, S)$ is integral if and only if $S_1 \in B(\mathcal{F}(\langle a \rangle))$ and $S_2 = b\langle a \rangle$, $\{ba^i\}$ or $b\langle a \rangle \setminus \{ba^i\}$ for some $0 \le i \le p - 1$.*

The statement in part (d) of Theorem 3.26 was also proved in [13, Theorem 3.5.5] independently.

3.3.2 Cayley graphs on dicyclic groups

In this section

$$\mathrm{Dic}_n = \langle a, b \mid a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1} \rangle$$

is the *dicyclic group* of order $4n \ge 8$.

Theorem 3.27. ([123, Theorem 4.6]) *Let S be a subset of $\mathrm{Dic}_n \setminus \{1\}$ with $S^{-1} = S$.*

(a) *If S contains no element of $\{a^i b : 0 \le i \le 2n - 1\}$, then $\mathrm{Cay}(\mathrm{Dic}_n, S)$ has eigenvalues $\sum_{a^i \in S} \omega_n^{ir}$ each with multiplicity 2, for $0 \le r \le 2n - 1$.*

(b) *If S contains an element of $\{a^i b : 0 \le i \le 2n - 1\}$, say $a^{i_0} b \in S$, then $\mathrm{Cay}(\mathrm{Dic}_n, S)$ has eigenvalues $\sum_{a^i \in S} \omega_n^{ir} \pm \left| \sum_{a^i b \in S} \omega_n^{(i_0 - i)r} \right|$, for $0 \le r \le 2n - 1$.*

Since $\left| \sum_{a^i b \in S} \omega_n^{(i_0 - i)r} \right| = \left| \sum_{a^i b \in S} \omega_n^{-ir} \right|$, this implies the following characterization of integral Cayley graphs on the dicyclic group of order $4n$.

Theorem 3.28. *Let S be a subset of $\mathrm{Dic}_n \setminus \{1\}$ with $S^{-1} = S$. Then $\mathrm{Cay}(\mathrm{Dic}_n, S)$ is integral if and only if $\sum_{a^i \in S} \omega_n^{ir}$ and $\left| \sum_{a^i b \in S} \omega_n^{-ir} \right|$ are all integers for $0 \le r \le 2n - 1$.*

Theorem 3.27 also implies the following result.

Theorem 3.29. ([7, Theorem 1.3]) *If $n \ge 3$ is odd and $S = \{a^k : 1 \le k \le 2n - 1, k \ne n\} \cup \{ab, a^{n+1}b\}$, then $\mathrm{Cay}(\mathrm{Dic}_n, S)$ is integral and its spectrum is*

$$\begin{pmatrix} 2n & 2n - 4 & 0 & -4 \\ 1 & 1 & 3n - 1 & n - 1 \end{pmatrix}.$$

Theorem 3.28 implies the following result.

Theorem 3.30. *Let S be a subset of $\mathrm{Dic}_n \setminus \{1\}$ with $S^{-1} = S$. The following hold:*

(a) *$\mathrm{Cay}(\mathrm{Dic}_n, S)$ is integral provided that each of $\{i : a^i \in S\}$ and $\{i : a^i b \in S\}$ is a union of $S_n(d)$ for some divisors d with $d < n$;*

(b) $\mathrm{Cay}(\mathrm{Dic}_n, S)$ *is integral provided that* $\{i : a^i \in S\}$ *is a union of* $S_n(d)$ *for some divisors* d *with* $d < n$ *and* $|\{i : a^i b \in S\}| = 1$.

Recently, integral Cayley graphs on Dic_n were further studied by Cheng et al. in [80]. All characters of Dic_n are known (see, for example, [162]), and among them the following irreducible characters ψ_j, $1 \le j \le n-1$ are most relevant to the integrality of Cayley graphs on Dic_n: $\psi_j(1) = 2$, $\psi_j(a^n) = 2(-1)^j$, $\psi_j(a^r) = \omega_n^{jr/2} + \omega_n^{-jr/2}$ $(1 \le r \le n-1)$, $\psi_j(b) = 0$, $\psi_j(ab) = 0$. Set $\psi_j(X^2) = \sum_{x_1, x_2 \in X} \psi_j(x_1 x_2)$ for any subset X of Dic_n.

Theorem 3.31. ([80, Theorem 3.2]) *Let* $S = H_1 \cup H_2$ *be a subset of* $\mathrm{Dic}_n \setminus \{1\}$ *such that* $S^{-1} = S$, *where* $H_1 \subseteq \langle a \rangle \setminus \{1\}$ *and* $H_2 \subseteq b\langle a \rangle$. *Then* $\mathrm{Cay}(\mathrm{Dic}_n, S)$ *is integral if and only if the following hold for* $1 \le j \le n-1$:

(a) $\psi_j(H_1)$ *and* $\psi_j(H_1^2) + \psi_j(H_2^2)$ *are integers;*

(b) $2\left(\psi_j(H_1^2) + \psi_j(H_2^2)\right) - \psi_j(H_1)^2$ *is a square number.*

Theorem 3.32. ([80, Theorem 4.3]) *Let* $S = H_1 \cup H_2$ *be a subset of* $\mathrm{Dic}_n \setminus \{1\}$ *such that* $S^{-1} = S$, *where* $H_1 \subseteq \langle a \rangle \setminus \{1\}$ *and* $H_2 \subseteq b\langle a \rangle$. *Then* $\mathrm{Cay}(\mathrm{Dic}_n, S)$ *is integral if and only if* H_1 *belongs to the Boolean algebra* $B(\mathcal{F}(\langle a \rangle))$ *and* $2\psi_j(H_2^2)$ *is a square number for all* $1 \le j \le n-1$.

In particular, by Theorem 3.32, $\mathrm{Cay}(\mathrm{Dic}_n, S)$ is integral if both H_1 and bH_2 belong to the Boolean algebra $B(\mathcal{F}(\langle a \rangle))$ (see [80, Corollary 4.5]). Theorem 3.32 also implies a necessary condition involving the integral cone over $B(\mathcal{F}(\langle a \rangle))$ for a Cayley on Dic_n to be integral (see [80, Corollary 4.6]). With the help of this condition the following result was proved in [80].

Theorem 3.33. ([80, Theorem 4.3]) *Let* p *be an odd prime and* $S = H_1 \cup H_2$ *a subset of* $\mathrm{Dic}_p \setminus \{1\}$ *such that* $S^{-1} = S$, *where* $H_1 \subseteq \langle a \rangle \setminus \{1\}$ *and* $H_2 \subseteq b\langle a \rangle$. *Then* $\mathrm{Cay}(\mathrm{Dic}_n, S)$ *is integral if and only if* H_1 *belongs to* $B(\mathcal{F}(\langle a \rangle))$ *and* H_2 *is equal to* $b\langle a \rangle \setminus \{ba^k, ba^{p+k}\}$, $b\langle a \rangle$ *or* $\{ba^k, ba^{p+k}\}$ *for some* $0 \le k \le p-1$.

3.3.3 Cayley graphs on a family of groups with order $6n$

Let $U_{6n} = \langle a, b \mid a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle$, where $n \ge 1$. The character table of this group can be found in [162, p.187]. This together with Theorem 2.2 can be used to compute the spectrum of $\mathrm{Cay}(U_{6n}, S)$ for any subset S of $U_{6n} \setminus \{1\}$ with $S^{-1} = S$. In particular, this enabled Abdollahi and Vatandoost [7] to prove the following result.

Theorem 3.34. ([7, Theorem 1.4]) *If* $n \ge 3$ *is odd and* $S = \{a^{2k}b : 1 \le k \le n-1\} \cup \{a^{2k}b^2 : 1 \le k \le n-1\} \cup \{a^{2k+1}b : 0 \le k \le n-1\}$, *then* $\mathrm{Cay}(U_{6n}, S)$ *is integral and its spectrum is*

$$\begin{pmatrix} 3n-2 & n-2 & 1 & -2 & 1-2n \\ 1 & 1 & 4n-2 & 2n-2 & 2 \end{pmatrix}.$$

3.3.4 Cayley graphs on symmetric groups

As usual we use S_n to denote the symmetric group on $[n] = \{1, 2, \ldots, n\}$, $n \ge 2$. The Cayley graph $\mathrm{Cay}(S_n, S)$ on S_n with connection set $S = \{(12), (13), \ldots, (1n)\}$ is known as the *star graph* of degree $n-1$ in theoretical computer science and probability theory. Using GAP [291] and

the GRAPE package of L. H. Soicher, the spectra of $\text{Cay}(S_n, S)$ for $3 \leq n \leq 6$ was computed in the proof of [7, Lemma 2.13]:

$$\text{Spec}(\text{Cay}(S_n, S)) = \begin{cases} (-2, 2, (-1)^2, 1^2), & n = 3 \\ (-3, 3, (-2)^6, 2^6, (-1)^3, 1^3, 0^4), & n = 4 \\ (-4, 4, (-3)^{12}, 3^{12}, (-2)^{28}, 2^{28}, (-1)^4, 1^4, 0^{30}), & n = 5 \\ (-5, 5, (-4)^{20}, 4^{20}, (-3)^{105}, 3^{105}, (-2)^{120}, 2^{120}, (-1)^{30}, 1^{30}, 0^{168}), & n = 6. \end{cases}$$

In particular, this implies that $\text{Cay}(S_n, S)$ is integral for $3 \leq n \leq 6$. Suggested by this computational result, Abdollahi and Vatandoost posed the following conjecture.

Conjecture 3.35. ([7, Conjecture 2.14]) *Let $n \geq 4$ be an integer and $S = \{(12), (13), \ldots, (1n)\}$. Then the star graph $\text{Cay}(S_n, S)$ is integral. Moreover, $\{0, \pm 1, \ldots, \pm(n-1)\}$ is the set of all distinct eigenvalues of $\text{Cay}(S_n, S)$.*

The second statement of this conjecture was confirmed by Krakovski and Mohar in [183].

Theorem 3.36. ([183, Theorem 1]) *Let $n \geq 4$ be an integer and $S = \{(12), (13), \ldots, (1n)\}$. Then $n - l$ and $-(n - l)$ are eigenvalues of $\text{Cay}(S_n, S)$ with multiplicity at least $\binom{n-2}{l-1}$, for $1 \leq l \leq n - 1$, and 0 is an eigenvalue of $\text{Cay}(S_n, S)$ with multiplicity at least $\binom{n-1}{2}$.*

In [35, Theorem 1], analytic formulas for the multiplicities of eigenvalues $\pm(n - k)$ of $\text{Cay}(S_n, S)$ for $k = 2, 3, 4, 5$ were obtained. In [170], a Chapuy-Feray combinatorial approach was used to obtain the multiplicities of eigenvalues of $\text{Cay}(S_n, S)$, and the exact values are calculated for $n \leq 10$.

Later it was noted in [77] (see also [265] and [183]) that the truth of Conjecture 3.35 was implied by certain properties of the Jucys-Murphy elements, discovered by Jucys [165] and independently by Flatto, Odlyzko and Wales [115]. Let λ be a partition of n. A *standard Young tableau (SYT) of shape* λ is a filling of the Ferrers diagram of λ with elements $\{1, 2, \ldots, n\}$ in such a way that elements increase along rows and columns (in particular all elements are distinct, and each element appears exactly once). Denote by $\mathcal{T}(\lambda)$ the set of SYTs of shape λ and $f_\lambda = |\mathcal{T}(\lambda)|$. Given $T \in \mathcal{T}(\lambda)$ and a box $\square$ of the Ferrers diagram of T, the content of $\square$ is the difference between its ordinate and abscissa. Define $c_T(i)$ to be the content of the box in which label i appears in T, and set $I_\lambda(k) = |\{T \in \mathcal{T}(\lambda) : c_T(n) = k\}|$. As noticed in [77, Corollary 2.1], the above-mentioned result in [115, 165] implies the following result.

Theorem 3.37. ([77, Corollary 2.1]) *Let $n \geq 2$ be an integer and $S = \{(12), (13), \ldots, (1n)\}$. Then the star graph $\text{Cay}(S_n, S)$ is integral, and the multiplicity of any integer k as an eigenvalue of $\text{Cay}(S_n, S)$ is equal to $\sum_{\lambda \in \mathcal{P}(n)} f_\lambda I_\lambda(k)$, where $\mathcal{P}(n)$ is the set of partitions of n.*

For an integer r with $2 \leq r \leq n$, let $\text{Cy}(r)$ be the set of all r-cycles in S_n which do not fix 1. That is,

$$\text{Cy}(r) = \{\alpha \in S_n : \alpha(1) \neq 1 \text{ and } \alpha \text{ is an } r\text{-cycle}\}.$$

Note that $\text{Cy}(2) = \{(12), (13), \ldots, (1n)\}$ and so $\text{Cay}(S_n, \text{Cy}(r))$ can be thought as a generalization of $\text{Cay}(S_n, S)$ above. Recently, Chen et al. [79] proved that the graphs $\text{Cay}(S_n, \text{Cy}(r))$ are all integral.

Theorem 3.38. ([79, Theorem 1.4]) $\mathrm{Cay}(S_n, \mathrm{Cy}(r))$ *is integral for* $2 \le r \le n$.

In fact, this follows from a more general result [79, Theorem 3.4], which asserts that a Cayley graph $\mathrm{Cay}(S_n, S)$ on S_n is integral provided that S is "nicely separately" in some sense.

It is known that central characters are algebraic integers [159, Theorem 3.7] and the characters of S_n are integers [162, Corollary 22.17]. Combining these with (2.2), we obtain:

Theorem 3.39. ([79, Corollary 1.2]) *All normal Cayley graphs on* S_n *are integral.*

An interesting normal Cayley graph on S_n is the *derangement graph* on $[n]$, which is defined as the Cayley graph $\mathrm{Cay}(S_n, \mathscr{D}_n)$, where $\mathscr{D}_n = \{\sigma \in S_n : \sigma(i) \ne i \text{ for each } i \in [n]\}$ is the set of derangements of $[n]$. This Cayley graph is normal as $\mathscr{D}_n$ is closed under conjugation. Thus $\mathrm{Cay}(S_n, \mathscr{D}_n)$ is integral by Theorem 3.39. In [264], Renteln gave several interesting formulas for the eigenvalues of $\mathrm{Cay}(S_n, \mathscr{D}_n)$ (see Theorems 3.2, 3.3, 4.2, 6.1 and 6.5 in [264]), including the formulas in the next two theorems.

Theorem 3.40. ([264, Theorem 3.2]) *The eigenvalues of the derangement graph* $\mathrm{Cay}(S_n, \mathscr{D}_n)$ *are given by*

$$\eta_\lambda = \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{(n-k)!} \frac{f_{\lambda/(k)}}{f_\lambda},$$

where λ *runs over all partitions of* n.

The *complete factorial symmetric function* ω_k is defined as

$$\omega_k(z_1, \dots, z_r) = \sum_{1 \le i_1 \le \cdots \le i_k \le r} (z_{i_1} - i_1 + 2 - 1) \cdots (z_{i_k} - i_k + 2 - k).$$

Theorem 3.41. ([264, Theorem 6.1]) *The eigenvalues of the derangement graph* $\mathrm{Cay}(S_n, \mathscr{D}_n)$ *are given by*

$$\eta_\lambda = \sum_{k=0}^{n} (-1)^{n-k} \omega_k(\mu_1, \dots, \mu_r),$$

where $\lambda = (\lambda_1, \dots, \lambda_r)$ *runs over all partitions of* n *and* $\mu_i = \lambda_i + r - i$ *for each* i.

Using these formulas, Renteln settled affirmatively a conjecture of Ku and Wong [189]. Using the work in [264], Deng and Zhang [101] worked out the second largest eigenvalue of $\mathrm{Cay}(S_n, \mathscr{D}_n)$. We put these two results together in the following theorem.

Theorem 3.42. *The following hold:*

(a) *the smallest eigenvalue of* $\mathrm{Cay}(S_n, \mathscr{D}_n)$ *is equal to* $-\frac{1}{n-1}|\mathscr{D}_n|$ *([264, Theorem 7.1]);*

(b) *the second largest eigenvalue of* $\mathrm{Cay}(S_n, \mathscr{D}_n)$ *is equal to* $\frac{n-1}{n-3}|\mathscr{D}_{n-2}|$ *([101, Theorem 1.1]).*

Further spectral properties of $\mathrm{Cay}(S_n, \mathscr{D}_n)$ were obtained in [187, 188]. It is well known that the eigenvalues of this graph can be indexed by parititions of n. The question about how these eigenvalues are determined by the shape of their corresponding partitions was investigated in [187]. Lower and upper bounds on the absolute values of these eigenvalues were also obtained in [187]. In [188], a new recurrence formula for the eigenvalues of $\mathrm{Cay}(S_n, \mathscr{D}_n)$ was given and a

conjecture of Ku and Wales [187, Conjecture 1.1] was confirmed. In [101], a lower bound on the connectivity of $\mathrm{Cay}(S_n, \mathscr{D}_n)$ and lower and upper bounds on the Cheeger constant (see section 7) of $\mathrm{Cay}(S_n, \mathscr{D}_n)$ were obtained.

Given an integer k with $0 \le k \le n - 1$, let $S(n, k)$ be the set of elements σ of S_n such that σ fixes exactly k points in $[n]$. The *k-point-fixing graph* is defined [185] to be the Cayley graph $\mathrm{Cay}(S_n, S(n, k))$, that is, two vertices σ, τ are adjacent if and only if $\sigma\tau^{-1}$ fixes exactly k points. Clearly, the 0-point-fixing graph is the derangement graph. So k-point-fixing graphs can be regarded as a generalization of derangement graphs. Since $S(n, k)$ is closed under conjugation, Theorem 3.39 implies that all k-point-fixing graphs are integral. In [185], a recursive formula for the eigenvalues of $\mathrm{Cay}(S_n, S(n, k))$ was obtained, and this was used to determine the signs of the eigenvalues of $\mathrm{Cay}(S_n, S(n, 1))$. See [185] for more properties of the k-point-fixing graphs.

Let n and k be integers with $1 \le k \le n$. Let $\mathscr{D}_n^{(k)}$ be the set of permutations in S_n without any i-cycle for every i from 1 to k. In [186], the Cayley graph $\mathrm{Cay}(S_n, \mathscr{D}_n^{(k)})$ was studied. Obviously, this is a spanning subgraph of $\mathrm{Cay}(S_n, \mathscr{D}_n)$, and in particular $\mathrm{Cay}(S_n, \mathscr{D}_n^{(1)}) = \mathrm{Cay}(S_n, \mathscr{D}_n)$. The following result is a generalization of part (a) in Theorem 3.42.

Theorem 3.43. ([186, Theorem 1.3]) *Let $n \ge 2$ and k be integers such that $1 \le k \le n^\delta$ with $0 < \delta < \frac{2}{3}$. Then for sufficiently large n, the smallest eigenvalue of $\mathrm{Cay}(S_n, \mathscr{D}_n^{(k)})$ is equal to $-\frac{1}{n-1}|\mathscr{D}_n^{(k)}|$.*

In [186], it was also proved that under the same condition as in Theorem 3.43 the set of maximum independent sets of $\mathrm{Cay}(S_n, \mathscr{D}_n^{(k)})$ coincides the set of maximum independent sets of the derangement graph $\mathrm{Cay}(S_n, \mathscr{D}_n)$, and all such maximum independent sets have size $(n-1)!$.

The *transposition network* T_n is the Cayley graph on S_n with connection set consisting of all transpositions in S_n. With motivation from computing the bisection width of T_n, the following result was proved in [166] using Theorem 2.5.

Theorem 3.44. ([166, Lemma 3]) *Let $n \ge 2$ be an integer. Then T_n is an integral Cayley graph, and the algebraic and geometric multiplicity of each eigenvalue of T_n are equal. The largest eigenvalue of T_n is $n(n-1)/2$ with multiplicity 1; the second largest eigenvalue of T_n is $n(n-3)/2$ with multiplicity $(n-1)^2$; and for $1 \le k \le n$, $n(n-2k+1)/2$ is an eigenvalue of T_n with multiplicity at least $n!/(n(n-k)!(k-1)!)$.*

3.4 Integral Cayley graphs of small degrees

The following result due to Abdollahi and Vatandoost gives a characterization of connected cubic integral Cayley graphs.

Theorem 3.45. ([7, Theorem 1.1]) *There are exactly seven connected cubic integral Cayley graphs. In particular, a connected cubic Cayley graph $\mathrm{Cay}(\Gamma, S)$ is integral if and only if Γ is isomorphic to one the following groups:*

$$\mathbb{Z}_2^2, \mathbb{Z}_4, \mathbb{Z}_6, S_3, \mathbb{Z}_2^3, \mathbb{Z}_2 \times \mathbb{Z}_4, D_8, \mathbb{Z}_2 \times \mathbb{Z}_6, D_{12}, A_4, S_4, D_8 \times \mathbb{Z}_3, D_6 \times \mathbb{Z}_4, A_4 \times \mathbb{Z}_2.$$

The following theorem also due to Abdollahi and Vatandoost determines the orders of connected 4-regular integral Cayley graphs on finite abelian groups.

Theorem 3.46. ([8, Theorem 1.1]) *The order of any connected 4-regular integral Cayley graph on a finite abelian group must be one of the following:*

$$5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 25, 32, 36, 40, 48, 50, 60, 64, 72, 80, 96, 100, 120, 144.$$

As a side result it was also proved in [8, Theorem 1.2] that there are precisely 27 connected integral Cayley graphs of order up to 11.

In 2015, Minchenko and Wanless [243] proved that up to isomorphism there are precisely 32 connected 4-regular integral Cayley graphs, 17 of which are bipartite. They also proved that there are exactly 27 arc-transitive 4-regular integral graphs of degree 4, 16 of which are bipartite. See [243, Tables 3, 5 and 7] for the groups and connection sets for these graphs. From their results they noticed that integral Cayley graphs can be co-spectral to integral non-Cayley graphs, and integral arc-transitive graphs can be co-spectral to integral non-arc-transitive graphs.

Recently, Ghasemi [125] determined all possible orders of 5-regular integral Cayley graphs on abelian groups other than cyclic groups.

Theorem 3.47. ([125, Theorem 3.3]) *Let Γ be a finite abelian group which is not cyclic, and let $S \subseteq \Gamma \setminus \{1\}$ be such that $|S| = 5$, $S = S^{-1}$ and $\Gamma = \langle S \rangle$. If $\mathrm{Cay}(\Gamma, S)$ is integral, then the order of Γ must be one of the following:*

$$8, 16, 18, 24, 32, 36, 40, 48, 50, 64, 72, 80, 96, 100, 120, 128, 144, 160, 192, 200, 240, 288.$$

3.5 Automorphism groups of integral Cayley graphs

3.5.1 Cayley integral groups

As seen in Corollary 3.18, every Cayley graph on $\mathbb{Z}_2^n$ is integral. In general, a group Γ is called *Cayley integral* [175] if every Cayley graph on Γ is integral, or equivalently any graph admitting Γ as a regular group of automorphisms is integral. In [175, Theorem 13], Klotz and Sander gave a classification of all finite abelian Cayley integral groups, which can be regarded as a generalization of Corollary 3.18.

Theorem 3.48. ([175, Theorem 13]) *All nontrivial abelian Cayley integral groups are represented by*

$$\mathbb{Z}_2^n, \ \mathbb{Z}_3^n, \ \mathbb{Z}_4^n, \ \mathbb{Z}_2^m \oplus \mathbb{Z}_3^n, \ \mathbb{Z}_2^m \oplus \mathbb{Z}_4^n, \quad m \geq 1, \ n \geq 1.$$

In [175, Section 5], Klotz and Sander found three non-abelian Cayley integral groups, namely S_3, Q_8 and $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$, and in [175, Problem 3] they posed the problem of determining all non-abelian Cayley integral groups. This problem was completely solved by Abdollahi and Jazaeri [5] and independently by Ahmady et al. [14].

Theorem 3.49. ([5, Theorem 1.1] and [14, Theorem 4.2]) *A finite non-abelian group is Cayley integral if and only if it is isomorphic to one of the following groups:*

(a) *the symmetric group S_3 of degree 3;*

(b) *the nontrivial semidirect product $\mathbb{Z}_3 \rtimes \mathbb{Z}_4 = \langle x, y \mid x^3 = y^4 = 1, y^{-1}xy = x^{-1} \rangle$;*

(c) *$Q_8 \times \mathbb{Z}_2^n$ for some integer $n \geq 0$, where Q_8 is the quaternion group.*

Theorems 3.48 and 3.49 together give a complete classification of Cayley integral groups.

Let Γ be an abelian group of order at least 3 having a unique involution t. The group $\mathrm{Dic}(\Gamma) = \langle \Gamma, x \rangle$, where $x^2 = t$ and $x^{-1}ax = a^{-1}$ for every $a \in \Gamma$, is known as a *generalized dicyclic group*. In the case when $\Gamma \cong \mathbb{Z}_n$ it is the *dicyclic group* Dic_{2n} of order $2n$, and when $\Gamma \cong \mathbb{Z}_{2^n}$ it is known as the *generalized quaternion group* $Q_{2^{n+1}}$ of order 2^{n+1}.

Let $k \geq 1$ be an integer. In [113], Estélyi and Kovács studied the family $\mathcal{G}_k$ of finite groups Γ such that every Cayley graph on Γ with degree at most k is integral. Clearly, $\mathcal{G}_1$ is the family of all finite groups, and $\mathcal{G}_2$ consists of those groups whose non-identity elements are of order $2, 3, 4$ or 6, and contain no subgroup isomorphic to D_{2n} for any $n \geq 4$.

Theorem 3.50. ([113, Theorem 3]) *If $k \geq 6$, then $\mathcal{G}_k$ consists of all Cayley integral groups. Moreover, $\mathcal{G}_4$ and $\mathcal{G}_5$ are equal and consist of the following groups:*

(a) *the Cayley integral groups;*

(b) *the generalized dicyclic groups $\mathrm{Dic}(\mathbb{Z}_3^n \times \mathbb{Z}_6)$, where $n \geq 1$.*

In [113], Estélyi and Kovács also gave a characterization of non-abelian 2-groups in $\mathcal{G}_3$. As usual we use $[a, b]$ to denote the commutator of two elements a, b of a group.

Theorem 3.51. ([113, Proposition 12]) *Let Γ be a non-abelian 2-group of exponent 4. Then $\Gamma \in \mathcal{G}_3$ if and only if every minimal non-abelian subgroup of Γ is isomorphic to Q_8, the metacyclic group $\langle a, b \mid a^4 = b^4 = 1,\ b^{-1}ab = a^{-1} \rangle$, or the non-metacyclic group $\langle a, b, c \mid a^4 = b^4 = c^2 = 1,\ [a, b] = c,\ [c, a] = [c, b] = 1 \rangle$.*

A complete characterization of $\mathcal{G}_3$ was recently obtained by Ma and Wang in [228]. For an integer $k \geq 2$, let $\mathcal{A}_k$ be the family of finite groups Γ such that every Cayley graph on Γ with degree exactly k is integral.

Theorem 3.52. ([228, Theorem 2.6 and Corollary 2.7]) *The following hold:*

(a) *A finite group Γ belongs to $\mathcal{A}_3$ if and only if $\Gamma \cong S_3$, or for any involution x and element y of Γ, $\langle x, y \rangle$ is isomorphic to one of the following groups:*

$$\mathbb{Z}_2,\ \mathbb{Z}_2^2,\ \mathbb{Z}_4,\ \mathbb{Z}_6,\ \mathbb{Z}_2 \times \mathbb{Z}_4,\ \mathbb{Z}_2 \times \mathbb{Z}_6,\ A_4;$$

(b) *$\mathcal{G}_3$ consists of all finite 3-groups of exponent 3 and all groups in $\mathcal{A}_3$.*

A second proof of Theorem 3.50 was also given in [228].

3.5.2 Cayley integral simple groups

A finite group Γ is called *Cayley simple* if the only connected integral Cayley graph on Γ is the complete graph of order $|\Gamma|$. This definition was first introduced in [8], where the following question was posed ([8, Question 2.21]): Is any finite simple group, Cayley simple? A negative answer to this question is implied in the following result.

Theorem 3.53. ([4, Proposition 2.6]) *Let Γ be a finite group and S an inverse-closed subset of $\Gamma \setminus \{1\}$. Then $\Gamma \setminus S$ is a subgroup of Γ if and only if $\mathrm{Cay}(\Gamma, S)$ is a complete multipartite graph. Moreover, such a complete multipartite Cayley graph is integral, with $|\Gamma \setminus S|$ vertices in each part of the corresponding multipartition.*

Note that complete graphs form a special class of complete multipartite graphs. So in [4] Abdollahi and Jazaeri modified the definition of Cayley simple groups and introduced the notion of *Cayley integral simple groups* (or *CIS-group* for short) by replacing "complete graph" by "complete multipartite graph". They proved the next two results.

Theorem 3.54. ([4, Theorem 1.3]) *Let Γ be a finite non-simple group. Then Γ is a CIS-group if and only if $\Gamma \cong \mathbb{Z}_{p^2}$ for some prime p or $\Gamma \cong \mathbb{Z}_2^2$.*

Theorem 3.55. ([4, Theorem 4.10]) *Let Γ be a finite group. Let H and K be proper subgroups of Γ such that $HK = \Gamma$ and $H \cap K = \{1\}$. If there exists an integral Cayley graph $\mathrm{Cay}(H, S)$ such that $S \cup \{1\}$ is not a subgroup of H, then Γ is not a CIS-group.*

It is known that any cyclic group of prime order is a CIS-group (see [8, Corollary 2.19]). Since these are the only simple abelian groups, it is natural to ask the following question ([4, Question 1.4]): Which finite non-abelian simple groups are CIS-groups? Regarding this question, Abdollahi and Jazaeri proved the following result using Theorem 3.55 and the fact that $A_p = A_{p-1}\langle (1, 2, \ldots, p) \rangle$.

Theorem 3.56. ([4, Corollary 4.12]) *For every prime $p \geq 5$, the alternating group A_p is not a CIS-group.*

Thus the alternating groups A_p for primes $p \geq 5$ form an infinite family of finite non-abelian simple groups which are not CIS-groups. It turns out that this is not a coincidence since the following result shows that there does not exist any non-abelian finite CIS-group, answering the question mentioned above.

Theorem 3.57. ([14, Theorem 3.3]) *Let Γ be a CIS group. Then Γ is abelian and in particular is isomorphic to $\mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$ for some prime p or is isomorphic to $\mathbb{Z}_2^2$.*

In particular, this implies that every finite non-abelian group admits a nontrivial integral Cayley graph. This answers the following question ([8, Question 2.21]): For which nontrivial groups, there exists a nontrivial connected integral Cayley graph?

3.5.3   Automorphism groups of integral circulant graphs

Let r be a positive integer. A poset $([r], \preceq)$ on $[r] = \{1, 2, \ldots, r\}$ is called *increasing* if $i \preceq j$ implies $i \leq j$. Given $([r], \preceq)$ and positive integers $n_1, \ldots, n_r$, the generalized wreath product $\prod_{([r], \preceq)} S_{n_i}$ introduced in [173] is a certain permutation group acting on $[n_1] \times \cdots \times [n_r]$. The following result was proved in [173] using the techniques of Schur rings.

Theorem 3.58. ([173, Theorem 1.1]) *Let Γ be a permutation group acting on the cyclic group $\mathbb{Z}_n$, where $n \geq 2$. The following conditions are equivalent:*

(a) $\Gamma = \mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_n, S))$ *for some integral circulant graph* $\mathrm{Cay}(\mathbb{Z}_n, S)$;

(b) Γ *is a permutation group which is permuationally isomorphic to a generalized wreath product $\prod_{([r], \preceq)} S_{n_i}$, where $([r], \preceq)$ is an increasing poset and $n_1, \cdots, n_r$ are integers no less than 2 such that $n = n_1 \ldots n_r$ and $\gcd(n_i, n_j) = 1$ whenever $i \not\preceq j$.*

In [50], Bašić and Ilić determined the automorphism groups of certain classes of integral circulant graphs: the unitary Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^\times)$ and more generally the gcd graphs $\mathrm{ICG}(n, \{d\})$ for proper divisors d of n. They also determined the automorphism groups of those gcd graphs $\mathrm{ICG}(n, D)$ of $\mathbb{Z}_n$ for which $D = \{1, p^k\}$ and n is either square-free or a prime power.

Theorem 3.59. ([50, Theorem 3.4]) *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k} \geq 2$ be an integer in canonical factorization, and let $m = p_1 p_2 \ldots p_k$. Then*

$$\mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^\times)) \cong (S_{p_1} \times \cdots \times S_{p_k}) \,\mathrm{wr}\, S_{n/m}.$$

In particular,

$$|\mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^\times))| = p_1! p_2! \ldots p_k! \left(\left(\frac{n}{m} \right)! \right)^m.$$

3.6 Cayley digraphs integral over the Gauss field or other number fields

A circulant digraph is called *Gaussian integral* if all its eigenvalues are algebraic integers of the Gauss field $\mathbb{Q}(i)$ (that is, all eigenvalues are Gaussian integers $a + bi$, where $a, b \in \mathbb{Z}$). In [301], circulant digraphs which are Gaussian integral were studied, and those with order $n = k$, $2k$ or $4k$ for odd integers k were characterized. Among others the following results were proved in [301], where $S_n(D)$ is as defined in (3.4) and $D(n)$ is the set of positive divisors of n as defined in (3.2).

Theorem 3.60. ([301, Theorems 2.1 and 2.2]) *Let $n \geq 3$ be an integer not divisible by 4. Then a circulant digraph $\mathrm{Cay}(\mathbb{Z}_n, S)$ is Gaussian integral if and only if $S = S_n(D)$ for some $D \subseteq D(n) \setminus \{n\}$.*

This together with Corollary 3.3 implies that a circulant graph of order not divisible by 4 is Gaussian integral if and only if it is integral.

In general, given an algebraic number field K (that is, a finite extension of $\mathbb{Q}$), a circulant digraph is called *integral over K* if all its eigenvalues are algebraic integers of K. Denote $F = K \cap \mathbb{Q}(\omega_n)$. Using Galois theory, Li [202] obtained a characterization of circulant digraphs which are integral over K. He considered the orbits of a certain action of the Galois group $\mathrm{Gal}(\mathbb{Q}(\omega_n)/F)$ on $S_n(d)$. Setting $m = n/d$, there are exactly $r_d = [F \cap \mathbb{Q}(\omega_m) : \mathbb{Q}]$ such orbits $M_i(d)$, for $1 \leq i \leq r_d$, and each of them has size $[\mathbb{Q}(\omega_m) : F \cap \mathbb{Q}(\omega_m)]$. Since $\{1, 2, \ldots, n-1\} = \cup_{d \in D(n)} S_n(d)$, the set $\{1, 2, \ldots, n-1\}$ is partitioned into $r(n, K) = \sum_{d \in D(n)} r_d$ such orbits $M_i(d)$, with d running over $D(n)$ and $1 \leq i \leq r_d$.

Theorem 3.61. ([202, Theorem 1 and Corollary 1]) *Let K be an algebraic number field. A circulant digraph $\mathrm{Cay}(\mathbb{Z}_n, S)$, $n \geq 3$ is integral over K if and only if S is the union of some (not necessarily all) of the orbits $M_i(d), d \in D(n), 1 \leq i \leq r_d$, possibly from different proper divisors d of n. Therefore, there are at most $2^{r(n,K)}$ pairwise non-isomorphic circulant digraphs of order n which are integral over K.*

In the special case when n is a multiple of 8 and $K = \mathbb{Q}(i)$ is the Gauss field, Theorem 3.61 yields the following corollary which proves a conjecture in [301, Conjecture 3.3].

Corollary 3.62. ([202, Corollary 2]) *A circulant digraph $\mathrm{Cay}(\mathbb{Z}_n, S)$ with order n a multiple of 8 is Gaussian integral if and only if S is the union of some (not necessarily all) of the orbits $M_i(d), d \in D(n), 1 \leq i \leq r_d$, possibly from different proper divisors d of n.*

Consider an abelian group $\Gamma = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_d}$ of order $n = n_1 \ldots n_d$, where $n_1, \cdots, n_d$ are positive integers. Let K be an algebraic number field. Then $\mathrm{Gal}(\mathbb{Q}(\omega_n)/K) \subseteq \mathrm{Gal}(\mathbb{Q}(\omega_n)/\mathbb{Q}) \cong \mathbb{Z}_n^*$. Thus, for each $\sigma \in \mathrm{Gal}(\mathbb{Q}(\omega_n)/K)$, there exists an element $a \in \mathbb{Z}_n^*$ such that $\sigma(\omega_n) = \omega_n^a$. This element a gives rise to an action of H on each $\mathbb{Z}_i^*$ defined by $\pi_i(x) = ax \pmod{n_i}$ for $x \in \mathbb{Z}_i^*$. Hence $\mathrm{Gal}(\mathbb{Q}(\omega_n)/K)$ acts on Γ by $\pi(x_1, \ldots, x_d) = (\pi_1(x), \ldots, \pi_d(x)) = (ax \bmod n_1, \ldots, ax \bmod n_d)$ for $(x_1, \ldots, x_d) \in \Gamma$. The following result was proved by Li in [201].

Theorem 3.63. ([201, Theorem 1]) *Let $\Gamma = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_d}$ be an abelian group and K an algebraic number field, where each $n_i \geq 2$. A Cayley digraph $\mathrm{Cay}(\Gamma, S)$ on Γ is integral over K if and only if S is the union of some orbits under the above-mentioned action of $\mathrm{Gal}(\mathbb{Q}(\omega_n)/K)$ on Γ.*

In [201, Proposition 3], Li also gave an upper bound for the number of Cayley digraphs on Γ which are integral over K.

In the case when $K = \mathbb{Q}$, the theorem above gives a necessary and sufficient condition for a Cayley digraph on an abelian group to be integral in the usual sense. The reader is invited to compare Theorem 3.63 with Theorem 3.7.

4 Cospectral Cayley graphs

4.1 Cospectral Cayley graphs on non-abelian groups

Using Theorem 2.2, Babai proved the following result.

Theorem 4.1. ([36, Theorem 5.2]) *Given an integer $k \geq 2$ and a prime $p > 64k$, there exist k pairwise non-isomorphic cospectral Cayley graphs on the dihedral group D_{2p}.*

Recently, Abdollahi et al. [3, Theorem 1.3] proved that for any prime $p \geq 13$, there exist two non-isomorphic cospectral 6-regular Cayley graphs on D_{2p}. Generalizing the construction in [3], Abdollahi et al. [2] proved the next two results.

Theorem 4.2. ([2, Theorem 1.1]) *Let $p \geq 23$ be a prime. Then for every integer k between 6 and $2p - 7$ there exist at least two non-isomorphic cospectral k-regular Cayley graphs on D_{2p}.*

Theorem 4.3. ([2, Theorem 1.2]) *Let $p \geq 23$ be a prime. Then for every integer k between 6 and $p + 6$ there exist at least $\binom{\frac{p-1}{2}}{\lfloor k/2 \rfloor - 3}$ pairwise non-isomorphic cospectral k-regular ($(2p - k - 1)$-regular) Cayley graphs on D_{2p}.*

Thus the number of pairwise non-isomorphic cospectral Cayley graphs on D_{2p} is exponential in terms of p.

In [223], Lubotzky et al. proved the following result. (Note that in [223] Cayley graphs are not required to be undirected.) Denote by $\begin{bmatrix} d \\ i \end{bmatrix}_q$ the Gaussian coefficient which gives the number of subspaces of dimension i over $\mathbb{F}_q$ in the vector space $\mathbb{F}_q^d$.

Theorem 4.4. ([223, Theorem 1]) *For every integer $d \geq 5$ ($d \neq 6$), every prime power q, and every integer $e \geq 1$ such that $q^e > 4d^2 + 1$, there are two systems A, B of generators of the group $G = \mathrm{PSL}_d(\mathbb{F}_{q^e})$ such that $\mathrm{Cay}(G, A)$ and $\mathrm{Cay}(G, B)$ are cospectral but not isomorphic. Moreover, the number of generators in A and B can be chosen to be either $k = 2(q^d - 1)/(q - 1)$ or $k = \sum_{i=1}^{d-1} \begin{bmatrix} d \\ i \end{bmatrix}_q$.*

In particular, this implies that for fixed d and q there are infinitely many cospectral pairs of Cayley graphs which are k-regular with the same k.

Theorem 4.5. ([223, Proposition 7]) *Let Γ be a finite group and Γ' a proper subgroup of Γ. Suppose that Γ' has two inverse-closed generating sets A' and B' of size k such that $\mathrm{Cay}(\Gamma', A')$ and $\mathrm{Cay}(\Gamma', B')$ are cospectral but non-isomorphic. Then Γ has two generating sets A and B of size $|\Gamma| - k - 1$ such that $\mathrm{Cay}(\Gamma, A)$ and $\mathrm{Cay}(\Gamma, B)$ are cospectral but non-isomorphic.*

Combining this with Theorem 4.1, the following result was noted in [223, Corollary 8]: For large enough n, each of the groups $G = S_n$ and $G = \mathrm{PSL}_n(\mathbb{F}_q)$ has two subsets A and B of size $|G| - 3$ such that $\mathrm{Cay}(G, A)$ and $\mathrm{Cay}(G, B)$ are cospectral but non-isomorphic.

A few explicit constructions of cospectral but non-isomorphic Cayley graphs were also given in [223].

4.2 Cospectrality and isomorphism

Consider a Cayley graph $\mathrm{Cay}(\Gamma, S)$ on a group Γ with connection set S. It can be verified that every $\sigma \in \mathrm{Aut}(\Gamma)$ induces an isomorphism from $\mathrm{Cay}(\Gamma, S)$ to $\mathrm{Cay}(\Gamma, \sigma(S))$, called a *Cayley isomorphism* (CI). A Cayley graph $\mathrm{Cay}(\Gamma, S)$ is called a *CI-graph* of Γ if for any Cayley graph $\mathrm{Cay}(\Gamma, T)$ isomorphic to $\mathrm{Cay}(\Gamma, S)$ there exists $\sigma \in \mathrm{Aut}(\Gamma)$ such that $T = \sigma(S)$. Ádám [9, 106] conjectured that all circulant graphs are CI-graphs of the corresponding cyclic groups. Though disproved in [112], this conjecture inspired a lot of interest on CI-graphs in more than four decades. See [200] for a survey of CI-graphs.

Motivated by Ádám's conjecture, Mans et al. [231] proposed to study the following problem: "Which circulant graphs $\mathrm{Cay}(\mathbb{Z}_n, S)$ satisfy the *spectral Ádám property*, that is, for any $T \subseteq \mathbb{Z}_n$ satisfying $0 \notin T = -T$, $\mathrm{Spec}(\mathrm{Cay}(\mathbb{Z}_n, S)) = \mathrm{Spec}(\mathrm{Cay}(\mathbb{Z}_n, T))$ implies that there exists an integer α coprime with n such that $T = \alpha S = \{\alpha s : s \in S\}$?" Obviously, if $T = \alpha S$ for an integer α coprime to n, then $\mathrm{Cay}(\mathbb{Z}_n, S) \cong \mathrm{Cay}(\mathbb{Z}_n, T)$. So a circulant graph $\mathrm{Cay}(\mathbb{Z}_n, S)$ satisfies the spectral Ádám property if and only if, for any circulant graph $\mathrm{Cay}(\mathbb{Z}_n, T)$, $\mathrm{Spec}(\mathrm{Cay}(\mathbb{Z}_n, S)) = \mathrm{Spec}(\mathrm{Cay}(\mathbb{Z}_n, T))$ implies $\mathrm{Cay}(\mathbb{Z}_n, S) \cong \mathrm{Cay}(\mathbb{Z}_n, T)$.

A graph G is said to be *determined by its spectrum* (DS for short) if every graph cospectral with G is isomorphic to G. In particular, a Cayley graph $\mathrm{Cay}(\Gamma, S)$ is called *Cay-DS* if, for any Cayley graph $\mathrm{Cay}(\Gamma, T)$, $\mathrm{Spec}(\mathrm{Cay}(\Gamma, S)) = \mathrm{Spec}(\mathrm{Cay}(\Gamma, T))$ implies $\mathrm{Cay}(\Gamma, S) \cong \mathrm{Cay}(\Gamma, T)$. If all Cayley graphs on Γ are Cay-DS, then the group Γ is said to be *Cay-DS*. These concepts can be extended to Cayley digraphs in an obvious way.

It is natural to ask whether all circulant (di)graphs are Cay-DS. The answer is negative, as shown in [112, 231]. On the positive side, Turner proved in [294, Theorem 2] that any cyclic group of prime order is Cay-DS. In [231], Mans et al. proved the following result for circulants of degree 4.

Theorem 4.6. ([231, Theorem 9]) *Let $n \geq 3$ and let $S = \{\pm a, \pm b\} \subseteq \mathbb{Z}_n$ be such that $\gcd(a, b, n) = 1$. Then $\mathrm{Cay}(\mathbb{Z}_n, S)$ is Cay-DS.*

This generalizes an earlier result [210] which asserts that $\mathrm{Cay}(\mathbb{Z}_n, \{\pm 1, \pm d\})$ is Cay-DS provided that $2 \leq d < \min\{n/4, \varphi(n)/2\}$, where $\varphi(n)$ is Euler's function. In [231], it was also shown that for any fixed m the probability that a random m-element subset $S \subseteq \mathbb{Z}_n$ does not have the spectral Ádám property is $O(n^{-1})$.

It was proved in [112, Corollary 3] that the circulant digraph $\mathrm{Cay}(\mathbb{Z}_p, S)$ is Cay-DS for any prime p. In [151], Huang and Chang proved the following result.

Theorem 4.7. ([151, Theorems 1, 2 and 3]) *The circulant digraph* $\mathrm{Cay}(\mathbb{Z}_n, S)$, $n \geq 3$, *is Cay-DS if one of the following conditions holds:*

(a) $n = p^a$ *is a prime power, and* $S \subset \mathbb{Z}_n$ *does not contain any coset of* $\langle p^{a-1} \rangle$ *in* $\mathbb{Z}_n$, *where* $\langle p^{a-1} \rangle$ *is the subgroup of* $\mathbb{Z}_n$ *generated by* p^{a-1};

(b) $n = p^a q^b$, *where* p *and* q *are distinct odd primes and* a *and* b *are positive integers, and* $S \subset \mathbb{Z}_n$ *satisfies* $\max\{s : s \in S\} \leq \varphi(n)$;

(c) $n = 2^a q^b$, *where* q *is an odd prime and* a *and* b *are positive integers, and* S *is a generating subset of* $\mathbb{Z}_n$ *that satisfies* $\max\{s : s \in S\} \leq \varphi(n)$.

In [283, Section 7], So proposed the following conjecture.

Conjecture 4.8. ([283, Conjecture 7.3]) *Two integral circulant graphs are isomorphic if and only if they are cospectral; that is, all integral circulant graphs are Cay-DS.*

So [283, Section 7] also verified that this conjecture is true if n is a prime power or a product of two distinct primes. This conjecture has also been confirmed [158, Section 5] when n is square-free and the set D in the integral circulant graph $\mathrm{ICG}(n, D)$ contains at most two prime divisors of n, but it is still open in its general form.

Given non-empty sets $A_1, \ldots, A_s$ of integers, define

$$\prod_{i=1}^{s} A_i := \{a_1 \cdots a_s : a_i \in A_i, 1 \leq i \leq s\}. \tag{4.1}$$

Denote by $\mathbb{P}$ the set of primes. Let n be a positive integer and $p \in \mathbb{P}$. Recall that $e_p(n)$ denotes the exponent of p in n. For $\emptyset \neq X \subseteq \mathbb{N}$ and $p \in \mathbb{P}$, define

$$X_p = \left\{ p^{e_p(x)} : x \in X \right\}. \tag{4.2}$$

If $X = \prod_{p \in \mathbb{P}} X_p$, then X is called a *multiplicative set*.

Define the *spectral vector* of an integral circulant graph $\mathrm{ICG}(n, D)$ to be

$$\overrightarrow{\lambda}(n, D) := (\lambda_1(n, D), \ldots, \lambda_{n-1}(n, D), \lambda_0(n, D)),$$

where $\lambda_i(n, D)$ for $i = 0, 1, \ldots, n-1$ are the eigenvalues of $\mathrm{ICG}(n, D)$ as shown in (3.6). In [274], Sander and Sander proved the following weaker form of Conjecture 4.8.

Theorem 4.9. ([274, Theorem 1.2]) *Let* $n \geq 3$ *be an integer. Let* D *and* E *be multiplicative divisor sets of* n. *The integral circulant graphs* $\mathrm{ICG}(n, D)$ *and* $\mathrm{ICG}(n, E)$ *are isomorphic if and only if* $\overrightarrow{\lambda}(n, D) = \overrightarrow{\lambda}(n, E)$.

In addition, an explicit formula for the eigenvalues of any integral circulant graph with a multiplicative divisor set was given in [274, Theorem 1.1].

We say that a group Γ is *DS* if for any Cayley graph $G = \mathrm{Cay}(\Gamma, S)$ on Γ and any graph G' cospectral with G we have $G \cong G'$. Obviously, a DS group is Cay-DS, but the converse is not true. The following two theorems were proved by Abdollahi et al. in [3].

34

Theorem 4.10. ([3, Theorem 1.2]) *The following hold:*

(a) *for any prime $p > 5$, every Sylow p-subgroup of any finite Cay-DS group is cyclic; and every Sylow 5-subgroup of any finite DS group is cyclic;*

(b) *every Sylow 2-subgroup of any finite Cay-DS group is of order at most 16;*

(c) *every Sylow 3-subgroup of any finite Cay-DS group is either cyclic or is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

Theorem 4.11. ([3, Theorem 1.3]) *Let p be a prime. The dihedral group D_{2p} is Cay-DS if and only if $p \in \{2, 3, 5, 7, 11\}$. Moreover, there exist two cospectral non-isomorphic 6-regular Cayley graphs on D_{2p} for every prime $p \geq 13$.*

Theorem 4.12. ([3, Theorem 1.4]) *Every finite DS group is solvable.*

In [154], Huang et al. classified all cubic Cayley graphs on D_{2p} and enumerated them up to isomorphism by means of the spectral method, where p is an odd prime. They proved that two such graphs are isomorphic if and only if they are cospectral. In other words, they proved:

Theorem 4.13. ([154, Corollary 3.8]) *Let p be an odd prime. All cubic Cayley graphs on D_{2p} are Cay-DS.*

Huang et al. [154] also posed the question of classifying and enumerating connected cubic Cayley graphs on general dihedral groups and determining which of them have the Cay-DS property.

5 Cayley graphs on finite commutative rings

In this section we focus on two families of Cayley graphs on additive groups of finite commutative rings. As we will see shortly, all graphs in the first family and some graphs in the second family are integral.

A *local ring* [34] is a commutative ring with a unique maximal ideal. Denote by $R^{\times}$ the set of units of R. It is readily seen [34, 110] that the set of units of a local ring R with maximal ideal M is given by $R^{\times} = R \setminus M$. It is well known [34, 110] that every finite commutative ring can be expressed as a direct product of finite local rings, and this decomposition is unique up to permutations of such local rings. We make the following assumption throughout this section.

Assumption 5.1. Whenever we consider a finite commutative ring R with unit element $1 \neq 0$, we assume that it is decomposed as

$$R = R_1 \times R_2 \times \cdots \times R_s$$

such that

$$\frac{|R_1|}{m_1} \leq \frac{|R_2|}{m_2} \leq \cdots \leq \frac{|R_s|}{m_s},$$

where each R_i, $1 \leq i \leq s$, is a local ring with maximal ideal M_i of order m_i.

5.1   Unitary Cayley graphs of finite commutative rings

Let R be a finite commutative ring. The *unitary Cayley graph* of R is defined as the Cayley graph $G_R = \mathrm{Cay}(R, R^\times)$ of the additive group of R with respect to $R^\times$. That is, G_R has vertex set R such that $x, y \in R$ are adjacent if and only if $x - y \in R^\times$. This notion is a generalization of the unitary Cayley graph of $\mathbb{Z}_n$ because

$$G_{\mathbb{Z}_n} \cong \mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^\times) \cong \mathrm{ICG}(n, \{1\}).$$

Define

$$\lambda_C = (-1)^{|C|} \frac{|R^\times|}{\prod_{j \in C} \frac{|R_j^\times|}{m_j}}$$

for every subset C of $\{1, 2, \ldots, s\}$.

Theorem 5.2. ([171, Lemma 2.3]; see also [16]) *Let R be a finite commutative ring. The eigenvalues of G_R are*

(a) λ_C, *repeated* $\prod_{j \in C} \frac{|R_j^\times|}{m_j}$ *times, for* $C \subseteq \{1, 2, \ldots, s\}$; *and*

(b) 0 *with multiplicity* $|R| - \prod_{i=1}^{s} \left(1 + \frac{|R_i^\times|}{m_i} \right)$.

In particular, if R is a finite local ring and m is the order of its unique maximal ideal, then

$$\mathrm{Spec}(G_R) = \left(|R| - m, (-m)^{\frac{|R|}{m} - 1}, 0^{\frac{|R|}{m}(m-1)} \right).$$

In [214], Liu and Zhou determined the eigenvalues of the complement $\overline{G}_R$ and the line graph $L(G_R)$ of G_R.

Theorem 5.3. ([214, Corollary 6]) *Let R be a finite commutative ring. The eigenvalues of $\overline{G}_R$ are*

(a) $|R| - 1 - |R^\times|$;

(b) $-\lambda_C - 1$, *repeated* $\prod_{j \in C} \frac{|R_j^\times|}{m_j}$ *times, for* $\emptyset \neq C \subseteq \{1, 2, \ldots, s\}$; *and*

(c) -1 *with multiplicity* $|R| - \prod_{i=1}^{s} \left(1 + \frac{|R_i^\times|}{m_i} \right)$.

In particular, if R is a finite local ring and m is the order of its unique maximal ideal, then

$$\mathrm{Spec}(\overline{G}_R) = \left((m-1)^{\frac{|R|}{m}}, (-1)^{\frac{|R|}{m}(m-1)} \right).$$

Theorem 5.4. ([214, Corollary 7]) *Let R be a finite commutative ring. The eigenvalues of $L(G_R)$ are*

(a) $\lambda_C + |R^\times| - 2$, *repeated* $\prod_{j \in C} \frac{|R_j^\times|}{m_j}$ *times, for* $C \subseteq \{1, 2, \ldots, s\}$;

(b) $|R^\times| - 2$ *with multiplicity* $|R| - \prod_{i=1}^{s} \left(1 + \frac{|R_i^\times|}{m_i} \right)$; *and*

(c) -2, *repeated* $|R|\left(|R^{\times}|-2\right)/2$ *times.*

In particular, if R is a finite local ring and m is the order of its unique maximal ideal, then

$$\mathrm{Spec}(L(G_R)) = \left(2|R^{\times}|-2, (|R^{\times}|-m-2)^{\frac{|R^{\times}|}{m}}, (|R^{\times}|-2)^{(\frac{|R^{\times}|}{m}+1)(m-1)}, (-2)^{\frac{|R|(|R^{\times}|-2)}{2}}\right).$$

Theorems 5.2–5.4 together imply the following result.

Corollary 5.5. *Let R be a finite commutative ring. Then G_R, $\overline{G}_R$ and $L(G_R)$ are integral. In particular, G_R is an integral Cayley graph.*

5.2 Quadratic unitary Cayley graphs of finite commutative rings

Let R be a finite commutative ring. Let $Q_R = \{u^2 : u \in R^{\times}\}$ and set $T_R = Q_R \cup (-Q_R)$. The *quadratic unitary Cayley graph* of R, denoted by $\mathcal{G}_R$, is defined as the Cayley graph $\mathrm{Cay}(R, T_R)$ on the additive group of R with respect to T_R. That is, $\mathcal{G}_R$ has vertex set R such that $x, y \in R$ are adjacent if and only if $x - y \in T_R$. This notion introduced by Liu and Zhou [215] is a generalization of the quadratic unitary Cayley graph $\mathcal{G}_{\mathbb{Z}_n}$ of $\mathbb{Z}_n$ introduced in [96] as well as the well known Paley graphs. (The *Paley graph $P(q)$*, where $q \equiv 1 \bmod 4$ is a prime power, is the Cayley graph on the additive group of the finite field $\mathbb{F}_q$ with respect to the set of nonzero squares.) In [215], Liu and Zhou determined the spectra of $\mathcal{G}_R$ in the case when all finite fields R_i/M_i are of odd order and at most one of them has order congruent to 3 modulo 4.

Theorem 5.6. ([215, Theorem 2.4]) *Let R be a local ring with maximal ideal M of order m.*

(a) *If $|R|/m \equiv 1 \ (\mathrm{mod}\ 4)$, then the spectrum of $\mathcal{G}_R$ is*

$$\left(\frac{|R|-m}{2}, \left(\frac{1}{2}m\left(-1+\sqrt{\frac{|R|}{m}}\right)\right)^{\frac{1}{2}\left(\frac{|R|}{m}-1\right)}, \left(\frac{1}{2}m\left(-1-\sqrt{\frac{|R|}{m}}\right)\right)^{\frac{1}{2}\left(\frac{|R|}{m}-1\right)}, 0^{|R|-\frac{|R|}{m}}\right).$$

(b) *If $|R|/m \equiv 3 \ (\mathrm{mod}\ 4)$, then the spectrum of $\mathcal{G}_R$ is*

$$\left(|R|-m, (-m)^{\frac{|R|}{m}-1}, 0^{|R|-\frac{|R|}{m}}\right).$$

In particular, $\mathcal{G}_R$ is an integral Cayley graph.

Define

$$\lambda_{A,B} = (-1)^{|B|} \frac{|R^{\times}|}{2^s \prod_{i\in A}\left(\sqrt{\frac{|R_i|}{m_i}}+1\right) \prod_{j\in B}\left(\sqrt{\frac{|R_j|}{m_j}}-1\right)}$$

for disjoint subsets A, B of $\{1, 2, \ldots, s\}$.

Theorem 5.7. ([215, Theorem 2.6]) *Let R be a finite commutative ring such that $|R_i|/m_i \equiv 1 \ (\mathrm{mod}\ 4)$ for $1 \le i \le s$. Then the eigenvalues of $\mathcal{G}_R$ are*

(a) $\lambda_{A,B}$, *repeated* $\dfrac{1}{2^{|A|+|B|}} \displaystyle\prod_{k\in A\cup B}\left(\frac{|R_k|}{m_k}-1\right)$ *times, for all pairs of disjoint subsets A, B of $\{1, 2, \ldots, s\}$; and*

(b) 0 with multiplicity $|R| - \displaystyle\sum_{\substack{A,B \subseteq \{1,\ldots,s\} \\ A \cap B = \emptyset}} \left(\dfrac{1}{2^{|A|+|B|}} \prod_{k \in A \cup B} \left(\dfrac{|R_k|}{m_k} - 1 \right) \right)$.

Theorem 5.8. ([215, Theorem 2.7]) *Let R be a finite commutative ring such that $|R_i|/m_i \equiv 1 \pmod 4$ for $1 \le i \le s$. Let R_0 be a local ring with maximal ideal M_0 of order m_0 such that $|R_0|/m_0 \equiv 3 \pmod 4$. Then the eigenvalues of $\mathcal{G}_{R_0 \times R}$ are*

(a) $|R_0^\times| \cdot \lambda_{A,B}$, *repeated* $\dfrac{1}{2^{|A|+|B|}} \displaystyle\prod_{k \in A \cup B} \left(\dfrac{|R_k|}{m_k} - 1 \right)$ *times, for all pairs of disjoint subsets A, B of $\{1, 2, \ldots, s\}$;*

(b) $-\dfrac{|R_0^\times|}{|R_0|/m_0 - 1} \cdot \lambda_{A,B}$, *repeated* $\dfrac{1}{2^{|A|+|B|}} \left(\dfrac{|R_0|}{m_0} - 1 \right) \displaystyle\prod_{k \in A \cup B} \left(\dfrac{|R_k|}{m_k} - 1 \right)$ *times, for all pairs of disjoint subsets A, B of $\{1, 2, \ldots, s\}$; and*

(c) 0 *with multiplicity* $|R| - \displaystyle\sum_{\substack{A,B \subseteq \{1,\ldots,s\} \\ A \cap B = \emptyset}} \left(\dfrac{1}{2^{|A|+|B|}} \dfrac{|R_0|}{m_0} \prod_{k \in A \cup B} \left(\dfrac{|R_k|}{m_k} - 1 \right) \right)$.

Theorems 5.6, 5.7 and 5.8 can be specified to obtain the eigenvalues of $\mathcal{G}_{\mathbb{Z}_n}$.

Theorem 5.9. ([215, Corollary 2.8]) *Let $p \ge 5$ be an odd prime and $\alpha \ge 1$ an integer.*

(a) *If $p \equiv 1 \pmod 4$, then*

$$\mathrm{Spec}(\mathcal{G}_{\mathbb{Z}_{p^\alpha}}) = \begin{pmatrix} p^{\alpha-1}(p-1)/2 & p^{\alpha-1}\left(-1+\sqrt{p}\right)/2 & 0 & p^{\alpha-1}\left(-1-\sqrt{p}\right)/2 \\ 1 & (p-1)/2 & p^\alpha - p & (p-1)/2 \end{pmatrix}.$$

(b) *If $p \equiv 3 \pmod 4$, then*

$$\mathrm{Spec}(\mathcal{G}_{\mathbb{Z}_{p^\alpha}}) = \begin{pmatrix} p^{\alpha-1}(p-1) & -p^{\alpha-1} & 0 \\ 1 & p-1 & p^\alpha - p \end{pmatrix}.$$

Theorem 5.10. ([215, Corollary 2.9]) *Let $n = p_1^{\alpha_1} \ldots p_s^{\alpha_s}$ be an integer in canonical factorization such that $p_i \equiv 1 \pmod 4$ for each i. Then the eigenvalues of $\mathcal{G}_{\mathbb{Z}_n}$ are*

(a) $(-1)^{|B|} \cdot \dfrac{\varphi(n)}{2^s \prod_{i \in A} \left(\sqrt{p_i}+1\right) \prod_{j \in B} \left(\sqrt{p_j}-1\right)}$, *repeated* $\dfrac{1}{2^{|A|+|B|}} \displaystyle\prod_{k \in A \cup B} (p_k - 1)$ *times, for all pairs of disjoint subsets A, B of $\{1, 2, \ldots, s\}$;*

(b) 0 *with multiplicity* $n - \displaystyle\sum_{\substack{A, B \subseteq \{1,\ldots,s\} \\ A \cap B = \emptyset}} \left(\dfrac{1}{2^{|A|+|B|}} \prod_{k \in A \cup B} (p_k - 1) \right)$.

Theorem 5.11. ([215, Corollary 2.10]) *Let $n = p^\alpha p_1^{\alpha_1} \ldots p_s^{\alpha_s}$ be an integer in canonical factorization such that $p \equiv 3 \pmod 4$ and $p_i \equiv 1 \pmod 4$ for each i. Then the eigenvalues of $\mathcal{G}_{\mathbb{Z}_n}$ are*

(a) $(-1)^{|B|} \cdot \dfrac{\varphi(n)}{2^s \prod_{i \in A} \left(\sqrt{p_i}+1\right) \prod_{j \in B} \left(\sqrt{p_j}-1\right)}$, *repeated* $\dfrac{1}{2^{|A|+|B|}} \displaystyle\prod_{k \in A \cup B} (p_k - 1)$ *times, for all pairs of disjoint subsets A, B of $\{1, 2, \ldots, s\}$;*

(b) $(-1)^{|B|+1} \cdot \dfrac{\varphi(n)}{2^s(p-1)\prod_{i\in A}\left(\sqrt{p_i}+1\right)\prod_{j\in B}\left(\sqrt{p_j}-1\right)}$, *repeated* $\dfrac{p-1}{2^{|A|+|B|}}\displaystyle\prod_{k\in A\cup B}(p_k-1)$ *times,*
for all pairs of disjoint subsets A, B *of* $\{1, 2, \ldots, s\}$; *and*

(c) 0 *with multiplicity* $n - \displaystyle\sum_{\substack{A,\,B\subseteq\{1,\ldots,s\}\\ A\cap B=\emptyset}}\left(\dfrac{p}{2^{|A|+|B|}}\prod_{k\in A\cup B}(p_k-1)\right).$

5.3 A family of Cayley graphs on finite chain rings

A finite *chain ring* is a finite local ring R such that for any ideals I, J of R we have either $I \subseteq J$ or $J \subseteq I$. Consider a finite chain ring R with unique maximal ideal M and residue field of q elements. The *nilpotency* s of R is the smallest positive integer such that $M^s = \{0\}$. Then $\{0\} = M^s \subset M^{s-1} \subset \cdots \subset M^2 \subset M \subset M^0 = R$ and $|M^i| = q^{s-i}$ for $0 \le i \le s$. For integers $a_1, a_2, \ldots, a_r$ with $0 \le a_1 < a_2 < \cdots < a_r \le s-1$, set

$$\mathcal{C} = \cup_{i=1}^r (M^{a_i} \setminus M^{a_i+1}). \tag{5.1}$$

The Cayley graph $\mathrm{Cay}(R, \mathcal{C})$ was studied in [285]. In the special case when $R = \mathbb{Z}_{p^s}$ and $\mathcal{C} = \{p^{a_i} : 1 \le i \le r\}$ is a set of proper divisors of p^s, this graph is exactly the gcd graph of the cyclic group $\mathbb{Z}_{p^s}$ with respect to $\mathcal{C}$.

Theorem 5.12. ([285, Section 2]) *Let R be a finite chain ring and let $\mathcal{C}$ be as above. Then* $\mathrm{Cay}(R, \mathcal{C})$ *is integral. Moreover, the following hold:*

(a) *if $a_r = s-1$, then the eigenvalues of $\mathrm{Cay}(R, \mathcal{C})$ are: $(q-1)\sum_{i=1}^r q^{s-a_i-1}$, with multiplicity q^{a_1}; $-q^{s-a_{k-1}-1} + (q-1)\sum_{i=k}^r q^{s-a_i-1}$, with multiplicity $q^{a_{k-1}}(q-1)$, for $2 \le k \le r$; $(q-1)\sum_{i=k}^r q^{s-a_i-1}$, with multiplicity $q^{a_k-a_{k-1}-1} - q^{a_{k-1}+1}$, for $2 \le k \le r$; -1, with multiplicity $q^{a_r}(q-1)$;*

(b) *if $a_r \ne s-1$, then the eigenvalues of $\mathrm{Cay}(R, \mathcal{C})$ are: $(q-1)\sum_{i=1}^r q^{s-a_i-1}$, with multiplicity q^{a_1}; $-q^{s-a_{k-1}-1} + (q-1)\sum_{i=k}^r q^{s-a_i-1}$, with multiplicity $q^{a_{k-1}}(q-1)$, for $2 \le k \le r$; $(q-1)\sum_{i=k}^r q^{s-a_i-1}$, with multiplicity $q^{a_k} - q^{a_{k-1}+1}$, for $2 \le k \le r$; $-q^{s-a_r-1}$, with multiplicity $q^{a_r}(q-1)$; 0, with multiplicity $q^{a_r+1}(q^{s-a_r-1} - 1)$.*

6 Energies of Cayley graphs

The *energy* of a graph G with n vertices and eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ is defined as

$$\mathcal{E}(G) = \sum_{i=1}^n |\lambda_i|.$$

This concept was introduced by Gutman [140] in the study of mathematical chemistry. In the past four decades a number of results on energies of various families of graphs have been obtained; see [206] for a monograph on this topic. In [182] it was proved that, for any graph G with n vertices,

$$\mathcal{E}(G) \le \frac{n}{2}\left(\sqrt{n}+1\right), \tag{6.1}$$

and the bound can be achieved by infinitely many graphs. It can be easily verified that $\mathcal{E}(K_n) = 2(n-1)$. A graph G with n vertices is called *hyperenergetic* [141] if $\mathcal{E}(G) > 2(n-1)$.

6.1 Energies of integral circulant graphs

In this section we assume that

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s} \geq 2 \tag{6.2}$$

is an integer in canonical factorization into prime powers, where $p_1 < p_2 < \ldots < p_s$ are primes and each $\alpha_i \geq 1$ is an integer. Recall that the circulant integral graphs are precisely the gcd graphs $\mathrm{ICG}(n, D)$ of cyclic groups (Corollary 3.3) and that $\mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^\times) = \mathrm{ICG}(n, \{1\})$ is the unitary Cayley graph of $\mathbb{Z}_n$. As before $\varphi(n)$ is Euler's totient function and $D(n)$ denotes the set of positive divisors of n.

Theorem 6.1. *Let $n \geq 3$ be an integer.*

(a) *The energy of $\mathrm{ICG}(n, \{1\})$ is given by*

$$\mathcal{E}(\mathrm{ICG}(n, \{1\})) = 2^s \varphi(n).$$

([156, Theorem 2.3] or [262, Theorem 3.7])

(b) *$\mathrm{ICG}(n, \{1\})$ is hyperenergetic if and only if $s > 2$ or $s = 2$ and $p_1 > 2$.* ([156, Theorem 2.4] or [262, Theorem 3.10])

(c) *The energy of $\mathrm{ICG}(n, \{1\})$ satisfies*

$$\mathcal{E}(\mathrm{ICG}(n, \{1\})) > \frac{2^{s-1}(n-1)}{s}.$$

([262, Theorem 3.11])

Part (a) of Theorem 6.1 was also obtained in [39] in the special case when n is a prime power.

Theorem 6.2. *Let $n \geq 3$ be an integer.*

(a) *The energy of the complement of $\mathrm{ICG}(n, \{1\})$ is equal to*

$$2n - 2 + (2^s - 2)\varphi(n) - \prod_{i=1}^{s} p_i + \prod_{i=1}^{s}(2 - p_i).$$

([156, Theorem 3.1])

(b) *The complement of $\mathrm{ICG}(n, \{1\})$ is hyperenergetic if and only if $s \geq 2$ and $n \neq 2p$, where p is a prime.* ([156, Theorem 3.2])

The next two results were obtained by Ilić and Bašić in [158]. Denote $p_i^\alpha \parallel n$ if $p_i^\alpha \mid n$ but $p_i^{\alpha+1} \nmid n$.

Theorem 6.3. ([158, Theorem 4.1]) *If $n \geq 4$, then for each i and any integer γ with $1 \leq \gamma \leq \alpha_i$,*

$$\mathcal{E}(\mathrm{ICG}(n, \{1, p_i^\gamma\})) = \begin{cases} 2^{s-1}(\varphi(n) + \varphi(n/p_i)), & \text{if } p_i \parallel n \\ 2^{s-1}(2\varphi(n) + (p_i^\gamma - 2p + 2)\varphi(n/p_i)), & \text{if } p_i^\gamma \parallel n \text{ with } \gamma \geq 2 \\ 2^s(\varphi(n) + (p_i^\gamma - p_i + 1)\varphi(n/p_i)), & \text{if } p_i^\gamma \nmid n. \end{cases}$$

Theorem 6.4. ([158, Theorem 4.2]) *If $n \geq 4$, then for $1 \leq i < j \leq s$,*

$$\mathcal{E}(\mathrm{ICG}(n, \{p_i, p_j\})) = \begin{cases} 2^s \varphi(n), & \text{if } p_i \parallel n \text{ and } p_j \parallel n \\ 3 \cdot 2^{s-1} \varphi(n), & \text{if } 2 \parallel n \text{ and } p_j^2 | n \\ 2^{s-1}(2\varphi(n) + \varphi(n/p_j)\varphi(p_j)), & \text{if } p_i \parallel n \text{ with } p_i \neq 2 \\ & \text{and } p_j^2 | n \\ 2^{s-1}(2\varphi(n) + \varphi(n/p_i)\varphi(p_i)), & \text{if } p_i^2 | n \text{ and } p_j \parallel n \\ 2^{s-1}(2\varphi(n) + \varphi(n/p_i)\varphi(p_i) + \varphi(n/p_j)\varphi(p_j)), & \text{if } p_i^2 | n \text{ and } p_j^2 | n. \end{cases}$$

The first formula in Theorem 6.3 was obtained earlier in [156, Theorem 4.1] (see also [248, Corollary 3.8]), and in the special case when $n = p_1 p_2 \ldots p_s$ the first formula in Theorem 6.4 was also obtained in [156, Theorem 4.2].

Theorem 6.4 implies that there are a large number of non-cospectral regular hyperenergetic graphs of the same order and same energy. See [158, Section 5] for more details.

The next three results were obtained by Mollahajiaghaei in [248].

Theorem 6.5. ([248, Theorem 3.10]) *Let $n = p^k m$, where p is a prime, $k > 1$ and $\gcd(m, p) = 1$. Then*

$$\mathcal{E}(\mathrm{ICG}(n; \{1, p\})) = 2(p^k - p^{k-2})\mathcal{E}(\mathrm{ICG}(m, \{1\})).$$

Theorem 6.6. ([248, Theorem 3.11]) *Let $n = p^k m$, where p is a prime, $k > 1$ and $\gcd(m, p) = 1$. Then $\mathrm{ICG}(n; \{1, p\})$ is hyperenergetic if and only if m has at least two distinct prime factors or m is an odd integer.*

Theorem 6.7. ([248, Theorems 4.2 and 4.7]) *Let $n = p_1 p_2 \ldots p_s m \geq 3$ and $D = \{p_1, p_2, \ldots, p_s\}$, where $p_1, p_2, \ldots, p_s$ are distinct primes each of which is greater than s, and m is a positive integer with $\gcd(p_1 p_2 \ldots p_s, m) = 1$. Let t be the number of prime factors of m. Then*

$$\mathcal{E}(\mathrm{ICG}(n, D)) = 2^t \left(s 2^{s-1} - (2^{s-1} - 2) \sum_{i=1}^{s} \frac{1}{p_i - 1} \right) \varphi(n).$$

Moreover, $\mathrm{ICG}(n, D)$ is hyperenergetic.

The next result tells us when the energy of an integral circulant graph is divisible by 4.

Theorem 6.8. ([158, Theorems 3.2 and 3.3]) *Let $n \geq 3$ be an integer and $D \subseteq D(n) \setminus \{n\}$. If n is odd, then $\mathcal{E}(\mathrm{ICG}(n, D))$ is divisible by 4; if n is even, then $\mathcal{E}(\mathrm{ICG}(n, D))$ is not divisible by 4 if and only if $n/2 \notin D$ and $\sum_{d \in D} (-1)^d \varphi(n/d) < 0$.*

In fact, when n is odd, $\sum_{d \in D} (-1)^d \varphi(n/d)$ is an eigenvalue of $\mathrm{ICG}(n, D)$.

In [196], Le and Sander found a connection between A-convolutions satisfying a weak form of regularity and the spectra of integral circulant graphs. Using this, they obtained a multiplicative decomposition of the energies of integral circulant graphs with multiplicative divisor sets. A few definitions are in order before presenting their results. Call $A = (A(n))_{n \in \mathbb{N}}$ a *divisor system* if $\emptyset \neq A(n) \subseteq D(n)$ for each $n \in \mathbb{N}$. Denote by $\mathbb{C}^{\mathbb{N}}$ the set of functions from $\mathbb{N}$ to $\mathbb{C}$. The A-*convolution* of two functions $f, g \in \mathbb{C}^{\mathbb{N}}$ is the function $f *_A g \in \mathbb{C}^{\mathbb{N}}$ defined by

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d) g\left(\frac{n}{d}\right), \ n \in \mathbb{N}.$$

The A-convolution is *regular* if it satisfies the following conditions:

- $(\mathbb{C}^{\mathbb{N}}, +, *_A)$ is a commutative ring with unity;

- if $f, g \in \mathbb{C}^{\mathbb{N}}$ are both multiplicative, so is $f *_A g$;

- the constant function 1 has an inverse μ_A with respect to the operation $*_A$ such that $\mu_A(p^s) \in \{-1, 0\}$ for all prime powers p^s (generalization of Möbius' μ function).

In [253], Narkiewicz proved that any regular A-convolution has the multiplicative property, in the sense that
$$A(mn) = A(m)A(n) := \{ab : a \in A(m), b \in A(n)\}$$
for all coprime integers $m, n \in \mathbb{N}$. This implies that

$$A\left(\prod_{i=1}^{l} p_i^{s_i}\right) = \prod_{i=1}^{l} A(p_i^{s_i}) = \left\{\prod_{i=1}^{l} a_i : a_i \in A(p_i^{s_i}) \text{ for } 1 \le i \le l\right\} \tag{6.3}$$

for any distinct primes p_i and non-negative integers s_i.

A divisor system $A = (A(n))_{n \in \mathbb{N}}$ is called *multiplicative* if for all $n \in \mathbb{N}$,

$$A(n) = \prod_{p \in \mathbb{P}, \, p|n} A(p^{e_p(n)}),$$

where $e_p(n)$ is the exponent of p in n, $\mathbb{P}$ is the set of primes, and the product on the right-hand side is defined as in (6.3). It follows that A is multiplicative if and only if each $A(n)$ is multiplicative. Recall that the Ramanujan sum $c(k, n)$ was defined in (3.5).

Theorem 6.9. ([196, Theorem 4.1]) *Let $A = (A(n))_{n \in \mathbb{N}}$ be a multiplicative divisor system. Then for any integer $n \ge 3$,*

$$\mathcal{E}(\mathrm{ICG}(n, A(n))) = \sum_{k=1}^{n} |\lambda_k(n, A(n))| = \prod_{p \in \mathbb{P}, \, p|n} \mathcal{E}(\mathrm{ICG}(p^{e_p(n)}, A(p^{e_p(n)}))),$$

where $\lambda_k(n, A(n)) := (1 *_A c(k, \cdot))(n)$ with 1 the constant function taking value 1.

Recall from (4.2) that $X_p = \{p^{e_p(x)} : x \in X\}$ for any prime p and non-empty set X of positive integers. Using Theorem 6.9, Le and Sander obtained the following result.

Theorem 6.10. ([196, Corollary 4.1]) *Let $A = (A(n))_{n \in \mathbb{N}}$ be a multiplicative divisor system. Then, for any $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s} \ge 3$ as in (6.2), we have $A(p_j^{\alpha_j}) = A(n)_{p_j}$ for $1 \le j \le s$, and moreover*

$$\mathcal{E}(\mathrm{ICG}(n, A(n))) = \prod_{j=1}^{s} \mathcal{E}(\mathrm{ICG}(p_j^{\alpha_j}, A(n)_{p_j})).$$

This implies the following result.

Theorem 6.11. ([196, Theorem 4.2]) *Let $n \ge 3$ and $D \subseteq D(n)$. Suppose that D can be factorized as $D = \{g\} \cdot X$ for a positive integer g and a multiplicative set $X \subseteq D(n/g)$. Then*

$$\mathcal{E}(\mathrm{ICG}(n, D)) = g \prod_{p \in \mathbb{P}, \, p|n} \mathcal{E}(\mathrm{ICG}(p^{e_p(n/g)}, X_p)).$$

Given $D \subseteq D(n)$, set $\overline{D} = D(n) \setminus D$ and

$$\Phi(n, D) = \sum_{d \in D} \varphi\left(\frac{n}{d}\right). \qquad (6.4)$$

Theorem 6.12. ([197, Theorem 6.1]) *Let $n \geq 3$ and let $D \subset D(n)$ be a multiplicative set containing n. Then*

$$\mathcal{E}(\mathrm{ICG}(n, \overline{D})) = \prod_{p \in \mathbb{P},\, p \mid n} \mathcal{E}(\mathrm{ICG}(p^{e_p(n)}, D_p)) + n - 2\Phi(n, D).$$

As mentioned in [196, Section 5], Theorem 6.10 reduces the computation of the energies of all integral circulant graphs with respect to multiplicative divisor sets to that of $\mathrm{ICG}(p^a, D)$ for an arbitrary prime power p^a and any divisor set $D \subseteq D(p^a)$. The latter was achieved in [270], where the following result was proved.

Theorem 6.13. ([270, Theorem 2.1; Corollary 2.2]) *Let p be a prime and let a be a positive integer. Let $D = \{p^{a_1}, p^{a_2}, \ldots, p^{a_r}\}$, where $0 \leq a_1 < a_2 < \cdots < a_r < a$. Then*

$$\mathcal{E}(\mathrm{ICG}(p^a, D)) = 2(p-1)\left(p^{a-1}r - (p-1)\sum_{k=1}^{r-1}\sum_{i=k+1}^{r} p^{a-a_i+a_k-1}\right).$$

Moreover, $\mathrm{ICG}(p^a, D)$ is hyperenergetic if and only if

$$\sum_{k=1}^{r-1}\sum_{i=k+1}^{r} \frac{1}{p^{a_i-a_k}} < \frac{1}{p-1}\left(r - \frac{p^a-1}{p^{a-1}(p-1)}\right).$$

We will see in Theorem 6.38 that this result can be generalized to the Cayley graph $\mathrm{Cay}(R, \mathcal{C})$ on a finite chain ring R as in section 5.3.

The condition $a_r < a$ in Theorem 6.13 is required for otherwise $\mathrm{ICG}(p^a, D)$ would have a loop. In the case when $a_r = a$, we have the following result.

Theorem 6.14. ([196, Proposition 5.1]) *Let p be a prime and let a be a positive integer. Let $D = \{p^{a_1}, p^{a_2}, \ldots, p^{a_r}\}$, where $0 \leq a_1 < a_2 < \cdots < a_r = a$. Then*

$$\mathcal{E}(\mathrm{ICG}(p^a, D)) = 2(p-1)\left(p^{a-1}(r-1) - (p-1)\sum_{k=1}^{r-2}\sum_{i=k+1}^{r-1} p^{a-a_i+a_k-1} - \sum_{k=1}^{r-1} p^{a_k}\right) + p^a.$$

Given $D = \{d_1, d_2, \ldots, d_r\} \subseteq D(p^a)$ with $d_1 < d_2 < \cdots < d_r$, define

$$\eta(p^a, D) := \begin{cases} 2(p-1)p^{a-1}\left(r - (p-1)\sum_{k=1}^{r-1}\sum_{i=k+1}^{r} \dfrac{d_k}{d_i}\right), & \text{if } d_r < p^a, \\[4ex] 2(p-1)p^{a-1}\left(r - 1 - (p-1)\sum_{k=1}^{r-2}\sum_{i=k+1}^{r-1} \dfrac{d_k}{d_i} - \dfrac{1}{p^{a-1}}\sum_{k=1}^{r-1} d_k\right) + p^a, & \text{if } d_r = p^a. \end{cases}$$

Theorems 6.10, 6.13 and 6.14 together imply the following result.

Theorem 6.15. ([196, Theorem 5.1]) *Let $n \geq 3$ and $D \subseteq D(n)$. Suppose that D can be factorized into $D = \{g\} \cdot X$ for a positive integer g and a multiplicative set $X \subseteq D(n/g)$. Then*

$$\mathcal{E}(\mathrm{ICG}(n, D)) = g \prod_{p \in \mathbb{P},\, p \mid n} \eta(p^{e_p(n/g)}, X_p).$$

In view of Theorem 3.8, all distance powers of the unitary Cayley graph $G_{\mathbb{Z}_n}$ $(= \mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^\times) = \mathrm{ICG}(n, \{1\}))$ are integral circulant graphs. The energies of some distance powers of $G_{\mathbb{Z}_n}$ were computed by Liu and Li in [212] and as a consequence when such a distance power is hyperenergetic was determined.

Theorem 6.16. ([212, Theorem 1.1]) *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} \geq 2$, where $p_1 < p_2 < \cdots < p_s$ are distinct primes and $\alpha_i \geq 1$.*

(a) *If n is a prime, then $\mathcal{E}(G_{\mathbb{Z}_n}^{\{1\}}) = 2(n-1)$.*

(b) *If $n = 2^\alpha$ with $\alpha > 1$ or n is an odd composite number, then*

$$\mathcal{E}\left(G_{\mathbb{Z}_n}^{\{1\}}\right) = 2^s \varphi(n), \quad \mathcal{E}\left(G_{\mathbb{Z}_n}^{\{2\}}\right) = 2n - 2 + (2^s - 2)\varphi(n) - \prod_{i=1}^s p_i + \prod_{i=1}^s (2 - p_i),$$

$$\mathcal{E}\left(G_{\mathbb{Z}_n}^{\{1,2\}}\right) = 2(n-1).$$

(c) *If n is even but has an odd prime divisor, then*

$$\mathcal{E}\left(G_{\mathbb{Z}_n}^{\{1\}}\right) = 2^s \varphi(n), \quad \mathcal{E}\left(G_{\mathbb{Z}_n}^{\{2\}}\right) = 2(n-2), \quad \mathcal{E}\left(G_{\mathbb{Z}_n}^{\{3\}}\right) = n - 4\varphi(n) + 2^s \varphi(n),$$

$$\mathcal{E}\left(G_{\mathbb{Z}_n}^{\{1,2\}}\right) = 2n - 2 + (2^s - 2)\varphi(n) - \prod_{i=1}^s p_i, \quad \mathcal{E}\left(G_{\mathbb{Z}_n}^{\{1,3\}}\right) = n,$$

$$\mathcal{E}\left(G_{\mathbb{Z}_n}^{\{2,3\}}\right) = 2n - 2 + (2^s - 2)\varphi(n) - \prod_{i=1}^s p_i, \quad \mathcal{E}\left(G_{\mathbb{Z}_n}^{\{1,2,3\}}\right) = 2(n-1).$$

Theorem 6.17. ([212, Corollary 2.10]) *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} \geq 2$, where $p_1 < p_2 < \cdots < p_s$ are distinct primes and $\alpha_i \geq 1$. Then the following hold:*

(a) *$G_{\mathbb{Z}_n}^{\{1\}}$ is hyperenergetic if and only if $s \geq 3$ or $s = 2$ and n is odd;*

(b) *$G_{\mathbb{Z}_n}^{\{2\}}$ is hyperenergetic if and only if n is an odd composite number;*

(c) *$G_{\mathbb{Z}_n}^{\{3\}}$ is hyperenergetic if and only if $p_1 = 2$ and $s \geq 3$;*

(d) *$G_{\mathbb{Z}_n}^{\{1,2\}}$ is hyperenergetic if and only if $p_1 = 2$ and $s \geq 2$ except when $n = 2p_2$;*

(e) *$G_{\mathbb{Z}_n}^{\{2,3\}}$ is hyperenergetic if and only if $p_1 = 2$ and $s \geq 2$ except when $n = 2p_2$;*

(f) *$G_{\mathbb{Z}_n}^{\{1,3\}}$ is not hyperenergetic;*

(g) *$G_{\mathbb{Z}_n}^{\{1,2,3\}}$ is not hyperenergetic.*

In [305], hyperenergetic integral circulant graphs were discussed and a method for constructing hyperenergetic integral circulant graphs using the Cartesian product of graphs was given.

6.2 Extremal energies of integral circulant graphs

In this section we survey some known results on the characterization of those graphs having maximal or minimal energy among all integral circulant graphs with a given order. We adopt the following notation from [270]:

$$\mathcal{E}_{\min}(n) := \min\{\mathcal{E}(\mathrm{ICG}(n, D)) : D \subseteq D(n) \setminus \{n\}\}$$

$$\mathcal{E}_{\max}(n) := \max\{\mathcal{E}(\mathrm{ICG}(n, D)) : D \subseteq D(n) \setminus \{n\}\}.$$

A subset $D \subseteq D(n) \setminus \{n\}$ is called *n-minimal* or *n-maximal* if $\mathcal{E}(\mathrm{ICG}(n, D)) = \mathcal{E}_{\min}(n)$ or $\mathcal{E}(\mathrm{ICG}(n, D)) = \mathcal{E}_{\max}(n)$, respectively.

Theorem 6.18. ([270, Theorem 3.1]) *Let p be a prime and s a positive integer. Then*

$$\mathcal{E}_{\min}(p^s) = 2(p - 1)p^{s-1}.$$

Moreover, the p^s-minimal sets are exactly the sets $\{p^t\}$ for $0 \leq t \leq s - 1$.

Theorem 6.19. ([270, Theorem 3.2]) *Let p be a prime. Then the following hold:*

(a) $\mathcal{E}_{\max}(p) = 2(p - 1)$, *the only p-maximal set being $D = \{1\}$;*

(b) $\mathcal{E}_{\max}(p^2) = 2(p - 1)(p + 1)$, *the only p^2-maximal set being $D = \{1, p\}$;*

(c) $\mathcal{E}_{\max}(p^3) = 2(p - 1)(2p^2 - p + 1)$, *the only p^3-maximal set being $D = \{1, p^2\}$, except when $p = 2$ for which $D = \{1, 2, 4\}$ is also 2^3-maximal;*

(d) $\mathcal{E}_{\max}(p^4) = 2(p - 1)(2p^3 + 1)$, *the only p^4-maximal sets being $D = \{1, p, p^3\}$ and $D = \{1, p^2, p^3\}$.*

Using tools from convex optimization, it was proved in [271, Theorem 4.2] that $\mathcal{E}_{\max}(p^s)$ lies between $s(p - 1)p^{s-1}$ and $2s(p - 1)p^{s-1}$ approximately.

In [270, Theorem 2.1], it was proved that, for $0 \leq a_1 < a_2 < \cdots < a_{r-1} < a_r \leq s - 1$,

$$\mathcal{E}(\mathrm{ICG}(p^s, \{p^{a_1}, \ldots, p^{a_r}\})) = 2(p - 1)p^{s-1}\left(r - (p - 1)h_p(a_1, \ldots, a_r)\right),$$

where

$$h_p(a_1, \ldots, a_r) = \sum_{k=1}^{r-1} \sum_{i=k+1}^{r} \frac{1}{p^{a_i - a_k}}.$$

Thus $\mathcal{E}_{\max}(p^s)$ can be obtained by minimizing $h_p(a_1, \ldots, a_r)$. As observed in [273], the minimum of $h_p(a_1, \ldots, a_r)$ occurs only when $a_1 = 0$ and $a_r = s - 1$. This approach was used in [272, Theorems 2.1 and 2.2], where the minimum value of $h_p(0, a_2, \ldots, a_{r-1}, s - 1)$ was determined when $s \equiv 1 \bmod (r - 1)$ or $s \equiv 0 \bmod (r - 1)$. Finally, in [273, Theorem 1.1], the following formula for $\mathcal{E}_{\max}(p^s)$ was obtained and all p^s-maximal sets were determined.

Theorem 6.20. ([273, Theorem 1.1]) *Let p be a prime and let r and s be positive integers.*

(a) *If s is odd, then*

$$\mathcal{E}_{\max}(p^s) = \frac{1}{(p + 1)^2}\left((s + 1)(p^2 - 1)p^s + 2(p^{s+1} - 1)\right).$$

Moreover, if $p \geq 3$ then $\{1, p^2, p^4, \ldots, p^{s-3}, p^{s-1}\}$ is the only p^s-maximal subset, and if $p = 2$ then $\{1, p^2, p^4, \ldots, p^{s-3}, p^{s-1}\}$ and $\{1, p^1, p^3, p^5, \ldots, p^{s-4}, p^{s-2}, p^{s-1}\}$ are the only p^s-maximal subsets.

(b) *If s is even, then*

$$\mathcal{E}_{\max}(p^s) = \frac{1}{(p+1)^2} \left(s(p^2-1)p^s + 2(2p^{s+1} - p^{s-1} + p^2 - p - 1)\right)$$

and $\{1, p^2, p^4, \ldots, p^{s-2}, p^{s-1}\}$ *and* $\{1, p^1, p^3, p^5, \ldots, p^{s-3}, p^{s-1}\}$ *are the only* p^s*-maximal subsets.*

In general, it seems difficult to obtain the exact values of $\mathcal{E}_{\min}(n)$ and $\mathcal{E}_{\max}(n)$ for arbitrary integers $n \geq 3$ due to the lack of explicit formulas for $\mathcal{E}(\mathrm{ICG}(n, D))$. The following relaxed forms of $\mathcal{E}_{\min}(n)$ and $\mathcal{E}_{\max}(n)$ were introduced in [197]:

$$\widetilde{\mathcal{E}}_{\min}(n) := \min\{\mathcal{E}(\mathrm{ICG}(n, D)) : D \subseteq D(n) \setminus \{n\} \text{ multiplicative}\}$$

$$\widetilde{\mathcal{E}}_{\max}(n) := \max\{\mathcal{E}(\mathrm{ICG}(n, D)) : D \subseteq D(n) \setminus \{n\} \text{ multiplicative}\}.$$

Of course,

$$\mathcal{E}_{\min}(n) \leq \widetilde{\mathcal{E}}_{\min}(n) \leq \widetilde{\mathcal{E}}_{\max}(n) \leq \mathcal{E}_{\max}(n).$$

For any prime power p^s, set

$$\theta(p^s) = \begin{cases} \dfrac{1}{(p+1)^2} \left((s+1)(p^2-1)p^s + 2(p^{s+1}-1)\right), & \text{if } 2 \nmid s, \\ \dfrac{1}{(p+1)^2} \left(s(p^2-1)p^s + 2(2p^{s+1} - p^{s-1} + p^2 - p - 1)\right), & \text{if } 2 \mid s. \end{cases}$$

Theorem 6.21. ([197, Theorem 2.1]) *Let* $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s} \geq 3$ *be an integer in canonical factorization. Then*

$$\widetilde{\mathcal{E}}_{\max}(n) = \prod_{i=1}^{s} \theta(p_i^{\alpha_i}).$$

Moreover, a multiplicative set $D \subseteq D(n) \setminus \{n\}$ *satisfies* $\mathcal{E}(\mathrm{ICG}(n, D)) = \widetilde{\mathcal{E}}_{\max}(n)$ *if and only if* $D = \prod_{i=1}^{s} D^{(i)}$*, where*

$$D^{(i)} = \begin{cases} \{1, p_i^2, p_i^4, \ldots, p_i^{\alpha_i-3}, p_i^{\alpha_i-1}\}, & \text{if } 2 \nmid \alpha_i, \ p_i \geq 3, \\ \{1, 2^2, 2^4, \ldots, 2^{\alpha_i-3}, 2^{\alpha_i-1}\} \text{ or } \{1, 2, 2^3, \ldots, 2^{\alpha_i-4}, 2^{\alpha_i-2}, 2^{\alpha_i-1}\}, & \text{if } 2 \nmid \alpha_i, \ p_i = 2, \\ \{1, p_i^2, p_i^4, \ldots, p_i^{\alpha_i-4}, p_i^{\alpha_i-2}, p_i^{\alpha_i-1}\} \text{ or } \{1, p_i, p_i^3, \ldots, p_i^{\alpha_i-3}, p_i^{\alpha_i-1}\}, & \text{if } 2 \mid \alpha_i. \end{cases}$$

A set $D \subseteq D(p^s)$ is called *uni-regular* if $D = \{p^i, p^{i+1}, \ldots, p^{j-1}, p^j\}$ for some integers i, j with $0 \leq i \leq j \leq s$.

Theorem 6.22. ([197, Theorem 2.2]) *Let* $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s} \geq 3$ *be an integer in canonical factorization, where* $p_1 < p_2 < \ldots < p_s$*. Then*

$$\widetilde{\mathcal{E}}_{\min}(n) = 2n \left(1 - \frac{1}{p_1}\right).$$

Moreover, a multiplicative set $D \subseteq D(n) \setminus \{n\}$ *satisfies* $\mathcal{E}(\mathrm{ICG}(n, D)) = \widetilde{\mathcal{E}}_{\min}(n)$ *if and only if* $D = \prod_{i=1}^{s} D^{(i)}$*, where* $D^{(1)} = \{p_1^u\}$ *for some* $u \in \{0, 1, \ldots, \alpha_1 - 1\}$*, and* $D^{(i)}$ *is an arbitrary uni-regular set with* $p_i^{\alpha_i} \in D^{(i)}$ *for* $2 \leq i \leq s$*.*

Theorems 6.21 and 6.22 together imply the following result.

Theorem 6.23. ([197, Corollary 2.1]) *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s} \geq 3$ be an integer in canonical factorization, where $p_1 < p_2 < \dots < p_s$. Then*

(a) $\mathcal{E}_{\max}(n) \geq \widetilde{\mathcal{E}}_{\max}(n) = \prod_{i=1}^{s} \theta(p_i^{\alpha_i})$;

(b) $\mathcal{E}_{\min}(n) \leq \widetilde{\mathcal{E}}_{\min}(n) = 2n \left(1 - \dfrac{1}{p_1} \right)$.

Denote by $\tau(n)$ the number of positive divisors of n and by $\omega(n)$ the number of distinct prime factors of n.

Theorem 6.24. ([197, Theorem 2.3]) *Let $n \geq 3$ be an integer. Then*

(a) $\mathcal{E}_{\max}(n) \leq n \sum_{d|n} \dfrac{\varphi(n)\tau(d)}{d} = n \prod_{p \in \mathbb{P},\, p|n} \left(\dfrac{1}{2} \left(1 - \dfrac{1}{p} \right) (e_p(n) + 1)(e_p(n) + 2) + \dfrac{1}{p} \right)$;

(b) $\mathcal{E}_{\max}(n) < \left(\dfrac{3}{4} \right)^{\omega(n)} n\tau(n)^2$;

(c) $\mathcal{E}_{\max}(n) \leq \widetilde{\mathcal{E}}_{\max}(n)\tau(n)$.

We finish this section by mentioning three conjectures from [197]. As far as we know, all these conjectures are still open.

Conjecture 6.25. ([197, Conjecture 6.1]) *For any integer $n \geq 3$,*

$$\mathcal{E}_{\min}(n) = 2n \left(1 - \frac{1}{p_1} \right),$$

where p_1 denotes the smallest prime factor of n.

Conjecture 6.26. ([197, Conjecture 6.2]) *For any integer $n \geq 3$ and $D \subseteq D(n) \setminus \{n\}$, if $\mathcal{E}(\mathrm{ICG}(n, D)) = \mathcal{E}_{\min}(n)$, then D is a multiplicative divisor set.*

Conjecture 6.27. ([197, Conjecture 6.3]) *Let $n \geq 3$ be an integer, and let $D_1, D_2 \subseteq D(n) \setminus \{n\}$ be multiplicative sets. If $\mathrm{ICG}(n, D_1)$ and $\mathrm{ICG}(n, D_2)$ are cospectral, then $D_1 = D_2$.*

6.3   Energies of circulant graphs

In [281], Shparlinski gave a construction of circulant graphs of very high energy using Gauss sums. Given a prime p, let Q_p denote the set of all quadratic residues modulo p in the set $\{1, \dots, (p-1)/2\}$.

Theorem 6.28. ([281, Theorem 1]) *For any prime $p \equiv 1 \pmod 4$,*

$$\mathcal{E}(\mathrm{Cay}(\mathbb{Z}_p, Q_p \cup (-Q_p))) \geq \frac{(p-1)(\sqrt{p}+1)}{2}.$$

Thus $\mathrm{Cay}(\mathbb{Z}_p, Q_p \cup (-Q_p))$ has energy close to the upper bound in (6.1).

Let n and d be integers with $1 \leq d \leq n - 1$. The average energy of all circulant graphs of order n and degree d is given by

$$\mathcal{E}(n, d) = \frac{1}{\binom{\lceil n/2 \rceil - 1}{\lfloor d/2 \rfloor}} \sum \mathcal{E}(\mathrm{Cay}(\mathbb{Z}_n, S)),$$

where the sum is running over all subsets S of $\mathbb{Z}_n \setminus \{0\}$ with $S = -S$ and $|S| = d$. Using (6.1), it can be verified [58] that

$$\mathcal{E}(n, d) \leq d + \sqrt{d(n-1)(n-d)}.$$

On the other hand, the following asymptotic lower bound was obtained by Blackburn and Shparlinski in [58].

Theorem 6.29. ([58, Theorem 5]) *As $n \to \infty$, for any integer d such that $4 \leq d = o(n^{1/2})$ and dn is even, we have*

$$\mathcal{E}(n, d) \geq \begin{cases} \dfrac{dn}{\sqrt{3d - 3 + o(1)}}, & \text{if } d \text{ is even} \\ \dfrac{dn}{\sqrt{3d - 3 + (1/d) + o(1)}}, & \text{if } d \text{ is odd.} \end{cases}$$

6.4 Energies of unitary Cayley graphs and quadratic unitary Cayley graphs of finite commutative rings

As mentioned in section 6.1, the energies of the unitary Cayley graph $G_{\mathbb{Z}_n}$ ($= \text{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^\times) = \text{ICG}(n, \{1\})$) and its complement have been determined. A generalization of these results obtained by Kiani et al. is as follows.

Theorem 6.30. ([171, Theorems 2.4, 2.5 and 4.1]) *Let R be a finite commutative ring as in Assumption 5.1. Then*

$$\mathcal{E}(G_R) = 2^s |R^\times|$$

and

$$\mathcal{E}(\overline{G}_R) = 2|R| - 2 + (2^s - 2)|R^\times| - \prod_{i=1}^{s} |R_i|/m_i + \prod_{i=1}^{s} (2 - |R_i|/m_i).$$

Moreover,

(a) *if $s = 1$, then G_R is not hyperenergetic;*

(b) *if $s = 2$, then G_R is hyperenergetic if and only if $|R_1|/m_1 \geq 3$ and $|R_2|/m_2 \geq 4$;*

(c) *if $s \geq 3$, then G_R is hyperenergetic if and only if $|R_{s-2}|/m_{s-2} \geq 3$, or $|R_{s-1}|/m_{s-1} \geq 3$ and $|R_s|/m_s \geq 4$.*

In [214, Theorem 18], Liu and Zhou determined the energy of the line graph $L(G_R)$ of G_R.

Theorem 6.31. ([214, Theorem 18]) *Let R be a finite commutative ring as in Assumption 5.1. Then*

$$\mathcal{E}(L(G_R)) = \begin{cases} 2^{s+1}(|R^\times| - 1)^2, & \text{if } 2 = |R_1|/m_1 = \cdots = |R_s|/m_s, \\ & \text{or } R = \underbrace{\mathbb{F}_2 \times \cdots \times \mathbb{F}_2}_{s-1} \times \mathbb{F}_3; \\ 2^{t+1} + 2|R|(|R^\times| - 2), & \text{if } 2 = |R_1|/m_1 = \cdots = |R_t|/m_t < |R_{t+1}|/m_{t+1} \\ & \text{with } 1 \leq t < s \text{ and } R \not\cong \underbrace{\mathbb{F}_2 \times \cdots \times \mathbb{F}_2}_{s-1} \times \mathbb{F}_3; \\ 2|R|(|R^\times| - 2), & \text{if } 3 \leq |R_1|/m_1 \leq \cdots \leq |R_s|/m_s \text{ and } R \not\cong \mathbb{F}_3. \end{cases}$$

In the special case where $R = \mathbb{Z}_n$, Theorem 6.31 yields the following result.

Corollary 6.32. ([214, Corollary 19]) *Let* $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s} \geq 3$ *be an integer in canonical factorization, where* $p_1 < p_2 < \ldots < p_s$ *are primes and each* $\alpha_i \geq 1$ *is an integer. Then*

$$
\mathcal{E}(L(G_{\mathbb{Z}_n})) =
\begin{cases}
4, & \text{if } n = 3; \\
8, & \text{if } n = 6; \\
4\left(2^{\alpha_1 - 1} - 1\right)^2, & \text{if } n = 2^{\alpha_1}; \\
4 + 2n\left(\left(\prod_{i=1}^{s} p_i^{\alpha_1 - 1}(p_i - 1)\right) - 2\right), & \text{if } 2 = p_1 \text{ and } n \neq 6; \\
2n\left(\left(\prod_{i=1}^{s} p_i^{\alpha_1 - 1}(p_i - 1)\right) - 2\right), & \text{if } 3 \leq p_1 \text{ and } n \neq 3.
\end{cases}
$$

By Theorem 6.31, we know exactly when $L(G_R)$ is hyperenergetic.

Corollary 6.33. ([214, Corollary 20]) *Let R be as in Assumption 5.1. Then $L(G_R)$ is hyperenergetic if and only if one of the following holds:*

(a) $|R^\times| \geq 4$;

(b) $s = 1$ *and* $|R| = 2m \geq 8$;

(c) $s \geq 2$, $2 = |R_1|/m_1 = \cdots = |R_s|/m_s$, *and* $|R^\times| \geq 2$.

Recall from section 5.2 that $\mathcal{G}_R$ is the quadratic unitary Cayley graph of a finite commutative ring R. The next two results were obtained by Liu and Zhou in [215].

Theorem 6.34. ([215, Theorem 3.1]) *Let R be a local ring with maximal ideal M of order m. Then the following hold:*

(a) *if* $|R|/m \equiv 1 \pmod 4$, *then* $\mathcal{E}(\mathcal{G}_R) = \left(\sqrt{|R|/m} + 1\right)|R^\times|/2$;

(b) *if* $|R|/m \equiv 3 \pmod 4$, *then* $\mathcal{E}(\mathcal{G}_R) = 2|R^\times|$.

Theorem 6.35. ([215, Theorem 3.2]) *Let R be as in Assumption 5.1 such that* $|R_i|/m_i \equiv 1 \pmod 4$ *for* $1 \leq i \leq s$, *and let R_0 be a local ring with maximal ideal M_0 of order m_0 such that* $|R_0|/m_0 \equiv 3 \pmod 4$. *Then the following hold:*

(a) $\mathcal{E}(\mathcal{G}_R) = \dfrac{|R^\times|}{2^s} \displaystyle\prod_{i=1}^{s}\left(\sqrt{|R_i|/m_i} + 1\right)$;

(b) $\mathcal{E}(\mathcal{G}_{R_0 \times R}) = \dfrac{|R_0^\times||R^\times|}{2^{s-1}} \displaystyle\prod_{i=1}^{s}\left(\sqrt{|R_i|/m_i} + 1\right)$.

Theorem 6.35 implies the following result.

Corollary 6.36. ([215, Corollary 3.3]) *Let R and R_0 be as in Theorem 6.35. Then the following hold:*

(a) $\mathcal{G}_R$ *is hyperenergetic except when* $R = R_1$ *with* $|R_1|/m_1 = 5$ *or* $R = R_1 \times R_2$ *with* $|R_1|/m_1 = |R_2|/m_2 = 5$;

(b) $\mathcal{G}_{R_0 \times R}$ *is hyperenergetic except when* $|R_0|/m_0 = 3$ *and* $R = R_1$ *with* $|R_1|/m_1 = 5$.

In the special case where $R = \mathbb{Z}_n$, Theorems 6.34 and 6.35 and Corollary 6.36 together imply the following result.

Corollary 6.37. ([215, Corollary 3.4]) *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s}$ be an integer in canonical factorization such that $p_i \equiv 1 \, (\mathrm{mod}\, 4)$ for $1 \leq i \leq s$. Let $p \equiv 3 \, (\mathrm{mod}\, 4)$ be a prime and $\alpha \geq 1$ an integer. Then the followng hold:*

(a) $\mathcal{E}(\mathcal{G}_{\mathbb{Z}_{p^\alpha}}) = 2\varphi(p^\alpha);$

(b) $\mathcal{E}(\mathcal{G}_{\mathbb{Z}_n}) = \dfrac{\varphi(n)}{2^s} \prod\limits_{i=1}^{s} \left(\sqrt{p_i} + 1\right);$

(c) $\mathcal{E}(\mathcal{G}_{\mathbb{Z}_{np^\alpha}}) = \dfrac{\varphi(n)\varphi(p^\alpha)}{2^{s-1}} \prod\limits_{i=1}^{s} \left(\sqrt{p_i} + 1\right);$

(d) $\mathcal{G}_{\mathbb{Z}_n}$ *is hyperenergetic except when $n = 5^{\alpha_1};$*

(e) $\mathcal{G}_{\mathbb{Z}_{np^\alpha}}$ *is hyperenergetic except when $np^\alpha = 3^\alpha \cdot 5^{\alpha_1}.$*

6.5 Energies of Cayley graphs on finite chain rings and gcd graphs of unique factorization domains

Recall from section 5.3 the definition of $\mathrm{Cay}(R, \mathcal{C})$ for a finite chain ring R. The following result was proved by Suntornpoch and Meemark in [285]. In the special case when $R = \mathbb{Z}_{p^s}$, it gives exactly Theorem 6.13.

Theorem 6.38. ([285, Section 2]) *Let R be a finite chain ring and $\mathcal{C}$ be as in (5.1). Then*

$$\mathcal{E}(\mathrm{Cay}(R, \mathcal{C})) = 2(q-1)\left(q^{s-1}r - (q-1)\sum_{k=1}^{r-1}\sum_{i=k+1}^{r} q^{s-a_i+a_k-1}\right).$$

Let R be a unique factorization domain (UFD). Let c be a nonzero nonunit element of R. Then the quotient $R/(c) = \{x + (c) : x \in R\}$ is a commutative ring. Assume that this ring is finite. Let D be a set of proper divisors of c. In [171], the *gcd graph* of R with respect to D, denoted by $\mathrm{ICG}(R/(c), D)$ (note that another notation was used in [171]), was defined as the graph with vertex set $R/(c)$ such that $x + (c)$ and $y + (c)$ are adjacent if and only if $\gcd(x - y, c) \in D$ (the gcd considered here is unique up to associate). This notion is a generalization of the gcd graphs in section 3.1.1: If $R = \mathbb{Z}$, $n \geq 3$ is an integer and D is a set of positive proper divisors of n, then $\mathrm{ICG}(R/(n), D)$ is exactly the gcd graph $\mathrm{ICG}(n, D)$ of the cyclic group $\mathbb{Z}_n$. Moreover, in the special case when $D = \{1\}$ (where 1 is the multiplicative identity of R), $\mathrm{ICG}(R/(c), \{1\}) = G_{R/(c)}$ is the unitary Cayley graph of the finite commutative ring $R/(c)$. In [171], Kiani et al. obtained the following result.

Theorem 6.39. ([171, Theorems 3.1 and 3.4]) *Let R be a unique factorization domain.*

(a) *Let $c = p_1^{a_1} \ldots p_n^{a_n} \in R$ be factorized into a product of irreducible elements. Assume that $R/(c)$ is finite. For $1 \leq i \leq n$, if $a_i = 1$, then*

$$\mathcal{E}(\mathrm{ICG}(R/(c), \{1, p_i\})) = 2^{n-1}|R/(p_i)||R/(c/p_i)^\times|.$$

(b) *Let $c = p_1 \ldots p_k p_{k+1}^{a_{k+1}} \ldots p_n^{a_n} \in R$ be factorized into a product of irreducible elements, where $a_i > 1$ for $k+1 \le i \le n$. Assume that $R/(c)$ is finite. Then*

$$\mathcal{E}(\mathrm{ICG}(R/(c), \{p_i, p_j\})) = 2^{n-1}|R/(c)^\times|$$

for $1 \le i < j \le k$.

The following result was proved by Suntornpoch and Meemark with the help of NEPS.

Theorem 6.40. ([285, Theorem 3.4]) *Let R be a unique factorization domain and $c = p_1^{s_1} \ldots p_k^{s_k}$ a nonzero nonunit element of R which is factorized as a product of irreducible elements. Assume that $R/(c)$ is finite. Assume further that for some l with $1 \le l \le k$ and each i with $1 \le i \le l$ there exists a set $D_i = \{p_i^{a_{i1}}, p_i^{a_{i2}}, \ldots, p_i^{a_{ir_i}}\}$ such that $0 \le a_{i1} < a_{i2} < \cdots < a_{ir_i} \le s_i - 1$. Set*

$$D = \{p_1^{a_{1t_1}} \cdots p_l^{a_{lt_l}} p_{l+1}^{s_{l+1}} \cdots p_k^{s_k} : t_i \in \{1, 2, \ldots, r_i\} \text{ for } 1 \le i \le l\}.$$

Then

$$\mathcal{E}(\mathrm{ICG}(R/(c), D)) = \mathcal{E}(\mathrm{ICG}(R/(p_1^{s_1}), D_1)) \cdots \mathcal{E}(\mathrm{ICG}(R/(p_l^{s_l}), D_l)) \prod_{j=l+1}^{k} |R/(p_j^{s_j})|.$$

6.6 Skew energy of orientations of hypercubes

Let G be a digraph of order n with no loops or parallel arcs such that at most one of (u, v) and (v, u) occurs as an arc for each pair of distinct vertices $u, v \in V(G)$. The *skew-adjacency matrix* of G is the $n \times n$ matrix $A_S(G)$ whose (u, v)-entry is 1 if (u, v) is an arc of G, -1 if (v, u) is an arc of G, and 0 if there is no arc between u and v. Since $A_S(G)$ is skew-symmetric, its eigenvalues are all purely imaginary numbers, and the collection of them with multiplicities is called the *skew spectrum* of G. The *skew energy* $\mathcal{E}_S(G)$ of G is defined as the sum of the moduli of the eigenvalues of G. It is known that $\mathcal{E}_S(G) \le n\sqrt{\Delta}$, where Δ is the maximum degree of the underlying graph of G. In particular, the skew energy of any orientation of any k-regular graph of order n is at most $n\sqrt{k}$. In [293], it was shown that for any $d \ge 1$ there exists an orientation of the hypercube $H(d, 2)$ which achieves this bound, that is, with skew energy $2^d \sqrt{d}$. It was also proved [293] that there exists an orientation of $H(d, 2)$ whose skew spectrum is obtained from the spectrum of $H(d, 2)$ by multiplying each eigenvalue by the imaginary unit i. Both orientations were given algorithmically in [293].

6.7 Others

In [127], estimates of the energy and Estrada index of Cayley graphs $\mathrm{Cay}(G, S)$ were obtained, where G is D_{2n} or U_{6n} (see section 3.3.3) and S is a normal generating set of G.

A proper colouring of a graph G is an assignment of colours to its vertices such that adjacent vertices receive distinct colours. Let G be a graph and c a proper colouring of G. Define $A_c(G)$ to be the matrix with rows and columns indexed by the vertices of G such that the (u, v)-entry is equal to 1 if u and v are adjacent (so that $c(u) \ne c(v)$), -1 if u and v are non-adjacent and $c(u) = c(v)$, and 0 if $u = v$ or u and v are non-adjacent and $c(u) \ne c(v)$. The eigenvalues of $A_c(G)$ are called the *colour eigenvalues* of (G, c). The *colour energy* $E_c(G)$ of G with respect to c is the sum of the absolute values of the colour eigenvalues of (G, c). In [12], Adiga et al. derived

formulas for the colour energies of the unitary Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^\times)$ and its complement. They also obtained explicit formulas for the colour energies of the gcd graphs $\mathrm{ICG}(n, D)$ of $\mathbb{Z}_n$ in the case when $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{i-1}^{\alpha_{i-1}} p_i^1 \ p_{i+1}^{\alpha_{i+1}} \ldots p_s^{\alpha_s}$ and $D = \{1, p_i\}$ or $n = p_1 p_2 \ldots p_s$ is a square-free number and $D = \{p_i, p_j\}$.

7 Ramanujan Cayley graphs

The *isoperimetric number* (or Cheeger constant) of a graph G is defined as

$$h(G) = \min \left\{ \frac{|\partial(S)|}{|S|} : S \subset V(G), \ 0 < |S| \leq \frac{1}{2}|V(G)| \right\}$$

where $\partial(S)$ is the set of edges of G with one end-vertex in S and the other end-vertex in $V(G) \setminus S$. This important parameter measures how well the graph expands and how strong the graph is connected in some sense. An infinite sequence of finite regular graphs with fixed degree but increasing orders is called a family of expanders if the isoperimetric numbers of such graphs are all bounded from below by a common positive constant. A well-known result of Alon-Milman [18, 20] and Dodziuk [107] asserts that, for any connected k-regular graph G,

$$\frac{k - \lambda_2}{2} \leq h(G) \leq \sqrt{2k(k - \lambda_2)},$$

where λ_2 is the second largest eigenvalue of G. So we may also define expander families using the *spectral gap* $k - \lambda_2$: A sequence of graphs $G_1, G_2, G_3, \ldots$ is called a family of ε-*expander graphs* [150], where $\varepsilon > 0$ is a fixed constant, if (i) all these graphs are k-regular for a fixed integer $k \geq 3$; (ii) $k - \lambda_2(G_i) \geq \varepsilon$ for each i; and (iii) $|V(G_i)| \to \infty$ as $i \to \infty$.

The well known Alon-Boppana bound (see, for example, [94, Theorem 0.8.8]) asserts that, for any family of finite connected k-regular graphs $\{G_i\}_{i \geq 1}$ such that $|V(G_i)| \to \infty$ as $i \to \infty$, we have $\liminf_{i \to \infty} \lambda(G_i) \geq 2\sqrt{k-1}$, where $\lambda(G_i)$ is the maximum in absolute value of an eigenvalue of G_i other than $\pm k$. This motivated the following definition: A finite k-regular graph G is called *Ramanujan* [150, 252] if $\lambda(G) \leq 2\sqrt{k-1}$.

Over the years a significant amount of work has been done in constructing expander families as well as Ramanujan graphs of a fixed degree, and Cayley graphs play an important role in this area of research. The reader is referred to three survey papers [252, 150, 222] and two books [94, 184] on expanders and Ramanujan graphs.

In this section we give an account of known results on the characterization of Ramanujan Cayley graphs. Note that, since abelian groups can never yield expander families (see [184, Chapter 4]), all Ramanujan Cayley graphs on abelian groups surveyed in this section cannot form expander families.

7.1 Ramanujan integral circulant graphs

In [108], Droll gave the following characterization of Ramanujan unitary Cayley graphs.

Theorem 7.1. ([108, Theorem 1.2]) *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s} \geq 2$ with $p_1 < p_2 < \ldots < p_s$ be an integer in canonical factorization. The unitary Cayley graph $G_{\mathbb{Z}_n}$ is Ramanujan if and only if one of the following holds:*

(a) $n = 2^{\alpha_1}$ with $\alpha_1 \geq 1$;

(b) $n = p_1^{\alpha_1}$ with p_1 odd and $\alpha_1 = 1, 2$;

(c) $n = 4p_2p_3$ with $p_2 < p_3 \leq 2p_2 - 3$;

(d) $n = p_1p_2$ with $3 \leq p_1 < p_2 \leq 4p_1 - 5$, or $n = 2p_2p_3$ with $3 \leq p_2 < p_3 \leq 4p_2 - 5$;

(e) $n = 2p_2^2$, $4p_2^2$ with p_2 odd, or $n = 2^{\alpha_1}p_2$ with $p_2 > 2^{\alpha_1 - 3} + 1$.

Since $G_{\mathbb{Z}_n} = \mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^{\times}) = \mathrm{ICG}(n, \{1\})$ is a special unitary Cayley graph as well as a special integral circulant graph, it is natural to ask the following questions: Besides the graphs in Theorem 7.1, which unitary Cayley graphs of finite commutative rings are Ramanujan? And which integral circulant graphs $\mathrm{ICG}(n, D)$ are Ramanujan? The former question was answered by Liu and Zhou in [214], and their result will be presented in Theorem 7.11. The latter question was studied by Le and Sander in [198] when n is a prime power but D is arbitrary, and by Sander in [269] when D is multiplicative but n is arbitrary. Set

$$D(n; m) := \{d \in D(n) : d \geq m\}$$

for any positive integers m, n with $m \leq n$.

Theorem 7.2. ([198, Theorem 1.1]) *Let $p^s \geq 3$ be a prime power and $D \subseteq D(p^s) \setminus \{p^s\}$. Then $\mathrm{ICG}(p^s, D)$ is Ramanujan if and only if one of the following holds:*

(a) $D = D\left(p^{\lceil \frac{s}{2} \rceil - 1}\right) \cup D'$ for some $D' \subseteq D\left(p^{s-1}; p^{\lceil \frac{s}{2} \rceil}\right)$;

(b) $D = \{1\}$ when $p = 2$ and $s \geq 3$;

(c) $D = D\left(p^{\frac{s-3}{2}}\right) \cup D'$ such that $|D| \geq 2$ for some $D' \subseteq D\left(p^{s-1}; p^{\frac{s+1}{2}}\right)$ when $p \in \{2, 3\}$ and $s \geq 3$ is odd;

(d) $D = D\left(2^{\frac{s-4}{2}}\right) \cup D'$ for some $D' \subseteq D\left(2^{s-1}; 2^{\frac{s}{2}}\right)$ satisfying $\emptyset \neq D' \neq \{2^{s-1}\}$ when $p = 2$ and $s \geq 4$ is even;

(e) $D = \{1, 2^2, 2^3, 2^4\}$ when $p = 2$ and $s = 5$;

(f) $D = D\left(5^{\frac{s-3}{2}}\right) \cup \left\{5^{\frac{s+1}{2}}\right\} \cup D'$ for some $D' \subseteq D\left(5^{s-1}; 5^{\frac{s+3}{2}}\right)$ when $p = 5$ and $s \geq 5$ is odd;

(g) $D = D\left(2^{\frac{s-5}{2}}\right) \cup \left\{2^{\frac{s-1}{2}}\right\} \cup D'$ for some $D' \subseteq D\left(2^{s-1}; 2^{\frac{s+1}{2}}\right)$ satisfying

$$3 - 2\sqrt{2} + \frac{1}{2^{\frac{s-3}{2}}} \leq 2^{\frac{s-1}{2}} \sum_{d' \in D'} \frac{1}{d'}$$

when $p = 2$ and $s \geq 5$ is odd.

Part (a) of Theorem 7.2 implies the following result.

Corollary 7.3. ([198, Corollary 1.1]) *Let $p^s \geq 3$ be a prime power and let $D = \{1, p, \ldots, p^{r-1}\}$, where $s/2 \leq r \leq s$. Then $\mathrm{ICG}(p^s, D)$ is Ramanujan. In particular, there is a Ramanujan integral circulant graph $\mathrm{ICG}(p^s, D)$ for every prime power $p^s \geq 3$.*

The next four results were proved by Sander in [269].

Theorem 7.4. ([269, Theorem 2.1]) $\mathrm{ICG}(n, D(n) \setminus \{n\})$ *is Ramanujan for every integer* $n \geq 3$.

Recall from (6.4) that $\Phi(n, D) = \sum_{d \in D} \varphi\left(\frac{n}{d}\right)$ for $D \subseteq D(n)$.

Theorem 7.5. ([269, Theorem 2.2]) *Let* $n \geq 3$ *be an integer and let* $D \subseteq D(n) \setminus \{n\}$ *be multiplicative. Write* $D = \prod_{p \in \mathbb{P}, p|n} D^*(p)$ *for suitable* $D^*(p) \subseteq D(p^{e_p(n)})$. *Then* $\mathrm{ICG}(n, D)$ *is Ramanujan if and only if* $1 \in D$ *and*

$$\max_{p|n, p \in \mathbb{P}}^* \frac{p^{e_p(n)}}{\Phi(p^{e_p(n)}, D^*(p))} \leq 1 + \frac{2\sqrt{\Phi(n, D) - 1}}{\Phi(n, D)},$$

where the operator $\max^*$ *is defined by*

$$\max_{p|n, p \in \mathbb{P}}^* := \begin{cases} \max_{p|n, p \in \mathbb{P}}, & \text{if } 4|n \text{ and } D^*(2) \neq \{1\}, \\ \max_{p|n, p \in \mathbb{P}, p>2}, & \text{otherwise.} \end{cases}$$

Theorem 7.6. ([269, Theorem 2.3]) *For every even integer* $n \geq 4$, *there exists a multiplicative divisor set* $D \subseteq D(n) \setminus \{n\}$ *such that* $\mathrm{ICG}(n, D)$ *is a Ramanujan graph.*

Theorem 7.7. ([269, Theorem 2.4]) *Let* $n \geq 3$ *be an odd integer.*

(a) *If* n *satisfies* $\max_{p \in \mathbb{P}, \, p|n} p^{e_p(n)} \geq \frac{n^{3/2} + n}{2(n-1)}$, *then there is a multiplicative divisor set* $D \subseteq D(n) \setminus \{n\}$ *such that* $\mathrm{ICG}(n, D)$ *is a Ramanujan graph.*

(b) *If* $n > 8295$ *satisfies* $\max_{p \in \mathbb{P}, \, p|n} p^{e_p(n)} < \frac{n^{3/2} + n}{2(n-1)}$, *then there is no multiplicative divisor set* $D \subseteq D(n) \setminus \{n\}$ *such that* $\mathrm{ICG}(n, D)$ *is a Ramanujan graph.*

Recently, Liu and Li [212] considered some distance powers of the unitary Cayley graph $G_{\mathbb{Z}_n}$ and determined when they are Ramanujan.

Theorem 7.8. ([212, Theorem 1.2]) *Let* $n \geq 2$ *be an integer.*

(a) $G_{\mathbb{Z}_n}^{\{1\}}$ *is Ramanujan if and only if one of the following holds: (i)* n *is a prime; (ii)* $n = 2^\alpha$ *for some* $\alpha \geq 1$; *(iii)* $n = p^2$ *for an odd prime* p; *(iv)* $n = pq$ *for some primes* p, q *with* $3 \leq p < q \leq 4p - 5$; *(v)* $n = 2pq$ *for some primes* p, q *with* $3 \leq p < q \leq 4p - 5$; *(vi)* $n = 4pq$ *for some primes* p, q *with* $p < q \leq 2p - 3$; *(vii)* $n = 2p^2$ *or* $4p^2$ *for an odd prime* p, *or* $n = 2^\alpha p$ *for some integer* α *and prime* $p > 2^{\alpha-3} + 1$.

(b) $G_{\mathbb{Z}_n}^{\{2\}}$ *is Ramanujan if and only if one of the following holds: (i)* $n = 2^\alpha$ *for some* $\alpha \geq 1$; *(ii)* $n = p^\alpha$ *for some odd prime* p *and integer* $\alpha \geq 2$; *(iii)* $n = 15, 21$ *or* 35; *(iv)* n *is even but has an odd prime divisor.*

(c) $G_{\mathbb{Z}_n}^{\{3\}}$ *is Ramanujan if and only if* $n = 30, 70$ *or* $4p$ *for a prime* p.

(d) $G_{\mathbb{Z}_n}^{\{1,2\}}$ *is Ramanujan if and only if one of the following holds: (i)* $n = 2^\alpha$ *for some* $\alpha \geq 1$; *(ii)* $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ *for* $s \geq 2$ *odd primes* $p_1, p_2, \ldots, p_s$ *and positive integers* $\alpha_1, \alpha_2, \ldots, \alpha_s$; *(iii)* $n = 2^\alpha p$ *for some* α *and odd prime* $p \geq 2^{\alpha-4} + 1$; *(iv)* $n = 2p^2, 4p^2$ *or* $8p^2$ *for some odd prime* p; *(v)* $n = 54, 250$ *or* 686; *(vi)* $n = 2^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ *with* $s \geq 3$ *for odd primes* $p_2, \ldots, p_s$ *and positive integers* $\alpha_1, \alpha_2, \ldots, \alpha_s$ *such that* $\varphi(n) \geq n/2 + 1 - 2\sqrt{n-2}$.

(e) $G_{\mathbb{Z}_n}^{\{2,3\}}$ is Ramanujan if and only if $n = 6$, 10, 12, 18, 24 or 30.

(f) $G_{\mathbb{Z}_n}^{\{1,3\}}$ is Ramanujan.

(g) $G_{\mathbb{Z}_n}^{\{1,2,3\}}$ is Ramanujan.

7.2 Ramanujan Cayley graphs on dihedral groups

Recall from (3.1) that $S_n(1) = \{k : 1 \le k \le n-1, \gcd(k,n) = 1\}$. Recall also that $D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ is the dihedral group of order $2n$. The following result due to Liu and Zhou identifies all Ramanujan graphs in two special families of Cayley graphs on dihedral groups.

Theorem 7.9. ([213]) *Let $n \ge 2$ be an integer.*

(a) *Let $S = \{a^k, a^k b : k \in S_n(1)\}$. Then $\mathrm{Cay}(D_{2n}, S)$ is Ramanujan if and only if one of the following holds: (i) $n = 2^\alpha$ for some integer $\alpha \ge 1$; (ii) $n = pq$ or $2pq$ for some primes p, q with $p < q \le 2p - 3$; (iii) $n = p^2$ or $2p^2$ for some prime p; (iv) $n = p$ for some prime p; (v) $n = 2^\alpha p$ for some integer $\alpha \ge 1$ and prime p with $p > 2^{\alpha-2} + 1$.*

(b) *Let $S = \{a^k : k \in S_n(1)\} \cup \{a^j b\}$ for some $j \in \mathbb{Z}_n$. Then $\mathrm{Cay}(D_{2n}, S)$ is Ramanujan if and only if $n \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}$.*

7.3 Ramanujan Cayley graphs on finite commutative rings

Recall that G_R is the unitary Cayley graph of a finite commutative ring R. In [214], Liu and Zhou determined when G_R or its complement $\overline{G}_R$ is Ramanujan using knowledge of the spectra of these graphs (see section 5.1).

Theorem 7.10. ([214, Theorem 11]) *Let R be a finite local ring with maximal ideal M of order m. Then G_R is Ramanujan if and only if either $|R| = 2m$ or $|R| \ge \left(\dfrac{m}{2} + 1\right)^2$ and $m \ne 2$.*

Theorem 7.11. ([214, Theorem 12]) *Let R be as in Assumption 5.1 with $s \ge 2$. Then G_R is Ramanujan if and only if R satisfies one of the following conditions:*

(a) $R_i/M_i \cong \mathbb{F}_2$ *for $i = 1, 2, \ldots, s$;*

(b) $R_i \cong \mathbb{F}_2$ *for $i = 1, 2, \ldots, s-3$, and $R_i \cong \mathbb{F}_3$ for $i = s-2, s-1, s$;*

(c) $R_i \cong \mathbb{F}_2$ *for $i = 1, 2, \ldots, s-3$, $R_i \cong \mathbb{F}_3$ for $i = s-2, s-1$, and $R_s \cong \mathbb{F}_4$;*

(d) $R_i \cong \mathbb{F}_2$ *for $i = 1, 2, \ldots, s-3$, and $R_i \cong \mathbb{F}_4$ for $i = s-2, s-1, s$;*

(e) $R_i \cong \mathbb{F}_2$ *for $i = 1, 2, \ldots, s-2$, $R_{s-1} \cong \mathbb{F}_3$, and $R_s \cong \mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$;*

(f) $R_1 \cong \mathbb{Z}_4$ *or $\mathbb{Z}_2[X]/(X^2)$, $R_i \cong \mathbb{F}_2$ for $i = 2, 3, \ldots, s-2$, and $R_{s-1} \cong \mathbb{F}_{q_1}$ and $R_s \cong \mathbb{F}_{q_2}$ for some prime powers $q_1, q_2 \ge 3$ such that $q_1 \le q_2 \le q_1 + \sqrt{(q_1 - 2)q_1}$;*

(g) $R_i \cong \mathbb{F}_2$ *for $i = 1, 2, \ldots, s-2$, and $R_{s-1} \cong \mathbb{F}_{q_1}$ and $R_s \cong \mathbb{F}_{q_2}$ for some prime powers $q_1, q_2 \ge 3$ such that $q_1 \le q_2 \le 2\left(q_1 + \sqrt{(q_1 - 2)q_1}\right) - 1$;*

(h) $R_i/M_i \cong \mathbb{F}_2$ for $i = 1, 2, \ldots, s - 1$, $R_s/M_s \cong \mathbb{F}_q$ for some prime power $q \geq 3$, and $\prod_{i=1}^{s} m_i \leq 2 \left(q - 1 + \sqrt{(q-2)q} \right)$.

In the special case when $R = \mathbb{Z}_n$, Theorem 7.11 gives rise to Theorem 7.1.

Theorem 7.12. ([214, Theorem 15]) *Let R be a finite local ring. Then $\overline{G}_R$ is Ramanujan.*

Theorem 7.13. ([214, Theorem 16]) *Let R be as in Assumption 5.1 with $s \geq 2$. Then $\overline{G}_R$ is Ramanujan if and only if R satisfies one of the following conditions:*

(a) $|R_i|/m_i = 2$ for $i = 1, 2, \ldots, s$, and $\prod_{i=1}^{s} m_i \leq 2^{s+1} - 3 + 2\sqrt{2^s(2^s - 3)}$;

(b) $2 = |R_1|/m_1 = \cdots = |R_t|/m_t < |R_{t+1}|/m_{t+1}$ for some t with $2 \leq t < s$, and $|R^\times| \leq 2\sqrt{|R|} - 3$;

(c) $2 = |R_1|/m_1 < |R_2|/m_2$ and $|R^\times| \leq 2\sqrt{|R| - 2} - 1$;

(d) $3 \leq |R_1|/m_1$ and $\frac{|R^\times|}{(|R_1|/m_1) - 1} \leq -(2(|R_1|/m_1) - 3) + \sqrt{(2(|R_1|/m_1) - 3)^2 + (4|R| - 9)}$.

Applying Theorems 7.12 and 7.13 to $\mathbb{Z}_n$, we obtain the following result on the complement of the unitary Cayley graph $G_{\mathbb{Z}_n}$ of $\mathbb{Z}_n$.

Corollary 7.14. ([214, Corollary 17]) *Let $n \geq 2$ be an integer. Then $\overline{G}_{\mathbb{Z}_n}$ is Ramanujan if and only if either n is a prime power or $n \in \{6, 10, 12, 15, 18, 21, 24, 30, 35\}$.*

Ramanujan quadratic unitary Cayley graphs were characterized by Liu and Zhou in [215] using knowledge of the spectra of quadratic unitary Cayley graphs (see section 5.2).

Theorem 7.15. ([215, Theorem 5.1]) *Let R be as in Assumption 5.1 such that $|R_i|/m_i \equiv 1 \pmod 4$ for $1 \leq i \leq s$, and let R_0 be a local ring with maximal ideal M_0 of order m_0 such that $|R_0|/m_0 \equiv 3 \pmod 4$. Then the following hold:*

(a) $\mathcal{G}_{R_0}$ *is Ramanujan if and only if* $|R_0| \geq (m_0 + 2)^2/4$;

(b) $\mathcal{G}_R$ *is Ramanujan if and only if R is isomorphic to $\mathbb{F}_5 \times \mathbb{F}_5$ or $\mathbb{F}_q$ for a prime power $q \equiv 1 \pmod 4$;*

(c) $\mathcal{G}_{R_0 \times R}$ *is Ramanujan if and only if $R_0 \times R$ is isomorphic to $\mathbb{F}_3 \times \mathbb{F}_5$, $\mathbb{F}_3 \times \mathbb{F}_9$ or $\mathbb{F}_3 \times \mathbb{F}_{13}$.*

In the special case where $R = \mathbb{Z}_n$, Theorem 7.15 yields the following result.

Corollary 7.16. ([215, Corollary 5.2]) *Let $n \geq 5$ be an integer such that each of its prime divisors is congruent to 1 modulo 4. Let $p \equiv 3 \pmod 4$ be a prime and $\alpha \geq 1$ an integer. Then the following hold:*

(a) $\mathcal{G}_{\mathbb{Z}_n}$ *is Ramanujan if and only if n is a prime;*

(b) $\mathcal{G}_{\mathbb{Z}_{p^\alpha}}$ *is Ramanujan if and only if $p^\alpha = p$ or p^2;*

(c) $\mathcal{G}_{\mathbb{Z}_{np^\alpha}}$ *is Ramanujan if and only if $np^\alpha = 15$ or 39.*

7.4 Ramanujan Euclidean graphs

Let $q = p^r$ be an odd prime power and $n \geq 1$ an integer. Define

$$d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} (x_i - y_i)^2$$

for (column) vectors $\mathbf{x}, \mathbf{y}$ in the linear space $\mathbb{F}_q^n$. Note that this is not a metric in the sense of analysis as $d(\mathbf{x}, \mathbf{y})$ is not real-valued but takes values in $\mathbb{F}_q$. Given $a \in \mathbb{F}_q$, the *Euclidean graph* $E_q(n, a)$, introduced by Medrano et al. in [235], is defined to be the graph with vertex set $\mathbb{F}_q^n$ such that two vertices $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n$ are adjacent if and only if $d(\mathbf{x}, \mathbf{y}) = a$. (Loops are allowed when $a = 0$.) Assume that $(q, n, a) \neq (q, 2, 0)$ and -1 is not a square in $\mathbb{F}_q$. Then

$$E_q(n, a) = \mathrm{Cay}(\mathbb{F}_q^n, S_q(n, a))$$

is a connected Cayley graph on the additive group of $\mathbb{F}_q^n$, where

$$S_q(n, a) = \{\mathbf{x} \in \mathbb{F}_q^n : d(\mathbf{x}, \mathbf{0}) = a\}.$$

As shown in [235, Theorem 1], the degree $|S_q(n, a)|$ of this graph is approximately q^{n-1}, with an error depending on the value of the quadratic character at $(-1)^{(n-1)/2} a$ or $(-1)^{n/2}$ and the parity of n. Define

$$e_{\mathbf{b}}(\mathbf{x}) = \exp\{2\pi i \mathrm{Tr}(\mathbf{b}^t \mathbf{x})/p\}, \ \mathbf{b}, \mathbf{x} \in \mathbb{F}_q^n,$$

where $\mathbf{b}^t$ is the transpose of $\mathbf{b}$ and

$$\mathrm{Tr}(u) = \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(u) = u + u^p + \cdots + u^{p^{r-1}}$$

is the trace of $u \in \mathbb{F}_q$. As shown in [235, Proposition 2], the eigenvalues of $E_q(n, a)$ are given by

$$\lambda_{\mathbf{b}} = \sum_{\mathbf{y} \in \mathbb{F}_q^n, \ d(\mathbf{y}, \mathbf{0})=1} e_{\mathbf{b}}(\mathbf{y}), \ \mathbf{b} \in \mathbb{F}_q^n,$$

and moreover $e_{\mathbf{b}}$ is an eigenfunction corresponding to $\lambda_{\mathbf{b}}$. Based on this the following was proved by Medrano et al. in [235].

Theorem 7.17. ([235, Theorem 3]) *The eigenvalues $\lambda_{\mathbf{b}}$ of $E_q(n, a)$ for $\mathbf{b} \in \mathbb{F}_q^n \setminus \{\mathbf{0}\}$ satisfy*

$$|\lambda_{\mathbf{b}}| \leq 2q^{(n-1)/2}$$

and can be expressed as some generalized Kloosterman sums.

Since the degree of $E_q(n, a)$ is approximately q^{n-1}, the bound in Theorem 7.17 is asymptotic to the Ramanujan bound as q approaches infinity. So Theorem 7.17 implies that $E_q(n, a)$ is asymptotically Ramanujan for large q (see also [290, Theorem 1, Chapter 5]).

It was proved in [235, Proposition 4] that, for fixed q and n, all graphs $E_q(n, a)$ for squares $a \neq 0$ are isomorphic, and all graphs $E_q(n, a)$ for non-squares $a \neq 0$ are isomorphic. It was further proved in [235, Theorem 5] that when n is even these graphs for $a \neq 0$ are isomorphic to each other, and so there are exactly two Euclidean graphs up to isomorphism, namely $E_q(n, 0)$ and $E_q(n, 1)$ for even n.

Similar to $E_q(n, a)$, for an odd prime power q and a positive integer n, the *Euclidean graph* associated with $\mathbb{Z}_q^n$ and $a \in \mathbb{Z}_q$ is defined [236] to be the Cayley graph

$$X_q(n, a) = \text{Cay}(\mathbb{Z}_q^n, S_q'(n, a)),$$

where

$$S_q'(n, a) = \left\{ \mathbf{x} \in \mathbb{Z}_q^n : \sum_{i=1}^n x_i^2 = a \right\}.$$

In [236], Medrano et al. determined the spectra and Ramanujancy of $X_q(n, a)$. See also [100] and [298]. The following result is a combination of [236, Theorem 2.5], [100, Theorem 11] and [298, Theorems 8 and 10].

Theorem 7.18. ([100, 236, 298]) *Let $r, d \geq 2$ and $a \in \mathbb{Z}$ be integers, and let p be a prime not dividing a. Then $X_{p^r}(n, a)$ is not Ramanujan except when $(p, r, n) = (2, 2, 2), (2, 2, 3)$ or $(3, 2, 2)$.*

In 2009, Bannai et al. [43] introduced finite Euclidean graphs in a more general setting. Let q be a prime power, $n \geq 1$ an integer, a an elment of $\mathbb{F}_q$, and Q a non-degenerate quadratic form on $\mathbb{F}_q^n$. Define $E_q(n, Q, a)$ to be the graph with vertex set $\mathbb{F}_q^n$ in which distinct $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n$ are adjacent if and only if $Q(\mathbf{x} - \mathbf{y}) = a$. In the special case where $Q(\mathbf{x}) = \sum_{i=1}^n x_i^2$, $E_q(n, Q, a)$ is exactly $E_q(n, a)$.

Consider a non-degenerate quadratic form Q on $V = V_n(q) = \mathbb{F}_q^n$. The *orthogonal group* associated with Q is the group of all linear transformations that fix Q; that is,

$$O(V, Q) = \{\sigma \in \text{GL}(V) : Q(\sigma(\mathbf{x})) = Q(\mathbf{x}) \text{ for all } \mathbf{x} \in V\}.$$

Natually, the semi-direct product of V (translations) by $O(V, Q)$, $V \rtimes O(V, Q)$, acts transitively on V. Let $\mathcal{O}_0 = \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in V\}, \mathcal{O}_1, \ldots, \mathcal{O}_d$ be the orbital of $V \rtimes O(V, Q)$ on V. Treating $\mathcal{O}_i$ as a relation on V, we obtain an association scheme $\mathcal{X}(O(V, Q), V) = (V, \{\mathcal{O}_i\}_{0 \leq i \leq d})$.

Let ρ be a primitive element of $\mathbb{F}_q$. If $n = 2m$ is even, then there are two inequivalent non-degenerate quadratic forms, denoted by Q^+ and Q^-, respectively. The corresponding orthogonal groups are $\text{GO}_{2m}^+(q) = O(V, Q^+)$ and $\text{GO}_{2m}^-(q) = O(V, Q^-)$. So we have the Euclidean graphs $E_q(2m, Q^+, \rho^i)$ and $E_q(2m, Q^-, \rho^i)$ for $1 \leq i \leq q$. The arc sets of $E_q(2m, Q^+, 0)$ and $E_q(2m, Q^+, \rho^i)$ $(1 \leq i \leq q-1)$, together with the identity relation $R_0 = \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in V_{2m}(q)\}$, are the relations of $\mathcal{X}(\text{GO}_{2m}^+(q), V_{2m}(q))$. Similarly, the arc sets of $E_q(2m, Q^-, 0)$ and $E_q(2m, Q^-, \rho^i)$ $(1 \leq i \leq q-1)$ together with R_0 are the relations of $\mathcal{X}(\text{GO}_{2m}^-(q), V_{2m}(q))$.

If $n = 2m + 1$ and q is odd, then there are two inequivalent non-degenerate quadratic forms Q and Q', but the groups $O(V, Q)$ and $O(V, Q')$ are isomorphic. This group is denoted by $\text{GO}_{2m+1}(q)$. In this case we have the Euclidean graphs $E_q(2m+1, Q, \rho^{2i})$ and $E_q(2m+1, Q, \rho^{2i-1})$ for $1 \leq i \leq (q-1)/2$, as well as $E_q(2m+1, Q, 0)$, whose arc sets together with the identity relation are the relations of $\mathcal{X}(\text{GO}_{2m+1}(q), V_{2m+1}(q))$.

If $n = 2m+1$ and q is even, then there is exactly one inequivalent non-degenerate quadratic form Q. In this case we have the Euclidean graphs $E_q(2m+1, Q, 0)$ and $E_q(2m+1, Q, \rho^i)$ for $1 \leq i \leq q-1$, and the arc set of $E_q(2m+1, Q, \rho^i)$ is the union of two relations R_i and R_{q+i} of $\mathcal{X}(\text{GO}_{2m+1}(q), V_{2m+1}(q))$, where $\text{GO}_{2m+1}(q) = O(V, Q)$.

The next four theorems were proved by Bannai et al. in [43].

Theorem 7.19. ([43, Theorem 3.1]) *The graphs $E_q(2m, Q^-, \rho^i)$ $(1 \leq i \leq q-1)$ are Ramanujan.*

Theorem 7.20. ([43, Theorem 3.2]) *The graphs $E_q(2m, Q^+, \rho^i)$ ($1 \le i \le q-1$) are Ramanujan if m is sufficiently large (that is, larger than a certain integer determined by q).*

Theorem 7.21. ([43, Theorem 3.3]) *Let q be an odd prime power.*

(a) *The graphs $E_q(2m+1, Q, \rho^{2i})$ ($1 \le i \le (q-1)/2$) and $E_q(2m+1, Q, 0)$ are Ramanujan.*

(b) *If q is a prime, then the graphs $E_q(2m+1, Q, \rho^{2i-1})$ ($1 \le i \le (q-1)/2$) are Ramanujan if m is sufficiently large (that is, larger than a certain integer determined by q).*

(c) *If q is not a prime, then the graphs $E_q(2m+1, Q, \rho^{2i-1})$ ($1 \le i \le (q-1)/2$) are not Ramanujan.*

Theorem 7.22. ([43, Theorem 3.4]) *Let q be an even prime power.*

(a) *The graphs (V, R_{q+i}) ($1 \le i \le q-1$) and $E_q(2m+1, Q, 0)$ are Ramanujan unless $q = 2$ and $m = 1$.*

(b) *For $a = \rho^i \ne 0$, the graph $E_q(2m+1, Q, a) = (V, R_i \cup R_{q+i})$ is Ramanujan if and only if $m \ge 1$.*

In a companion of [43], Bannai et al. [42] constructed many other Ramanujan graphs from association schemes obtained from the following actions of orthogonal groups over finite fields: (i) $O_{2m+1}(q)$ (q odd) acting on the set of non-square-type non-isotropic 1-dimensional subspaces of $V_{2m+1}(q)$ with respect to the non-degenerate quadratic form $Q(\mathbf{x}) = 2(x_1 x_{m+1} + \cdots + x_m x_{2m}) + x_{2m+1}^2$; (ii) $O_{2m+1}(q)$ (q odd) acting on the set of square-type non-isotropic 1-dimensional subspaces of $V_{2m+1}(q)$ with respect to the same quadratic form; (iii) $O_{2m+1}(q)$ (q even) acting on the set of negative-type hyperplanes of $V_{2m+1}(q)$ with respect to the non-degenerate quadratic form $Q(\mathbf{x}) = x_1 x_{m+1} + \cdots + x_m x_{2m} + x_{2m+1}^2$; (iv) $O_{2m+1}(q)$ (q even) acting on the set of positive-type hyperplanes of $V_{2m+1}(q)$ with respect to the same quadratic form; (v) $O_{2m}^{\pm}(q)$ acting on the set of non-isotropic points of $V_{2m}(q)$; (vi) $U_n(q)$ acting on the set of 1-dimensional non-isotropic subspaces of $V_n(q^2)$ with respect to the canonical non-singular Hermitian form. The reader is referred to [42] for details.

7.5 Ramanujan finite upper half plane graphs

Let $q = p^r$ be an odd prime power and δ a non-square of the finite field $\mathbb{F}_q$. The *finite upper half plane* over $\mathbb{F}_q$ is defined as

$$\mathbf{H}_q = \{z = x + y\sqrt{\delta} : x, y \in \mathbb{F}_q, \ y \ne 0\}.$$

For $z = x + y\sqrt{\delta} \in \mathbf{H}_q$ and $w = u + v\sqrt{\delta} \in \mathbf{H}_q$, define

$$d(z, w) = \frac{(x - u)^2 - \delta(y - v)^2}{yv}.$$

Let $a \in \mathbf{H}_q$. Define $X_q(\delta, a)$ to be the graph with vertex set $\mathbf{H}_q$ such that $z, w \in \mathbf{H}_q$ are adjacent if and only if $d(z, w) = a$. This graph is called [290] a *finite upper half plane graph* on $\mathbb{F}_q$. As shown in [290, Theorem 2(4), Chapter 19], this is a Cayley graph on the affine group

$$\mathrm{Aff}(q) = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : x, y \in \mathbb{F}_q, \ y \ne 0 \right\},$$

namely,

$$X_q(\delta, a) \cong \mathrm{Cay}(\mathrm{Aff}(q), S_q(\delta, a)),$$

where

$$S_q(\delta, a) = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : x, y \in \mathbb{F}_q, \ y \neq 0, \ x^2 = ay + \delta(y-1)^2 \right\}.$$

Theorem 7.23. ([290, Theorem 4, Chapter 20]) *The finite upper half plane graphs $X_q(\delta, a)$, for $a \neq 0, 4\delta$, are Ramanujan.*

The reader is referred to [75] and [290, Theorem 2, Chapter 19] for more properties of $X_q(\delta, a)$ (mostly due to Angel [30, 26], Angel et al. [27, 28], Celniker [71], Celniker et al. [72], Poulos [259], and Terras [288, 289]).

Let $q = p^r$ be an odd prime power and δ a positive integral generator of the group $\mathbb{Z}_q^\times$. The *finite upper half plane* over $\mathbb{Z}_q$ is defined as

$$\mathbf{H}_q' = \{z = x + y\sqrt{\delta} : x \in \mathbb{Z}_q, \ y \in \mathbb{Z}_q^\times\}.$$

For $z = x + y\sqrt{\delta} \in \mathbf{H}_q'$ and $w = u + v\sqrt{\delta} \in \mathbf{H}_q'$, define

$$d'(z, w) = \frac{(x-u)^2 - \delta(y-v)^2}{yv}.$$

Fix $a \in \mathbb{Z}_q$. In [29], Angel et al. defined the *finite upper half plane graph* $X_q'(\delta, a)$ on $\mathbb{Z}_q$ to be the graph with vertex set $\mathbf{H}_q'$ such that $z, w \in \mathbf{H}_q'$ are adjacent if and only if $d'(z, w) = a$. This graph was proved (see [29, Theorem 2] or [30, Proposition 3.2]) to be a Cayley graph on the affine group

$$\mathrm{Aff}(\mathbb{Z}_q) = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{Z}_q, \ y \in \mathbb{Z}_q^\times \right\},$$

namely,

$$X_q'(\delta, a) \cong \mathrm{Cay}(\mathrm{Aff}(\mathbb{Z}_q), S_{\mathbb{Z}_q}(\delta, a)),$$

where

$$S_{\mathbb{Z}_q}(\delta, a) = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{Z}_q, \ y \in \mathbb{Z}_q^\times, \ x + y\sqrt{\delta} \in \mathbb{Z}_q, \ d'\left(x + y\sqrt{\delta}, \sqrt{\delta}\right) = a \right\}.$$

If $a \not\equiv 0, 4\delta \bmod p$, then $X_q'(\delta, a)$ is a connected $(p^r + p^{r-1})$-regular graph with $p^r(p^r - p^{r-1})$ vertices (see [29, Theorems 1 and 2]). The following result was proved by Angel et al. in [29].

Theorem 7.24. ([29, Theorem 6]) *Let p be an odd prime, δ a generator of $\mathbb{Z}_{p^2}^\times$, and $a \in \mathbb{Z}_{p^2}$. If $p \geq 5$, then $X_{p^2}'(\delta, a)$ is not Ramunujan; if $p = 3$, then $X_{p^2}'(\delta, a)$ is Ramanujan.*

The following result due to Bell and Minei [51] gives a class of Ramanujan Cayley graphs on $\mathrm{Aff}(\mathbb{Z}_p)$.

Theorem 7.25. ([51, Theorem 13]) *Let p be a prime and let a and d be integers with $\gcd(p-1, d) = 1$ and $\gcd(p-1, d-1) = 2$. Let g be a primitive root of p. Set*

$$S = \left\{ \begin{pmatrix} i^d & g^a i \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i^d & -g^a i^{d-1} \\ 0 & 1 \end{pmatrix} : i = 1, 2, \dots, p-1 \right\}.$$

Then $\mathrm{Cay}(\mathrm{Aff}(\mathbb{Z}_p), S)$ is a Ramanujan graph.

Let $p \geq 5$ be an odd prime. Consider the following subgroup of $\mathrm{GL}(3, p)$:

$$\Gamma' = \left\{ \begin{pmatrix} y & x & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : x, z \in \mathbb{F}_p,\ y \in \mathbb{F}_p^\times \right\}.$$

In [17], Allen studied the Cayley graph $X_p(\delta, a, c)$ on Γ' with connection set

$$S_p(\delta, a, c) = \left\{ \begin{pmatrix} y & x & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma' : x, z \in \mathbb{F}_p,\ y, \delta, a, c \in \mathbb{F}_p^\times,\ x^2 + cz^2 = ay + \delta(y-1)^2 \right\}.$$

It was proved [17] that this is a connected non-bipartite graph with girth 3 or 4 and degree $p^2 - p + p \left(\frac{-c}{p} \right) \left(\frac{a(a-4\delta)}{p} \right) + \left(\frac{-c}{p} \right)$, where $\left(\frac{x}{y} \right)$ denotes the Legendre symbol. Allen [17] made the following conjectures:

Conjecture 7.26. ([17, Conjecture 1]) *If* $\left(\frac{-c}{p} \right) = \left(\frac{a(a-4\delta)}{p} \right) = 1$, *then* $X_p(\delta, a, c)$ *is Ramanujan.*

Conjecture 7.27. ([17, Conjecture 2]) *If* $a = 4\delta$, *then* $X_p(\delta, a, c)$ *is Ramanujan.*

Conjecture 7.26 has been confirmed by Allen for all odd primes less than 225 (for a total of over 2000 graphs).

In a similar fashion, Allen also studied a family of Cayley graphs on $\mathrm{GL}(n, p)$ for $n \geq 4$. In general, these graphs are not Ramanujan as shown in [17, Section 3].

7.6 Heisenberg graphs

The *Heisenberg group* $H(R)$ over a ring R is the multiplicative group of 3×3 upper triangular matrices with entries in R and ones on the diagonal. Denote such a matrix $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ by (x, y, z). Call

$$G(p^n) = \mathrm{Cay}(H(\mathbb{Z}_{p^n}), \{(\pm 1, 0, 0), (0, \pm 1, 0)\})$$

and

$$G(2^n)' = \mathrm{Cay}(H(\mathbb{Z}_{2^n}), \{(\pm 1, 0, 0), (1, 1, 0)^{\pm 1}\})$$

the *Heisenberg graphs* [99] of $\mathbb{Z}_{p^n}$. Obviously, these are 4-regular graphs. In [99], DeDeo et al. proved the following result.

Theorem 7.28. ([99, Theorems 2-3]) *Let p be a prime and $n \geq 1$ an integer. If p is odd, then $G(p^n)$ is the only connected Cayley graph on $H(\mathbb{Z}_{p^n})$ up to isomorphism; if $p = 2$, then $G(2^n)$ and $G(2^n)'$ are the only connected Cayley graphs on $H(\mathbb{Z}_{2^n})$ up to isomorphism. Moreover, the spectra of these Heisenberg graphs approach a continuous interval $[-4, 4]$ as $p^n \to \infty$.*

7.7 Platonic graphs and beyond

Let $q = p^r$ be an odd prime power. Define $G^*(q)$ to be the graph with vertex set $V(G^*(q)) = (\mathbb{F}_q^2 \setminus \{(0,0)\})/\langle \pm 1 \rangle$ in which $(a, b), (c, d) \in V(G^*(q))$ are adjacent if and only if the determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is equal to ± 1. In particular, for an odd prime p, $G^*(p)$ is called the pth *Platonic*

graph. This name was coined as $G^*(3)$ and $G^*(5)$ are isomorphic to the 1-skeletons of the tetrahedron and icosahedron, respectively.

Theorem 7.29. ([98, Theorem 1]; also [205]) *Let $q = p^r$ be an odd prime power. Then*

$$\text{Spec}(G^*(q)) = \begin{pmatrix} q & -1 & \sqrt{q} & -\sqrt{q} \\ 1 & q & (q+1)(q-3)/4 & (q+1)(q-3)/4 \end{pmatrix}$$

In particular, $G^*(q)$ is a Ramanujan graph.

The proof of this result in [205] by Li and Meemark used representation-theoretic methods, while the proof of it in [98] by DeDeo et al. was done by relating $G^*(q)$ to a certain quotient graph of a Cayley graph $G(q)$ on $\text{PSL}(2,q)$ with degree 4 (when $p \equiv 1 \bmod 4$) or 5 (when $p \equiv 3 \bmod 4$). Lower and upper bounds for the isoperimetric number of $G^*(q)$ were also given in [98].

7.8 A family of Ramanujan Cayley graphs on $\mathbb{Z}_p[i]$

Solving a conjecture in [68, Conjecture 31], Bibak et al. [55] proved the following result using results from number theory.

Theorem 7.30. ([55, Theorem 9]) *Let p be a prime with $p \equiv 3$ (mod 4) and S the set of units of ring $\mathbb{Z}_p[i]$. Then the Cayley graph $\text{Cay}(\mathbb{Z}_p[i], S)$ on the additive group of $\mathbb{Z}_p[i]$ is a $(p+1)$-regular Ramanujan graph.*

7.9 Bounds for the degrees of Ramanujan Cayley graphs

Given an odd integer $m \geq 1$, define $\hat{l}_m$ to be the maximum odd positive integer l such that for every even integer k between $m - l$ and $m - 1$, all k-regular circulant graphs of order m are Ramanujan. The following result is due to Hirano et al. [148].

Theorem 7.31. ([148, Theorem 1.1]) *Let $m \geq 15$ be an odd integer. Then*

$$\hat{l}_m = 2\left\lfloor \sqrt{m} - \frac{3}{2} \right\rfloor + \varepsilon_m + 1,$$

where $\varepsilon_m \in \{0, 2\}$. Moreover, the case $\varepsilon_m = 2$ occurs only if m is represented by one of the quadratic polynomials $k^2 + 5k + c$ for some $c \in \{\pm1, \pm3, \pm5\}$ and is either a prime or the product of two primes p, q with $p < q < 4p$.

A parameter $\hat{l}$ similar to $\hat{l}_m$ can be defined [148] for any finite abelian group of odd order m. It was noted in [148] that this parameter satisfies $\hat{l} \geq 2\left\lfloor \sqrt{m} - \frac{3}{2} \right\rfloor + 1$. Moreover, for most finite abelian groups of odd order the equality holds, as proved in [148].

Theorem 7.32. ([148, Theorem 7.1]) *Let Γ be a finite abelian group of odd order other than a cyclic group. Then*

$$\hat{l} = 2\left\lfloor \sqrt{m} - \frac{3}{2} \right\rfloor + 1$$

except when $\Gamma = \mathbb{Z}_p \oplus \mathbb{Z}_p$ for an odd prime p between 3 and 17, and in these exceptional cases we have

$$\hat{l} = \begin{cases} 2\left\lfloor \sqrt{m} - \frac{3}{2} \right\rfloor + 3, \ p = 7, 11, 13, 17 \\ 2\left\lfloor \sqrt{m} - \frac{3}{2} \right\rfloor + 5, \ p = 5. \end{cases}$$

In general, let Γ be a finite group and $\mathcal{S}$ a family of connection sets of Γ (that is, inverse-closed subsets of $\Gamma \setminus \{1\}$). Define $\hat{l}(\Gamma, \mathcal{S})$ to be the maximum positive integer l such that for every $S \in \mathcal{S}$ such that $1 \leq |\Gamma| - |S| \leq l$ the Cayley graph $\text{Cay}(\Gamma, S)$ is Ramanujan (that is, every Cayley graph $\text{Cay}(\Gamma, S)$ with $S \in \mathcal{S}$ and degree between $|\Gamma| - l$ and $|\Gamma| - 1$ is Ramanujan). In [149], this parameter was introduced and studied for two families $\mathcal{S}$ of connection sets of D_{2p}, where p is an odd prime. As mentioned above, earlier this parameter was studied in [148] for cyclic groups of odd order and finite abelian groups of odd order, with $\mathcal{S}$ the set of all connection sets.

The following families were considered by Hirano et al. in [149]: $\mathcal{S}_A$, the family of all connection sets of Γ; $\mathcal{S}_N$, the family of all normal connection sets of Γ, where a connection set is called *normal* if it is the union of some conjugacy classes. Define

$$l_0(\Gamma) = \max\{|\Gamma| - |S| : S \in \mathcal{S}_N, |\Gamma| - |S| \leq 2(\sqrt{|\Gamma|} - 1)\}.$$

Then $l_0(\Gamma) \leq \hat{l}(\Gamma, \mathcal{S}_N)$ as shown in [149, Lemma 2.2].

A *Frobenius group* is a transitive permutation group which is not regular but only the identity element can fix two points. It is well known that any finite Frobenius group can be expressed as $N \rtimes H$ with N a nilpotent normal subgroup, where $N \rtimes H$ acts on N in such a way that N acts on itself by right multiplication and H acts on N by conjugation.

The next three results are due to Hirano et al.

Theorem 7.33. ([149, Theorem 3.3]) *Let $\Gamma = N \rtimes H$ be a finite Frobenius group with Frobenius kernel N and complement H. Suppose that $(|N| - 1)/|H| \geq 4$. Then*

$$\hat{l}(\Gamma, \mathcal{S}_N) = l_0(\Gamma) < |N|.$$

It was noted that the condition $(|N|-1)/|H| \geq 4$ cannot be removed for otherwise the result may not be true. Applying Theorem 7.33 to $D_{2p} = \mathbb{Z}_p \rtimes \mathbb{Z}_2$, where p is a prime, the following result for normal dihedrants (that is, Cayley graphs on dihedral groups with respect to normal connection sets) was obtained.

Theorem 7.34. ([149, Corollary 3.4]) *Let $p \geq 11$ be a prime. Then*

$$\hat{l}(D_{2p}, \mathcal{S}_N) = l_0(D_{2p}) = 2\left\lfloor \sqrt{2p} - \frac{1}{2} \right\rfloor - 1.$$

Theorem 7.35. ([149, Theorem 4.3]) *Let $p \geq 29$ be a prime. Then the following hold:*

(a) *if $\lfloor 2\sqrt{2p} \rfloor$ is even, then $\hat{l}(D_{2p}, \mathcal{S}_A) = 2\left\lfloor \sqrt{2p} - \frac{1}{2} \right\rfloor$;*

(b) *if $\lfloor 2\sqrt{2p} \rfloor$ is odd, then $\hat{l}(D_{2p}, \mathcal{S}_A) = 2\left\lfloor \sqrt{2p} - \frac{1}{2} \right\rfloor - 1$ or $2\left\lfloor \sqrt{2p} - \frac{1}{2} \right\rfloor$.*

In the case when $\lfloor 2\sqrt{2p} \rfloor$ is odd and $\hat{l}(D_{2p}, \mathcal{S}_A) = 2\left\lfloor \sqrt{2p} - \frac{1}{2} \right\rfloor$, the prime p is called *exceptional* [149]. A characterization of exceptional primes was given in [149, Theorem 4.5] and connections with the well-known Hardy-Littlewood conjecture was discussed in [149, Corollary 4.6].

In [302], Yamasaki applied the approach above to generalized quaternion groups

$$Q_{4m} = \langle x, y \mid x^{2m} = 1, x^m = y^2, y^{-1}xy = x^{-1} \rangle$$

and proved the following result.

Theorem 7.36. ([302, Theorem 4.3]) *Let $m \geq 1$ be an integer. Then*

$$\hat{l}(Q_{4m}, \mathcal{S}_A) = l_0(Q_{4m}) = \lfloor 4\sqrt{m} \rfloor - 2.$$

Denote by $\mathcal{S}'$ the family of connection sets S of Q_{4m} such that $\langle x \rangle y \not\subseteq S$. In [302], the parameter $\hat{l}(Q_{4m}, \mathcal{S}')$ was also studied and its relation to the Hardy-Littlewood conjecture was discussed.

8 Second largest eigenvalue of Cayley graphs

In this section we use λ_2 to denote the second largest eigenvalue of the graph under consideration. This invariant has been a focus of research in spectral graph theory over many years. In particular, the second largest eigenvalue of Cayley graphs has been studied extensively in the context of expander graphs, owning to the basic result (see section 7) that a k-regular graph is a good expander if and only if the spectral gap $k - \lambda_2$ is large. In this section we review some of the known results on the second largest eigenvalue of several families of Cayley graphs.

8.1 Cayley graphs on Coxeter groups

As before, we use S_n to denote the symmetric group on $[n] = \{1, 2, \ldots, n\}$, where $n \geq 2$. In this subsection we focus on the second largest eigenvalue of Cayley graphs on S_n and other Coxeter groups. (See [95] for terminology and notation on Coxeter groups.) The following result was obtained by Flatto et al. in [115].

Theorem 8.1. ([115]) *Let $T = \{(1, n), (2, n), \ldots, (n-1, n)\}$. Then $\lambda_2(\mathrm{Cay}(S_n, T)) = n - 2$.*

Note that this graph $\mathrm{Cay}(S_n, T)$ is exactly the star graph of degree $n-1$ discussed in section 3.3.4 and Theorem 8.1 can be derived from Theorem 3.36.

The following result due to Bacher gives the second largest eigenvalue of another well-known Cayley graph on S_n, namely the *bubble sort graph*.

Theorem 8.2. ([38]) *Let $T_1 = \{(1, 2), (2, 3), \ldots, (n-1, n)\}$. Then $\lambda_2(\mathrm{Cay}(S_n, T_1)) = n - 3 + 2\cos(\pi/n)$ and its multiplicity is equal to $n - 1$.*

The proof of Theorem 8.2 as given in [38] made use of irreducible representations of S_n. A similar approach was used by Friedman [118] to prove the following result.

Theorem 8.3. ([118, Theorem 1.1]) *Let T be a set of $n-1$ transpositions in S_n. Then $\lambda_2(\mathrm{Cay}(S_n, T)) \geq n - 2$, with equality if and only if $T = \{(i, j) : j \neq i\}$ for some fixed i.*

Given a set T of transpositions in S_n, define G_T to be the graph with vertex set $[n]$ such that $i, j \in [n]$ are adjacent if and only if $(i, j) \in T$. Theorem 8.3 was proved by establishing the following result.

Theorem 8.4. ([118, Theorem 1.2]) *Let T be a set of $n-1$ transpositions in S_n. If G_T is bipartite, then $\lambda_2(G_T)$ occurs as an eigenvalue of $\mathrm{Cay}(S_n, T)$ with multiplicity at least $n-1$ times its multiplicity in G_T.*

Friedman further conjectured that $\mathrm{Cay}(S_n, T)$ and G_T have the same second smallest Laplacian eigenvalue as long as G_T is bipartite (see [118, Conjecture 1.1]). It turns out that this is a special case of the following more general conjecture of Aldous, known as Aldous' spectral gap conjecture.

Conjecture 8.5. *For any set T of transpositions in S_n such that G_T is connected, $\mathrm{Cay}(S_n, T)$ and G_T have the same second smallest Laplacian eigenvalue.*

In the language of probability theory, this conjecture asserts that the random walk and the interchange process on the graph have the same spectral gap. See [73] for more information about Aldous's conjecture.

Theorem 8.6. *Aldous' spectral gap conjecture is true if G_T is*

(a) *a star ([115]),*

(b) *a complete graph ([104]), or*

(c) *a complete multipartite graph ([73, Theorem 3.1]).*

Major progresses on Aldous' spectral gap conjecture were made in [142] and [181], where irreducible representations of S_n were heavily used. The conjecture in its general form was finally proved by Caputo et al. in [70].

Theorem 8.7. *([70]) Aldous' spectral gap conjecture is true for any set T of transpositions in S_n such that G_T is connected.*

A key ingredient in the proof of this result was an inequality called the octopus inequality. In [74], Cesi gave a simpler and more transparent proof of this inequality by looking at Aldous' spectral gap conjecture from an algebraic perspective. In the same paper Cesi also gave a self-contained algebraic proof of Aldous' spectral gap conjecture, showing how the conjecture follows from the octopus inequality.

The following result for general Coexter groups was obtained by Akhiezer in [15].

Theorem 8.8. *([15, Theorem]) Let (W, S) be a finite Coxeter system, $S = \{s_1, s_2, \ldots, s_l\}$ the set of Coxeter generators, h the Coxeter number of (W, S), and $m_1, m_2, \ldots, m_l$ the exponents, $0 < m_1 \leq m_2 \leq \cdots \leq m_l < h$. Then the numbers*

$$l - 2 + 2\cos\frac{\pi m_i}{h}, \ i = 1, 2, \ldots, l$$

are eigenvalues of $\mathrm{Cay}(W, S)$. Moreover, if W is irreducible, then each of these eigenvalues has multiplicity at least l.

The following result follows from Theorem 8.8 immediately.

Corollary 8.9. *([15, Corollary]) Let (W, S) be a finite irreducible Coxeter system with $l = |S|$ generators and Coxeter number h. Then*

$$\lambda_2(\mathrm{Cay}(W, S)) \geq l - 2 + 2\cos\frac{\pi}{h}.$$

Eigenvalues of Cayley graphs on Coxeter groups have also been studied in [161] in the context of spectral representations associated with random walks on vertex-transitive graphs.

Recall from section 3.3.4 that the transposition network T_n is the Cayley graph on the symmetric group S_n with connection set consisting of all transpositions in S_n. The eigenvalues of T_n were obtained in [166]; see Theorem 3.44. The value of the second largest eigenvalue of T_n was used to obtain the exact value of the bisection width of T_n, answering an open question posed by F. T. Leighton.

8.2 Normal Cayley graphs on highly transitive groups

A partition $\mathcal{P} = \{V_1, \dots, V_m\}$ of the vertex set of a graph G is called an *equitable partition* of G if for each pair i, j with $1 \le i, j \le m$, every vertex of V_i has the same number (say, b_{ij}) of neighbours in V_j; the matrix $B_{\mathcal{P}} = (b_{ij})$ is called the *quotient matrix* of G with respect to $\mathcal{P}$. It is known that the eigenvalues of $B_{\mathcal{P}}$ are eigenvalues of G (see, for example, [133]).

Let Γ be a group acting transitively on $[n] = \{1, 2, \dots, n\}$. Denote by Γ_i the stabilizer of $i \in [n]$ in Γ. Then $|\Gamma : \Gamma_i| = n$ as Γ is transitive. Let $\Gamma = \Gamma_{1,i} \cup \dots \cup \Gamma_{n,i}$ be the left coset decomposition of Γ, where each $\Gamma_{j,i}$ is a left coset of Γ_i in Γ. Then for any Cayley graph $G = \text{Cay}(\Gamma, S)$ on Γ, $\mathcal{P}_i = \{\Gamma_{1,i}, \dots, \Gamma_{n,i}\}$ is an equitable partition of G with corresponding quotient matrix $B_{\mathcal{P}_i} = (b_{ij})$ given by $b_{ij} = |S \cap \Gamma_{j,i}|$. Fix $k \in [n]$. It is not difficult to see that for all $i \in [n]$ the subgraph $G[\Gamma_{k,i}]$ of G induced by $\Gamma_{k,i}$ is isomorphic to the Cayley graph $\text{Cay}(\Gamma_k, S \cap \Gamma_k)$, and the spanning subgraph of G induced by those edges with end-vertices in distinct $\Gamma_{k,i}, \Gamma_{k,i'}, i \ne i'$, is isomorphic to the Cayley graph $\text{Cay}(\Gamma, S \backslash (S \cap \Gamma_k))$ (see [152, Lemma 2.5]). Based on these observations the following result was proved by Huang et al. in [152].

Theorem 8.10. ([152, Theorem 2.6]) *Let Γ be a group acting transitively on $[n] = \{1, 2, \dots, n\}$, and let $G = \text{Cay}(\Gamma, S)$ be a Cayley graph on Γ. Then $\mathcal{P}_i$ is an equitable partition of G and the corresponding quotient matrix $B_{\mathcal{P}_i}$ is symmetric and independent of the choice of $i \in [n]$. Moreover, if λ is an eigenvalue of G which is not an eigenvalue of $B_{\mathcal{P}_i}$, then for each $k \in [n]$ we have*

$$\lambda \le \lambda_2(\text{Cay}(\Gamma_k, S \cap \Gamma_k)) + \lambda_2(\text{Cay}(\Gamma, S \setminus (S \cap \Gamma_k))).$$

Using this result, Huang et al. computed explicitly the second largest eigenvalue of a normal Cayley graph on a highly transitive group under a certain condition; see [152, Theorem 3.5] for details. This result was then used to obtain the second largest eigenvalues of a majority of normal Cayley graphs on S_n such that the connection set coontains no element with support of size greater than 5; see [152, Theorem 4.1]. As corolloaries, the following known results for $n \ge 3$ were recovered (see parts (a) and (b) of Theorem 8.6): If G_T is a complete graph, then the spectral gap of $\text{Cay}(S_n, T)$ is equal to n [104]; if G_T is a star, then the spectral gap of $\text{Cay}(S_n, T)$ is equal to 1 [115].

8.3 Three Cayley graphs on alternating groups

In [153], Huang and Huang determined the second largest eigenvalues of three Cayley graphs on the alternating group A_n, where $n \ge 3$. Set $T_1 = \{(1, 2, i), (1, i, 2) : 3 \le i \le n\}$, $T_2 = \{(1, i, j), (1, j, i) : 2 \le i < j \le n\}$ and $T_3 = \{(i, j, k), (i, k, j) : 1 \le i < j < k \le n\}$. Call $AG_n = \text{Cay}(A_n, T_1)$, $EAG_n = \text{Cay}(A_n, T_2)$ and $CAG_n = \text{Cay}(A_n, T_3)$ the *alternating group*

graph, *extended alternating group graph* and *complete alternating group graph*, respectively. Since $T_1 \subseteq T_2 \subseteq T_3$, AG_n is a spanning subgraph of EAG_n and in turn EAG_n is a spanning subgraph of CAG_n, and all three graphs are connected as T_1 is a generating set of A_n.

Theorem 8.11. ([153, Theorems 2.3, 3.2 and 3.4]) *Let $n \geq 4$. Then $\lambda_2(AG_n) = 2n - 6$, $\lambda_2(EAG_n) = n^2 - 5n + 5$ and $\lambda_2(CAG_n) = n(n-2)(n-4)/3$.*

8.4 Cayley graphs on abelian groups

In [119], it was proved that for any abelian group Γ with order n and any k-regular Cayley graph $\text{Cay}(\Gamma, S)$ on Γ we have $\lambda_2(\text{Cay}(\Gamma, S)) \geq k - O(kn^{-\frac{4}{k}})$, where the constant in the big O term does not depend on k and n. More explicitly, the following result was proved by Friedman et al. in [119].

Theorem 8.12. ([119, Theorem 6]) *Let $k \geq 1$ be a fixed integer and let $\mu \in [0, 1]$. Then there exists a constant C_k satisfying*

$$C_k \leq \frac{1}{2}(1 - \mu)^{-1}\mu^{-\frac{4}{k}}\pi^2$$

such that for any abelian group Γ of order n and any k-regular Cayley graph $\text{Cay}(\Gamma, S)$ on Γ we have

$$\lambda_2(\text{Cay}(\Gamma, S)) \geq k - C_k k n^{-\frac{4}{k}} + o(n^{-\frac{4}{k}}).$$

In particular, as $k \to \infty$ we can take $C_k \preccurlyeq \pi^2/2$.

It was also shown in [119] that for any fixed k, the lower bound above cannot be improved for large odd primes n; that is, if n is an odd prime, then most k-regular graphs on n vertices have the second largest eigenvalue at most $k - \Omega(kn^{-\frac{4}{k}})$.

A well-known result of J.-P. Serre [94] asserts that for any $\varepsilon > 0$ and integer $k \geq 1$ there exists a constant $c = c(\varepsilon, k) > 0$ such that any k-regular graph G of order n has at least cn eigenvalues no less than $(2 - \varepsilon)\sqrt{k - 1}$. A result of this type for Cayley graphs on abelian groups was established by Cioabă in [87].

Theorem 8.13. ([87, Theorem 1.2]) *For any $\varepsilon > 0$ and integer $k \geq 3$, there exists a constant $C = C(\varepsilon, k) > 0$ such that any Cayley graph $\text{Cay}(\Gamma, S)$ on any abelian group Γ with order n has at least Cn eigenvalues no less than $k - \varepsilon$.*

8.5 A family of Cayley graphs with small second largest eigenvalues

In [117], Friedman studied the second largest eigenvalue of two families of directed Cayley graphs and one family of Cayley graphs. We only mention the third family here and leave the other two to section 12.2. Let $p \equiv 3 \pmod 4$ be a prime. Let $\text{AGL}(1, p)$ be the group of affine linear transformations $t_{a,b}$ of $\mathbb{Z}_p$, where $t_{a,b} : \mathbb{Z}_p \to \mathbb{Z}_p, x \mapsto ax + b$ for $a \in \mathbb{Z}_p^*$ and $b \in \mathbb{Z}_p$. In [117], $\text{SQRT}(p)$ was defined to be the Cayley graph on $\text{AGL}(1, p)$ with respect to the connection set $\{t_{r^2,r}, t_{-r^2,r} : r \in \mathbb{Z}_p^*\}$. Clearly, this is an undirected graph of order $p(p-1)$ and degree $2(p-1)$. In [117], Friedman proved the following result.

Theorem 8.14. ([117, Theorem 1.3]) *Let $p \equiv 3 \pmod 4$ be a prime. The graph $\text{SQRT}(p)$ above has second largest eigenvalue in absolute value at most $2\sqrt{p}$.*

8.6 First type Frobenius graphs

Let $N \rtimes H$ be a finite Frobenius group (see section 7.9). A Cayley graph $\mathrm{Cay}(N, S)$ on N is called [114] a *first type Frobenius graph* if $S = a^H$ for some $a \in N$ such that $\langle a^H \rangle = N$ and either $|H|$ is even or a is an involution, where $x^H = \{h^{-1}xh : h \in H\}$ for $x \in N$. In [307], it was shown that first type Frobenius graphs exhibit "perfect" routing properties in some sense. To be precise, let us define a *shortest path routing* of a graph G to be a set of oriented shortest paths in G which contains exactly one oriented shortest path from u to v for each ordered pair of distinct vertices (u, v) of G. The *load* of an arc under such a routing is the number of paths in the routing that traverse the arc in its direction, and the maximum load among all arcs is called the maximum-arc-load of the routing. The *minimal arc-forwarding index* [147] of a graph G, $\overrightarrow{\pi}_m(G))$, is the minimum of the maximum-arc-loads over all shortest path routings of G. Obviously,

$$\overrightarrow{\pi}_m(G)) \geq \frac{\sum_{u,v \in V(G)} d(u, v)}{2|E(G)|},$$

where $d(u, v)$ is the distance in G between u and v. In [307, Theorem 6.1], it was proved that any first type Frobenius graph attains this lower bound and thus has the smallest possible minimal arc-forwarding index.

On the other hand, it is known [105, Corollary 1] that the second largest eigenvalue of random walks on a connected graph G is bounded from above by $1 - \left(2|E(G)|/(\Delta(G)^2 \mathrm{diam}(G) \overrightarrow{\pi}_m(G))\right)$, where $\mathrm{diam}(G)$ and $\Delta(G)$ are the diameter and maximum degree of G, respectively. In the case when G is regular, this can be translated into an upper bound on the second largest eigenvalue $\lambda_2(G)$ of G. This upper bound, together with [307, Theorem 6.1] and [114, Theorem 1.6], implies the following result.

Theorem 8.15. ([307, Corollary 6.4]) *Let $N \rtimes H$ be a finite Frobenius group. Let $\mathrm{Cay}(N, S)$ be a first kind Frobenius graph of $N \rtimes H$. Let d be the diameter of $\mathrm{Cay}(N, S)$ and n_i the number of H-orbits on N at distance i in $\mathrm{Cay}(N, S)$ to the identity element of N, $1 \leq i \leq d$. Then*

$$\lambda_2(\mathrm{Cay}(N, S)) \leq |H| - \frac{|N|}{d \sum_{i=1}^d in_i}.$$

8.7 Distance-j Hamming graphs and distance-j Johnson graphs

Let $q \geq 2$ and $d \geq 1$ be integers. The *distance-j Hamming graph* $H(d, q, j)$, where $0 \leq j \leq d$, is the graph with vertex set $\mathbb{Z}_q^d$ in which two vectors are adjacent if and only if they differ in exactly j coordinates. Obviously, $H(d, q, j)$ is a Cayley graph for each j and $H(d, q, 1)$ is the Hamming graph $H(d, q)$ in the usual sense. The eigenvalues of $H(d, q, j)$ are given by

$$K_j(i) = \sum_{h=0}^{j} (-1)^h (q - 1)^{j-h} \binom{i}{h} \binom{d - i}{j - h}, \ 0 \leq i \leq d.$$

Solving a conjecture of van Dam and Sotirov [297], Brouwer et al. [64] proved the following result.

Theorem 8.16. ([64]) *Let $q \geq 2$, $d \geq 1$ and $j \geq d - \frac{d-1}{q}$, where j is even when $q = 2$. Then the smallest eigenvalue of $H(d, q, j)$ is $K_j(1)$.*

In [64], Brouwer et al. also proved that in most cases $K_j(1)$ is not only the smallest eigenvalue, but also the second largest eigenvalue in absolute value, the only exception being the case when $d = 4$ and $q = 3$. Earlier van Dam and Sotirov's conjecture was confirmed for $q = 2$ by other researchers (see [64] for details).

Let $n \geq 2$ and $d \geq 1$ be integers. The *distance-j Johnson graph* $J(n, d, j)$, where $0 \leq j \leq d$, is the graph with vertices the d-subsets of a fixed n-set such that two d-subsets are adjacent if and only if they meet at a $(d - j)$-subset. In particular, $J(n, d, 1)$ is the Johnson graph $J(n, d)$ in the usual sense and $J(n, d, d)$ is commonly known as the Kneser graph. The eigenvalues of $J(n, d, j)$ are given by

$$E_j(i) = \sum_{h=0}^{i} (-1)^{i-h} \binom{i}{h} \binom{d - h}{j} \binom{n - d - i + h}{n - d - j}, \ 0 \leq i \leq d.$$

In [64], Brouwer et al. proved the following result.

Theorem 8.17. ([64, Theorem 3.10]) *Let $j > 0$. Then $E_j(1)$ is the smallest eigenvalue of $J(n, d, j)$ if and only if $j(n - 1) \geq d(n - d)$. In this case $E_j(1)$ is also the second largest eigenvalue in absolute value among the eigenvalues of $J(n, d, j)$.*

In particular, if $n = 2d$ and $j > \frac{d}{2}$, then the smallest eigenvalue of $J(n, d, j)$, and the second largest eigenvalue of $J(n, d, j)$ in absolute value, is $E_j(1)$, solving a conjecture of Karloff [168].

Theorem 8.17 is relevant to this survey as some (but not all) distance-j Johnson graphs are Cayley graphs; see [164] for when this happens exactly.

8.8 Algebraically defined graphs

Let $q = p^e$ be a prime power, where p is a prime and $e \geq 1$ an integer. Let $k \geq 2$ be an integer. For $2 \leq i \leq k$, take an arbitrary polynomial h_i in $2(i - 1)$ indeterminants over $\mathbb{F}_q$. Define $B\Gamma_k = B\Gamma(q; h_2, \ldots, h_k)$ to be the bipartite graph with biparition $\{P_k, L_k\}$, where each of P_k and L_k is a copy of $\mathbb{F}_q^k$, such that $(p_1, p_2, \ldots, p_k) \in P_k$ and $(l_1, l_2, \ldots, l_k) \in L_k$ are adjacent if and only if their coordinates satisfy

$$p_i + l_i = h_i(p_1, l_1, p_2, l_2, \ldots, p_{i-1}, l_{i-1}), \ 2 \leq i \leq k.$$

This is a q-regular graph with $2q^k$ vertices. It was introduced by Lazebnik and Woldar as a generalization of graphs introduced earlier by Lazebnik and Ustimenko (see the survey papers [194, 195] for details).

Three specifications of graphs $B\Gamma_k$ have received special attention in recent years. In the case when $h_i = p_1 l_1^{i-1}$, $2 \leq i \leq k + 1$, $B\Gamma(q; h_2, \ldots, h_{k+1})$ is called the *Wenger graph*, denoted by $W_k(q)$; the spectra of such graphs have been determined completely by Cioabă et al. in [88]. In the case when $h_i = p_1^{p^{i-2}} l_1$, $2 \leq i \leq k + 1$, $B\Gamma(q; h_2, \ldots, h_{k+1})$ is called the *linearized Wenger graph*, denoted by $L_k(q)$; the eigenvalues of such graphs have been determined by Cao et al. [69] and their multiplicities have been determined by Yan and Liu [303]. In [250], Moorhouse et al. studied the spectra of graphs $D(4, q) = B\Gamma(q; p_1 l_1, p_1 l_2, p_2 l_1)$ and proved in particular that the second largest eigenvalues of these graphs are bounded from above by $2\sqrt{q}$. It turns out that the distance-two graph $G^{(2)}$ of each of $G = W_k(q), L_k(q), D(4, q)$ is a Cayley graph, and this fact was used to bound the second largest eigenvalue of G. (In general, given a bipartite graph

G with bipartition $\{V_1, V_2\}$, the *distance-two graph* $G^{(2)}$ of G on V_1 is the graph with vertex set V_1 in which two vertices are adjacent if and only if they are at distance two in G.)

In a recent paper [89], Cioabă et al. introduced a family of Cayley graphs on certain abelian groups by directly generalizing the defining system of equations for $W_k(q)^{(2)}$ and $L_k(q)^{(2)}$. Let $k \geq 3$ and let $f_i, g_i \in \mathbb{F}_q[x]$, $3 \leq i \leq k$, be polynomials of degrees at most $q-1$ such that $g_i(-x) = -g_i(x)$ for each i. Define $S(k,q) = S(k,q; f_3, g_3, \ldots, f_k, g_k)$ to be the graph with vertex set $\mathbb{F}_q^k$ such that $(a_1, a_2, \ldots, a_k) \in \mathbb{F}_q^k$ and $(b_1, b_2, \ldots, b_k) \in \mathbb{F}_q^k$ are adjacent if and only if their coordinates satisfy

$$b_i - a_i = g_i(b_1 - a_1) f_i \left(\frac{b_2 - a_2}{b_1 - a_1} \right), \ 3 \leq i \leq k.$$

The requirement $g_i(-x) = -g_i(x)$ ensures that this is an undirected graph. It was noticed [89] that $S(k,q)$ is a Cayley graph on the additive group of the vector space $\mathbb{F}_q^k$ with connection set $\{(a, au, g_3(a) f_3(u), \ldots, g_k(a) f_k(u)) : a \in \mathbb{F}_q^*, u \in \mathbb{F}_q\}$. Thus $S(k,q)$ is a $q(q-1)$-regular graph with q^k vertices. With the help of Theorem 2.2, Cioabă et al. [89, Theorem 2.1] determined the spectrum of $S(k,q)$. Using this they gave the following sufficient condition for the graphs $S(k,q)$ to form a family of expander graphs.

Theorem 8.18. ([89, Theorem 2.2]) *Let $k \geq 3$ and let $(q_m)_{m \geq 1}$ be an increasing sequence of prime powers, where $q_m = p_m^{e_m}$ with p_m a prime and $e_m \geq 1$ an integer. Let*

$$S(k, q_m) = S(k, q_m; f_{3,m}, g_{3,m}, \ldots, f_{k,m}, g_{k,m}).$$

Let $d_f^{(m)}$ be the maximum degree of polynomials $f_{3,m}, \ldots, f_{k,m}$ and $d_g^{(m)}$ the maximum degree of polynomials $g_{3,m}, \ldots, g_{k,m}$. Suppose that $1 \leq d_f^{(m)} = o_m(q_m)$, $1 \leq d_g^{(m)} = o_m(\sqrt{q_m})$, $d_g^{(m)} < p_m$, and for all m at least one of the following conditions is satisfied:

(a) *the polynomials $1, x, f_{3,m}, \ldots, f_{k,m}$ are $\mathbb{F}_q$-linearly independent, and $g_{i,m}$ has linear term for all i, $3 \leq i \leq k$;*

(b) *the polynomials $f_{3,m}, \ldots, f_{k,m}$ are $\mathbb{F}_q$-linearly independent, and there exists some j, $2 \leq j \leq d_g^{(m)}$, such that each $g_{i,m}$, $3 \leq i \leq k$, contains a term $c_{i,j}^{(m)} x^j$ with $c_{i,j}^{(m)} \neq 0$.*

Then $S(k, q_m)$ is connected and

$$\lambda_2(S(k, q_m)) = o_m(q_m^2).$$

In [89], Cioabă et al. also obtained better upper bounds on the second largest eigenvalue of $S(k,q)$ for two specifications of polynomials f_i, g_i. Define the *exponential sum* of $f \in \mathbb{F}_q[x]$ by $\varepsilon_f = \sum_{x \in \mathbb{F}_q} \omega_p^{\mathrm{Tr}(f(x))}$, where $\mathrm{Tr}(\alpha) = \alpha + \alpha^p + \cdots + \alpha^{p^{e-1}}$ is the *trace* of $\alpha \in \mathbb{F}_q$ over $\mathbb{F}_p$.

Theorem 8.19. ([89, Theorem 2.3]) *Let q be an odd prime power with $q \equiv 2 \bmod 3$ and $4 \leq k \leq q+1$. Let $g_i(x) = x^3$ and $f_i(x) = x^{i-1}$ for each i, $3 \leq i \leq k$. Then $S(k,q)$ is connected and*

$$\lambda_2(S(k,q)) = \max\{q(k-3), (q-1)M_q\},$$

where $M_q = \max_{a,b \in \mathbb{F}_q^*} \varepsilon_{ax^3 + bx} \leq 2\sqrt{q}$.

In particular, when $(q-1)M_q \leq q(k-3)$, we have $\lambda_2(S(k,q)) = q(k-3) < q(k-2) = \lambda_2(W_{k-1}(q)^{(2)})$.

Theorem 8.20. ([89, Theorem 2.4]) *Let q be an odd prime power with $q \equiv 2$ mod 3 and $3 \leq k \leq e+2$. Let $g_i(x) = x^3$ and $f_i(x) = x^{p^{i-2}}$ for each i, $3 \leq i \leq k$. Then $S(k,q)$ is connected and*

$$\lambda_2(S(k,q)) \leq \max\{q(p^{k-3}-1), (q-1)M_q\},$$

where $M_q = \max_{a,b \in \mathbb{F}_q^*} \varepsilon_{ax^3+bx} \leq 2\sqrt{q}$.

In particular, when $(q-1)M_q \leq q(p^{k-3}-1)$, we have $\lambda_2(S(k,q)) = q(p^{k-3}-1) < q(p^{k-2}-1) = \lambda_2(L_{k-1}(q)^{(2)})$.

9 Perfect state transfer in Cayley graphs

Let G be a graph. Define

$$H_G(t) = \exp(itA(G)) = \sum_{k=0}^{\infty} \frac{i^k t^k A(G)^k}{k!} \tag{9.1}$$

and call it the *transition matrix* of G, where $A(G)$ is the adjacency matrix of G and $i = \sqrt{-1}$. We say that *perfect state transfer* occurs from a vertex u to another vertex v in G if there exists a time τ such that $|H_G(\tau)_{u,v}| = 1$. If there exists a time τ such that $|H_G(\tau)_{u,u}| = 1$, then G is *periodic* at vertex u with period τ. A graph is called *periodic* if it is periodic at every vertex with the same period. As seen in [83, 82, 169], these two concepts arise from quantum computing, in which a continuous-time quantum walk on a graph G with transition matrix $H_G(t)$ plays a significant role. The reader is referred to [91, 130, 132] and the surveys [131, 169, 284] for background information and results on perfect state transfer and periodicity in graphs.

It is known that if G has perfect state transfer from u to v at time τ then G is periodic at both u and v with period 2τ (see [82] or [130, Lemma 2.1]). Moreover, there is an interesting relation between the eigenvalues of a graph G and its periodicity (see [130, Corollary 3.3] or [306, Theorem 2.3]), which asserts that G is periodic if and only if the eigenvalues of G are all integers or all rational multiples of $\sqrt{\Delta}$ for some square-free integer Δ. Furthermore, if the second alternative holds, then G is bipartite. This criterion shows that perfect state transfer can occur in integral graphs but not always (see [83]). In addition, if G is regular, then its largest eigenvalue is an integer and so the second alternative above cannot occur. Thus a regular graph is periodic if and only if it is integral (see [131, Section 5]). Hence all integral Cayley graphs are periodic. In particular, all integral Cayley graphs mentioned in section 3 are periodic. Moreover, from the discussion above one can see that integral Cayley graphs are good candidates if one wants to construct regular graphs having perfect state transfer. In fact, as noted in [286, Lemma 2.2] and [255, Theorem 3.5], if a Cayley graph on an abelian group has perfect state transfer, then it must be integral.

In [91, Theorem 4.3], necessary and sufficient conditions for a graph belonging to an association scheme to admit perfect state transfer were obtained. In particular, this result applies to distance-regular graphs (see [91, Corollaries 4.5 and 5.1]), and was used in [91] to determine which distance-regular graphs listed in [65, Chapter 14] admit perfect state transfer. Note that many distance-regular graphs are Cayley graphs, and for them [91, Corollaries 4.5 and 5.1] can be used to test whether they admit perfect state transfer.

9.1 Perfect state transfer in Cayley graphs on abelian groups

Let $\Gamma = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_d}$ be an abelian group, where each $n_t \geq 2$. It is well known that for any $\mathbf{x} = (x_1, \ldots, x_d) \in \Gamma$ the mapping $\chi_{\mathbf{x}} : \Gamma \to \mathbb{C}$ defined by

$$\chi_{\mathbf{x}}(\mathbf{g}) = \exp\left(2\pi\mathrm{i} \sum_{j=1}^{d} \frac{x_j g_j}{n_j}\right), \ \mathbf{g} = (g_1, g_2, \ldots, g_d) \in \Gamma \tag{9.2}$$

is a character of Γ. Let S be a subset of Γ not containing the identity element of Γ such that $S^{-1} = S$. By Corollary 2.3, the eigenvalues of $\mathrm{Cay}(\Gamma, S)$ are

$$\alpha_{\mathbf{x}} = \sum_{\mathbf{g} \in S} \chi_{\mathbf{x}}(\mathbf{g}), \ \mathbf{x} \in \Gamma.$$

Thus $\mathrm{Cay}(\Gamma, S)$ is integral if and only if $\alpha_{\mathbf{x}} \in \mathbb{Z}$ for all $\mathbf{x} \in \Gamma$.

The following result due to Tan et al. gives a sufficient condition for an integral Cayley graph on an abelian group to be periodic.

Theorem 9.1. ([286, Theorem 2.3]) *Let $\Gamma = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_d}$ be an abelian group and $\mathrm{Cay}(\Gamma, S)$ an integral Cayley graph on Γ, where each $n_t \geq 2$. Set $c = \gcd(|S| - \alpha_{\mathbf{x}} : \mathbf{x} \in \Gamma)$. Then $\mathrm{Cay}(\Gamma, S)$ is periodic at every vertex with period $2\pi l/c$ for each $l = 1, 2, \ldots$.*

The 2-*adic exponential valuation* of rational numbers is the mapping $v_2 : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ defined by

$$v_2(0) = \infty, \ v_2\left(\frac{2^l a}{b}\right) = l, \ \text{where } a, b, l \in \mathbb{Z} \text{ with } 2 \nmid ab.$$

Note that if Γ contains involutions then $|\Gamma|$ is even. Moreover, $\chi_{\mathbf{a}}(\mathbf{g}) = \pm 1$ for any involution $\mathbf{a}$ and any element $\mathbf{g}$ of Γ. The following result gives a necessary and sufficient condition for a Cayley graph on an abelian group to admit perfect state transfer.

Theorem 9.2. ([286, Theorem 2.4]) *Let $\Gamma = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_d}$ be an abelian group of order at least 3 and $\mathrm{Cay}(\Gamma, S)$ a connected Cayley graph on Γ, where each $n_t \geq 2$. Then for any distinct $\mathbf{g}, \mathbf{h} \in \Gamma$, $\mathrm{Cay}(\Gamma, S)$ has perfect state transfer from $\mathbf{g}$ to $\mathbf{h}$ if and only if the following conditions are satisfied:*

(a) $\mathrm{Cay}(\Gamma, S)$ *is an integral graph (that is, $\alpha_{\mathbf{x}} \in \mathbb{Z}$ for all $\mathbf{x} \in \Gamma$);*

(b) *the element $\mathbf{a} := \mathbf{g} - \mathbf{h}$ is an involution of Γ;*

(c) *the values of $v_2(|S| - \alpha_{\mathbf{x}})$ for all $\mathbf{x} \in \chi_{\mathbf{a}}^{-1}(-1)$ are the same, say, ρ, and moreover $v_2(c_{\mathbf{a}}) \geq \rho + 1$, where $c_{\mathbf{a}} = \gcd(|S| - \alpha_{\mathbf{x}} : \mathbf{x} \in \chi_{\mathbf{a}}^{-1}(1))$.*

Moreover, if conditions (a)-(c) *are satisfied, then $\mathrm{Cay}(\Gamma, S)$ has perfect state transfer from $\mathbf{g}$ to $\mathbf{h}$ at time $(\pi/c) + (2\pi l/c)$, for $l = 1, 2, \ldots$, where $c = \gcd(|S| - \alpha_{\mathbf{x}} : \mathbf{x} \in \Gamma)$.*

By condition (b) in Theorem 9.2, if $\mathrm{Cay}(\Gamma, S)$ admits perfect state transfer, then $|\Gamma|$ must be even. It was proved in [258] that for circulant graphs the stronger necessary condition that n is a multiple of 4 should be satisfied. Recently, Tan et al. proved that the same condition should be respected for all integral Cayley graphs on abelian groups.

Theorem 9.3. ([286, Theorem 3.5]) *Let $\mathrm{Cay}(\Gamma, S)$ be an integral Cayley graph on an abelian group Γ with $|\Gamma| \geq 6$ and $|\Gamma| \equiv 2 \pmod 4$. Then $\mathrm{Cay}(\Gamma, S)$ has no perfect state transfer between any two distinct vertices.*

9.2 A few families of Cayley graphs on abelian groups admitting perfect state transfer

9.2.1 Perfect state transfer in integral circulant graphs

Perfect state transfer in integral circulant graphs has been studied extensively. The main problem, as raised by Angeles-Canul et al. in [31], is to characterize those integral circulant graphs which admit perfect state transfer. Partial results on this problem were obtained in several papers, including [31, 44, 45, 46, 49, 258, 279], [131, Section 7] and [284, Section 9]. A complete characterization, stated below, was finally obtained by Bašić in [47]. Recall from Corollary 3.3 that a circulant graph is integral if and only if it is isomorphic to a gcd graph $\mathrm{ICG}(n, D)$ as defined in (3.3), where $D \subseteq D(n) \setminus \{n\}$ with $D(n)$ as defined in (3.2). Denote $kD = \{kd : d \in D\}$ for any positive integer k.

Theorem 9.4. ([47, Theorem 22]) *The integral circulant graph* $\mathrm{ICG}(n, D)$ *has perfect state transfer if and only if* $n \in 4\mathbb{N}$ *and* $D = D_2 \cup 2D_2 \cup 4D_2 \cup D_3 \cup \{n/2^a\}$, *where* $D_2 = \{d \in D : n/d \in 8\mathbb{N} + 4\} \setminus \{n/4\}$, $D_3 = \{d \in D : n/d \in 8\mathbb{N}\}$, *and* $a \in \{1, 2\}$.

A weighted graph whose weighted adjacency matrix is circulant is called a *weighted circulant graph*. In [48], Bašić studied perfect state transfer and periodicity in weighted circulant graphs. He proved that a weighted circulant graph is periodic if and only if it is integral. He also gave a criterion for the existence of perfect state transfer in a weighted circulant graph in terms of its eigenvalues (see [48, Theorem 8]). As applications he found some classes of weighted circulant graphs having perfect state transfer (see [48, Theorems 10]) as well as ones admitting no perfect state transfer (see [48, Theorems 13-15]).

9.2.2 Perfect state transfer in cubelike graphs

Recall from section 3.2.2 that cubelike graphs are precisely Cayley graphs on elementary abelian 2-groups and are exactly NEPS graphs of copies of K_2. Specifically, every subset S of $\mathbb{Z}_2^d \setminus \{\mathbf{0}\}$ gives rise to a cubelike graph $\mathrm{Cay}(\mathbb{Z}_2^d, S)$, where $\mathbf{0}$ is the zero element of $\mathbb{Z}_2^d$. The existence of perfect state transfer in cubelike graphs was studied in [52, 81, 83]. See also [131, Section 7] and [284, Section 8].

Theorem 9.5. ([52, Theorem 1], or [81, Theorem 2.3]) *Let* $d \geq 1$ *be an integer. Let* S *be a subset of* $\mathbb{Z}_2^d \setminus \{\mathbf{0}\}$ *and let* $\mathbf{a}$ *be the sum of the elements of* S. *If* $\mathbf{a} \neq 0$, *then perfect state transfer occurs in* $\mathrm{Cay}(\mathbb{Z}_2^d, S)$ *from* $\mathbf{u}$ *to* $\mathbf{u} + \mathbf{a}$ *at time* $\pi/2$, *for any* $\mathbf{u} \in \mathbb{Z}_2^d$. *If* $\mathbf{a} = 0$, *then* $\mathrm{Cay}(\mathbb{Z}_2^d, S)$ *is periodic with period* $\pi/2$.

The elements of $\mathbb{Z}_2^d$ can be treated as column vectors of the linear space $\mathbb{F}_2^d$. Given $S \subseteq \mathbb{Z}_2^d \setminus \{\mathbf{0}\}$, let $M(S)$ be the matrix with vectors in S as its columns. The *code* of S, $C(S)$, is defined as the row space of $M(S)$ in the linear space $\mathbb{F}_2^d$. The *support* of $\mathbf{x} \in C(S)$ is defined as $\mathrm{supp}(\mathbf{x}) = \{i : \text{the } i\text{th entry of } \mathbf{x} \text{ is nonzero}\}$.

Theorem 9.6. ([81, Theorem 4.1]) *Let* $d \geq 1$ *be an integer. Let* $S \subseteq \mathbb{Z}_2^d \setminus \{\mathbf{0}\}$ *and* $\mathbf{u} \in \mathbb{Z}_2^d \setminus \{\mathbf{0}\}$. *Then the following conditions are equivalent:*

(a) *there is perfect state transfer from* $\mathbf{0}$ *to* $\mathbf{u}$ *at time* $\pi/2\Delta$ *in the cubelike graph* $\mathrm{Cay}(\mathbb{Z}_2^d, S)$;

(b) *all codewords in the code of S have weight divisible by Δ, and $\Delta^{-1} w(\mathbf{a}^T M(S))$ and $\mathbf{a}^T \mathbf{u}$ have the same parity for all vectors $\mathbf{a} \in \mathbb{Z}_2^d$, where $w(\cdot)$ denotes the Hamming weight;*

(c) Δ *divides $|\operatorname{supp}(\mathbf{x}) \cap \operatorname{supp}(\mathbf{y})|$ for any two codewords $\mathbf{x}$ and $\mathbf{y}$ in the code of S.*

In particular, if $\operatorname{Cay}(\mathbb{Z}_2^d, S)$ admits perfect state transfer from $\mathbf{0}$ to $\mathbf{u}$ at time $\pi/2\Delta$, then Δ must be the greatest common divisor of the weights of the codewords in the code of S (see [81, Corollary 4.2]). As an application of Theorem 9.6, Cheung and Godsil [81] presented a cubelike graph over $\mathbb{Z}_2^5$ having perfect state transfer at time $\pi/4$. They asked [81] further whether the minimum time can be less than $\pi/4$.

In [131], Godsil asked the following: "Are there cubelike graphs having perfect state transfer at time τ, where τ is arbitrarily small?" Recently, this question has been positively answered by Tan et al. in [286].

Theorem 9.7. ([286, Theorem 4.3]) *Let $d \geq 2$ be an integer. Let S be a subset of $\mathbb{Z}_2^d \setminus \{\mathbf{0}\}$ that generates $\mathbb{Z}_2^d$, and let $0 \neq \mathbf{a} \in \mathbb{Z}_2^d$. If the cubelike graph $\operatorname{Cay}(\mathbb{Z}_2^d, S)$ has perfect state transfer from $\mathbf{u}$ to $\mathbf{u} + \mathbf{a}$ at time t, for some $\mathbf{u} \in \mathbb{Z}_2^d$, then the minimum time t is $\pi/2^l$ for some l with $1 \leq l \leq \lfloor d/2 \rfloor$.*

Given a subset S of $\mathbb{Z}_2^d$, define the Boolean function $f : \mathbb{Z}_2^d \to \mathbb{Z}_2$ with respect to S by

$$f(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \in S, \\ 0, & \text{otherwise.} \end{cases}$$

We call S the *support* of f and write $S = \operatorname{supp}(f)$.

Let B_d be the ring of Boolean functions with d variables. For $f \in B_d$, we have the function $(-1)^f : \mathbb{Z}_2^d \to \{\pm 1\}$ which maps $\mathbf{x}$ to $(-1)^{f(\mathbf{x})}$. The *Fourier transformation of $(-1)^f$* over $(\mathbb{Z}_2^d, +)$, also called the *Walsh transformation of f*, is the function $W_f : \mathbb{Z}_2^d \to \mathbb{Z}$ defined by

$$W_f(\mathbf{y}) = \sum_{\mathbf{x} \in \mathbb{Z}_2^d} (-1)^{f(\mathbf{x}) + \mathbf{x} \cdot \mathbf{y}}, \ \mathbf{y} \in \mathbb{Z}_2^d,$$

where $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^d x_i y_i \in \mathbb{Z}_2$ for $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{Z}_2^d$ and $\mathbf{y} = (y_1, \ldots, y_d) \in \mathbb{Z}_2^d$. For $d = 2m$ $(m \geq 2)$, $f \in B_d$ is called a *bent function* in B_d if $|W_f(\mathbf{y})| = 2^m$ for all $\mathbf{y} \in \mathbb{Z}_2^d$.

The next two results due to Tan et al. tell us when the lower bound given in Theorem 9.7 can be achieved.

Theorem 9.8. ([286, Theorem 4.5]) *Let $d = 2m + 1$, where $m \geq 2$. Let f be a bent function in B_{d-1} with $0 \notin \operatorname{supp}(f) = \{\mathbf{z} \in \mathbb{Z}_2^{d-1} : f(\mathbf{z}) = 1\}$ and $S = \{(0, \mathbf{z}), (1, \mathbf{z}) : \mathbf{z} \in \operatorname{supp}(f)\} \subset \mathbb{Z}_2^d$. Then the following hold:*

(a) *the cubelike graph $\operatorname{Cay}(\mathbb{Z}_2^d, S)$ is connected;*

(b) *for $\mathbf{a} = (1, 0, 0, \ldots, 0) \in \mathbb{Z}_2^d$, $\operatorname{Cay}(\mathbb{Z}_2^d, S)$ has perfect state transfer from $\mathbf{u}$ to $\mathbf{u} + \mathbf{a}$ at time $\pi/2^m$, for any $\mathbf{u} \in \mathbb{Z}_2^d$;*

(c) *the minimum period with which $\operatorname{Cay}(\mathbb{Z}_2^d, S)$ is periodic at any vertex is $\pi/2^m$.*

Theorem 9.9. ([286, Theorem 4.5]) *Let $d = 2m$, where $m \geq 2$. Let f be a bent function in B_d and $S = \operatorname{supp}(f) = \{\mathbf{z} \in \mathbb{Z}_2^d : f(\mathbf{z}) = 1\}$. Then the cubelike graph $\operatorname{Cay}(\mathbb{Z}_2^d, S)$ is connected and the minimum period with which $\operatorname{Cay}(\mathbb{Z}_2^d, S)$ is periodic at any vertex is equal to $\pi/2^m$.*

In [304], Zheng et al. proved that $G_{d,k} := \text{NEPS}(H(k,2),\ldots,H(k,2);\mathcal{B})$ is isomorphic to a cubelike graph, where $d \geq 1$, $k \geq 2$, $\emptyset \neq \mathcal{B} \subseteq \mathbb{Z}_2^d \setminus \{\mathbf{0}\}$ and $H(k,2)$ is the hypercube of dimension k. In the same paper they also proved the following statements: If $k \geq 3$ is odd and the sum of the elements of $\mathcal{B}$ is not equal to $\mathbf{0}$, then $G_{d,k}$ admits perfect state transfer at time $\pi/2$; if k is even, then $G_{d,k}$ admits perfect state transfer if and only if $\mathcal{B}$ contains at least one element β with Hamming weight $w(\beta) = 1$, and in this case the perfect state transfer occurs at time $\pi/2$.

9.2.3 Perfect state transfer in gcd graphs of abelian groups

Let $\Gamma = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_d}$ be an abelian group, where each $n_t \geq 2$. Set $\mathbf{n} = (n_1,\ldots,n_d)$ and let $\mathbf{D} \subseteq \mathbf{D}(\mathbf{n}) \setminus \{\mathbf{n}\}$, where $\mathbf{D}(\mathbf{n})$ is as defined in (3.8). Recall from section 3.2.2 that the gcd graph $\text{ICG}(\mathbf{n},\mathbf{D})$ of Γ with respect to $\mathbf{D}$ is the Cayley graph $\text{Cay}(\Gamma, S_\Gamma(\mathbf{D}))$, where $S_\Gamma(\mathbf{D})$ is the set of elements $\mathbf{x} = (x_1,\ldots,x_d)$ of Γ such that $\gcd(\mathbf{x},\mathbf{n}) \in \mathbf{D}$.

It is known [179, Corollary 2.7] that every gcd graph of any abelian group with order n is isomorphic to a gcd graph of $\Gamma = \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_k}$, where $n = p_1 p_2 \ldots p_k$ is the factorization of n into the product of primes but $p_1, p_2, \ldots, p_k$ are not necessarily distinct. In particular, if n is a power of 2, then the gcd graph is actually a cubelike graph [256, Theorem 4.5]. Recall that $G \otimes H$ denotes the tensor product of graphs G and H.

Theorem 9.10. ([256, Lemma 4.6]) *Let $\Gamma = \Gamma_1 \oplus \Gamma_2$ be an abelian group, where $\Gamma_1 = \mathbb{Z}_{2^{a_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{a_r}}$ and $\Gamma_2 = \mathbb{Z}_{p_1^{b_1}} \oplus \cdots \oplus \mathbb{Z}_{p_s^{b_s}}$, with $p_1,\ldots,p_s$ odd primes. Set $\mathbf{n} = (2^{a_1},\ldots,2^{a_r},p_1^{b_1},\ldots,p_s^{b_s})$ and $\mathbf{n}_2 = (p_1^{b_1},\ldots,p_s^{b_s})$. Assume that $d_{r+1},\ldots,d_{r+s}$ are fixed divisors of $p_1^{b_1},\ldots,p_s^{b_s}$, respectively. Let $\mathbf{D} \subseteq \mathbf{D}(\mathbf{n}) \setminus \{\mathbf{n}\}$ be such that the last s components of each member of $\mathbf{D}$ are $d_{r+1},\ldots,d_{r+s}$, respectively. Set $\mathbf{D}_2 = \{(d_{r+1},\ldots,d_{r+s})\}$. Then there exists a cubelike graph $\text{Cay}(\Gamma_1, S_1)$ such that*

$$\text{ICG}(\mathbf{n},\mathbf{D}) \cong \text{Cay}(\Gamma_1, S_1) \otimes \text{ICG}(\mathbf{n}_2,\mathbf{D}_2).$$

As shown in [256], the connection set S_1 (which relies on $\mathbf{D}$) in Theorem 9.10 can be constructed explicitly from $\mathbf{D}$. Let $\mathscr{D}$ be the collection of all subsets $\mathbf{D} \subseteq \mathbf{D}(\mathbf{n})$ satisfying the conditions of Theorem 9.10 as well as the following two conditions: (i) the sum of the elements in S_1 is equal to $\mathbf{0}$; (ii) $|S_1| \equiv 0 \pmod 4$ if there exists i with $1 \leq i \leq s$ such that $d_{r+i} < p_i^{b_i}$. Define $\mathscr{D}_1$ to be the collection of all possible unions of pairwise disjoint members of $\mathscr{D}$. That is, each member of $\mathscr{D}_1$ is of the form $\cup_{i=1}^t \mathbf{D}_i$ for some pairwise disjoint members $\mathbf{D}_1,\ldots,\mathbf{D}_t$ of $\mathscr{D}$. Define $\mathscr{D}_2$ to be the collection of all subsets $\mathbf{D} \subseteq \mathbf{D}(\mathbf{n})$ such that the last s components of each member of $\mathbf{D}$ are $p_1^{b_1},\ldots,p_s^{b_s}$, respectively, and the sum of the elements in the corresponding S_1 is not equal to $\mathbf{0}$.

The following result due to Pal and Bhattacharjya gives a sufficient condition for a gcd graph of an abelian group to admit perfect state transfer.

Theorem 9.11. ([256, Theorem 4.8]) *Let $\Gamma = \Gamma_1 \oplus \Gamma_2$ be an abelian group, where $\Gamma_1 = \mathbb{Z}_{2^{a_1}} \oplus \cdots \oplus \mathbb{Z}_{2^{a_r}}$ and $\Gamma_2 = \mathbb{Z}_{p_1^{b_1}} \oplus \cdots \oplus \mathbb{Z}_{p_s^{b_s}}$, with $p_1,\ldots,p_s$ odd primes. Let $\mathbf{D}_1 \in \mathscr{D}_1$ and $\mathbf{D}_2 \in \mathscr{D}_2$. If $\mathbf{D}_1 \cap \mathbf{D}_2 = \emptyset$ and $\mathbf{D} = \mathbf{D}_1 \cup \mathbf{D}_2$ generates Γ, then $\text{ICG}(\mathbf{n},\mathbf{D})$ is connected and has perfect state transfer at time $\pi/2$, where $\mathbf{n} = (2^{a_1},\ldots,2^{a_r},p_1^{b_1},\ldots,p_s^{b_s})$.*

In [256], Pal and Bhattacharjya also gave the following sufficient condition for the existence of gcd graphs of abelian groups having perfect state transfer at time $\pi/2$ (see also [255]).

Theorem 9.12. ([256, Theorem 4.12]) *Let Γ be a finite abelian group with order $|\Gamma| \equiv 0 \pmod 4$. Then there exists a connected gcd graph of Γ that admits perfect state transfer at time $\pi/2$.*

It was shown in [179, Theorems 2.5 and 2.6] that a gcd graph of an abelian group is isomorphic to an NEPS of complete graphs and vice versa (see [208] for more details). Thus Theorems 9.11 and 9.12 can also be presented using the language of NEPS.

9.3 Perfect state transfer in Cayley graphs on finite commutative (chain) rings and gcd graphs of unique factorization domains

In [292], Thongsomnuk and Meemark determined when the unitary Cayley graph G_R of a finite commutative ring R admits perfect state transfer.

Theorem 9.13. ([292, Theorem 2.5]) *Let $R \cong R_0 \times R_1 \times \cdots \times R_s$ be a finite commutative ring with $1 \neq 0$, where $s \geq 0$ and each R_i, $0 \leq i \leq s$, is a local ring with maximal ideal M_i of size m_i. Set $m = m_0 m_1 \ldots m_s$. The unitary Cayley graph G_R has perfect state transfer if and only if R satisfies one of the following conditions:*

(a) *$m = 1$ and $R \cong \mathbb{F}_2 \times \mathbb{F}_{2^{a_1}} \times \mathbb{F}_{2^{a_2}} \times \cdots \times \mathbb{F}_{2^{a_s}}$ for some integers $a_1, a_2, \ldots, a_s \geq 1$;*

(b) *$m = 2$ and $R \cong R_0 \times \mathbb{F}_{2^{a_1}} \times \mathbb{F}_{2^{a_2}} \times \cdots \times \mathbb{F}_{2^{a_s}}$ for some integers $a_1, a_2, \ldots, a_s \geq 1$, where R_0 is $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.*

This result yields [237, Corollary 2.6] in the special case when R is a finite local ring.

The following result characterizes when perfect state transfer occurs in unitary Cayley graphs of finite chain rings.

Theorem 9.14. ([292, Theorem 4.1]) *Let R be a finite chain ring with nilpotency s. Let $\mathrm{Cay}(R, \mathcal{C})$ be the graph defined in section 5.3, where $\mathcal{C}$ depending on a sequence of integers $0 \leq a_1 < a_2 < \cdots < a_r \leq s - 1$ is as defined in (5.1). Then $\mathrm{Cay}(R, \mathcal{C})$ has perfect state transfer if and only if $q = 2$ and one of the following holds:*

(a) *$s = 1$ or 2, and $a_1 = 1$;*

(b) *$s \geq 2$, and $a_r = s - 2$.*

Let R be a unique factorization domain (UFD) and c a nonzero nonunit element of R. Assume that the commutative ring $R/(c)$ is finite. Let D be a set of proper divisors of c. Recall from section 6.5 that the gcd graph $\mathrm{ICG}(R/(c), D)$ is the graph with vertex set the quotient ring $R/(c)$ such that $x + (c), y + (c) \in R/(c)$ are adjacent if and only if $\gcd(x - y, c)$ belongs to D up to associate. As applications of Theorem 9.13, Thongsomnuk and Meemark [292] determined the existence of perfect state transfer in some gcd graphs of a unique factorization domain.

Theorem 9.15. ([292, Theorem 3.1]) *Let R be a unique factorization domain. Let*

$$c = p_1 p_2^{a_2} \ldots p_n^{a_n}$$

be an element of R which is factorized into a product of non-associate irreducible elements, where $a_i > 1$ for $2 \leq i \leq n$. Then $\mathrm{ICG}(R/(c), \{1, p_1\})$ has perfect state transfer if and only if there exists $i \in \{2, \ldots, n\}$ such that $a_j = 1$ and $R/(p_j)$ is a finite field of characteristic 2 for each $j \in \{1, 2, \ldots, n\} \setminus \{i\}$, and either (i) $a_i = 1$ and $R/(p_i) \cong \mathbb{F}_2$, or (ii) $a_i = 2$ and $R/(p_i^2)$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.

Theorem 9.16. ([292, Theorem 3.2]) *Let R be a unique factorization domain. Let*

$$c = p_1 p_2 \ldots p_k q_1^{a_1} \ldots q_l^{a_l}$$

be an element of R which is factorized as a product of non-associate irreducible elements, where $a_i > 1$ for $1 \leq i \leq l$. Then $\mathrm{ICG}(R/(c), \{p_i, p_j\})$ with $1 \leq i < j \leq k$ has perfect state transfer if and only if one of the following occurs:

(a) $c = p_1 p_2 \ldots p_k$, $R/(p_s)$ *is a finite field of characteristic* 2 *for* $1 \leq s \leq k$, *and* $R/(p_i) \cong R/(p_j) \cong \mathbb{F}_2$ *or* $R/(p_t) \cong \mathbb{F}_2$ *for some* $t \neq i, j$;

(b) $c = p_1 p_2 \ldots p_k q_1^2$, $R/(p_s)$ *is a finite field of characteristic* 2 *for* $1 \leq s \leq k$, *and* $R/(q_1^2)$ *is isomorphic to* $\mathbb{Z}_4$ *or* $\mathbb{Z}_2[x]/(x^2)$.

Let R be a UFD and $c = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$ a nonzero nonunit element of R factorized as a product of irreducible elements. Assume that $R/(c)$ is finite. Assume further that for some $k \geq 2$ and each i with $1 \leq i \leq k$ there exists a set

$$D_i = \{p_i^{a_{i1}}, p_i^{a_{i2}}, \ldots, p_i^{a_{ir_i}}\}$$

such that $0 \leq a_{i1} < a_{i2} < \cdots < a_{ir_i} \leq a_i - 1$. Set

$$D = \{p_1^{a_{1t_1}} \ldots p_k^{a_{kt_k}} : t_i \in \{1, 2, \ldots, r_i\} \text{ for } 1 \leq i \leq k\}.$$

Then

$$\mathrm{ICG}(R/(c), D) \cong \mathrm{Cay}(R/(p_1^{a_1}), D_1) \otimes \cdots \otimes \mathrm{Cay}(R/(p_k^{a_k}), D_k),$$

where each factor on the right hand side is the Cayley graph $\mathrm{Cay}(R/(p_i^{a_i}), D_i)$ over the finite chain ring $R/(p_i^{s_i})$ with respect to the divisor set D_i. Using this isomorphism, Thongsomnuk and Meemark established the following result.

Theorem 9.17. ([292, Theorem 4.2]) *Under the assumption above, the following hold:*

(a) *if $\mathrm{ICG}(R/(c), D)$ has perfect state transfer, then $\mathrm{Cay}(R/(p_i^{a_i}), D_i)$ has perfect state transfer for some $i \in \{1, 2, \ldots, k\}$;*

(b) *if $\mathrm{Cay}(R/(p_1^{a_1}), D_1)$ has perfect state transfer, and for all $i = 2, \ldots, k$, $R/(p_i^{a_i})$ is of even characteristic and $a_{ir_i} = a_i - 1$, then $\mathrm{ICG}(R/(c), D)$ admits perfect state transfer.*

10 Distance-regular Cayley graphs

A connected graph G with diameter d is called *distance-regular* (see [92, Section 3.7]) if there exist non-negative integers $b_0, b_1, \ldots, b_{d-1}$ and $c_1, c_2, \ldots, c_d$ such that for any pair of vertices u, v at distance i, we have

$$b_i = |N_{i+1}(u) \cap N_1(v)|, \ 0 \leq i \leq d-1; \quad c_i = |N_{i-1}(u) \cap N_1(v)|, \ 1 \leq i \leq d,$$

where $N_i(u)$ is the set of vertices of G at distance i from u. It is well known (see [92, Theorem 3.7.3]) that the eigenvalues of a distance-regular graph are determined by its *intersection array* $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$. A distance-regular graph of diameter two is called a *strongly regular graph*. More explicitly, a *strongly regular graph* with parameters (n, r, s, t) is a connected

r-regular graph with n vertices in which any two adjacent vertices have exactly s common neighbours and any two non-adjacent vertices have exactly t common neighbours (see [92, Section 1.2]). The eigenvalues of such a strongly regular graph are (see [92, Theorem 3.6.5])

$$r, \quad \frac{1}{2}\left((s-t)+\sqrt{\Delta}\right), \quad \frac{1}{2}\left((s-t)-\sqrt{\Delta}\right),$$

where $\Delta = (s-t)^2 + 4(r-t)$, with the corresponding multiplicities

$$1, \quad \frac{1}{2}\left(n-1-\frac{2r+(n-1)(s-t)}{\sqrt{\Delta}}\right), \quad \frac{1}{2}\left(n-1+\frac{2r+(n-1)(s-t)}{\sqrt{\Delta}}\right).$$

The reader is referred to the monograph [65] and the survey paper [296] for comprehensive treatments of distance-regular graphs.

It is well known that strongly regular Cayley graphs are essentially partial difference sets. An r-subset S of a group Γ with order n is called an (n, r, s, t)-*partial difference set* (PDS) in Γ if the expressions gh^{-1}, for $g, h \in S$ with $g \neq h$, represent each nonidentity element in S exactly s times and represent each nonidentity element not in S exactly t times. It follows immediately that a Cayley graph $\mathrm{Cay}(\Gamma, S)$ is strongly regular if and only if S is a PDS in Γ such that $1 \notin S$ and $S^{-1} = S$.

In this section we restrict ourselves to distance-regular Cayley graphs. We barely touch strongly regular Cayley graphs since we believe that a survey on partial difference sets should be a separate treatise written by experts in the area (see in [226] for such a survey published in 1994).

10.1 Distance-regular Cayley graphs

A well-known strongly regular graph with parameters $\left(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4}\right)$ is the *Paley graph* $P(q)$, which is defined as the Cayley graph on the additive group of $\mathbb{F}_q$ with connection set the set of nonzero squares of $\mathbb{F}_q$, where q is a prime power with $q \equiv 1 \pmod 4$. The following result, obtained independently by Bridges and Mena [61], Hughes et al. [155], Ma [225], and partially by Marušič [234], gives a complete classification of strongly regular circulant graphs.

Theorem 10.1. ([61, 155, 225, 234]) *If G is a nontrivial strongly regular circulant graph, then G is isomorphic to a Paley graph $P(p)$ for some prime $p \equiv 1 \pmod 4$.*

In [240], Miklavič and Potočnik generalized this result to distance-regular circulant graphs. Denote by $K_{n,n} - nK_2$ the complete bipartite graph $K_{n,n}$ with a perfect matching removed.

Theorem 10.2. ([240, Theorem 1.2]) *Let G be a circulant graph with $n \geq 3$ vertices. Then G is distance-regular if and only if it is isomorphic to one of the following graphs:*

(a) *cycle C_n;*

(b) *complete graph K_n;*

(c) *complete t-partite graph $K_{m,\dots,m}$, where $tm = n$;*

(d) *$K_{m,m} - mK_2$, where $2m = n$ and m is odd;*

(e) *Paley graph $P(n)$, where $n \equiv 1 \pmod 4$ is a prime.*

In [242], Miklavič and Šparl classified all distance-regular Cayley graphs on abelian groups with respect to "maximal" inverse-closed generating sets. A few definitions are in order before presenting their result. The *Shrikhande graph* is the Cayley graph on $\mathbb{Z}_4 \times \mathbb{Z}_4$ with connection set $\{\pm(1,0), \pm(0,1), \pm(1,1)\}$; this is a strongly regular graph with parameters $(16, 6, 2, 2)$. The *Doob graph $D(m,n)$* (where $n, m \geq 1$) is the Cartesian product of the Hamming graph $H(n,4)$ with m copies of the Shrikhande graph. A distance-regular graph G of diameter d is called *antipodal* if the relation R on $V(G)$ defined by $xRy \Leftrightarrow d(x,y) \in \{0,d\}$ is an equivalence relation, and *non-antipodal* otherwise. In the former case the *antipodal quotient* of G is the graph with vertices the equivalence classes of R such that two equivalence classes are adjacent if and only if there is at least one edge between them in G. For example, the hypercube $H(d,2)$ is antipodal since each vertex x has a unique antipodal vertex $\bar{x}$ whose distance to x is equal to d. The antipodal quotient of $H(d,2)$ is the quotient of $H(d,2)$ with respect to the partition $\{\{x, \bar{x}\} : x \in V(H(d,2))\}$ of $V(H(d,2))$.

Theorem 10.3. ([242, Theorem 1.1]) *Let Γ be an abelian group with identity 1 and let S be an inverse-closed subset of $\Gamma \setminus \{1\}$ which generates Γ such that $S \setminus \{s, s^{-1}\}$ does not generate Γ for at least one element $s \in S$. Then $\mathrm{Cay}(\Gamma, S)$ is distance-regular if and only if it is isomorphic to one of the following graphs:*

(a) *complete bipartite graph $K_{3,3}$;*

(b) *complete tripartite graph $K_{2,2,2}$;*

(c) *$K_{6,6} - 6K_2$;*

(d) *cycle C_n for $n \geq 3$;*

(e) *Hamming graph $H(d,n)$, where $d \geq 1$ and $n \in \{2,3,4\}$;*

(f) *Doob graph $D(m,n)$, where $n, m \geq 1$;*

(g) *antipodal quotient of the hypercube $H(d,2)$, where $d \geq 2$.*

Let n, k and μ be non-negative integers and Γ a group of order n with identity 1_Γ. A k-subset S of Γ satisfying $S \cdot S^{-1} = (k - \mu)1_\Gamma + \mu\Gamma$ is called a *(n, k, μ)-difference set*; S is *trivial* if $|S| \in \{0, 1, n-1, n\}$ and *nontrivial* otherwise. Recall that Cayley graphs on dihedral groups $D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ are called *dihedrants*. Given $S, T \subseteq \mathbb{Z}_n$, let $a^S = \{a^i : i \in S\}$ and $a^T b = \{a^i b : i \in T\}$. Given $A \subseteq \mathbb{Z}_n$ and $i \in \mathbb{Z}_n$, let $i + A = \{i + a : a \in A\}$ and $iA = \{ia : a \in A\}$. Distance-regular dihedrants have been classified by Miklavič and Potočnik in [241].

Theorem 10.4. ([241, Theorem 4.1]) *Let $S, T \subseteq \mathbb{Z}_n$, where $n \geq 2$. Let $G = \mathrm{Cay}(D_{2n}, a^S \cup a^T b)$ be a connected dihedrant other than C_{2n}, K_{2n}, the complete t-partite graph $K_{m,\dots,m}$ (where $tm = 2n$), or $K_{n,n} - nK_2$. Then G is distance-regular if and only if one of the following holds:*

(a) *$S = \emptyset$ and T is a nontrivial difference set in $\mathbb{Z}_n$;*

(b) *n is even, S is a non-empty subset of $1 + 2\mathbb{Z}_n$, and either*

(i) $T \subseteq 1 + 2\mathbb{Z}_n$ and $a^{-1+S} \cup a^{-1+T}b$ is a nontrivial difference set in the dihedral group $\langle a^2, b \rangle$, or

(ii) $T \subseteq 2\mathbb{Z}_n$ and $a^{-1+S} \cup a^T b$ is a nontrivial difference set in the dihedral group $\langle a^2, b \rangle$.

Moreover, if either (a) or (b) occurs, then G is bipartite, non-antipodal and has diameter 3.

Given a prime power q and an integer $n \geq 1$, let $M(n, q)$ be the ring of $n \times n$ matrices over $\mathbb{F}_q$. Then the unitary Cayley graph of $M(n, q)$ is $G_{M(n,q)} = \text{Cay}(M(n, q), \text{GL}(n, q))$. In [172, Theorem 2.3], Kiani and Mollahajiaghaei proved that this graph is strongly regular with parameters $(q^4, q^4 - q^3 - q^2 + q, q^4 - 2q^3 - q^2 + 3q, q^4 - 2q^3 + q)$.

In [249], Momihara gave a construction of strongly regular Cayley graphs on the additive groups of finite fields based on three-valued Gauss periods. As corollaries, he obtained two infinite families and one sporadic example of new strongly regular Cayley graphs. The construction in [249] can be viewed as a generalization of the construction of strongly regular Cayley graphs given by Bamberg et al. [41].

Theorem 10.5. ([249, Theorem 1.1]) *Let q be a prime power. There exists a strongly regular Cayley graph on the additive group of $\mathbb{F}_{q^6}$ with negative Latin square type parameters $(q^6, r(q^3 + 1), q^3 + r^2 - 3r, r^2 - r)$, where $r = M(q^2 - 1)/2$, in the following cases:*

(a) *$M = 3$ and $q \equiv 7 \pmod{24}$;*

(b) *$M = 7$ and $q \equiv 11, 51 \pmod{56}$.*

Theorem 10.6. ([1, Theorem 15]) *Let R be a finite commutative ring. Let $Z^*(R)$ be the set of nonzero zero divisors of R and $\text{Cay}(R, Z^*(R))$ the Cayley graph on the additive group of R with connection set $Z^*(R)$. Then the following statements are equivalent:*

(a) *$\text{Cay}(R, Z^*(R))$ is edge-transitive;*

(b) *$\text{Cay}(R, Z^*(R))$ is strongly regular;*

(c) *R is a local ring, or $R = \mathbb{Z}_2^d$ for some $d \geq 2$, or $R = \mathbb{F}_q \times \mathbb{F}_q$ for some $q \geq 3$.*

Moreover, if R is not a local ring, then each of these statements is equivalent to that $\text{Cay}(R, Z^*(R))$ is distance-regular.

It was also noted in [1, Corollary 16] that G_R is strongly regular if and only if R is a local ring, or $R = \mathbb{Z}_2^d$ for some $d \geq 2$, or $R = \mathbb{F}_q \times \mathbb{F}_q$ for some $q \geq 3$.

All distance-regular graphs with degree 3 are known [65, Theorem 7.5.1]; all intersection arrays for distance-regular graphs with degree 4 are known [66]; and all intersection arrays for distance-regular graphs with girth 3 and degree 6 or 7 are known. With the help of these results, van Dam and Jazaeri [295] determined all Cayley graphs among such distance-regular graphs with small degree, as well as all Cayley graphs among distance-regular graphs of degree 5 with one of the known feasible intersection arrays. See [295] for details.

10.2 Distance-regular Cayley graphs with least eigenvalues -2

All graphs with least eigenvalue -2 have been classified (see [65, Section 3.12]). In particular, a distance-regular graph with least eigenvalue -2 is strongly regular or the line graph of a regular graph with girth at least five. In [280], Seidel classified all strongly regular graphs with least eigenvalue -2.

In [6], Abdollahi et al. classified all distance-regular Cayley graphs with least eigenvalue -2 and diameter at most three. We present their results in Theorems 10.7 and 10.8, but before that let us first mention a few well-known strongly regular graphs. The 5-regular *Clebsch graph* is the graph obtained from the 4-dimensional hypercube $H(4,2)$ by adding an edge between each pair of antipodal vertices. Its complement is called the 10-regular *Clebsch graph*. Both Clebsch graphs are strongly regular, with parameters $(16, 5, 0, 2)$ and $(16, 10, 6, 6)$, respectively. The *Schläfli graph* is the graph obtained from $G = L(K_8)$ (the line graph of K_8) by applying the following operations (see [92, Example 1.2.5]): Select a vertex v of G; switch edges and non-edges of G between $N_G(v)$ and $V(G) \setminus N_G(v)$, where $N_G(v)$ is the neighbourhood of v in G; and delete v from the resultant graph. This is a strongly regular graph with parameters $(27, 16, 10, 8)$. The *cocktail party graph* $CP(n)$ is the complete n-partite graph $K_{2,\ldots,2}$ with each part of size two. The *triangular graph* $T(n)$ is the line graph of K_n, and the *lattice graph* $L_2(n)$ is the line graph of $K_{n,n}$.

Theorem 10.7. ([6, Theorem 4.8]) *A graph G is a strongly regular Cayley graph with least eigenvalue at least -2 if and only if G is isomorphic to one of the following graphs:*

(a) *cycle C_5, Clebsch graph, Shrikhande graph, or Schläfli graph;*

(b) *cocktail party graph $CP(n)$ with $n \geq 2$;*

(c) *triangular graph $T(n)$, with $n = 4$, or $n \equiv 3 \pmod 4$ and $n > 4$ a prime power;*

(d) *lattice graph $L_2(n)$, with $n \geq 2$.*

The *dual* of an incidence structure D is the incidence structure obtained from D by interchanging the roles of points and blocks but retaining the incidence relation. The *incidence graph* of D is the bipartite graph with points in one part and blocks in the other part such that two vertices are adjacent if and only if they are incident in D. An incident point-block pair of D is usually called a *flag* of D. An *isomorphism* from an incidence structure D to an incidence structure D' is a bijection that maps points to points, blocks to blocks, and preserves the incidence relation. An isomorphism from D to itself is called an *automorphism* of D.

A *projective plane* is a point-line incidence structure such that any two distinct points are joined by exactly one line, any two distinct lines intersect in a unique point, and there exists a set of four points no three of which are on a common line. It is well known that for any finite projective plane π there exists an integer $n \geq 1$, called the order of π, such that each line contains $n + 1$ points, each point is on $n + 1$ lines, and the number of points and the number of lines are both equal to $n^2 + n + 1$. In other words, π is a symmetric 2-$(n^2 + n + 1, n + 1, 1)$ design. An automorphism of π is usually called a *collineation* of π. An isomorphism between π and its dual is called a *correlation* of π. A projective plane is Desarguesian or non-Desarguesian depending on whether it satisfies Desargues' Theorem.

Theorem 10.8. ([6, Theorem 5.8]) *Let G be a distance-regular Cayley graph with diameter three and least eigenvalue at least -2. Then G is isomorphic to one of the following graphs:*

(a) *cycle C_6 or C_7;*

(b) *line graph of the incidence graph of the Desarguesian projective plane of order 2 or 8;*

(c) *line graph of the incidence graph of a non-Desarguesian projective plane of order q, where $q^2 + q + 1$ is prime and q is even and at least 2×10^{11};*

(d) *line graph of the incidence graph of a projective plane of odd order with a group of collineations and correlations acting regularly on its flags.*

11 Generalizations of Cayley graphs

The concept of a Cayley graph can be generalized or modified in various ways. In this section we discuss three avenues of generalization, with an emphasis on eigenvalues of such generalized Cayley graphs.

11.1 Eigenvalues of n-Cayley graphs

Let Γ be a finite group and R, S, T be (not necessarily non-empty) subsets of Γ such that $R^{-1} = R, S^{-1} = S$ and $1 \notin R \cup S$. The *bi-Cayley graph* $\mathrm{Cay}(\Gamma; R, S, T)$ on Γ is the graph with vertex set $\Gamma \times \{0, 1\}$ such that $(g, i), (h, j)$ are adjacent if and only if one of the following holds: (i) $i = j = 0$ and $g^{-1}h \in R$; (ii) $i = j = 1$ and $g^{-1}h \in S$; (iii) $i = 0, j = 1$ and $g^{-1}h \in T$. Bi-Cayley graphs are also called *semi-Cayley graphs* in the literature (see, for example, [97]), and a bi-Cayley graph on a cyclic group is called a *bicirculant graph* or *bicirculant*. A bi-Cayley graph $\mathrm{Cay}(\Gamma; R, S, T)$ such that $T = \{1\}$ is called a *one-matching bi-Cayley graph* on Γ.

In [123], Gao and Luo proved the following result, which shows that the computation of eigenvalues of bi-Cayley graphs on abelian groups can be reduced to that of three Cayley graphs on the same group.

Theorem 11.1. ([123, Theorem 3.2]) *Let $\Gamma = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_d}$ be an abelian group, where each $n_i \geq 2$, and let $G = \mathrm{Cay}(\Gamma; R, S, T)$ be a bi-Cayley graph on Γ. Then the eigenvalues of G are given by*

$$\frac{1}{2}\left(\lambda^R_{r_1 \cdots r_d} + \lambda^S_{r_1 \cdots r_d} \pm \sqrt{(\lambda^R_{r_1 \cdots r_d} - \lambda^S_{r_1 \cdots r_d})^2 + 4|\lambda^T_{r_1 \cdots r_d}|^2}\right),$$

for $r_j = 0, 1, \ldots, n_j - 1$, $1 \leq j \leq d$, where $\lambda^R_{r_1 \cdots r_d}$, $\lambda^S_{r_1 \cdots r_d}$ and $\lambda^T_{r_1 \cdots r_d}$ are the eigenvalues of $\mathrm{Cay}(\Gamma, R)$, $\mathrm{Cay}(\Gamma, S)$ and $\mathrm{Cay}(\Gamma, T)$, respectively.

In particular, this implies that for abelian groups Γ, $\mathrm{Cay}(\Gamma; R, R, T)$ is integral provided that $\mathrm{Cay}(\Gamma, R)$ is integral (see [123, Corollary 3.5]).

In [122], Gao et al. obtained formulas for the Laplacian and signless Laplacian spectra of bi-Cayley graphs on abelian groups. As applications of their main result, special formulas for the Laplacian and signless Laplacian spectra are also given for two classes of bi-Cayley graphs, namely one-matching bi-Cayley graphs and the join of two Cayley graphs over isomorphic abelian groups. In particular, a method for constructing Laplacian and signless Laplacian integral bi-Cayley graphs was given.

Theorem 11.2. ([122, Theorem 1]) *Let* $\Gamma = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_d}$ *be an abelian group, where each* $n_i \geq 2$, *and let* $G = \mathrm{Cay}(\Gamma; R, S, T)$ *be a bi-Cayley graph on* Γ. *Then the Laplacian eigenvalues (respectively, signless Laplacian eigenvalues) of* G *are given by*

$$\frac{1}{2} \left(\mu^R_{r_1 \cdots r_d} + \mu^S_{r_1 \cdots r_d} + 2|T| \pm \sqrt{(\mu^R_{r_1 \cdots r_d} - \mu^S_{r_1 \cdots r_d})^2 + 4|\lambda^T_{r_1 \cdots r_d}|^2} \right),$$

for $r_j = 0, 1, \ldots, n_j - 1$, $1 \leq j \leq d$, *where* $\lambda^T_{r_1 \cdots r_d}$ *are the eigenvalues of* $\mathrm{Cay}(\Gamma, T)$, *and* $\mu^R_{r_1 \cdots r_d}$ *and* $\mu^S_{r_1 \cdots r_d}$ *are the Laplacian eigenvalues (respectively, signless Laplacian eigenvalues) of* $\mathrm{Cay}(\Gamma, R)$ *and* $\mathrm{Cay}(\Gamma, S)$, *respectively.*

In particular, this implies that for abelian groups Γ, $\mathrm{Cay}(\Gamma; R, R, T)$ is a Laplacian and signless Laplacian integral graph provided that $\mathrm{Cay}(\Gamma, R)$ and $\mathrm{Cay}(\Gamma, T)$ are both integral ([122, Corollary 4.6]).

A few special cases and applications of Theorem 11.2 were also given in [122]. In particular, the Laplacian and signless Laplacian eigenvalues of an interesting family of bi-Cayley graphs, namely the *I-graphs* $I(n, j, k) := \mathrm{Cay}(\mathbb{Z}_n, \{\pm j\}, \{\pm k\}, \{0\})$, were obtained. I-graphs are generalizations of generalized Petersen graphs, and Theorem 11.2 implies the following result about their Laplacian and signless Laplacian eigenvalues.

Corollary 11.3. ([122, Corollary 4.2]) *The Laplacian eigenvalues (respectively, signless Laplacian eigenvalues) of the I-graph* $I(n, j, k)$ *are given by*

$$\frac{1}{2} \left(\mu^j_r + \mu^k_r + 2 \pm \sqrt{(\mu^j_r - \mu^k_r)^2 + 4} \right),$$

for $r = 0, 1, \ldots, n-1$, *where* μ^j_r *and* μ^k_r *are the Laplacian eigenvalues (respectively, signless Laplacian eigenvalues) of* $\mathrm{Cay}(\Gamma, \{\pm j\})$ *and* $\mathrm{Cay}(\Gamma, \{\pm k\})$, *respectively.*

In [309], Zou and Meng determined the eigenvalues of the bi-circulants $\mathrm{Cay}(\mathbb{Z}_n; \emptyset, \emptyset, T)$. This was achieved through investigating connections between the eigenvalues of the Cayley digraph $\mathrm{Cay}(\Gamma, T)$ and those of the bi-Cayley graph $\mathrm{Cay}(\Gamma; \emptyset, \emptyset, T)$ for any finite abelian group Γ. They also obtained asymptotic results on the number of spanning trees of bi-circulants.

A finite group Γ is called *bi-Cayley integral* if $\mathrm{Cay}(\Gamma; \emptyset, \emptyset, T)$ is an integral graph for any $T \subseteq \Gamma$. This concept was introduced in [33], where Arezoomand and Taeri proved that a finite group is a bi-Cayley integral group if and only if it is isomorphic to $\mathbb{Z}_3$, S_3 or $\mathbb{Z}_2^d$ for some positive integer d.

Let R be a finite commutative ring with unit element $1 \neq 0$, and let $R^\times$ be the set of units of R. The one-matching bi-Cayley graph $\mathrm{Cay}(R; R^\times, R^\times, \{0\})$ on the additive group of R is called [211] the *unitary one-matching bi-Cayley graph* of R. In [211], Liu computed the energies of this graph and its line graph and determined when these graphs are hyperenergetic. He also proved [211] that neither of these graphs can be *hypoenergetic* (that is, with energy strictly less than the number of vertices).

In general, a graph (or digraph) G is called an *n-Cayley graph* (or *n*-Cayley digraph) if $\mathrm{Aut}(G)$ contains a semiregular subgroup Γ that has exactly n orbits on $V(G)$; we also say that G is an *n*-Cayley graph (or digraph) on the group Γ. A 1-Cayley graph is precisely a Cayley graph and a 2-Cayley graph is exactly a bi-Cayley graph. A 3-Cayley graph is often called a *tri-Cayley graph*, and in particular a 3-Cayley graph on a cyclic group is called a *tricirculant graph*

or *tricirculant.* In [32], a factorization of the characteristic polynomials of n-Cayley digraphs on an arbitrary group in terms of linear representations of the group was given, and eigenvalues together with the corresponding eigenspaces of any n-Cayley digraph were determined for $n \geq 2$. We omit details of these technical results.

The concept of an n-Cayley graph was also studied by Sjogren in [282] under a different name. In [282], a Γ-*graph* was defined as a graph G whose automorphism group contains a semiregular subgroup which is isomorphic to Γ. Note that this is exactly an n-Cayley graph on Γ, where n is the number of orbits of Γ on $V(G)$. (The term "regular" in [282] is meant "semiregular" by our definition in section 1.) In [282], among other things it was proved that the Laplacian spectrum of any n-Cayley graph G can be expressed in terms of the Laplacian spectrum of the quotient graph of G with respect to the partition of $V(G)$ into such orbits. As a corollary it was proved that the Laplacian spectrum of any Cayley graph of odd order consists of 0 and a set of eigenvalues each with an even multiplicity.

11.2   Strongly regular n-Cayley graphs

In [233], Marušič obtained necessary conditions for a bicirculant or tricirculant to be strongly regular, initiating the study of strongly regular n-Cayley graphs. In [97], de Resmini and Jungnickel studied strongly regular bi-Cayley graphs on finite groups G and gave a representation of such graphs in terms of suitable triples of elements in the group ring $\mathbb{Z}[G]$. Similar to the case of strongly regular Cayley graphs, a strongly regular bi-Cayley graph on G is determined by a triple (C, D, D') of subsets of G satisfying certain conditions; such a triple is called a *partial difference triple.* In [97], de Resmini and Jungnickel also proved some nonexistence results when G is abelian, especially when G is cyclic. This line of research was continued in [199], where Leung and Ma studied the case when $D \cup D'$ is contained in a proper normal subgroup of G and determined all possible partial difference triples in this case. They also investigated partial difference triples over cyclic groups, giving all possible parameters in this case, and solved a problem raised in [97].

In [191], Kutnar et al. gave a necessary condition for a tri-Cayley graph on an abelian group to be strongly regular (see [191, Proposition 4.1]) and a structural description of strongly regular tri-Cayley graphs on cyclic groups (see [191, Proposition 5.3]).

In [190], Kutnar et al. introduced a new class of graphs, called *quasi n-Cayley graphs.* Such graphs admit a group of automorphisms that fixes one vertex of the graph and acts semiregularly on the set of the rest vertices. They determined when these graphs are strongly regular, leading to the definition of quasi-partial difference family (or QPDF for short), and gave several infinite families and sporadic examples of QPDFs. They also studies properties of QPDFs and determined, under several conditions, the form of the parameters of QPDFs when the group involved is cyclic.

In [230], Malnič et al. gave a necessary condition for the existence of a strongly regular vertex-transitive bicirculant of order twice a prime and constructed three new strongly regular bicirculants having 50, 82 and 122 vertices, respectively. These graphs together with their complements form the first known pairs of complementary strongly regular bicirculants which are vertex-transitive but not edge-transitive.

A brief survey of strongly regular graphs and digraphs admitting a semiregular cyclic group of automorphisms can be found in the first part of the article [232] by Martínez. In the second

part of the same paper, Martínez studied some new types of such digraphs. By using partial sum families, he determined the form of the parameters and obtained some directed strongly regular graphs derived from these partial sum families with previously unknown parameters.

Cayley graphs on semigroups are generalizations of Cayley graphs on groups. Given a semigroup Γ and a subset S of Γ, the Cayley (di)graph $\mathrm{Cay}(\Gamma, S)$ is defined to be the digraph with vertex set Γ such that there is an arc (x, y) from x to y, where $x \neq y$, if and only if $y = sx$ for some $s \in S$. Cayley (di)graphs on semigroups have been studied extensively due to their wide applications in various areas including automata theory, but here we only mention one result relating to strongly regular graphs. A completely 0-simple inverse semigroup is called a *Brandt semigroup*. In [143, Theorem 2.12], Hao et al. obtained a necessary and sufficient condition for the connected components of a Cayley graph on a Brandt semigroup to be strongly regular, and as corollaries recovered two known results on strongly regular Cayley graphs on groups [226] and strongly regular bi-Cayley graphs on groups [97].

11.3   Cayley sum graphs

Let Γ be a finite abelian group and S a subset of Γ. The *Cayley sum graph* $\mathrm{Cay}^+(\Gamma, S)$ on Γ with respect to S is the graph with vertex set Γ in which $x, y \in \Gamma$ are adjacent if and only if $x + y \in S$. Since Γ is abelian, this is an undirected graph of degree $|S|$, with a loop at x if $x + x \in S$. If S is required to be *square-free* (that is, $x + x \notin S$ for any $x \in \Gamma$), then $\mathrm{Cay}^+(\Gamma, S)$ is an undirected graph without loops. Cayley sum graphs are also known as *addition Cayley graphs*, *addition graphs*, and *sum graphs* [86] in the literature.

Let $\Gamma = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_d}$ be a finite abelian group, where each $n_i \geq 2$. Recall that the characters of Γ are $\chi_{\mathbf{x}}$, for $\mathbf{x} = (x_1, \ldots, x_d) \in \Gamma$, where $\chi_{\mathbf{x}}$ is given in (9.2). The set of real-valued characters is $R = \{\chi_{\mathbf{x}} : \mathbf{x} \in \Gamma, \mathbf{x} + \mathbf{x} = \mathbf{0}\}$. Let C be a set containing exactly one character from each conjugate pair $\{\chi_{\mathbf{x}}, \chi_{-\mathbf{x}}\}$ (where $\mathbf{x} \in \Gamma$ and $\mathbf{x} + \mathbf{x} \neq \mathbf{0}$). Then the set of characters of Γ is $R \cup \{\chi, \overline{\chi} : \chi \in C\}$. The following result was observed by a few authors [19, 86, 102].

Theorem 11.4. ([102, Theorem 2.1]) *Let $\Gamma = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_d}$ be a finite abelian group, where each $n_i \geq 2$, and let R and C be as above. Let S be a subset of Γ. Then the multiset of eigenvalues of the Cayley sum graph $\mathrm{Cay}^+(\Gamma, S)$ is*

$$\{\chi(S) : \chi \in R\} \cup \{\pm|\chi(S)| : \chi \in C\}.$$

Moreover, the corresponding eigenvectors are χ (for $\chi \in R$) and the real and imaginary parts of $\alpha\chi$ (for $\chi \in C$ with a suitable complex scalar α which depends on $\chi(S)$ only).

A finite abelian group Γ is called a *Cayley sum integral group* [25] if for every subset S of Γ the Cayley sum graph $\mathrm{Cay}^+(\Gamma, S)$ is integral. In [25], Amooshahi and Taeri proved that all Cayley sum integral groups are represented by $\mathbb{Z}_3$ and $\mathbb{Z}_2^d$, where $d \geq 1$. In the same paper they also classified all simple connected cubic integral Cayley sum graphs.

A finite abelian group Γ is called *Cayley sum integral* if all Cayley sum graphs on any subgroup of Γ with respect to square-free subsets are integral. For an integer $k \geq 2$, let $\mathcal{A}_k$ be the class of finite abelian groups Γ such that for any subgroup Σ of Γ all Cayley sum graphs $\mathrm{Cay}^+(\Sigma, S)$ on Σ with $|S| = k$ are integral.

Theorem 11.5. ([227, Theorems 1-3]) *The following hold:*

(a) $\mathcal{A}_2$ *consists of the following groups:* $\mathbb{Z}_2^n$, $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_3^m$, $n \geq 2, m \geq 1$;

(b) $\mathcal{A}_3$ *consists of the following groups:* $\mathbb{Z}_2^n$, $\mathbb{Z}_6$, $\mathbb{Z}_8$, $n \geq 2$;

(c) *all finite groups which are Cayley sum integral are represented by:* $\mathbb{Z}_2^n$, $\mathbb{Z}_4$, $\mathbb{Z}_6$, $n \geq 1$.

The notion of a Cayley sum graph was generalized to all finite groups in [24]. Let Γ be a finite group (with multiplicative operation) and S a subset of Γ. The *Cayley sum graph* $\mathrm{Cay}^+(\Gamma, S)$ on Γ with respect to S is the digraph with vertex set Γ such that there is an arc from x to y if and only if $xy \in S$. In the case when $x^2 \in S$ for some $x \in \Gamma$, this digraph contains a semi-edge from x; such a semi-edge contributes only one to the degree of its end-vertex and the corresponding diagonal entry in the adjacency matrix. We can extend the definition of a Cayley sum graph to the case when S is a multi-set of Γ, in which case $\mathrm{Cay}^+(\Gamma, S)$ has parallel arcs. If S is normal (that is, the union of some conjugacy classes of Γ), then $\mathrm{Cay}^+(\Gamma, S)$ can be considered as an undirected graph.

Theorem 11.6. ([24, Theorem 7]) *Let Γ be a finite non-abelian group. There exists a square-free normal subset S of Γ with $|S| = 3$ such that $\mathrm{Cay}^+(\Gamma, S)$ is connected and integral if and only if Γ is isomorphic to the dihedral group D_6 of order 6.*

Given a finite group Γ and a real-valued function $\alpha : \Gamma \to \mathbb{R}$, the *Cayley sum colour graph* $\mathrm{Cay}^+(\Gamma, \alpha)$ is the complete directed graph with vertex set Γ in which each arc $(x, y) \in \Gamma \times \Gamma$ is associated with *color* $\alpha(xy)$ (see [23] and [24]). In the case when α is the characteristic function of a subset S of Γ, $\mathrm{Cay}^+(\Gamma, \alpha)$ becomes the Cayley sum graph $\mathrm{Cay}^+(\Gamma, S)$ when arcs of colour 0 are omitted. In [23] and [24, Section 3], Cayley sum colour graphs and their eigenvalues via irreducible representations of the underlying groups have been studied. The following result gives Theorem 11.4 in the special case when α is the characteristic function of $S \subseteq \Gamma$.

Theorem 11.7. ([23, Theorem 3.4]) *Let $\mathrm{Cay}^+(\Gamma, \alpha)$ be a Cayley sum colour graph of a finite abelian group Γ and $\{\chi_1, \chi_2, \ldots, \chi_n\}$ a complete set of irreducible inequivalent characters of Γ. If χ_k is real-valued, then $\sum_{g \in \Gamma} \alpha(g)\chi_k(g)$ is an eigenvalue of $\mathrm{Cay}^+(\Gamma, \alpha)$; if χ_k is not real-valued, then $\pm|\sum_{g \in \Gamma} \alpha(g)\chi_k(g)|$ are two eigenvalues of $\mathrm{Cay}^+(\Gamma, \alpha)$.*

The *anti-shift operator* maps every vector $(c_0, c_1, \ldots, c_{n-1})$ to $(c_1, c_2, \ldots, c_{n-1}, c_0)$. An $n \times n$ *anti-circulant matrix* is a matrix whose ith row is obtained from the first row by applying anti-shift operator $i - 1$ times. A graph is called *anti-circulant* if it has an anti-circulant adjacency matrix. It was observed in [23, Lemma 1.1] that a graph is anti-circulant if and only if it is a Cayley sum graph of a cyclic group. Using this and Theorem 11.7, the eigenvalues of any anti-circulant graph were computed by Amooshahi and Taeri in [23, Theorem 3.6].

Sidon sets in groups are important objects of study which have numerous applications. Let Γ be a finite abelian group and S a subset of Γ. If for every nonzero element x of Γ, there is at most one pair (a, b) of elements of Γ such that $x = a - b$, then S is called a *Sidon set* in Γ. It can be seen that $|S| < \sqrt{|\Gamma|} + (1/2)$ for any Sidon set S of Γ. Sidon sets of size close to this bound, namely those with size $\sqrt{|\Gamma|} - \delta$ for a small number δ, are the most interesting. The following result shows that, if S is a Sidon set of size $\sqrt{|\Gamma|} - \delta$, then the Cayley sum graph $\mathrm{Cay}^+(\Gamma, S)$ is pseudo-random. As before, we use $\lambda(G)$ to denote the maximum absolute value of the eigenvalues of a graph G other than the largest one.

Theorem 11.8. ([299, Theorem 2.3]) *Let Γ be a finite abelian group and S a Sidon set of Γ with $|S| = \sqrt{|\Gamma|} - \delta$. Then the Cayley sum graph $\mathrm{Cay}^+(\Gamma, S)$ satisfies*

$$\lambda(\mathrm{Cay}^+(\Gamma, S)) \leq \sqrt{2(1+\delta)|X|^{1/2}}.$$

Finally, the Cayley sum graph $\mathrm{Cay}^+(R, Z^*(R))$ on the additive group of a finite commutative ring R was studied in [1], where $Z^*(R)$ is the set of nonzero zero divisors of R.

11.4 Group-subgroup pair graphs

The following generalization of Cayley graphs was introduced in [266]. Let Γ be a group, Σ a subgroup of Γ, and S a subset of Γ such that $S \cap \Sigma$ is closed under taking inverse elements. The *group-subgroup pair graph* (or a *pair-graph*) $\mathrm{Cay}(\Gamma, \Sigma, S)$ is the graph with vertex set Γ and edges $\{x, xs\}$ for $x \in \Sigma$ and $s \in S$. Obviously, $\mathrm{Cay}(\Gamma, \Gamma, S)$ is the usual Cayley graph $\mathrm{Cay}(\Gamma, S)$. In [266], Reyes-Bustos investigated several combinatorial properties of pair-graphs and determined some of the eigenvalues of $\mathrm{Cay}(\Gamma, \Sigma, S)$, including the largest eigenvalue. Among others the following result was proved in [266].

Theorem 11.9. ([266, Theorem 5.1]) *Let Γ be a group, Σ a subgroup of Γ, and S a subset of Γ such that $S_\Sigma = S \cap \Sigma$ is inverse-closed and $S \setminus \Sigma \neq \emptyset$. Suppose that $|\Gamma : \Sigma| = k + 1 \geq 2$, with coset representatives $1, x_1, \ldots, x_k$. Then*

$$\mu^\pm = \frac{1}{2}\left(|S_\Sigma| \pm \left(|S_\Sigma|^2 + 4\sum_{i=1}^k |S \cap \Sigma x_i|^2\right)^{\frac{1}{2}}\right)$$

are eigenvalues of $\mathrm{Cay}(\Gamma, \Sigma, S)$. Moreover, the corresponding eigenfunctions $f^\pm$ are given by $f^\pm(y) = \mu^\pm$ for $y \in \Sigma$ and $f^\pm(y) = |S \cap \Sigma x_i|$ for $y \in \Sigma x_i$, $1 \leq i \leq k$.

A few other results on the spectra of pair-graphs and their applications in constructing Ramanujan graphs can be found in [266, Sections 5-6].

12 Directed Cayley graphs

Eigenvalues of directed Cayley graphs appeared at several places before this point. In this section we review some of the known results about them which were not mentioned in previous sections.

12.1 Eigenvalues of directed Cayley graphs

Let Γ be a finite group and S a subset of $\Gamma \setminus \{1\}$. Consider the Cayley digraph $\mathrm{Cay}(\Gamma, S)$ on Γ (which is undirected if and only if S is inverse-closed). Denote by $\mathrm{Irred}(\Gamma)$ the set of irreducible complex characters of Γ and by A_S the adjacency matrix of $\mathrm{Cay}(\Gamma, S)$. The following result was first noted by MacWilliams and Mann in [229].

Theorem 12.1. ([229]; see also [109, Theorem 3.1]) *Let Γ be a finite abelian group and S a subset of $\Gamma \setminus \{1\}$. Let M denote the character table of Γ, with rows indexed by $\mathrm{Irred}(\Gamma)$ and*

columns indexed by Γ in the same order as the one underlying A_S. Then

$$\frac{1}{|\Gamma|} M A_S^t \overline{M}^t = \operatorname{diag}\left(\sum_{s \in S} \chi(s)\right)_{\chi \in \operatorname{Irred}(\Gamma)}.$$

Thus the eigenvalues of $\operatorname{Cay}(\Gamma, S)$ take the form $\sum_{s \in S} \chi(s)$, for $\chi \in \operatorname{Irred}(\Gamma)$.

A matrix with real entries is called *normal* if it commutes with its transpose. This is equivalent to saying that the sum of the squares of the absolute values of its eigenvalues equals the sum of the squares of the absolute values of its entries. It can be shown (see [224, Lemma 1]) that for a finite group Γ and a subset $S \subseteq \Gamma \setminus \{1\}$, the adjacency matrix of $\operatorname{Cay}(\Gamma, S)$ is normal if and only if $S^{-1}S = SS^{-1}$. The next two results were obtained by Lyubshin and Savchenko.

Theorem 12.2. ([224, Corollaries 3 and 4]) *Let Γ be a finite group and S a minimal generating set of Γ. Then the adjacency matrix of $\operatorname{Cay}(\Gamma, S)$ is normal if and only if any two elements of S have at least one common out-neighbour in $\operatorname{Cay}(\Gamma, S)$. Moreover, if Γ is of odd order, then the adjacency matrix of $\operatorname{Cay}(\Gamma, S)$ is normal if and only if Γ is abelian.*

Theorem 12.3. ([224, Theorem 1]) *Let Γ be a finite group. If the adjacency matrix of every Cayley digraph of degree two on Γ is normal, then Γ is abelian or $\Gamma \cong Q_8 \times \mathbb{Z}_2^d$ for some positive integer d, where Q_8 is the quaternion group.*

A digraph is called *strongly connected* if it contains a directed path from any vertex to any other vertex.

Theorem 12.4. ([224, Theorem 2]) *Let Γ be a finite group. If there exists a strongly connected Cayley digraph of degree two on Γ whose adjacency matrix is normal, then one of the following holds:*

(a) *Γ is an abelian group of rank at most two;*

(b) *$\Gamma \cong \langle a, c \mid a^{2k} = c^n = 1, a^{-1}ca = c^{-1} \rangle$, for some $k \geq 1$ and $n \geq 3$;*

(c) *$\Gamma \cong \langle a, c \mid a^{4p} = 1, c^{2q} = a^{2p}, a^{-1}ca = c^{-1} \rangle$, for some $p \geq 1$ and $q \geq 3$.*

The next four theorems were proved by Godsil in [136].

Theorem 12.5. ([136, Theorem 3.1]) *Let G be a digraph with maximum degree greater than one. Then there is a Cayley digraph H such that the minimal polynomial of G divides that of H.*

Theorem 12.6. ([136, Corollary 3.2]) *The following hold:*

(a) *every algebraic integer is an eigenvalue of some Cayley digraph;*

(b) *there exist Cayley digraphs whose adjacency matrices are not diagonalizable.*

Theorem 12.7. ([136, Theorem 3.3]) *Let G be a graph with minimum degree greater than one. Then there is a Cayley graph H such that the minimal polynomial of G divides that of H.*

Theorem 12.8. ([136, Corollary 3.4]) *If θ is an eigenvalue of a symmetric integral matrix, then it is an eigenvalue of some Cayley graph.*

The last result above was sharpened by Babai in the following way.

Theorem 12.9. ([37, Theorem 1.2]) *Let A be an integral matrix. Then there exists an arc-transitive digraph G such that the minimal polynomial of A divides that of the adjacency matrix of G. Moreover, if A is symmetric, then G may be required to be a 2-arc-transitive graph.*

Part (b) of this theorem is related to a question of Cameron who asked whether there exists an arc-transitive digraph whose adjacency matrix is not diagonalizable. A negative answer to this question follows from Theorem 12.9: There exist arc-transitive digraphs with non-diagonalizable adjacency matrices.

A major tool used in the proof of Theorem 12.9 is the following result (see [37, Theorem 1.3]) which is of interest for its own sake: Every finite regular multidigraph has a finite arc-transitive covering digraph, and every finite regular multigraph has a finite 2-arc-transitive covering graph.

Cospectral Cayley digraphs were studied in [239].

12.2 Two families of directed Cayley graphs

In this section we focus on two families of Cayley digraphs introduced by Friedman in [117] (see section 8.5 for a related family of Cayley graphs). Let p be a prime. Define [117] SUMPROD(p) to be the digraph with vertex set $\mathbb{Z}_p \times \mathbb{Z}_p^\times$, with each vertex (x, y) having an arc to $(x + a, ya)$ for each $a = 1, 2, \ldots, p - 1$. In other words, SUMPROD(p) is the Cayley digraph on $\mathbb{Z}_p \times \mathbb{Z}_p^\times$ with connection set $\{(a, a) : a \in \mathbb{Z}_p^\times\}$. Obviously, this digraph has order $p(p - 1)$, with each vertex having in-degree and out-degree $p - 1$.

Let $d \geq 2$ an integer. Define [117] POWER(p, d) to be the digraph with vertex set $\mathbb{Z}_p^d$, with each vertex $(x_1, x_2, \ldots, x_d)$ having an arc to each of $(x_1 + a, x_2 + a^2, \ldots, x_d + a^d)$ for $a = 0, 1, \ldots, p - 1$. In other words, POWER(p, d) is the Cayley digraph on $\mathbb{Z}_p^d$ with connection set $\{(a, a^2, \ldots, a^d) : a \in \mathbb{Z}_p\}$. This graph has order p^d, with each vertex having in-degree and out-degree p.

The following results show that both digraphs above have relatively small second largest eigenvalue in absolute value.

Theorem 12.10. ([117, Theorem 1.1]) *The graph SUMPROD(p) has second largest eigenvalue in absolute value at most $\sqrt{p}$.*

Theorem 12.11. ([117, Theorem 1.2]) *The graph POWER(p, d) has second largest eigenvalue in absolute value at most $(d - 1)\sqrt{p}$.*

12.3 Two more families of directed Cayley graphs

Consider a finite field $\mathbb{F}_q$ of characteristic p. Take an irreducible polynomial $f(x)$ of degree $n \geq 2$ over $\mathbb{F}_q$. Let $\Gamma_f = (\mathbb{F}_q[x]/((f(x)))^* = (\mathbb{F}_q[\alpha])^* = \mathbb{F}_{q^n}^*$, where $\alpha = \bar{x}$. Of course Γ_f is a cyclic group of order $q^n - 1$. For $1 \leq d < n$, let P_d be the set of monic primary polynomials of degree d in $\mathbb{F}_q[x]$, where a polynomial in $\mathbb{F}_q[x]$ is called *primary* if it is a power of an irreducible polynomial. Set $E_d = \{g(\alpha) : g \in P_d\}$. Then E_d is a proper subset of Γ_f. In [220], Lu et al. studied the Cayley digraph $G_d(n, q, \alpha) = \text{Cay}(\Gamma_f, E_d)$. In the special case when $d = 1$, we have $E_1 = \alpha + \mathbb{F}_q$ and $G_1(n, q, \alpha)$ is Chung's difference graph [86]. In general, $G_d(n, q, \alpha)$ is a regular digraph of order $q^n - 1$ with degree $|P_d|$ approximately q^d/d. Among other things the following

result was proved by Lu et al. in [220], where an expander graph is understood as a regular digraph such that the modulus of every nontrivial eigenvalue is "much" less than the degree in some sense.

Theorem 12.12. ([220, Theorem 6]) *Let δ be a constant with $0 < \delta < 1$ such that $n + d - 1 \leq q^{d/2}(1-\delta)$. Then every nontrivial eigenvalue λ of the adjacency operator for $G_d(n, q, \alpha)$ satisfies*

$$|\lambda| \leq \frac{q^d}{d}(1 - \delta) \leq |P_d|(1 - \delta).$$

In particular, $G_d(n, q, \alpha)$ is an expander graph.

A weighted digraph $G_d^*(n, q, \alpha)$ obtained from $G_d(n, q, \alpha)$ by assigning a specific weight to each of its arcs was also considered in [220]. It was proved in [220] that $G_d^*(n, q, \alpha)$ is also an expander graph.

Let R be a finite local commutative ring with maximal ideal M and residue field $K = R/M$ equipped with the reduction map $^- : R \to K$. A polynomial in $R[x]$ is called *regular* if its reduction is not zero in the residue field. Let $f(x)$ be a primary regular non-unit polynomial in $R[x]$. That is, its reduction $\bar{f}(x)$ is a power of an irreducible polynomial in $K[x]$. By Hensel's Lemma,

$$f(x) = \delta(x)\pi(x)^s + \beta(x),$$

where $s \geq 1$ is an integer, $\delta(x)$ is a unit, $\beta(x) \in M[x]$, and $\pi(x)$ is a monic irreducible polynomial in $R[x]$ of degree $n \geq 1$ such that $\bar{\pi}(x)$ is irreducible in $K[x]$. Let $I = \langle f(x) \rangle$ be the principal ideal generated by $f(x)$. Then $R[x]/I$ is a local ring of order $|R|^{ns}$. Denote by $\Gamma_f(R)$ the unit group $(R[x]/I)^\times$. For each integer d with $1 \leq d < n$, let $P_d(R)$ be the set of monic primary polynomials of degree d in $R[x]$. Define $G_d(R, f)$ to be the Cayley digraph on $\Gamma_f(R)$ with respect to the set $P_d(R) + I$ of cosets of I with representatives in $P_d(R)$. That is, $G_d(R, f)$ has vertex set $\Gamma_f(R)$ and there is an arc from $g_1(x) + I$ to $g_2(x) + I$ if and only if $(g_2(x) + I)(g_1(x) + I)^{-1} \in P_d(R) + I$. This graph was introduced by Rasri and Meemark in [263] as a generalization of the digraph $G_d(n, q, \alpha)$ above, the latter being the digraph obtained when R is a finite field. Among other results, the following was proved in [263].

Theorem 12.13. ([263, Theorem 3.1]) *Let $G_d(R, f)$, n and s be as above. Let δ be a constant with $0 < \delta < 1$ such that*

$$ns - 1 + d|M|^d \leq |M|^{d/2}|R|^{d/2}(1 - \delta).$$

Then every nontrivial eigenvalue λ of the adjacency matrix of $G_d(R, f)$ satisfies

$$|\lambda| \leq \frac{|R|^d}{d}(1 - \delta) \leq |P_d(R)|(1 - \delta).$$

In particular, $G_d(R, f)$ is an expander graph.

It was also proved in [263, Theorem 3.2] that the weighted digraph obtained from $G_d(R, f)$ by assigning a specific weight to each arc is also an expander graph. In [263], the construction of $G_d(R, f)$ was generalized to the case when f is any regular polynomial in $R[x]$ and a counterpart of Theorem 12.13 under this general setting was given in [263, Theorem 5.3].

13 Miscellaneous

13.1 Random Cayley graphs

The well-known Alon-Roichman theorem [21] states that for $\varepsilon > 0$, with high probability, $O(\log|\Gamma|/\varepsilon^2)$ elements chosen independently and uniformly from a finite group Γ give rise to a Cayley graph with second eigenvalue no more than ε. In particular, such a graph is an expander with high probability. Landau and Russell [193], and independently Loh and Schulman [216], improved the bounds in the theorem. Following Landau and Russell, in [84] Christofides and Markström gave a new proof of the result, improving the bounds even further, and proved a generalization of the Alon-Roichman theorem to random coset graphs. See also [254] for an improvement of the Alon-Roichman theorem.

A k-regular graph G of order n is called a *c-expander* if for every set of vertices S the number of neighbours of S exceeds $c|S|(n-|S|)/n$. In [21], Alon and Roichman proved that for every δ, $0 < \delta < 1$, there exists a $c(\delta) > 0$ such that for every group Γ of order n and for every set S of $c(\delta)\log n$ random elements of Γ, the expected value of the second largest eigenvalue in absolute value of the normalized adjacency matrix of $\mathrm{Cay}(\Gamma, S \cup S^{-1})$ is at most $1 - \delta$. Moreover, the probability that such graph is a δ-expander tends to 1 as $n \to \infty$.

In [218], Lovett et al. proved that there exists a family of groups Γ_n and nontrivial irreducible representations ρ_n such that, for any constant t, the average of ρ_n over t uniformly random elements $g_1, \ldots, g_t \in \Gamma_n$ has operator norm 1 with probability approaching 1 as $n \to \infty$. More explicitly, settling a conjecture of Wigderson, they proved that there exist families of finite groups Γ for which $\Omega(\log\log|\Gamma|)$ random elements are required to bound the norm of a typical representation below 1.

The following concepts were introduced in the study of quasirandomness and expansion of graphs. An n-vertex k-regular graph G is called ε-*uniform* if, for all $S, T \subseteq V(G)$,

$$\left| e(S,T) - \frac{k}{n}|S||T| \right| \le \varepsilon k n,$$

where $e(S,T)$ is the number of edges of G between S and T. The graph G is called an (n, k, λ)-*graph* if all eigenvalues of G except k are bounded from above in absolute value by λ. The well-known expander mixing lemma asserts that, if G is an (n, k, λ)-graph, then

$$\left| e(S,T) - \frac{k}{n}|S||T| \right| \le \lambda\sqrt{|S||T|}$$

for all $S, T \subseteq V(G)$. Thus, if the second largest eigenvalue $\lambda_2(G)$ of a k-regular graph G satisfies $|\lambda_2(G)| \le \varepsilon k$, then G is ε-uniform. In [180], Kohayakawa et al. proved the following result.

Theorem 13.1. ([180, Theorem 1.6]) *Let Γ be an abelian group. Then every ε-uniform Cayley graph $\mathrm{Cay}(\Gamma, S)$ is an (n, k, λ)-graph with $n = |\Gamma|$, $k = |S|$ and $\lambda \le C\varepsilon k$ for some absolute constant C.*

In other words, having small discrepancy and having large eigenvalue gap are equivalent properties for Cayley graphs on abelian groups, even if they are sparse. This answers a question of Chung and Graham [85] for the particular case of Cayley graphs on abelian groups, while in general the answer is negative.

In [90], Conlon and Zhao generalized Theorem 13.1 to any finite group.

Theorem 13.2. ([90, Theorem 1.4]) *Let Γ be a finite group. Then every ε-uniform Cayley graph $\mathrm{Cay}(\Gamma, S)$ is an (n, k, λ)-graph with $n = |\Gamma|$, $k = |S|$ and $\lambda \leq 8\varepsilon k$.*

As a corollary, Conlon and Zhao [90] also proved that the same result holds for all vertex-transitive graphs. That is, every n-vertex k-regular ε-uniform vertex-transitive graph is an (n, k, λ)-graph with $\lambda \leq 8\varepsilon k$ (see [90, Corollary 1.5]).

13.2 Distance eigenvalues of Cayley graphs

The *distance matrix* of a graph of order n is the $n \times n$ matrix whose (u, v)-entry is equal to the distance between vertices u and v in the graph. The eigenvalues of this matrix are called the *distance eigenvalues* of the graph. A graph whose distance eigenvalues are all integers is called *distance integral*. The *distance energy* of a graph is defined as the sum of the absolute values of its distance eigenvalues.

In [157], Ilić characterized the distance eigenvalues of integral circulant graphs $\mathrm{ICG}(n, D)$ and proved that these graphs have integral distance eigenvalues. In the same paper he also computed the distance eigenvalues and distance energy of the unitary Cayley graphs $\mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^\times)$.

Let (W, S) be a finite Coxeter system, and let $T = \{wsw^{-1} : w \in W, s \in S\}$ be the set of all reflections of W. The Cayley graphs $\mathrm{Cay}(W, S)$ and $\mathrm{Cay}(W, T)$ are called the *weak order graph* and *absolute order graph* of (W, S), respectively. It is known that every $w \in W$ can be written as a word in the simple reflections S (respectively, reflections T), and the minimum number of such reflections that must be used is the length $\ell_S(w)$ (respectively, absolute length $\ell_T(w)$) of w. It is also known that the graph distance between two vertices $u, v \in W$ in these two graphs is equal to $\ell_S(uv^{-1})$ and $\ell_T(uv^{-1})$, respectively. In other words, the distance matrices of $\mathrm{Cay}(W, S)$ and $\mathrm{Cay}(W, T)$ are $(\ell_S(uv^{-1}))_{u,v \in \Gamma}$ and $(\ell_T(uv^{-1}))_{u,v \in \Gamma}$, respectively. The following result was obtained by Renteln [265].

Theorem 13.3. ([265, Theorems 1 and 6]) *Let (W, S) be a finite Coxeter system and T the set of all reflections of W. Then the distance eigenvalues of the absolute order graph $\mathrm{Cay}(W, T)$ are given by*

$$\frac{1}{\chi(1)} \sum_K |K| \ell_T(w_K) \chi(w_K), \;\; \text{with multiplicity } \chi(1)^2,$$

where K runs over all conjugacy classes of W, w_K is any element of K, and χ ranges over all irreducible characters of W. Moreover, $\mathrm{Cay}(W, T)$ is distance integral.

Using this result, Renteln computed explicitly the distance spectra of some absolute order graphs (see [265, Section 3.3]) and in particular proved the following result for the dihedral group D_{2n} of order $2n$.

Theorem 13.4. ([265, Theorem 9]) *The characteristic polynomial of the distance matrix of the absolute order graph $\mathrm{Cay}(D_{2n}, T)$ of D_{2n} is given by*

$$(x - 3n + 2)(x - n + 2)(x + 2)^{2n-2}.$$

The weak order graph of D_{2n} is simply the cycle of length $2n$ and so its distance eigenvalues can be easily computed (see [265, Theorem 16]). However, it was noted in [265] that in general the computation of the distance spectrum of the weak order graphs is more complicated and that

there appear to be far fewer nontrivial eigenvalues in the weak order case than the absolute order case. Computational results in [265] indicate that in most cases the majority of the distance eigenvalues of weak order graphs are zero, and in some cases the distance spectra are integral. In [265], the distance eigenvalues of the weak order graphs of Coxeter groups of a few types were given, a variant of the distance matrix of $\mathrm{Cay}(W, S)$ was introduced, the eigenvalues of the former were related to that of the latter, and a few open questions were posed.

A real (respectively, complex) *reflection group* is a finite group generated by reflections of a Euclidean (respectively, unitary) vector space. A reflection group is *irreducible* if that reflection representation is irreducible.

Let W be a reflection group on V. For $w \in V$, define $\mathrm{codim}(w)$ to be the codimension of the fixed point space $V^w = \{v \in W : wv = v\}$. If W is a real reflection group, then $\ell_T(w) = \mathrm{codim}(w)$ for all $w \in W$, but this is not true in general for complex reflection groups. The *codimension matrix* of W is defined to be the matrix $(\mathrm{codim}(uv^{-1}))_{u,v \in W}$. The spetrum of this matrix is called the *codimension spectrum* of W.

Let δ_T denote the characteristic function of T. Since δ_T, ℓ_T and codim are all class functions for W, Theorem 2.6 can be used to prove the following result.

Theorem 13.5. ([116, Section 4.3]) *Let W be a finite complex reflection group and T the set of all reflections of W. Then the eigenvalues of $\mathrm{Cay}(W, T)$, the distance eigenvalues of $\mathrm{Cay}(W, T)$ and the eigenvalues of the codimension matrix of W are given by*

$$\theta_\chi(f) = \frac{1}{\chi(1)} \sum_{w \in W} f(w)\chi(w), \ \ with \ multiplicity \ \chi(1)^2,$$

with χ ranging over the irredducible characters of W, for $f = \delta_T, \ell_T, \mathrm{codim}$, respectively.

Moreover, the largest eigenvalue of $\mathrm{Cay}(W, T)$, the largest distance eigenvalue of $\mathrm{Cay}(W, T)$ and the largest eigenvalue of the codimension matrix of W are $\theta_1(f)$ corresponding to the trivial character, for $f = \delta_T, \ell_T, \mathrm{codim}$, respectively.

Note that for $f = \delta_T$ the formula above is the same as the one in Theorem 13.3.

Theorem 13.6. ([116, Corollary 4.11 and Theorem 5.4]) *Let W be a finite irreducible complex reflection group and T the set of all reflections of W. Then the Cayley graph $\mathrm{Cay}(W, T)$ is integral and distance integral, and the codimension spectrum of W is integral as well.*

Note that the distance integrality of $\mathrm{Cay}(W, T)$ in Theorem 13.6 extends Theorem 13.4 from irreducible real reflection groups to irreducible complex reflection groups.

13.3   Others

Fullerenes are of great importance for chemistry as evidenced by the well-known Buckminis-terfullerene C_{60}. A *fullerene* can be represented by a 3-regular graph on a closed surface with pentagonal and hexagonal faces such that its vertices are carbon atoms of the molecule and two vertices are adjacent if there is a bond between the corresponding atoms. Fullerenes exist in the sphere, torus, projective plane, and the Klein bottle. A fullerene is called *toroidal* if it lies on the torus. In [167], Kang classified all fullerenes which are Cayley graphs and determined their eigenvalues.

The *power graph* of a group Γ is the graph with vertex set Γ in which two distinct elements x, y are adjacent if and only if $x^m = y$ or $y^m = x$ for some positive integer m. In [78], Chattopadhyay and Panigrahi studied relations between the power graph and the unitary Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^\times)$ of the cyclic group $\mathbb{Z}_n$.

Given a graph G of order n, say, with vertex set $\{1, 2, \ldots, n\}$, define $S(G) = \{e_i + e_j : ij \in E(G)\} \subseteq \mathbb{Z}_2^n$, where $\{e_1, e_2, \ldots, e_n\}$ is the standard basis of $\mathbb{Z}_2^n$. In [260], Qin et al. observed that G can be isometrically embedded as a subgraph of the cubelike graph $\mathrm{Cay}(\mathbb{Z}_2^n, S(G))$. Among other things they found some relations between the spectrum of G and that of $\mathrm{Cay}(\mathbb{Z}_2^n, S(G))$.

A Cayley graph $G = \mathrm{Cay}(\Gamma, S)$ is called *normal edge-transitive* if the normalizer of Γ in $\mathrm{Aut}(G)$ is transitive on the set of edges of G. In [126], the eigenvalues of normal edge-transitive Cayley graphs on D_{2n} and T_{4n} were given by Ghorbani, where D_{2n} is the dihedral group of order $2n$ and $T_{4n} = \langle a, b \mid a^{2n} = 1, b^2 = a^n, bab^{-1} = a^{-1} \rangle$.

Denote by Γ_n the multiplicative group of the upper unitriangular 2×2 matrices over $\mathbb{Z}$ mod n. In [139], Gupta et al. studied $\mathrm{Cay}(\Gamma_n, S)$ for a certain set S of two generators of Γ_n. Among their findings are the spectra and energies of the adjacency, Laplacian, normalized Laplacian and signless Laplacian matrices of $\mathrm{Cay}(\Gamma_n, S)$ when n is odd.

It is known that for any vertex-transitive graph G and any eigenvalue λ of G the multiplicity of λ is decreased by one as an eigenvalue of $G - v$ for any $v \in V(G)$. In [278], Savchenko proved that the same statement holds for vertex-transitive digraphs.

Let p be an odd prime and $m \geq 2$ an integer such that $d = \gcd(p - 1, m)$ divides $(p - 1)/2$. The *generalized Paley graph* X_p^m was defined by Johnson et al. [163] to be the Cayley graph on the additive group of $\mathbb{F}_p$ with respect to $\{a^m : a \in \mathbb{F}_p^*\}$. In [163], it was shown that the isoperimetric number $h(X_p^m)$ of X_p^m satisfies $(p + (1 - d)\sqrt{p})/2d \leq h(X_p^m) \leq (4 \sum_{i=1}^k \gamma_i)/(p - 1)$, where $\{a^m : a \in \mathbb{F}_p^*\} = \{\pm\gamma_1, \ldots, \pm\gamma_k\}$ with $k = (p - 1)/2d$ and $0 \leq \gamma_i \leq (p - 1)/2$ for each i. Note that $\{a^m : a \in \mathbb{F}_p^*\} = \langle \omega^m \rangle$ is a subgroup of the multiplicative group $\mathbb{F}_p^*$ of $\mathbb{F}_p$, where ω is a pth primitive root of $\mathbb{F}_p$. Since $d = \gcd(p - 1, m)$, we have $\langle \omega^m \rangle = \langle \omega^d \rangle$. Hence X_p^m is a special generalized Paley graph $\mathrm{GPaley}(p, (p - 1)/d)$ introduced by Lim and Praeger in [209]. In general, for any prime power q and divisor $k \geq 2$ of $q - 1$ such that either q or $(q - 1)/k$ is even, the *generalized Paley graph* $\mathrm{GPaley}(q, (q - 1)/k)$ was defined [209] as the Cayley graph on the additive group of $\mathbb{F}_q$ with respect to the unique subgroup of the multiplicative group $\mathbb{F}_q^*$ of order $(q - 1)/k$.

A graph G is called a *t-existentially closed* (*t-e.c.*) *graph*, where $t \geq 1$ is an integer, if for any disjoint (but not necessarily non-empty) subsets A, B of $V(G)$ satisfying $|A \cup B| = t$, there exists a vertex $z \in V(G) \setminus (A \cup B)$ adjacent to all vertices of A but none of B. It is known that random graphs are t-e.c. for any $t \geq 1$. Another property shared by random graphs is the one of being best pseudo-random. In the case of regular graphs, the expander mixing lemma implies that this property can be defined using eigenvalues: A regular graph of order n and degree $d(n)$ is *best pseudo-random* if $\lambda(G) = O(\sqrt{d(n)})$ as $n \to \infty$, where $\lambda(G)$ is the largest eigenvalue of G other than $d(n)$ in absolute value. Many known t-e.c. graphs are best pseudo-random, and it was not clear whether there are t-e.c. graphs which are not best pseudo-random. In [277], Satake answered this question affirmatively using a special family of quadratic unitary Cayley graphs (see section 5.2). He proved that for any $t \geq 1$, any prime power $q \equiv 1 \pmod{4}$ and any odd integer $e \geq 1$, if $q^e - (t2^{t-1} - 2^t + 1)q^{e-\frac{1}{2}} - t2^t q^{e-1} + t2^{t-1} > 0$, then the quadratic unitary Cayley graph $\mathcal{G}_{\mathbb{Z}_{q^e}}$ is a t-e.c. graph. This together with Theorem 5.9 gives an infinite family of

t-e.c. graphs which are not best pseudo-random for any $t \geq 1$.

A *mixed graph* is a graph in which both edges and arcs may present. The *mixed adjacency matrix* $M(G)$ of a mixed graph G of order n is the matrix whose (u, v)-entry is 1 if there is an edge between u and v or an arc from u to v, -1 if there is an arc from v to u, and 0 otherwise. In [11], Adiga et al. studied various spectral properties of $M(G)$ and in particular obtained bounds on the mixed energy of G. In the same paper they introduced the *mixed unitary Cayley graph* M_n: The vertex set of M_n is $\{0, 1, \ldots, n-1\}$; for $0 \leq i < j \leq n-1$ with $\gcd(j-i, n) = 1$, there is an edge between i and j if $\left(\frac{j-i}{n}\right) = 1$, there is an arc from i to j if $\left(\frac{j-i}{n}\right) = -1$ and $j - i < \lfloor n/2 \rfloor$, and there is an arc from j to i if $\left(\frac{j-i}{n}\right) = -1$ and $j - i > \lfloor n/2 \rfloor$, where $\left(\frac{a}{n}\right)$ denotes the Jacobi symbol. With the help of Ramanujan sums, they determined the eigenvalues of M_n in [11, Theorem 4.15] (see also [10]) in the case when n has an even number of prime factors congruent to 3 modulo 4. The energies of some mixed unitary Cayley graphs were computed by Adiga and Rakshith in [10].

A *signed graph* is a graph in which each edge is labelled as positive or negative. A signed graph is called *balanced* if the number of negative edges in each cycle is even. The *adjacency matrix* of a signed graph G is obtained from that of the underlying graph of G by changing the entries corresponding to negative edges from 1 to -1, and the *energy* $\mathcal{E}(G)$ of G is the sum of the absolute values of the eigenvalues of the adjacency matrix of G. A signed graph G of order n is said to be *hyperenergetic* if $\mathcal{E}(G) > 2n - 2$. If G is a signed graph, define its *line signed graph* $L(G)$ to be the signed graph whose underlying graph is that of the line graph of the underlying graph of G, such that the sign of an edge $\{e, f\}$ of $L(G)$ is negative if and only if both e and f are negative edges of G. Let R be a finite commutative ring. In [238], Meemark and Suntornpoch defined the *unitary Cayley signed graph* $G(R)$ of R to be the signed graph whose underlying graph is the unitary graph $\mathrm{Cay}(R, R^\times)$ such that an edge xy (where $x, y \in R, x - y \in R^\times$) is negative if and only if $\{x, y\} \cap R^\times = \emptyset$. In the same paper they also determined when $G(R)$ is balanced, when $L(G(R))$ is balanced, when $G(R)$ is hyperenergetic and balanced, and when $L(G(R))$ is hyperenergetic and balanced.

The *resistance distance* r_{uv} between two vertices u, v in a connected graph G is the effective electrical resistance between them when unit resistors are placed on every edge of G. The *Kirchhoff index* of G is the quantity $Kf(G) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} r_{uv}$. It is known that $Kf(G)$ is equal to the order of G times the sum of the reciprocals of the nonzero Laplacian eigenvalues of G. In [124], Gao et al. obtained closed-form formulas for the Kirchhoff index and resistance distances of Cayley graphs on finite abelian groups in terms of the Laplacian eigenvalues and eigenvectors, respectively. In particular, they gave formulas for the Kirchhoff index of the hexagonal torus network, the multidimensional torus and the t-dimensional cube, as well as formulas for the Kirchhoff index and resistance distances of complete multipartite graphs.

In [138], Grigorchuk and Nowak studied the relation between the diameter, the first positive eigenvalue of the discrete p-Laplacian, and the ℓ_p-distortion of a finite graph. They proved an inequality connecting these three quantities, and apply it to families of Cayley and Schreier graphs. They also showed that the ℓ_p-distortion of Pascal graphs is bounded, which allows one to obtain estimates for the convergence to zero of the spectral gap.

In [244], Minei and Skogman presented a block diagonalization method for the adjacency matrices of two types of covering graphs, using the irreducible representations of the Galois group of the covering graph over the base graph. The first type of covering graph is the Cayley

graph over the finite ring $\mathbb{Z}_{p^n}$, and the second one resembles lattices with vertices $\mathbb{Z}_n \times \mathbb{Z}_n$ for large n. Using the block diagonalization method, they obtained explicit formulas for the eigenvalues of one lattice and nontrivial bounds on the eigenvalues of another lattice.

In [192], Lafferty and Rockmore described numerical computation of eigenvalue spacings of 4-regular Cayley graphs on cyclic and symmetric groups, and of two-dimensional special linear groups over prime fields. Denote by $P(s)$ the number of indices where the gap between two consecutive distinct eigenvalues of a Cayley graph is at most s. It was observed [192] that for the above-mentioned groups, after a linear transformation the function $P(s)$ approximates $1 - e^{-s}$. In contrast, in random 4-regular graphs a linear transformation of $P(s)$ seems to approximate $1 - e^{-\pi s^2/4}$.

Consider a continuous-time quantum walk on a graph G with transition matrix $H_G(t)$ as defined in (9.1). The probability that at time τ the quantum walk with initial state u is in state v is given by $|H_G(\tau)_{u,v}|^2$. We say that G admits *uniform mixing* at time τ if this probability is the same for all vertices u and v. As noticed in [135], uniform mixing on graphs is rare, and most known examples are integral Cayley graphs on abelian groups. In [135], Godsil and Zhan constructed among other things infinite families of Cayley graphs on $\mathbb{Z}_n^d$ admitting uniform mixing. Several other results about uniform mixing on Cayley graphs can also be found in [135] and the references therein.

Given integers $k \geq 2$, $N \geq 0$ and $M \geq 1$, the *spider-web graph* $\mathcal{S}_{k,N,M}$ is the graph with vertex set $\{0, 1, \ldots, k-1\}^N \times \mathbb{Z}_M$ such that each vertex $((x_1, \ldots, x_N), i)$ is adjacent to $((x_2, \ldots, x_N, y), i+1)$, for $y \in \{0, 1, \ldots, k-1\}$. In [137], Grigorchuk et al. proved that, as $N, M \to \infty$, $\mathcal{S}_{k,N,M}$ converges in some sense to the Cayley graph on the lamplighter group $\mathbb{Z}_k \operatorname{wr} \mathbb{Z}$. In the proof of this result, they realised $\mathcal{S}_{k,N,M}$ as the tensor product of the de Bruijn graphs $B_{k,N}$ with cycle C_M. Since the spectra of de Bruijn graphs are known, this enabled them to compute the spectra of spider-web graphs by relating to the Laplacian of the Cayley graph on the lamplighter group.

Let n and t be integers with $n \geq 2$ and $1 \leq t \leq n$. As before, let S_n be the symmetric group over $[n] = \{1, 2, \ldots, n\}$. A subset $A \subset S_n$ is called *t-set-intersecting* if for any $\sigma, \pi \in A$ there exists some t-set $T \subset [n]$ such that $\sigma(T) = \pi(T)$. Generalizing a result of Frankl and Deza (for $t = 1$) and settling a conjecture by Körner (for $t = 2$), Ellis [111] proved that, if n is sufficiently large depending on t, and $A \subset S_n$ is t-set-intersecting, then $|A| \leq t!(n-t)!$ and equality holds only if A is a coset of the stabilizer of a t-set. In the proof of this result the Cayley graph $G_{(t)}$ on S_n with connection set generated by the set $D_{(t)}$ of t-derangements in S_n played a key role, where an element of S_n is called a *t-derangement* if it fixes no t-set of $[n]$ setwise. This graph is called the *t-derangement graph* on $[n]$ and can be defined equivalently as the graph with vertex set S_n in which $\sigma, \pi \in S_n$ are adjacent if and only if $\sigma(T) \neq \pi(T)$ for any t-set $T \subset [n]$. In particular, $G_{(1)}$ is the derangement graph on $[n]$ as seen in section 3.3.4. A t-set-intersecting family of permutations in S_n is precisely an independent set in $G_{(t)}$. Thus the task is to bound the independence number of $G_{(t)}$ and analyze maximum independent sets in $G_{(t)}$. A key step towards this goal is to construct a 'pseudo-adjacency matrix' A for $G_{(t)}$, which is a suitable real linear combination of the adjacency matrices of certain subgraphs of $G_{(t)}$; the subgraphs used are normal Cayley graphs. This enabled Ellis [111] to apply a weighted version of Hoffman's theorem to obtain the desired independence number using the spectra of matrix A.

Finally, let us mention three families of Cayley graphs from the domain of interconnection networks. Let $d \geq 1$ be an integer. The *cube-connected-cycle graph* $CCC(d)$ is obtained

from the hypercube $H(d, 2)$ by replacing each vertex of $H(d, 2)$ by a cycle of length d and can be defined as a certain Cayley graph on the group $\mathbb{Z}_2^n \rtimes \mathbb{Z}_n$ (see [147, 247] for details). The *shuffle-exchange graph* $SE(d)$ is defined to have vertex set $\mathbb{Z}_2^d$ such that $(a_1, a_2, \ldots, a_d)$ and $(b_1, b_2, \ldots, b_d)$ are adjacent if and only if either they differ at only the last coordinate, or $(b_1, b_2, \ldots, b_d) = (a_2, a_3, \ldots, a_d, a_1)$, or $(b_1, b_2, \ldots, b_d) = (a_d, a_1, \ldots, a_{d-1})$. Both $CCC(d)$ and $SE(d)$ are popular networks in parallel computing [147]. The *spectral set* of a graph is the set of its eigenvalues with multiplicities ignored. In [267], Riess et al. determined the spectral sets of $CCC(d)$ and $SE(d)$ for $d \geq 3$. It was noted that for odd integers $d \geq 3$ the spectral sets of these two graphs are identical, and for even integers $d \geq 4$ the spectral set of $SE(d)$ is a proper subset of that of $CCC(d)$.

Let $n \geq 2$ be an even integer and Δ an integer with $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$. The *Knödel graph* $W_{\Delta,n}$ is the graph with vertex set $\{(i, j) : i = 1, 2,\ 0 \leq j \leq \frac{n}{2} - 1\}$ and edges joining $(1, j)$ and $(2, j + 2^k - 1 \bmod \frac{n}{2})$ for $0 \leq j \leq \frac{n}{2} - 1$ and $0 \leq k \leq \Delta - 1$. It is not difficult to see that $W_{\Delta,n}$ is the Cayley graph on the semidirect product $\mathbb{Z}_{n/2} \rtimes \mathbb{Z}_2$ (with the underlying action given by $y^x = (-1)^x y$, $x \in \mathbb{Z}_2$, $y \in \mathbb{Z}_{n/2}$) with respect to the connection set $\{(1, 2^k - 1) : 0 \leq k \leq \Delta - 1\}$. Knödel graphs have been studied extensively as a topological structure for interconnection networks. Computational issues pertaining to the eigenvalues of Knödel graphs were considered by Harutyunyan and Morosan in [146]. Recently, Balakrishnan et al. [261] noted that the eigenvalues of $W_{\Delta,n}$ are $\pm |\sum_{k=0}^{\Delta-1} \omega_n^{2^{k+1}t}|$, $0 \leq t \leq \frac{n}{2} - 1$, and using this fact they determined the eigenvalues of $W_{\Delta,2^\Delta}$ together with their multiplicities, for all $\Delta \geq 4$.

References

[1] G. Aalipour and S. Akbari. Some properties of a Cayley graph of a commutative ring. *Comm. Algebra*, 42(4):1582–1593, 2014.

[2] A. Abdollahi, S. Janbaz, and M. Ghahramani. A large family of cospectral Cayley graphs over dihedral groups. *Discrete Math.*, 340(5):1116–1121, 2017.

[3] A. Abdollahi, S. Janbaz, and M. Jazaeri. Groups all of whose undirected Cayley graphs are determined by their spectra. *J. Algebra Appl.*, 15(9):1650175, 15 pages, 2016.

[4] A. Abdollahi and M. Jazaeri. On groups admitting no integral Cayley graphs besides complete multipartite graphs. *Appl. Anal. Discrete Math.*, 7(1):119–128, 2013.

[5] A. Abdollahi and M. Jazaeri. Groups all of whose undirected Cayley graphs are integral. *European J. Combin.*, 38:102–109, 2014.

[6] A. Abdollahi, E. R. van Dam, and M. Jazaeri. Distance-regular Cayley graphs with least eigenvalue -2. *Des. Codes Cryptogr.*, 84(1-2):73–85, 2017.

[7] A. Abdollahi and E. Vatandoost. Which Cayley graphs are integral? *Electron. J. Combin.*, 16(1):R122, 17 pages, 2009.

[8] A. Abdollahi and E. Vatandoost. Integral quartic Cayley graphs on abelian groups. *Electron. J. Combin.*, 18(1):P89, 14 pages, 2011.

[9] A. Ádám. Research problems 2–10. *Journal of Combinatorical Theory*, 2(3):393, 1967.

[10] C. Adiga and B. R. Rakshith. On spectra of unitary Cayley mixed graph. *Trans. Comb.*, 5(2):1–9, 2016.

[11] C. Adiga, B. R. Rakshith, and W. So. On the mixed adjacency matrix of a mixed graph. *Linear Algebra Appl.*, 495:223–241, 2016.

[12] C. Adiga, E. Sampathkumar, and M. A. Sriraj. Color energy of a unitary Cayley graph. *Discuss. Math. Graph Theory*, 34(4):707–721, 2014.

[13] A. Ahmady. *Integral Cayley graphs*. ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)– Simon Fraser University (Canada).

[14] A. Ahmady, J. P. Bell, and B. Mohar. Integral Cayley graphs and groups. *SIAM J. Discrete Math.*, 28(2):685–701, 2014.

[15] D. N. Akhiezer. On the eigenvalues of the Coxeter Laplacian. *J. Algebra*, 313(1):4–7, 2007.

[16] R. Akhtar, T. Jackson-Henderson, R. Karpman, M. Boggess, I. Jiménez, A. Kinzel, and D. Pritikin. On the unitary Cayley graph of a finite ring. *Electron. J. Combin.*, 16(1):R117, 13 pages, 2009.

[17] N. T. Allen. On the spectra of certain graphs arising from finite fields. *Finite Fields Appl.*, 4(4):393–440, 1998.

[18] N. Alon. Eigenvalues and expanders. *Combinatorica*, 6(2):83–96, 1986.

[19] N. Alon. Large sets in finite fields are sumsets. *J. Number Theory*, 126(1):110–118, 2007.

[20] N. Alon and V. D. Milman. λ_1, isoperimetric inequalities for graphs, and superconcentrators. *J. Combin. Theory Ser. B*, 38(1):73–88, 1985.

[21] N. Alon and Y. Roichman. Random Cayley graphs and expanders. *Random Structures Algorithms*, 5(2):271–284, 1994.

[22] R. C. Alperin and B. L. Peterson. Integral sets and Cayley graphs of finite groups. *Electron. J. Combin.*, 19(1):P44, 12 pages, 2012.

[23] M. Amooshahi and B. Taeri. Cayley sum color and anti-circulant graphs. *Linear Algebra Appl.*, 466:409–420, 2015.

[24] M. Amooshahi and B. Taeri. On Cayley sum graphs of non-abelian groups. *Graphs Combin.*, 32(1):17–29, 2016.

[25] M. Amooshahi and B. Taeri. On integral Cayley sum graphs. *Indian J. Pure Appl. Math.*, 47(4):583–601, 2016.

[26] J. Angel. Finite upper half planes over finite fields. *Finite Fields Appl.*, 2(1):62–86, 1996.

[27] J. Angel, N. Celniker, S. Poulos, A. Terras, C. Trimble, and E. Velasquez. Special functions on finite upper half planes. In *Hypergeometric functions on domains of positivity, Jack polynomials, and applications (Tampa, FL, 1991)*, volume 138 of *Contemp. Math.*, pages 1–26. Amer. Math. Soc., Providence, RI, 1992.

[28] J. Angel, S. Poulos, A. Terras, C. Trimble, and E. Velasquez. Spherical functions and transforms on finite upper half planes: eigenvalues of the combinatorial Laplacian, uncertainty, traces. In *Geometry of the spectrum (Seattle, WA, 1993)*, volume 173 of *Contemp. Math.*, pages 15–70. Amer. Math. Soc., Providence, RI, 1994.

[29] J. Angel, C. Trimble, B. Shook, and A. Terras. Graph spectra for finite upper half planes over rings. *Linear Algebra Appl.*, 226/228:423–457, 1995.

[30] J. P. Angel. *Finite Upper Half Planes over Finite Rings and Their Associated Graphs.* ProQuest LLC, Ann Arbor, MI, 1993. Thesis (Ph.D.)–University of California, San Diego.

[31] R. J. Angeles-Canul, R. M. Norton, M. C. Opperman, C. C. Paribello, M. C. Russell, and C. Tamon. Perfect state transfer, integral circulants, and join of graphs. *Quantum Inf. Comput.*, 10(3-4):325–342, 2010.

[32] M. Arezoomand and B. Taeri. On the characteristic polynomial of n-Cayley digraphs. *Electron. J. Combin.*, 20(3):Paper 57, 14, 2013.

[33] M. Arezoomand and B. Taeri. A classification of finite groups with integral bi-Cayley graphs. *Trans. Comb.*, 4(4):55–61, 2015.

[34] M. F. Atiyah and I. G. Macdonald. *Introduction to Commutative Algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[35] S. V. Avgustinovich, E. N. Khomyakova, and E. V. Konstantinova. Multiplicities of eigenvalues of the star graph. *Sib. Èlektron. Mat. Izv.*, 13:1258–1270, 2016.

[36] L. Babai. Spectra of Cayley graphs. *J. Combin. Theory Ser. B*, 27(2):180–189, 1979.

[37] L. Babai. Arc transitive covering digraphs and their eigenvalues. *J. Graph Theory*, 9(3):363–370, 1985.

[38] R. Bacher. Valeur propre minimale du laplacien de Coxeter pour le groupe symétrique. *J. Algebra*, 167(2):460–472, 1994.

[39] R. Balakrishnan. The energy of a graph. *Linear Algebra Appl.*, 387:287–295, 2004.

[40] K. Balińska, D. Cvetković, Z. Radosavljević, S. Simić, and D. Stevanović. A survey on integral graphs. *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.*, 13:42–65, 2002.

[41] J. Bamberg, M. Lee, K. Momihara, and Q. Xiang. A New Infinite Family of Hemisystems of the Hermitian Surface. *Combinatorica*, 38(1):43–66, 2018.

[42] E. Bannai, O. Shimabukuro, and H. Tanaka. Finite analogues of non-Euclidean spaces and Ramanujan graphs. *European J. Combin.*, 25(2):243–259, 2004.

[43] E. Bannai, O. Shimabukuro, and H. Tanaka. Finite Euclidean graphs and Ramanujan graphs. *Discrete Math.*, 309(20):6126–6134, 2009.

[44] M. Bašić and M. D. Petković. Some classes of integral circulant graphs either allowing or not allowing perfect state transfer. *Appl. Math. Lett.*, 22(10):1609–1615, 2009.

[45] M. Bašić and M. D. Petković. Perfect state transfer in integral circulant graphs of non-square-free order. *Linear Algebra Appl.*, 433(1):149–163, 2010.

[46] M. Bašić, M. D. Petković, and D. Stevanović. Perfect state transfer in integral circulant graphs. *Appl. Math. Lett.*, 22(7):1117–1121, 2009.

[47] M. Bašić. Characterization of quantum circulant networks having perfect state transfer. *Quantum Inf. Process.*, 12(1):345–364, 2013.

[48] M. Bašić. Which weighted circulant networks have perfect state transfer? *Inform. Sci.*, 257:193–209, 2014.

[49] M. Bašić. Perfect state transfer between non-antipodal vertices in integral circulant graphs. *Ars Combin.*, 122:65–78, 2015.

[50] M. Bašić and A. Ilić. On the automorphism group of integral circulant graphs. *Electron. J. Combin.*, 18(1):Paper 68, 21 pages, 2011.

[51] J. Bell and M. Minei. Spectral analysis of the affine graph over the finite ring. *Linear Algebra Appl.*, 414(1):244–265, 2006.

[52] A. Bernasconi, C. Godsil, and S. Severini. Quantum networks on cubelike graphs. *Phys. Rev. A (3)*, 78(5):052320, 5 pages, 2008.

[53] P. Berrizbeitia and R. E. Giudici. Counting pure k-cycles in sequences of Cayley graphs. *Discrete Math.*, 149(1-3):11–18, 1996.

[54] P. Berrizbeitia and R. E. Giudici. On cycles in the sequence of unitary Cayley graphs. *Discrete Math.*, 282(1-3):239–243, 2004.

[55] K. Bibak, B. M. Kapron, and V. Srinivasan. The Cayley graphs associated with some quasi-perfect Lee codes are Ramanujan graphs. *IEEE Trans. Inform. Theory*, 62(11):6355–6358, 2016.

[56] N. Biggs. *Algebraic Graph Theory*. Cambridge University Press, Cambridge, second edition, 1993.

[57] G. Bini and F. Flamini. *Finite Commutative Rings and Their Applications*, volume 680 of *The Kluwer International Series in Engineering and Computer Science*. Kluwer Academic Publishers, Boston, MA, 2002. With a foreword by Dieter Jungnickel.

[58] S. R. Blackburn and I. E. Shparlinski. On the average energy of circulant graphs. *Linear Algebra Appl.*, 428(8-9):1956–1963, 2008.

[59] J. A. Bondy and U. S. R. Murty. *Graph Theory*, volume 244 of *Graduate Texts in Mathematics*. Springer, New York, 2008.

[60] J. Bourgain and A. Gamburd. Uniform expansion bounds for Cayley graphs of $\mathrm{SL}_2(\mathbb{F}_p)$. *Ann. of Math. (2)*, 167(2):625–642, 2008.

[61] W. G. Bridges and R. A. Mena. Rational circulants with rational spectra and cyclic strongly regular graphs. *Ars Combin.*, 8:143–161, 1979.

[62] W. G. Bridges and R. A. Mena. On the rational spectra of graphs with abelian Singer groups. *Linear Algebra Appl.*, 46:51–60, 1982.

[63] W. G. Bridges and R. A. Mena. Rational G-matrices with rational eigenvalues. *J. Combin. Theory Ser. A*, 32(2):264–280, 1982.

[64] A. E. Brouwer, S. M. Cioabă, F. Ihringer, and M. McGinnis. The smallest eigenvalues of Hamming graphs, Johnson graphs and other distance-regular graphs with classical parameters. *J. Combin. Theory Ser. B*, 133:88–121, 2018.

[65] A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance-Regular Graphs*, volume 18 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1989.

[66] A. E. Brouwer and J. H. Koolen. The distance-regular graphs of valency four. *J. Algebraic Combin.*, 10(1):5–24, 1999.

[67] R. A. Brualdi. Spectra of digraphs. *Linear Algebra Appl.*, 432(9):2181–2213, 2010.

[68] C. Camarero and C. Martínez. Quasi-perfect Lee codes of radius 2 and arbitrarily large dimension. *IEEE Trans. Inform. Theory*, 62(3):1183–1192, 2016.

[69] X. Cao, M. Lu, D. Wan, L.-P. Wang, and Q. Wang. Linearized Wenger graphs. *Discrete Math.*, 338(9):1595–1602, 2015.

[70] P. Caputo, T. M. Liggett, and T. Richthammer. Proof of Aldous' spectral gap conjecture. *J. Amer. Math. Soc.*, 23(3):831–851, 2010.

[71] N. Celniker. Eigenvalue bounds and girths of graphs of finite, upper half-planes. *Pacific J. Math.*, 166(1):1–21, 1994.

[72] N. Celniker, S. Poulos, A. Terras, C. Trimble, and E. Velasquez. Is there life on finite upper half planes? In *A tribute to Emil Grosswald: number theory and related analysis*, volume 143 of *Contemp. Math.*, pages 65–88. Amer. Math. Soc., Providence, RI, 1993.

[73] F. Cesi. On the eigenvalues of Cayley graphs on the symmetric group generated by a complete multipartite set of transpositions. *J. Algebraic Combin.*, 32(2):155–185, 2010.

[74] F. Cesi. A few remarks on the octopus inequality and Aldous' spectral gap conjecture. *Comm. Algebra*, 44(1):279–302, 2016.

[75] C.-L. Chai and W.-C. W. Li. Character sums, automorphic forms, equidistribution, and Ramanujan graphs. II. Eigenvalues of Terras graphs. *Forum Math.*, 16(5):631–661, 2004.

[76] A. Chan and C. D. Godsil. Symmetry and eigenvectors. In *Graph symmetry (Montreal, PQ, 1996)*, volume 497 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 75–106. Kluwer Acad. Publ., Dordrecht, 1997.

[77] G. Chapuy and V. Féray. A note on a Cayley graph of S_n. *arXiv preprint arXiv:1202.4976*, 2012.

[78] S. Chattopadhyay and P. Panigrahi. Some relations between power graphs and Cayley graphs. *J. Egyptian Math. Soc.*, 23(3):457–462, 2015.

[79] B. F. Chen, E. Ghorbani, and K. B. Wong. On the eigenvalues of certain Cayley graphs and arrangement graphs. *Linear Algebra Appl.*, 444:246–253, 2014.

[80] T. Cheng, L. Feng, and H. Huang. Integral Cayley graphs over dicyclic group. *Linear Algebra Appl.*, 566:121–137, 2019.

[81] W.-C. Cheung and C. Godsil. Perfect state transfer in cubelike graphs. *Linear Algebra Appl.*, 435(10):2468–2474, 2011.

[82] M. Christandl, N. Datta, T. C. Dorlas, A. Ekert, A. Kay, and A. J. Landahl. Perfect transfer of arbitrary states in quantum spin networks. *Physical Review A*, 71(3):032312, 2005.

[83] M. Christandl, N. Datta, A. Ekert, and A. J. Landahl. Perfect state transfer in quantum spin networks. *Physical Review Letters*, 92(18):187902, 2004.

[84] D. Christofides and K. Markström. Expansion properties of random Cayley graphs and vertex transitive graphs via matrix martingales. *Random Structures Algorithms*, 32(1):88–100, 2008.

[85] F. Chung and R. Graham. Sparse quasi-random graphs. *Combinatorica*, 22(2):217–244, 2002.

[86] F. R. K. Chung. Diameters and eigenvalues. *J. Amer. Math. Soc.*, 2(2):187–196, 1989.

[87] S. M. Cioabă. Closed walks and eigenvalues of abelian Cayley graphs. *C. R. Math. Acad. Sci. Paris*, 342(9):635–638, 2006.

[88] S. M. Cioabă, F. Lazebnik, and W. Li. On the spectrum of Wenger graphs. *J. Combin. Theory Ser. B*, 107:132–139, 2014.

[89] S. M. Cioabă, F. Lazebnik, and S. Sun. Spectral and combinatorial properties of some algebraically defined graphs. *arXiv:1708.07597v1*, 2017.

[90] D. Conlon and Y. Zhao. Quasirandom Cayley graphs. *Discrete Anal.*, pages Paper No. 6, 14 pages, 2017.

[91] G. Coutinho, C. Godsil, K. Guo, and F. Vanhove. Perfect state transfer on distance-regular graphs and association schemes. *Linear Algebra Appl.*, 478:108–130, 2015.

[92] D. Cvetković, P. Rowlinson, and S. Simić. *An Introduction to the Theory of Graph Spectra*, volume 75 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2010.

[93] N. N. Dai, N. M. Hai, D. D. Hieu, and L. A. Vinh. Integral Cayley graphs generated by distance sets in vector spaces over finite fields. *Electron. J. Combin.*, 20(1):Paper 29, 6 pages, 2013.

[94] G. Davidoff, P. Sarnak, and A. Valette. *Elementary Number Theory, Group Theory, and Ramanujan Graphs*, volume 55 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2003.

[95] M. W. Davis. *The Geometry and Topology of Coxeter Groups*, volume 32 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2008.

[96] N. de Beaudrap. On restricted unitary Cayley graphs and symplectic transformations modulo n. *Electron. J. Combin.*, 17(1):R69, 26 pages, 2010.

[97] M. J. de Resmini and D. Jungnickel. Strongly regular semi-Cayley graphs. *J. Algebraic Combin.*, 1(2):171–195, 1992.

[98] M. DeDeo, D. Lanphier, and M. Minei. The spectrum of Platonic graphs over finite fields. *Discrete Math.*, 307(9-10):1074–1081, 2007.

[99] M. DeDeo, M. Martínez, A. Medrano, M. Minei, H. Stark, and A. Terras. Spectra of Heisenberg graphs over finite rings. *Discrete Contin. Dyn. Syst.*, (suppl.):213–222, 2003. Dynamical systems and differential equations (Wilmington, NC, 2002).

[100] M. R. DeDeo. Non-Ramanujancy of Euclidean graphs of order 2^r. *Discrete Math.*, 265(1-3):45–57, 2003.

[101] Y.-P. Deng and X.-D. Zhang. A note on eigenvalues of the derangement graph. *Ars Combin.*, 101:289–299, 2011.

[102] M. DeVos, L. Goddyn, B. Mohar, and R. Šámal. Cayley sum graphs and eigenvalues of $(3,6)$-fullerenes. *J. Combin. Theory Ser. B*, 99(2):358–369, 2009.

[103] M. DeVos, R. Krakovski, B. Mohar, and A. S. Ahmady. Integral Cayley multigraphs over Abelian and Hamiltonian groups. *Electron. J. Combin.*, 20(2):Paper 63, 16 pages, 2013.

[104] P. Diaconis and M. Shahshahani. Generating a random permutation with random transpositions. *Z. Wahrsch. Verw. Gebiete*, 57(2):159–179, 1981.

[105] P. Diaconis and D. Stroock. Geometric bounds for eigenvalues of Markov chains. *Ann. Appl. Probab.*, 1(1):36–61, 1991.

[106] D. Z. Djoković. Isomorphism problem for a special class of graphs. *Acta Math. Acad. Sci. Hungar.*, 21:267–270, 1970.

[107] J. Dodziuk. Difference equations, isoperimetric inequality and transience of certain random walks. *Trans. Amer. Math. Soc.*, 284(2):787–794, 1984.

[108] A. Droll. A classification of Ramanujan unitary Cayley graphs. *Electron. J. Combin.*, 17(1):N29, 6 pages, 2010.

[109] J. E. Ducey and D. M. Jalil. Integer invariants of abelian Cayley graphs. *Linear Algebra Appl.*, 445:316–325, 2014.

[110] D. S. Dummit and R. M. Foote. *Abstract Algebra*. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004.

[111] D. Ellis. Setwise intersecting families of permutations. *J. Combin. Theory Ser. A*, 119(4):825–849, 2012.

[112] B. Elspas and J. Turner. Graphs with circulant adjacency matrices. *J. Combinatorial Theory*, 9:297–307, 1970.

[113] I. Estélyi and I. Kovács. On groups all of whose undirected Cayley graphs of bounded valency are integral. *Electron. J. Combin.*, 21(4):Paper 4.45, 11, 2014.

[114] X. G. Fang, C. H. Li, and C. E. Praeger. On orbital regular graphs and Frobenius graphs. *Discrete Math.*, 182(1-3):85–99, 1998.

[115] L. Flatto, A. M. Odlyzko, and D. B. Wales. Random shuffles and group representations. *Ann. Probab.*, 13(1):154–178, 1985.

[116] B. Foster-Greenwood and C. Kriloff. Spectra of Cayley graphs of complex reflection groups. *J. Algebraic Combin.*, 44(1):33–57, 2016.

[117] J. Friedman. Some graphs with small second eigenvalue. *Combinatorica*, 15(1):31–42, 1995.

[118] J. Friedman. On Cayley graphs on the symmetric group generated by transpositions. *Combinatorica*, 20(4):505–519, 2000.

[119] J. Friedman, R. Murty, and J.-P. Tillich. Spectral estimates for abelian Cayley graphs. *J. Combin. Theory Ser. B*, 96(1):111–121, 2006.

[120] E. Fuchs and J. Sinz. Longest induced cycles in Cayley graphs. *arXiv preprint math/0410308*, 2004.

[121] E. D. Fuchs. Longest induced cycles in circulant graphs. *Electron. J. Combin*, 12(1):R52, 12 pages, 2005.

[122] X. Gao, H. Lü, and Y. Hao. The Laplacian and signless Laplacian spectrum of semi-Cayley graphs over abelian groups. *J. Appl. Math. Comput.*, 51(1-2):383–395, 2016.

[123] X. Gao and Y. Luo. The spectrum of semi-Cayley graphs over abelian groups. *Linear Algebra Appl.*, 432(11):2974–2983, 2010.

[124] X. Gao, Y. Luo, and W. Liu. Resistance distances and the Kirchhoff index in Cayley graphs. *Discrete Appl. Math.*, 159(17):2050–2057, 2011.

[125] M. Ghasemi. Integral pentavalent Cayley graphs on abelian or dihedral groups. *Proc. Indian Acad. Sci. Math. Sci.*, 127(2):219–224, 2017.

[126] M. Ghorbani. On the eigenvalues of normal edge-transitive Cayley graphs. *Bull. Iranian Math. Soc.*, 41(1):101–107, 2015.

[127] M. Ghorbani. On the energy and Estrada index of Cayley graphs. *Discrete Math. Algorithms Appl.*, 7(1):1550005, 8 pages, 2015.

[128] M. Ghorbani and F. N. Larki. On the spectrum of Cayley graphs. *Sib. Èlektron. Mat. Izv.*, 13:1283–1289, 2016.

[129] M. Ghorbani and F. N. Larki. On the spectrum of finite Cayley graphs. *J. Discrete Math. Sci. Cryptogr.*, 21(1):83–112, 2018.

[130] C. Godsil. Periodic graphs. *Electron. J. Combin.*, 18(1):P23, 15 pages, 2011.

[131] C. Godsil. State transfer on graphs. *Discrete Math.*, 312(1):129–147, 2012.

[132] C. Godsil. When can perfect state transfer occur? *Electron. J. Linear Algebra*, 23:877–890, 2012.

[133] C. Godsil and G. Royle. *Algebraic Graph Theory*, volume 207 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[134] C. Godsil and P. Spiga. Rationality conditions for the eigenvalues of normal finite Cayley graphs. *arXiv preprint arXiv:1402.5494*, 2014.

[135] C. Godsil and H. Zhan. Uniform mixing on Cayley graphs. *Electron. J. Combin.*, 24(3):Paper 3.20, 27, 2017.

[136] C. D. Godsil. Eigenvalues of graphs and digraphs. *Linear Algebra Appl.*, 46:43–50, 1982.

[137] R. Grigorchuk, P.-H. Leemann, and T. Nagnibeda. Lamplighter groups, de Brujin graphs, spider-web graphs and their spectra. *J. Phys. A*, 49(20):205004, 35 pages, 2016.

[138] R. I. Grigorchuk and P. W. Nowak. Diameters, distortion, and eigenvalues. *European J. Combin.*, 33(7):1574–1587, 2012.

[139] C. K. Gupta, B. Shwetha Shetty, and V. Lokesha. On the graph of nilpotent matrix group of length one. *Discrete Math. Algorithms Appl.*, 8(1):1650009, 31 pages, 2016.

[140] I. Gutman. The energy of a graph. *Ber. Math.-Statist. Sekt. Forsch. Graz*, (100-105):Ber. No. 103, 22, 1978. 10. Steiermärkisches Mathematisches Symposium (Stift Rein, Graz, 1978).

[141] I. Gutman. Hyperenergetic molecular graphs. *Journal of the Serbian Chemical Society*, 64(3):199–205, 1999.

[142] S. Handjani and D. Jungreis. Rate of convergence for shuffling cards by transpositions. *J. Theoret. Probab.*, 9(4):983–993, 1996.

[143] Y. Hao, X. Gao, and Y. Luo. On the Cayley graphs of Brandt semigroups. *Comm. Algebra*, 39(8):2874–2883, 2011.

[144] F. Harary and A. J. Schwenk. Which graphs have integral spectra? In *Graphs and Combinatorics (Proc. Capital Conf., George Washington Univ., Washington, D.C., 1973)*, pages 45–51. Lecture Notes in Math., Vol. 406. Springer, Berlin, 1974.

[145] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers.* Oxford University Press, Oxford, sixth edition, 2008. Revised by D. R. Heath-Brown and J. H. Silverman, With a foreword by Andrew Wiles.

[146] H. A. Harutyunyan and C. D. Morosan. The spectra of Knödel graphs. *Informatica (Ljubl.)*, 30(3):295–299, 2006.

[147] M.-C. Heydemann. Cayley graphs and interconnection networks. In *Graph symmetry (Montreal, PQ, 1996)*, volume 497 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 167–224. Kluwer Acad. Publ., Dordrecht, 1997.

[148] M. Hirano, K. Katata, and Y. Yamasaki. Ramanujan circulant graphs and the conjecture of Hardy-Littlewood and Bateman-Horn. *arXiv preprint arXiv:1310.2130*, 2013.

[149] M. Hirano, K. Katata, and Y. Yamasaki. Ramanujan Cayley graphs of Frobenius groups. *Bull. Aust. Math. Soc.*, 94(3):373–383, 2016.

[150] S. Hoory, N. Linial, and A. Wigderson. Expander graphs and their applications. *Bull. Amer. Math. Soc. (N.S.)*, 43(4):439–561, 2006.

[151] Q. Huang and A. Chang. Circulant digraphs determined by their spectra. *Discrete Math.*, 240(1-3):261–270, 2001.

[152] X. Huang, Q. Huang, , and S. M. Cioabă. The second eigenvalue of some normal cayley graphs of high transitive groups. *arXiv:1808.01118v2*, 2018.

[153] X. Huang and Q. Huang. The second largest eigenvalues of some cayley graphs on alternating groups. *arXiv:1711.08944v2*, 2018.

[154] X. Y. Huang, Q. X. Huang, and L. Lu. Enumeration of cubic Cayley graphs on dihedral groups. *Acta Math. Sin. (Engl. Ser.)*, 33(7):996–1010, 2017.

[155] D. R. Hughes, J. V. van Lint, and R. M. Wilson. *Announcement at the Seventh British Combinatorial Conference*. Cambridge, 1979 (unpublished).

[156] A. Ilić. The energy of unitary Cayley graphs. *Linear Algebra Appl.*, 431(10):1881–1889, 2009.

[157] A. Ilić. Distance spectra and distance energy of integral circulant graphs. *Linear Algebra Appl.*, 433(5):1005–1014, 2010.

[158] A. Ilić and M. Bašić. New results on the energy of integral circulant graphs. *Appl. Math. Comput.*, 218(7):3470–3482, 2011.

[159] I. M. Isaacs. *Character Theory of Finite Groups*. AMS Chelsea Publishing, Providence, RI, 2006.

[160] I. M. Isaacs. *Finite Group Theory*, volume 92 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[161] I. Ivrissimtzis and N. Peyerimhoff. Spectral representations of vertex transitive graphs, Archimedean solids and finite Coxeter groups. *Groups Geom. Dyn.*, 7(3):591–615, 2013.

[162] G. James and M. Liebeck. *Representations and Characters of Groups*. Cambridge University Press, New York, second edition, 2001.

[163] S. Johnson, A. Shaheen, and G. Subuyuj. The isoperimetric number of a generalized Paley graph. *Serdica Math. J.*, 42(3-4):235–250, 2016.

[164] G. A. Jones and R. Jajcay. Cayley properties of merged Johnson graphs. *J. Algebraic Combin.*, 44(4):1047–1067, 2016.

[165] A.-A. A. Jucys. Symmetric polynomials and the center of the symmetric group ring. *Rep. Mathematical Phys.*, 5(1):107–112, 1974.

[166] K. Kalpakis and Y. Yesha. On the bisection width of the transposition network. *Networks*, 29(1):69–76, 1997.

[167] M.-H. Kang. Toroidal fullerenes with the Cayley graph structures. *Discrete Math.*, 311(21):2384–2395, 2011.

[168] H. Karloff. How good is the Goemans-Williamson MAX CUT algorithm? *SIAM J. Comput.*, 29(1):336–350, 1999.

[169] A. Kay. Perfect, efficient, state transfer and its application as a constructive tool. *International Journal of Quantum Information*, 8(04):641–676, 2010.

[170] E. N. Khomyakova and E. V. Konstantinova. Note on exact values of multiplicities of eigenvalues of the star graph. *Sib. Èlektron. Mat. Izv.*, 12:92–100, 2015.

[171] D. Kiani, M. M. H. Aghaei, Y. Meemark, and B. Suntornpoch. Energy of unitary Cayley graphs and gcd-graphs. *Linear Algebra Appl.*, 435(6):1336–1343, 2011.

[172] D. Kiani and M. Mollahajiaghaei. On the unitary Cayley graphs of matrix algebras. *Linear Algebra Appl.*, 466:421–428, 2015.

[173] M. Klin and I. Kovács. Automorphism groups of rational circulant graphs. *Electron. J. Combin.*, 19(1):Paper 35, 52 pages, 2012.

[174] W. Klotz and T. Sander. Some properties of unitary Cayley graphs. *Electron. J. Combin.*, 14(1):R45, 12 pages, 2007.

[175] W. Klotz and T. Sander. Integral Cayley graphs over abelian groups. *Electron. J. Combin.*, 17(1):R81, 13 pages, 2010.

[176] W. Klotz and T. Sander. Wie kommt Sudoku zu ganzzahligen Eigenwerten? *Math. Semesterber.*, 57(2):169–183, 2010.

[177] W. Klotz and T. Sander. Integral Cayley graphs defined by greatest common divisors. *Electron. J. Combin.*, 18(1):P94, 15 pages, 2011.

[178] W. Klotz and T. Sander. Distance powers and distance matrices of integral Cayley graphs over abelian groups. *Electron. J. Combin.*, 19(4):Paper 25, 8 pages, 2012.

[179] W. Klotz and T. Sander. GCD-graphs and NEPS of complete graphs. *Ars Math. Contemp.*, 6(2):289–299, 2013.

[180] Y. Kohayakawa, V. e. Rödl, and M. Schacht. Discrepancy and eigenvalues of Cayley graphs. *Czechoslovak Math. J.*, 66(141)(3):941–954, 2016.

[181] T. Koma and B. Nachtergaele. The spectral gap of the ferromagnetic XXZ chain. *Lett. Math. Phys.*, 40(1):1–16, 1997.

[182] J. H. Koolen and V. Moulton. Maximal energy graphs. *Adv. in Appl. Math.*, 26(1):47–52, 2001.

[183] R. Krakovski and B. Mohar. Spectrum of Cayley graphs on the symmetric group generated by transpositions. *Linear Algebra Appl.*, 437(3):1033–1039, 2012.

[184] M. Krebs and A. Shaheen. *Expander Families and Cayley Graphs – A Beginner's Guide.* Oxford University Press, Oxford, 2011.

[185] C. Y. Ku, T. Lau, and K. B. Wong. Cayley graph on symmetric group generated by elements fixing k points. *Linear Algebra Appl.*, 471:405–426, 2015.

[186] C. Y. Ku, T. Lau, and K. B. Wong. Largest independent sets of certain regular subgraphs of the derangement graph. *J. Algebraic Combin.*, 44(1):81–98, 2016.

[187] C. Y. Ku and D. B. Wales. Eigenvalues of the derangement graph. *J. Combin. Theory Ser. A*, 117(3):289–312, 2010.

[188] C. Y. Ku and K. B. Wong. Solving the Ku-Wales conjecture on the eigenvalues of the derangement graph. *European J. Combin.*, 34(6):941–956, 2013.

[189] C. Y. Ku and T. W. H. Wong. Intersecting families in the alternating group and direct product of symmetric groups. *Electron. J. Combin.*, 14(1):Research Paper 25, 15 pages, 2007.

[190] K. Kutnar, A. Malnič, L. Martínez, and D. Marušič. Quasi m-Cayley strongly regular graphs. *J. Korean Math. Soc.*, 50(6):1199–1211, 2013.

[191] K. Kutnar, D. Marušič, v. Miklavič, and P. Šparl. Strongly regular tri-Cayley graphs. *European J. Combin.*, 30(4):822–832, 2009.

[192] J. D. Lafferty and D. N. Rockmore. Level spacings for Cayley graphs. In *Emerging applications of number theory (Minneapolis, MN, 1996)*, volume 109 of *IMA Vol. Math. Appl.*, pages 373–386. Springer, New York, 1999.

[193] Z. Landau and A. Russell. Random Cayley graphs are expanders: a simple proof of the Alon-Roichman theorem. *Electron. J. Combin.*, 11(1):Research Paper 62, 6, 2004.

[194] F. Lazebnik, S. Sun, and Y. Wang. Some families of graphs, hypergraphs and digraphs defined by systems of equations: a survey. In *Selected topics in graph theory and its applications*, volume 14 of *Lect. Notes Semin. Interdiscip. Mat.*, pages 105–142. Semin. Interdiscip. Mat. (S.I.M.), Potenza, 2017.

[195] F. Lazebnik and A. J. Woldar. General properties of some families of graphs defined by systems of equations. *J. Graph Theory*, 38(2):65–86, 2001.

[196] T. A. Le and J. W. Sander. Convolutions of Ramanujan sums and integral circulant graphs. *Int. J. Number Theory*, 8(7):1777–1788, 2012.

[197] T. A. Le and J. W. Sander. Extremal energies of integral circulant graphs via multiplicativity. *Linear Algebra Appl.*, 437(6):1408–1421, 2012.

[198] T. A. Le and J. W. Sander. Integral circulant Ramanujan graphs of prime power order. *Electron. J. Combin.*, 20(3):P35, 12 pages, 2013.

[199] K. H. Leung and S. L. Ma. Partial difference triples. *J. Algebraic Combin.*, 2(4):397–409, 1993.

[200] C. H. Li. On isomorphisms of finite Cayley graphs – a survey. *Discrete Math.*, 256(1-2):301–334, 2002.

[201] F. Li. Determination of integral Cayley graphs on finite abelian groups. *https://arxiv.org/abs/1206.0799*, 2012.

[202] F. Li. Circulant digraphs integral over number fields. *Discrete Math.*, 313(6):821–823, 2013.

[203] S. Li and L. A. Vinh. On the spectrum of unitary Euclidean graphs. *Ars Combin.*, 110:275–287, 2013.

[204] W.-C. W. Li. A survey of Ramanujan graphs. In *Arithmetic, geometry and coding theory (Luminy, 1993)*, pages 127–143. de Gruyter, Berlin, 1996.

[205] W.-C. W. Li and Y. Meemark. Ramanujan graphs on cosets of $\mathrm{PGL}_2(\mathbb{F}_q)$. *Finite Fields Appl.*, 11(3):511–543, 2005.

[206] X. Li, Y. Shi, and I. Gutman. *Graph Energy*. Springer, New York, 2012.

[207] Y. Li, X. Liu, S. Zhang, and S. Zhou. The 2nd power of generalized Hamming graphs. In preparation.

[208] Y. Li, X. Liu, S. Zhang, and S. Zhou. Perfect state transfer in the non-complete extended p-sums of complete graphs. In preparation.

[209] T. K. Lim and C. E. Praeger. On generalized Paley graphs and their automorphism groups. *Michigan Math. J.*, 58(1):293–308, 2009.

[210] B. Litow and B. Mans. A note on the Ádám conjecture for double loops. *Inform. Process. Lett.*, 66(3):149–153, 1998.

[211] X. Liu. Energies of unitary one-matching bi-cayley graphs over finite commutative rings. *Discrete Appl. Math.*, https://doi.org/10.1016/j.dam.2018.12.027, 2019.

[212] X. Liu and B. Li. Distance powers of unitary Cayley graphs. *Appl. Math. Comput.*, 289:272–280, 2016.

[213] X. Liu and S. Zhou. Spectral properties of Cayley graphs over dihedral groups. In preparation.

[214] X. Liu and S. Zhou. Spectral properties of unitary Cayley graphs of finite commutative rings. *Electron. J. Combin.*, 19(4):P13, 19 pages, 2012.

[215] X. Liu and S. Zhou. Quadratic unitary Cayley graphs of finite commutative rings. *Linear Algebra Appl.*, 479:73–90, 2015.

[216] P.-S. Loh and L. J. Schulman. Improved expansion of random Cayley graphs. *Discrete Math. Theor. Comput. Sci.*, 6(2):523–528, 2004.

[217] L. Lovász. Spectra of graphs with transitive groups. *Period. Math. Hungar.*, 6(2):191–195, 1975.

[218] S. Lovett, C. Moore, and A. Russell. Group representations that resist random sampling. *Random Structures Algorithms*, 47(3):605–614, 2015.

[219] L. Lu, Q. Huang, and X. Huang. Integral cayley graphs over dihedral groups. *Journal of Algebraic Combinatorics*, 47(4):585–601, 2018.

[220] M. Lu, D. Wan, L.-P. Wang, and X.-D. Zhang. Algebraic Cayley graphs over finite fields. *Finite Fields Appl.*, 28:43–56, 2014.

[221] A. Lubotzky. Cayley graphs: eigenvalues, expanders and random walks. In *Surveys in combinatorics, 1995 (Stirling)*, volume 218 of *London Math. Soc. Lecture Note Ser.*, pages 155–189. Cambridge Univ. Press, Cambridge, 1995.

[222] A. Lubotzky. Expander graphs in pure and applied mathematics. *Bull. Amer. Math. Soc. (N.S.)*, 49(1):113–162, 2012.

[223] A. Lubotzky, B. Samuels, and U. Vishne. Isospectral Cayley graphs of some finite simple groups. *Duke Math. J.*, 135(2):381–393, 2006.

[224] D. S. Lyubshin and S. V. Savchenko. Cayley digraphs with normal adjacency matrices. *Discrete Math.*, 309(13):4343–4348, 2009.

[225] S. L. Ma. Partial difference sets. *Discrete Math.*, 52(1):75–89, 1984.

[226] S. L. Ma. A survey of partial difference sets. *Des. Codes Cryptogr.*, 4(3):221–261, 1994.

[227] X. Ma and K. Wang. Integral Cayley sum graphs and groups. *Discuss. Math. Graph Theory*, 36(4):797–803, 2016.

[228] X. Ma and K. Wang. On finite groups all of whose cubic Cayley graphs are integral. *J. Algebra Appl.*, 15(6):1650105, 10 pages, 2016.

[229] F. J. MacWilliams and H. B. Mann. On the p-rank of the design matrix of a difference set. *Information and Control*, 12:474–488, 1968.

[230] A. Malnič, D. Marušič, and P. Šparl. On strongly regular bicirculants. *European J. Combin.*, 28(3):891–900, 2007.

[231] B. Mans, F. Pappalardi, and I. Shparlinski. On the spectral Ádám property for circulant graphs. *Discrete Math.*, 254(1-3):309–329, 2002.

[232] L. Martínez. Strongly regular m-Cayley circulant graphs and digraphs. *Ars Math. Contemp.*, 8(1):195–213, 2015.

[233] D. Marušič. Strongly regular bicirculants and tricirculants. *Ars Combin.*, 25(C):11–15, 1988.

[234] D. Marušič. Strong regularity and circulant graphs. *Discrete Math.*, 78(1-2):119–125, 1989.

[235] A. Medrano, P. Myers, H. M. Stark, and A. Terras. Finite analogues of Euclidean space. *J. Comput. Appl. Math.*, 68(1-2):221–238, 1996.

[236] A. Medrano, P. Myers, H. M. Stark, and A. Terras. Finite Euclidean graphs over rings. *Proc. Amer. Math. Soc.*, 126(3):701–710, 1998.

[237] Y. Meemark and S. Sriwongsa. Perfect state transfer in unitary Cayley graphs over local rings. *Trans. Comb.*, 3(4):43–54, 2014.

[238] Y. Meemark and B. Suntornpoch. Balanced unitary Cayley sigraphs over finite commutative rings. *J. Algebra Appl.*, 13(5):1350152, 12, 2014.

[239] J. Meng. Non-isomorphic cospectral Cayley digraphs. *Graph Theory Notes N. Y.*, 35:51–53, 1998. New York Graph Theory Day, 35 (1998).

[240] Š. Miklavič and P. Potočnik. Distance-regular circulants. *European J. Combin.*, 24(7):777–784, 2003.

[241] Š. Miklavič and P. Potočnik. Distance-regular Cayley graphs on dihedral groups. *J. Combin. Theory Ser. B*, 97(1):14–33, 2007.

[242] Š. Miklavič and P. Šparl. On distance-regular Cayley graphs on abelian groups. *J. Combin. Theory Ser. B*, 108:102–122, 2014.

[243] M. Minchenko and I. M. Wanless. Quartic integral Cayley graphs. *Ars Math. Contemp.*, 8(2):381–408, 2015.

[244] M. Minei and H. Skogman. Block diagonalization method for the covering graph. *Ars Combin.*, 92:321–352, 2009.

[245] B. Mohar. Laplace eigenvalues of graphs—a survey. *Discrete Math.*, 109(1-3):171–183, 1992.

[246] B. Mohar and W. Woess. A survey on spectra of infinite graphs. *Bull. London Math. Soc.*, 21(3):209–234, 1989.

[247] H. Mokhtar and S. Zhou. Recursive cubes of rings as models for interconnection networks. *Discrete Appl. Math.*, 217(part 3):639–662, 2017.

[248] M. Mollahajiaghaei. The eigenvalues and energy of integral circulant graphs. *Transactions on Combinatorics*, 1(3):47–56, 2012.

[249] K. Momihara. Construction of strongly regular Cayley graphs based on three-valued Gauss periods. *European J. Combin.*, 70:232–250, 2018.

[250] G. E. Moorhouse, S. Sun, and J. Williford. The eigenvalues of the graphs $D(4, q)$. *J. Combin. Theory Ser. B*, 125:1–20, 2017.

[251] C. J. Moreno. Harmonic analysis on finite rings and applications. In *Algebra and its Applications (Athens, OH, 1999)*, volume 259 of *Contemp. Math.*, pages 409–430. Amer. Math. Soc., Providence, RI, 2000.

[252] M. R. Murty. Ramanujan graphs. *J. Ramanujan Math. Soc.*, 18(1):33–52, 2003.

[253] W. Narkiewicz. On a class of arithmetical convolutions. *Colloq. Math.*, 10:81–94, 1963.

[254] I. Pak. Random Cayley graphs with $O(\log |G|)$ generators are expanders. In *Algorithms—ESA '99 (Prague)*, volume 1643 of *Lecture Notes in Comput. Sci.*, pages 521–526. Springer, Berlin, 1999.

[255] H. Pal and B. Bhattacharjya. Perfect state transfer on Cayley graphs over finite abelian groups. *arXiv:1601.07647v1 [math.CO]*, 2016.

[256] H. Pal and B. Bhattacharjya. Perfect state transfer on gcd-graphs. *Linear Multilinear Algebra*, 65(11):2245–2256, 2017.

[257] M. Petersdorf and H. Sachs. Spektrum und Automorphismengruppe eines Graphen. *Combinatorial Theory and its Applications, III (Proc. Colloq., Balatonfüred, 1969)*, pages 891–907, 1970.

[258] M. D. Petković and M. Bašić. Further results on the perfect state transfer in integral circulant graphs. *Comput. Math. Appl.*, 61(2):300–312, 2011.

[259] S. C. Poulos. *Graph Theoretic and Spectral Properties of Finite Upper Half-Planes*. ProQuest LLC, Ann Arbor, MI, 1991. Thesis (Ph.D.)–University of California, San Diego.

[260] Y. Qin, W. Xiao, and Š. Miklavič. Connected graphs as subgraphs of Cayley graphs: conditions on Hamiltonicity. *Discrete Math.*, 309(17):5426–5431, 2009.

[261] R. R. Balakrishnan, P. Paulraja, W. So, and M. Vinay. Some properties of the Knödel graph $W(k, 2^k)$, $k \geq 4$. *Australas. J. Combin.*, submitted.

[262] H. N. Ramaswamy and C. R. Veena. On the energy of unitary Cayley graphs. *Electron. J. Combin.*, 16(1):N24, 8 pages, 2009.

[263] A. Rasri and Y. Meemark. Algebraic Cayley graphs over finite local rings. *Finite Fields Appl.*, 48:227–240, 2017.

[264] P. Renteln. On the spectrum of the derangement graph. *Electron. J. Combin.*, 14(1):R82, 17 pages, 2007.

[265] P. Renteln. The distance spectra of Cayley graphs of Coxeter groups. *Discrete Math.*, 311(8-9):738–755, 2011.

[266] C. Reyes-Bustos. Cayley-type graphs for group-subgroup pairs. *Linear Algebra Appl.*, 488:320–349, 2016.

[267] C. Riess, V. Strehl, and R. Wanka. The spectral relation between the cube-connected cycles and the shuffle-exchange network. *ARCS 2012*, pages 1–6, 2012.

[268] H. Sachs and M. Stiebitz. Simple eigenvalues of transitive graphs. *Studia Sci. Math. Hungar.*, 17(1-4):77–90, 1982.

[269] J. W. Sander. Integral circulant Ramanujan graphs via multiplicativity and ultrafriable integers. *Linear Algebra Appl.*, 477:21–41, 2015.

[270] J. W. Sander and T. Sander. The energy of integral circulant graphs with prime power order. *Appl. Anal. Discrete Math.*, 5(1):22–36, 2011.

[271] J. W. Sander and T. Sander. Integral circulant graphs of prime power order with maximal energy. *Linear Algebra Appl.*, 435(12):3212–3232, 2011.

[272] J. W. Sander and T. Sander. The maximal energy of classes of integral circulant graphs. *Discrete Appl. Math.*, 160(13-14):2015–2029, 2012.

[273] J. W. Sander and T. Sander. The exact maximal energy of integral circulant graphs with prime power order. *Contrib. Discrete Math.*, 8(2):19–40, 2013.

[274] J. W. Sander and T. Sander. On So's conjecture for integral circulant graphs. *Appl. Anal. Discrete Math.*, 9(1):59–72, 2015.

[275] T. Sander. Sudoku graphs are integral. *Electron. J. Combin.*, 16(1):N25, 7 pages, 2009.

[276] T. Sander. Eigenspaces of Hamming graphs and unitary Cayley graphs. *Ars Math. Contemp.*, 3(1):13–19, 2010.

[277] S. Satake. A note on the relation between two properties of random graphs. *personal communication*, 2019.

[278] S. V. Savchenko. On the change of the Jordan form under the transition from the adjacency matrix of a vertex-transitive digraph to its principal submatrix of co-order one. *Linear Algebra Appl.*, 394:225–235, 2005.

[279] N. Saxena, S. Severini, and I. E. Shparlinski. Parameters of integral circulant graphs and periodic quantum dynamics. *International Journal of Quantum Information*, 5(03):417–430, 2007.

[280] J. J. Seidel. Strongly regular graphs with $(-1, 1, 0)$ adjacency matrix having eigenvalue 3. *Linear Algebra Appl.*, 1:281–298, 1968.

[281] I. Shparlinski. On the energy of some circulant graphs. *Linear Algebra Appl.*, 414(1):378–382, 2006.

[282] J. A. Sjogren. Connectivity and spectrum in a graph with a regular automorphism group of odd order. *Internat. J. Algebra Comput.*, 4(4):529–560, 1994.

[283] W. So. Integral circulant graphs. *Discrete Math.*, 306(1):153–158, 2005.

[284] D. Stevanović. Applications of graph spectra in quantum physics. *Zb. Rad. (Beogr.)*, 14(22)(Selected topics on applications of graph spectra):85–111, 2011.

[285] B. Suntornpoch and Y. Meemark. Cayley graphs over a finite chain ring and GCD-graphs. *Bull. Aust. Math. Soc.*, 93(3):353–363, 2016.

[286] Y. Tan, K. Feng, and X. Cao. Perfect state transfer on abelian Cayley graphs. *Linear Algebra Appl.*, 563:331–352, 2019.

[287] T. Tao. *Expansion in finite simple groups of Lie type*, volume 164 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015.

[288] A. Terras. Eigenvalue problems related to finite analogues of upper half planes. In *Forty more years of ramifications: spectral asymptotics and its applications*, volume 1 of *Discourses Math. Appl.*, pages 237–263. Texas A & M Univ., College Station, TX, 1991.

[289] A. Terras. Survey of spectra of Laplacians on finite symmetric spaces. *Experiment. Math.*, 5(1):15–32, 1996.

[290] A. Terras. *Fourier Analysis on Finite Groups and Applications*, volume 43 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1999.

[291] The GAP Group. GAP-Groups, Algorithms, and Programming - a System for Computational Discrete Algebra, Version 4.7.8. available online at `http://gap-system.org`, 2015.

[292] I. Thongsomnuk and Y. Meemark. Perfect state transfer in unitary Cayley graphs and gcd-graphs. *Linear Multilinear Algebra*, available online at `https://doi.org/10.1080/03081087.2017.1410091`, 2017.

[293] G.-X. Tian. On the skew energy of orientations of hypercubes. *Linear Algebra Appl.*, 435(9):2140–2149, 2011.

[294] J. Turner. Point-symmetric graphs with a prime number of points. *J. Combinatorial Theory*, 3:136–145, 1967.

[295] E. R. van Dam and M. Jazaeri. Distance-regular cayley graphs with small valency. *arXiv:1808.01428v1*, 2018.

[296] E. R. van Dam, J. H. Koolen, and H. Tanaka. Distance-regular graphs. *Electron. J. Combin.*, DS 22:1–156, 2016.

[297] E. R. van Dam and R. Sotirov. New bounds for the max-k-cut and chromatic number of a graph. *Linear Algebra Appl.*, 488:216–234, 2016.

[298] L. A. Vinh. Finite Euclidean graphs over $\mathbb{Z}_{2^r}$ are non-Ramanujan. *Australas. J. Combin.*, 43:295–305, 2009.

[299] L. A. Vinh. Graphs generated by Sidon sets and algebraic equations over finite fields. *J. Combin. Theory Ser. B*, 103(6):651–657, 2013.

[300] L. Wang. A survey of results on integral trees and integral graphs. Memorandum 1763, Department of Applied Mathematics, University of Twente, Enschede, 2005.

[301] Y. Xu and J. Meng. Gaussian integral circulant digraphs. *Discrete Math.*, 311(1):45–50, 2011.

[302] Y. Yamasaki. Ramanujan Cayley graphs of the generalized quaternion groups and the Hardy-Littlewood conjecture. In *Mathematical modelling for next-generation cryptography*, volume 29 of *Math. Ind. (Tokyo)*, pages 159–175. Springer, Singapore, 2018.

[303] H. Yan and C. Liu. A note on the spectrum of linearized Wenger graphs. *Discrete Math.*, 340(5):1050–1053, 2017.

[304] S. Zheng, X. Liu, and S. Zhang. Perfect state transfer in NEPS of some graphs. *Linear Multilinear Algebra*, available online at `https://doi.org/doi:10.1080/03081087.2018.1548555`, 2018.

[305] H. Q. Zhou and Y. Z. Xu. Some constructions of hyperenergetic integral circulant graphs. *J. Zhejiang Univ. Sci. Ed.*, 42(2):127–132, 2015.

[306] J. Zhou, C. Bu, and J. Shen. Some results for the periodicity and perfect state transfer. *Electron. J. Combin.*, 18(1):Paper 184, 7 pages, 2011.

[307] S. Zhou. A class of arc-transitive Cayley graphs as models for interconnection networks. *SIAM J. Discrete Math.*, 23(2):694–714, 2009.

[308] P.-H. Zieschang. Cayley graphs of finite groups. *J. Algebra*, 118(2):447–454, 1988.

[309] H. Zou and J. X. Meng. Some algebraic properties of bi-Cayley graphs. *Acta Math. Sinica (Chin. Ser.)*, 50(5):1075–1080, 2007.

[310] A. Zuk. Analysis and geometry on groups. In *Discrete geometric analysis*, volume 34 of *MSJ Mem.*, pages 113–157. Math. Soc. Japan, Tokyo, 2016.