

SOLITON RESOLUTION FOR THE MODIFIED KDV EQUATION

GONG CHEN AND JIAQI LIU

ABSTRACT. The soliton resolution for the focusing modified Korteweg-de vries (mKdV) equation is established for initial conditions in some weighted Sobolev spaces. Our approach is based on the nonlinear steepest descent method and its reformulation through $\bar{\partial}$ -derivatives. From the view of stationary points, we give precise asymptotic formulas along trajectory $x = vt$ for any fixed v . To extend the asymptotics to solutions with initial data in low regularity spaces, we apply a global approximation via PDE techniques. As byproducts of our long-time asymptotics, we also obtain the asymptotic stability of nonlinear structures involving solitons and breathers.

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1. INTRODUCTION

In this paper, we study the long-time dynamics of the focusing modified Korteweg-de vries equation (mKdV)

$$(1.1) \quad \partial_t u + \partial_{xxx} u + 6u^2 \partial_x u = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+.$$

There is a vast body of literature regarding the mKdV equation, in particular with the local and global well-posedness of the Cauchy problem. For a summary of known results we refer the reader to Linares-Ponce [41]. For the focusing mKdV equation on the line, we mention the works on the

local and global well-posedness by Kato [34], Kenig-Ponce-Vega [35], Colliander-Keel-Staffilani-Takaoka-Tao [12], Guo [27] and Kishimoto [38]. In particular, it is proven by see Kenig, Ponce and Vega in [35] that the equation is locally well-posed. And global well-posedness is proven by Colliander, Keel, Staffilani, Takaoka and Tao in [12]. Finally Guo [27] and Kishimoto [38] established well-posedness in $H^s(\mathbb{R})$ for $s \geq \frac{1}{4}$. These results are complemented by several ill-posedness results; see Kenig-Ponce-Vega [36], Christ-Colliander-Tao [11] and references therein.

Besides global regularity, another fundamental question for dispersive PDEs concerns the asymptotic behavior for large time. For small data, the defocusing and the focusing mKdV have similar asymptotics. For example, using the complete integrability of the mKdV, in the seminal work of Deift-Zhou [16], the global existence and asymptotic behavior can be studied for the defocusing case using inverse scattering transforms and the nonlinear steepest descent approach to oscillatory Riemann-Hilbert problems. Our recent work [10] extends these analysis to low regularity data. For small data, one can also study the asymptotics without using completely integrability. A proof of global existence and a (partial) derivation of the asymptotic behavior for small localized solutions, without making use of complete integrability, was later given by Hayashi and Naumkin [29,30] using the method of factorization of operators. Recently, Germain-Pusateri-Rousset [24] use ideas based on the space-time resonance to study the long-time asymptotics of small data and the stability of solitary waves. A precise derivation of asymptotics and a proof of asymptotic completeness, was given by Harrop-Griffiths [28] using wave packets analysis.

Compared with the defocusing modified Korteweg-de vries equation

$$\partial_t u + \partial_{xxx} u - 6u^2 \partial_x u = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+$$

studied in Deift-Zhou [16] and our earlier work [10], the first striking feature is the existence of solitons and breathers which do not decay in time (up to translations). This is a remarkable consequence of the focusing interaction between the nonlinearity and the dispersion.

Equation (1.1) admits a solution of the following form: for $c > 0$

$$(1.2) \quad u = Q_c(x - ct)$$

where

$$Q_c(x) := \sqrt{c} Q(\sqrt{c}x).$$

solves

$$Q_c'' - cQ_c + Q_c^3 = 0, \quad Q_c \in H^1(\mathbb{R}).$$

and one can write down the solution explicitly

$$Q(x) := 2\sqrt{2}\partial_x(\arctan(e^x)).$$

With these solitons, more complicated solutions are present, such as multi-soliton solutions [31] [57] [54].

In the context of the focusing mKdV, there exist other nonlinear structures which do not decay in time. These nonlinear structures, of oscillatory character, are known as breathers [40] [56]. They are periodic in time but spatially localized (after a suitable space shift) real-valued functions. They are of the following form: For $\alpha, \beta \in \mathbb{R} \setminus \{0\}$,

$$(1.3) \quad B_{\alpha,\beta}(x, t) = 2\sqrt{2}\partial_x \left[\arctan \left(\frac{\beta \sin(\alpha(x + \delta t))}{\alpha \cosh(\beta(x + \gamma t))} \right) \right]$$

where

$$\delta := \alpha^2 - 3\beta^2, \quad \gamma := 3\alpha^2 - \beta^2.$$

Notice that $-\gamma$ plays the role of velocity, which can be positive or negative. Therefore, compared with a soliton which only moves to the right, a breather can travel in both directions. (We will use slightly different notations later on to be consistent with the inverse scattering literature.)

If we assume there are no breathers nor solitons, the pure radiation will behave similarly to the defocusing mKdV. In [16] and [10] it has been shown that the defocusing mKdV has asymptotic behaviors in different space-time regions. These include the soliton region, the self-similar region and the oscillatory region.

From our brief discussion above, one should realize that general solution to the focusing mKdV will consist of solitons moving to the right, breathers traveling to both directions and a radiation term. Our goal in this paper is to give detailed asymptotic analysis for the focusing mKdV with generic data. To achieve this, we need to understand the interaction among solitons, breathers and radiation in different regions precisely. In the generic setting, finitely many breathers and solitons can appear and they interact with the radiation. One might expect that a consequence of the integrability, these nonlinear modes interact elastically but the way they influence the radiation are remarkably different. To illustrate the complicated behavior of the solution to the mKdV, we compare the dynamics here with the cubic NLS and the KdV equation.

- (i) The KdV equation has solitons but no breathers. Like the KdV solitons, mKdV solitons travel in the opposite direction of the radiation. So one might expect the interactions between them are weak. Interactions among solitons cause the shift of centers of solitons and the soliton influence on the radiation can be seen from matrix conjugation. Meanwhile, mKdV breathers can travel in both directions. For those traveling in the same direction with solitons, again, the results of interactions are similar to that of solitons. More importantly, there are breathers traveling in the same direction with radiation. As for the behavior of the radiation, both the KdV and the mKdV have soliton region, Painlevé region and the oscillatory region but the KdV has one extra part, the collisionless shock region.
- (ii) For those breathers traveling in the same direction with the radiation, the interactions are strong. They are always coupled with the radiations like the NLS. If the stationary phase point we choose is close to the velocity of some breather, the model Riemann-Hilbert problem is significantly different from the defocusing problem. In particular, there are eigenvalues of the direct scattering transform located on the critical curve with respect to the stationary point. This is the place where the interactions among breathers and radiation is seen from matrix conjugation. More explicitly, we are conjugating matrices obtained from solving a one-breather Riemann-Hilbert problem with matrices resulting from solving a parabolic cylinder model problem.

Our long-time asymptotics will also provide a proof for soliton resolution conjecture for the mKdV with generic data. This conjecture asserts, roughly speaking, that any reasonable solution eventually resolves into a superposition of a radiation component plus a finite number of “nonlinear bound states” or “solitons”. Without using integrability, in Duyckaerts-Kenig-Merle [21] and Duyckaerts-Jia-Kenig-Merle [22] establish this conjecture for the energy critical wave equation in high dimensions (along a sequence of time for the non-radial case). For integrable systems, this resolution phenomenon is studied by Borghese, Jenkins and McLaughlin in [9] for the cubic NLS and by Jenkins, Liu, Perry and Sulem in [33] for the the derivative NLS and more recently by Saalman [51] for the massive Thirring model. For the mKdV equation we mention that in [54], the author only allows solitons to appear and the analysis is also restricted to the soliton region for smooth initial condition. Our analysis include *all kinds* of solitons and breathers and establish the long-time asymptotics on the *full line*. We also lower the regularity condition to be *almost optimal*. We will show that for any generic data in the sense of Definition 1.5, the solution u to the focusing mKdV (1.1), can be written as a superposition of solitons, breathers and a radiation term:

$$u(x, t) = \sum_{\ell=1}^{N_2} u_{\ell}^{(br)}(x, t) + \sum_{\ell=1}^{N_1} u_{\ell}^{(so)}(x, t) + R(x, t).$$

For the detailed description of the formula above, we refer the reader to Theorem 1.10.

As a byproduct of our analysis, we obtain the asymptotic stability of some nonlinear structures. More precisely, we obtain the *full asymptotic stability* of soliton, multi-soliton, breather, multi-breathers and the combination of them.

Without trying to be exhaustive, we discuss the historical progress of the stability analysis. Indeed, H^1 -stability of mKdV solitons and multi-solitons have been considered e.g. in Bona-Souganidis-Strauss [7], Pego-Weinstein [49], Martel-Merle-Tsai [46] and Martel-Merle [44, 45]. For the stability of breathers, see Alejo-Muñoz [3, 4]. To understand the (asymptotic) stability of soliton or breathers, for those traveling in different direction with radiation, one can use the energy method with the Lyapunov functional as in, for example Martel-Merle [44, 45] and Alejo-Muñoz [3, 4] after restricting to the soliton region. Understanding the radiation requires more refined analysis, see Germain-Pusateri-Rousset [24] and Mizumachi [48]. In particular, in [24], for the perturbation of the mKdV soliton, the authors give detailed descriptions of the radiation in terms of Painlevé function and modified scattering. Here, we illustrate explicitly the influence of solitons/breathers on the radiation. More importantly, in the context of mKdV, there are breathers traveling alongside the radiation to the *left*. As we point out above, the interaction here behaves like the interaction between solitons and radiation in NLS. To understand the asymptotic stability of them, one can always attempt to linearize the equation near breathers. But the spectral analysis here is much more involved compared with the NLS equation since breathers oscillates periodically in time. Moreover, even for the integrable cubic NLS, to the best of our knowledge, there is no PDE proof the asymptotic stability of the soliton without invoking the inverse scattering transform.

To study the long-time asymptotics of integrable system, in the pioneering work of Deift-Zhou [16], a key step in the nonlinear steepest descent method consists of deforming the contour associated to the RHP in such a way that the phase function with oscillatory dependence on parameters become exponential decay. In general the entries of the jump matrix are not analytic, so direct analytic extension off the real axis is not possible. Instead they must be approximated by rational functions and this results in some error term in the recovered solution. Therefore, in the context of nonlinear steepest descent, most works are carried out under the assumptions that the initial data belong to the Schwartz space.

In [59], Xin Zhou developed a rigorous analysis of the direct and inverse scattering transform of the AKNS system for a class of initial conditions $u_0(x) = u(x, t = 0)$ belonging to the space $H^{i,j}(\mathbb{R})$. Here, $H^{i,j}(\mathbb{R})$ denotes the completion of $C_0^\infty(\mathbb{R})$ in the norm

$$\|u\|_{H^{i,j}(\mathbb{R})} = \left(\|(1 + |x|^j)u\|_2^2 + \|u^{(i)}\|_2^2 \right)^{1/2}.$$

Recently, much effort has been devoted to relax the regularities of the initial data. In particular, among the most celebrated results concerning nonlinear Schrödinger equations, we point out the work of Deift-Zhou [18] where they provide the asymptotics for the NLS in the weighted space $H^{1,1}$. Dieng and McLaughlin in [19] (see also an extended version [20]) developed a variant of Deift-Zhou method. In their approach rational approximation of the reflection coefficient is replaced by some non-analytic extension of the jump matrices off the real axis, which leads to a $\bar{\partial}$ -problem to be solved in some regions of the complex plane. The new $\bar{\partial}$ -problem can be reduced to an integral equation and is solvable through Neumann series. These ideas were originally implemented by Miller and McLaughlin [47] to the study the asymptotic stability of orthogonal polynomials. This method has shown its robustness in its application to other integrable models. Notably, for focusing NLS and derivative NLS, they were successfully applied to address the soliton resolution in [9] and [33] respectively. In this paper, we incorporate this

approach into the framework of [16] to calculate the long time behavior of the focusing mKdV equation in weighted Sobolev spaces.

Also in Deift-Zhou [18], they apply an approximation argument to extend the long-time asymptotics of the cubic NLS to the weighted space $L^{2,1}$. This topology is more or less optimal from the views of PDE and inverse scattering transformations. The global L^2 existence of the cubic NLS can be carried out by the $L_t^4 L_x^\infty$ Strichartz estimate and the conservation of the L^2 norm. But in order to obtain the precise asymptotics, one needs to “pay the price of weights”, i.e. working with the weighted space $L^{2,1}$. Recently, in our earlier work Chen-Liu [10], we establish the long-time asymptotics for the defocusing mKdV in $H^{1/4,s}$, $s > 1/2$ using a global approximation argument based on contractions in the spirit of Kenig-Ponce-Vega [35]. In Deift-Zhou [18], due to the $L_t^4 L_x^\infty$ Strichartz estimates for the linear Schrödinger equation and the conservation of the L^2 norm, the authors can globally approximate the solution to the nonlinear Schrödinger equation with data in $L^{2,1}$ using the Beals-Coifman representation of solutions directly. Unlike the Schrödinger equation, the smoothing estimates and Strichartz estimates for the Airy equation and the mKdV are much more involved. For example, one needs $L_x^4 L_t^\infty$ which behaves like a maximal operator. To directly work on the Beals-Coifman solution to the mKdV to establish the smoothing estimates and Strichartz estimates, one needs estimates for pseudo-differential operators with very rough symbols. To avoid these technicalities, we first identify the Beals-Coifman solution with the solution given by the Duhamel formula which we call a strong solution for smooth data. Since the strong solutions by construction enjoy Strichartz estimates and smoothing estimates, by our identification, the Beals-Coifman solutions also satisfy these estimates. Then we combine Strichartz estimates and smoothing estimates with the L^2 Sobolev bijectivity result by Zhou [59] to pass limits of Beals-Coifman solutions to obtain the asymptotics for rougher initial data in $H^{\frac{s}{2},1}(\mathbb{R})$ with $s \geq 1/4$. In contrast to our earlier work [10], in this paper, we use the recent work on low regularity conservation law due to Kilip-Visan-Zhang [37] and Koch-Tataru [39] to perform the approximation argument for $H^{\frac{s}{2},1}(\mathbb{R})$ with $s \geq 1/4$ in the unified manner. To deal with the focusing problem here, we need some refined analysis on the discrete scattering data since the Beals-Coifman representation is more complicated. Then again, via the approximation argument adapted to the focusing problem, we extend the soliton resolution to generic data in $H^{1/4,s}$, $s > 1/2$.

Finally, we would like point out that similar to Deift-Zhou [16], our method is general and algorithmic and does not require an a priori ansatz for the form of the solution of the asymptotic problem. We only assume the number of zeros of $a(z)$ and $\check{a}(z)$ are finite, see Section 1.2.1 for the definition. This condition is generic which means that the initial data satisfying this condition is an open dense set in the space of the initial data. For the KdV problem, if certain norms of the initial data is bounded, then automatically, this spectral condition holds, see Deift-Trubowitz [15]. If the reflection coefficient is zero, these finite number of zeros will correspond to a pure multi-soliton solution. When the radiation appears, with the interaction of reflection coefficients, the resulting phenomenon is more delicate and complicated. A-priori, just knowing there are finitely many number of zeros, it is not clear at all that under the influence of the radiation, the initial data will evolve into a sequence of solitons. To establish the soliton resolution, we go through reductions step by step via $\bar{\partial}$ -derivatives analysis and nonlinear steepest descent to reduce our Riemann-Hilbert problems (RHPs) to some solvable models. We make sure that only controllable error terms are introduced through these reduction. It is from these exactly solvable model problems that we are going to illustrate the interaction between solitary waves and radiation and the leading asymptotics of the solution.

We begin with some notations:

1.1. **Notations.** Let σ_3 be the third Pauli matrix:

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and define the matrix operation

$$e^{\text{ad } \sigma_3} A = \begin{pmatrix} a & e^2 b \\ e^{-2} c & d \end{pmatrix}.$$

Given any contour Σ , C^\pm is the Cauchy projection:

$$(1.4) \quad (C^\pm f)(z) = \lim_{z \rightarrow \Sigma_\pm} \frac{1}{2\pi i} \int_\Sigma \frac{f(s)}{s-z} ds.$$

Here $+(-)$ denotes taking limit from the positive (negative) side of the oriented contour. We define Fourier transform as

$$(1.5) \quad \hat{h}(\xi) = \mathcal{F}[h](\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ix\xi} h(x) dx.$$

Using the Fourier transform, one can define the fractional weighted Sobolev spaces:

$$(1.6) \quad H^{k,s}(\mathbb{R}) := \left\{ h : \langle 1 + |\xi|^2 \rangle^{\frac{k}{2}} \hat{h}(\xi) \in L^2(\mathbb{R}), \langle 1 + x^2 \rangle^{\frac{s}{2}} h \in L^2(\mathbb{R}) \right\}.$$

As usual, " $A := B$ " or " $B =: A$ " is the definition of A by means of the expression B . We use the notation $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$. For positive quantities a and b , we write $a \lesssim b$ for $a \leq Cb$ where C is some prescribed constant. Also $a \simeq b$ for $a \lesssim b$ and $b \lesssim a$. Throughout, we use $u_t := \frac{\partial}{\partial t} u$, $u_x := \frac{\partial}{\partial x} u$.

1.2. **The Riemann–Hilbert problem and inverse scattering.** To describe our approach, we recall that (1.1) generates an isospectral flow for the problem

$$(1.7) \quad \frac{d}{dx} \Psi = -iz\sigma_3 \Psi + U(x)\Psi$$

where

$$U(x) = \begin{pmatrix} 0 & iu(x) \\ iu(x) & 0 \end{pmatrix}.$$

This is a standard AKNS system [2]. If $u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, equation (1.7) admits bounded solutions for $z \in \mathbb{R}$. There exist unique solutions Ψ^\pm of (1.7) obeying the the following space asymptotic conditions

$$\lim_{x \rightarrow \pm\infty} \Psi^\pm(x, z) e^{-ixz\sigma_3} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and there is a matrix $T(z)$, the transition matrix, with

$$(1.8) \quad \Psi^+(x, z) = \Psi^-(x, z)T(z).$$

The matrix $T(z)$ takes the form

$$(1.9) \quad T(z) = \begin{pmatrix} a(z) & \check{b}(z) \\ b(z) & \check{a}(z) \end{pmatrix}$$

and the determinant relation gives

$$a(z)\check{a}(z) - b(z)\check{b}(z) = 1.$$

By uniqueness we have

$$(1.10) \quad \psi_{11}^\pm(z) = \overline{\psi_{22}^\pm(\bar{z})}, \quad \psi_{12}^\pm(z) = -\overline{\psi_{21}^\pm(\bar{z})},$$

$$(1.11) \quad \psi_{11}^\pm(z) = \psi_{22}^\pm(-z), \quad \psi_{12}^\pm(z) = \psi_{21}^\pm(-z).$$

This leads to the symmetry relation of the entries of T :

$$(1.12) \quad \check{a}(z) = \overline{a(\bar{z})}, \quad \check{b}(z) = -\overline{b(z)}.$$

On \mathbb{R} , the determinant of $T(z)$ is given by

$$|a(z)|^2 + |b(z)|^2 = 1.$$

Making the change of variable

$$\Psi = m e^{ixz\sigma_3}$$

the system (1.7) then becomes

$$(1.13) \quad m_x = -iz \operatorname{ad} \sigma_3 m + Um.$$

The standard AKNS method starts with the following two Volterra integral equations for real z :

$$(1.14) \quad m^{(\pm)}(x, z) = I + \int_{\pm\infty}^x e^{i(y-x)z \operatorname{ad} \sigma_3} U(y) m^{(\pm)}(y, z) dy.$$

By the standard inverse scattering theory, we formulate the reflection coefficient:

$$(1.15) \quad r(z) = -b(z)/\check{a}(z), \quad z \in \mathbb{R}.$$

Also from the symmetry conditions (1.10)-(1.11) we deduce that

$$(1.16) \quad r(-z) = -\overline{r(z)}.$$

1.2.1. *Eigenvalues.* It is important to notice that $\check{a}(z)$ and $a(z)$ has analytic continuation into the \mathbb{C}^+ and \mathbb{C}^- half planes respectively. From (1.8) we deduce that

$$(1.17) \quad \check{a}(z) = \det \begin{pmatrix} \psi_{11}^-(x, z) & \psi_{12}^+(x, z) \\ \psi_{21}^-(x, z) & \psi_{22}^+(x, z) \end{pmatrix}.$$

$$(1.18) \quad a(z) = \det \begin{pmatrix} \psi_{11}^+(x, z) & \psi_{12}^-(x, z) \\ \psi_{21}^+(x, z) & \psi_{22}^-(x, z) \end{pmatrix}.$$

From (1.11)-(1.12) we read off directly that if $\check{a}(z_i) = 0$ for some $z_i \in \mathbb{C}^+$, then $\overline{\check{a}(-\bar{z}_i)} = 0$ by symmetry. Thus if $\check{a}(z_i) = 0$, then either

(i) z_i is purely imaginary;

or

(ii) $-\bar{z}_i$ is also a zero \check{a} .

When $r \equiv 0$, Case (i) above corresponds to solitons while Case (ii) introduces breathers.

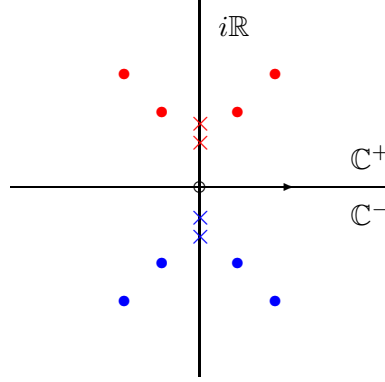
Remark 1.1. It is proven in [5] that there is an open and dense subset $U_0 \subset L^1(\mathbb{R})$ such that if $u \in U_0$, then the zeros of \check{a} (a) are finite and simple and off the real axis. We restrict the initial data to such set in this paper.

Suppose that $\check{a}(z_i) = 0$ for some $z_i \in \mathbb{C}^+$, $i = 1, 2, \dots, N$, then we have the linear dependence of the columns :

$$(1.19) \quad \begin{bmatrix} \psi_{11}^-(x, z_i) \\ \psi_{21}^-(x, z_i) \end{bmatrix} = b_i \begin{bmatrix} \psi_{12}^+(x, z_i) \\ \psi_{22}^+(x, z_i) \end{bmatrix}$$

$$(1.20) \quad \begin{bmatrix} m_{11}^-(x, z_i) \\ m_{21}^-(x, z_i) \end{bmatrix} = b_i \begin{bmatrix} m_{12}^+(x, z_i) \\ m_{22}^+(x, z_i) \end{bmatrix} e^{2iaz_i}.$$

Remark 1.2. As the zeros of \check{a} are of order one, $\check{a}'(z_i) \neq 0$.

FIGURE 1.1. Zeros of \check{a} and a 

Origin (o) zeros of \check{a} (x •) zeros of a (x •)

1.2.2. *Inverse Problem.* In this subsection we construct the Beals-Coifman solutions needed for the RHP. We need to find certain piecewise analytic matrix functions. An obvious choice is

$$(1.21) \quad \begin{cases} (m_1^{(-)}, m_2^{(+)}) & \text{Im } z > 0 \\ (m_1^{(+)}, m_2^{(-)}) & \text{Im } z < 0. \end{cases}$$

We want the solution to the RHP normalized as $x \rightarrow +\infty$, so we set

$$(1.22) \quad M(x, z) = \begin{cases} (m_1^{(-)}, m_2^{(+)}) \begin{pmatrix} \check{a}^{-1} & 0 \\ 0 & 1 \end{pmatrix}, & \text{Im } z > 0 \\ (m_1^{(+)}, m_2^{(-)}) \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix}, & \text{Im } z < 0. \end{cases}$$

We assume $a(z) \neq 0$ for all $z \in \mathbb{R}$ and recall

$$(1.23) \quad r(z) = -\frac{b(z)}{\check{a}(z)}$$

and by symmetry

$$\frac{\check{b}(z)}{a(z)} = \overline{r(z)}.$$

Using the asymptotic condition of m^\pm and (1.9), we conclude that for $z \in \mathbb{R}$

$$(1.24a) \quad \lim_{x \rightarrow +\infty} (m_1^{(-)}, m_2^{(+)}) \begin{pmatrix} \check{a}^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -e^{2ixz} \frac{b(z)}{\check{a}(z)} & 1 \end{pmatrix},$$

$$(1.24b) \quad \lim_{x \rightarrow +\infty} (m_1^{(+)}, m_2^{(-)}) \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & -e^{-2ixz} \frac{\check{b}(z)}{a(z)} \\ 0 & 1 \end{pmatrix}.$$

Setting $M_\pm(x, z) = \lim_{\epsilon \rightarrow 0} M(x, z \pm i\epsilon)$, then M_\pm satisfy the following jump condition on \mathbb{R} :

$$M_+(x, z) = M_-(x, z) \begin{pmatrix} 1 + |r(z)|^2 & e^{-2ixz} \overline{r(z)} \\ e^{2ixz} r(z) & 1 \end{pmatrix}.$$

We now calculate the residue at the pole z_i :

$$(1.25) \quad \begin{aligned} \operatorname{Res}_{z=z_i} M_{+,1}(x, z) &= \frac{1}{\check{a}'(z_i)} \begin{pmatrix} m_{11}^-(x, z_i) & 0 \\ m_{21}^-(x, z_i) & 0 \end{pmatrix} \\ &= \frac{e^{2ixz_i} b_i}{\check{a}'(z_i)} \begin{pmatrix} m_{12}^+(x, z_i) & 0 \\ m_{22}^+(x, z_i) & 0 \end{pmatrix}. \end{aligned}$$

Similarly we can calculate the residues at the pole \bar{z}_i :

$$(1.26) \quad \operatorname{Res}_{z=\bar{z}_i} M_{-,2}(x, z) = -\frac{e^{-2ix\bar{z}_i} \bar{b}_i}{a'(\bar{z}_i)} \begin{pmatrix} 0 & m_{11}^+(x, \bar{z}_i) \\ 0 & m_{21}^+(x, \bar{z}_i) \end{pmatrix}.$$

If z_i is not purely imaginary, we also have

$$(1.27) \quad \begin{aligned} \operatorname{Res}_{z=-\bar{z}_i} M_{+,1}(x, z) &= \frac{1}{\check{a}'(-\bar{z}_i)} \begin{pmatrix} m_{11}^-(x, -\bar{z}_i) & 0 \\ m_{21}^-(x, -\bar{z}_i) & 0 \end{pmatrix} \\ &= -\frac{e^{-2ix\bar{z}_i} \bar{b}_i}{\check{a}'(-\bar{z}_i)} \begin{pmatrix} m_{12}^+(x, -\bar{z}_i) & 0 \\ m_{22}^+(x, -\bar{z}_i) & 0 \end{pmatrix} \end{aligned}$$

and

$$(1.28) \quad \operatorname{Res}_{z=-z_i} M_{-,2}(x, z) = \frac{e^{2ixz_i} b_i}{a'(-z_i)} \begin{pmatrix} 0 & m_{11}^+(x, -z_i) \\ 0 & m_{21}^+(x, -z_i) \end{pmatrix}.$$

Using symmetry reduction we have that $\check{a}'(z_i) = \overline{a'(\bar{z}_i)}$ so we can define norming constant

$$c_i = \frac{b_i}{\check{a}'(z_i)}.$$

The following result is proven in [59]:

Proposition 1.3. *If $u_0 \in H^{2,1}(\mathbb{R})$, then $r(z) \in H^{1,2}(\mathbb{R})$.*

Thus we arrive at the following set of scattering data

$$(1.29) \quad \mathcal{S} = \{r(z), \{z_k, c_k\}_{k=1}^{N_1}, \{z_j, c_j\}_{j=1}^{N_2}\} \subset H^{1,2}(\mathbb{R}) \oplus \mathbb{C}^{2N_1} \oplus \mathbb{C}^{2N_2}.$$

Here $z_k = i\zeta_k$ for $\zeta_k > 0$ while $z_j = \xi_j + i\eta_j$ with $\xi_j > 0$ and $\eta_j > 0$.

It is well-known that $r(z)$, c_j and c_k have linear time evolution:

$$r(z, t) = e^{8itz^3} r(z), \quad c_j(t) = e^{8itz_j^3} c_j, \quad c_k(t) = e^{8itz_k^3} c_k.$$

In Appendix A, we will show that the maps $u_0 \mapsto \mathcal{S}$ is Lipschitz continuous from $H^{2,1}(\mathbb{R})$ into a subset of $H^{1,2}(\mathbb{R}) \oplus \mathbb{C}^{2N_1} \oplus \mathbb{C}^{2N_2}$. The long time asymptotics of mKdV is obtained through a sequence of transformations of the following RHP:

Problem 1.4. For fixed $x \in \mathbb{R}$ and $r(z) \in H^{1,2}(\mathbb{R})$, find a meromorphic matrix $M(x, t; z)$ satisfying the following conditions:

- (i) (Normalization) $M(x, t; z) \rightarrow I + \mathcal{O}(z^{-1})$ as $z \rightarrow \infty$.
- (ii) (Jump relation) For each $z \in \mathbb{R}$, $M(x, t; z)$ has continuous non-tangential boundary value $M_{\pm}(x, t; z)$ as z approaches \mathbb{R} from \mathbb{C}^{\pm} and the following jump relation holds

$$(1.30) \quad M_+(x, t; z) = M_-(x, t; z) e^{-i\theta(x, t; z) \operatorname{ad} \sigma_3} v(z)$$

$$(1.31) \quad = M_-(x, t; z) v_{\theta}(z)$$

where

$$v(z) = \begin{pmatrix} 1 + |r(z)|^2 & \overline{r(z)} \\ r(z) & 1 \end{pmatrix}$$

and

$$(1.32) \quad \theta(x, t; z) = 4t(z^3 - 3z_0^2 z) = 4tz^3 + xz$$

where

$$(1.33) \quad \pm z_0 = \pm \sqrt{\frac{-x}{12t}}$$

are the two stationary points.

(iii) (Residue condition) For $k = 1, 2, \dots, N_1$, $M(x, t; z)$ has simple poles at each $z_k, \overline{z_k}$ with

$$(1.34) \quad \text{Res}_{z_k} M = \lim_{z \rightarrow z_k} M \begin{pmatrix} 0 & 0 \\ e^{2i\theta} c_k & 0 \end{pmatrix}$$

$$(1.35) \quad \text{Res}_{\overline{z_k}} M = \lim_{z \rightarrow \overline{z_k}} M \begin{pmatrix} 0 & -e^{-2i\theta} \overline{c_k} \\ 0 & 0 \end{pmatrix}.$$

For $j = 1, 2, \dots, N_2$, $M(x, t; z)$ has simple poles at each $\pm z_j, \pm \overline{z_j}$ with

$$(1.36) \quad \text{Res}_{z_j} M = \lim_{z \rightarrow z_j} M \begin{pmatrix} 0 & 0 \\ e^{2i\theta} c_j & 0 \end{pmatrix},$$

$$(1.37) \quad \text{Res}_{\overline{z_j}} M = \lim_{z \rightarrow \overline{z_j}} M \begin{pmatrix} 0 & -e^{-2i\theta} \overline{c_j} \\ 0 & 0 \end{pmatrix},$$

$$(1.38) \quad \text{Res}_{-z_j} M = \lim_{z \rightarrow -z_j} M \begin{pmatrix} 0 & e^{-2i\theta} c_j \\ 0 & 0 \end{pmatrix},$$

$$(1.39) \quad \text{Res}_{-\overline{z_j}} M = \lim_{z \rightarrow -\overline{z_j}} M \begin{pmatrix} 0 & 0 \\ -e^{2i\theta} \overline{c_j} & 0 \end{pmatrix}.$$

Definition 1.5. We say that the initial condition u_0 is *generic* if

1. $\check{a}(z)$ ($a(z)$) associated to u_0 satisfies the simpleness and finiteness assumptions stated in Remark 1.1.
2. For all $\{z_k\}_{k=1}^{N_1}$ and $\{z_j\}_{j=1}^{N_2}$ where $z_k = i\zeta_k$ and $z_j = \xi_j + i\eta_j$,

$$4\zeta_k^2 \neq 4\eta_j^2 - 12\xi_j^2, \quad 4\eta_{j_1}^2 - 12\xi_{j_2}^2 \neq 4\eta_{j_2}^2 - 12\xi_{j_2}^2$$

for all j, k and $z_{j_1} \neq z_{j_2}$.

Remark 1.6. We arrange eigenvalues $\{z_k\}_{k=1}^{N_1}$ and $\{z_j\}_{j=1}^{N_2}$ in the following way:

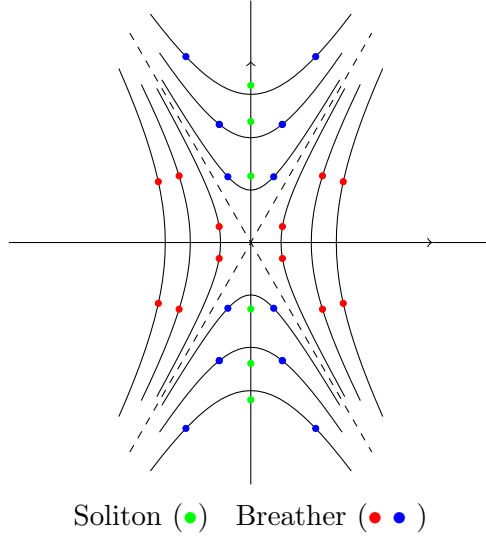
- (1) For $z_k = i\zeta_k$, $\zeta_k > 0$, we have $\zeta_1 < \zeta_2 < \dots < \zeta_k < \dots < \zeta_{N_1}$.
- (2) For $z_j = \xi_j + i\eta_j$, $\xi_j, \eta_j > 0$, we have

$$4\eta_1^2 - 12\xi_1^2 < \dots < 4\eta_j^2 - 12\xi_j^2 < \dots < 4\eta_{N_2}^2 - 12\xi_{N_2}^2.$$

Remark 1.7. For each pole $z_k(z_j) \in \mathbb{C}^+$, let $\gamma_k, (\gamma_j)$ be a circle centered at $z_k(z_j)$ of sufficiently small radius to be lie in the open upper half-plane and to be disjoint from all other circles. By doing so we replace the residue conditions (1.34)-(1.39) of the Riemann-Hilbert problem with Schwarz invariant jump conditions across closed contours (see Figure 1.3). The equivalence of this new RHP on augmented contours with the original one is a well-established result (see [58] Sec 6). The purpose of this replacement is to

- (1) make use of the *vanishing lemma* from [58, Theorem 9.3].
- (2) Formulate the Beals-Coifman representation of the solution of (1.1).

FIGURE 1.2. solitons and breathers



We now rewrite the the jump conditions of Problem 1.4: $M(x, z)$ is analytic in $\mathbb{C} \setminus \Sigma$ and has continuous boundary values M_{\pm} on Σ and M_{\pm} satisfy

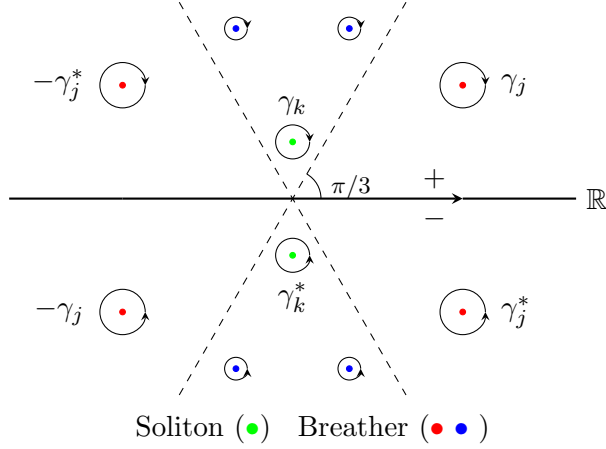
$$M_+(x, t; z) = M_-(x, t; z) e^{-i\theta(x, t; z) \text{ad } \sigma_3} v(z)$$

where

$$v(z) = \begin{pmatrix} 1 + |r(z)|^2 & \overline{r(z)} \\ r(z) & 1 \end{pmatrix}, \quad z \in \mathbb{R}$$

and

$$v(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{c_k}{z - z_k} & 1 \end{pmatrix} & z \in \gamma_k, \\ \begin{pmatrix} 1 & \overline{c_k} \\ 0 & z - \overline{z_k} \end{pmatrix} & z \in \gamma_k^* \end{cases}$$

FIGURE 1.3. The Augmented Contour Σ 

and

$$v(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{c_j}{z - z_j} & 1 \end{pmatrix} & z \in \gamma_j, \\ \begin{pmatrix} 1 & \frac{\bar{c}_j}{z - \bar{z}_j} \\ 0 & 1 \end{pmatrix} & z \in \gamma_j^*, \\ \begin{pmatrix} 1 & \frac{-c_j}{z + z_j} \\ 0 & 1 \end{pmatrix} & z \in -\gamma_j, \\ \begin{pmatrix} 1 & 0 \\ \frac{-\bar{c}_j}{z + \bar{z}_j} & 1 \end{pmatrix} & z \in -\gamma_j^* \end{cases}$$

It is well-known that v_θ admits triangular factorization:

$$v_\theta = (1 - w_{\theta^-})^{-1}(1 + w_{\theta^+}).$$

We define

$$\mu = m_+(1 - w_\theta^-)^{-1} = m_-(1 + w_\theta^+)$$

then the solvability of the RHP above is equivalent to the solvability of the following Beals-Coifman integral equation:

$$(1.40) \quad \begin{aligned} \mu(z; x, t) &= I + C_\Sigma^+ \mu w_\theta^- + C_\Sigma^- \mu w_\theta^+ \\ &= I + C_{\mathbb{R}}^+ \mu w_\theta^- + C_{\mathbb{R}}^- \mu w_\theta^+ \\ &+ \left(\begin{array}{c} \sum_{k=1}^{N_1} \frac{\mu_{12}(z_k) c_k e^{2i\theta(z_k)}}{z - z_k} - \sum_{k=1}^{N_1} \frac{\mu_{11}(\bar{z}_k) \bar{c}_k e^{-2i\theta(\bar{z}_k)}}{z - \bar{z}_k} \\ \sum_{k=1}^{N_1} \frac{\mu_{22}(z_k) c_k e^{2i\theta(z_k)}}{z - z_k} - \sum_{k=1}^{N_1} \frac{\mu_{21}(\bar{z}_k) \bar{c}_k e^{-2i\theta(\bar{z}_k)}}{z - \bar{z}_k} \end{array} \right) \end{aligned}$$

$$\begin{aligned}
& + \left(\begin{array}{c} \sum_{j=1}^{N_2} \frac{\mu_{12}(z_j)c_j e^{2i\theta(z_j)}}{z - z_j} - \sum_{j=1}^{N_2} \frac{\mu_{11}(\bar{z}_j)\bar{c}_j e^{-2i\theta(\bar{z}_j)}}{z - \bar{z}_j} \\ \sum_{j=1}^{N_2} \frac{\mu_{22}(z_j)c_j e^{2i\theta(z_j)}}{z - z_j} - \sum_{j=1}^{N_2} \frac{\mu_{21}(\bar{z}_j)\bar{c}_j e^{-2i\theta(\bar{z}_j)}}{z - \bar{z}_j} \end{array} \right) \\
& + \left(\begin{array}{c} -\sum_{j=1}^{N_2} \frac{\mu_{12}(-\bar{z}_j)\bar{c}_j e^{2i\theta(-\bar{z}_j)}}{z + \bar{z}_j} \quad \sum_{j=1}^{N_2} \frac{\mu_{11}(-z_j)c_j e^{-2i\theta(-z_j)}}{z + z_j} \\ -\sum_{j=1}^{N_2} \frac{\mu_{22}(-\bar{z}_j)\bar{c}_j e^{2i\theta(-\bar{z}_j)}}{z + \bar{z}_j} \quad \sum_{j=1}^{N_2} \frac{\mu_{21}(-z_j)c_j e^{-2i\theta(-z_j)}}{z + z_j} \end{array} \right).
\end{aligned}$$

From the solution of Problem 1.4, we recover

$$(1.41) \quad u(x, t) = \lim_{z \rightarrow \infty} 2zm_{12}(x, t, z)$$

$$(1.42) \quad = \left(\frac{1}{\pi} \int_{\Sigma} \mu(w_{\theta}^{-} + w_{\theta}^{+}) \right)_{12}$$

$$(1.43) \quad = \frac{1}{\pi} \int_{\mathbb{R}} \mu_{11}(x, t; z) \bar{r}(z) e^{-2i\theta} dz + \sum_{k=1}^{N_1} \mu_{11}(\bar{z}_k) \bar{c}_k e^{-2i\theta(\bar{z}_k)}$$

$$(1.44) \quad - \sum_{j=1}^{N_2} \mu_{11}(\bar{z}_j) \bar{c}_j e^{-2i\theta(\bar{z}_j)} + \sum_{j=1}^{N_2} \mu_{11}(-z_j) c_j e^{-2i\theta(-z_j)}$$

where the limit is taken in $\mathbb{C} \setminus \Sigma$ along any direction not tangent to Σ .

1.3. Single soliton and single breather solution. If we assume $r = 0$ and \check{a} has exactly one simple zero at $z = i\zeta$, $\zeta > 0$ and let c be the norming constant. Notice that c is purely imaginary, then we let

$$\varepsilon_{\pm} = \begin{cases} 1, & \text{Im } c > 0 \\ -1, & \text{Im } c < 0 \end{cases}$$

then equation (1.1) admits the following single-soliton solution [56] :

$$(1.45) \quad u(x, t) = 2\zeta \varepsilon_{\pm} \operatorname{sech}(-2\zeta(x - 4\zeta^2 t) + \omega).$$

where

$$\omega = \log \left(\frac{|c|}{2\zeta} \right)$$

If we assume $r = (0)$ and \check{a} has exactly two simple zeros at $z = \pm\xi + i\eta$, $\eta > 0$ and let $c = A + iB$ be the norming constant, then Equation (1.1) admits the following one-breather solution [56] :

$$(1.46) \quad u(x, t) = -4 \frac{\eta \xi \cosh(\nu_2 + \omega_2) \sin(\nu_1 + \omega_1) + \eta \sinh(\nu_2 + \omega_2) \cos(\nu_1 + \omega_1)}{\xi \cosh^2(\nu_2 + \omega_2) + (\eta/\xi)^2 \cos^2(\nu_1 + \omega_1)}$$

with

$$\nu_1 = 2\xi(x + 4(\xi^2 - 3\eta^2)t)$$

$$\nu_2 = 2\eta(x - 4(\eta^2 - 3\xi^2)t)$$

and

$$(1.47) \quad \tan \omega_1 = \frac{B\xi - A\eta}{A\xi + B\eta}$$

$$(1.48) \quad e^{-\omega_2} = \left| \frac{\xi}{2\eta} \right| \sqrt{\frac{A^2 + B^2}{\xi^2 + \eta^2}}.$$

From above we observe that soliton has velocity $v_s = 4\eta^2$, always traveling in the positive direction and breather has velocity $v_b = 4\eta^2 - 12\xi^2$, which means breather can travel in both directions. Also notice that

$$(1.49) \quad \operatorname{Re}i\theta = (4\eta^2 - 12\xi^2 + 12z_0^2)\eta t$$

If we fix the velocity $v_b = x/t$, then $4\eta^2 - 12\xi^2 = v_b$ implies that the hyperbola pass through the stationary points

$$(1.50) \quad \pm z_0 = \pm \sqrt{\frac{-x}{12t}}$$

Conversely, if $a(z)$ has zeros on the hyperbola $4\eta^2 - 12\xi^2 = x/t$, we expect breathers moving with velocity x/t .

Remark 1.8. Rewrite (1.49) as

$$\operatorname{Re}i\theta(x, t; z) = t^{1/3} \left(4(-3u^2v + v^3)t^{2/3} - \frac{x}{t^{1/3}}v \right).$$

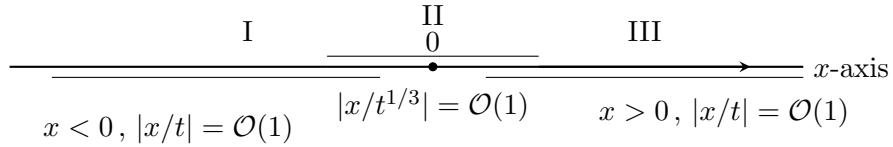
In the Painlevé region where we set $|x/t^{1/3}| \leq c$, it is easy to see that for $\sqrt{3}u > v$ ($\sqrt{3}u < v$), we have $\operatorname{Re}i\theta < 0$ ($\operatorname{Re}i\theta > 0$). In the soliton region where $x > 0$, $x/t = \mathcal{O}(1)$ we write

$$\operatorname{Re}i\theta(x, t; z) = t \left(4(-3u^2v + v^3) - \frac{x}{t}v \right).$$

It is now clear that if we set $x/t = v_{b_j} = 4\eta_j^2 - 12\xi_j^2$, then $\operatorname{Re}i\theta(x, t; z_j) = 0$.

1.4. Main results. The central result of this paper is to describe the long-time behavior of the solutions u of (1.1) in different regions respectively.

FIGURE 1.4. Three Regions



We are mainly interested in the long time asymptotics of mKdV in the following three regions:

- oscillatory region: $x < 0$, $|x/t| = \mathcal{O}(1)$ as $t \rightarrow \infty$. In this region, we can observe breathers traveling in the left direction.
- self-similar region: $|x/t^{1/3}| \leq c$ as $t \rightarrow \infty$. This region does not have breathers and solitons as $t \rightarrow \infty$.
- soliton region: $x > 0$, $|x/t| = \mathcal{O}(1)$ as $t \rightarrow \infty$. In this region, we can observe breathers and solitons traveling in the right direction.

Remark 1.9. The long time asymptotics for overlaps of the regions have been studied in the previous paper [10, Theorem 1.6]. There are no solitons and breathers in those overlap regions.

1.4.1. *Long-time asymptotics.* Our main results is the following detailed long-time asymptotics of the solution to the focusing mKdV. This also verifies the soliton resolution for generic data.

Theorem 1.10. *Given initial the data $u_0 \in H^{2,1}(\mathbb{R})$ and assume u_0 is generic in the sense of Definition 1.5. Suppose the initial data produce the scattering data*

$$S = \left\{ r(z), \{z_k, c_k\}_{k=1}^{N_1}, \{z_j, c_j\}_{j=1}^{N_2} \right\} \in H^{1,2}(\mathbb{R}) \oplus \mathbb{C}^{2N_1} \oplus \mathbb{C}^{2N_2}$$

as in Subsection 1.2.2. We first arrange $z_j = \xi_j + i\eta_j$, $\xi_j, \eta_j > 0$ and suppose for some $1 \leq \ell_0 \leq N_2$, one has

$$4\eta_1^2 - 12\xi_1^2 < \dots < 4\eta_{\ell_0}^2 - 12\xi_{\ell_0}^2 < 0 < 4\eta_{\ell_0+1}^2 - 12\xi_{\ell_0+1}^2 < \dots < 4\eta_{N_2}^2 - 12\xi_{N_2}^2.$$

Secondly, we list $z_k = i\zeta_k$, $\zeta_k > 0$ as

$$0 < \zeta_1 < \dots < \zeta_{N_1}.$$

Let u be the solution the focusing mKdV

$$\partial_t u + \partial_{xxx} u + 6u^2 \partial_x u = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+$$

with initial data u_0 given by the reconstruction formula (the Beals-Coifman solution). Denote

$$\tau = z_0^3 t, \quad \pm z_0 = \pm \sqrt{\frac{-x}{12t}}.$$

Then the solution u can be written as the superposition of breathers, solitons and the radiation as following:

$$u(x, t) = \sum_{\ell=1}^{N_2} u_{\ell}^{(br)}(x, t) + \sum_{\ell=1}^{N_1} u_{\ell}^{(so)}(x, t) + R(x, t).$$

(1). For the breather part,

(i) if $\ell \leq \ell_0$,

$$(1.51) \quad u_{\ell}^{(br)}(x, t) = -4 \frac{\eta_{\ell} \xi_{\ell} \cosh(\nu_2 + \tilde{\omega}_2) \sin(\nu_1 + \tilde{\omega}_1) + \eta_{\ell} \sinh(\nu_2 + \tilde{\omega}_2) \cos(\nu_1 + \tilde{\omega}_1)}{\xi_{\ell} \cosh^2(\nu_2 + \tilde{\omega}_2) + (\eta_{\ell}/\xi_{\ell})^2 \cos^2(\nu_1 + \tilde{\omega}_1)}$$

where

$$(1.52) \quad \nu_1 = 2\xi_{\ell} (x + 4(\xi_{\ell}^2 - 3\eta_{\ell}^2)t),$$

$$(1.53) \quad \nu_2 = 2\eta_{\ell} (x - 4(\eta_{\ell}^2 - 3\xi_{\ell}^2)t),$$

and

$$(1.54) \quad \tan(\tilde{\omega}_1) = \frac{\tilde{B}\xi_{\ell} - \tilde{A}\eta_{\ell}}{\tilde{A}\xi_{\ell} + \tilde{B}\eta_{\ell}}, \quad e^{-\tilde{\omega}_2} = \left| \frac{\xi_{\ell}}{2\eta_{\ell}} \right| \sqrt{\frac{\tilde{A}^2 + \tilde{B}^2}{\xi_{\ell}^2 + \eta_{\ell}^2}}$$

here \tilde{A} and \tilde{B} are given as

$$(1.55) \quad \tilde{c}_{\ell} = c_{\ell} \delta(z_{\ell})^{-2} = \tilde{A} + i\tilde{B}.$$

where the scalar function $\delta(z)$ is given by

$$\delta(z) = \left(\prod_{k=1}^{N_1} \frac{z - \bar{z}_k}{z - z_k} \right) \left(\prod_{z_j \in \mathcal{B}_{\ell}} \frac{z - \bar{z}_j}{z - z_j} \right) \left(\prod_{z_j \in \mathcal{B}_{\ell}} \frac{z + z_j}{z + \bar{z}_j} \right) \left(\frac{z - z_0}{z + z_0} \right)^{i\kappa} e^{\chi(z)}$$

with

$$\chi(z) = \frac{1}{2\pi i} \int_{-z_0}^{z_0} \log \left(\frac{1 + |r(\zeta)|^2}{1 + |r(z_0)|^2} \right) \frac{d\zeta}{\zeta - z},$$

$$\kappa = -\frac{1}{2\pi} \log \left(1 + |r(z_0)|^2 \right),$$

and

$$(1.56) \quad \mathcal{B}_{\ell} = \{z_j = \xi_j + i\eta_j : 4\eta_j^2 - 12\xi_j^2 > 4\eta_{\ell}^2 - 12\xi_{\ell}^2\}.$$

- (ii) If $\ell_0 + 1 \leq \ell$, we have the same expressions as (1.51), (1.52), (1.53), (1.54) and (1.55) but with \tilde{A} and \tilde{B} are given as

$$\tilde{c}_\ell = c_\ell \psi (z_\ell)^{-2} = \tilde{A} + i\tilde{B}$$

where the scalar function ψ is defined by

$$\psi(z) = \left(\prod_{z_k \in \mathcal{B}_{\ell,s}} \frac{z - \bar{z}_k}{z - z_k} \right) \left(\prod_{z_j \in \mathcal{B}_{\ell,b}} \frac{z - \bar{z}_j}{z - z_j} \right) \left(\prod_{z_j \in \mathcal{B}_{\ell,b}} \frac{z_\ell + z_j}{z_\ell + \bar{z}_j} \right)$$

with

$$\begin{aligned} \mathcal{B}_\ell &= \{z_j = \xi_j + i\eta_j : 4\eta_j^2 - 12\xi_j^2 > 4\eta_\ell^2 - 12\xi_\ell^2\} \cup \{z_k = i\zeta_k : 4\zeta_k^2 > 4\eta_\ell^2 - 12\xi_\ell^2\} \\ &=: \mathcal{B}_{\ell,b} \cup \mathcal{B}_{\ell,s}. \end{aligned}$$

- (2). For the soliton part, we have

$$(1.57) \quad u_\ell^{(so)}(x, t) = 2\zeta_\ell \varepsilon_{\pm, \ell} \operatorname{sech}(-2\zeta_\ell(x - 4\zeta_\ell^2 t) + \omega_\ell)$$

with

$$\omega_\ell = \log\left(\frac{|c_\ell|}{2\zeta_\ell}\right) + 2 \sum_{z_k \in \mathcal{S}_{\ell,s}} \log\left|\frac{z_\ell - z_k}{z_\ell - \bar{z}_k}\right| + 2 \sum_{z_j \in \mathcal{S}_{\ell,b}} \log\left|\frac{z_\ell - z_j}{z_\ell - \bar{z}_j}\right| + 2 \sum_{z_j \in \mathcal{S}_{\ell,b}} \log\left|\frac{z_\ell + \bar{z}_j}{z_\ell + z_j}\right|.$$

where

$$\begin{aligned} \mathcal{S}_\ell &= \{z_j = \xi_j + i\eta_j : 4\eta_j^2 - 12\xi_j^2 > 4\zeta_\ell^2\} \cup \{z_k = i\zeta_k : 4\zeta_k^2 > 4\zeta_\ell^2\} \\ &=: \mathcal{S}_{\ell,b} \cup \mathcal{S}_{\ell,s}. \end{aligned}$$

- (3). Finally, the radiation term, we have the following asymptotics.

- (i) In the soliton region, i.e., Region III, we have

$$(1.58) \quad |R(x, t)| \lesssim \frac{1}{t}.$$

- (ii) In the self-similar region, i.e., Region II, for $4 < p < \infty$, one has

$$(1.59) \quad R(x, t) = \frac{1}{(3t)^{\frac{1}{3}}} P\left(\frac{x}{(3t)^{\frac{1}{3}}}\right) + \mathcal{O}\left(t^{\frac{2}{3p} - \frac{1}{2}}\right)$$

where P is a solution to the Painlevé II equation

$$P''(s) - sP'(s) + 2P^3(s) = 0$$

determined by $r(0)$.

- (iii) In the oscillatory region, i.e. Region I, there are two separate cases.

- (a) If we choose the frame $x = vt$ with $v = 4\eta_\ell^2 - 12\xi_\ell^2 < 0$, then one can see the influence of the breather on the radiation strongly and explicitly as the following:

$$(1.60) \quad R(x, t) = u_{as}(x, t) + \mathcal{O}\left((z_0 t)^{-\frac{3}{4}}\right)$$

where

$$\begin{aligned} u_{as}(x, t) &= \frac{1}{\sqrt{48tz_0}} \left(\left(m_{11}^{(br)}(-z_0)^2 (i\delta_A^0)^2 \bar{\beta}_{12} \right) + m_{12}^{(br)}(-z_0)^2 \left((i\delta_A^0)^2 \bar{\beta}_{21} \right) \right) \\ &\quad + \frac{1}{\sqrt{48tz_0}} \left(\left(m_{11}^{(br)}(-z_0)^2 (i\delta_B^0)^2 \beta_{12} \right) - m_{12}^{(br)}(-z_0)^2 \left((i\delta_B^0)^2 \beta_{21} \right) \right) \end{aligned}$$

with some explicit constants $m_{11}^{(br)}(-z_0)$, $m_{12}^{(br)}(-z_0)$, $m_{11}^{(br)}(z_0)$, $m_{12}^{(br)}(z_0)$ from the breather matrix, see Section 4,

$$\beta_{12} = \frac{\sqrt{2\pi}e^{i\pi/4}e^{-\pi\kappa}}{r(z_0)\Gamma(-i\kappa)}, \quad \beta_{21} = \frac{-\sqrt{2\pi}e^{-i\pi/4}e^{-\pi\kappa}}{r(z_0)\Gamma(i\kappa)},$$

$$\delta_A^0 = (192\tau)^{i\kappa/2} e^{-8i\tau} e^{\chi(-z_0)} \eta_0(-z_0),$$

$$\delta_B^0 = (192\tau)^{-i\kappa/2} e^{8i\tau} e^{\chi(z_0)} \eta_0(z_0)$$

and

$$\eta_0(\pm z_0) = \left(\prod_{z_k=1}^{N_1} \frac{\pm z_0 - \bar{z}_k}{\pm z_0 - z_k} \right) \left(\prod_{z_j \in \mathcal{B}_\ell} \frac{\pm z_0 - \bar{z}_j}{\pm z_0 - z_j} \right) \left(\prod_{z_j \in \mathcal{B}_\ell} \frac{\pm z_0 + z_j}{\pm z_0 + \bar{z}_j} \right)$$

where \mathcal{B}_ℓ as (1.56).

(b) If $v \neq 4\eta_j^2 - 12\xi_j^2$ for $1 \leq j \leq N_2$, then we have

$$(1.61) \quad R(x, t) = u_{as}(x, t) + \mathcal{O}\left((z_0 t)^{-\frac{3}{4}}\right)$$

where

$$u_{as}(x, t) = \left(\frac{\kappa}{3tz_0} \right)^{\frac{1}{2}} \cos(16tz_0^3 - \kappa \log(192tz_0^3) + \phi(z_0))$$

with

$$\begin{aligned} \phi(z_0) &= \arg \Gamma(i\kappa) - \frac{\pi}{4} - \arg r(z_0) + \frac{1}{\pi} \int_{-z_0}^{z_0} \log \left(\frac{1 + |r(\zeta)|^2}{1 + |r(z_0)|^2} \right) \frac{d\zeta}{\zeta - z_0} \\ &\quad - 4 \left(\sum_{k=1}^{N_1} \arg(z_0 - z_k) + \sum_{z_j \in \mathcal{B}_\ell} \arg(z_0 - z_j) + \sum_{z_j \in \mathcal{B}_\ell} \arg(z_0 + \bar{z}_j) \right). \end{aligned}$$

Remark 1.11. The two expressions (1.60) and (1.61) above match each other since as the velocity of the frame moving away from the the velocity of the breather, $m_{12}^{(br)}(z_0)$ provides the exponential decay in time and the remain terms combined together give the same asymptotics as the later expression up to terms exponential decay in time. These exponential decay rates depend on the gap between the velocity of the frame the the velocities of breathers.

One can trace all the details in our analysis and notice that actually it suffices to require the weights in x to be $\langle x \rangle^s$ with $s > \frac{1}{2}$. More precisely, note that first $s = 1$ is used in the construction of Jost functions. But actually, in that construction, we just need $L^{2,s}(\mathbb{R})$, $s > \frac{1}{2}$ and the potential in $L^1(\mathbb{R})$. One can simply check that $L^1(\mathbb{R}) \subset L^{2,s}(\mathbb{R})$ for $s > \frac{1}{2}$. Secondly, $s = 1$ is used in the analysis of asymptotics of the Riemann-Hilbert problem but note that reflection coefficient in H^s for $s > \frac{1}{2}$ is sufficient for us due to Sobolev's embedding and the estimate of modulus of continuity. To estimate the H^s norm of the reflection coefficient, by bijectivity, in terms of the initial data, $L^{2,s}(\mathbb{R})$ for $s > \frac{1}{2}$ is enough for us. Although in Zhou's work, he only deals with $s \in \mathbb{N}$, the fractional results can be obtained simply by interpolation.

Then using the lowest regularity for the local well-posedness in $H^k(\mathbb{R})$ with $k \geq \frac{1}{4}$ via contraction obtained by Kenig-Ponce-Vega [35] and the recent low regularity conservation laws due to Killip-Visan-Zhang [37] and Koch-Tataru [39], we can use a global approximation argument to extend our long-time asymptotics to $H^{k,s}$ with $k \geq \frac{1}{4}$ and $\ell > \frac{1}{2}$.

Theorem 1.12. *Suppose that $u_0 \in H^{k,s}(\mathbb{R})$ with $s > \frac{1}{2}$ and $k \geq \frac{1}{4}$ is generic in the sense of Definition 1.5, the long-time asymptotics as in Theorem 1.10 hold for the solution to the focusing mKdV*

$$(1.62) \quad \partial_t u + \partial_{xxx} u + 6u^2 \partial_x u = 0, \quad u(0) = u_0 \in H^{k,s}(\mathbb{R})$$

given by the Duhamel representation

$$(1.63) \quad u = W(t) u_0 - \int_0^t W(t-s) (6u^2 \partial_x u(s)) ds$$

where

$$W(t) u_0 = e^{-t\partial_{xxx}} u_0 \quad \text{and} \quad \mathcal{F}_x [W(t) u_0](\xi) = e^{it\xi^3} \hat{u}_0(\xi)$$

The key point is that Beals-Coifman solutions have asymptotics and strong solutions in the sense of Duhamel can be used to pass to limits. For smooth data, Beals-Coifman solutions and strong solutions are the same. Our computations for Beals-Coifman solutions show that the error estimates only depend on weights but not the regularity of initial data. To illustrate this philosophy, we have the follow diagram:

$$\begin{array}{ccc} \mathcal{S} \ni u_0(x) & \xrightarrow{IST} & \text{asymptotics} \\ \text{Duhamel} \downarrow & & \uparrow \\ \mathcal{S} \ni u(x,t) & \xrightarrow{Approx} & u(x,t) \in H^{1/4,1} \end{array}$$

Remark 1.13. In order to get precise behavior of the radiation, the weights in the Sobolev norms are necessary. From the inverse scattering point of view, these weights are used to construct Jost functions and in the $\bar{\partial}$ -interpolation argument. On the other hand, from the stationary phase point of view, to obtain the precise asymptotics of the oscillatory integral, we need the function which is multiplied by an oscillatory factor to be defined pointwise so that we can localize the leading order behavior to the stationary point. The weights precisely give us the pointwise meaning of the function which is integrated again an oscillatory factor via Sobolev embedding. For the linear scattering theory, one can probably conclude the long-time behavior of the nonlinear equation matches a linear flow using unweighted norms. But in our setting, the scattering behavior is nonlinear, so we have to carry out the precise asymptotics and hence the weights can not be avoided.

Hereinafter, for the sake of simplicity, we focus on the case $s = 1$.

1.4.2. *Asymptotic stability.* As by products of our long-time asymptotics, the full asymptotic stability of solitons/breathers of the mKdV follows naturally. First of all, we state the asymptotic the stability of a breather traveling to the left separately. Recall that the stability of a breather traveling to the right restricted to the solitary region by energy method is analyzed in Alejo-Muñoz [3, 4]. The stability of a breather traveling to the right via our approach is given in Corollary 1.15 as a special case.

Corollary 1.14. *Let $u^{(br)}(x, t; z_0)$ be a breather with discrete scattering data (z_0, c_0) such that $z_0 = \xi_0 + i\eta_0$, $\xi_0, \eta_0 > 0$ with $4\eta_0^2 - 12\xi_0^2 < 0$. Suppose $\|\mathbf{R}(0)\|_{H^{\frac{1}{4},1}(\mathbb{R})} < \epsilon$ for $0 \leq \epsilon \ll 1$ small enough, consider the solution u to the focusing mKdV (1.62) with the initial data*

$$u_0 = u^{(br)}(x, 0; z_0, c_0) + \mathbf{R}(0).$$

Then there exist $z_1 = \xi_1 + i\eta_1$ and the norming constant c_1 such that

$$(1.64) \quad |z_1 - z_0| + |c_1 - c_0| \lesssim \epsilon.$$

Let $r(z)$ be the reflection coefficient computed from u_0 . Then, as $t \rightarrow \infty$

$$u = u^{(br)}(x, t; z_1, c_1) + R(x, t)$$

where the radiation term $R(x, t)$ has the following asymptotics:

(1). In the soliton region, i.e., Region III, we have

$$(1.65) \quad |R(x, t)| \lesssim \frac{1}{t}.$$

(2). In the self-similar region, (Region II), for $4 < p < \infty$, one has

$$(1.66) \quad R(x, t) = \frac{1}{(3t)^{\frac{1}{3}}} P\left(\frac{x}{(3t)^{\frac{1}{3}}}\right) + \mathcal{O}\left(t^{\frac{2}{3p}-\frac{1}{2}}\right)$$

where P is a solution to the Painlevé II equation

$$P''(s) - sP'(s) + 2P^3(s) = 0$$

determined by $r(0)$.

(3). In the oscillatory region (Region I), the asymptotics for $R(x, t)$ are more involved.

(i) If we choose the frame $x = vt$ with $v = 4\eta_1^2 - 12\xi_1^2 < 0$, one has

$$R(x, t) = u_{as}(x, t) + \mathcal{O}\left((z_0 t)^{-\frac{3}{4}}\right)$$

where

$$(1.67) \quad u_{as}(x, t) = \frac{1}{\sqrt{48tz_0}} \left(\left(m_{11}^{(br)}(-z_0)^2 (i\delta_A^0)^2 \bar{\beta}_{12} \right) + m_{12}^{(br)}(-z_0)^2 \left((i\delta_A^0)^2 \bar{\beta}_{21} \right) \right) \\ + \frac{1}{\sqrt{48tz_0}} \left(\left(m_{11}^{(br)}(z_0)^2 (i\delta_B^0)^2 \beta_{12} \right) - m_{12}^{(br)}(z_0)^2 \left((i\delta_B^0)^2 \beta_{21} \right) \right)$$

with some explicit constants $m_{11}^{(br)}(-z_0)$, $m_{12}^{(br)}(-z_0)$, $m_{11}^{(br)}(z_0)$, $m_{12}^{(br)}(z_0)$ from the breather matrix,

$$\beta_{12} = \frac{\sqrt{2\pi} e^{i\pi/4} e^{-\pi\kappa}}{r(z_0) \Gamma(-i\kappa)}, \quad \beta_{21} = \frac{-\sqrt{2\pi} e^{-i\pi/4} e^{-\pi\kappa}}{r(z_0) \Gamma(i\kappa)},$$

$$\delta_A^0 = (192\tau)^{i\kappa/2} e^{-8i\tau} e^{\chi(-z_0)},$$

$$\delta_B^0 = (192\tau)^{-i\kappa/2} e^{8i\tau} e^{\chi(z_0)}$$

(ii) If $v \neq 4\eta_1^2 - 12\xi_1^2$, then we have

$$R(x, t) = u_{as}(x, t) + \mathcal{O}\left((z_0 t)^{-\frac{3}{4}}\right)$$

where

$$(1.68) \quad u_{as}(x, t) = \left(\frac{\kappa}{3tz_0} \right)^{\frac{1}{2}} \cos(16tz_0^3 - \kappa \log(192tz_0^3) + \phi(z_0))$$

with

$$\phi(z_0) = \arg \Gamma(i\kappa) - \frac{\pi}{4} - \arg r(z_0) + \frac{1}{\pi} \int_{-z_0}^{z_0} \log \left(\frac{1 + |r(\zeta)|^2}{1 + |r(z_0)|^2} \right) \frac{d\zeta}{\zeta - z_0}.$$

Proof. The above results follow from Theorem 1.12 and the Lipschitz continuity of the direct scattering map see the Appendix A and Zhou [59]. \square

To conclude our stability discussion, one can also consider the full asymptotic stability of a complicated radiationless nonlinear structure. To construct the reflectionless solution, suppose we have the following discrete scattering data

$$S_D = \left\{ \{z_{0,k}, c_{0,k}\}_{k=1}^{N_1}, \{z_{0,j}, c_{0,j}\}_{j=1}^{N_2} \right\} \in \mathbb{C}^{2N_1} \oplus \mathbb{C}^{2N_2}.$$

Assume that $z_{0,j} = \xi_{0,j} + i\eta_{0,j}$, $\xi_{0,j}, \eta_{0,j} > 0$ and $z_{0,k} = \zeta_{0,k}i$ and for some $1 \leq \ell_0 \leq N_2$, one has

$$4\eta_{0,1}^2 - 12\xi_{0,1}^2 < \dots < 4\eta_{0,\ell_0}^2 - 12\xi_{0,\ell_0}^2 < 0 < 4\eta_{0,\ell_0+1}^2 - 12\xi_{0,\ell_0+1}^2 < \dots < 4\eta_{0,N_2}^2 - 12\xi_{0,N_2}^2.$$

Secondly, we list the eigenvalues of $\check{a}(z)$ on the upper-half imaginary axis as $z_{0,k} = \zeta_{0,k}i$,

$$\zeta_{0,1} < \dots < \zeta_{0,N_1}.$$

Then one can construct a reflectionless solution $u_N(x, t)$ using

$$S_D = \left\{ \{z_{0,k}, c_{0,k}\}_{k=1}^{N_1}, \{z_{0,j}, c_{0,j}\}_{j=1}^{N_2} \right\}$$

as

$$u_N(x, t) = \sum_{\ell=1}^{N_2} u_{\ell}^{(br)}(x, t; z_{0,\ell}, c_{0,\ell}) + \sum_{\ell=1}^{N_1} u_{\ell}^{(so)}(x, t; z_{0,\ell}, c_{0,\ell}).$$

where

$$u_{\ell}^{(br)}(x, t; z_{0,\ell}) = -4 \frac{\eta_{0,\ell} \xi_{0,\ell} \cosh(\nu_2 + \tilde{\omega}_2) \sin(\nu_1 + \tilde{\omega}_1) + \eta_{0,\ell} \sinh(\nu_2 + \tilde{\omega}_2) \cos(\nu_1 + \tilde{\omega}_1)}{\xi_{0,\ell} \cosh^2(\nu_2 + \tilde{\omega}_2) + (\eta_{0,\ell}/\xi_{0,\ell})^2 \cos^2(\nu_1 + \tilde{\omega}_1)}$$

where

$$\begin{aligned} \nu_1 &= 2\xi_{0,\ell} (x + 4(\xi_{0,\ell}^2 - 3\eta_{0,\ell}^2)t), \\ \nu_2 &= 2\eta_{0,\ell} (x - 4(\eta_{0,\ell}^2 - 3\xi_{0,\ell}^2)t), \end{aligned}$$

and

$$\tan(\tilde{\omega}_1) = \frac{\tilde{B}\xi_{0,\ell} - \tilde{A}\eta_{0,\ell}}{\tilde{A}\xi_{0,\ell} + \tilde{B}\eta_{0,\ell}}, \quad e^{-\tilde{\omega}_2} = \left| \frac{\xi_{0,\ell}}{2\eta_{0,\ell}} \right| \sqrt{\frac{\tilde{A}^2 + \tilde{B}^2}{\xi_{0,\ell}^2 + \eta_{0,\ell}^2}}$$

here \tilde{A} and \tilde{B} are given as

$$\tilde{c}_{0,\ell} = c_{0,\ell} \psi(z_{0,\ell})^{-2} = \tilde{A} + i\tilde{B}$$

where the scalar function as before is given has

$$\psi(z) = \left(\prod_{z_{0,k} \in \mathcal{B}_{\ell,s}} \frac{z - \overline{z_{0,k}}}{z - z_{0,k}} \right) \left(\prod_{z_{0,j} \in \mathcal{B}_{\ell,b}} \frac{z - \overline{z_{0,j}}}{z - z_{0,j}} \right) \left(\prod_{z_{0,j} \in \mathcal{B}_{\ell,b}} \frac{z + z_{0,j}}{z + \overline{z_{0,j}}} \right).$$

We also define

$$\begin{aligned} \mathcal{B}_{\ell} &= \{z_{0,j} = \xi_{0,j} + i\eta_{0,j} : 4\eta_{0,j}^2 - 12\xi_{0,j}^2 > 4\eta_{0,\ell}^2 - 12\xi_{0,\ell}^2\} \cup \{z_{0,k} = i\zeta_{0,k} : 4\zeta_{0,k}^2 > 4\eta_{0,\ell}^2 - 12\xi_{0,\ell}^2\} \\ &=: \mathcal{B}_{\ell,b} \cup \mathcal{B}_{\ell,s}. \end{aligned}$$

For the soliton part,

$$u_{\ell}^{(so)}(x, t; z_{0,\ell}) = 2\zeta_{0,\ell} \varepsilon_{\pm, \ell} \operatorname{sech}(-2\zeta_{0,\ell}(x - 4\zeta_{0,\ell}^2 t) + \omega_{0,\ell})$$

where

$$\omega_{0,\ell} = \log\left(\frac{|c_{0,\ell}|}{2\zeta_{0,\ell}}\right) + 2 \sum_{z_{0,k} \in \mathcal{S}_{\ell,s}} \log\left|\frac{z_{0,\ell} - z_{0,k}}{z_{0,\ell} - \overline{z_{0,k}}}\right| + 2 \sum_{z_{0,j} \in \mathcal{S}_{\ell,b}} \log\left|\frac{z_{0,\ell} - z_{0,j}}{z_{0,\ell} - \overline{z_{0,j}}}\right| + 2 \sum_{z_{0,j} \in \mathcal{S}_{\ell,b}} \log\left|\frac{z_{0,\ell} + \overline{z_{0,j}}}{z_{0,\ell} + z_{0,j}}\right|.$$

We also define

$$\mathcal{S}_{\ell} = \{z_{0,j} = \xi_{0,j} + i\eta_{0,j} : 4\eta_{0,j}^2 - 12\xi_{0,j}^2 > 4\zeta_{0,\ell}^2\} \cup \{z_{0,k} = i\zeta_{0,k} : 4\zeta_{0,k}^2 > 4\zeta_{0,\ell}^2\}$$

$$=: \mathcal{S}_{\ell,b} \cup \mathcal{S}_{\ell,s}.$$

Finally, we state a corollary regarding the full asymptotic stability of $u_N(x, t)$.

Corollary 1.15. *Consider the reflectionless solution $u_N(x, t)$ to the focusing mKdV (1.62). Suppose $\|R(0)\|_{H^{\frac{1}{4},1}(\mathbb{R})} < \epsilon$ for $0 \leq \epsilon \ll 1$ small enough, then consider the solution u to the focusing mKdV (1.62) with the initial data*

$$u_0 = u_N(x, t) + R(0),$$

then there exist scattering data

$$S = \left\{ r(z), \{z_{1,k}, c_{1,k}\}_{k=1}^{N_1}, \{z_{1,j}, c_{1,j}\}_{j=1}^{N_2} \right\} \in H^{1,\frac{1}{4}}(\mathbb{R}) \oplus \mathbb{C}^{2N_1} \oplus \mathbb{C}^{2N_2}$$

computed in terms of u_0 such that

$$\sum_{k=1}^{N_1} (|z_{0,k} - z_{1,k}| + |c_{0,k} - c_{1,k}|) + \sum_{j=1}^{N_2} (|z_{0,j} - z_{1,j}| + |c_{0,j} - c_{1,j}|) \lesssim \epsilon.$$

Then with the scattering data S , one can write the solution u to the focusing mKdV with the initial data u_0 as

$$u = \sum_{\ell=1}^{N_2} u_{\ell}^{(br)}(x, t; z_{1,\ell}, c_{1,\ell}) + \sum_{\ell=1}^{N_1} u_{\ell}^{(so)}(x, t; z_{1,\ell}, c_{1,\ell}) + R(x, t)$$

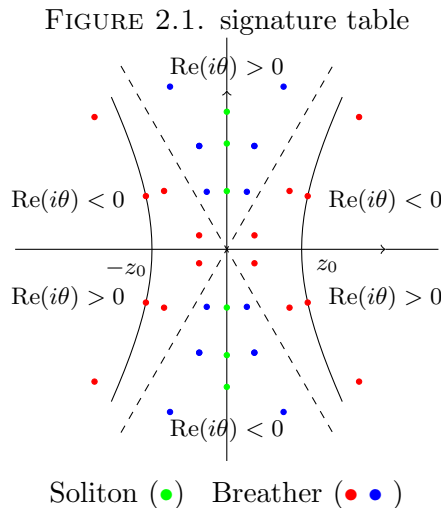
where the radiation term $R(x, t)$ has the asymptotics in Theorem 1.12 and Theorem 1.10 using the scattering data S .

Remark 1.16. Notice that for $N_2 = 0$ and $N_1 = 1$, $u_N(x, t)$ is simply a solitary wave and for $N_2 = 1$ and $N_1 = 0$, $u_N(x, t)$ is a breather. Corollary 1.15 in particular gives the full asymptotic stability of soliton and breather. Also this corollary covers the full asymptotic stability of multi-soliton solution, multi-breather solution and the mixed structure of them.

1.5. Acknowledgement. We thank Prof. Catherine Sulem for her detailed comments and helpful remarks.

2. CONJUGATION

Along a characteristic line $x = vt$ for $v < 0$ we have the following signature table:



In the figure above, we have chosen

$$v = \frac{x}{t} = 4\eta_\ell^2 - 12\xi_\ell^2$$

where $\{z_j\}_{j=1}^{N_2} \ni z_\ell = \xi_\ell + i\eta_\ell$ with $1 \leq \ell \leq N_2$. Define the following sets:

$$(2.1) \quad \mathcal{B}_\ell = \{z_j = \xi_j + i\eta_j : 4\eta_j^2 - 12\xi_j^2 > 4\eta_\ell^2 - 12\xi_\ell^2\}.$$

and

$$(2.2) \quad \mathcal{Z}_k = \{z_k\}_{k=1}^{N_1}, \quad \mathcal{Z}_j = \{z_j\}_{k=1}^{N_2}, \quad \mathcal{Z} = \mathcal{Z}_j \cup \mathcal{Z}_k.$$

Also define

$$(2.3) \quad \lambda = \min\{\min_{z, z' \in \mathcal{Z}} |z - z'|, \quad \text{dist}(\mathcal{Z}, \mathbb{R})\}.$$

We observe that for all $z_k \in \mathcal{Z}_k$ and $z_j \in \mathcal{B}_\ell$,

$$\text{Re}(i\theta(z_k)) > 0, \quad \text{Re}(i\theta(z_j)) > 0.$$

Then we introduce a new matrix-valued function

$$(2.4) \quad m^{(1)}(z; x, t) = m(z; x, t)\delta(z)^{-\sigma_3}$$

where $\delta(z)$ solves the scalar RHP Problem 2.1 below:

Problem 2.1. Given $\pm z_0 \in \mathbb{R}$ and $r \in H^1(\mathbb{R})$, find a scalar function $\delta(z) = \delta(z; z_0)$, meromorphic for $z \in \mathbb{C} \setminus [-z_0, z_0]$ with the following properties:

- (1) $\delta(z) \rightarrow 1$ as $z \rightarrow \infty$,
- (2) $\delta(z)$ has continuous boundary values $\delta_\pm(z) = \lim_{\varepsilon \downarrow 0} \delta(z \pm i\varepsilon)$ for $z \in (-z_0, z_0)$,
- (3) δ_\pm obey the jump relation

$$\delta_+(z) = \begin{cases} \delta_-(z) \left(1 + |r(z)|^2\right), & z \in (-z_0, z_0), \\ \delta_-(z), & z \in \mathbb{R} \setminus (-z_0, z_0), \end{cases}$$

- (4) $\delta(z)$ has simple pole at z_k for $k = 1 \dots N_1$ and at $z_j, -\bar{z}_j$ for $j \in \mathcal{B}_\ell$.

Lemma 2.2. Suppose $r \in H^1(\mathbb{R})$ and that $\kappa(s)$ is given by

$$(2.5) \quad \kappa = -\frac{1}{2\pi} \log \left(1 + |r(z_0)|^2\right),$$

Then

- (i) Problem 2.1 has the unique solution

$$(2.6) \quad \delta(z) = \left(\prod_{k=1}^{N_1} \frac{z - \bar{z}_k}{z - z_k} \right) \left(\prod_{z_j \in \mathcal{B}_\ell} \frac{z - \bar{z}_j}{z - z_j} \right) \left(\prod_{z_j \in \mathcal{B}_\ell} \frac{z + z_j}{z + \bar{z}_j} \right) \left(\frac{z - z_0}{z + z_0} \right)^{i\kappa} e^{\chi(z)}$$

where κ is given by equation (2.5) and

$$(2.7) \quad \chi(z) = \frac{1}{2\pi i} \int_{-z_0}^{z_0} \log \left(\frac{1 + |r(\zeta)|^2}{1 + |r(z_0)|^2} \right) \frac{d\zeta}{\zeta - z}$$

$$\left(\frac{z - z_0}{z + z_0} \right)^{i\kappa} = \exp \left(i\kappa \left(\log \left| \frac{z - z_0}{z + z_0} \right| + i \arg(z - z_0) - i \arg(z + z_0) \right) \right).$$

Here we have chosen the branch of the logarithm with $-\pi < \arg(z) < \pi$.

- (ii) For $z \in \mathbb{C} \setminus [-z_0, z_0]$

$$\delta(z) = (\overline{\delta(\bar{z})})^{-1}$$

(iii) As $z \rightarrow \infty$,

$$\delta(z) = 1 + \frac{\delta_1}{z} + \mathcal{O}(z^{-2}).$$

(iv) Along any ray of the form $\pm z_0 + e^{i\phi}\mathbb{R}^+$ with $0 < \phi < \pi$ or $\pi < \phi < 2\pi$,

$$\left| \delta(z) - \left(\frac{z - z_0}{z + z_0} \right)^{i\kappa} \delta_0(\pm z_0) \right| \leq C_r |z \mp z_0|^{1/2}$$

where

$$\delta_0(\pm z_0) = \left(\prod_{k=1}^{N_1} \frac{\pm z_0 - \bar{z}_k}{\pm z_0 - z_k} \right) \left(\prod_{z_j \in B_\ell} \frac{\pm z_0 - \bar{z}_j}{\pm z_0 - z_j} \right) \left(\prod_{z_j \in B_\ell} \frac{\pm z_0 + z_j}{\pm z_0 + \bar{z}_j} \right) e^{\chi(\pm z_0)}$$

and the implied constant depends on r through its $H^1(\mathbb{R})$ -norm and is independent of $\pm z_0 \in \mathbb{R}$.

Proof. The proofs of (i)-(ii) can be found in [16]. For (iii), we use the fact that as $z \rightarrow \infty$

$$\begin{aligned} \frac{z - \bar{z}_k}{z - z_k} &= \frac{z - z_k + z_k - \bar{z}_k}{z - z_k} \\ &= 1 + \frac{2i \operatorname{Im}(z_k)}{z} + \mathcal{O}(z^{-2}) \end{aligned}$$

and

$$\exp\left(\frac{1}{2\pi i} \int_{-z_0}^{z_0} \frac{\log(1 + |r(\zeta)|^2)}{\zeta - z} d\zeta \right) = 1 - \frac{1}{2\pi i z} \int_{-z_0}^{z_0} \log(1 + |r(\zeta)|^2) d\zeta + \mathcal{O}(z^{-2}).$$

To establish (iv), we first note that

$$\left| \left(\frac{z - z_0}{z + z_0} \right)^{i\kappa} \right| \leq e^{\pi\kappa}.$$

To bound the difference $e^{\chi(z)} - e^{\chi(\pm z_0)}$, notice that

$$\begin{aligned} \left| e^{\chi(z)} - e^{\chi(\pm z_0)} \right| &\leq \left| e^{\chi(\pm z_0)} \right| \left| e^{\chi(z) - \chi(\pm z_0)} - 1 \right| \\ &\lesssim \left| \int_0^1 \frac{d}{ds} e^{s(\chi(z) - \chi(\pm z_0))} ds \right| \\ &\lesssim |z \mp z_0|^{1/2} \sup_{0 \leq s \leq 1} \left| e^{s(\chi(z) - \chi(\pm z_0))} \right| \\ &\lesssim |z \mp z_0|^{1/2} \end{aligned}$$

where the third inequality follows from [8, Lemma 23]. \square

It is straightforward to check that if $m(z; x, t)$ solves Problem 1.4, then the new matrix-valued function $m^{(1)}(z; x, t) = m(z; x, t)\delta(z)^{\sigma_3}$ is the solution to the following RHP.

Problem 2.3. Given

$$\mathcal{S} = \{r(z), \{z_k, c_k\}_{k=1}^{N_1}, \{z_j, c_j\}_{j=1}^{N_2}\} \subset H^1(\mathbb{R}) \oplus \mathbb{C}^{2N_1} \oplus \mathbb{C}^{2N_2}$$

and the augmented contour Σ in Figure 2.2 and set

$$\frac{x}{t} = 4\eta_\ell^2 - 12\xi_\ell^2$$

where $\{z_j\}_{j=1}^{N_2} \ni z_\ell = \xi_\ell + i\eta_\ell$, find a matrix-valued function $m^{(1)}(z; x, t)$ on $\mathbb{C} \setminus \Sigma$ with the following properties:

$$(1) \quad m^{(1)}(z; x, t) \rightarrow I \text{ as } |z| \rightarrow \infty,$$

- (2) $m^{(1)}(z; x, t)$ is analytic for $z \in \mathbb{C} \setminus \Sigma$ with continuous boundary values $m_{\pm}^{(1)}(z; x, t)$.
(3) On \mathbb{R} , the jump relation

$$m_{+}^{(1)}(z; x, t) = m_{-}^{(1)}(z; x, t)e^{-i\theta \operatorname{ad} \sigma_3 v^{(1)}(z)}$$

holds, where

$$v^{(1)}(z) = \delta_{-}(z)^{\sigma_3} v(z) \delta_{+}(z)^{-\sigma_3}.$$

The jump matrix $e^{-i\theta \operatorname{ad} \sigma_3 v^{(1)}}$ is factorized as

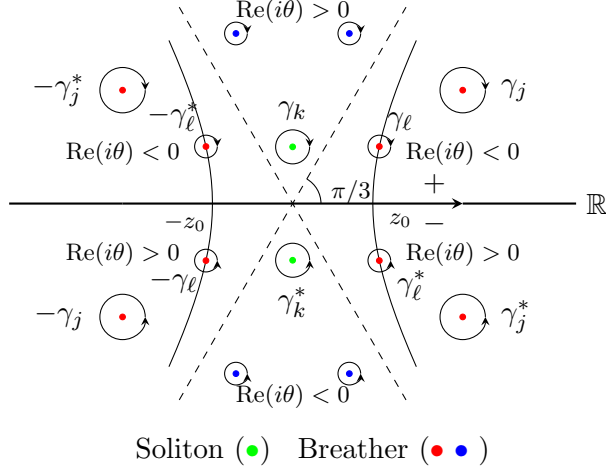
$$(2.8) \quad e^{-i\theta \operatorname{ad} \sigma_3 v^{(1)}}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{\delta_{-}^{-2} r}{1+|r|^2} e^{2i\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\delta_{+}^2 \bar{r}}{1+|r|^2} e^{-2i\theta} \\ 0 & 1 \end{pmatrix}, & z \in (-z_0, z_0), \\ \begin{pmatrix} 1 & \bar{r} \delta^2 e^{-2i\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r \delta^{-2} e^{2i\theta} & 1 \end{pmatrix}, & z \in (-\infty, -z_0) \cup (z_0, \infty). \end{cases}$$

- (4) On $(\bigcup_{k=1}^{N_1} \gamma_k) \cup (\bigcup_{j=1}^{N_2} \gamma_j)$, let $\delta(z)$ be the solution to Problem 2.1 we have the following jump conditions $m_{+}^{(1)}(z; x, t) = m_{-}^{(1)}(z; x, t)e^{-i\theta \operatorname{ad} \sigma_3 v^{(1)}(z)}$ where

$$e^{-i\theta \operatorname{ad} \sigma_3 v^{(1)}}(z) = \begin{cases} \begin{pmatrix} 1 & \frac{(1/\delta)'(z_k)^{-2}}{c_k(z-z_k)} e^{-2i\theta} \\ 0 & 1 \end{pmatrix} & z \in \gamma_k, \\ \begin{pmatrix} 1 & 0 \\ \frac{\delta'(\bar{z}_k)^{-2}}{c_k(z-\bar{z}_k)} e^{2i\theta} & 1 \end{pmatrix} & z \in \gamma_k^* \end{cases}$$

and for $z_j \in \mathcal{B}_\ell$

$$e^{-i\theta \operatorname{ad} \sigma_3 v^{(1)}}(z) = \begin{cases} \begin{pmatrix} 1 & \frac{(1/\delta)'(z_j)^{-2}}{c_j(z-z_j)} e^{-2i\theta} \\ 0 & 1 \end{pmatrix} & z \in \gamma_j, \\ \begin{pmatrix} 1 & 0 \\ \frac{\delta'(\bar{z}_j)^{-2}}{c_j(z-\bar{z}_j)} e^{2i\theta} & 1 \end{pmatrix} & z \in \gamma_j^* \\ \begin{pmatrix} 1 & 0 \\ -\frac{\delta'(-z_j)^{-2}}{c_j(z+z_j)} e^{2i\theta} & 1 \end{pmatrix} & z \in -\gamma_j, \\ \begin{pmatrix} 1 & -\frac{(1/\delta)'(-\bar{z}_j)^{-2}}{c_j(z+\bar{z}_j)} e^{-2i\theta} \\ 0 & 1 \end{pmatrix} & z \in -\gamma_j^* \end{cases}$$

FIGURE 2.2. The Augmented Contour Σ


and for $z_j \in \{z_j\}_{j=1}^{N_2} \setminus \mathcal{B}_\ell$

$$e^{-i\theta \text{ad}_{\sigma_3} v^{(1)}}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{c_j \delta(z_j)^{-2}}{z - z_j} e^{2i\theta} & 1 \end{pmatrix} & z \in \gamma_j, \\ \begin{pmatrix} 1 & \frac{\bar{c}_j \delta(\bar{z}_j)^2}{z - \bar{z}_j} e^{-2i\theta} \\ 0 & 1 \end{pmatrix} & z \in \gamma_j^*, \\ \begin{pmatrix} 1 & \frac{-c_j \delta(-z_j)^2}{z + z_j} e^{-2i\theta} \\ 0 & 1 \end{pmatrix} & z \in -\gamma_j, \\ \begin{pmatrix} 1 & 0 \\ \frac{-\bar{c}_j \delta(-\bar{z}_j)^{-2} e^{2i\theta}}{z + \bar{z}_j} & 1 \end{pmatrix} & z \in -\gamma_j^* \end{cases}$$

Remark 2.4. We set

$$(2.9) \quad \Gamma = \left(\bigcup_{k=1}^{N_1} \gamma_k \right) \cup \left(\bigcup_{k=1}^{N_1} \gamma_k^* \right) \cup \left(\bigcup_{j=1}^{N_2} \pm \gamma_j \right) \cup \left(\bigcup_{j=1}^{N_2} \pm \gamma_j^* \right).$$

From the signature table Figure 2 and the triangularities of the jump matrices, we observe that along the characteristic line $x = vt$ where $v = 4\eta_\ell^2 - 12\xi_\ell^2$, by choosing the radius of each element of Γ small enough, we have for $z \in \Gamma \setminus (\pm\gamma_\ell \cup \pm\gamma_\ell^*)$

$$e^{-i\theta \text{ad}_{\sigma_3} v^{(1)}}(z) \lesssim e^{-ct}, \quad t \rightarrow \infty.$$

For technical purpose which will become clear later, we want that the radius of each element of Γ less than $\lambda/3$ where λ is given by (2.3). Also we make each element of Γ is invariant under Schwarz reflection.

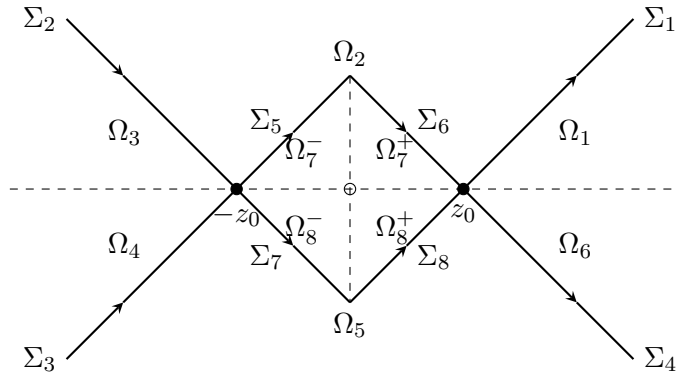
3. CONTOUR DEFORMATION

We now perform contour deformation on Problem 2.3, following the standard procedure outlined in [43, Section 4]. Since the phase function (1.32) has two critical points at $\pm z_0$, our new contour is chosen to be

$$(3.1) \quad \Sigma^{(2)} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4 \cup \Sigma_5 \cup \Sigma_6 \cup \Sigma_7 \cup \Sigma_8$$

shown in Figure 3.1 and consists of rays of the form $\pm z_0 + e^{i\phi} \mathbb{R}^+$ where $\phi = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

FIGURE 3.1. Deformation from \mathbb{R} to $\Sigma^{(2)}$



For technical reasons (see Remark 3.2), we define the following smooth cutoff function:

$$(3.2) \quad \Xi_{\mathcal{Z}}(z) = \begin{cases} 1 & \text{dist}(z, \mathcal{Z} \cup \mathcal{Z}^*) \leq \lambda/3 \\ 0 & \text{dist}(z, \mathcal{Z} \cup \mathcal{Z}^*) > 2\lambda/3. \end{cases}$$

Here recall that \mathcal{Z} is given by (2.2) and λ is defined in (2.3). We now introduce another matrix-valued function $m^{(2)}$:

$$m^{(2)}(z) = m^{(1)}(z) \mathcal{R}^{(2)}(z).$$

Here $\mathcal{R}^{(2)}$ will be chosen to remove the jump on the real axis and bring about new analytic jump matrices with the desired exponential decay along the contour $\Sigma^{(2)}$. Straight forward computation gives

$$\begin{aligned} m_+^{(2)} &= m_+^{(1)} \mathcal{R}_+^{(2)} \\ &= m_-^{(1)} \left(e^{-i\theta \text{ad } \sigma_3 v(1)} \right) \mathcal{R}_+^{(2)} \\ &= m_-^{(2)} \left(\mathcal{R}_-^{(2)} \right)^{-1} \left(e^{-i\theta \text{ad } \sigma_3 v(1)} \right) \mathcal{R}_+^{(2)}. \end{aligned}$$

We want to make sure that the following condition is satisfied

$$\left(\mathcal{R}_-^{(2)} \right)^{-1} \left(e^{-i\theta \text{ad } \sigma_3 v(1)} \right) \mathcal{R}_+^{(2)} = I$$

where $\mathcal{R}_{\pm}^{(2)}$ are the boundary values of $\mathcal{R}^{(2)}(z)$ as $\pm \text{Im}(z) \downarrow 0$. In this case the jump matrix associated to $m_{\pm}^{(2)}$ will be the identity matrix on \mathbb{R} .

From the signature table [16, Figure 0.1] we find that the function $e^{2i\theta}$ is exponentially decreasing on $\Sigma_3, \Sigma_4, \Sigma_5, \Sigma_6$ and increasing on $\Sigma_1, \Sigma_2, \Sigma_7, \Sigma_8$ while the reverse is true for $e^{-2i\theta}$.

Letting

$$(3.3) \quad \eta(z; \pm z_0) = \left(\prod_{k=1}^{N_1} \frac{\pm z_0 - \bar{z}_k}{\pm z_0 - z_k} \right) \left(\prod_{z_j \in B_\ell} \frac{\pm z_0 - \bar{z}_j}{\pm z_0 - z_j} \right) \left(\prod_{z_j \in B_\ell} \frac{\pm z_0 + z_j}{\pm z_0 + \bar{z}_j} \right) \left(\frac{z - z_0}{z + z_0} \right)^{i\kappa}$$

$$(3.4) \quad \eta_0(\pm z_0) = \left(\prod_{k=1}^{N_1} \frac{\pm z_0 - \bar{z}_k}{\pm z_0 - z_k} \right) \left(\prod_{z_j \in B_\ell} \frac{\pm z_0 - \bar{z}_j}{\pm z_0 - z_j} \right) \left(\prod_{z_j \in B_\ell} \frac{\pm z_0 + z_j}{\pm z_0 + \bar{z}_j} \right)$$

and we define $\mathcal{R}^{(2)}$ as follows (Figure 3.2-3.3): the functions $R_1, R_3, R_4, R_6, R_7^+, R_8^+, R_7^-, R_8^-$ satisfy

$$(3.5) \quad R_1(z) = \begin{cases} -r(z)\delta(z)^{-2} & z \in (z_0, \infty) \\ -r(z_0)e^{-2\chi(z_0)}\eta(z; z_0)^{-2}(1 - \Xi_{\mathcal{Z}}) & z \in \Sigma_1, \end{cases}$$

$$(3.6) \quad R_3(z) = \begin{cases} -r(z)\delta(z)^{-2} & z \in (-\infty, -z_0) \\ -r(-z_0)e^{-2\chi(-z_0)}\eta(z; -z_0)^{-2}(1 - \Xi_{\mathcal{Z}}) & z \in \Sigma_2, \end{cases}$$

$$(3.7) \quad R_4(z) = \begin{cases} \overline{r(z)}\delta(z)^2 & z \in (-\infty, -z_0) \\ \overline{r(-z_0)}e^{2\chi(-z_0)}\eta(z; z_0)^2(1 - \Xi_{\mathcal{Z}}) & z \in \Sigma_3, \end{cases}$$

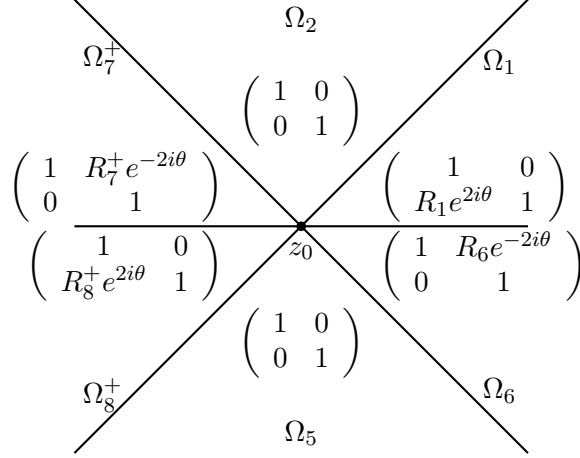
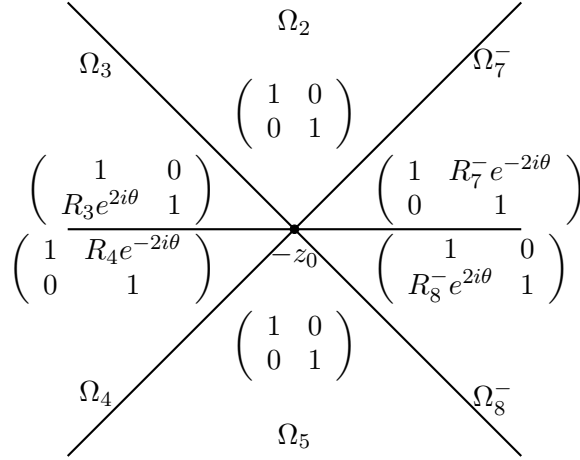
$$(3.8) \quad R_6(z) = \begin{cases} \overline{r(z)}\delta(z)^2 & z \in (z_0, \infty) \\ \overline{r(z_0)}e^{2\chi(z_0)}\eta(z; z_0)^2(1 - \Xi_{\mathcal{Z}}) & z \in \Sigma_4, \end{cases}$$

$$(3.9) \quad R_7^+(z) = \begin{cases} -\frac{\delta_+^2(z)\overline{r(z)}}{1 + |r(z)|^2} & z \in (-z_0, z_0) \\ -\frac{e^{2\chi(z_0)}\eta(z; z_0)^2\overline{r(z_0)}}{1 + |r(z_0)|^2}(1 - \Xi_{\mathcal{Z}}) & z \in \Sigma_6, \end{cases}$$

$$(3.10) \quad R_8^+(z) = \begin{cases} \frac{\delta_-^{-2}(z)r(z)}{1 + |r(z)|^2} & z \in (-z_0, z_0) \\ \frac{e^{-2\chi(z_0)}\eta(z; z_0)^{-2}r(z_0)}{1 + |r(z_0)|^2}(1 - \Xi_{\mathcal{Z}}) & z \in \Sigma_8, \end{cases}$$

$$(3.11) \quad R_7^-(z) = \begin{cases} -\frac{\delta_+^2(z)\overline{r(z)}}{1 + |r(z)|^2} & z \in (-z_0, z_0) \\ -\frac{e^{2\chi(-z_0)}\eta(z; z_0)^2\overline{r(-z_0)}}{1 + |r(-z_0)|^2}(1 - \Xi_{\mathcal{Z}}) & z \in \Sigma_5 \end{cases}$$

$$(3.12) \quad R_8^-(z) = \begin{cases} \frac{\delta_-^{-2}(z)r(z)}{1 + |r(z)|^2} & z \in (-z_0, z_0) \\ \frac{e^{-2\chi(-z_0)}\eta(z; z_0)^{-2}r(-z_0)}{1 + |r(-z_0)|^2}(1 - \Xi_{\mathcal{Z}}) & z \in \Sigma_7. \end{cases}$$

FIGURE 3.2. The Matrix $\mathcal{R}^{(2)}$ for Region I, near z_0 FIGURE 3.3. The Matrix $\mathcal{R}^{(2)}$ for Region I, near $-z_0$ 

Each $R_i(z)$ in Ω_i is constructed in such a way that the jump matrices on the contour and $\bar{\partial}R_i(z)$ enjoys the property of exponential decay as $t \rightarrow \infty$. We formulate Problem 2.3 into a mixed RHP- $\bar{\partial}$ problem. In the following sections we will separate this mixed problem into a localized RHP and a pure $\bar{\partial}$ problem whose long-time contribution to the asymptotics of $u(x, t)$ is of higher order than the leading term.

The following lemma ([19, Proposition 2.1]) will be used in the error estimates of $\bar{\partial}$ -problem in Section 5.

We first denote the entries that appear in (3.5)–(3.12) by

$$\begin{aligned}
 p_1(z) &= p_3(z) = r(z), & p_4(z) &= p_6(z) = -\overline{r(z)}, \\
 p_{7-}(z) &= p_{7+}(z) = -\frac{\overline{r(z)}}{1 + |r(z)|^2}, & p_{8-}(z) &= p_{8+}(z) = \frac{r(z)}{1 + |r(z)|^2}.
 \end{aligned}$$

Lemma 3.1. *Suppose $r \in H^1(\mathbb{R})$. There exist functions R_i on Ω_i , $i = 1, 3, 4, 6, 7^\pm, 8^\pm$ satisfying (3.5)–(3.12), so that*

$$|\bar{\partial}R_i(z)| \lesssim |p'_i(\operatorname{Re}(z))| + |z - \xi|^{-1/2} + \bar{\partial}(\Xi_{\mathcal{Z}}(z)), \quad z \in \Omega_i$$

where $\xi = \pm z_0$ and the implied constants are uniform for r in a bounded subset of $H^1(\mathbb{R})$.

Proof. We only prove the lemma for R_1 . Define $f_1(z)$ on Ω_1 by

$$f_1(z) = p_1(z_0)e^{-2\chi(z_0)}\eta(z; z_0)^{-2}\delta(z)^2$$

and let

$$(3.13) \quad R_1(z) = (f_1(z) + [p_1(\operatorname{Re}(z)) - f_1(z)]\mathcal{K}(\phi))\delta(z)^{-2}(1 - \Xi_{\mathcal{Z}})$$

where $\phi = \arg(z - \xi)$ and \mathcal{K} is a smooth function on $(0, \pi/4)$ with

$$\mathcal{K}(\phi) = \begin{cases} 1 & z \in [0, \pi/12], \\ 0 & z \in [\pi/6, \pi/4] \end{cases}$$

It is easy to see that R_1 as constructed has the boundary values (3.5). Writing $z - z_0 = \rho e^{i\phi}$, we have

$$\bar{\partial} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) = \frac{1}{2} e^{i\phi} \left(\frac{\partial}{\partial \rho} + \frac{i}{\rho} \frac{\partial}{\partial \phi} \right).$$

We calculate

$$\begin{aligned} \bar{\partial}R_1(z) &= \left(\frac{1}{2} p'_1(\operatorname{Re} z) \mathcal{K}(\phi) \delta(z)^{-2} - [p_1(\operatorname{Re} z) - f_1(z)] \delta(z)^{-2} \frac{ie^{i\phi}}{|z - \xi|} \mathcal{K}'(\phi) \right) \\ &\quad \times (1 - \Xi_{\mathcal{Z}}) - (f_1(z) + [p_1(\operatorname{Re}(z)) - f_1(z)]\mathcal{K}(\phi))\delta(z)^{-2}\bar{\partial}(\Xi_{\mathcal{Z}}(z)). \end{aligned}$$

Given that $\Xi(z)$ is infinitely smooth and compactly supported, it follows from Lemma 2.2 (iv) that

$$|(\bar{\partial}R_1)(z)| \lesssim |p'_1(\operatorname{Re} z)| + |z - \xi|^{-1/2} + \bar{\partial}(\Xi_{\mathcal{Z}}(z))$$

where the implied constants depend on $\|r\|_{H^1}$ and the smooth function \mathcal{K} . The estimates in the remaining sectors are identical. \square

The unknown $m^{(2)}$ satisfies a mixed $\bar{\partial}$ -RHP. We first identify the jumps of $m^{(2)}$ along the contour $\Sigma^{(2)}$. Recall that $m^{(1)}$ is analytic along the contour, the jumps are determined entirely by $\mathcal{R}^{(2)}$, see (3.5)–(3.12). Away from $\Sigma^{(2)}$, using the triangularity of $\mathcal{R}^{(2)}$, we have that

$$(3.14) \quad \bar{\partial}m^{(2)} = m^{(2)} \left(\mathcal{R}^{(2)} \right)^{-1} \bar{\partial}\mathcal{R}^{(2)} = m^{(2)} \bar{\partial}\mathcal{R}^{(2)}.$$

Remark 3.2. By construction of $\mathcal{R}^{(2)}$ (see (3.5)–(3.12) and (3.13)) and the choice of the radius of the circles in the set Γ (see Remark 2.4), the right multiplication of $\mathcal{R}^{(2)}$ to $m^{(1)}$ will not change the jump conditions on circles in the set Γ . Thus over circles in the set Γ , $m^{(2)}$ has the same jump matrices as given by (4) of Problem 2.3.

Problem 3.3. Given $r \in H^1(\mathbb{R})$, find a matrix-valued function $m^{(2)}(z; x, t)$ on $\mathbb{C} \setminus \mathbb{R}$ with the following properties:

- (1) $m^{(2)}(z; x, t) \rightarrow I$ as $z \rightarrow \infty$ in $\mathbb{C} \setminus (\Sigma^{(2)} \cup \Gamma)$,
- (2) $m^{(2)}(z; x, t)$ is continuous for $z \in \mathbb{C} \setminus (\Sigma^{(2)} \cup \Gamma)$ with continuous boundary values $m_{\pm}^{(2)}(z; x, t)$ (where \pm is defined by the orientation in Figure 4.1)
- (3) The jump relation $m_{+}^{(2)}(z; x, t) = m_{-}^{(2)}(z; x, t)e^{-i\theta \operatorname{ad} \sigma_3 v^{(2)}(z)}$ holds, where $e^{-i\theta \operatorname{ad} \sigma_3 v^{(2)}(z)}$ is given in Figure 3.4–3.5 and part (4) of Problem 2.3.

(4) The equation

$$\bar{\partial}m^{(2)} = m^{(2)}\bar{\partial}\mathcal{R}^{(2)}$$

holds in $\mathbb{C} \setminus \Sigma^{(2)}$, where

$$\bar{\partial}\mathcal{R}^{(2)} = \begin{cases} \begin{pmatrix} 0 & 0 \\ (\bar{\partial}R_1)e^{2i\theta} & 0 \end{pmatrix}, & z \in \Omega_1 & \begin{pmatrix} 0 & (\bar{\partial}R_7^+)e^{-2i\theta} \\ 0 & 0 \end{pmatrix}, & z \in \Omega_7^+ \\ \begin{pmatrix} 0 & 0 \\ (\bar{\partial}R_8^+)e^{2i\theta} & 0 \end{pmatrix}, & z \in \Omega_8^+ & \begin{pmatrix} 0 & (\bar{\partial}R_6)e^{-2i\theta} \\ 0 & 0 \end{pmatrix}, & z \in \Omega_6 \\ \begin{pmatrix} 0 & 0 \\ (\bar{\partial}R_3)e^{2i\theta} & 0 \end{pmatrix}, & z \in \Omega_3 & \begin{pmatrix} 0 & (\bar{\partial}R_4)e^{-2i\theta} \\ 0 & 0 \end{pmatrix}, & z \in \Omega_4 \\ \begin{pmatrix} 0 & 0 \\ (\bar{\partial}R_8^-)e^{2i\theta} & 0 \end{pmatrix}, & z \in \Omega_8^- & \begin{pmatrix} 0 & (\bar{\partial}R_7^-)e^{-2i\theta} \\ 0 & 0 \end{pmatrix}, & z \in \Omega_7^- \\ 0 & z \in \Omega_2 \cup \Omega_5 \end{cases}$$

The following picture is an illustration of the jump matrices of RHP Problem 3.3. For brevity we ignore the discrete scattering data.

FIGURE 3.4. Jump Matrices $v^{(2)}$ for $m^{(2)}$ near z_0

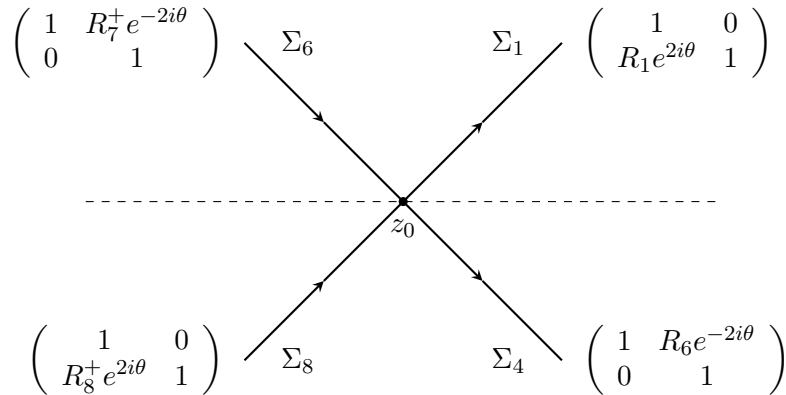
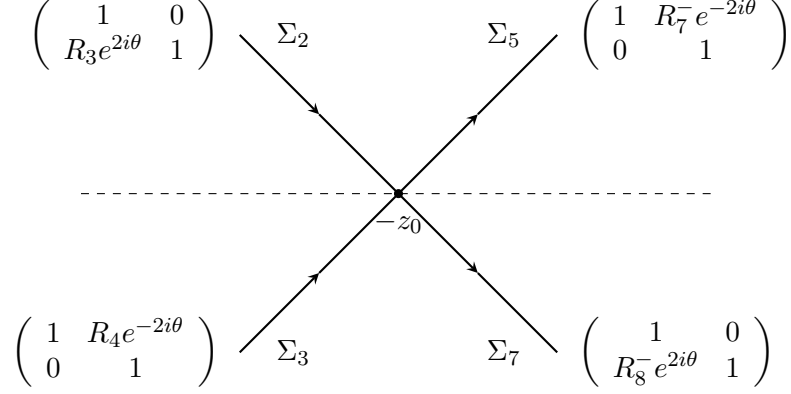


FIGURE 3.5. Jump Matrices $v^{(2)}$ for $m^{(2)}$ near $-z_0$ 

4. THE LOCALIZED RIEMANN-HILBERT PROBLEM

We perform the following factorization of $m^{(2)}$:

$$(4.1) \quad m^{(2)} = m^{(3)} m^{\text{LC}}.$$

Here we require that $m^{(3)}$ to be the solution of the pure $\bar{\partial}$ -problem, hence no jump, and m^{LC} solution of the localized RHP Problem 4.1 below with the jump matrix $v^{\text{LC}} = v^{(3)}$. The current section focuses on finding m^{LC} .

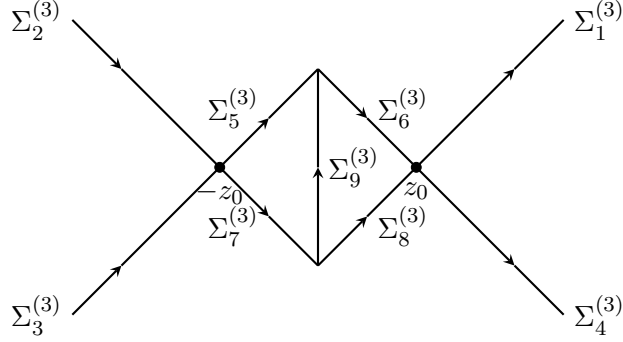
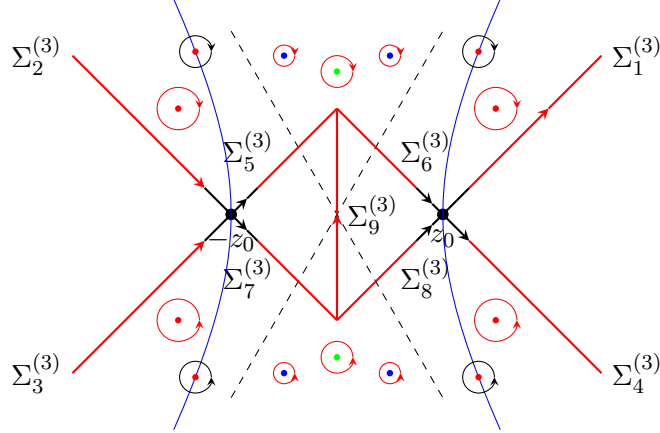
Problem 4.1. Find a 2×2 matrix-valued function $m^{\text{LC}}(z; x, t)$, analytic on $\mathbb{C} \setminus \Sigma^{(3)}$, with the following properties:

- (1) $m^{\text{LC}}(z; x, t) \rightarrow I$ as $|z| \rightarrow \infty$ in $\mathbb{C} \setminus (\Sigma^{(3)} \cup \Gamma)$, where I is the 2×2 identity matrix,
- (2) $m^{\text{LC}}(z; x, t)$ is analytic for $z \in \mathbb{C} \setminus (\Sigma^{(3)} \cup \Gamma)$ with continuous boundary values m_{\pm}^{LC} on $\Sigma^{(3)} \cup \Gamma$,
- (3) The jump relation $m_{+}^{\text{LC}}(z; x, t) = m_{-}^{\text{LC}}(z; x, t) v^{\text{LC}}(z)$ holds on $\Sigma^{(3)} \cup \Gamma$, where

$$v^{\text{LC}}(z) = v^{(3)}(z).$$

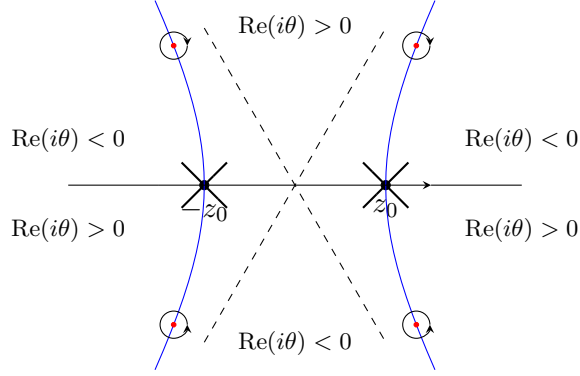
Remark 4.2. Comparing the jump condition on $\Sigma^{(2)}$ and $\Sigma^{(3)}$, we note that the interpolation defined through (3.13) introduce new jump on $\Sigma_9^{(3)}$ with jump matrix given by

$$(4.2) \quad v_9 = \begin{cases} I, & z \in (-iz_0 \tan(\pi/12), iz_0 \tan(\pi/12)) \\ \begin{pmatrix} 1 & (R_7^- - R_7^+) e^{-2i\theta} \\ 0 & 1 \end{pmatrix}, & z \in (iz_0 \tan(\pi/12), iz_0) \\ \begin{pmatrix} 1 & 0 \\ (R_8^- - R_8^+) e^{2i\theta} & 1 \end{pmatrix}, & z \in (-iz_0, -iz_0 \tan(\pi/12)). \end{cases}$$

FIGURE 4.1. $\Sigma^{(3)}$ FIGURE 4.2. $\Sigma^{(3)} \cup \Gamma$ 

For some fixed $\varepsilon > 0$, we define

$$\begin{aligned}
 L_\varepsilon &= \{z : z = z_0 + uz_0 e^{3i\pi/4}, \varepsilon \leq u \leq \sqrt{2}\} \\
 &\cup \{z : z = z_0 + uz_0 e^{i\pi/4}, \varepsilon \leq u \leq +\infty\} \\
 &\cup \{z : z = -z_0 + uz_0 e^{i\pi/4}, \varepsilon \leq u \leq \sqrt{2}\} \\
 &\cup \{z : z = -z_0 + uz_0 e^{3i\pi/4}, \varepsilon \leq u \leq +\infty\} \\
 \Sigma' &= \left(\Sigma^{(3)} \setminus (L_\varepsilon \cup L_\varepsilon^* \cup \Sigma_9^{(3)}) \right) \cup (\pm\gamma_\ell) \cup (\pm\gamma_\ell^*).
 \end{aligned}$$

FIGURE 4.3. Σ' 

Here Σ' is the black portion of the contour $\Sigma^{(3)} \cup \Gamma$ given in Figure 4.2. Now we decompose $w_\theta^{(3)} = v_\theta^{(3)} - I$ into two parts:

$$(4.3) \quad w_\theta^{(3)} = w^e + w'$$

where $w' = w_\theta^{(3)} \upharpoonright_{\Sigma'}$ and $w^e = w_\theta^{(3)} \upharpoonright_{(\Sigma^{(3)} \cup \Gamma) \setminus \Sigma'}$.

Near $\pm z_0$, we write

$$i\theta(z; x, t) = 4it \left((z \mp z_0)^3 \pm 3z_0(z \mp z_0)^2 \pm 2z_0^3 \right)$$

and on L_ε , away from $\pm z_0$, we estimate:

$$(4.4) \quad \left| R_1 e^{2i\theta} \right| \leq C_r e^{-24tz_0^3 u^2} \leq C_r e^{-24\varepsilon^2 \tau},$$

$$(4.5) \quad \left| R_3 e^{2i\theta} \right| \leq C_r e^{-24tz_0^3 u^2} \leq C_r e^{-24\varepsilon^2 \tau},$$

$$(4.6) \quad \left| R_7^\pm e^{-2i\theta} \right| \leq C_r e^{-16tz_0^3 u^2} \leq C_r e^{-16\varepsilon^2 \tau}$$

where the constant C_r depends on the H^1 norm of r . Similarly, on L_ε^*

$$(4.7) \quad \left| R_4 e^{-2i\theta} \right| \leq C_r e^{-24tz_0^3 u^2} \leq C_r e^{-24\varepsilon^2 \tau},$$

$$(4.8) \quad \left| R_6 e^{-2i\theta} \right| \leq C_r e^{-24tz_0^3 u^2} \leq C_r e^{-24\varepsilon^2 \tau},$$

$$(4.9) \quad \left| R_8^\pm e^{2i\theta} \right| \leq C_r e^{-16tz_0^3 u^2} \leq C_r e^{-16\varepsilon^2 \tau}.$$

Also notice that On $\Sigma_9^{(3)}$, by the construction of $\mathcal{K}(\phi)$ and v_9 , one obtains

$$(4.10) \quad |v_9 - I| \lesssim e^{-ct}.$$

Combining Remark 2.4 with the discussion above we conclude that

$$(4.11) \quad |w^e| \lesssim e^{-ct}$$

Proposition 4.3. *There exists a 2×2 matrix $E_1(x, t; z)$ with*

$$E_1(x, t; z) = I + \mathcal{O}\left(\frac{e^{-ct}}{z}\right),$$

such that

$$(4.12) \quad m^{\text{LC}}(x, t; z) = E_1(x, t; z) m_*^{\text{LC}}(x, t; z)$$

where $m_*^{\text{LC}}(x, t; z)$ solves the RHP with jump contour Σ' given in Figure 4.3 and jump matrices

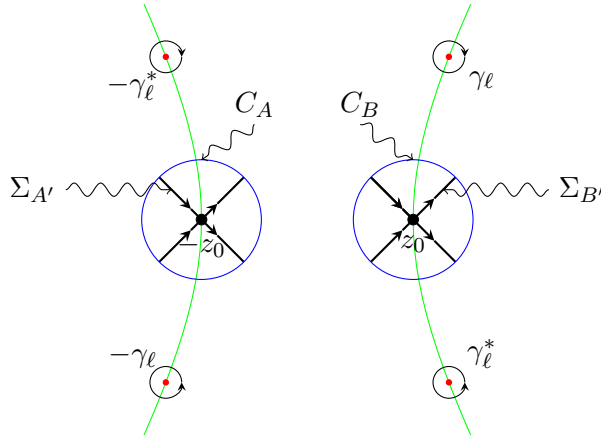
$$v' = I + w'.$$

Proof. We will later show the existence of $m_*^{\text{LC}}(x, t; z)$ and $\|m_*^{\text{LC}}(x, t; z)\|_{L^\infty}$ is finite. Assuming this, it is easy to see that on $(\Sigma^{(3)} \cup \Gamma) \setminus \Sigma'$, E_1 satisfies the following jump condition:

$$E_{1+} = E_{1-} \left(m_*^{\text{LC}}(1 + w^e) (m_*^{\text{LC}})^{-1} \right).$$

Using (4.11) the conclusion follows from solving a small norm Riemann-Hilbert problem (see the solution to Problem 4.11 for detail). \square

FIGURE 4.4. $\Sigma' = \Sigma_{A'} \cup \Sigma_{B'} \cup \pm\gamma_\ell \cup \gamma_\ell^*$



4.1. Construction of parametrix. In this subsection we construct m_*^{LC} needed in the proof of Proposition 4.3. To achieve this, we need the solutions of the following three exactly solvable RHPs:

Problem 4.4. Find a matrix-valued function $m^{(br)}(z; x, t)$ on $\mathbb{C} \setminus \Sigma$ with the following properties:

- (1) $m^{(br)}(z; x, t) \rightarrow I$ as $|z| \rightarrow \infty$,
- (2) $m^{(br)}(z; x, t)$ is analytic for $z \in \mathbb{C} \setminus (\pm\gamma_\ell \cup \pm\gamma_\ell^*)$ with continuous boundary values $m_\pm^{(br)}(z; x, t)$.

- (3) On $\pm\gamma_\ell \cup \pm\gamma_\ell^*$, let $\delta(z)$ be the solution to Problem 2.1 and we have the following jump conditions $m_+^{(br)}(z; x, t) = m_-^{(br)}(z; x, t)e^{-i\theta \text{ad } \sigma_3 v^{(br)}(z)}$ where

$$e^{-i\theta \text{ad } \sigma_3 v^{(br)}(z)} = \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{c_\ell \delta(z_\ell)^{-2}}{z - z_\ell} e^{2i\theta} & 1 \end{pmatrix} & z \in \gamma_\ell, \\ \begin{pmatrix} 1 & \frac{\bar{c}_\ell \delta(\bar{z}_\ell)^2}{z - \bar{z}_\ell} e^{-2i\theta} \\ 0 & 1 \end{pmatrix} & z \in \gamma_\ell^* \\ \begin{pmatrix} 1 & \frac{-c_\ell \delta(-z_\ell)^2}{z + z_\ell} e^{-2i\theta} \\ 0 & 1 \end{pmatrix} & z \in -\gamma_\ell, \\ \begin{pmatrix} 1 & 0 \\ \frac{-\bar{c}_\ell \delta(-\bar{z}_\ell)^{-2} e^{2i\theta}}{z + \bar{z}_\ell} & 1 \end{pmatrix} & z \in -\gamma_\ell^*. \end{cases}$$

Problem 4.5. Find a matrix-valued function $m^{A'}(z; x, t)$ on $\mathbb{C} \setminus \Sigma'_A$ with the following properties:

- (1) $m^{A'}(z; x, t) \rightarrow I$ as $z \rightarrow \infty$.
- (2) $m^{A'}(z; x, t)$ is analytic for $z \in \mathbb{C} \setminus \Sigma'_A$ with continuous boundary values $m_\pm^{A'}(z; x, t)$.
- (3) On Σ'_A we have the following jump conditions

$$m_+^{A'}(z; x, t) = m_-^{A'}(z; x, t)e^{-i\theta \text{ad } \sigma_3 v^{A'}(z)}$$

where $v^{A'} = v^{(2)} \upharpoonright_{\Sigma'_A}$.

Problem 4.6. Find a matrix-valued function $m^{B'}(z; x, t)$ on $\mathbb{C} \setminus \Sigma'_B$ with the following properties:

- (1) $m^{B'}(z; x, t) \rightarrow I$ as $z \rightarrow \infty$.
- (2) $m^{B'}(z; x, t)$ is analytic for $z \in \mathbb{C} \setminus \Sigma'_B$ with continuous boundary values $m_\pm^{B'}(z; x, t)$.
- (3) On Σ'_B we have the following jump conditions

$$m_+^{B'}(z; x, t) = m_-^{B'}(z; x, t)e^{-i\theta \text{ad } \sigma_3 v^{B'}(z)}$$

where $v^B = v^{(2)} \upharpoonright_{\Sigma'_B}$.

We first study the solution to Problem 4.4. Since this problem consists of only discrete data, (1.40) reduces to a linear system. More explicitly, we have a closed system:

(4.13)

$$\begin{pmatrix} \mu_{11}(\bar{z}_l) & \mu_{12}(z_l) \\ \mu_{21}(\bar{z}_l) & \mu_{22}(z_l) \end{pmatrix} = I + \begin{pmatrix} \frac{\mu_{12}(z_l)c_l\delta(z_\ell)^{-2}e^{2i\theta(z_l)}}{\bar{z}_l - z_l} & -\frac{\mu_{11}(\bar{z}_l)\bar{c}_l\delta(\bar{z}_\ell)^2e^{-2i\theta(\bar{z}_l)}}{z_l - \bar{z}_l} \\ \frac{\mu_{22}(z_l)c_l\delta(z_\ell)^{-2}e^{2i\theta(z_l)}}{\bar{z}_l - z_l} & -\frac{\mu_{21}(\bar{z}_l)\bar{c}_l\delta(\bar{z}_\ell)^2e^{-2i\theta(\bar{z}_l)}}{z_l - \bar{z}_l} \end{pmatrix} + \begin{pmatrix} -\frac{\mu_{12}(-\bar{z}_l)\bar{c}_l\delta(-\bar{z}_\ell)^{-2}e^{2i\theta(-\bar{z}_l)}}{\bar{z}_l + \bar{z}_l} & \frac{\mu_{11}(-z_l)c_l\delta(-z_\ell)^2e^{-2i\theta(-z_l)}}{z_l + z_l} \\ -\frac{\mu_{22}(-\bar{z}_l)\bar{c}_l\delta(-\bar{z}_\ell)^{-2}e^{2i\theta(-\bar{z}_l)}}{\bar{z}_l + \bar{z}_l} & \frac{\mu_{21}(-z_l)c_l\delta(-z_\ell)^2e^{-2i\theta(-z_l)}}{z_l + z_l} \end{pmatrix},$$

(4.14)

$$\begin{pmatrix} \mu_{11}(-z_l) & \mu_{12}(-\bar{z}_l) \\ \mu_{21}(-z_l) & \mu_{22}(-\bar{z}_l) \end{pmatrix} = I + \begin{pmatrix} \frac{\mu_{12}(z_l)c_l\delta(z_l)^{-2}e^{2i\theta(z_l)}}{-z_l - z_l} & -\frac{\mu_{11}(\bar{z}_l)\bar{c}_l\delta(\bar{z}_l)^2e^{-2i\theta(\bar{z}_l)}}{-\bar{z}_l - \bar{z}_l} \\ \frac{\mu_{22}(z_l)c_l\delta(z_l)^{-2}e^{2i\theta(z_l)}}{-z_l - z_l} & -\frac{\mu_{21}(\bar{z}_l)\bar{c}_l\delta(\bar{z}_l)^2e^{-2i\theta(\bar{z}_l)}}{-\bar{z}_l - \bar{z}_l} \end{pmatrix} \\ + \begin{pmatrix} -\frac{\mu_{12}(-\bar{z}_l)\bar{c}_l\delta(-\bar{z}_l)^{-2}e^{2i\theta(-\bar{z}_l)}}{-z_l + \bar{z}_l} & \frac{\mu_{11}(-z_l)c_l\delta(-z_l)^2e^{-2i\theta(-z_l)}}{-\bar{z}_l + z_l} \\ -\frac{\mu_{22}(-\bar{z}_l)\bar{c}_l\delta(-\bar{z}_l)^{-2}e^{2i\theta(-\bar{z}_l)}}{-z_l + \bar{z}_l} & \frac{\mu_{21}(-z_l)c_l\delta(-z_l)^2e^{-2i\theta(-z_l)}}{-\bar{z}_l + z_l} \end{pmatrix}.$$

Given that

$$\delta(z) = \left(\delta(\bar{z})\right)^{-1},$$

the Schwarz invariant condition of the jump matrices $e^{-i\theta \text{ad } \sigma_3 v^{(br)}}(z)$ is satisfied and the solvability of this linear system (4.13)-(4.14) follows. Moreover, we find the single breather solution:

$$(4.15) \quad \begin{aligned} u^{(br)}(x, t) &= 2z \lim_{z \rightarrow \infty} m_{12}^{(br)} \\ &= -4 \frac{\eta_\ell \xi_\ell \cosh(\nu_2 + \tilde{\omega}_2) \sin(\nu_1 + \tilde{\omega}_1) + (\eta_\ell) \sinh(\nu_2 + \tilde{\omega}_2) \cos(\nu_1 + \tilde{\omega}_1)}{\xi_\ell \cosh^2(\nu_2 + \tilde{\omega}_2) + (\eta_\ell/\xi_\ell)^2 \cos^2(\nu_1 + \tilde{\omega}_1)} \end{aligned}$$

with

$$\begin{aligned} \nu_1 &= 2\xi_\ell(x + 4(\xi_\ell^2 - 3\eta_\ell^2)t) \\ \nu_2 &= 2\eta_\ell(x - 4(\eta_\ell^2 - 3\xi_\ell^2)t). \end{aligned}$$

And

$$(4.16) \quad \tan \tilde{\omega}_1 = \frac{\tilde{B}\xi_\ell - \tilde{A}\eta_\ell}{\tilde{A}\xi_\ell + \tilde{B}\eta_\ell}$$

$$(4.17) \quad e^{-\tilde{\omega}_2} = \left| \frac{\xi_\ell}{2\eta_\ell} \right| \sqrt{\frac{\tilde{A}^2 + \tilde{B}^2}{\xi_\ell^2 + \eta_\ell^2}}$$

where we set

$$\tilde{c}_\ell = c_\ell \delta(z_\ell)^{-2} = \tilde{A} + i\tilde{B}.$$

We then study the solution to Problem 4.5 and Problem 4.6. Extend the contours $\Sigma_{A'}$ and $\Sigma_{B'}$ to

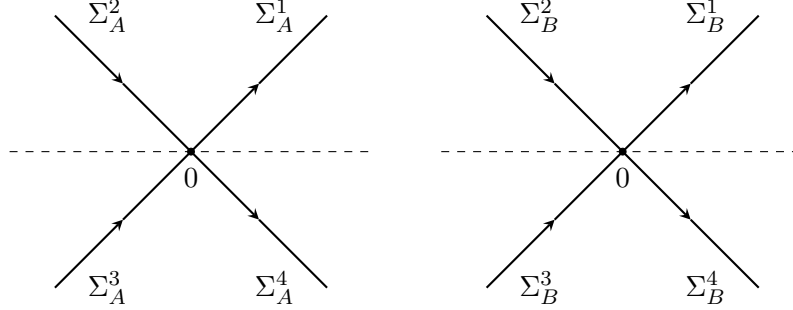
$$(4.18a) \quad \widehat{\Sigma}_{A'} = \{z = -z_0 + z_0 u e^{\pm i\pi/4} : -\infty < u < \infty\},$$

$$(4.18b) \quad \widehat{\Sigma}_{B'} = \{z = z_0 + z_0 u e^{\pm i3\pi/4} : -\infty < u < \infty\}$$

respectively and define $\hat{v}^{A'}$, $\hat{v}^{B'}$ on $\widehat{\Sigma}_{A'}$, $\widehat{\Sigma}_{B'}$ through

$$(4.19a) \quad \hat{v}^A = \begin{cases} v^{A'}(z), & z \in \Sigma_{A'} \subset \widehat{\Sigma}_{A'}, \\ 0, & z \in \widehat{\Sigma}_{A'} \setminus \Sigma_{A'}, \end{cases}$$

$$(4.19b) \quad \hat{v}^{B'} = \begin{cases} v^{B'}(z), & z \in \Sigma_{B'} \subset \widehat{\Sigma}_{B'} \\ 0, & z \in \widehat{\Sigma}_{B'} \setminus \Sigma_{B'}. \end{cases}$$

FIGURE 4.5. Σ_A, Σ_B 

Let Σ_A and Σ_B denote the contours

$$\{z = z_0 u e^{\pm i\pi/4} : -\infty < u < \infty\}$$

with the same orientation as those of $\Sigma_{A'}$ and $\Sigma_{B'}$ respectively. On $\widehat{\Sigma}_{A'}$ ($\widehat{\Sigma}_{B'}$) we carry out the following change of variable

$$z \mapsto \zeta = \sqrt{48z_0 t}(z \pm z_0)$$

and introduce the scaling operators

$$(4.20a) \quad \begin{cases} N_A : L^2(\widehat{\Sigma}_{A'}) \rightarrow L^2(\Sigma_A) \\ f(z) \mapsto (N_A f)(z) = f\left(\frac{\zeta}{\sqrt{48z_0 t}} - z_0\right), \end{cases}$$

$$(4.20b) \quad \begin{cases} N_B : L^2(\widehat{\Sigma}_{B'}) \rightarrow L^2(\Sigma_B) \\ f(z) \mapsto (N_B f)(z) = f\left(\frac{\zeta}{\sqrt{48z_0 t}} + z_0\right). \end{cases}$$

We also define

$$(4.21) \quad 1_A = 1 \upharpoonright_{\Sigma_A}, \quad 1_B = 1 \upharpoonright_{\Sigma_B}$$

We first consider the case Σ_B . The rescaling gives

$$N_B \left(e^{\chi(z_0)} \eta(z; z_0) e^{-it\theta} \right) = \delta_B^0 \delta_B^1(\zeta)$$

with

$$\begin{aligned} \delta_B^0 &= (192\tau)^{-i\kappa/2} e^{8i\tau} e^{\chi(z_0)} \eta_0(z_0) \\ \delta_B^1(\zeta) &= \zeta^{i\kappa} \left(\frac{2z_0}{\zeta/\sqrt{48t z_0} + 2z_0} \right)^{i\kappa} e^{(-i\zeta^2/4)(1+\zeta(432\tau)^{-1/2})}. \end{aligned}$$

Note that $\delta_B^0(z)$ is independent of z and that $|\delta_B^0(z)| = 1$. Set

$$\begin{aligned} \Delta_B^0 &= (\delta_B^0)^{\sigma_3} \\ w^B(\zeta) &= (\Delta_B^0)^{-1} (N_B \hat{w}^{B'}) \Delta_B^0 \end{aligned}$$

and define the operator $B : L^2(\Sigma_B) \rightarrow L^2(\Sigma_B)$

$$\begin{aligned} B &= C_{w^B} \\ &= C^+ \left(\cdot (\Delta_B^0)^{-1} (N_B \hat{w}_-^{B'}) \Delta_B^0 \right) + C^- \left(\cdot (\Delta_B^0)^{-1} (N_B \hat{w}_+^{B'}) \Delta_B^0 \right). \end{aligned}$$

On

$$\begin{aligned} L_B \cup \bar{L}_B &= \{z = uz_0\sqrt{48tz_0}e^{i\pi/4} : -\varepsilon < u < \varepsilon\} \\ &\cup \{z = uz_0\sqrt{48tz_0}e^{-i\pi/4} : -\varepsilon < u < \varepsilon\} \end{aligned}$$

From the list of entries stated in (3.5), (3.7), (3.9) and (3.10), we have

$$(4.22) \quad \left((\Delta_B^0)^{-1} \left(N_B \hat{w}_-^{B'} \right) \Delta_B^0 \right) (\zeta) = \begin{pmatrix} 0 & 0 \\ r(z_0)\delta_B^1(\zeta)^{-2} & 0 \end{pmatrix},$$

$$(4.23) \quad \left((\Delta_B^0)^{-1} \left(N_B \hat{w}_-^{B'} \right) \Delta_B^0 \right) (\zeta) = \begin{pmatrix} 0 & 0 \\ \frac{r(z_0)}{1 + |r(z_0)|^2} \delta_B^1(\zeta)^{-2} & 0 \end{pmatrix},$$

$$(4.24) \quad \left((\Delta_B^0)^{-1} \left(N_B \hat{w}_+^{B'} \right) \Delta_B^0 \right) (\zeta) = \begin{pmatrix} 0 & \overline{r(z_0)} \delta_B^1(\zeta)^2 \\ 0 & 0 \end{pmatrix},$$

$$(4.25) \quad \left((\Delta_B^0)^{-1} \left(N_B \hat{w}_+^{B'} \right) \Delta_B^0 \right) (\zeta) = \begin{pmatrix} 0 & \frac{\overline{r(z_0)}}{1 + |r(z_0)|^2} \delta_B^1(\zeta)^2 \\ 0 & 0 \end{pmatrix}.$$

Lemma 4.7. *Let γ be a small but fixed positive number with $0 < 2\gamma < 1$. Then*

$$\left| \delta_B^1(\zeta)^{\pm 2} - \zeta^{\pm 2i\kappa} e^{\mp i\zeta^2/2} \right| \leq c |e^{\mp i\gamma\zeta^2/2}| \tau^{-1/2}$$

and as a consequence

$$(4.26) \quad \left\| \delta_B^1(\zeta)^{\pm 2} - \zeta^{\pm 2i\kappa} e^{\mp i\zeta^2/2} \right\|_{L^1 \cap L^2 \cap L^\infty} \leq c\tau^{-1/2}$$

where the \pm sign corresponds to $\zeta \in L_B$ and $\zeta \in \bar{L}_B$ respectively. Moreover,

$$(4.27) \quad \left| \zeta^{\pm 2i\kappa} e^{\mp i\zeta^2/2} \right| \lesssim \left| e^{\mp i\gamma\zeta^2/2} \right| e^{-\varepsilon^2(1-\gamma)24\tau} \lesssim \left| e^{\mp i\gamma\zeta^2/2} \right| \tau^{-1/2}$$

where the \pm sign corresponds to $\zeta \in (\Sigma_B^1 \cup \Sigma_B^3) \setminus L_B$ and $\zeta \in (\Sigma_B^2 \cup \Sigma_B^4) \setminus \bar{L}_B$ respectively.

Proof. We only deal with the $-$ sign. One can write

$$\begin{aligned} &\delta_B^1(\zeta)^{-2} - \zeta^{-2i\kappa} e^{i\zeta^2/2} \\ &= e^{i\gamma\zeta^2/2} \left(e^{i\gamma\zeta^2/2} \left[\left(\frac{2z_0}{\zeta/\sqrt{48tz_0} + 2z_0} \right)^{-2i\kappa} \zeta^{-2i\kappa} e^{i(1-2\gamma)(\zeta^2/2)(1+\zeta/[(1-2\gamma)(432\tau)^{1/2}]})} \right. \right. \\ &\quad \left. \left. - \zeta^{-2i\kappa} e^{i(1-2\gamma)\zeta^2/2} \right] \right). \end{aligned}$$

Each of the terms in the expression above is uniformly bounded for $x < 0$ and $t > 0$ ([16, p 334]). Following the proof of [16, Lemma 3.35], we estimate

$$\left| e^{i\gamma\zeta^2/2} \left(\left(\frac{2z_0}{\zeta/\sqrt{48tz_0} + 2z_0} \right)^{-2i\kappa} - 1 \right) \right| \leq c |e^{i\gamma\zeta^2/2}| \tau^{-1/2}$$

and

$$\begin{aligned} &\left| e^{i\gamma\zeta^2/2} \zeta^{-2i\kappa} \left(e^{i(1-2\gamma)(\zeta^2/2)(1+\zeta/[(1-2\gamma)(432\tau)^{1/2}]})} - e^{i(1-2\gamma)\zeta^2/2} \right) \right| \\ &\leq c |e^{i\gamma\zeta^2/2}| \tau^{-1/2} \end{aligned}$$

as desired. And the inequality in (4.27) is an easy consequence of (4.4)-(4.9). \square

We then consider the case Σ_A . Again the rescaling gives

$$N_A \left(e^{\chi(z_0)} \eta(z; z_0) e^{-it\theta} \right) = \delta_A^0 \delta_A^1(\zeta)$$

with

$$\begin{aligned} \delta_A^0 &= (192\tau)^{i\kappa/2} e^{-8i\tau} e^{\chi(-z_0)} \eta_0(-z_0) \\ \delta_A^1(\zeta) &= (-\zeta)^{-i\kappa} \left(\frac{-2z_0}{\zeta/\sqrt{48tz_0} - 2z_0} \right)^{-i\kappa} e^{(i\zeta^2/4)(1-\zeta(432\tau))^{-1/2}}. \end{aligned}$$

Note that δ_A^0 is independent of ζ and that $|\delta_A^0| = 1$. Set

$$\begin{aligned} \Delta_A^0 &= (\delta_A^0)^{\sigma_3} \\ w^A(\zeta) &= (\Delta_A^0)^{-1} (N_A \hat{w}^{A'}) \Delta_A^0 \end{aligned}$$

and define the operator $A : L^2(\Sigma_A) \rightarrow L^2(\Sigma_A)$

$$\begin{aligned} A &= C_{(\Delta_A^0)^{-1} (N_A \hat{w}^{A'}) \Delta_A^0} \\ &= C^+ \left(\cdot (\Delta_A^0)^{-1} (N_A \hat{w}_-^{A'}) \Delta_A^0 \right) + C^- \left(\cdot (\Delta_A^0)^{-1} (N_A \hat{w}_+^{A'}) \Delta_A^0 \right). \end{aligned}$$

On

$$\begin{aligned} L_A \cup \bar{L}_A &= \{z = uz_0 \sqrt{48tz_0} e^{-i3\pi/4} : -\varepsilon < u < \varepsilon\} \\ &\cup \{z = uz_0 \sqrt{48tz_0} e^{-i\pi/4} : -\varepsilon < u < \varepsilon\} \end{aligned}$$

we have from the list of entries stated in (3.5), (3.7), (3.9) and (3.10)

$$(4.28) \quad \left((\Delta_A^0)^{-1} (N_A \hat{w}_-^{A'}) \Delta_A^0 \right) (z) = \begin{pmatrix} 0 & 0 \\ r(-z_0) \delta_A^1(z)^{-2} & 0 \end{pmatrix},$$

$$(4.29) \quad \left((\Delta_A^0)^{-1} (N_A \hat{w}_-^{A'}) \Delta_A^0 \right) (z) = \begin{pmatrix} 0 & 0 \\ \frac{r(-z_0)}{1 + |r(z_0)|^2} \delta_A^1(z)^{-2} & 0 \end{pmatrix},$$

$$(4.30) \quad \left((\Delta_A^0)^{-1} (N_A \hat{w}_+^{A'}) \Delta_A^0 \right) (z) = \begin{pmatrix} 0 & \overline{r(-z_0)} \delta_A^1(z)^2 \\ 0 & 0 \end{pmatrix},$$

$$(4.31) \quad \left((\Delta_A^0)^{-1} (N_A \hat{w}_+^{A'}) \Delta_A^0 \right) (z) = \begin{pmatrix} 0 & \frac{\overline{r(-z_0)}}{1 + |r(z_0)|^2} \delta_A^1(z)^2 \\ 0 & 0 \end{pmatrix}.$$

Lemma 4.8. *Let γ be a small but fixed positive number with $0 < 2\gamma < 1$. Then*

$$\left| \delta_A^1(\zeta)^{\pm 2} - (-\zeta)^{\mp 2i\kappa} e^{\pm i\zeta^2/2} \right| \leq c |e^{\pm i\gamma\zeta^2/2}| \tau^{-1/2}$$

and as a consequence,

$$(4.32) \quad \left\| \delta_A^1(\zeta)^{\pm 2} - (-\zeta)^{\mp 2i\kappa} e^{\pm i\zeta^2/2} \right\|_{L^1 \cap L^2 \cap L^\infty} \leq c \tau^{-1/2}$$

where the \pm sign corresponds to $\zeta \in L_A$ and $\zeta \in \bar{L}_A$ respectively. Moreover,

$$(4.33) \quad \left| (-\zeta)^{\pm 2i\kappa} e^{\mp i\zeta^2/2} \right| \lesssim \left| e^{\mp i\gamma\zeta^2/2} \right| e^{-\varepsilon^2(1-\gamma)24\tau} \lesssim \left| e^{\mp i\gamma\zeta^2/2} \right| \tau^{-1/2}$$

where the \pm sign corresponds to $\zeta \in (\Sigma_A^2 \cup \Sigma_A^3) \setminus L_A$ and $\zeta \in (\Sigma_A^1 \cup \Sigma_A^4) \setminus \bar{L}_A$ respectively.

We now define

$$\begin{aligned} w^{A^0}(\zeta) &= \lim_{\tau \rightarrow \infty} (\Delta_A^0)^{-1} (N_A \hat{w}^{A'}) \Delta_A^0(\zeta), \\ w^{B^0}(\zeta) &= \lim_{\tau \rightarrow \infty} (\Delta_B^0)^{-1} (N_B \hat{w}^{B'}) \Delta_B^0(\zeta), \end{aligned}$$

$$\begin{aligned} A^0 &= C^+(\cdot w_-^{A^0}) + C^-(\cdot w_+^{A^0}), \\ B^0 &= C^+(\cdot w_-^{B^0}) + C^-(\cdot w_+^{B^0}). \end{aligned}$$

Proposition 4.9.

$$(4.34) \quad \|(1_A - A)^{-1}\|_{L^2(\Sigma_A)}, \|(1_B - B)^{-1}\|_{L^2(\Sigma_B)} \leq c$$

as $\tau \rightarrow \infty$.

Proof. From Lemma 4.7 and Lemma 4.8, it is easily seen that

$$(4.35) \quad \|A - A^0\|_{L^2(\Sigma_A)}, \|B - B^0\|_{L^2(\Sigma_B)} \leq c\tau^{-1/2}.$$

We will only establish the boundedness of $(1_B - B)^{-1}$ since the case for $(1_A - A)^{-1}$ is similar. From Lemma 4.7 we deduce that on Σ_B

$$(4.36) \quad w^{B^0}(\zeta) = \begin{cases} \begin{pmatrix} 0 & 0 \\ r(z_0)\zeta^{-2i\kappa}e^{i\zeta^2/2} & 0 \end{pmatrix}, & \zeta \in \Sigma_B^1, \\ \begin{pmatrix} 0 & \frac{\overline{r(z_0)}}{1+|r(z_0)|^2}\zeta^{2i\kappa}e^{-i\zeta^2/2} \\ 0 & 0 \end{pmatrix}, & \zeta \in \Sigma_B^2, \\ \begin{pmatrix} 0 & 0 \\ \frac{r(z_0)}{1+|r(z_0)|^2}\zeta^{-2i\kappa}e^{i\zeta^2/2} & 0 \end{pmatrix}, & \zeta \in \Sigma_B^3, \\ \begin{pmatrix} 0 & \overline{r(z_0)}\zeta^{2i\kappa}e^{-i\zeta^2/2} \\ 0 & 0 \end{pmatrix}, & \zeta \in \Sigma_B^4. \end{cases}$$

Setting

$$v^{B^0}(\zeta) = I + w^{B^0}(\zeta)$$

we first notice that $v^{B^0}(\zeta)$ is precisely the jumps of the exactly solvable parabolic cylinder problem. The solution of this problem is standard and can be found in [9, Appendix A]. More importantly, $v^{B^0}(\zeta)$ satisfies the Schwarz invariant condition:

$$v^{B^0}(\zeta) = v^{B^0}(\bar{\zeta})^\dagger$$

which will guarantee the uniqueness of the solution. By standard arguments in [58] and [13, Sec 7.5], this implies the existence and boundedness of the resolvent operator $(1_B - B^0)^{-1}$. And the boundedness of $(1_B - B)^{-1}$ is a consequence of (4.35) and the second resolvent identity. \square

Indeed, for $\zeta \in \Sigma_B$ we let

$$(4.37) \quad m^{B^0}(\zeta) = I + \frac{1}{2\pi i} \int_{\Sigma_B} \frac{((1_B - B^0)^{-1}I)(s)w^{B^0}(s)}{s - \zeta} ds$$

then $m^{B^0}(\zeta)$ solves the following Riemann-Hilbert problem

$$(4.38) \quad \begin{cases} m_+^{B^0}(\zeta) = m_-^{B^0}(\zeta)v^{B^0}(\zeta), & \zeta \in \Sigma_B \\ m^{B^0}(\zeta) \rightarrow I, & \zeta \rightarrow \infty \end{cases}$$

In the large ζ expansion,

$$m^{B^0}(\zeta) = I - \frac{m_1^{B^0}}{\zeta} + O(\zeta^{-2}), \quad \zeta \rightarrow \infty$$

thus

$$m_1^{B^0} = \frac{1}{2\pi i} \int_{\Sigma_B} ((1_B - B)^{-1}I)(s)w^{B^0}(s)ds.$$

Similarly, setting

$$(4.39) \quad m^B(\zeta) = I + \frac{1}{2\pi i} \int_{\Sigma_B} \frac{((1_B - B)^{-1}I)(s)w^B(s)}{s - \zeta} ds$$

then $m^B(\zeta)$ solves the following Riemann-Hilbert problem

$$(4.40) \quad \begin{cases} m_+^B(\zeta) &= m_-^B(\zeta)v^B(\zeta), \quad \zeta \in \Sigma_B \\ m^B(\zeta) &\rightarrow I, \quad \zeta \rightarrow \infty \end{cases}$$

Here $v^B(\zeta) = I + w^B(\zeta)$ where $w^B(\zeta)$ is given by (4.22)-(4.25). In the large ζ expansion,

$$m^B(\zeta) = I - \frac{m_1^B}{\zeta} + O(\zeta^{-2}), \quad \zeta \rightarrow \infty$$

thus

$$m_1^B = \frac{1}{2\pi i} \int_{\Sigma_B} ((1_B - B)^{-1}I)(s)w^B(s)ds.$$

Setting $w^d = w^B - w^{B^0}$, a simple computation shows that

$$\begin{aligned} \int_{\Sigma_B} ((1_B - B)^{-1}I)w^B - \int_{\Sigma_B} ((1_B - B^0)^{-1}I)w^{B^0} &= \int_{\Sigma_B} w^d + \int_{\Sigma_B} ((1_B - B^0)^{-1}(C_{w^d}I))w^B \\ &\quad + \int_{\Sigma_B} ((1_B - B^0)^{-1}(B^0I))w^d \\ &\quad + \int_{\Sigma_B} ((1_B - B^0)^{-1}C_{w^d}(1_B - B)^{-1})(B(I))w^B \\ &= \text{I} + \text{II} + \text{III} + \text{IV}. \end{aligned}$$

From Lemma 4.7 and Proposition 4.9, it is clear that

$$\begin{aligned} |\text{I}| &\lesssim \tau^{-1/2}, \\ |\text{II}| &\leq \| (1_B - B^0)^{-1} \|_{L^2(\Sigma_B)} \| C_{w^d}I \|_{L^2(\Sigma_B)} \| w^B \|_{L^2(\Sigma_B)} \\ &\lesssim \tau^{-1/2}, \\ |\text{III}| &\leq \| (1_B - B^0)^{-1} \|_{L^2(\Sigma_B)} \| B^0I \|_{L^2(\Sigma_B)} \| w^d \|_{L^2(\Sigma_B)} \\ &\lesssim \tau^{-1/2}. \end{aligned}$$

For the last term

$$\begin{aligned} |\text{IV}| &\leq \| (1_B - B^0)^{-1} \|_{L^2(\Sigma_B)} \| (1_B - B)^{-1} \|_{L^2(\Sigma_B)} \| C_{w^d} \|_{L^2(\Sigma_B)} \\ &\quad \times \| B(I) \|_{L^2(\Sigma_B)} \| w^B \|_{L^2(\Sigma_B)} \\ &\leq c \| w^d \|_{L^\infty(\Sigma_B)} \| w^B \|_{L^2(\Sigma^3)}^2 \\ &\lesssim \tau^{-1/2}. \end{aligned}$$

So we conclude that

$$(4.41) \quad \left| \int_{\Sigma_B} ((1_B - B)^{-1}I)w^B - \int_{\Sigma_B} ((1_B - B^0)^{-1}I)w^{B^0} \right| \lesssim \tau^{-1/2}.$$

Clearly there is a parallel case for Σ_A :

$$(4.42) \quad \left| \int_{\Sigma_A} ((1_A - A)^{-1}I) w^A - \int_{\Sigma_A} ((1_A - A^0)^{-1}I) w^{A^0} \right| \lesssim \tau^{-1/2}.$$

The explicit form of $m_1^{B^0}$ is given as follows (see [9, Appendix A]) :

$$(4.43) \quad m_1^{B^0} = \begin{pmatrix} 0 & -i\beta_{12} \\ i\beta_{21} & 0 \end{pmatrix}$$

where

$$\beta_{12} = \frac{\sqrt{2\pi}e^{i\pi/4}e^{-\pi\kappa}}{r(z_0)\Gamma(-i\kappa)}, \quad \beta_{21} = \frac{-\sqrt{2\pi}e^{-i\pi/4}e^{-\pi\kappa}}{r(z_0)\Gamma(i\kappa)}$$

and $\Gamma(z)$ is the *Gamma* function. Recall that on Σ_B , $\zeta = \sqrt{48z_0t}(z - z_0)$, thus by (4.41), we have

$$(4.44) \quad \left| \frac{m_1^B}{\zeta} - \frac{m_1^{B^0}}{\zeta} \right| \lesssim \frac{1}{t(z - z_0)}.$$

Using the explicit form of w^{B^0} given by (4.36), symmetry reduction given by (1.16) and their analogue for w^{A^0} , we verify that

$$(4.45) \quad v^{A^0}(z) = \sigma_3 \overline{v^{B^0}(-\bar{z})} \sigma_3$$

which in turn implies by uniqueness that

$$(4.46) \quad m^{A^0}(z) = \sigma_3 \overline{m^{B^0}(-\bar{z})} \sigma_3$$

and from this we deduce that

$$(4.47) \quad \begin{aligned} m_1^{A^0} &= -\sigma_3 \overline{m_1^{B^0}} \sigma_3 \\ &= \begin{pmatrix} 0 & i\bar{\beta}_{12} \\ -i\bar{\beta}_{21} & 0 \end{pmatrix}. \end{aligned}$$

We also have an analogue of (4.44) for $m_1^{A^0}$:

$$(4.48) \quad \left| \frac{m_1^A}{\zeta} - \frac{m_1^{A^0}}{\zeta} \right| \lesssim \frac{1}{t(z + z_0)}.$$

Collecting all the computations above, we write down the asymptotic expansions of solutions to Problem 4.5 and Problem 4.6 respectively.

Proposition 4.10. *Setting $\zeta = \sqrt{48z_0t}(z + z_0)$, the solution to RHP Problem 4.5 $m^{A'}$ admits the following expansion:*

$$(4.49) \quad m^{A'}(z(\zeta); x, t) = I + \frac{1}{\zeta} \begin{pmatrix} 0 & i(\delta_A^0)^2 \bar{\beta}_{12} \\ -i(\delta_A^0)^{-2} \bar{\beta}_{21} & 0 \end{pmatrix} + \mathcal{O}(t^{-1}).$$

Similarly, setting $\zeta = \sqrt{48z_0t}(z - z_0)$, the solution to RHP Problem 4.6 $m^{B'}$ admits the following expansion:

$$(4.50) \quad m^{B'}(z(\zeta); x, t) = I + \frac{1}{\zeta} \begin{pmatrix} 0 & -i(\delta_B^0)^2 \beta_{12} \\ i(\delta_B^0)^{-2} \beta_{21} & 0 \end{pmatrix} + \mathcal{O}(t^{-1}).$$

Now we construct m_1^{LC} needed in the proof of Proposition 4.3. In Figure 4.4, we let ρ be the radius of the circle C_A (C_B) centered at z_0 ($-z_0$). We seek a solution of the form

$$(4.51) \quad m_*^{\text{LC}}(z) = \begin{cases} E_2(z)m^{(br)}(z) & |z \pm z_0| > \rho \\ E_2(z)m^{(br)}(z)m^{A'}(z) & |z + z_0| \leq \rho \\ E_2(z)m^{(br)}(z)m^{B'}(z) & |z - z_0| \leq \rho \end{cases}$$

Since $m^{(br)}$, $m^{A'}$ and $m^{B'}$ solve Problem 4.4, Problem 4.5 and Problem 4.6 respectively, we can construct the solution $m_*^{LC}(z)$ if we find $E_2(z)$. Indeed, E_2 solves the following Riemann-Hilbert problem:

Problem 4.11. Find a matrix-valued function $E_2(z)$ on $\mathbb{C} \setminus (C_A \cup C_B)$ with the following properties:

- (1) $E_2(z) \rightarrow I$ as $z \rightarrow \infty$,
- (2) $E_2(z)$ is analytic for $z \in \mathbb{C} \setminus (C_A \cup C_B)$ with continuous boundary values $E_{2\pm}(z)$.
- (3) On $C_A \cup C_B$ we have the following jump conditions

$$E_{2+}(z) = E_{2-}(z)v^{(E)}(z)$$

where

$$(4.52) \quad v^{(E)}(z) = \begin{cases} m^{(br)}(z)m^{A'}(z(\zeta))m^{(br)}(z)^{-1}, & z \in C_A \\ m^{(br)}(z)m^{B'}(z(\zeta))m^{(br)}(z)^{-1}, & z \in C_B \end{cases}$$

Setting

$$\eta(z) = E_{2-}(z)$$

then by standard theory, we have the following singular integral equation

$$\eta = I + C_{v^{(E)}}\eta$$

where the singular integral operator is defined by:

$$C_{v^{(E)}}\eta = C^- \left(\eta \left(v^{(E)} - I \right) \right).$$

We first deduce from (4.49)-(4.50) that

$$(4.53) \quad \left\| v^{(E)} - I \right\|_{L^\infty} \lesssim t^{-1/2}$$

hence the operator norm of $C_{v^{(E)}}$

$$(4.54) \quad \left\| C_{v^{(E)}}f \right\|_{L^2} \leq \|f\|_{L^2} \left\| v^{(E)} - I \right\|_{L^\infty} \lesssim t^{-1/2}.$$

Then the resolvent operator $(1 - C_{v^{(E)}})^{-1}$ can be obtained through *Neumann* series and we obtain the unique solution to Problem 4.11:

$$(4.55) \quad E_2(z) = I + \frac{1}{2\pi i} \int_{C_A \cup C_B} \frac{(1 + \eta(s))(v^{(E)}(s) - I)}{s - z} ds$$

which admits the following asymptotic expansion in z :

$$(4.56) \quad E_2(z) = I + \frac{E_{2,1}}{z} + \mathcal{O}\left(\frac{1}{z^2}\right).$$

Using the bound on the operator norm (4.54), we obtain

$$(4.57) \quad E_{2,1}(z) = -\frac{1}{2\pi i} \int_{C_A \cup C_B} (1 + \eta(s))(v^{(E)}(s) - I) ds$$

$$(4.58) \quad = -\frac{1}{2\pi i} \int_{C_A \cup C_B} (v^{(E)}(s) - I) ds + \mathcal{O}(t^{-1}).$$

Given the form of $v^{(E)}$ in (4.52) and the asymptotic expansions (4.49)-(4.50), an application of Cauchy's integral formula leads to

$$(4.59) \quad E_{2,1} = \frac{1}{\sqrt{48z_0t}} m^{(br)}(z_0) \begin{pmatrix} 0 & -i(\delta_B^0)^2 \beta_{12} \\ i(\delta_B^0)^{-2} \beta_{21} & 0 \end{pmatrix} m^{(br)}(z_0)^{-1} \\ + \frac{1}{\sqrt{48z_0t}} m^{(br)}(-z_0) \begin{pmatrix} 0 & i(\delta_A^0)^2 \bar{\beta}_{12} \\ -i(\delta_A^0)^{-2} \bar{\beta}_{21} & 0 \end{pmatrix} m^{(br)}(-z_0)^{-1}$$

$$+ \mathcal{O}(t^{-1}).$$

We now completed the construction of the matrix-valued function $E_2(z)$ hence $m_*^{\text{LC}}(x, t; z)$. Combining this with Proposition 4.11, we obtain $m^{\text{LC}}(z)$ in (4.1).

5. THE $\bar{\partial}$ -PROBLEM

From (4.1) we have matrix-valued function

$$(5.1) \quad m^{(3)}(z; x, t) = m^{(2)}(z; x, t)m^{\text{LC}}(z; x, t)^{-1}.$$

The goal of this section is to show that $m^{(3)}$ only results in an error term E with higher order decay rate than the leading order term of the asymptotic formula. The computations and proofs are standard. We follow [10, Section 5] with slight modifications.

Since $m^{\text{LC}}(z; x, t)$ is analytic in $\mathbb{C} \setminus (\Sigma^{(3)} \cup \Gamma)$, we may compute

$$\begin{aligned} \bar{\partial}m^{(3)}(z; x, t) &= \bar{\partial}m^{(2)}(z; x, t)m^{\text{LC}}(z; x, t)^{-1} \\ &= m^{(2)}(z; x, t)\bar{\partial}\mathcal{R}^{(2)}(z)m^{\text{LC}}(z; x, t)^{-1} && \text{(by (3.14))} \\ &= m^{(3)}(z; x, t)m^{\text{LC}}(z; x, t)\bar{\partial}\mathcal{R}^{(2)}(z)m^{\text{LC}}(z; x, t)^{-1} && \text{(by (5.1))} \\ &= m^{(3)}(z; x, t)W(z; x, t) \end{aligned}$$

where

$$(5.2) \quad W(z; x, t) = m^{\text{LC}}(z; x, t)\bar{\partial}\mathcal{R}^{(2)}(z)m^{\text{LC}}(z; x, t)^{-1}.$$

We thus arrive at the following pure $\bar{\partial}$ -problem:

Problem 5.1. Give $r \in H^1(\mathbb{R})$, find a continuous matrix-valued function $m^{(3)}(z; x, t)$ on \mathbb{C} with the following properties:

- (1) $m^{(3)}(z; x, t) \rightarrow I$ as $|z| \rightarrow \infty$.
- (2) $\bar{\partial}m^{(3)}(z; x, t) = m^{(3)}(z; x, t)W(z; x, t)$.

It is well understood (see for example [1, Chapter 7]) that the solution to this $\bar{\partial}$ problem is equivalent to the solution of a Fredholm-type integral equation involving the solid Cauchy transform

$$(Pf)(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{\zeta - z} f(\zeta) d\zeta$$

where d denotes Lebesgue measure on \mathbb{C} . Also throughout this section, ζ refers to complex numbers, not to be confused with $\zeta = \sqrt{48z_0t}(z \pm z_0)$ in the previous section.

Lemma 5.2. *A bounded and continuous matrix-valued function $m^{(3)}(z; x, t)$ solves Problem (5.1) if and only if*

$$(5.3) \quad m^{(3)}(z; x, t) = I + \frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{\zeta - z} m^{(3)}(\zeta; x, t)W(\zeta; x, t) d\zeta.$$

Using the integral equation formulation (5.3), we will prove:

Proposition 5.3. *Suppose that $r \in H^1(\mathbb{R})$. Then, for $t \gg 1$, there exists a unique solution $m^{(3)}(z; x, t)$ for Problem 5.1 with the property that*

$$(5.4) \quad m^{(3)}(z; x, t) = I + \frac{1}{z}m_1^{(3)}(x, t) + o\left(\frac{1}{z}\right)$$

for $z = i\sigma$ with $\sigma \rightarrow +\infty$. Here

$$(5.5) \quad \left| m_1^{(3)}(x, t) \right| \lesssim (z_0t)^{-3/4}$$

where the implicit constant in (5.5) is uniform for r in a bounded subset of $H^1(\mathbb{R})$.

Proof. Given Lemmas 5.4–5.8, as in [43], we first show that, for large t , the integral operator K_W defined by

$$(K_W f)(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{\zeta - z} f(\zeta) W(\zeta) d\zeta$$

is bounded by

$$(5.6) \quad \|K_W\|_{L^\infty \rightarrow L^\infty} \lesssim (z_0 t)^{-1/4}$$

where the implied constants depend only on $\|r\|_{H^1}$. This is the goal of Lemma 5.6. It implies that

$$(5.7) \quad m^{(3)} = (I - K_W)^{-1} I$$

exists as an L^∞ solution of (5.3).

We then show in Lemma 5.7 that the solution $m^{(3)}(z; x, t)$ has a large- z asymptotic expansion of the form (5.4) where $z \rightarrow \infty$ along the *positive imaginary axis*. Note that, for such z , we can bound $|z - \zeta|$ below by a constant times $|z| + |\zeta|$. Finally, in Lemma 5.8 we prove estimate (5.5) where the constants are uniform in r belonging to a bounded subset of $H^1(\mathbb{R})$. Estimates (5.4), (5.5), and (5.6) result from the bounds obtained in the next four lemmas. \square

Lemma 5.4. *Set $z = (u \mp \xi) + iv$. We have*

$$(5.8) \quad \left| \bar{\partial} \mathcal{R}^{(2)} e^{\pm 2i\theta} \right| \lesssim \left(|p'_i(\operatorname{Re}(z))| + |z \mp \xi|^{-1/2} + |\Xi_{\mathcal{Z}}(z)| \right) e^{-z_0 t |u||v|}.$$

Proof. We only show the inequalities above in Ω_1 and Ω_7^+ . Recall that near z_0

$$i\theta(z; x, t) = 4it \left((z - z_0)^3 + 3z_0(z - z_0)^2 - 2z_0^3 \right).$$

In Ω_1 , we use the facts that $u \geq 0$, $v \geq 0$ and $|u| \geq |v|$ to deduce

$$\begin{aligned} \operatorname{Re}(2i\theta) &= 8it(3iu^2v - iv^3 + 6iuvz_0) \\ &= 8t(-3u^2v + v^3 - 6uvz_0) \\ &\leq 8t(-3u^2v + u^2v - 6uvz_0) \\ &\leq 8t(-2u^2v - 6uvz_0) \\ &\leq -8|u||v|z_0t. \end{aligned}$$

Similarly, in Ω_7^+ , we have $u \leq 0$, $v \geq 0$ and $|u| \geq |v|$, hence

$$\begin{aligned} \operatorname{Re}(-2i\theta) &= -8it(3iu^2v - iv^3 + 6iuvz_0) \\ &= 8t(3u^2v + 6uvz_0) \\ &\leq 8t(-3uz_0v + 6uvz_0) \\ &\leq -8|u||v|z_0t. \end{aligned}$$

Estimate (5.8) then follows from Lemma 3.1. The quantities $p'_i(\operatorname{Re} z)$ are all bounded uniformly for r in a bounded subset of $H^1(\mathbb{R})$. \square

Lemma 5.5. *For the localized Riemann-Hilbert problem from Problem 4.1, we have*

$$(5.9) \quad \|m^{\text{LC}}(\cdot; x, t)\|_\infty \lesssim 1,$$

$$(5.10) \quad \|m^{\text{LC}}(\cdot; x, t)^{-1}\|_\infty \lesssim 1.$$

All implied constants are uniform for r in a bounded subset of $H^1(\mathbb{R})$.

The proof of this lemma is a consequence of the previous section.

Lemma 5.6. *Suppose that $r \in H^1(\mathbb{R})$. Then, the estimate (5.6) holds, where the implied constants depend on $\|r\|_{H^1}$.*

Proof. To prove (5.6), first note that

$$(5.11) \quad \|K_W f\|_\infty \leq \|f\|_\infty \int_{\mathbb{C}} \frac{1}{|z - \zeta|} |W(\zeta)| dm(\zeta)$$

so that we need only estimate the right-hand integral. We will prove the estimate in the region $z \in \Omega_1$ since estimates for the remaining regions are identical. From (5.2), it follows

$$|W(\zeta)| \leq \|m^{\text{LC}}\|_\infty \|(m^{\text{LC}})^{-1}\|_\infty |\bar{\partial} R_1| |e^{2i\theta}|.$$

Setting $z = \alpha + i\beta$ and $\zeta = (u + z_0) + iv$, the region Ω_1 corresponds to $u \geq v \geq 0$. We then have from (5.8) (5.9), and (5.10) that

$$\int_{\Omega_1} \frac{1}{|z - \zeta|} |W(\zeta)| d\zeta \lesssim I_1 + I_2 + I_3$$

where

$$\begin{aligned} I_1 &= \int_0^\infty \int_v^\infty \frac{1}{|z - \zeta|} |p_1'(u)| e^{-tz_0 uv} du dv, \\ I_2 &= \int_0^\infty \int_v^\infty \frac{1}{|z - \zeta|} |u + iv|^{-1/2} e^{-tz_0 uv} du dv, \\ I_3 &= \int_0^\infty \int_v^\infty \frac{1}{|z - \zeta|} |\bar{\partial}(\Xi_Z(\zeta))| e^{-tz_0 uv} du dv. \end{aligned}$$

It now follows from [9, proof of Proposition D.1] that

$$|I_1|, |I_2|, |I_3| \lesssim (z_0 t)^{-1/4}.$$

It then follows that

$$\int_{\Omega_1} \frac{1}{|z - z_0|} |W(\zeta)| d\zeta \lesssim (z_0 t)^{-1/4}$$

which, together with similar estimates for the integrations over the remaining Ω_i s, proves (5.6). \square

Lemma 5.7. *For $z = i\sigma$ with $\sigma \rightarrow +\infty$, the expansion (5.4) holds with*

$$(5.12) \quad m_1^{(3)}(x, t) = \frac{1}{\pi} \int_{\mathbb{C}} m^{(3)}(\zeta; x, t) W(\zeta; x, t) d\zeta.$$

Proof. We write (5.3) as

$$m^{(3)}(z; x, t) = I + \frac{1}{z} m_1^{(3)}(x, t) + \frac{1}{\pi z} \int_{\mathbb{C}} \frac{\zeta}{z - \zeta} m^{(3)}(\zeta; x, t) W(\zeta; x, t) dm(\zeta)$$

where $m_1^{(3)}$ is given by (5.12). If $z = i\sigma$, it is easy to see that $|\zeta|/|z - \zeta|$ is bounded above by a fixed constant independent of z , while $|m^{(3)}(\zeta; x, t)| \lesssim 1$ by the remarks following (5.7). If we can show that $\int_{\mathbb{C}} |W(\zeta; x, t)| d\zeta$ is finite, it will follow from the Dominated Convergence Theorem that

$$\lim_{\sigma \rightarrow \infty} \int_{\mathbb{C}} \frac{\zeta}{i\sigma - \zeta} m^{(3)}(\zeta; x, t) W(\zeta; x, t) d\zeta = 0$$

which implies the required asymptotic estimate. We will estimate $\int_{\Omega_1} |W(\zeta)| dm(\zeta)$ since the other estimates are identical. One can write

$$\Omega_1 = \{(u + z_0, v) : v \geq 0, v \leq u < \infty\}.$$

Using (5.8), (5.9), and (5.10), we may then estimate

$$\int_{\Omega_1} |W(\zeta; x, t)| d\zeta \lesssim I_1 + I_2 + I_3$$

where

$$\begin{aligned} I_1 &= \int_0^\infty \int_v^\infty |p'_1(u + z_0)| e^{-tz_0uv} du dv \\ I_2 &= \int_0^\infty \int_v^\infty |u^2 + v^2|^{-1/2} e^{-tz_0uv} du dv \\ I_3 &= \int_0^\infty \int_v^\infty |\bar{\partial}(\Xi_{\mathcal{Z}}(\zeta))| e^{-tz_0uv} du dv \end{aligned}$$

It now follows from [9, Proposition D.2] that

$$I_1, I_2, I_3 \lesssim (z_0t)^{-3/4}.$$

These estimates together show that

$$(5.13) \quad \int_{\Omega_1} |W(\zeta; x, t)| d\zeta \lesssim (z_0t)^{-3/4}$$

and that the implied constant depends only on $\|r\|_{H^1}$. In particular, the integral (5.13) is bounded uniformly as $t \rightarrow \infty$. \square

Lemma 5.8. *The estimate (5.5) holds with constants uniform in r in a bounded subset of $H^1(\mathbb{R})$.*

Proof. From the representation formula (5.12), Lemma 5.6, and the remarks following, we have

$$\left| m_1^{(3)}(x, t) \right| \lesssim \int_{\mathbb{C}} |W(\zeta; x, t)| d\zeta.$$

In the proof of Lemma 5.7, we bounded this integral by $(z_0t)^{-3/4}$ modulo constants with the required uniformities. \square

6. LONG-TIME ASYMPTOTICS

We now put together our previous results and formulate the long-time asymptotics of $u(x, t)$ in Region I. Undoing all transformations we carried out previously, we get back m :

$$(6.1) \quad m(z; x, t) = m^{(3)}(z; x, t) m^{\text{LC}}(z; z_0) \mathcal{R}^{(2)}(z)^{-1} \delta(z)^{\sigma_3}.$$

By stand inverse scattering theory, the coefficient of z^{-1} in the large- z expansion for $m(z; x, t)$ will be the solution to the mKdV.

Lemma 6.1. *For $z = i\sigma$ and $\sigma \rightarrow +\infty$, the asymptotic relations*

$$(6.2) \quad m(z; x, t) = I + \frac{1}{z} m_1(x, t) + o\left(\frac{1}{z}\right)$$

$$(6.3) \quad m^{\text{LC}}(z; x, t) = I + \frac{1}{z} m_1^{\text{LC}}(x, t) + o\left(\frac{1}{z}\right)$$

hold. Moreover,

$$(6.4) \quad (m_1(x, t))_{12} = (m_1^{\text{LC}}(x, t))_{12} + \mathcal{O}\left((z_0t)^{-3/4}\right).$$

Proof. By Lemma 2.2 (iii), the expansion

$$(6.5) \quad \delta(z)^{\sigma_3} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{z} \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_1^{-1} \end{pmatrix} + \mathcal{O}(z^{-2})$$

holds, with the remainder in (6.5) uniform in r in a bounded subset of H^1 . (6.2) follows from (6.1), (6.3), the fact that $\mathcal{R}^{(2)} \equiv I$ in Ω_2 , and (6.5). Notice the fact that the diagonal matrix in (6.5) does not affect the 12-component of m . Hence, for $z = i\sigma$,

$$(m(z; x, t))_{12} = \frac{1}{z} \left(m_1^{(3)}(x, t) \right)_{12} + \frac{1}{z} \left(m_1^{\text{LC}}(x, t) \right)_{12} + o\left(\frac{1}{z}\right)$$

and result now follows from (5.5). \square

From previous results (see Proposition 4.3, Problem 4.4 and Problem 4.11) we have:

$$(6.6) \quad \begin{aligned} m^{\text{LC}}(z) &= E_1(z) m_*^{\text{LC}}(z) \\ &= E_1(z) E_2(z) m^{(br)}(z) \\ &= \left(I + \frac{E_{1,1}}{z} + \dots \right) \left(I + \frac{E_{2,1}}{z} + \dots \right) \left(I + \frac{m_1^{(br)}}{z} + \dots \right) \end{aligned}$$

as $z \rightarrow \infty$.

Together with Lemma 6.1, we arrive at the asymptotic formula in Region I:

Proposition 6.2. *The function*

$$(6.7) \quad u(x, t) = 2 \lim_{z \rightarrow \infty} z m_{12}(z; x, t)$$

takes the form

$$u(x, t) = u^{(br)}(x, t) + u_{as}(x, t) + \mathcal{O}\left((z_0 t)^{-3/4}\right)$$

where $u^{(br)}(x, t)$ is given by (4.15) and

$$(6.8) \quad \begin{aligned} u_{as}(x, t) &= \frac{1}{\sqrt{48tz_0}} \left(m_{11}^{(br)}(-z_0)^2 (i(\delta_A^0)^2 \bar{\beta}_{12}) + m_{12}^{(br)}(-z_0)^2 (i(\delta_A^0)^{-2} \bar{\beta}_{21}) \right) \\ &\quad + \frac{1}{\sqrt{48tz_0}} \left(m_{11}^{(br)}(z_0)^2 (-i(\delta_B^0)^2 \beta_{12}) - m_{12}^{(br)}(z_0)^2 (i(\delta_B^0)^{-2} \beta_{21}) \right). \end{aligned}$$

Proposition 6.3. *If we choose the frame $x = vt$ with $v < 0$ and $v \neq 4\eta_j^2 - 12\xi_j^2$ for all $1 \leq j \leq N_2$, then*

$$u(x, t) = u_{as}(x, t) + \mathcal{O}\left((z_0 t)^{-3/4}\right)$$

where

$$(6.9) \quad u_{as}(x, t) = \left(\frac{\kappa}{3tz_0} \right)^{1/2} \cos\left(16tz_0^3 - \kappa \log(192tz_0^3) + \phi(z_0)\right)$$

with

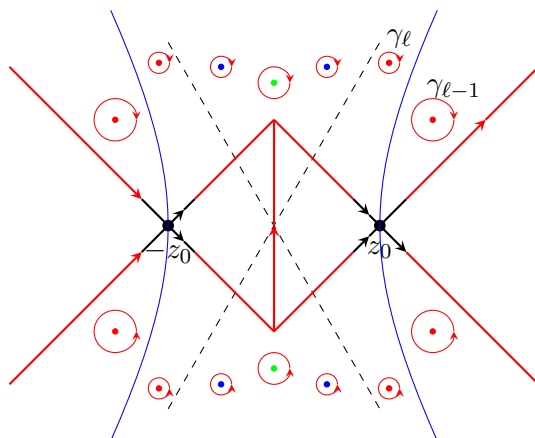
$$\begin{aligned} \phi(z_0) &= \arg \Gamma(i\kappa) - \frac{\pi}{4} - \arg r(z_0) + \frac{1}{\pi} \int_{-z_0}^{z_0} \log \left(\frac{1 + |r(\zeta)|^2}{1 + |r(z_0)|^2} \right) \frac{d\zeta}{\zeta - z_0} \\ &\quad - 4 \left(\sum_{k=1}^{N_1} \arg(z_0 - z_k) + \sum_{z_j \in \mathcal{B}_\ell} \arg(z_0 - z_j) + \sum_{z_j \in \mathcal{B}_\ell} \arg(z_0 + \bar{z}_j) \right) \end{aligned}$$

Proof. Indeed, if we choose v such that

$$4\eta_1^2 - 12\xi_1^2 < \dots < 4\eta_{\ell-1}^2 - 12\xi_{\ell-1}^2 < v < 4\eta_\ell^2 - 12\xi_\ell^2 < \dots < 4\eta_{N_2}^2 - 12\xi_{N_2}^2$$

and define the same δ function as (2.6). We follow the same procedure as in Section 3 and arrive at the following set of deformed contours and conclude that on the red portion of the contour all jump matrices decay exponentially as $t \rightarrow \infty$. Thus the localized RHP reduces to Problem 4.5 and Problem 4.6. We then follow [10] and [16, Section 4] to derive the explicit formula of u_{as} in (6.9). \square

FIGURE 6.1. $\Sigma^{(3)} \cup \Gamma$



7. REGIONS II-III

We now turn to the study of the Regions II-III. We first study Region II. Our starting point is RHP Problem 1.4 and the strategy of the proof is as follows:

1. We conjugate the jump matrices of Problem 1.4 of by a scalar function ψ .
2. We scale the conjugated jump matrices by a factor determined by the region.
3. We use $\bar{\partial}$ -steepest descent to study the scaled RHP and obtain both leading term and error term.
4. We multiply by the scaling factor to get the asymptotic formula.

We then study Region III. We mention that in both regions the application of $\bar{\partial}$ steepest descent method is analogous to that in Section 2-6. For the purpose of brevity we are only going to display the calculations directly related to the leading order term and error terms.

7.1. Region II. In this region, $|x/t^{-1/3}| = \mathcal{O}(1)$ as $t \rightarrow \infty$. We first mention that for $x > 0$, we have the stationary points

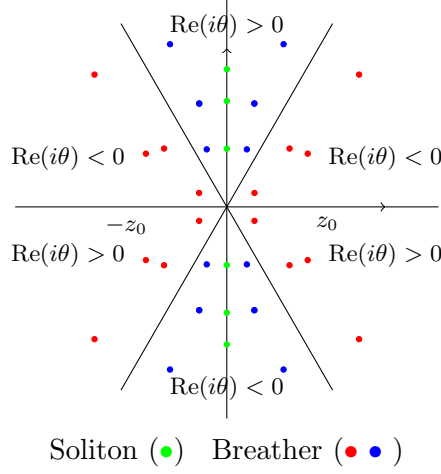
$$\pm z_0 = \pm \sqrt{\frac{-x}{12t}} = \pm i \sqrt{\frac{|x|}{12t}}$$

stay on the imaginary axis. So we are only going to study the case for $x < 0$ since for $x > 0$ the asymptotic formula will follow from a similar (and simpler) computation. For $x < 0$, we first notice that

$$z_0 = \sqrt{\frac{-x}{12t}} = \sqrt{\frac{-x}{12t^{1/3}}} t^{-1/3} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

By Remark 1.8, for the phase function $e^{i\theta(z;x,t)}$ we have the following signature table:

FIGURE 7.1. signature table-Painleve



So we only need the following upper/lower factorization on \mathbb{R} :

$$(7.1) \quad e^{-i\theta \operatorname{ad} \sigma_3} v(z) = \begin{pmatrix} 1 & \overline{r(z)} e^{-2i\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(z) e^{2i\theta} & 1 \end{pmatrix}, \quad z \in \mathbb{R}$$

Define the following set:

$$(7.2) \quad \mathcal{B}_0 = \{z_j = \xi_j + i\eta_j : 4\eta_j^2 - 12\xi_j^2 > 0\}$$

and the scalar function:

$$(7.3) \quad \psi(z) = \left(\prod_{k=1}^{N_1} \frac{z - \overline{z_k}}{z - z_k} \right) \left(\prod_{z_j \in \mathcal{B}_0} \frac{z - \overline{z_j}}{z - z_j} \right) \left(\prod_{z_j \in \mathcal{B}_0} \frac{z + z_j}{z + \overline{z_j}} \right).$$

It is straightforward to check that if $m(z; x, t)$ solves Problem 1.4, then the new matrix-valued function $m^{(1)}(z; x, t) = m(z; x, t) \psi(z)^{\sigma_3}$ has the following jump matrices:

$$(7.4) \quad e^{-i\theta \operatorname{ad} \sigma_3} v^{(1)}(z) = \begin{pmatrix} 1 & \overline{r(z)} \psi^2 e^{-2i\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(z) \psi^{-2} e^{2i\theta} & 1 \end{pmatrix}, \quad z \in \mathbb{R}$$

$$(7.5) \quad e^{-i\theta \operatorname{ad} \sigma_3} v^{(1)}(z) = \begin{cases} \begin{pmatrix} 1 & \frac{(1/\psi)'(z_k)^{-2}}{c_k(z - z_k)} e^{-2i\theta} \\ 0 & 1 \end{pmatrix} & z \in \gamma_k, \\ \begin{pmatrix} 1 & 0 \\ \frac{\psi'(\overline{z_k})^{-2}}{\overline{c_k}(z - \overline{z_k})} e^{2i\theta} & 1 \end{pmatrix} & z \in \gamma_k^*, \end{cases}$$

and for $z_j \in \mathcal{B}_0$

$$(7.6) \quad e^{-i\theta \operatorname{ad} \sigma_3 v^{(1)}}(z) = \begin{cases} \begin{pmatrix} 1 & \frac{(1/\psi)'(z_j)^{-2}}{c_j(z-z_j)} e^{-2i\theta} \\ 0 & 1 \end{pmatrix} & z \in \gamma_j, \\ \begin{pmatrix} 1 & 0 \\ \frac{\psi'(\bar{z}_j)^{-2}}{\bar{c}_j(z-\bar{z}_j)} e^{2i\theta} & 1 \end{pmatrix} & z \in \gamma_j^*, \\ \begin{pmatrix} 1 & 0 \\ -\frac{\psi'(-z_j)^{-2}}{c_j(z+z_j)} e^{2i\theta} & 1 \end{pmatrix} & z \in -\gamma_j, \\ \begin{pmatrix} 1 & -\frac{(1/\psi)'(-\bar{z}_j)^{-2}}{\bar{c}_j(z+\bar{z}_j)} e^{-2i\theta} \\ 0 & 1 \end{pmatrix} & z \in -\gamma_j^* \end{cases}$$

and for $z_j \in \{z_j\}_{j=1}^{N_2} \setminus \mathcal{B}_0$

$$(7.7) \quad e^{-i\theta \operatorname{ad} \sigma_3 v^{(1)}}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{c_j \psi(z_j)^{-2}}{z-z_j} e^{2i\theta} & 1 \end{pmatrix} & z \in \gamma_j, \\ \begin{pmatrix} 1 & \frac{\bar{c}_j \psi(\bar{z}_j)^2}{z-\bar{z}_j} e^{-2i\theta} \\ 0 & 1 \end{pmatrix} & z \in \gamma_j^*, \\ \begin{pmatrix} 1 & \frac{-c_j \psi(-z_j)^2}{z+z_j} e^{-2i\theta} \\ 0 & 1 \end{pmatrix} & z \in -\gamma_j, \\ \begin{pmatrix} 1 & 0 \\ \frac{-\bar{c}_j \psi(-\bar{z}_j)^{-2} e^{2i\theta}}{z+\bar{z}_j} & 1 \end{pmatrix} & z \in -\gamma_j^*. \end{cases}$$

By the signature table Figure 7.1, we see that all entries in (7.6)-(7.7) decay exponentially as $t \rightarrow \infty$, so we are allowed to reduce the RHP to a problem on \mathbb{R} following the same argument in the proof of Proposition 4.3. Now we carry out the following scaling:

$$(7.8) \quad z \rightarrow \zeta t^{-1/3}$$

and (7.1) becomes

$$(7.9) \quad \begin{pmatrix} 1 & \overline{r(\zeta t^{-1/3})} \psi^2(\zeta t^{-1/3}) e^{-2i\theta(\zeta t^{-1/3})} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(\zeta t^{-1/3}) \psi^{-2}(\zeta t^{-1/3}) e^{2i\theta(\zeta t^{-1/3})} & 1 \end{pmatrix}, \quad z \in \mathbb{R}$$

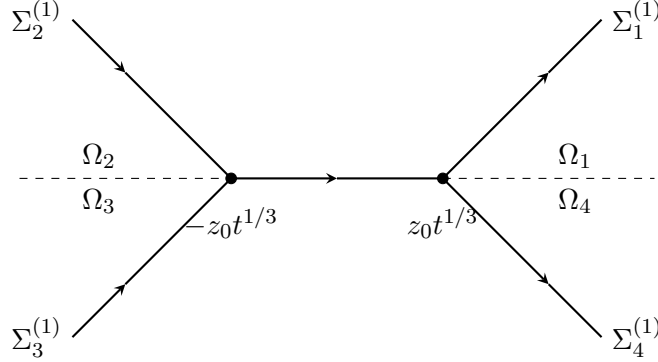
where

$$\theta(\zeta t^{-1/3}) = 4\zeta^3 + x\zeta t^{-1/3} = 4(\zeta^3 - 3\tau^{2/3}\zeta).$$

Note that the stationary points now become $\pm z_0 t^{1/3}$.

We then study the scaled Riemann-Hilbert problem with jump matrix (7.9). We will again perform contour deformation and write the solution as a product of solution to a $\bar{\partial}$ -problem and a "localized" Riemann-Hilbert problem.

FIGURE 7.2. $\Sigma^{(1)}$ – scale



For brevity, we only discuss the $\bar{\partial}$ -problem in Ω_1 . In Ω_1 , we write

$$\zeta = u + z_0 t^{1/3} + iv$$

then

$$\begin{aligned} \operatorname{Re}(2i\theta(\zeta t^{-1/3})) &= 8 \left(-3(u + z_0 t^{1/3})^2 v + v^3 + 3\tau^{2/3} v \right) \\ &\leq 8 \left(-3u^2 v - 6uvz_0 t^{1/3} + v^3 \right) \\ &\leq -16u^2 v \end{aligned}$$

$$R_1 = \begin{cases} \begin{pmatrix} 0 & 0 \\ r(\zeta t^{-1/3})\psi^{-2}(\zeta t^{-1/3})e^{2i\theta(\zeta t^{-1/3})} & 0 \end{pmatrix} & \zeta \in (z_0 t^{1/3}, \infty) \\ \begin{pmatrix} 0 & 0 \\ r(z_0)\psi^{-2}(\zeta t^{-1/3})(1 - \Xi_{\mathcal{Z}})e^{2i\theta(\zeta t^{-1/3})} & 0 \end{pmatrix} & \zeta \in \Sigma_1 \end{cases}$$

and the interpolation is given by

$$\left(r(z_0) + \left(r(\operatorname{Re}\zeta t^{-1/3}) - r(z_0) \right) \cos 2\phi \right) \psi^{-2}(\zeta t^{-1/3})(1 - \Xi_{\mathcal{Z}})$$

So we arrive at the $\bar{\partial}$ -derivative in Ω_1 in the ζ variable:

$$(7.10) \quad \bar{\partial}R_1 = \left(t^{-1/3}r'(ut^{-1/3}) \cos 2\phi - 2 \frac{r(ut^{-1/3}) - r(z_0)}{|\zeta - z_0 t^{1/3}|} e^{i\phi} \sin 2\phi \right) \psi^{-2}(\zeta t^{-1/3})e^{2i\theta}$$

$$(7.11) \quad \times (1 - \Xi_{\mathcal{Z}})$$

$$(7.12) \quad - \left(r(z_0) + \left(r(\operatorname{Re}\zeta t^{-1/3}) - r(z_0) \right) \cos 2\phi \right) t^{-1/3} \bar{\partial}(\Xi_{\mathcal{Z}}(\zeta t^{-1/3})) \psi^{-2}(\zeta t^{-1/3})e^{2i\theta}$$

$$(7.13) \quad \left| \bar{\partial}R_1 e^{2i\theta} \right| \lesssim \left(\left| t^{-1/3}r'(ut^{-1/3}) \right| + \frac{\|r'\|_{L^2}}{t^{1/3}|\zeta t^{-1/3} - z_0|^{1/2}} + t^{-1/3} \bar{\partial}(\Xi_{\mathcal{Z}}(\zeta t^{-1/3})) \right) e^{-16u^2 v}.$$

We proceed as in the previous section and study the integral equation related to the $\bar{\partial}$ problem. Setting $z = \alpha + i\beta$ and $\zeta = (u + z_0 t^{1/3}) + iv$, the region Ω_1 corresponds to $u \geq v \geq 0$. We decompose the integral operator into three parts:

$$\int_{\Omega_1} \frac{1}{|z - \zeta|} |W(\zeta)| d\zeta \lesssim I_1 + I_2 + I_3$$

where

$$\begin{aligned} I_1 &= \int_0^\infty \int_v^\infty \frac{1}{|z - \zeta|} \left| t^{-1/3} r' \left(ut^{-1/3} \right) \right| e^{-16u^2 v} du dv, \\ I_2 &= \int_0^\infty \int_v^\infty \frac{1}{|z - \zeta|} \frac{1}{t^{1/3} |ut^{-1/3} + ivt^{-1/3}|^{1/2}} e^{-16u^2 v} du dv, \\ I_3 &= \int_0^\infty \int_v^\infty \frac{1}{|z - \zeta|} \left| t^{-1/3} \bar{\partial} \left(\Xi_{\mathcal{Z}}(\zeta t^{-1/3}) \right) \right| e^{-16u^2 v} du dv. \end{aligned}$$

We first note that

$$\left(\int_{\mathbb{R}} \left| t^{-1/3} r' \left(ut^{-1/3} \right) \right|^2 du \right)^{1/2} = t^{-1/6} \|r'\|_{L^2}$$

Using this and the following estimate from [9, proof of Proposition D.1]

$$(7.14) \quad \left\| \frac{1}{|z - \zeta|} \right\|_{L^2(v, \infty)} \leq \frac{\pi^{1/2}}{|v - \beta|^{1/2}}.$$

and Schwarz's inequality on the u -integration we may bound I_1 by constants times

$$t^{-1/6} \|r'\|_2 \int_0^\infty \frac{1}{|v - \beta|^{1/2}} e^{-v^3} dv \lesssim t^{-1/6}.$$

For I_2 , taking $p > 4$ and q with $1/p + 1/q = 1$, we estimate

$$\begin{aligned} \left\| \frac{1}{t^{1/3} |ut^{-1/3} + ivt^{-1/3}|^{1/2}} \right\|_{L^p(v, \infty)} &\leq \left(\int_v^\infty t^{-p/3} \left(\frac{1}{(ut^{-1/3})^2 + (vt^{-1/3})^2} \right)^{p/4} du \right)^{1/p} \\ &= t^{(3-p)/(3p)} \left(\int_v^\infty \left(\frac{1}{(ut^{-1/3})^2 + (vt^{-1/3})^2} \right)^{p/4} d(ut^{-1/3}) \right)^{1/p} \\ &= t^{(3-p)/(3p)} \left(\int_{v'}^\infty \left(\frac{1}{(u')^2 + (v')^2} \right)^{p/4} du' \right)^{1/p} \\ &\leq ct^{(3-p)/(3p)} v^{1/p-1/2} \\ &= ct^{(2/(3p)-1/6)} v^{1/p-1/2}. \end{aligned}$$

Now by (7.14) and an application of the Hölder inequality we get

$$\begin{aligned} |I_2| &\leq \int_0^\infty \left\| \frac{1}{t^{1/3} |ut^{-1/3} + ivt^{-1/3}|^{1/2}} \right\|_{L^p(v, \infty)} \left\| \frac{1}{|z - \zeta|} \right\|_{L^q(v, \infty)} e^{-16v^3} dv \\ &\leq c \int_0^\infty t^{(2/(3p)-1/6)} v^{1/p-1/2} |v - \beta|^{1/q-1} e^{-16v^3} dv \\ &\leq ct^{(2/(3p)-1/6)}. \end{aligned}$$

The estimate on I_3 is similar to that of I_1 and

$$|I_3| \leq ct^{-1/6}.$$

This proves that

$$\int_{\Omega_1} \frac{1}{|z - \zeta|} |W(\zeta)| d\zeta \lesssim t^{(2/(3p)-1/6)}$$

for all $4 < p < \infty$. We now show that

$$\int_{\Omega_1} |W(\zeta)| d\zeta \lesssim t^{(2/(3p)-1/6)}.$$

Again we decompose the integral above into three parts

$$\begin{aligned} I_1 &= \int_0^\infty \int_v^\infty \left| t^{-1/3} r' \left(ut^{-1/3} \right) \right| e^{-16u^2v} du dv \\ I_2 &= \int_0^\infty \int_v^\infty \frac{1}{t^{1/3} |ut^{-1/3} + ivt^{-1/3}|^{1/2}} e^{-16u^2v} du dv. \\ I_3 &= \int_0^\infty \int_v^\infty \left| t^{-1/3} \bar{\partial} \Xi_Z \left(\zeta t^{-1/3} \right) \right| e^{-16u^2v} du dv \end{aligned}$$

By Cauchy-Schwarz inequality:

$$\begin{aligned} I_1 &\leq \int_0^\infty t^{-1/6} \|r'\|_2 \left(\int_v^\infty e^{-16u^2v} du \right)^{1/2} dv \\ &\leq ct^{-1/6} \int_0^\infty \frac{e^{-16v^3}}{\sqrt[4]{v}} dv \\ &\leq ct^{-1/6}. \end{aligned}$$

By Hölder's inequality:

$$\begin{aligned} I_2 &\leq ct^{(2/(3p)-1/6)} \int_0^\infty v^{1/p-1/2} \left(\int_v^\infty e^{-16qu^2v} du \right)^{1/q} dv \\ &\leq ct^{(2/(3p)-1/6)} \int_0^\infty v^{3/(2p)-1} e^{-16v^3} dv \\ &\leq ct^{(2/(3p)-1/6)}. \end{aligned}$$

Again the estimate on I_3 is similar to that of I_1 and

$$I_3 \leq ct^{-1/6}.$$

We can apply the fundamental theorem of calculus to get

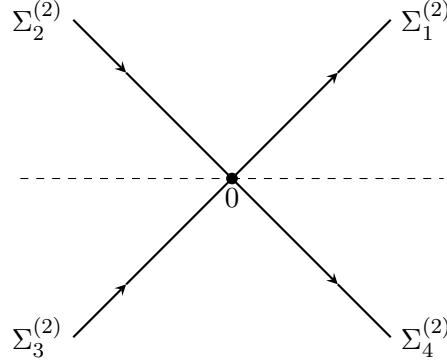
$$r(\zeta t^{-1/3}) \psi(\zeta t^{-1/3})^{-2} e^{2i\theta} - r(0) e^{2i\theta} \leq \left| \frac{\zeta}{t^{1/6}} e^{8i(\zeta^3 - 3\tau^{2/3}\zeta)} \right|.$$

Given the fact that $z_0 t^{1/3} = \tau^{1/3} \leq (M')^{1/3}$, we have that

$$\left\| \frac{\zeta}{t^{1/6}} e^{8i(\zeta^3 - 3\tau^{2/3}\zeta)} \right\|_{L^1 \cap L^2 \cap L^\infty} \lesssim t^{-1/6}.$$

Also notice that $\psi(0)^\pm = 1$ so we have reduce the problem to a problem on the following contour

FIGURE 7.3. $\Sigma^{(2)}$ -Scale



with jump matrices:

$$(7.15) \quad \begin{aligned} e^{-i\theta \operatorname{ad} \sigma_3} v^{(2)}(\zeta) &= e^{-4i(\zeta^3 + (x/(4t^{1/3}))\zeta) \operatorname{ad} \sigma_3} \begin{pmatrix} 1 & 0 \\ r(0) & 1 \end{pmatrix}, \quad \zeta \in \Sigma_1^{(2)} \cup \Sigma_2^{(2)} \\ &= e^{-4i(\zeta^3 + (x/(4t^{1/3}))\zeta) \operatorname{ad} \sigma_3} \begin{pmatrix} 1 & \overline{r(0)} \\ 0 & 1 \end{pmatrix}, \quad \zeta \in \Sigma_3^{(2)} \cup \Sigma_4^{(2)} \end{aligned}$$

In [24] and [29] the leading order term of the focusing mKdV is RHP is given by the solution to the Painlevé II equation:

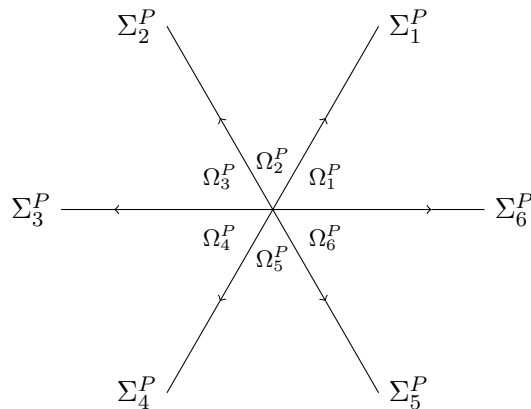
$$(7.16) \quad P''(s) - sP(s) + 2P^3(s) = 0.$$

In fact, by changing $P(s) \mapsto -iP(s)$, we have the solution to the following Painlevé II equation:

$$(7.17) \quad P''(s) - sP(s) - 2P^3(s) = 0.$$

The RHP on $\Sigma^{(2)}$ is related to Equation (7.17). We now follows the argument of [16, Section 5] and [17] to obtain the long-time asymptotic formula in Region II ($x < 0$).

FIGURE 7.4. Painlevé six ray



Associate to each ray Σ_i^P , $i = 1, 2, \dots, 6$, a jump matrix independent of ζ

$$(7.18) \quad \begin{cases} S_1 = \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}, & S_2 = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, & S_3 = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \\ S_4 = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}, & S_5 = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}, & S_6 = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \end{cases}$$

where p, q, r satisfies the constraint

$$(7.19) \quad p + q + r + pqr = 0.$$

To construct (7.18), we let $m^{(2)}(\zeta)$ denote the solution to the RHP on Figure 7.3 and set

$$(7.20) \quad m^{(3)}(\zeta) = m^{(2)}(\zeta), \quad \zeta \in \Omega_1^P \cup \Omega_2^P \cup \Omega_4^P \cup \Omega_4^P$$

$$(7.21) \quad = m^{(2)}(\zeta)e^{-i\theta \operatorname{ad} \sigma_3 v^{(2)}(\zeta)}, \quad \zeta \in \Omega_3^P$$

$$(7.22) \quad = m^{(2)}(\zeta) \left(e^{-i\theta \operatorname{ad} \sigma_3 v^{(2)}(\zeta)} \right)^{-1}, \quad \zeta \in \Omega_6^P.$$

From (1.16) we deduce that $r(0) = -\overline{r(0)}$, so $r(0)$ is purely imaginary. Setting

$$p = r(0), \quad q = \overline{r(0)}$$

from (7.19) we deduce

$$r = -(p + q)/(1 + pq) = 0.$$

We then observe that $\Psi = m^{(3)}(\zeta/3^{1/3})e^{-((4i/3)\zeta^3 + is\zeta)\sigma}$ satisfies the jump (7.18) on Σ^P given by Figure 7.4 with

$$\Psi_{i+1}(s, \zeta) = \Psi_i(s, \zeta)S_i, \quad 1 \leq i \leq 6$$

with $s = x/t^{1/3}$. From [17] we know that Ψ can be uniquely obtained and Ψ solves the following linear problem

$$\frac{d\Psi}{d\zeta} = \begin{pmatrix} -4i\zeta^2 - is - 2iP^2 & 4iP\zeta - 2P' \\ -4iP\zeta - 2P' & 4i\zeta^2 + is + 2iP^2 \end{pmatrix} \Psi$$

where $P(s)$ is a *purely imaginary* solution to (7.17). Indeed, we have

$$P = P(x/t^{1/3}, r(0)) = \lim_{\zeta \rightarrow \infty} 2i\zeta \left(\Psi e^{((4i/3)\zeta^3 + is\zeta)\sigma} - I \right)_{12}.$$

Finally, by sending $P \mapsto -iP$ and recalling the scaling (7.8) and combining the error term resulting from the $\bar{\partial}$ -extension, we arrive at the long time asymptotics in Region II:

$$(7.23) \quad u(x, t) = \frac{1}{(3t)^{1/3}} P \left(\frac{x}{(3t)^{1/3}} \right) + \mathcal{O} \left(t^{2/(3p)-1/2} \right)$$

where $4 < p < \infty$ and P is a *real* solution of the Painlevé II equation

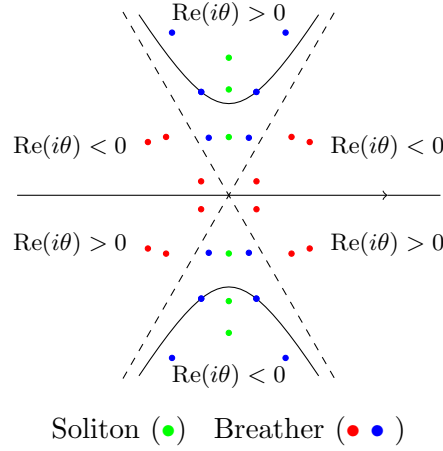
$$P''(s) - sP(s) + 2P^3(s) = 0.$$

7.2. Region III. In this region, $|x/t| = \mathcal{O}(1)$ as $t \rightarrow \infty$ and $x > 0$. We have the stationary points

$$\pm z_0 = \pm \sqrt{\frac{-x}{12t}} = \pm i \sqrt{\frac{|x|}{12t}}$$

which is purely imaginary and have a fixed distance from the real axis. The signature table of the phase function $i\theta$ is as follows:

FIGURE 7.5. Signature-solitons



We write

$$\operatorname{Re}i\theta(x, t; z) = t \left(4(-3u^2v + v^3) - \frac{x}{t}v \right)$$

then it is clear that if we set $x/t = v_{b_j} = 4\eta_j^2 - 12\xi_j^2$, then $\operatorname{Re}i\theta(x, t; z_j) = 0$. We again choose the frame of a single breather/soliton:

$$x/t = 4\eta_\ell^2 - 12\xi_\ell^2.$$

Define the following set

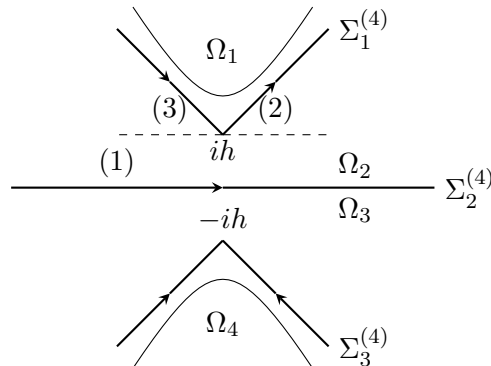
$$(7.24) \quad \mathcal{B}_\ell = \{z_j = \xi_j + i\eta_j : 4\eta_j^2 - 12\xi_j^2 > 4\eta_\ell^2 - 12\xi_\ell^2\} \cup \{z_k = i\zeta_k : 4\zeta_k^2 > 4\eta_\ell^2 - 12\xi_\ell^2\}.$$

and scalar function

$$(7.25) \quad \psi(z) = \left(\prod_{z_k \in \mathcal{B}_\ell} \frac{z - \bar{z}_k}{z - z_k} \right) \left(\prod_{z_j \in \mathcal{B}_\ell} \frac{z - \bar{z}_j}{z - z_j} \right) \left(\prod_{z_j \in \mathcal{B}_\ell} \frac{z + z_j}{z + \bar{z}_j} \right).$$

We follow the strategy of the previous subsection. If $m(z; x, t)$ solves Problem 1.4, then the new matrix-valued function $m^{(1)}(z; x, t) = m(z; x, t)\psi(z)^{\sigma_3}$ has exponentially decaying jumps across all small circles except $\pm\gamma_\ell \cup \pm\gamma_\ell^*$. Also we can deform \mathbb{R} as follows:

FIGURE 7.6. $\Sigma^{(4)}$ – solitons



where h is chosen such that $h > 0$ and

$$(7.26) \quad 4\eta_\ell^2 - 12\xi_\ell^2 - 12h^2 = c > 0.$$

On $\Sigma_1^{(4)}$ we set

$$R_1 = \begin{cases} \begin{pmatrix} 0 & 0 \\ r(z)\psi^{-2}(z)e^{2i\theta} & 0 \end{pmatrix} & z \in \mathbb{R} \\ \begin{pmatrix} 0 & 0 \\ r(0)\psi^{-2}(z)(1 - \Xi_{\mathcal{Z}})e^{2i\theta} & 0 \end{pmatrix} & z \in \Sigma_1^{(4)}. \end{cases}$$

We only study Ω_2 . In Part (1) of Ω_2 , we extend $r(z) = r(\operatorname{Re}z)$. Also in this region, by (7.26)

$$\begin{aligned} \operatorname{Re}(2i\theta(z)) &= 2t \left(4(-3u^2v + v^3) - \frac{x}{t}v \right) \\ &\leq -24u^2vt + 2 \left(4h^2 - \frac{x}{t} \right) vt \\ &\leq -24u^2vt - 2cvt. \end{aligned}$$

We now integrate and find that

$$(7.27) \quad \int_{(1)} \left| r'(u)e^{2i\theta(z)} \right| dz = \int_0^\eta \int_{-\infty}^\infty \left| r'(u)e^{-(24u^2v+2cv)t} \right| dudv$$

$$(7.28) \quad \begin{aligned} &\lesssim \int_0^\infty \frac{e^{-2cvt}}{\sqrt{vt}} dv \\ &\lesssim t^{-1}. \end{aligned}$$

In Part (2) of Ω_2 , we interpolate

$$(r(0) + (r(\operatorname{Re}z) - r(0)) \cos 2\phi) \psi^{-2}(z)(1 - \Xi_{\mathcal{Z}})$$

and calculate

$$\begin{aligned} \bar{\partial}R_1 &= \left(r'(u) \cos 2\phi - 2 \frac{r(u) - r(0)}{|z|} e^{i\phi} \sin 2\phi \right) \psi^{-2}(z)e^{2i\theta} \\ &\quad \times (1 - \Xi_{\mathcal{Z}}) \\ &\quad - (r(0) + (r(u) - r(0)) \cos 2\phi) \bar{\partial}(\Xi_{\mathcal{Z}}(z)) \psi^{-2}(z)e^{2i\theta} \end{aligned}$$

Also in this region, changing variable $v \mapsto v + h$, from (7.26) and the fact that $u \geq v \geq 0$,

$$\begin{aligned} \operatorname{Re}(2i\theta(z)) &= 2t \left(4(-3u^2(v+h) + (v+h)^3) - \frac{x}{t}(v+h) \right) \\ &\leq 2t \left(-8u^2v + \left(12h^2 - \frac{x}{t} \right) v + \left(4h^2 - \frac{x}{t} \right) h \right) \\ &\leq -16u^2vt - 2cvt. \end{aligned}$$

Since

$$(7.29) \quad |W(z)| = \left| \bar{\partial}R_1 e^{2i\theta} \right| \lesssim \left(|r'(u)| + \frac{\|r'\|_{L^2}}{|z|^{1/2}} + \bar{\partial}(\Xi_{\mathcal{Z}}(z)) \right) e^{-16u^2vt - 2cvt}$$

we still have

$$\int_{(2)} |W(z)| dz = I_1 + I_2 + I_3$$

with

$$I_1 = \int_0^\infty \int_v^\infty |r'(u)| e^{-16u^2vt - 2cvt} du dv,$$

$$I_2 = \int_0^\infty \int_v^\infty \frac{1}{|u + i(v+h)|^{1/2}} e^{-16u^2vt - 2cvt} du dv,$$

$$I_3 = \int_0^\infty \int_v^\infty |\bar{\partial}\Xi_{\mathcal{Z}}(u + i(v+h))| e^{-16u^2vt - 2cvt} du dv.$$

Direct calculation gives

$$(7.30) \quad \int_{(2)} |W(z)| dz \lesssim t^{-1}.$$

Notice that in Figure 7.6, we have deformed \mathbb{R} into \mathbb{C}^\pm . So the jump matrices across $\Sigma_1^{(4)} \cup \Sigma_3^{(4)}$ enjoy the property of exponential decay as $t \rightarrow +\infty$. Thus the reflection coefficient r makes no contribution to the leading order term in Region III. If we choose the frame $x/t = 4\eta_\ell^2 - 12\xi_\ell^2 > 0$, then by solving Problem 4.4 (with δ replaced by ψ) we obtain the following asymptotic formula in Region III:

$$(7.31) \quad u(x, t) = -4 \frac{\eta_\ell \xi_\ell \cosh(\nu_2 + \tilde{\omega}_2) \sin(\nu_1 + \tilde{\omega}_1) + \eta_\ell \sinh(\nu_2 + \tilde{\omega}_2) \cos(\nu_1 + \omega_1)}{\xi_\ell \cosh^2(\nu_2 + \tilde{\omega}_2) + (\eta_\ell/\xi_\ell)^2 \cos^2(\nu_1 + \tilde{\omega}_1)} + \mathcal{O}(t^{-1})$$

with

$$\nu_1 = 2\xi_\ell(x + 4(\xi_\ell^2 - 3\eta_\ell^2)t)$$

$$\nu_2 = 2\eta_\ell(x - 4(\eta_\ell^2 - 3\xi_\ell^2)t).$$

And

$$(7.32) \quad \tan \tilde{\omega}_1 = \frac{\tilde{B}\xi_\ell - \tilde{A}\eta_\ell}{\tilde{A}\xi_\ell + \tilde{B}\eta_\ell}$$

$$(7.33) \quad e^{-\tilde{\omega}_2} = \left| \frac{\xi_\ell}{2\eta_\ell} \right| \sqrt{\frac{\tilde{A}^2 + \tilde{B}^2}{\xi_\ell^2 + \eta_\ell^2}}$$

where we set

$$\tilde{c}_\ell = c_\ell \psi(z_\ell)^{-2} = \tilde{A} + i\tilde{B}.$$

Similarly, if we choose the frame $x/t = 4\zeta_\ell^2$, the velocity of the l -th soliton, then

$$(7.34) \quad u(x, t) = 2\zeta_\ell \varepsilon_{\pm, \ell} \operatorname{sech}(-2\zeta_\ell(x - 4\zeta_\ell^2 t) + \omega) + \mathcal{O}(t^{-1})$$

where

$$\omega = \log \left(\frac{|c_\ell|}{2\zeta_\ell} \right) + 2 \sum_{z_k \in S_\ell} \log \left| \frac{z_k - z_\ell}{z_\ell - \bar{z}_k} \right| + 2 \sum_{z_j \in S_\ell} \log \left| \frac{z_\ell - z_j}{z_\ell - \bar{z}_j} \right| + 2 \sum_{z_j \in S_\ell} \log \left| \frac{z_\ell + \bar{z}_j}{z_\ell + z_j} \right|.$$

Here S_ℓ is defined by

$$S_\ell = \{z_j = \xi_j + i\eta_j : 4\eta_j^2 - 12\xi_j^2 > 4\zeta_\ell^2\} \cup \{z_k = i\zeta_k : 4\zeta_k^2 > 4\zeta_\ell^2\}.$$

8. GLOBAL APPROXIMATIONS OF SOLUTIONS

In this section, as in our earlier work, [10], we apply a global approximation arguments to extend our long-time asymptotics to the focusing mKdV with rougher data. Again two important spaces are $H^1(\mathbb{R})$ and $H^{\frac{1}{4}}(\mathbb{R})$. In $H^1(\mathbb{R})$, the mKdV enjoys the natural conservation. For $H^{\frac{1}{4}}(\mathbb{R})$, this space is the lowest regularity that the solution can be constructed by iterations, see Theorem 8.2. We will show that the long-time asymptotics and soliton resolution remain valid in spaces $H^s(\mathbb{R})$ for $s \geq \frac{1}{4}$ after we pay the price of weights. Unlike in [10], here we deal with $H^s(\mathbb{R})$ with $s \geq \frac{1}{4}$ in a uniform manner using the local well-posedness due to Kenig-Ponce-Vega [35], see Theorem 8.2, and the new construction of conservation laws in H^s with $s > -\frac{1}{2}$ due to the recent work of Killip-Visan-Zhang [37] and Koch-Tataru [39].

In contrast to the defocusing problem in our earlier paper, [10], in the focusing problem, the Beals-Coifman solutions are more complicated and need to take care of solitons and breathers. To implement our approximation arguments, not only we need to ensure the radiation terms converge well but also have to make sure the discrete scattering data including the number and locations of singularities have meaningful limits. More refined analysis of Jost functions is necessary. See Appendix A.

8.1. Strong solutions. For the sake of completeness, as in [10], we first sketch the uniqueness of the strong solution and the local existence of the strong solution for the focusing mKdV in H^s for $s \geq \frac{1}{4}$. The following discussion will be similar to our earlier work [10] up to the change of the sign of the nonlinearity. We mainly follow Kenig-Ponce-Vega [35] and Linares-Ponce [41].

First of all, we define the solution operator to the linear Airy function as

$$W(t)u_0 = e^{-t\partial_{xxx}}u_0.$$

In other words, using the Fourier transform, one has

$$\mathcal{F}_x[W(t)u_0](\xi) = e^{it\xi^3}\hat{u}_0(\xi).$$

The *strong solution* is defined as the following integral sense:

Definition 8.1. We say the the function $u(t, x)$ is a *strong solution* in $H^s(\mathbb{R})$ to the focusing mKdV

$$(8.1) \quad \partial_t u + \partial_{xxx}u + 6u^2\partial_x u = 0, \quad u(0) = u_0 \in H^s(\mathbb{R})$$

if and only if $u \in C(I, H^s(\mathbb{R}))$ satisfies

$$(8.2) \quad u = W(t)u_0 - \int_0^t W(t-s)(6u^2\partial_x u(s)) ds.$$

We also define

$$\mathcal{D}_x^s h(x) = \mathcal{F}^{-1}\left[|\xi|^s \hat{h}(\xi)\right](x).$$

Theorem 8.2 (Kenig-Ponce-Vega). *Let $s \geq \frac{1}{4}$. Then for any $u_0 \in H^s(\mathbb{R})$ there is $T = T\left(\left\|\mathcal{D}_x^{\frac{1}{4}}u_0\right\|_{L^2}\right) \sim \left\|\mathcal{D}_x^{\frac{1}{4}}u_0\right\|_{L^2}^{-4}$ such that there exists a unique strong solution $u(t)$ to the initial-value problem*

$$\partial_t u + \partial_{xxx}u + 6u^2\partial_x u = 0, \quad u(0) = u_0$$

satisfying

$$(8.3) \quad u \in C([-T, T] : H^s(\mathbb{R}))$$

$$(8.4) \quad \|\mathcal{D}_x^s \partial_x u\|_{L_x^\infty(\mathbb{R}; L_t^2[-T, T])} < \infty,$$

$$(8.5) \quad \left\|\mathcal{D}_x^{s-\frac{1}{4}}\partial_x u\right\|_{L_x^{20}(\mathbb{R}; L_t^{\frac{5}{2}}[-T, T])} < \infty,$$

$$(8.6) \quad \|\mathcal{D}_x^s u\|_{L_x^5(\mathbb{R}; L_t^{10}[-T, T])} < \infty,$$

and

$$(8.7) \quad \|u\|_{L_x^4(\mathbb{R}; L_t^\infty[-T, T])} < \infty.$$

Moreover, there exists a neighborhood \mathcal{N} of u_0 in $H^s(\mathbb{R})$ such that the solution map: $\tilde{u}_0 \in \mathcal{N} \mapsto \tilde{u}$ is smooth with respect to the norms given by (8.3), (8.4), (8.5), (8.6) and (8.7).

Proof. Given T and \mathcal{C} , define the space

$$(8.8) \quad \mathcal{X}_T^s = \left\{ v \in C([-T, T] : H^s(\mathbb{R})) : \|v\|_{\mathcal{X}_T^s} < \infty \right\}$$

and

$$(8.9) \quad \mathcal{X}_{T, \mathcal{C}}^s = \left\{ v \in C([-T, T] : H^s(\mathbb{R})) : \|v\|_{\mathcal{X}_T^s} \leq \mathcal{C} \right\}$$

where

$$\begin{aligned} \|v\|_{\mathcal{X}_T^s} &= \|\mathcal{D}_x^s v\|_{L_t^\infty([-T, T] : H^s(\mathbb{R}))} + \|v\|_{L_x^4(\mathbb{R} : L_t^\infty[-T, T])} \\ &\quad + \|\mathcal{D}_x^s v\|_{L_x^5(\mathbb{R} : L_t^{10}[-T, T])} + \left\| \mathcal{D}_x^{s-\frac{1}{4}} \partial_x v \right\|_{L_x^{20}(\mathbb{R} : L_t^{\frac{5}{2}}[-T, T])} \\ &\quad + \|\mathcal{D}_x^s \partial_x v\|_{L_x^\infty(\mathbb{R} : L_t^2[-T, T])}. \end{aligned}$$

To obtain a strong solution to the initial-value problem we need find appropriate T and \mathcal{C} such that the operator

$$\mathcal{S}(v, u_0) = \mathcal{S}(v) = W(t) u_0 - \int_0^t W(t-s) (6v^2 \partial_x v(s)) ds$$

is a contraction map on $\mathcal{X}_{T, \mathcal{C}}^s$.

Using linear estimates for $W(t)$ and the Leibniz rule for fractional derivatives one can show that

$$\|\mathcal{S}(v)\|_{\mathcal{X}_T^s} \leq c \|u_0\|_{H^s} + cT^{\frac{1}{2}} \|v\|_{\mathcal{X}_T^s}^3$$

where c is from linear estimates etc independent of the initial data. See Kenig-Ponce-Vega [35] and Linares-Ponce [41] for details. Then choose $\mathcal{C} = 2c \|u_0\|_{H^s}$ and T such that $c\mathcal{C}^2 T^{\frac{1}{2}} < \frac{1}{4}$, we obtain that

$$\mathcal{S}(\cdot, u_0) : \mathcal{X}_{T, \mathcal{C}}^s \rightarrow \mathcal{X}_{T, \mathcal{C}}^s.$$

Similarly, one can also show

$$\begin{aligned} \|\mathcal{S}(v_1) - \mathcal{S}(v_2)\|_{\mathcal{X}_T^s} &\leq cT^{\frac{1}{2}} \left(\|v_1\|_{\mathcal{X}_T^s}^2 + \|v_2\|_{\mathcal{X}_T^s}^2 \right) \|v_1 - v_2\|_{\mathcal{X}_T^s} \\ &\leq 2cT^{\frac{1}{2}} \mathcal{C}^2 \|v_1 - v_2\|_{\mathcal{X}_T^s}. \end{aligned}$$

Therefore, with our choice of T and \mathcal{C} , $\mathcal{S}(\cdot, u_0)$ is a contraction on $\mathcal{X}_{T, \mathcal{C}}^s$. So there is a unique fixed point of this $\mathcal{S}(\cdot, u_0)$ in $\mathcal{X}_{T, \mathcal{C}}^s$ so we obtain the unique strong solution:

$$u = \mathcal{S}(u) = W(t) u_0 - \int_0^t W(t-s) (6u^2 \partial_x u(s)) ds.$$

To check the dependence on the initial data, using similar arguments as above, one can show that

$$\begin{aligned} \|\mathcal{S}(u_1, u_1(0)) - \mathcal{S}(u_2, u_2(0))\|_{\mathcal{X}_{T_1}^s} &\leq c \|u_1(0) - u_2(0)\|_{H^s} \\ &\quad + cT_1^{\frac{1}{2}} \left(\|u_1\|_{\mathcal{X}_{T_1}^s}^2 + \|u_2\|_{\mathcal{X}_{T_1}^s}^2 \right) \|u_1 - u_2\|_{\mathcal{X}_{T_1}^s}. \end{aligned}$$

This can be used to show that for $T_1 \in (0, T)$, the solution map from a neighborhood \mathcal{N} of u_0 depending on T_1 to $\mathcal{X}_{T_1, \mathcal{C}}^s$ is Lipschitz. Further work can be used to show actually the solution map is smooth.

Again for more details, see Kenig-Ponce-Vega [35] and Linares-Ponce [41]. \square

Finally, we notice that if u_0 is smooth, say, Schwartz, then the solution u to the initial-value problem is also smooth and hence is a classical solution. The uniqueness of the classical solution is well-known, see for example Bona-Smith [6] and Saut [52].

Next, we recall the consequences of low regularity conservation laws from Killip-Visan-Zhang [37] and Koch-Tataru [39]. Here we formulate the Corollary 1.2 from Koch-Tataru [39].

Theorem 8.3 (Koch-Tataru). *Let $s > -\frac{1}{2}$, let $R > 0$, and let u_0 be an initial datum for the focusing mKdV*

$$(8.10) \quad \partial_t u + \partial_{xxx} u + 6u^2 \partial_x u = 0,$$

so that

$$\|u_0\|_{H^s} \leq R.$$

Then the corresponding solution u satisfies that global bound

$$\|u(t)\|_{H^s} \lesssim F(R, s) = \begin{cases} R + R^{1+2s} & s \geq 0 \\ R + R^{\frac{1+4s}{1+2s}} & s < 0 \end{cases}.$$

We will use the above results for $s \geq \frac{1}{4}$.

8.2. Approximation. As before given $u_0 \in H^{2,1}(\mathbb{R})$, one can solve the focusing mKdV using the inverse scattering transform. By the Beals-Coifman representation, one can write the solution as

$$\begin{aligned} u(x, t) &= \lim_{z \rightarrow \infty} 2zm_{12}(x, t, z) \\ &= \left(\int_{\Sigma} \mu(w_{\theta}^{-} + w_{\theta}^{+}) \right)_{12} \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \mu_{11}(x, t; z) \bar{r}(z) e^{-2i\theta} dz + \sum_{k=1}^{N_1} \mu_{11}(\bar{z}_k) \bar{c}_k e^{-2i\theta(\bar{z}_k)} \\ &\quad - \sum_{j=1}^{N_2} \mu_{11}(\bar{z}_j) \bar{c}_j e^{-2i\theta(\bar{z}_j)} + \sum_{j=1}^{N_2} \mu_{11}(-z_j) c_j e^{-2i\theta(-z_j)}. \end{aligned}$$

But as we discussed above using PDE techniques, one can construct solutions with rougher data at least locally. Motivated by Deift-Zhou [18], as in our earlier paper [10], we try to understand the relations between Beals-Coifman solutions and strong solutions. Again first of all, if u_0 is Schwartz, one can also show u is Schwartz, see for example Deift-Zhou [16]. So in this case, the strong solution is surely the same as the Beals-Coifman solution.

The leading logic of our approximation argument is that Beals-Coifman solutions gives asymptotic formulae and the strong solutions can be used to pass to the limit. Using the bijectivity between the initial data and scattering data due to Zhou [59], one can show in low regularity spaces with weights, one can always find a limit of a sequence of smooth sequence of Beals-Coifman solutions which has asymptotic formula. But due to low regularity, one can not make sense of this limiting function using the inverse scattering mechanism. On the other hand, due to the uniqueness of smooth solutions, these smooth Beals-Coifman solutions are also strong solutions. Then one can use strong solutions to pass to the limit which again is a strong solution to the mKdV. Combining these together, we conclude that the asymptotic formula remains valid for low regularity solutions. In the reaming part of our paper, we make the philosophy rigorous. Compared with our earlier work [10] for the defocusing problem, we have discrete scattering data associated to solitons and breathers in our current setting.

Theorem 8.4. *For $u_0 \in H^{s,1}(\mathbb{R})$ with $s \geq \frac{1}{4}$, the strong solution given by the Duhamel formulation (8.2) with initial data u_0 has the same asymptotics as in our main Theorem 1.10 (up to null sets).*

Proof. Suppose $u_0 \in H^{s,1}(\mathbb{R})$ with $s \geq \frac{1}{4}$, we can find a sequence $\{u_{0,q}\}$ of Schwartz functions such that it is a Cauchy sequence in $H^{s,1}(\mathbb{R})$ and $u_{0,q} \rightarrow u_0$ in $H^{s,1}(\mathbb{R})$ and

$$(8.11) \quad \sup_q \|u_{0,q} - u_0\|_{H^{s,1}} \leq \epsilon \ll 1.$$

Moreover, we assume that for all q , there is a uniform bound

$$\|u_{0,q}\|_{\dot{H}^s(\mathbb{R})} \lesssim \|u_{0,q}\|_{H^s(\mathbb{R})} \lesssim \|u_{0,q}\|_{H^{s,1}(\mathbb{R})} \leq C.$$

Then applying Theorem 8.2 we can find a strong solution u_q with initial data $u_{0,q}$ in $\mathcal{X}_{T,C}^s$ where T and C are chosen as in Theorem 8.2.

By Theorem 8.2, we also have

$$\|u_q - u_\ell\|_{\mathcal{X}_{T,C}^s} \lesssim \|u_{0,q} - u_{0,\ell}\|_{H^s(\mathbb{R})}.$$

So in $\mathcal{X}_{T,C}^s$, u_q converges to a limit u_∞ which is a strong solution. Using the notation above, we have

$$u_\infty = \mathcal{S}(u_\infty, u_0) \in \mathcal{X}_{T,C}^s.$$

From the inverse scattering transform, we can also have the Beals-Coifman solutions

$$\begin{aligned} \tilde{u}_q &= \frac{1}{\pi} \int_{\mathbb{R}} \mu_{q,11}(x, t; z) \bar{r}_q(z) e^{-2i\theta} dz + \sum_{k=1}^{N_1} \mu_{q,11}(\bar{z}_{q,k}) \bar{c}_{q,k} e^{-2i\theta(\bar{z}_{q,k})} \\ &\quad - \sum_{j=1}^{N_2} \mu_{q,11}(\bar{z}_{q,j}) \bar{c}_{q,j} e^{-2i\theta(\bar{z}_{q,j})} + \sum_{j=1}^{N_2} \mu_{q,11}(-z_{q,j}) c_{q,j} e^{-2i\theta(-z_{q,j})} \end{aligned}$$

with initial data $u_{0,q}$. Note that due to Proposition A.2 and the smallness condition (8.11), the numbers of zeros of $\check{a}_q(z)$ and $\check{a}(z)$ are the same, in particular, the numbers of the imaginary zeros are N_1 and the numbers of zeros of the imaginary axis are N_2 (up to symmetry reduction).

Since $u_{0,q}$ is Schwartz, so u_q and \tilde{u}_q are also Schwartz. Therefore we have $u_q = \tilde{u}_q$.

Using the bijectivity of the direct transformation, see Zhou [59], in terms of reflection coefficients,

$$r_q = \mathcal{R}(u_{0,q}) \in H^{1,s},$$

we have

$$\|r_q - r_\ell\|_{H^{1,s}(\mathbb{R})} \lesssim \|u_{0,q} - u_{0,\ell}\|_{H^{s,1}(\mathbb{R})}.$$

By the Lipschitz continuity of discrete scattering data, it follows that

$$|c_{q,k} - c_{\ell,k}| + |z_{q,k} - z_{\ell,k}| \lesssim \|u_{0,q} - u_{0,\ell}\|_{H^{s,1}(\mathbb{R})}, \quad 1 \leq k \leq N_1,$$

and

$$|c_{q,j} - c_{\ell,j}| + |z_{q,j} - z_{\ell,j}| \lesssim \|u_{0,q} - u_{0,\ell}\|_{H^{s,1}(\mathbb{R})}, \quad 1 \leq j \leq N_2.$$

Combining with the resolvent estimates, one also has

$$\|\tilde{u}_\ell - \tilde{u}_q\|_{L^\infty(\mathbb{R})} \lesssim \|u_{0,q} - u_{0,\ell}\|_{H^{s,1}(\mathbb{R})}.$$

Hence as r_q converges to a function r_∞ in $H^1(\mathbb{R})$ and with the convergence of discrete scattering data $\{c_{q,k}, z_{q,k}, c_{q,j}, z_{q,j}\}$ to $\{c_{\infty,k}, z_{\infty,k}, c_{\infty,j}, z_{\infty,j}\}$, the corresponding Beals-Coifman solution converges to a limit

$$\tilde{u}_\infty = \lim_{q \rightarrow \infty} \tilde{u}_q$$

in the sense of the L^∞ norm. Indeed, we can write

$$\begin{aligned}
\tilde{u}_q &= \frac{1}{\pi} \int_{\mathbb{R}} (\mu_{q,11}(x, t; z) - I) \bar{r}_q(z) e^{-2i\theta} dz \\
&+ \frac{1}{\pi} \int_{\mathbb{R}} \bar{r}_q(z) e^{-2i\theta} dz \\
&+ \sum_{k=1}^{N_1} \mu_{q,11}(\bar{z}_{q,k}) \bar{c}_{q,k} e^{-2i\theta(\bar{z}_{q,k})} \\
&- \sum_{j=1}^{N_2} \mu_{q,11}(\bar{z}_{q,j}) \bar{c}_{q,j} e^{-2i\theta(\bar{z}_{q,j})} + \sum_{j=1}^{N_2} \mu_{q,11}(-z_{q,j}) c_{q,j} e^{-2i\theta(-z_{q,j})} \\
&= \text{I}_q + \text{II}_q + \text{III}_q.
\end{aligned}$$

where

$$\begin{aligned}
\text{I}_q &= \frac{1}{\pi} \int_{\mathbb{R}} (\mu_{q,11}(x, t; z) - I) \bar{r}_q(z) e^{-2i\theta} dz \\
\text{II}_q &= \frac{1}{\pi} \int_{\mathbb{R}} \bar{r}_q(z) e^{-2i\theta} dz
\end{aligned}$$

and

$$\begin{aligned}
\text{III}_q &= \sum_{k=1}^{N_1} \mu_{q,11}(\bar{z}_{q,k}) \bar{c}_{q,k} e^{-2i\theta(\bar{z}_{q,k})} \\
&- \sum_{j=1}^{N_2} \mu_{q,11}(\bar{z}_{q,j}) \bar{c}_{q,j} e^{-2i\theta(\bar{z}_{q,j})} + \sum_{j=1}^{N_2} \mu_{q,11}(-z_{q,j}) c_{q,j} e^{-2i\theta(-z_{q,j})}.
\end{aligned}$$

Then due to the resolvent estimate, $(\mu_q - I)$ has the L^2 estimate and the L^2 estimate for $\bar{r}_q(z) e^{-2i\theta}$ is straightforward, so I_q can be made sense pointwise. For II_q , one simply notices that $\int \frac{1}{\pi} \int_{\mathbb{R}} \bar{r}_q(z) e^{-2i\theta} dz$ is proportional to $W(t) \check{r}_q = e^{-t\partial_{xxx}} \check{r}_q$, by the standard stationary phase analysis, for $r_q \in H^1$, II_q is a function in $L^\infty(\mathbb{R})$ with the standard pointwise decay estimates for the Airy equation. Finally, since $\mu_q(x, z) \in H^1$, by Sobolev's embedding, we can evaluate III_q pointwise.

Hence as

$$\begin{aligned}
|c_{q,k} - c_{\infty,k}| + |z_{q,k} - z_{\infty,k}| &\rightarrow 0, \quad 1 \leq k \leq N_1, \\
|c_{q,j} - c_{\infty,j}| + |z_{q,j} - z_{\infty,j}| &\rightarrow 0, \quad 1 \leq j \leq N_2
\end{aligned}$$

and

$$\|r_q - r_\infty\|_{H^1} \rightarrow 0$$

one has

$$\|\tilde{u}_q - \tilde{u}_\infty\|_{L^\infty} \rightarrow 0 \text{ as } q \rightarrow \infty.$$

It follows that as $q \rightarrow \infty$, we have

$$\|u_q - u_\infty\|_{\mathcal{X}_{T,C}^s} = \|\tilde{u}_q - u_\infty\|_{\mathcal{X}_{T,C}^s} \rightarrow 0.$$

In particular, as $q \rightarrow \infty$, one has

$$\sup_{t \in [-T, T]} \|u_q - u_\infty\|_{H^s(\mathbb{R})} = \sup_{t \in [-T, T]} \|\tilde{u}_q - u_\infty\|_{H^s(\mathbb{R})} \rightarrow 0.$$

By construction, as $q \rightarrow \infty$,

$$\sup_{t \in [-T, T]} \|\tilde{u}_q - \tilde{u}_\infty\|_{L^\infty(\mathbb{R})} \rightarrow 0.$$

Hence for given $t \in [-T, T]$, one has

$$u_\infty(t) = \tilde{u}_\infty(t)$$

up to a measure zero set. By construction, we moreover have

$$u(t) = u_\infty(t) = \tilde{u}_\infty(t)$$

up to null sets.

Next, applying the consequence of the conservation law for H^s , Theorem 8.3, we can repeat the above construct infinity many times to extend the interval $[-T, T]$ to \mathbb{R} and conclude that for $t \in \mathbb{R}_+$,

$$u(t) = u_\infty(t) = \tilde{u}_\infty(t).$$

For fixed t , by our resolution formula, one can write the solution as a superposition of breathers, solitons, radiation

$$\tilde{u}_q(t) = \sum_{j=1}^{N_2} u_{q,j}^{(br)}(x, t) + \sum_{k=1}^{N_1} u_{q,k}^{(so)}(x, t) + R_q(x, t).$$

Moreover, for the radiation, we can write

$$R_q(x, t) = L_q(x, t) + E_q(x, t)$$

where $L_q(x, t)$ gives the leading order behavior and $E_q(x, t)$ collects the error term, see Theorem 1.10. By the convergence of scattering data, we know

$$\sum_{j=1}^{N_2} u_{q,j}^{(br)}(x, t) + \sum_{k=1}^{N_1} u_{q,k}^{(so)}(x, t) + L_q(x, t) \rightarrow \sum_{j=1}^{N_2} u_{\infty,j}^{(br)}(x, t) + \sum_{k=1}^{N_1} u_{\infty,k}^{(so)}(x, t) + L_\infty(x, t)$$

pointwise. Also note that by our computations, the error terms estimates only depend on the H^1 norm of r_q which by bijectivity only depend on the $\|u_{0,q}\|_{H^{0,1}(\mathbb{R})} \leq C$ uniformly.

Therefore for an arbitrary fixed t , as the pointwise limit of $\tilde{u}_q(t)$, one can write

$$\tilde{u}_\infty(t) = \sum_{\ell=1}^{N_2} u_{\infty,\ell}^{(br)}(x, t) + \sum_{\ell=1}^{N_1} u_{\infty,\ell}^{(so)}(x, t) + L_\infty(x, t) + E_\infty(x, t)$$

where the decay estimates for $E_\infty(x, t)$ is the same as $E_q(x, t)$ due to the uniform error estimates.

Hence up the null sets, one can write

$$u(t) = \sum_{\ell=1}^{N_2} u_{\infty,\ell}^{(br)}(x, t) + \sum_{\ell=1}^{N_1} u_{\infty,\ell}^{(so)}(x, t) + L_\infty(x, t) + E_\infty(x, t)$$

which has the same form as Theorem 1.10. \square

Remark 8.5. Finally, we should point out that one essential step in our approximation argument is that the convergence of initial data gives the convergence of scattering data due to Zhou's bijectivity results and the leading order terms of solutions can be computed precisely using these scattering data. All of these computations are independent of t . Finally, since the error terms have estimates uniformly depending on the weighted norms of initial data, one can conclude the asymptotics of the limit. All of these use the machinery of the inverse scattering. From the view of PDEs, one can also compute the leading order terms using some ODE arguments see Hayashi-Naumkin [29, 30] and Germain-Pusateri-Rousset [24]. But to find these leading order terms, it introduces extra error terms. If one try to use approximation argument in this setting, it is not clear these extra error terms give decay fast enough.

APPENDIX A. CONTINUITY OF THE DISCRETE SCATTERING DATA

In this Appendix, we recall some results concerning the continuity of Jost functions with respect to the potentials. The generic condition for initial data will also be briefly discussed.

Recall that we have

$$a(z) = 1 - \int_{\mathbb{R}} iu(y) m_{21}^+(y, z) dy,$$

$$\check{a}(z) = 1 - \int_{\mathbb{R}} iu(y) m_{12}^+(y, z) dy,$$

$$\check{b}(z) = 1 - \int_{\mathbb{R}} iu(y) m_{11}^+(y, z) dy,$$

$$b(z) = 1 - \int_{\mathbb{R}} iu(y) m_{21}^+(y, z) dy.$$

Note that $a(z)$ and $\check{a}(z)$ are independent of t . So actually, we can replace the u in the above formulae by u_0 and m_{21}^+, m_{12}^+ by the corresponding solutions constructed with respect to u_0 .

Denote $a(z, u_1)$ and $a(z, u_2)$ as the a component of the scattering matrix constructed using u_1 and u_2 respectively. Using the continuous dependence on the potential, we obtain that

$$|a(z, u_1) - a(z, u_2)| \lesssim \|u_1 - u_2\|_{L^{2,1}}.$$

Therefore, we notice that if we have a sequence $u_n \in L^{2,1}$ converges to a function $u_0 \in L^{2,1}$, then the zeros of $a(z, u_j)$ will converge pointwise to $z_{\infty, \ell}$. (here we have two options, one is since u_n has a uniform upper bound, then $z_{j, \ell}$ is bounded sequence in the complex plane. Then there is a subsequence converges to a limit. Secondly, we can use the continuity to get the convergence and moreover, the limit is unique).

The same arguments apply to $\check{a}(z)$, $b(z)$ and $\check{b}(z)$ associated with the initial data. By similar arguments, one can also get the initial norming constant for each singularity also enjoys the similar properties.

We record the following proposition to show there is a dense subset of $u \in L^{2,1}(\mathbb{R})$ such that $a(z; u)$ has at most finitely many simple zeros in \mathbb{C}^- and no zeros on \mathbb{R} .

Proposition A.1. *Suppose $R > 0$ and $u \in C_0^\infty([-R, R])$. Let $a(z; u)$ be the $(1, 1)$ entry of the scattering matrix for u . For $\varphi \in C_0^\infty(\mathbb{R})$, denote $a(z, \mu)$ as the $(1, 1)$ entry of the scattering matrix for $u + \mu\varphi$. By construction, $a(z, 0) = a(z; u)$.*

(1) *Suppose $S = \{z_i\}_{i=1}^N$ are the isolated zeros of $a(z; u)$ in $\mathbb{C}^- \cup \mathbb{R}$ and for some i such that $z_i \neq 0$ is one of the zeros of $a(z; u)$ of multiplicity $\lambda \geq 2$, in other words, $a(z; u) = (z - z_i)^\lambda g(z)$ for some analytic function $g(z)$ with $g(z_i) \neq 0$. Then for some $\varphi \in C_0^\infty(\mathbb{R})$ and all sufficiently small $\mu \neq 0$, $a(z, \mu)$ has λ simple zeros in the disc $D_{r_i}(z_i)$ with r_i small enough.*

(2) *Suppose that after the perturbation in (1), for small j , Z_j is a simple zero of $a(z, \mu)$ on the real axis such that $Z_j \neq 0$. Then for some $\varphi \in C_0^\infty(\mathbb{R})$ and all sufficiently small $\mu' \neq 0$, $a(z, \mu')$ has no zeros on the real axis near Z_j .*

In each case, one can choose φ to have support in $(-2R, -R) \cup (R, 2R)$.

Since as we discussed above, $a(z; u)$ is continuous with respect to u and $a(z; u)$ is analytic in \mathbb{C}^- , this set is also open.

Note that due to symmetry, a breather corresponds two simple zeros off the imaginary axis. A simple zero on the imaginary axis results a soliton. These structure is fairly stable since the zero of the soliton is simple. It will not bifurcate into two simple zeros under small perturbations.

More precisely, we focus on $\check{a}(z; u_0)$. This is analytic in the \mathbb{C}^+ . By our assumption, $\check{a}(z; u_0)$ has exactly N_1 simple zeros on the imaginary axis, $2N_2$ simple zeros off the imaginary axis and no zeros on \mathbb{R} . Focusing on one zero on the imaginary axis, say, $z_i(u_0)$, we consider the integral

of $\frac{\check{a}'(z)}{\check{a}(z)}$ over a small circle \mathcal{C}_i with radius small enough centered at $z_i(u_0)$, then by Cauchy's argument principle,

$$\frac{1}{2\pi} \int_{\mathcal{C}_i} \frac{\check{a}'(z; u_0)}{\check{a}(z; u_0)} dz = 1$$

due to the analyticity of $\check{a}(z; u_0)$ and the simplicity of $z_i(u_0)$. Under a sufficiently small perturbation, the zeros of $\check{a}(z; u)$ corresponding to $z_i(u_0)$ where $\|u - u_0\|_{L^{2,1}}$ is sufficiently small will be located in the disc \mathcal{D}_i surrounded by \mathcal{C}_i . By the continuity of $\check{a}(z; u_0)$, again we have

$$\frac{1}{2\pi} \int_{\mathcal{C}_i} \frac{\check{a}'(z; u)}{\check{a}(z; u)} dz = 1.$$

Therefore the number of zeros of $\check{a}(z; u)$ in \mathcal{D}_i should be one and located on the imaginary axis. Otherwise, if the zero is off the imaginary axis, it should come with a pair due to the symmetry of the mKdV and will give

$$\frac{1}{2\pi} \int_{\mathcal{C}_i} \frac{\check{a}'(z; u)}{\check{a}(z; u)} dz = 2$$

which is a contradiction.

Similar analysis can be applied to those simple zeros located near the imaginary axis. They will not degenerate to zeros on the imaginary axis.

Therefore, under the the simplicity assumption, any sufficiently small perturbation will not destroy the structures of breathers and solitons.

Proposition A.2. *Suppose that $u_0 \in L^{2,1}(\mathbb{R})$ and $a(z; u_0)$ has exactly N_1 simple zeros on the imaginary axis, N_2 simple zeros off the imaginary axis and no zeros on \mathbb{R} . There is a neighborhood \mathcal{N} of u_0 in $L^{2,1}(\mathbb{R})$ so that all $u \in \mathcal{N}$ have these same properties.*

Suppose that u_0 is a generic potential with n simple zeros of $\check{a}(z, u_0)$ in \mathbb{C}^+ . (Here we do not distinguish breathers and solitons). Let $S_1 = \{z_i\}_{i=1}^{N_1}$ and $S_2 = \{z_j\}_{j=1}^{N_2}$ be a list of the zeros of $\check{a}(z, u_0)$ in \mathbb{C}^+ . Set

$$d_1(u_0) = \min \left(\min_{1 \leq j \neq k \leq N_1} |z_j(u_0) - z_k(u_0)|, \min_{1 \leq j \neq k \leq N_2} |z_j(u_0) - z_k(u_0)| \right)$$

$$d_2(u_0) = \min \left(d(S_1(u_0), S_2(u_0)), \min_{1 \leq j \leq N_1} (\Im z_j), \min_{1 \leq j \leq N_1} (\Re z_j) \right)$$

and

$$d_S(u_0) = \min(d_1(u_0), d_2(u_0)).$$

There is a neighborhood \mathcal{N} of u_0 in $L^{2,1}$ so that

(1) For any $u \in \mathcal{N}$, $\check{a}(z, u)$ has exactly $N_1 + N_2$ zeros in \mathbb{C}^+ , no zeros in \mathbb{R} , and

$$|z_i(u) - z_i(u_0)| \leq \frac{1}{2} d_S(u_0)$$

$$|z_j(u) - z_j(u_0)| \leq \frac{1}{2} d_S(u_0).$$

(2) We also have

$$|z_i(u) - z_i(u_0)|, |z_j(u) - z_j(u_0)| \leq C \|u - u_0\|_{L^{2,1}}$$

hold for C uniform in $u \in \mathcal{N}$.

(3)

$$|b_i(u) - b_i(u_0)|, |b_j(u) - b_j(u_0)| \leq C \|u - u_0\|_{L^{2,1}}$$

hold for C uniform in $u \in \mathcal{N}$.

(4)

$$|C_i(u) - C_i(u_0)|, |C_j(u) - C_j(u_0)| \leq C \|u - u_0\|_{L^{2,1}}$$

hold for C uniform in $u \in \mathcal{N}$.

Proof. We can find the neighborhood \mathcal{N} of u_0 by our general argument. By construction, for any $u \in \mathcal{N}$, $\check{a}(z, u)$ has exactly $N_1 + N_2$ zeros in \mathbb{C}^+ , no zeros in \mathbb{R} , and

$$|z_i(u) - z_i(u_0)| \leq \frac{1}{2}d_S(u_0),$$

$$|z_j(u) - z_j(u_0)| \leq \frac{1}{2}d_S(u_0).$$

To establish other claims, we need to use the simplicity of zeros and the implicit function theorem. We will study the equation $a(z_j(u); u) = 0$ and regard it as a function on $\mathbb{C}^- \times L^{2,1}$. First of all, we know this function is analytic in $z \in \mathbb{C}^-$ and is differentiable in u from

$$a(z; u) = 1 - \int iu(y) m_{21}^+(y, z) dy$$

where $m_{21}^+(z, y)$ is analytic in \mathbb{C}^- and depends on u smoothly. Since $z_j(u_0)$'s and $z_i(u_0)$'s are simply zeros, we also have

$$a'(z_j(u_0); u_0) \neq 0$$

which is the condition for the implicit function theorem to be applied.

(2) The implicit function theorem also guarantees that the function $z_i(q)$ and $z_j(q)$ will be C^1 as a function of u , and hence surely Lipschitz continuous. See Pöschel-Trubowitz [50].

(3) The estimates for $b(z)$ can be obtained similarly as $a(z)$.

(4) Finally, for the norming constants, $a'(z_i)$ and $a'(z_j)$ can be expressed in terms of a via the Cauchy integral over a small circle around z_i and z_j respectively since a is analytic in \mathbb{C}^- . Since $a(z_i(u); u)$, $a(z_j(u); u)$, $b(z_i(u); u)$ and $b(z_j(u); u)$ are Lipschitz continuous with respect to u . Then we can extend the Lipschitz continuity to the norming constants. \square

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(Chen) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ONTARIO M5S 2E4, CANADA
E-mail address: gc@math.toronto.edu

(Liu) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ONTARIO M5S 2E4, CANADA
E-mail address: jliu@math.utoronto.ca