FINITENESS OF SEMIGROUPS OF OPERATORS IN UNIVERSAL ALGEBRA

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Introduction. This paper is a partial solution of problem 24 in **(2)** which suggests that the finiteness of the partially ordered semigroups generated by various combinations of operators on classes of universal algebras be investigated. The main result is that the semigroups generated by the following sets of operators (for definitions see § 2) are finite: $\{H, S, P, P_S\}$, $\{C, H, S, P, P_F\}$, $\{C, H, S, P_U, P_F\}$.

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1. Partially ordered semigroups. A partially ordered semigroup is a triple $(G, \leqslant, \phi)$ consisting of a set G, a partial order $\leqslant$ on G, and an associative binary operation ϕ on G such that, for all $a, b, c, d \in G$, if $a \leqslant c$ and $b \leqslant d$, then $\phi(a, c) \leqslant \phi(b, d)$. As usual we write ab for $\phi(a, b)$ and G for $(G, \leqslant, \phi)$. A positively ordered semigroup is a partially ordered semigroup G such that, for all $a, b \in G$, $ab \geqslant a$ and $ab \geqslant b$.

Let G be a partially ordered semigroup generated by a set S. The elements of G are products of finite sequences of elements of S. For any non-empty set $T \subseteq S$, let $[T]$ be the subsemigroup of G generated by T. The T-length $L_T(a)$ of an element $a \in [T]$ is defined to be the smallest natural number n for which there exists an n-element sequence in T, the product of which is a.

THEOREM 1. *If G is a positively ordered semigroup generated by n idempotent elements $x_1, \ldots, x_n$ such that if $i \leqslant j$ then $x_j x_i \leqslant x_i x_j$, then G is finite.*

Proof. We prove by induction on k that if $S_k = \{x_1, \ldots, x_k\}$, then for any $a \in [S_k]$, $L_{S_k}(a) \leqslant 2^k - 1$.

In the case $k = 1$, $S_1 = \{x_1\}$ and $[S_1] = \{x_1\}$ since x_1 is idempotent and $L_{S_1}(x_1) = 1 = 2^k - 1$.

Suppose there exists an $a \in [S_{k+1}]$ with $L_{S_{k+1}}(a) = m \geqslant 2^{k+1}$. Then a is equal to a product of an m-element sequence of elements from S_{k+1}, and, since $m \geqslant 2^{k+1}$, by the induction hypothesis, there are at least two occurrences of x_{k+1} in this m-element sequence. Thus $a = bx_{k+1}\, cx_{k+1}\, d$, where $b, c, d \in [S_{k+1}]$ and

$$L_{S_{k+1}}(b) + L_{S_{k+1}}(c) + L_{S_{k+1}}(d) + 2 = m.$$

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(Or $a = bx_{k+1}\,cx_{k+1}$ or $a = x_{k+1}\,cx_{k+1}d$; but these cases can be treated in the same way as the first.) Since for all $x_i \in S_{k+1}$, $x_{k+1}\,x_i \leqslant x_i\,x_{k+1}$, it follows that

$$bx_{k+1}\,cx_{k+1}^2\,d \leqslant bcx_{k+1}^2\,d = bcx_{k+1}\,d.$$

But $bcx_{k+1}\,d \leqslant bx_{k+1}\,cx_{k+1}\,d$ since G is positively ordered, and hence $a = bcx_{k+1}d$. This implies that

$$L_{S_{k+1}}(a) \leqslant L_{S_{k+1}}(b) + L_{S_{k+1}}(c) + L_{S_{k+1}}(d) + 1 = m - 1 < m,$$

and this contradicts the assumption that $L_{S_{k+1}}(a) = m$. Hence for all $a \in [S_{k+1}]$, $L_{S_{k+1}}(a) \leqslant 2^{k+1} - 1$.

In particular, for all $x \in G$, $L_S(x) \leqslant 2^n - 1$; hence G is finite.

It may be noted at this point that if x and y are idempotent elements in a positively ordered semigroup G, then $x \leqslant y$ implies $yx \leqslant xy$, since then $xy \leqslant yy = y \leqslant xy$ and $yx \leqslant yy = y \leqslant yx$, so that $xy = y = yx$.

2. Semigroups of operators in a universal algebra. A universal algebra is a set together with a family of finitary operations defined on that set. Two universal algebras are of the same type if their families of operations have the same indexing set and, if the two families are $(f_\lambda)_{\lambda \in L}$ and $(g_\lambda)_{\lambda \in L}$ and if f_λ is an n-ary operation, then g_λ is also n-ary. All algebras under consideration will be of the same type. We assume that the reader is familiar with the notion of subalgebra, homomorphic image, isomorphic image, direct product, congruence relation, and quotient of universal algebras; see **(1, 2)**.

A universal algebra U is a subdirect product of the family $(A_i)_{i \in I}$ of universal algebras if it is a subalgebra of the direct product $\prod A_i$ $(i \in I)$ and if, for each i, the restriction to U of the natural projection $P_j\colon \prod A_i \to A_j$ maps onto A_j.

If $(A_i)_{i \in I}$ is any family of universal algebras and if $\mathfrak{F}$ is a filter on I, then the relation $\equiv_{\mathfrak{F}}$ defined on $\prod A_i$ $(i \in I)$ by $f \equiv_{\mathfrak{F}} g$ if and only if

$$\{i \mid f(i) = g(i)\} \in \mathfrak{F}$$

(where the elements of a direct product $\prod A_i$ are denoted by choice functions on $(A_i)_{i \in I}$) is a congruence relation. The quotient algebra $\prod A_i/\mathfrak{F}$ of $\prod A_i$ modulo this congruence relation is called a filter product of the family $(A_i)_{i \in I}$. If $\mathfrak{F}$ is an ultrafilter, $\prod A_i/\mathfrak{F}$ is called an ultraproduct; see **(1)**. Note that if $\mathfrak{F} = \{I\}$, the filter product reduces to an ordinary product.

A universal algebra U is called a cover of the family $(A_i)_{i \in I}$ of universal algebras if, for each i, A_i is a subalgebra of U, and if $U = \cup A_i$ $(i \in I)$.

It may be noted at this point that all these operators are invariant under isomorphism, i.e., if $(A_i)_{i \in I}$ and $(B_i)_{i \in I}$ are families of universal algebras such that A_i is isomorphic to B_i for each $i \in I$, and if one of the above operations is applied to both of these families of algebras, the two resulting algebras will be isomorphic. For example, if $\mathfrak{F}$ is a filter on I, then the filter products $\prod_{i\in I} A_i/\mathfrak{F}$ and $\prod_{i\in I} B_i/\mathfrak{F}$ are isomorphic.

A class of universal algebras will be called algebraic if it contains all isomorphic copies of the algebras contained in it.

For an arbitrary class K of universal algebras we define $S(K)$ to be the smallest algebraic class containing all subalgebras of algebras in K. Then S is an operator on classes of universal algebras. Similarly the operators H, P, P_S, P_F, P_U, and C are defined, to correspond to homomorphic images, direct products, subdirect products, filter products, ultraproducts, and covers respectively.

For two operators X, Y we define XY by $XY(K) = X(Y(K))$ for an arbitrary algebraic class K of universal algebras. This defines an associative binary operation and so we can consider the semigroup G generated by these operators. The relation $\leqslant$ is defined on G by: $X \leqslant Y$ if and only if $X(K) \subseteq Y(K)$ for all algebraic classes K of universal algebras. Then $(G, \leqslant)$ is a partially ordered semigroup. Moreover, since if X is any one of S, H, P, P_S, P_F, P_U, or C, then $X(K) \supseteq K$ for all algebraic classes K of universal algebras, $(G, \leqslant)$ is a positively ordered semigroup.

LEMMA 1. *The operators S, H, P, P_S, P_F, P_U, and C are idempotent.*

Proof. The proof that $S^2 = S$, $H^2 = H$, $P^2 = P$, $P_S^2 = P_S$, $C^2 = C$ is trivial.

$P_F^2 = P_F$: Let K be an arbitrary algebraic class of universal algebras and let $A \in P_F^2(K)$. Since P_F is invariant under isomorphism, it is enough to consider $A = \prod_{i\in I} A_i/\mathfrak{F}$ where $\mathfrak{F}$ is a filter on I, $A_i = \prod_{j\in J_i} B_{ij}/\mathfrak{F}_i$ for each i, $\mathfrak{F}_i$ a filter on J_i, and $B_{ij} \in K$. Let

$$S = \prod_{i\in I} J_i = \{(i,j) \mid i \in I, j \in J_i\}.$$

The filtered sum $\mathfrak{A}$ of the family $(\mathfrak{F}_i)_{i\in I}$ of filters is defined as follows:

$$\mathfrak{A} = \{M \subseteq S \mid \{i \mid M(i) \in \mathfrak{F}_i\} \in \mathfrak{F}\},$$

where $M(i) = \{j \mid j \in J_i \text{ and } (i,j) \in M\}$. For more details on "filtered sums" see (**4**, pp. 330ff.). For each $f \in \prod B_{ij}\ ((i,j) \in S)$, and each $i \in I$, define $f(i,\ ) \in A_i$ by $f(i,\ )(j) = f(i,j)$ for $j \in J_i$. Define ϕ: $\prod_{(i,j)\in S} B_{ij} \to A$ by

$$\phi(f) = [([f(i,\ )]_{\mathfrak{F}_i})_{i\in I}]_{\mathfrak{F}},$$

where square brackets denote congruence classes, and subscripts the filters giving the congruence relations. For $f, g \in \prod B_{ij}\ ((i,j) \in S)$,

$$\begin{aligned}\phi(f) = \phi(g) \quad &\text{if and only if } \{i \mid [f(i,\ )]_{\mathfrak{F}_i} = [g(i,\ )]_{\mathfrak{F}_i}\} \in \mathfrak{F}\\ &\text{if and only if } \{i \mid \{j \in J_i \mid f(i,\ )(j) = g(i,\ )(j)\} \in \mathfrak{F}_i\} \in \mathfrak{F}\\ &\text{if and only if } \{(i,j) \mid f(i,j) = g(i,j)\} \in \mathfrak{A}\\ &\text{if and only if } f \equiv_{\mathfrak{A}} g.\end{aligned}$$

This, together with the fact that ϕ is an epimorphism, yields that

$$\prod_{(i,j)\in S} B_{ij}/\mathfrak{A}$$

is isomorphic to A. Thus $A \in P_F(K)$. Hence $P_F^2 \leqslant P_F$ and thus $P_F^2 = P_F$.

If $\mathfrak{F}$, $\mathfrak{F}_i$ are all ultrafilters, then $\mathfrak{A}$ is also an ultrafilter; see (**4**, p. 335). This, together with the above, yields $P_U^2 \leqslant P_U$, and hence $P_U^2 = P_U$.

LEMMA 2. *$P_F S \leqslant SP_F$, $P_U S \leqslant SP_U$, $PS \leqslant SP$.*

Proof. Let K be an arbitrary class of universal algebras and let $A \in P_F S(K)$. It is enough to consider $A = \prod_{i\in I} A_i/\mathfrak{F}$ where $\mathfrak{F}$ is a filter on I, and for each i, A_i is a subalgebra of B_i, and $B_i \in K$. Then the canonical homomorphism from $\prod_{i\in I} A_i/\mathfrak{F}$ to $\prod_{i\in I} B_i/\mathfrak{F}$ is a monomorphism, and hence $A \in SP_F(K)$. Thus $P_F S \leqslant SP_F$. The same argument shows that $P_U S \leqslant SP_U$ and, if $\mathfrak{F} = \{I\}$, that $PS \leqslant SP$.

LEMMA 3. *$P_F H \leqslant HP_F$, $P_U H \leqslant HP_U$, $PH \leqslant HP$.*

Proof. Let K be an arbitrary algebraic class of universal algebras and let $A \in P_F H(K)$. Since P_F is invariant under isomorphism, it is enough to consider $A = \prod_{i\in I} A_i/\mathfrak{F}$ where $\mathfrak{F}$ is a filter on I and, for each i, there is a $B_i \in K$ and an epimorphism $\phi_i: B_i \to A_i$. Let $B = \prod_{i\in I} B_i/\mathfrak{F}$, and define

$$\phi: \prod_{i\in I} B_i \to \prod_{i\in I} A_i$$

by $(\phi(f))(i) = \phi_i(f(i))$.

It is then obvious that, for any $f, g \in \prod B_i$ $(i \in I)$, $f \equiv_{\mathfrak{F}} g$ (in $\prod B_i$) implies that $\phi(f) \equiv_{\mathfrak{F}} \phi(g)$ (in $\prod A_i$). Hence ϕ induces an epimorphism ϕ' of

$$\prod_{i\in I} B_i/\mathfrak{F}$$

onto $\prod_{i\in I} A_i/\mathfrak{F}$. Hence $A \in HP_F(K)$, and thus $P_F H \leqslant HP_F$.

The same argument shows that $P_U H \leqslant HP_U$ and, if $\mathfrak{F} = \{I\}$, that $PH \leqslant HP$.

LEMMA 4. *$P_F C \leqslant CP_F$, $P_U C \leqslant CP_U$, $PC \leqslant CP$.*

Proof. Let K be an arbitrary algebraic class of universal algebras and let $A \in P_F C(K)$. We may consider $A = \prod_{i\in I} A_i/\mathfrak{F}$, where $\mathfrak{F}$ is a filter on I and, for each i, $A_i = \cup B_{ij}$ $(j \in J_i)$, where the B_{ij} are subalgebras of A_i and $B_{ij} \in K$. For each $r \in \prod_{i\in I} J_i = J$, let $A_r = \prod_{i\in I} B_{ir(i)}/\mathfrak{F}$. Then the canonical homomorphism $\phi_r: A_r \to A$ is a monomorphism. Moreover, $A = \bigcup_{r\in J} \phi_r(A_r)$. Hence $A \in CP_F(K)$, and so $P_F C \leqslant CP_F$. The same argument shows that $P_U C \leqslant CP_U$ and, in the special case where $\mathfrak{F} = \{I\}$, that $PC \leqslant CP$. ($PS \leqslant SP$, $PH \leqslant HP$ can be found in (**2**, p. 252); $PC \leqslant CP$ in (**3**, p. 1).)

THEOREM 2. *The partially ordered semigroups generated by the following sets of operators are finite:*

1. $\{H, S, P, P_S\}$.
2. $\{C, H, S, P, P_F\}$.
3. $\{C, H, S, P_F, P_U\}$.

Proof. These sets of operators generate positively ordered semigroups, and, by Lemma 1, each of the operators in these sets is idempotent. We apply Theorem 1:

1. It is sufficient to show that $SH \leqslant HS$, $PH \leqslant HP$, $PS \leqslant SP$, $P_S H \leqslant HP_S$, $P_S S \leqslant SP_S$, and $P_S P \leqslant PP_S$. The first, fourth, and fifth of these inequalities are proved in **(5**, pp. 44ff.**)**; $PH \leqslant HP$ and $PS \leqslant SP$ are proved above. Since $P \leqslant P_S$, $P_S P \leqslant PP_S$. Hence the semigroup generated by H, S, P, P_S is finite.

2. In addition to what has already been shown it is sufficient to show that $P_F H \leqslant HP_F$, $P_F S \leqslant SP_F$, $P_F C \leqslant CP_F$, $P_F P \leqslant PP_F$, $PC \leqslant CP$, $HC \leqslant CH$, and $SC \leqslant CS$. The first three and the fifth inequalities are proved in the lemmas above; $HC \leqslant CH$ and $SC \leqslant CS$ can be found in **(3)**, and, since $P \leqslant P_F$, $P_F P \leqslant PP_F$. Hence the semigroup generated by H, S, P, P_F, C is finite.

3. In addition to what is shown above, it is sufficient to show that $P_U H \leqslant HP_U$, $P_U S \leqslant SP_U$, $P_U C \leqslant CP_U$, and $P_U P_F \leqslant P_F P_U$. The proofs of the first three inequalities are in the lemmas above, and since $P_U \leqslant P_F$, $P_U P_F \leqslant P_F P_U$. Hence the semigroup generated by H, S, P_F, P_U, and C is finite.

The fact that H, S, and P generate a finite semigroup is stated in **(6)**.

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