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Spectral Super-resolution and Band-limited Extrapolation Using Slepian Series

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Abstract: Band-limited signal extrapolation and spectral super-resolutions are closely related. They both can be achieved using Slepian series. This method is often believed to be dependent on computing definite integrals. Instead of using numerical integration, a simple and reliable method, which can be implemented in MATLAB, is presented. This way, it is much easier to investigated the extent to which a signal can be extrapolated from its samples taken in a finite interval. The proposed method is tested with data from a real microphone array.

Keywords: Prolate spheroidal wave functions, Slepian series, Signal extrapolation, Sampled signals, Frequency spectrum, Fourier analysis, Bandwidth

1. INTRODUCTION

Sampled band-limited signals play an important role in most parts of everyday modern life. Measured signal is always finite in time and sampled at discrete time instants. Thus if one is interested in computing its spectrum, an additional assumption about the missing data has to be made. Fourier series is computed assuming that the signal is periodic on the whole real line of time; Fourier transform (sometimes referred to as Fourier integral) is computed assuming that the signal is identically zero outside the observation interval.

In general both these assumptions lead to presence of spurious harmonics in the obtained spectrum due to discontinuities at both ends of the observation interval. The problem can be mitigated by suppressing these discontinuities using a window function; however, by applying window functions of different shapes to the same signal, one is able to achieve almost arbitrary spectrum. There is no consensus which window is the 'best', unless some additional properties of the signal are a priori known. Furthermore by suppressing the signal amplitude at both ends of the observation interval, the interval is further shortened. This leads to lower spectral resolution due to the uncertainty principle.

If there was a mathematical tool capable of bypassing aforementioned problems, it would be greatly appreciated. Such a tool exists—it is called the Prolate spheroidal wave functions (PSWFs). They are also referred to as 'Slepian functions'. Solid theoretical foundations were laid by mathematicians Slepian, Pollak, and Landau in the 1960s, see Slepian (1961). Sadly, for many decades this tool has not attracted as much attention as it deserves. Even though the extrapolation method was simple, the main obstacle was the computational burden of PSWFs. In the 1980s the attention shifted towards iterative methods based on Gerchberg-Papoulis algorithm, see Papoulis (1978). Nowadays, there is no longer problem with computing PSWFs; the freely available software for computing their values with arbitrary precision was created by Adelman (2014) and the software for computing their discrete version, the Discrete prolate spheroidal sequences (DPSS's), is now a standard feature of MATLAB. New papers concerning signal extrapolation using Slepian functions, such as Gosse (2010) or Devasia (2013), begin to appear. However, these papers treat only idealized signal processing, expecting that no noise is present in the data.

The aim of this article is to investigate feasibility of extrapolation if the noise is present. It is organised as follows. Section 2 gives an overview of the mathematical framework needed for band-limited extrapolation. Section 3 deals with the regularization of Slepian series, since it is vital for dealing with the effects of noise. Instead of abrupt truncation, Tikhonov regularization is presented. Section 4 shows extrapolation of a simulated signal. Finally, in section 5, the proposed method is tested by extrapolating a two-dimensional signal of acoustic pressure, which was measured by a real microphone array.

2. THE PRINCIPLE OF BAND-LIMITED EXTRAPOLATION

2.1 Preliminaries

In many scientific fields, it is desired to obtain the Fourier transform

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t \qquad (1)$$

of a signal $f(t)$. However, in a real-world application, the signal is observed only at a finite interval of length T. Thus

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the measured signal is of the form

$$g(t) = \begin{cases} f(t), & \text{for} \quad |\,t\,| \le T/2, \\ \text{undefined}, & \text{for} \quad |\,t\,| > T/2; \end{cases} \tag{2}$$

the rest of the signal remains unknown.

In order to compute the Fourier transform (1), the measured signal $g(t)$ has to be defined for all real t. The most common approach is to assume

$$g(t) = 0 \quad \text{for all} \quad |t| > T/2 \tag{3}$$

or to periodically extend $g(t)$ for all $|t| > T/2$. However a much more interesting option is to assume that

$$F(\omega) = 0 \quad \text{for all} \quad |\,\omega\,| > \Omega. \tag{4}$$

Signals associated with spectrum of the form (4) are called band-limited (with band-limit Ω). This assumption is not uncommon, for instance, it is essential for Nyquist–Shannon sampling theorem and, according to Slepian (1976), there are many signals which can be (to a certain degree of precision) considered band-limited.

For the sake of simplicity, from here to the end of section 4, it is considered that the observation interval is $\langle -1, 1 \rangle$. This can be done without loss of generality, because (by the proper change of time-scale and time-shift) the observation interval can be normalized to $\langle -1, 1 \rangle$.

2.2 Slepian functions

PSWFs or 'Slepian functions', denoted $\psi_n(c,t)$, are special functions defined by the integral equation

$$\int_{-1}^{1} \frac{\sin c(t-s)}{\pi(t-s)} \psi_n(c,s)\,\mathrm{d}s = \lambda_n(c)\psi_n(c,t) \tag{5}$$

for all real t and non-negative integers n. The eigenvalues $\lambda_n(c)$ are real and they satisfy the inequality

$$1 > \lambda_0(c) > \lambda_1(c) > \lambda_2(c) > \cdots > 0. \tag{6}$$

Both the functions $\psi_n(c,t)$ and the eigenvalues $\lambda_n(c)$ are continuous functions of a parameter c. The convolution in (5) represents a time-limiting operation followed by a band-limiting operation. Therefore the c is the band-limit of Slepian functions.

When the order n is smaller than $2c/\pi$, the eigenvalues are close to 1, Slepian functions look similar to Hermite functions, and their energy outside the interval $\langle -1, 1 \rangle$ is negligible (Fig. 2, top). However, the eigenvalues fall off to zero rapidly with increasing n once n has exceeded the critical value $2c/\pi$. This behaviour is illustrated by the Fig. 1.

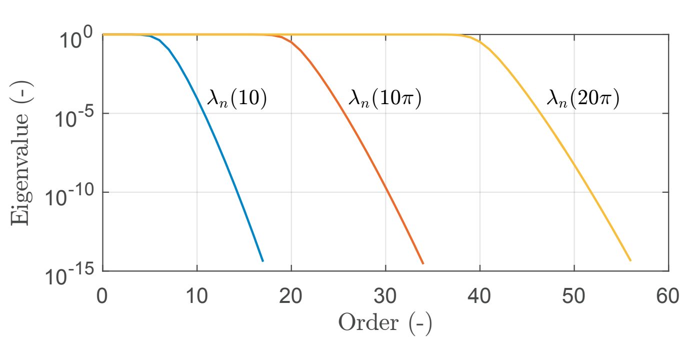

Fig. 1. The eigenvalues of integral equation (5).

If $n > 2c/\pi$, the amplitude of Slepian functions for $|t| > 1$ is significant. The amplitude further grows with increasing order.

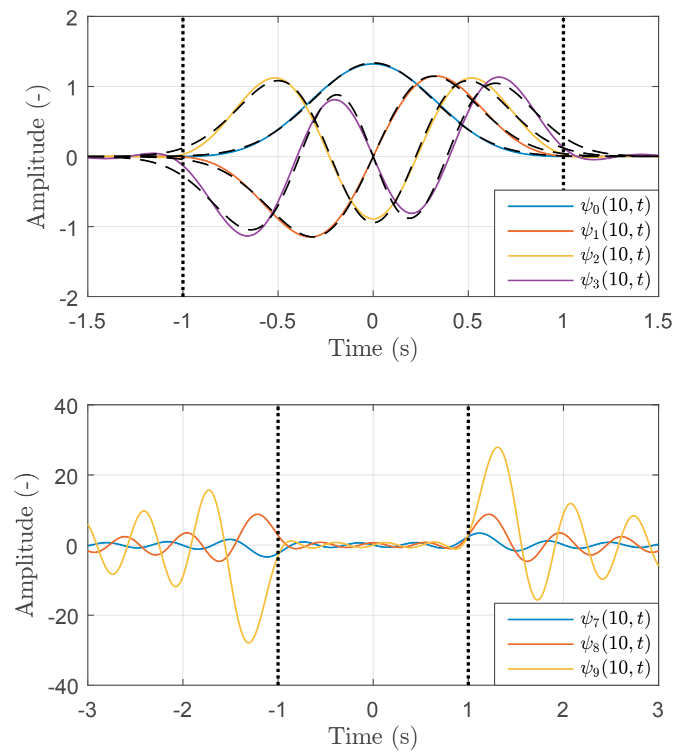

Fig. 2. Slepian functions for $c = 10$ and $T = 2$: Comparison of Slepian functions [solid] with corresponding Hermite functions [dashed] (top). Three Slepian functions of orders $n > 2c/\pi$ (bottom).

Slepian functions possess a unique and rather surprising property—they are orthogonal over a finite as well as infinite interval.

$$\int_{-1}^{1} \psi_n(c,t)\psi_m(c,t)\,\mathrm{d}t = \delta_{mn}. \tag{7}$$

$$\int_{-\infty}^{\infty} \psi_n(c,t)\psi_m(c,t)\,\mathrm{d}t = \frac{\delta_{mn}}{\lambda_n(c)}. \tag{8}$$

The symbol δ_{mn} stands for the Kronecker delta. Slepian functions are eigenfunctions of Fourier transform

$$\mathcal{F}\{\psi_n(c,t)\} = \begin{cases} \mathrm{j}^{-n}\sqrt{\dfrac{2\pi}{c\lambda_n(c)}}\,\psi_n\!\left(c, \dfrac{\omega}{c}\right), & |\omega| \le c, \\ 0, & |\omega| > c. \end{cases} \tag{9}$$

These and many other properties make them ideal for processing of band-limited signals.

2.3 The principle of band-limited extrapolation

Let $\mathcal{B}$ denote the class of all square-integrable band-limited functions. Slepian functions form a complete orthogonal basis with respect to the class $\mathcal{B}$, i.e. any function $f(t) \in \mathcal{B}$ can be expressed in the form of Slepian series

$$f(t) = \sum_{n=0}^{\infty} a_n \psi_n(c,t), \tag{10}$$

where a_n are the Slepian coefficients. The coefficients can be obtained using the orthogonality (7) or (8).

$$a_n = \lambda_n(c)\int_{-\infty}^{\infty} f(t)\psi_n(c,t)\,\mathrm{d}t = \int_{-1}^{1} f(t)\psi_n(c,t)\,\mathrm{d}t \quad (11)$$

Using the right-hand side of this equation, one is able to compute Slepian coefficients a_n using only the knowledge of the measured part of the signal (2). Then, by plugging these coefficients into (10), which is defined for all real t, the signal is extrapolated.

For a simple sine wave $\sin\omega_0 t$, observed at $|t| \leq 1$, the coefficients computed using the right-hand side of (11) are

$$a_n = \sin\left(n\frac{\pi}{2}\right)\sqrt{\frac{2\pi\lambda_n(c)}{c}}\,\psi_n\left(c, \frac{\omega_0}{c}\right). \quad (12)$$

It is noteworthy that, for the purpose of extrapolation, the coefficients are valid only if $\omega_0 \leq c$. If the condition is not met, the error $\|f(t) - f_N(t)\|_1$ still converges to zero; however, no convergence is guaranteed for $\|f(t) - f_N(t)\|_\infty$.

If one is more interested in super-resolution, rather than extrapolation, the Slepian series can be constructed in the frequency domain (instead of the time domain). The obtained spectrum is given by the series

$$F(\omega) = \sum_{n=0}^{\infty} a_n \mathcal{F}\{\psi_n(c,t)\}, \quad (13)$$

where the Fourier transform of Slepian function is given by (9). This way there is no need to compute the standard Fourier transform or to apply a window function.

2.4 Numerical computation of Slepian coefficients

When the signal is measured, only a finite number of samples is known. Let

$$\boldsymbol{g} = [g(t_1)\ g(t_2)\ \dots\ g(t_K)]^\mathsf{T} \quad (14)$$

denote the collection of known samples acquired at time instants $\{t_k\}_{k=1}^{K} \subset \langle -T/2, T/2\rangle$.

Clearly, (11) is not suitable for any practical data processing. It is impossible to compute the integral exactly; it has to be approximated by means of numerical integration—for instance using Newton-Cotes or Gaussian quadrature rules. The former is essentially the case of Devasia (2013). He proposes polynomial interpolation and integration of order 250. Obviously, such high-order interpolation reeks havoc due to instability if noise is present.

In case of noisy signal, these polynomial methods would require optimization of their order, which leads to unnecessary complication of Slepian's idea. In fact the solution to the problem does not lie in the proper choice of the approximating method, it lies in the choice of proper orthogonal basis—the Slepian sequences.

2.5 Slepian sequences

Discrete prolate spheroidal sequences (DPSS's) or simply 'Slepian sequences' are discrete versions of Slepian functions, see Slepian (1978). They are defined by the equation

$$\sum_{l=1}^{K} \frac{\sin 2\pi W(k-l)}{\pi(k-l)}\psi_n(K,W,l) = \lambda_n(K,W)\psi(K,W,k). \quad (15)$$

Both the sequences and the eigenvalues are functions of index-limit K and band-limit $2\pi W < \pi$. They are orthogonal with respect to summation over a finite as well as infinite interval

$$\sum_{k=1}^{K}\psi_n(K,W,k)\psi_m(K,W,k) = \delta_{mn}, \quad (16)$$

$$\sum_{k=-\infty}^{\infty}\psi_n(K,W,k)\psi_m(K,W,k) = \frac{\delta_{mn}}{\lambda_n(c)}. \quad (17)$$

This makes them ideal for processing of uniformly sampled signals. Slepian sequences can be easily computed—the software for generating Slepian sequences is implemented in MATLAB known as the `dpss`. The sequences are generated only for $1 \leq k \leq K$. The samples outside the observation interval can be computed using the convolution (15). Even if the signal is sampled non-uniformly, it can be still processed using Slepian functions. Then the only disadvantage is that the orthogonality cannot be used; the least-squares approximation has to be used in order to obtain Slepian coefficients.

3. SERIES TRUNCATION AND REGULARIZATION

The Slepian coefficients obtained from a noisy signal

$$\tilde{a}_n = a_n + \epsilon_n \quad (18)$$

can be regarded as a superposition of the deterministic part a_n and the noisy part ϵ_n. In a real-world application, the series (10) has to be truncated. Let

$$f_N(t) = \sum_{n=0}^{N}\tilde{a}_n\psi_n(c,t) \quad (19)$$

denote the Nth order approximation of signal $f(t)$. As often mentioned, with increasing n the method becomes unstable. In order to avoid the blow-up problem, Slepian series has to be truncated at certain order N. The optimal order is dependent on the parameter c and the Signal-to-noise ratio (SNR). Finding it by hand is not an easy task; it is admissible only in the absence of noise.

3.1 Tikhonov regularization

By applying Tikhonov regularization, the abrupt truncation is avoided. The regularized solution is given by

$$f_{N,\alpha}(t) = \sum_{n=0}^{N}\frac{\tilde{a}_n\lambda_n^2(c)}{\alpha + \lambda_n^2(c)}\psi_n(c,t), \quad (20)$$

where α denotes the regularization parameter. Some of the coefficients a_n, which would normally be discarded, now partially contribute to the extrapolation process. The parameter α may be determined using Morozov discrepancy principle—by solving the discrepancy equation

$$\frac{1}{K}\sum_{k=1}^{K}|f_{K,\alpha}(k) - g(k)|^2 = \sigma^2. \quad (21)$$

It requires knowledge of the noise variance σ. Fortunately, the variance can be computed quite easily.

3.2 Estimation of the variance of the signal noise

Here proposed method was already used by Williams (2000) in the field of acoustic holography for regularization of signal reconstruction using singular value decomposition.

Slepian coefficients usually decay very rapidly once n has exceeded the critical value $2c/\pi$. For example, for a sine wave, they decay with $\sqrt{\lambda_n(c)}$, see (12). Thanks to the rapid decay, the expected value of Slepian coefficient is very close to the variance of noise, when its order n is large.

$$\mathrm{EX}(|\tilde{a}_n|) \approx \sigma \quad \text{for} \quad n \gg 2c/\pi. \tag{22}$$

Using the coefficients of orders from M to N, the noise variance can be estimated by

$$\frac{1}{N-M+1} \sum_{n=M}^{N} \left| \sum_{k=1}^{K} f(k)\psi_n(K,W,k) \right|^2 \approx \sigma^2 \tag{23}$$

providing that $M, N \gg 2c/\pi$.

4. NUMERICAL EXPERIMENT

The example was chosen in accordance with Gosse (2010) and Devasia (2013) so the reader may easily notice the effect of noise. The signal

$$f(t) = \sin(\pi t)\cos(3\pi t)\mathrm{e}^{-2t^2+3t} + 0.5[\sin(5\pi t) - \cos(7\pi t)] \tag{24}$$

is observed for $|t| \leq 1$ s and sampled with period $T_\mathrm{s} = 10^{-3}$ s. The samples contain white Gaussian noise with variance $\sigma = 10^{-3}$.

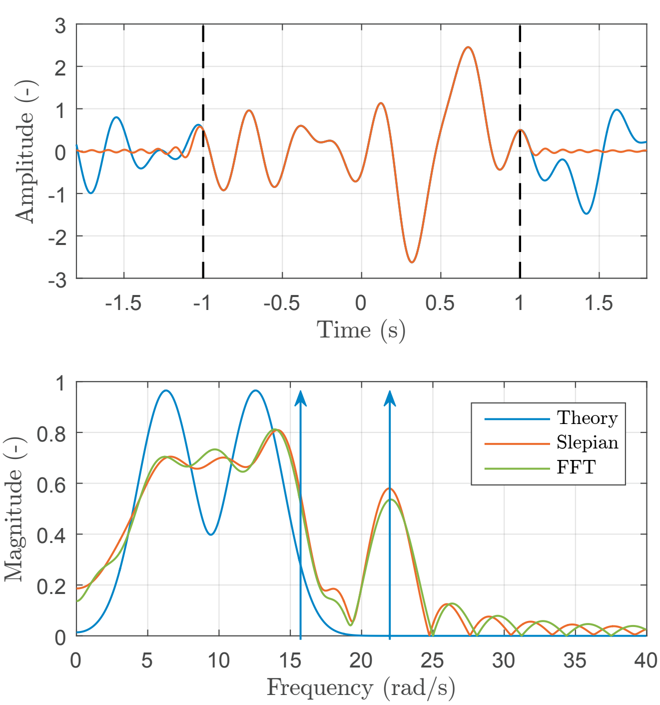

Fig. 3. Extrapolation for $c = 20\pi$: Original signal [blue] and extrapolated [red] in time domain (top) and comparison of the Fourier spectra (bottom).

Slepian sequences are the most suitable tool for extrapolation, when the signal is sampled uniformly. Slepian coefficients are shown in Fig. 4. Notice that the coefficients do not decay for $n > 48$. As expected, they fluctuate around $20\log(\sigma)$. These coefficients were used for estimation of the noise variance. Fig. 3 shows the extrapolation obtained using Tikhonov regularization in both time and frequency domain. The effects of noise are devastating: negligible part of signal was extrapolated; slight improvement of spectral resolution occurred at the frequency of 7π rad/s.

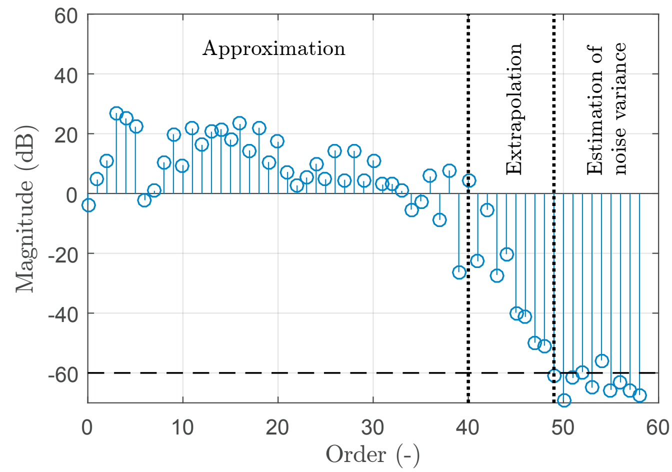

Fig. 4. Slepian coefficients: The coefficients play different roles in the process of signal extrapolation depending on their order.

4.1 *Importance of the correct choice of the band-limit*

In order to illuminate how crucial the choice of the c is, the experiment was repeated with $c = 7.5\pi$. Now, the spectral resolution is significantly increased in comparison to the standard Fourier transform—a 'true' super-resolution is achieved (Fig. 5, bottom).

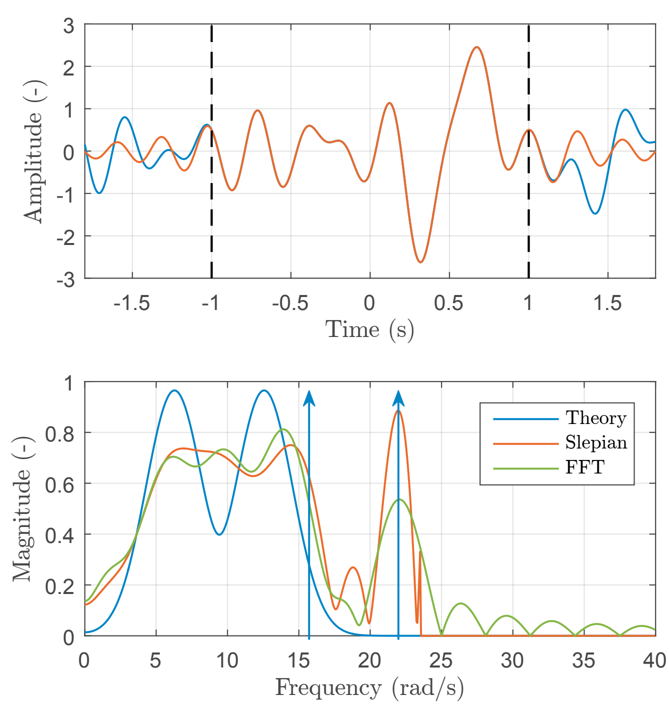

Fig. 5. Extrapolation for $c = 7.5\pi$: Original signal [blue] and extrapolated [red] in time domain (top) and comparison of the Fourier spectra (bottom).

When $c \to \pi/T_s$, no extrapolation is possible and the results of Slepian method are similar to those of Fourier transform. The only way to improve extrapolation (and spectral resolution) is by lowering the c as close to the signal's band-limit Ω as possible. In real data-processing the c has to be set according to the knowledge of a physical phenomena which causes that the measured signal is band-limited.

5. EXAMPLE OF APPLICATION

In the field of acoustic holography, see Williams (2000), a microphone array is used to measure the amplitude of sound pressure $\hat{p}(x, y)$ (a complex functions of spatial coordinates x and y). Further data processing often requires spatial Fourier transform of $\hat{p}(x, y)$; however, there are two factors limiting the transform: a) limited dimensions of the microphone array b) limited number of samples—given by the number of microphones. By using Slepian method instead of Fourier transform, data processing may be improved.

5.1 Extension to two dimensions

In order to apply the proposed method to a two-dimensional signal, a generalization has to be made. Combining the ideas of Slepian (1964, 1978), it is possible to seek discrete two-dimensional orthogonal sequences maximally concentrated in a rectangle consisting of $K \times K$ samples and band-limited in a circle of radius W. These sequences are solutions to the equation

$$\sum_{p=1}^{K}\sum_{q=1}^{K} \phi_n(K, W, p, q) \frac{W J_1\left[2\pi W\sqrt{(k-p)^2 + (l-q)^2}\right]}{\sqrt{(k-p)^2 + (l-q)^2}} = \\ = \lambda_n(K, W)\phi_n(K, W, k, l), \tag{25}$$

where $J_1(x)$ denotes Bessel function of order 1. Both the eigenfunctions and the eigenvalues are functions of index-limit K and band-limit $2\pi W < \pi$. These sequences can be regarded as a generalization of Slepian sequences to two dimensions.

5.2 Data from a microphone array

A harmonic sound source vibrating at the frequency $\omega_0 = 4\,000\pi$ rad/s was placed 35 cm above a microphone array. The sound source is a simple speaker and the array is a matrix of 8×8 microphones with spacing of 50 mm between the microphones. The sound pressure $p(x, y, t)$ was sampled at 51.2 kHz. The complex amplitude of sound pressure $\hat{\boldsymbol{p}}$ was obtained using FFT algorithm (Fig. 6, left). The matrix $\hat{\boldsymbol{p}}$ contains 64 samples of the $\hat{p}(x, y)$. In order to test Slepian method, only 6×6 samples (coloured) will be used for extrapolation; the remaining 26 samples (dark blue) are reserved for the verification of results.

In order to compute the spatial Fourier transform of $\hat{p}(x, y)$, an assumption about its band-limit has to be made. The c can be determined according to the physical phenomena of far-field holography. In the distance of 35 cm from the sound source, virtually all evanescent waves (the waves with spatial frequency exceeding ω_0/v, where v denotes the speed of sound) should be reliably eliminated by exponential decay of their amplitude with distance. Therefore the signal can be considered band-limited with band-limit ω_0/v. In our experiment, the band-limit should be approximately 37 rad/m.

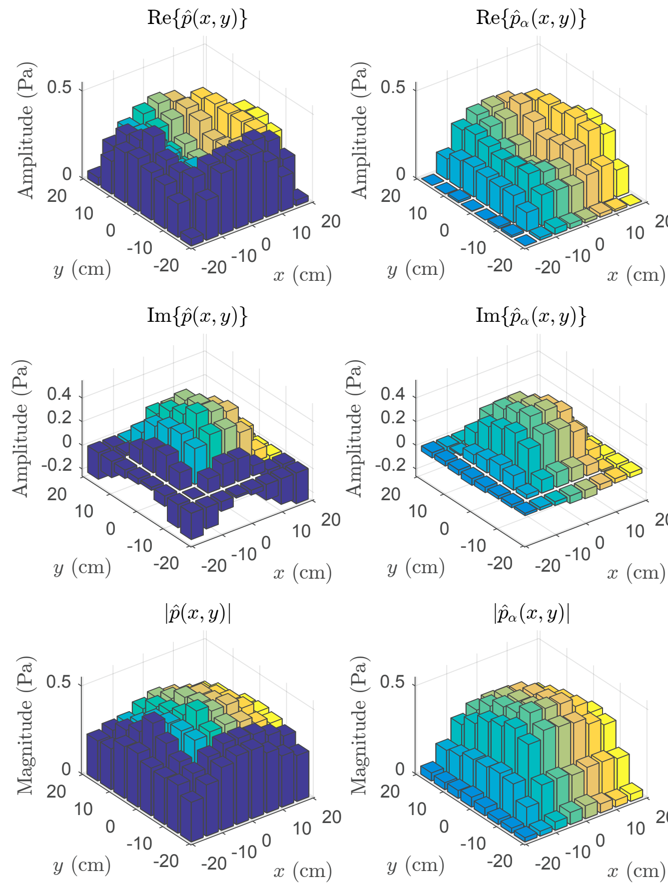

Fig. 6. Sound pressure: Measured spatial signal (left) is divided into 6×6 samples used for extrapolation [coloured] and 26 samples used for validation [dark blue]. Extrapolated signal (right). From top to bottom: real part, imaginary part, and absolute values.

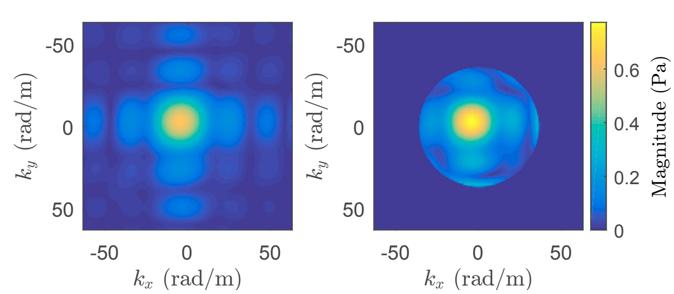

Fig. 7. Spatial Fourier transform: Plot of Fourier spectrum obtained by FFT (left, peak value 0.64 Pa) and by Slepian method (right, peak value 0.77 Pa).

The signal was extrapolated using Tikhonov regularization as in previous sections. Out of all 36 coefficients $\tilde{a}_n$ the last ten were used for estimation of the noise variance and the value $\sigma \approx 0.02$ Pa was obtained. For band-limited extrapolation, this value is relatively high. Nevertheless, there is no doubt that the signal was partially extrapolated (Fig. 6, right).

A slight super-resolution was achieved. The spectrum obtained from Slepian series (Fig. 7, right) is a bit sharper with higher peak value than the spectrum computed by FFT (Fig. 7, left). Had there been higher precision in data acquisition, better results could be achieved.

The experiment was repeated using a more restrictive band-limiting setting. The c was reduced by 25% and the results are shown in Fig. 8. There is no physical reason for such reduction of c; however, the quality of extrapolation has improved and the signal is visibly extended. The spectrum has peaked at 0.92 Pa.

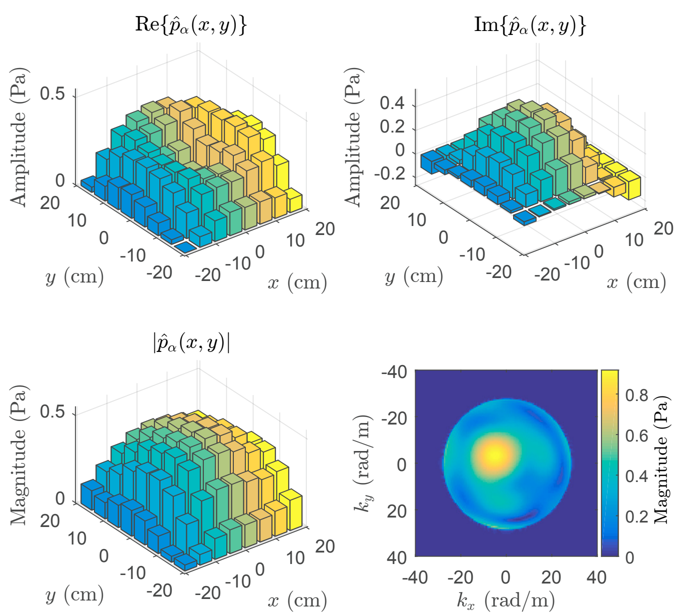

Fig. 8. Second extrapolation: Extrapolated sound pressure and its spatial Fourier transform obtained using band-limit of 27.75 rad/m.

If the limited data was treated by windowing, large part of the useful information would be lost and the k-space (spatial frequency domain) would show unsatisfactory resolution.

6. CONCLUSION

This paper shows how to implement a robust algorithm capable of practical band-limited signal extrapolation and spectral super-resolution. Thanks to the use of Slepian sequences (instead of Slepian functions), the unwieldy approximation chain involving numerical integration is successfully eliminated. The proposed method is relatively simple, can be implemented in MATLAB and, as it was demonstrated, it is suitable for processing of measured signals.

It was shown that the noise is an important limiting factor and it is noteworthy that the parameter c plays a much more important role than previously anticipated. In order to improve the performance of Slepian method, c should be chosen as close to the signal's band-limit as possible.

Extrapolation and super-resolution are often treated separately. Here, they were presented together, so that the reader can see their close relation. Compared to windowing, Slepian method extends signals instead of shortening them. It is an interesting alternative to the classical approach of windowing and FFT.

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