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An Incremental Path-Following Splitting Method for Linearly Constrained Nonconvex Nonsmooth Programs

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Abstract

The linearly constrained nonconvex nonsmooth program has drawn much attention over the last few years due to its ubiquitous power of modeling in the area of machine learning. A variety of important problems, including deep learning, matrix factorization and phase retrieval, can be reformulated as the problem of optimizing a highly nonconvex and nonsmooth objective function with some linear constraints. However, it is challenging to solve a linearly constrained nonconvex nonsmooth program, which is much complicated than its unconstrained counterpart. In fact, the feasible region is a polyhedron, where a simple projection is intractable in general, and moreover, the per-iteration cost is extremely expensive in real scenario, where the dimension of decision variable is high. Therefore, it has been recognized promising to develop a provable and practical algorithm for solving linearly constrained nonconvex nonsmooth programs.

In this paper, we develop an incremental path-following splitting algorithm, denoted as IPFS, with a theoretical guarantee and a low computational cost. In specific, we show that this algorithm converges to an ϵ-approximate stationary solution within $O(1/\epsilon)$ iterations with very low per-iteration cost. To the best of our knowledge, this is the first incremental method to solve linearly constrained nonconvex nonsmooth programs with a theoretical guarantee. Experiments conducted on the constrained concave penalized linear regression (CCPLR) and nonconvex support vector machine (NCSVM) demonstrate that the proposed algorithm is more effective and stable than other competing methods.

Keywords: Linearly constrained nonconvex nonsmooth program; increment; path-following; splitting method; iteration complexity.

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1 Introduction

In this paper, we are aiming at developing an effective optimization algorithm for solving the linearly constrained nonconvex nonsmooth program:

$$\min_{x\in\mathcal{X}} \ f(x)+r(x), \quad \text{s.t. } Ax\leq b, \tag{1}$$

where $x=[x_1,\ldots,x_n]\in\mathbb{R}^d$, $x_i\in\mathbb{R}^{d_i}$ and $\sum_{i=1}^n d_i=d$. The objective function $f:\mathbb{R}^d\to\mathbb{R}$ is a smooth but possibly nonconvex and $r(x)=\sum_{i=1}^n r_i(x_i)$, where r_i is nonsmooth and possibly nonconvex. $A\in\mathbb{R}^{p\times d}$, $b\in\mathbb{R}^p$ and $\mathcal{X}\subset\mathbb{R}^d$ is a closed and convex set.

Problem (1) abstracts a plethora of mathematical models arising from deep learning [1], distributed optimization and coordination [2], network utility maximization [29], resource allocation [33], statistical learning [11] and so on. Two typical problems widely used in practice are: 1) *the constrained concave penalized linear regression (CCPLR)* [19] and 2) *the nonconvex support vector machine (NCSVM)* [8].

Despite its effectiveness, it is very hard to find even a stationary solution of problem (1). The difficulty comes from two aspects. From the perspective of problem structure, the nonsmoothness of the objective function prohibits the use of the gradient while the projection onto the set $\{x:Ax\leq b\}$ is intractable in general. In another perspective of computational cost, the per-iteration cost is proportional to the full dimension and hence extremely expensive for high-dimensional data-driven applications. Therefore, an efficient algorithm with a theoretical guarantee is in demand. However, none of the existing algorithms meet these requirements.

In this paper, we propose a novel incremental path-following splitting algorithm, denoted as IPFS, to resolve problem (1). The key idea behind our approach is to construct a sequence of δ-smoothed problems with log-barrier functions and approximately solve each problem via a splitting method. In specific, we introduce a slack variable $s\geq 0$ to transform the inequality constraint into an equality constraint, and eliminate the non-negative constraint by using a log-barrier function. This leads to several following δ-smoothed problem, which are much easier than problem (1),

$$\begin{aligned}\min_{x\in\mathcal{X}} \quad & f(x)+r(x)-\delta\sum_{i=1}^p \log(s_i),\\ \text{s.t.} \quad & Ax+s=b.\end{aligned} \tag{2}$$

In fact, we are able to show that the stationary solution to a sequence of δ-smoothed problems constitute a path which converges to one stationary solution of problem (1). In this light, the proposed IPFS method approximately follow this path, and return an ϵ-approximate stationary solution of problem (1). Furthermore, either *cyclic* or *randomized* variable selection rule, which are both amenable to high-dimensional optimization, is assigned to alleviate the issue of computational cost. Finally, we present the detailed convergence and iteration complexity analysis in the appendix.

The contributions of our work are summarized as follows:

- We propose to construct a sequence of the δ-smoothed problems, where a path is constituted and converges to one stationary solution of problem (1) as $\delta\to 0$.

- We develop a novel incremental path-following splitting algorithm, denoted as IPFS, together with either *cyclic* or *randomized* variable selection rule, which are both amenable to high-dimensional optimization.
- We evaluate the efficacy of the proposed algorithm on *the constrained concave penalized linear regression* and *the nonconvex support vector machine*. Experimental results demonstrate that our method consistently outperforms other competing methods.

Related works: To the best of our knowledge, our approach is the first incremental algorithm developed for solving problem (1) with a theoretical guarantee and very low per-iteration cost. This is achieved through optimizing a sequence of a smoothed problem with decreasing parameter via an incremental splitting method. The following briefly discusses related work in literature.

The log-barrier function has been widely used in the path-following method for linear programming [14], and then generalized to the self-concordant function [26, 31] for convex programming. Recent years have witnessed the renewed interests of the path-following method in solving Lagrangian decomposition in separable convex optimization [6] and constrained convex minimization [5], where the idea behind is also strongly relevant to the continuation method, a standard technique in training neural network [17]. Very recently, a new continuation method proposed by Hazan et al. [15] has been proven globally convergent for a special class of unconstrained non-convex smooth programs. However, it remains unclear if the path-following method can be extended to solve problem (1) with a theoretical guarantee.

The splitting method [21, 7] is the standard tool for solving the linearly constrained convex programs. Much effort has been devoted to establishing the theoretical guarantee of the nonconvex splitting methods. However, the existing analysis is limited to a fraction of problems, or requires some strong assumptions. The iteration complexity analysis has been established for the nonconvex smooth consensus problem and sharing problem [18], the nonconvex problems with Kurdyka-Lojasiewicz (KL) condition [32] and the symmetric nonnegative matrix factorization [24, 23]. Very recently, Jiang *et al.* [20] presented a unified framework to define the ϵ-stationary solution of nonconvex nonsmooth problems, and presented the iteration complexity of the splitting method in terms of variational inequality. Melo and Monteiro [25] analyzed the nonconvex Jacobi-type non-Euclidean splitting method with an elegant iteration complexity analysis, while Gonçalves et al. [12] obtained the similar complexity result for the nonconvex proximal splitting method with over-relaxation step-size. On the other hand, Combettes and Pesquet [4] proposed block-coordinate fixed-point algorithms, which achieved very low per-iteration cost by incorporating random sweeping. However, problem (1) does not fall into the class of problems discussed before. Therefore, it still remains unclear whether problem (1) can be solved by the randomized splitting method with a theoretical guarantee.

2 Algorithm

We make the following assumptions throughout the paper:

Assumption 1. *The set of the stationary solutions of problem* (1) *is nonempty.*

Assumption 2. *The set* $\mathcal{X} = \prod_{i=1}^{n} \mathcal{X}_i$ *is bounded, and each* A_i *has full column rank.*

Assumption 3. *The objective f is differentiable and each partial derivative $\nabla_i f$ is Lipschitz continuous. In specific, there exists a constant $L_i > 0$ such that, for $i = 1, \ldots, n$,*

$$\|\nabla_i f(x) - \nabla_i f(y)\|_2 \leq L_i \|x - y\|_2, \ \forall x, y \in \mathbb{R}^d.$$

Assumption 3 is standard and satisfied by many loss functions in machine learning. For example, the least square or logistic loss, *i.e.*, $f(x) = \frac{1}{N}\sum_{i=1}^{N} l(x, \xi_i)$ where $\xi_i = (a_i, b_i)$ is a single data sample, and $l(x, \xi_i)$ is defined as:

$$\frac{1}{2}\left\|a_i^\top x - b_i\right\|_2^2 \text{ or } \log\left(1 + \exp\left(-b_i \cdot a_i^\top x\right)\right).$$

We proceed to the optimality of problem (1).

Definition 1. *Let $h : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous function. Suppose $h(\bar{x})$ is finite for a given $\bar{x}$. For $v \in \mathbb{R}^d$, we say that*

- *v is a regular sub-gradient of h at $\bar{x}$, written $v \in \partial\hat{h}(\bar{x})$, if*

$$\lim_{x \neq \bar{x}} \inf_{x \to \bar{x}} \frac{h(x) - h(\bar{x}) - \langle v, x - \bar{x}\rangle}{\|x - \bar{x}\|_2} \geq 0.$$

- *v is a general sub-gradient of h at $\bar{x}$, written $v \in \partial h(\bar{x})$, if there exist sequences $\{x^k\}$ and $\{v^k\}$ such that $x^k \to \bar{x}$ with $h(x^k) \to h(\bar{x})$, and $v^k \in \partial\hat{h}(x^k)$ with $v^k \to v$ when $k \to +\infty$.*

The following proposition lists some facts about the semi-continuous functions.

Proposition 4. *Let $h : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous functions. Then it holds that:*

1. *Fermats rule remains true: if $\bar{x}$ is a local minimum of h, then $0 \in \partial h(\bar{x})$.*
2. *If h is continuously differentiable at x, then $\partial(h + g)(x) = \partial h(x) + \partial g(x)$.*
3. *If h is locally Lipschitz continuous at x, then $\partial(h + g)(x) \subset \partial h(x) + \partial g(x)$.*
4. *Suppose $h(x)$ is locally Lipschitz continuous, $\mathcal{X}$ is a closed and convex set, and $\bar{x}$ is a local minimum of h on $\mathcal{X}$. Then there exists $v \in \partial h(\bar{x})$ such that $(x - \bar{x})^\top v \geq 0, \forall x \in \mathcal{X}$.*

Assumption 5. *The set of generalized gradient of r_i, denoted as ∂r_i, is assumed to be bounded. In addition, the proximal mapping of each r_i, defined as:*

$$\text{prox}_{\alpha r_i}(x) = \underset{y}{\text{argmin}}\left[r_i(y) + \frac{1}{2\alpha}\|y - x\|_2^2\right]$$

is easily obtained.

Remark 6. *Assumption 5 is standard and satisfied by ℓ_1-norm and the smoothly clipped absolute deviation (SCAD) [9], and also reasonable since the solution to $r_i(x) + \frac{1}{2\alpha}\|x\|_2^2$ is unique for $\alpha > 0$ sufficiently small.*

We are ready to introduce the notion of an ϵ-stationary solution of problem (1). After introducing the slack variable $s \geq 0$, the Lagrangian function is defined as:

$$\mathcal{L}(x, s, \lambda) = f(x) + r(x) + \langle \lambda, Ax + s - b \rangle .$$

Based on the first-order optimality condition, we define an ϵ-stationary solution of problem (1) as follows:

Definition 2. *We call $x^* \in \mathbb{R}^d$ to be an ϵ-stationary solution of problem (1) if there exists $s^* \geq 0$, and $\lambda^* \in \mathbb{R}^p$ such that the following holds true,*

$$\begin{aligned} \operatorname{dist}\left(-\nabla_i f(x^*) - A_i^\top \lambda^*, \partial r_i(x_i^*)\right) \leq \epsilon,\ i = 1, \ldots, n, \\ \left(s_j - s_j^*\right) \lambda_j^* \geq -\epsilon,\ j = 1, \ldots, p, \\ \|Ax^* + s^* - b\|_2 \leq \epsilon, \end{aligned}$$

where A_i is the i-th column of A, $s \geq 0$ and $\operatorname{dist}(x, \mathcal{H})$ is the standard Euclidean distance between x and a closed convex set $\mathcal{H}$. The solution x^* is a stationary solution of problem (1) if $\epsilon = 0$ holds true.

Remark 7. *Given a stationary solution of problem (1), it has been recognized as a significant reference for several first-order methods to obtain the iterates, which converge to this solution. Furthermore, the stationary solution attained by the proposed algorithm could be a local minimizer under some conditions, such as second-order sufficient condition [27].*

We propose an incremental path-following splitting method, denoted as IPFS, and analyze its iteration complexity with *cyclic* or *randomized* variable selection rule. The proposed algorithm is presented in Algorithm 1.

The algorithm is double-looped. In each iteration of the outer loop, we consider a δ-smoothed problem, *i.e.*, problem (2), where $\delta > 0$ is a smoothing parameter. We optimize this problem via a splitting method with a variable selection rule, and obtain an ϵ_δ-stationary solution to problem (2). Then we move to the next iteration of the outer loop, and decrease δ to $\gamma\delta$, where $0 < \gamma < 1$. Each inner loop is devoted to optimize problem (2) via the splitting method. In each iteration, we select a set of index $\mathcal{I} \subseteq \{1, 2, \ldots, n\}$, and update $\{x_i\}_{i \in \mathcal{I}}$ based on the last iterate, *i.e.*, x^k. In specific, we introduce $\beta > 0$ and define a function as:

$$\begin{aligned} \mathcal{L}^k\left(\{x_i\}_{i\in\mathcal{I}}, x^k, s, \lambda\right) &= \sum_{i\in\mathcal{I}} \left[\left\langle \nabla_i f(x^k), x_i - x_i^k \right\rangle + \frac{L_i + 1}{2} \left\| x_i - x_i^k \right\|_2^2 + r_i(x_i) \right] - \delta \sum_{i=1}^p \log(s_i) \quad (3) \\ &\quad + \left\langle \lambda, \sum_{i\in\mathcal{I}} A_i x_i + \sum_{i\notin\mathcal{I}} A_i x_i^k + s - b \right\rangle + \frac{\beta}{2} \left\| \sum_{i\in\mathcal{I}} A_i x_i + \sum_{i\notin\mathcal{I}} A_i x_i^k + s - b \right\|_2^2 . \end{aligned}$$

Given $s = s^k$ and $\lambda = \lambda^k$, this function is strongly convex for $\{x_i\}_{i\in\mathcal{I}}$ according to the boundedness of ∂r_i. To this end, we obtain $\{x_i^{k+1}\}_{i\in\mathcal{I}}$ by following:

$$\{x_i^{k+1}\}_{i\in\mathcal{I}} = \underset{\{x_i \in \mathcal{X}_i\}_{i\in\mathcal{I}}}{\operatorname{argmin}} \mathcal{L}^k\left(\{x_i\}_{i\in\mathcal{I}}, x^k, s^k, \lambda^k\right). \quad (4)$$

Algorithm 1 Incremental Path-Following Splitting Method (IPFS)

Initialize: the primal variable $\bar{x} \in \mathbb{R}^d$; the slack variable $\bar{s} \geq 0$; the dual variable $\bar{\lambda} \in \mathbb{R}^p$; the smoothing parameter $\delta > 0$.
Set: the ratio $\gamma \in (0, 1)$; the final tolerance $\epsilon > 0$.
while $\delta > \epsilon$ **do**
  Set $k \leftarrow 0$.
  Set $\beta > 0$ according to δ.
  Set $\left(x^0, s^0, \lambda^0\right) \leftarrow \left(\bar{x}, \bar{s}, \bar{\lambda}\right)$.
  while the stopping criterion is not satisfied **do**
    1. Pick up a set of index, *i.e.*, $\mathcal{I}$, according to a variable selection rule.
    2. Update $\{x_i^{k+1}\}_{i \in \mathcal{I}}$ via Eq. (4).
    3. Update $\{x_i^{k+1}\}_{i \notin \mathcal{I}} \leftarrow \{x_i^k\}_{i \notin \mathcal{I}}$.
    4. Update s^{k+1} via Eq. (5).
    5. Update λ^{k+1} via Eq. (6).
    6. Update $k \leftarrow k + 1$.
  end while
  $\delta \leftarrow \gamma\delta$.
  $\left(\bar{x}, \bar{s}, \bar{\lambda}\right) \leftarrow \left(x^{k+1}, s^{k+1}, \lambda^{k+1}\right)$.
end while

Furthermore, we set $\{x_i^{k+1}\}_{i \notin \mathcal{I}} = \{x_i^k\}_{i \notin \mathcal{I}}$. Finally, we obtain s^{k+1} and λ^{k+1} by following:

$$s^{k+1} = \underset{s \in \mathbb{R}^p}{\operatorname{argmin}} \left[-\delta \sum_{i=1}^{p} \log(s_i) + \frac{\beta}{2} \left\| s + Ax^{k+1} - b + \frac{1}{\beta}\lambda^k \right\|_2^2 \right], \tag{5}$$

and

$$\lambda^{k+1} = \lambda^k + \beta \left(Ax^{k+1} + s^{k+1} - b \right). \tag{6}$$

Remark 8. *We assume that problem* (4) *can be solved* ***exactly*** *in theoretical analysis for convenience since it has been shown in [13] that this subproblem admits a closed-form solution for a few special nonconvex regularization function r_i, e.g., SCAD [9] if A_i is an identity matrix and $\mathcal{X}_i = \mathbb{R}^{d_i}$. However, the closed-form solution is generally* ***intractable****. As an alternative, some iterative methods can be used to approximately solve it. We refer the interested readers to [13] for more details.*

In what follows, we discuss the variable selection rules: *cyclic* and *randomized*.

1. *cyclic rule.* Let $\mathcal{I}$ be the set of index selected at the k-th iteration of one inner loop, we have

$$\mathcal{I} = \{i_k = k \bmod n\},$$

where $k \bmod n$ is the remainder for k divided by n.

2. *randomized rule.* At each k-th iteration, the index $i \in \mathcal{I}$ is selected at random with probability $p_i > 0$, *i.e.*,

$$\text{Prob}\,(i \in \mathcal{I}) = p_i \geq p_{\min} > 0.$$

Finally, we design a reasonable procedure to identify the tolerance $\epsilon_\delta > 0$. This procedure, also known as *stopping criterion*, is crucial for the performance of the proposed algorithm. In specific, it is unnecessary to obtain very accurate local solution when δ is large, which might take a lot of iterations. In what follows, we clarify the connection between ϵ_δ used in the stopping criterion and δ.

Stopping Criterion: We repeat the inner loop of the splitting method until the following statements hold true,

$$\begin{aligned} \left\| x^{k+1} - x^k \right\|_2 &\leq C\delta, \\ \left\| Ax^{k+1} + s^{k+1} - b \right\|_2 &\leq C\delta, \end{aligned}$$

where $C > 0$ is set as a constant which is independent of δ.

2.1 Discussion

Firstly, we compare our method to some existing methods.

1. Our method is greatly different from stochastic alternating direction method (SADM) [28]. SADM randomly draw a subset of data samples at each iteration while our method randomly selects a subset of decision variables. In addition, the theoretical guarantee of SADM is established only when the objective is convex.

2. Our method is related to the algorithm presented in [18]. However, the theoretical guarantee of that algorithm is only established only when applied to solve the nonconvex smooth consensus and sharing problems.

3. Our method is related to the two variants of ADMM analyzed in [20], i.e., proximal ADMM-m and proximal ADMM-g. However, neither of these methods has a theoretical guarantee when applied to solve problem (1). In specific, problem (1) can be reformulated as

$$\min_{x \in \mathcal{X}} \; f(x) + r(x), \quad \text{s.t. } Ax + s = b, \; s \geq 0.$$

   The objective is nonsmooth with respect to x and s, which violates the conditions in [20]. In addition, the computational cost of proximal ADMM-m and proximal ADMM-g is very expensive on high-dimensional problems.

Secondly, we remark that the parameter setting varies from practical usage to theoretical analysis. For example, we prove in the next section that the iterates converge to an ϵ-stationary solution if β remains as a sufficiently large constant. However, β should be adapted to accelerate the method and improve the practical performance.

3 Main Result

In this section, we present the convergence and iteration complexity analysis of the proposed algorithm with *cyclic* or *randomized* variable selection rule.

3.1 Convergence

We define a set of stationary solutions of problem (2), and prove that $(\bar{x}, \bar{s})$ converges to one of the stationary solutions.

Definition 3. *We call* $\left(x^{\delta,*}, s^{\delta,*}\right) \in \mathbb{R}^d \times \mathbb{R}^p$ *to be an* ϵ_δ*-stationary solution of problem* (2) *if there exists* $\lambda^{\delta,*} \in \mathbb{R}^p$ *such that the following statement holds true,*

$$\begin{aligned} \text{dist}\left(-\nabla_i f(x^{\delta,*}) - A_i^\top \lambda^{\delta,*}, \partial r_i(x_i^{\delta,*})\right) &\leq \epsilon_\delta,\ i = 1, \ldots, n, \\ \left\| s_j^{\delta,*} \lambda_j^{\delta,*} - \delta \right\|_2 &\leq \epsilon_\delta,\ j = 1, \ldots, p, \\ \left\| Ax^{\delta,*} + s^{\delta,*} - b \right\|_2 &\leq \epsilon_\delta. \end{aligned}$$

The solution $\left(x^{\delta,*}, s^{\delta,*}\right)$ *is a stationary solution of problem* (2) *if* $\epsilon_\delta = 0$ *holds true.*

At a high-level, for any given $\delta > 0$, $(\bar{x}, \bar{s})$ can approximate $\left(x^{\delta,*}, s^{\delta,*}\right)$ well as the inner loop goes. Since $\left(x^{\delta,*}\right)$ converges to x^* as $\delta \to 0$, we can characterize the limiting behavior of our algorithm.

Theorem 9. *As* $\delta \to 0$*, we show that* $\bar{x} \to x^*$ *deterministically or in terms of expectation for cyclic and randomized variable selection rules, where* x^* *is a stationary solution of problem* (1)*.*

3.2 Iteration Complexity

In this subsection, we analyze the iteration complexity of the proposed algorithm. Specifically, we use the measure of optimality in terms of variational inequality, instead of the closeness to the optimal solution, which is intractable in nonconvex optimization. The ϵ-stationary solution of problem (1) is regarded reached if the following statements both hold true,

1. The smoothing parameter is smaller than the final tolerance, *i.e.*, $\delta < \epsilon$.
2. The stopping criterion in each inner loop is satisfied deterministically or in terms of expectation for cyclic and randomized variable selection rules, respectively.

Now we are ready to state our main result on the iteration complexity. Specifically, we show that the proposed algorithm returns ϵ-stationary solution of problem (1) within $O\left(\frac{1}{\epsilon}\right)$ iterations in term of the iteration number in inner loop, and $O(\log(\frac{1}{\epsilon}))$ iterations in term of the iteration number in outer loop.

Theorem 10. *Suppose either cyclic or randomized variable selection rule is employed, the proposed* IPFS *algorithm returns an* ϵ*-stationary solution of problem* (1) *deterministically or in terms of expectation for cyclic and randomized variable selection rules, respectively, within* $O\left(\frac{1}{\epsilon}\right)$ *iterations in term of the iteration number in inner loop, and* $O(\log(\frac{1}{\epsilon}))$ *iterations in term of the iteration number in outer loop.*

Table 1: Statistics of datasets.

Dataset	Number of Samples	Dimension
ohscal	11,162	11,465
w8a	64,700	300
a9a	32,561	123
20news	16,242	100
SUSY	5,000,000	19

Table 2: Performance comparison of the referred algorithms.

(n,d)	σ	Methods	$\lambda = 1.4$	$\lambda = 1.6$	$\lambda = 1.8$	$\lambda = 2.0$
			(objective,time)	(objective,time)	(objective,time)	(objective,time)
(2000, 5000)	$\sigma = 0.1$	IPFS-R	**(126.73, 7.31)**	**(120.12, 7.61)**	**(141.27, 7.25)**	**(160.17, 7.11)**
		IPFS-C	(720.09, 23.39)	(807.96, 23.52)	(942.65, 23.92)	(950.82, 23.79)
		IALM	(1048.47, 46.70)	(1146.34, 46.60)	(1171.69, 48.32)	(1170.91, 46.33)
		proximal ADMM-m	(1348.56, 30.61)	(1534.81, 30.39)	(1717.74, 30.66)	(1896.04, 30.64)
	$\sigma = 0.3$	IPFS-R	**(93.17, 8.15)**	**(119.38, 7.57)**	**(148.05, 7.39)**	**(173.73, 7.13)**
		IPFS-C	(679.48, 23.48)	(672.42, 24.47)	(627.10, 24.08)	(1158.11, 24.39)
		IALM	(1150.52, 47.84)	(1304.51, 47.51)	(1450.57, 47.67)	(1590.68, 46.76)
		proximal ADMM-m	(1370.99, 31.49)	(1560.77, 31.55)	(1746.86, 31.64)	(1931.58, 31.56)
(1000, 10000)	$\sigma = 0.1$	IPFS-R	(276.60, **15.12**)	(286.44, **15.04**)	(384.07, **14.45**)	(417.84, **15.00**)
		IPFS-C	(**84.84**, 33.19)	(**85.20**, 37.54)	(**57.03**, 37.11)	(**123.92**, 37.25)
		IALM	(3109.88, 64.10)	(3553.76, 61.16)	(3997.23, 60.27)	(4440.59, 56.81)
		proximal ADMM-m	(3761.72, 24.75)	(4299.11, 24.47)	(4836.50, 24.71)	(5373.87, 24.93)
	$\sigma = 0.3$	IPFS-R	(283.47, **14.34**)	(376.80, **14.92**)	(405.42, **14.41**)	(430.40, **15.85**)
		IPFS-C	(**134.16**, 38.10)	(**114.69**, 37.68)	(**58.43**, 40.84)	(**92.20**, 38.34)
		IALM	(3132.08, 111.95)	(3579.23, 122.77)	(4026.25, 140.42)	(4472.96, 126.79)
		proximal ADMM-m	(3774.29, 35.96)	(4313.47, 36.26)	(4852.62, 36.26)	(5391.79, 36.42)

Remark 11. *We highlight the iteration complexity in order of* $O\left(\frac{1}{\epsilon}\right)$ *is theoretical optimal for first-order methods when applied to solve several non-convex problems [3, 22]. In practice, a variety of large-scale artificial intelligence and machine learning applications requires the solution with low accuracy, i.e.,* $\epsilon \approx 10^{-3}$.

4 Experiments

In this section, we evaluate the efficacy of our algorithm on the linearly constrained nonconvex problem arising from machine learning. We compare our method with two well-known heuristic algorithms since the existing exact convergence method for problem (1) is **unknown**. More specifically, we consider the inexact augmented Lagrangian method (InexactALM) [16] and the proximal alternating direction method of multipliers (proximal ADMM-m) [20], which are referred to "heuristic" since the theoretical guarantees of these two methods are only valid on convex minimization and a class of nonconvex minimization.

We conduct our experiments on the synthetic data in the first task, named *Constrained Concave Penalized Linear Regression* [10], and on the real data in the second task, named *Nonconvex Support Vector Machine* [30]. The objective value is used as the metric in our experiments.

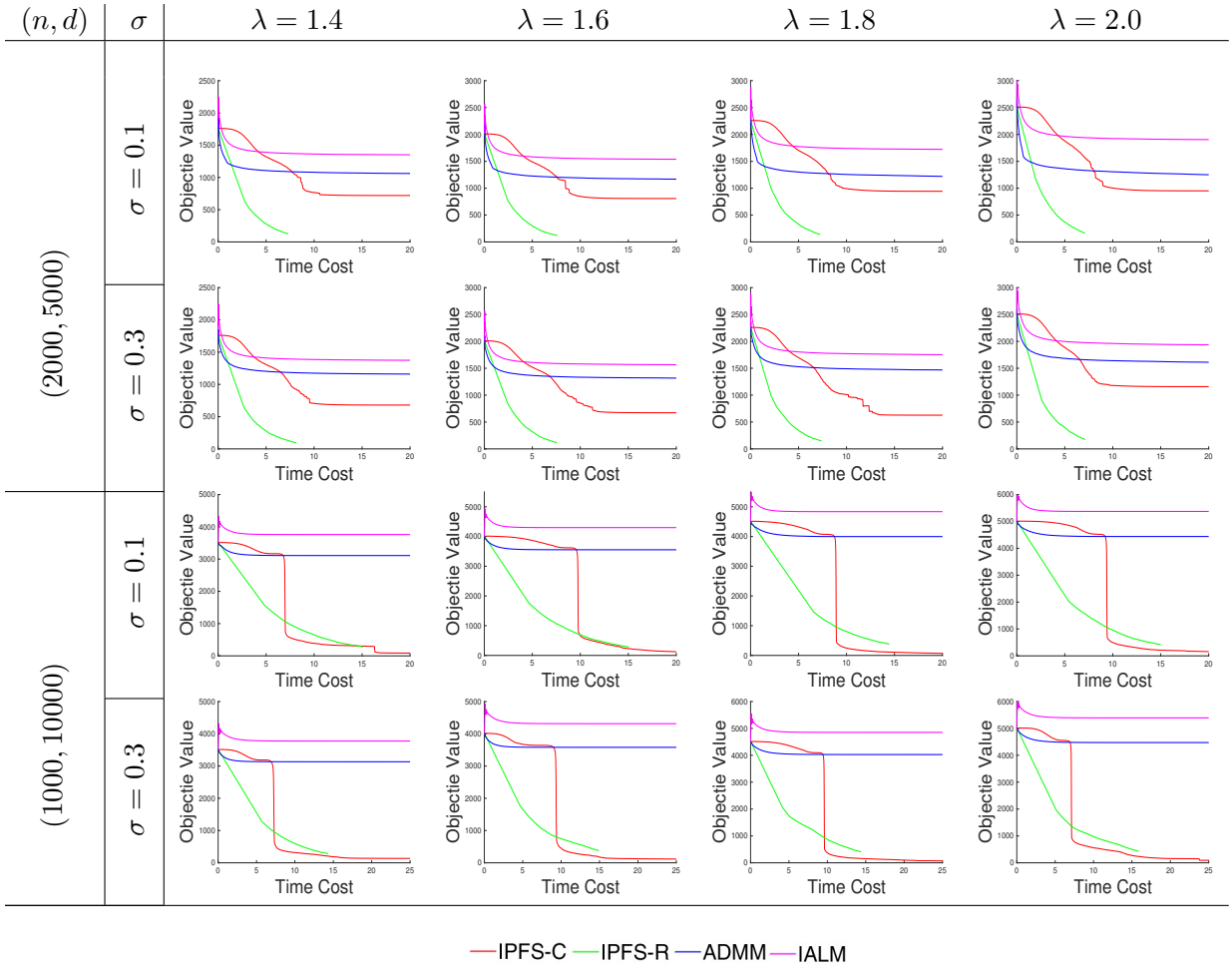

Figure 1: Comparison of IPFS-C (Cyclic variable selection rule) and IPFS-R (Randomized variable selection rule) with ADMM (proximal ADMM-m) and IALM (InexactALM) on Constrained Concave Penalized Linear Regression.

4.1 Constrained Concave Penalized Linear Regression

Problem: The problem of *Constrained Concave Penalized Linear Regression (CCPLR)* has been recognized as one linearly constrained nonconvex nonsmooth program, which covers a few interesting applications in statistical learning and image processing. In specific, it is aiming at recovering a sparse signal $x^* \in \mathbb{R}^d$ with $s \ll d$ non-zero components from the observation $y \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, which are defined as $y = Ax^* + \epsilon_1$ and $b = Cx^* + \epsilon_2$. Here $A \in \mathbb{R}^{n\times d}$ and $C \in \mathbb{R}^{m\times d}$ are measurement matrices, and $\epsilon_1 \in \mathbb{R}^m$ and $\epsilon_2 \in \mathbb{R}^n$ are white noises. Mathematically, it is defined as:

$$\min_{x\in\mathbb{R}^d} \ \frac{1}{2}\|Ax - y\|_2^2 + P_\lambda(x), \ \text{s.t. } Cx - b \leq 0, \tag{7}$$

where $P_\lambda(w)$ is defined as:

$$P_\lambda(w) = \lambda \|w\|_1 - \sum_{i=1}^{d} \left[\frac{w_i^2 - 2w_i + \lambda^2}{2(\theta - 1)} I(\lambda < w_i \leq \theta\lambda) + (\lambda w_i - \frac{(\theta+1)\lambda^2}{2}) I(w_i > \theta\lambda) \right].$$

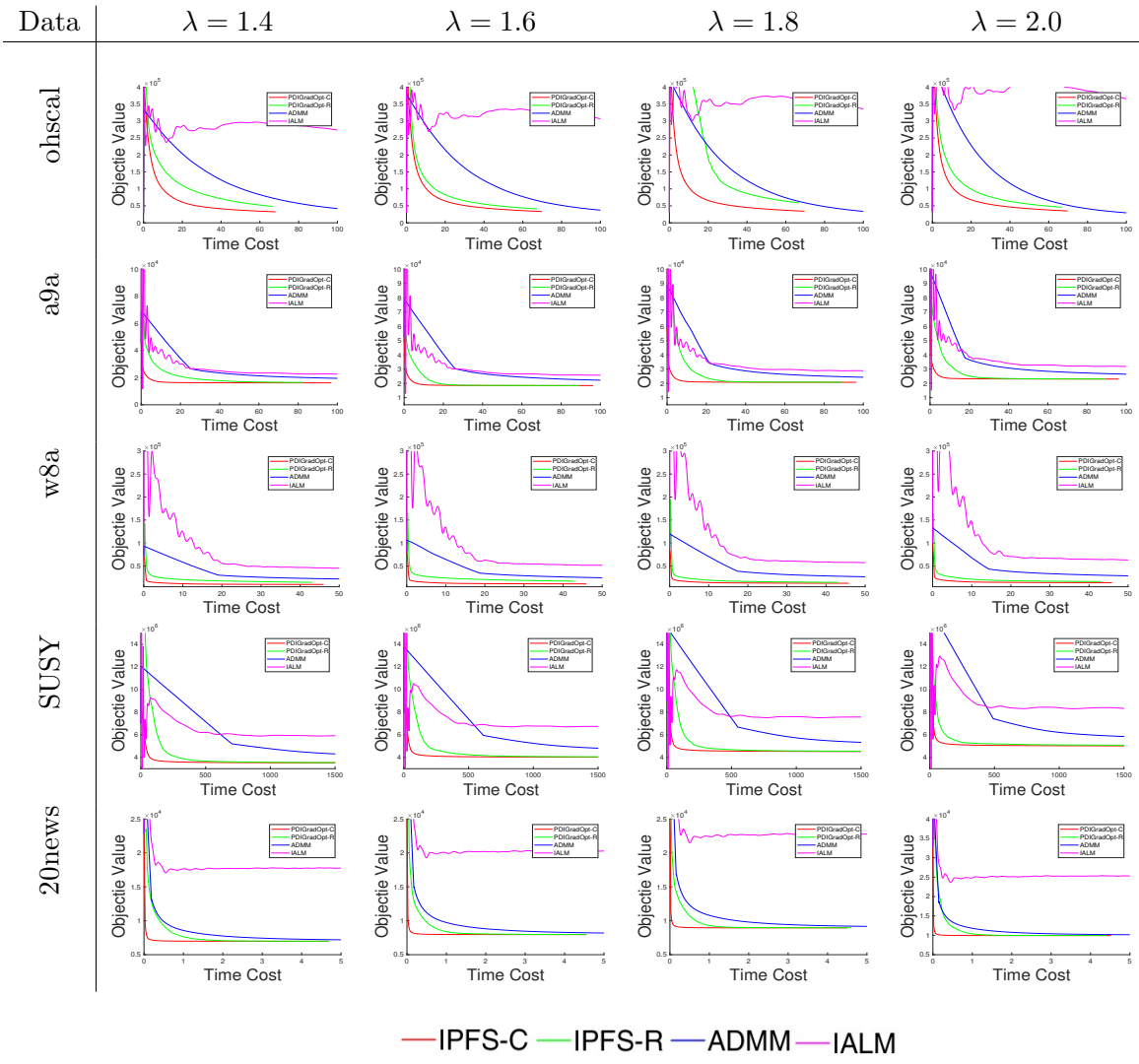

Figure 2: Comparison of IPFS-C (Cyclic variable selection rule) and IPFS-R (Randomized variable selection rule) with ADMM (proximal ADMM-m) and IALM (InexactALM) on Nonconvex Support Vector Machine.

Settings: We generate $A \in \mathbb{R}^{n\times d}$ with independent standard Gaussian entries and normalized it in column; We generate $C \in \mathbb{R}^{m\times d}$ with independent standard Gaussian entries; $\epsilon_1 \sim \mathcal{N}(0, \sigma^2 I_d)$, and ϵ_2 contains independent random entries uniformly distributed in $[0, \sigma]$, where $\sigma = 0.1$ or $\sigma = 0.3$. We select different regularization parameters in $\{1.4, 1.6, 1.8, 2.0\}$ to show that our algorithms are robust. We implement with the random initialization, and the terminate that the relative change of the consecutive objective function values is less than 10^{-7}.

Experimental results: Figure 1 shows the objective value as a function of time cost (in seconds) where $\sigma = 0.1$ and $\sigma = 0.3$. We observe that IPFS-C (Cyclic variable selection rule) and IPFS-R (Randomized variable selection rule) consistently outperform InexactALM and proximal ADMM-m on all datasets, especially when the dimension is high. The proposed methods could achieve a lower objective value in the non-convex optimization procedure. This confirms the advantage of our algorithms over InexactALM and proximal ADMM-m is the solid theoretical guarantee. Furthermore, IPFS-Randomized performs the best mainly because of its low per-iteration cost, strongly supporting the usage of randomized algorithms.

4.2 Nonconvex Support Vector Machine

Problem: The problem of *NonConvex Support Vector Machine (NCSVM)* is a very powerful binary classification tool with high accuracy and great flexibility. Mathematically, it is defined as:

$$\min_{x\in\mathbb{R}^d} \ \alpha \mathbf{1}^\top \xi + P_\lambda(x), \ \text{s.t.} \ \mathbf{1} - \xi - b \cdot A^\top x \leq 0, \ \xi \leq 0, \tag{8}$$

where $\mathbf{1} \in \mathbb{R}^n$, and $P_\lambda(w)$ is defined as before.

Settings: We set $\sigma = 1.4$, and vary the regularization parameter λ in $\{1.4, 1.6, 1.8, 2.0\}$. We use seven datasets[1][2][3] to evaluate the proposed algorithm, where the statistics is presented in Table 1. The remaining setting is the same as that used in the CCPLR problem.

Experimental results: Figure 2 shows the objective value as a funtion of time cost (in seconds). Indeed, our algorithms outperform InexactALM and proximal ADMM-m consistently by a significantly large margin on real data, which confirms the advantage of our algorithms over InexactALM and proximal ADMM-m with the solid theoretical guarantee.

5 Conclusions

In this paper, we proposed a novel incremental path-following splitting algorithm, denoted as IPFS, to solve the linearly constrained nonconvex nonsmooth program, which abstracts quite a few machine learning applications. To the best of our knowledge, this is the first incremental method developed for solving problem (1) with a theoretical guarantee. Furthermore, the *cyclic* and *randomized* block variable selection rules significantly improve the efficiency of the proposed algorithm on high-dimensional data, as confirmed by our experiments on nonconvex penalized linear regression and support vector machine tasks.

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A Proof of Theorem 9

We firstly construct a potential function $\Phi(x,s,\lambda)$ and present a key technical lemma which guarantees that, when optimizing δ-smoothed version, *i.e.*, problem (2), the sequence of $\left\{\Phi(x^k,s^k,\lambda^k)\right\}_{k=0}^{+\infty}$ is non-increasing and lower bounded under some conditions of the penalty parameter $\beta>0$. The potential function $\Phi(x,s,\lambda)$ is defined as

$$\Phi(x,s,\lambda)=f(x)+r(x)-\delta\sum_{j=1}^{p}\log(s_j)+\langle\lambda,Ax+s-b\rangle+\frac{\beta}{2}\left\|Ax+s-b\right\|_2^2,$$

where $\delta>0$ is a given smoothing parameter, and $\beta>0$ is a penalty parameter chosen according to δ.

A.1 Proof of Technical Lemmas

Lemma 12. *When optimizing the δ-smoothed version, there exists $\underline{s}>0$. If the following statement holds true,*

$$\beta>\frac{\sqrt{2}\delta}{\underline{s}^2},\tag{9}$$

then a sequence of $\left\{\Phi(x^k,s^k,\lambda^k)\right\}_{k=0}^{+\infty}$ is non-increasing and lower bounded. Therefore, we conclude that $\Phi(x^k,s^k,\lambda^k)\to\Phi^*$, and $\left\|Ax^{k+1}+s^{k+1}-b\right\|_2\to 0$, $\left\|x^{k+1}-x^k\right\|_2\to 0$ and $\left\|s^{k+1}-s^k\right\|_2\to 0$ as $k\to\infty$, deterministically and almost surely for cyclic and randomized variable selection rules, respectively. Furthermore, a sequence of $\left\{\left(x^k,s^k,\lambda^k\right)\right\}_{k=0}^{+\infty}$ remains bounded.

Proof. It follows from the update of $\{x_i^{k+1}\}_{i\in I}$, $\{x_i^{k+1}\}_{i\notin I}$ that

$$\Phi(x^k,s^k,\lambda^k)-\Phi(x^{k+1},s^k,\lambda^k)\geq\frac{1}{2}\sum_{i\in I}\left\|x_i^k-x_i^{k+1}\right\|_2^2.\tag{10}$$

Furthermore, $\Phi(x^{k+1},s,\lambda^k)$ is strongly convex with respect to s, and hence by using the update of s^{k+1}, we have

$$\Phi(x^{k+1},s^k,\lambda^k)-\Phi(x^{k+1},s^{k+1},\lambda^k)\geq\frac{\beta}{2}\left\|s^k-s^{k+1}\right\|_2^2.$$

Finally, by using the update of λ^{k+1}, we have

$$\Phi(x^{k+1},s^{k+1},\lambda^k)-\Phi(x^{k+1},s^{k+1},\lambda^{k+1})=-\frac{1}{\beta}\left\|\lambda^k-\lambda^{k+1}\right\|_2^2.\tag{11}$$

In what follows, we try to show that there exists $\underline{s}>0$ such that $s_j^k\geq\underline{s}$ for $i=1,2,\ldots,p$. It is clear that $\{s^k\}_{k=1}^{+\infty}$ is a bounded sequence since $-\sum_{j=1}^{p}\log(s_j)$ is strictly convex. Firstly, we obtain that $s_j^{k+1}\lambda_j^{k+1}=\delta$ through combining the update of s^{k+1} and λ^{k+1}. Then we suffice to show that there exists $\bar{\lambda}>0$ such that $\lambda_j^k\leq\bar{\lambda}$. In specific, we have

$$\begin{aligned}\partial r_i(x_i^{k+1}) &\ni -\nabla_i f(x_1^k,\ldots,x_N^k)-A_i^\top\lambda^k-(L_i+1)\left(x_i^{k+1}-x_i^k\right)-\beta A_i^\top\left[\sum_{j\leq i}A_jx_j^{k+1}+\sum_{j>i}A_jx_j^k+s^k-b\right]\\ &= -\nabla_i f(x_1^k,\ldots,x_N^k)-A_i^\top\lambda^{k+1}-(L_i+1)\left(x_i^{k+1}-x_i^k\right)-\beta A_i^\top\left(s^{k+1}-s^k\right)\end{aligned}$$

For $i = 1, 2, \ldots, n$, we obtain that $A_i^\top \lambda^k$ is bounded since $\{s^k\}_{k=1}^{+\infty}$ is a bounded sequence, $\mathcal{X}_i$ and the set of generalized gradient of r_i are both bounded sets. Without loss of generality, $A = [A_1\ A_2\ \ldots\ A_N]$ is assumed to be full row rank, which leads to the fact that $\lambda_j^k \leq \bar{\lambda}$ for some $\bar{\lambda} > 0$. Therefore, we have

$$\left\|\lambda^k - \lambda^{k+1}\right\|_2 = \left\|\frac{\delta}{s^k} - \frac{\delta}{s^{k+1}}\right\|_2 \leq \frac{\delta}{\underline{s}^2}\left\|s^k - s^{k+1}\right\|_2. \tag{12}$$

Finally, we combine (10), (11) and (12) to obtain that

$$\Phi(x^k, s^k, \lambda^k) - \Phi(x^{k+1}, s^{k+1}, \lambda^{k+1}) \geq \frac{1}{2}\sum_{i\in I}\left\|x_i^k - x_i^{k+1}\right\|_2^2 + \left(\frac{\beta}{2} - \frac{\mu^2}{\beta \underline{s}^4}\right)\left\|s^k - s^{k+1}\right\|_2^2. \tag{13}$$

Furthermore, the function $\Phi\left(x^k, s^k, \lambda^k\right)$ has been shown lower bounded in [20], *i.e.*, there exists $\Phi^* \in \mathbb{R}$ such that $\Phi(x^k, s^k, \lambda^k) \geq \Phi^*$. For cyclic variable selection rule, it holds deterministically that,

$$\left\|x^k - x^{k+1}\right\|_2 = \sqrt{\sum_{i\in I}\left\|x_i^k - x_i^{k+1}\right\|_2^2} \to 0, \quad \text{as } k \to +\infty.$$

For randomized variable selection rule, we take the conditional expectation over both sides of (13) and obtain that

$$\Phi(x^k, s^k, \lambda^k) - \mathbb{E}\left[\Phi(x^{k+1}, s^{k+1}, \lambda^{k+1}) \mid (x^k, s^k, \lambda^k)\right] \geq \frac{1}{2}\mathbb{E}\left[\sum_{i\in I}\left\|x_i^k - x_i^{k+1}\right\|_2^2\right].$$

Since $p_i \geq p_{\min}$ for $i = 1, 2, \ldots, N$, we have

$$\Phi(x^k, s^k, \lambda^k) - \mathbb{E}\left[\Phi(x^{k+1}, s^{k+1}, \lambda^{k+1}) \mid (x^k, s^k, \lambda^k)\right] \geq \frac{p_{\min}}{2}\sum_{i=1}^{N}\left[\left\|x_i^k - \tilde{x}_i^{k+1}\right\|_2^2\right] \geq 0,$$

where $\tilde{x}^{k+1}$ is a "virtual" iterate assuming that all variables are updated once. Thus $\left\{\Phi(x^k, s^k, \lambda^k)\right\}_{k=0}^{+\infty}$ is a super-martingale with respect to the natural history; and by the super-martingale convergence theorem, the sequence $\left\{\Phi(x^k, s^k, \lambda^k)\right\}_{k=0}^{+\infty}$ converges almost surely. Therefore, it also surely holds true that,

$$\left\|x^k - x^{k+1}\right\|_2 \to 0, \quad \text{as } k \to +\infty.$$

Similarly, we obtain that $\left\|s^{k+1} - s^k\right\|_2 \to 0$ and $\left\|Ax^{k+1} + s^{k+1} - b\right\|_2 = \frac{1}{\beta}\left\|\lambda^k - \lambda^{k+1}\right\|_2 \to 0$ as $k \to +\infty$ deterministically and almost surely for cyclic and randomized variable selection rules, respectively. This completes the proof. □

From Lemma 12, it is easy to see that the stopping criterion must be satisfied. Then we present a lemma which guarantees that, $\bar{x} \in \mathbb{R}^d$ approaches one local solution to problem (2), where the Lagrangian function is defined as

$$\mathcal{L}^\delta(x, s, \lambda) = f(x) + r(x) - \delta\sum_{i=1}^{p}\log(s_i) + \langle\lambda, Ax + s - b\rangle.$$

Lemma 13. *Let $(\bar{x}, \bar{s})$ satisfies the stopping criterion deterministically and almost surely for cyclic and randomized variable selection rules, respectively. Then $(\bar{x}, \bar{s})$ is an δ-stationary solution of problem (2). That is to say, there exists $\bar{\lambda} \in \mathbb{R}^p$ such that the following statement holds true,*

$$\begin{aligned} \operatorname{dist}\left(-\nabla_i f(\bar{x}) - A_i^\top \bar{\lambda}, \partial r_i(\bar{x}_i)\right) &\leq \delta,\ i = 1, \ldots, n, \\ \left\|\bar{s}_j \bar{\lambda}_j - \delta\right\|_2 &\leq \delta,\ j = 1, \ldots, p, \\ \|A\bar{x} + \bar{s} - b\|_2 &\leq \delta, \end{aligned}$$

deterministically for cyclic variable selection rule and

$$\begin{aligned} \mathbb{E}\left[\operatorname{dist}\left(-\nabla_i f(\bar{x}) - A_i^\top \bar{\lambda}, \partial r_i(\bar{x}_i)\right)\right] &\leq \delta,\ i = 1, \ldots, n, \\ \mathbb{E}\left[\left\|\bar{s}_j \bar{\lambda}_j - \delta\right\|_2\right] &\leq \delta,\ j = 1, \ldots, p, \\ \mathbb{E}\left[\|A\bar{x} + \bar{s} - b\|_2\right] &\leq \delta, \end{aligned}$$

for randomized variable selection rule.

Proof. When optimizing a δ-smoothed version, the first-order optimality condition for $(x^{k+1}, s^{k+1}, \lambda^{k+1})$ is

$$\operatorname{dist}\left(-\nabla_i f(x^{k+1}) - A_i^\top \lambda^{k+1}, \partial r_i(x_i^{k+1})\right) \leq D\left[\left(\sum_{j=1}^n \|A_j\|_2 \left\|x_j^k - x_j^{k+1}\right\|_2\right) + (2L_i + 1)\left\|x_i^k - x_i^{k+1}\right\|_2\right],$$

where $i \in \mathcal{I}$ and $D = \max\limits_{1 \leq i \leq n} \{\operatorname{diam}(\mathcal{X}_i)\}$.

Furthermore, we have $\left\|Ax^{k+1} + s^{k+1} - b\right\|_2 = \frac{1}{\beta}\left\|\lambda^k - \lambda^{k+1}\right\|_2$ and $s_j^{k+1}\lambda_j^{k+1} = \delta$ for $1 \leq j \leq p$. It follows from Lemma 12 that there exists sufficiently large $\bar{K} > 0$ such that, for $k \geq \bar{K}$, we have

$$\begin{aligned} D\left[\left(\sum_{j=1}^n \|A_j\|_2 \left\|x_j^k - x_j^{k+1}\right\|_2\right) + (L_i + 1)\left\|x_i^k - x_i^{k+1}\right\|_2\right] &\leq \delta, \\ \frac{1}{\beta}\left\|\lambda^k - \lambda^{k+1}\right\|_2 &\leq \delta, \end{aligned}$$

for cyclic variable selection rule, and

$$\begin{aligned} \mathbb{E}\left[D\left[\left(\sum_{j=1}^n \|A_j\|_2 \left\|x_j^k - x_j^{k+1}\right\|_2\right) + (L_i + 1)\left\|x_i^k - x_i^{k+1}\right\|_2\right]\right] &\leq \delta, \\ \frac{1}{\beta}\mathbb{E}\left\|\lambda^k - \lambda^{k+1}\right\|_2 &\leq \delta, \end{aligned}$$

for randomized variable selection rule. In this case, the stopping criterion is satisfied, and hence the above inequality holds for $\left(\bar{x}, \bar{s}, \bar{\lambda}\right)$ when optimizing δ-smoothed version.

In conclusion, we obtain that

$$\begin{aligned}\operatorname{dist}\left(-\nabla_i f(\bar{x}) - A_i^\top \bar{\lambda}, \partial r_i(\bar{x}_i)\right) &\leq \delta,\ i = 1, \ldots, n,\\ \bar{s}_j \bar{\lambda}_j - \delta &= 0,\ j = 1, \ldots, p,\\ \|A\bar{x} + \bar{s} - b\|_2 &\leq \delta,\end{aligned}$$

for cyclic variable selection rule, and

$$\begin{aligned}\mathbb{E}\left[\operatorname{dist}\left(-\nabla_i f(\bar{x}) - A_i^\top \bar{\lambda}, \partial r_i(\bar{x}_i)\right)\right] &\leq \delta,\ i = 1, \ldots, n,\\ \mathbb{E}\left[\bar{s}_j \bar{\lambda}_j - \delta\right] &= 0,\ j = 1, \ldots, p,\\ \mathbb{E}\left[\|A\bar{x} + \bar{s} - b\|_2\right] &\leq \delta,\end{aligned}$$

for randomized variable selection rule. □

A.2 Proof of Theorem

Since a sequence of $(\bar{x}, \bar{s})$ remains bounded as δ decreases, the set of the limiting points is non-empty. We consider a sub-sequence of $(\bar{x}, \bar{s})$ indexed by $\{k_l\}_{l=1}^{+\infty}$ which converges to (x^*, s^*). By using Lemma 13, we conclude that, there exists $\bar{\lambda} \in \mathbb{R}^p$ such that $(\bar{x}, \bar{s})$ indexed by k_l satisfies that

$$\begin{aligned}\operatorname{dist}\left(-\nabla_i f(\bar{x}) - A_i^\top \bar{\lambda}, \partial r_i(\bar{x}_i)\right) &\leq (\gamma)^{k_l}\delta_0,\ i = 1, \ldots, n,\\ \left\|\bar{s}_j \bar{\lambda}_j - (\gamma)^{k_l}\delta_0\right\|_2 &\leq (\gamma)^{k_l}\delta_0,\ j = 1, \ldots, p,\\ \|A\bar{x} + \bar{s} - b\|_2 &\leq (\gamma)^{k_l}\delta_0,\end{aligned}$$

deterministically for cyclic variable selection rule and

$$\begin{aligned}\mathbb{E}\left[\operatorname{dist}\left(-\nabla_i f(\bar{x}) - A_i^\top \bar{\lambda}, \partial r_i(\bar{x}_i)\right)\right] &\leq (\gamma)^{k_l}\delta_0,\ i = 1, \ldots, n,\\ \mathbb{E}\left[\left\|\bar{s}_j \bar{\lambda}_j - (\gamma)^{k_l}\delta_0\right\|_2\right] &\leq (\gamma)^{k_l}\delta_0,\ j = 1, \ldots, p,\\ \mathbb{E}\left[\|A\bar{x} + \bar{s} - b\|_2\right] &\leq (\gamma)^{k_l}\delta_0,\end{aligned}$$

for randomized variable selection rule. Here δ_0 is the initial smoothing parameter. We know that $(\gamma)^{k_l} \to 0$ as $l \to +\infty$ since $\gamma \in (0, 1)$. Therefore, we conclude that x^* is a stationary solution of problem (1) deterministically for cyclic variable selection rule and in terms of expectation for randomized variable selection rule. This completes the proof.

B Proof of Theorem 10

B.1 Proof of Technical Lemma

We present a technical lemma to show the number of iterations required to reach the stopping criterion when optimizing δ-smoothed version of problem (1), *i.e.*, problem (2). Specifically, this number is

disproportionate to the value of δ, which makes a lot of sense since problem (2) becomes harder as $\delta \to 0$.

Lemma 14. *Suppose cyclic or randomized variable selection rule is employed for optimizing δ-smoothed version, i.e., problem (2), the stopping criterion is satisfied deterministically or in terms of expectation, respectively, within $O(\frac{1}{\delta})$ iterations.*

Proof. From Lemma 13, it suffices to show that, there exists $C > 0$ such that, if $k \geq \frac{C}{\delta}$, the following statement holds true,

$$\begin{aligned} D\left[\left(\sum_{j=1}^{n} \|A_j\|_2 \left\|x_j^k - x_j^{k+1}\right\|_2\right) + (L_i+1)\left\|x_i^k - x_i^{k+1}\right\|_2\right] &\leq \delta, \\ \frac{1}{\beta}\left\|\lambda^k - \lambda^{k+1}\right\|_2 &\leq \delta, \end{aligned}$$

for cyclic variable selection rule, and

$$\begin{aligned} \mathbb{E}\left[D\left[\left(\sum_{j=1}^{n} \|A_j\|_2 \left\|x_j^k - x_j^{k+1}\right\|_2\right) + (L_i+1)\left\|x_i^k - x_i^{k+1}\right\|_2\right]\right] &\leq \delta, \\ \frac{1}{\beta}\mathbb{E}\left\|\lambda^k - \lambda^{k+1}\right\|_2 &\leq \delta, \end{aligned}$$

for randomized variable selection rule.

It follows from Lemma 12 that, for $\forall K \geq 1$, we have

$$\Phi\left(x^0, s^0, \lambda^0\right) - \Phi^* \geq \sum_{k=0}^{K}\left[\frac{1}{2}\sum_{i\in\mathcal{I}_{k+1}}\left\|x_i^k - x_i^{k+1}\right\|_2^2 + \left(\frac{\beta\underline{s}^2}{2\mu} - \frac{\mu}{\beta\underline{s}^2}\right)\left\|\lambda^k - \lambda^{k+1}\right\|_2^2\right], \tag{14}$$

where $\mathcal{I}_k$ is denoted as the active set chosen at the k-th iteration of the inner loop when optimizing δ-smoothed version.

Combining the fact that

$$\left\|x^k - x^{k+1}\right\|_2^2 = \sum_{i\in\mathcal{I}_{k+1}}\left\|x_i^k - x_i^{k+1}\right\|_2^2,$$

yields that the iteration complexity is $O(\frac{1}{\delta})$. By using similar technique, we can obtain the same complexity for randomized variable selection rule, where it holds true that

$$\begin{aligned} \mathbb{E}\left[D\left[\left(\sum_{j=1}^{n} \|A_j\|_2 \left\|x_j^k - x_j^{k+1}\right\|_2\right) + (L_i+1)\left\|x_i^k - x_i^{k+1}\right\|_2\right]\right] &\leq \delta, \\ \frac{1}{\beta}\mathbb{E}\left\|\lambda^k - \lambda^{k+1}\right\|_2 &\leq \delta. \end{aligned}$$

This completes the proof. □

B.2 Proof of Theorem

On one hand, it is clear to derive from $\delta \leftarrow \gamma\delta$ that $\delta \leq \epsilon$ is satisfied within $O(\log(\frac{1}{\epsilon}))$ iterations in term of the iteration number in outer loop. On the other hand, by using Lemma 14, we obtain the iterations required in term of the iteration number in inner loop is

$$T = \sum_{k=0}^{\log(\frac{1}{\epsilon})} \frac{C}{\gamma^k \delta_0} = \frac{C}{\delta_0} \frac{\frac{1}{\epsilon} - 1}{\frac{1}{\gamma} - 1} \leq \frac{C\gamma}{\delta_0 - \gamma\delta_0} \frac{1}{\epsilon},$$

which implies that the iterations required in term of the iteration number in inner loop is $O(\frac{1}{\epsilon})$. This completes the proof.