Goldbach's conjecture

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ABSTRACT

This paper presents an elementary and short proof of the strong Goldbach conjecture. Whereas the traditional approaches focus on the control over the distribution of the primes by means of circle method and sieve theory, the proof is based on the constructive properties of the prime numbers, reflecting their multiplicative character within the natural numbers. With an equivalent but more convenient form of the conjecture in mind, we create a structure on the natural numbers. That structure leads to arithmetic identities which immediately imply the conjecture, more precisely, an even strengthened form of it. Moreover, we can achieve further results by generalizing the structuring. Thus, it turns out that the statement of the strong Goldbach conjecture is the special case of a general principle.

1. INTRODUCTION

In the course of the various attempts to solve the strong and the weak Goldbach conjecture – both formulated by Goldbach and Euler in their correspondence in 1742 – a substantially wrong-headed route was taken, mainly due to the fact that two underlying aspects of the strong (or *binary*) conjecture were overlooked. First, that focusing exclusively on the additive character of the statement does not take into account its real content, and second, that a principle known as *emergence* lies beneath the statement, a principle any existing proof of the conjecture must consider.

Let us discuss some of the most important milestones in the different approaches to the problem.

When a proof could not be achieved even for the sum of three primes (the weak conjecture for odd numbers) without additional assumptions, in the twenties of the previous century mathematicians began to search for the maximum number of primes necessary to represent any natural number greater than 1 as their sum. At the beginning, there were proofs that required hundreds of thousands (!) of primes (L. Schnirelmann [2]). In 1937 the weak conjecture was proven (I. Vinogradov [4]), but only above a constant large enough to make available sufficient primes as summands.

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Almost an entire century passed before the representation for all integers > 1 could be reduced to the maximum of five or six summands of primes, respectively (T. Tao [3]). In 2013 the huge gap of numbers for the weak Goldbach version was closed, using numerical verification combined with a complex estimative proof (H. Helfgott [1]).

The so-called Hardy-Littlewood circle method in combination with sophisticated techniques of sieve theory was employed and constantly improved upon in those approaches. However, these methods do not reflect the primes' actual role in the problem as originally formulated by Goldbach and Euler, by continuously examining 'how many' prime numbers are available as summands. As this does not work for the binary Goldbach conjecture, concern for that original problem has gradually been sidelined up to the present day, even though a solution would definitively resolve the issue of integers represented as the sum of primes.

We will show that the solution lies in the constructive characteristics of the prime numbers and not in their distribution.

2. THE STRONG GOLDBACH CONJECTURE

Theorem 2.1 (Strong Goldbach conjecture (SGB)). *Every even integer greater than 2 can be expressed as the sum of two primes.*

Moreover, we claim

Theorem 2.2 (SSGB). *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Proof. After the initial cases 4 = 2+2 and 6 = 3+3, it suffices to prove SSGB. The basic idea of the proof is as follows: SSGB is equivalent to saying that every natural number greater than 3 is the arithmetic mean of two different odd primes. Correspondingly, SGB means that every composite number is the arithmetic mean of two odd primes. We achieve this result by using the constructive properties of the prime numbers within the natural numbers. For this we provide a structured representation of the natural numbers starting from 3 and we deduce arithmetic identities from the properties of this representation. Then, we show that the above reformulation of SSGB contradicts these identities when it is not true.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3. Furthermore, we denote the projections from $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ onto the i-th factor by π_i, $1 \le i \le 3$.

At first, we replace SGB and SSGB with the following equivalent representations:

Every integer greater than 1 is prime or is the arithmetic mean of two different primes, p_1 and p_2.

and

Every integer greater than 3 is the arithmetic mean of two different primes, p_1 and p_2.

$$\text{SGB} \Leftrightarrow \forall n \in \mathbb{N},\ n > 1 : (\ n \text{ prime } \vee\ \exists\, p_1, p_2 \in \mathbb{P}_3\ \exists\, d \in \mathbb{N} \text{ with } p_1 + d = n = p_2 - d\ ) \tag{2.1}$$

$$\text{SSGB} \Leftrightarrow \forall n \in \mathbb{N},\ n > 3 : (\ \exists\, p_1, p_2 \in \mathbb{P}_3\ \exists\, d \in \mathbb{N} \text{ with } p_1 + d = n = p_2 - d\ ) \tag{2.2}$$

Now, we define

$$S_g := \{\ (pk, mk, qk) \mid k, m \in \mathbb{N};\ p, q \in \mathbb{P}_3,\ p < q;\ m = (p + q)/2\ \}$$

and call S_g the g-structure on $\mathbb{N}_3$.

According to (2.2), SSGB is equivalent to saying that all $n \in \mathbb{N}_4$ occur as arithmetic mean m in a triple of S_g.

We note that the whole range of $\mathbb{N}_3$ is redundantly represented (we say '*structured* ' or '*covered* ') by the triple components of S_g. This is a simple consequence of prime factorization and is easily verified through the following three cases:

The primes p in $\mathbb{N}_3$ are represented by components pk with k = 1; the composite numbers in $\mathbb{N}_3$, different from the powers of 2, are represented by pk with $p \in \mathbb{P}_3$ and $k \in \mathbb{N}$; the powers of 2 in $\mathbb{N}_3$ are represented by mk with m = 4 and k = 1, 2, 4, 8, 16, … .

The following examples for the number 42 illustrate the redundant covering through S_g:

(**42**, 54, 66) = (**7·6**, 9·6, 11·6)

(18, **42**, 66) = (3·6, **7·6**, 11·6)

(30, 36, **42**) = (5·6, 6·6, **7·6**)

(**42**, 70, 98) = (**3·14**, 5·14, 7·14)

(33, **42**, 51) = (11·3, **14·3**, 17·3)

(38, **42**, 46) = (19·2, **21·2**, 23·2)

(41, **42**, 43) = (41·1, **42·1**, 43·1)

(37, **42**, 47) = (37·1, **42·1**, 47·1)

(5, **42**, 79) = (5·1, **42·1**, 79·1)

Due to the complete covering of $\mathbb{N}_3$ through S_g, all arithmetic relations within $\mathbb{N}_3$ are exclusively based on numbers represented by pk, mk, qk. Particularly, equations between terms with values in $\mathbb{N}_3$ can be expressed by pk, mk, qk. We realize that all these equations are invariant to the existence of any $x \in \mathbb{N}_4$ with $x \neq m$ for all m generated in S_g, because such an x would be equal to pk, mk or qk for some (pk, mk, qk) $\in S_g$.

This invariance means that equations in $\mathbb{N}_3$ consisting of terms represented by pk, mk, qk behave equally in either case, that is, if such an x exists or not. For example, if we have n, n' $\in \mathbb{N}_3$ with n = pk $\vee$ n = mk $\vee$ n = qk and n' = p'k' $\vee$ n' = m'k' $\vee$ n' = q'k', then the validity of n = n' does not depend on whether there exists such an x above or not.

On the other hand, the numbers m in S_g have two crucial properties: m is always the arithmetic mean of two odd primes p, q and all generating pairs (p, q) are used in S_g. So, by choosing the non-existence of such an x in the above invariance, these two properties imply that every pk ≥ 5 equals $m_{all}k'$ and every power of 2 starting from 8, given by $(p_{pow} + q_{pow})k_{pow}$, equals $m_{pow}k_{pow}'$ where the arithmetic means m_{all} and m_{pow} run through all divisors ≥ 4 of pk and $(p_{pow} + q_{pow})k_{pow}$, respectively; k', $k_{pow}' \geq 1$.

Let us assume now that SSGB is not true. This leads to a contradiction, because the existence of at least one $n \in \mathbb{N}_4$ with $n \neq m$ for all m, where n is represented by some pk or by some $(p_{pow} + q_{pow})k_{pow}/2$, violates one of the above identities.

□

Alternative view**.** In the following we give an alternative view on the formal proof above.

Apart from the <u>covering</u> of $\mathbb{N}_3$, we observe the following two properties of S_g that we used in the proof:

<u>Equidistance</u>: The successive components in the triples of S_g are always equidistant. So, we call these triples as well as the structure S_g equidistant. We note that the numbers m in the triples are uniquely determined by the pairs (p, q) as the arithmetic mean of p and q.

<u>Maximality</u>: Actually, for a complete covering of $\mathbb{N}_3$ it would be sufficient if we chose (3k, 4k, 5k) together with triples (pk, mk, qk) in which all other odd primes occur as p, q or m. However, for our purpose we use the structure S_g that is based on all pairs (p, q) of odd primes with p < q. We call this the maximality of the structure S_g.

The structure S_g can be written as a ((i, j) x k) - matrix, where each row is formed by the triple components $p_i \cdot k$, $m_{ij} \cdot k$, $q_j \cdot k$ with $p_i < q_j$ running through $\mathbb{P}_3$ and $m_{ij} = (p_i + q_j)/2$ for a fixed k ≥ 1. The matrix starts as follows:

(3·1, 4·1, 5·1), (3·1, 5·1, 7·1) … (5·1, 6·1, 7·1), (5·1, 8·1, 11·1) ...
(3·2, 4·2, 5·2), (3·2, 5·2, 7·2) … (5·2, 6·2, 7·2), (5·2, 8·2, 11·2) ...
…
…
…

Written down the complete matrix, what we see is the whole $\mathbb{N}_3$ in redundant form, i.e. a structured $\mathbb{N}_3$.

Now, we check if it is possible to insert an additional nk, n ≥ 3, for any fixed k ≥ 1 in the k-th row of the matrix or if this is not possible. If not, according to (2.1), SGB is proved.

(a) Due to the complete covering through the S_g matrix, it is already excluded the option to place nk in a subset of $\mathbb{N}_3$ that is not covered by the matrix.

Note: In case of complete covering, for a composite n > 5 a representation of nk as nk = n'k', k' ≠ k, in the matrix (with n' ∈ $\mathbb{P}_3$ when nk not a power of 2; with n' = 4 when nk a power of 2) is always possible. However, inserting an additional nk in the k-th row of the matrix means that it is based on the fixed factor k in the decomposition nk.

(b) Due to the equidistance of all triples in S_g, also the option is excluded that we can place the nk (like the mk too in a non-equidistant structure) anywhere between pk and qk.

(c) Due to the maximality of S_g, also the option is excluded that we can place the additional nk as mk in a triple (pk, mk, qk) that is based on a pair of primes (p, q) not used in S_g.

Due to the construction of the S_g matrix, i.e. the pairs (p, q) are the exclusive parameters in the triples for each k, we don't have any further option to place the nk and since the whole matrix represents $\mathbb{N}_3$, any nk added to the matrix would be left over for that fixed k. In other words: The structure S_g leaves no space in $\mathbb{N}_3$ for that decomposition nk with fixed factor k. Therefore, such additional nk can not exist.

So, we realize that the three properties, covering, equidistance and maximality, that we identified in our structure S_g, lead to the following consequence:

The multiples in the triple form (pk, mk, qk) already represent all multiples nk, n ≥ 3, of a fixed k ≥ 1. More specifically, the triples are divided into two types: First, all triples (pk, mk, qk) where m is composite, and second, all remaining triples (pk, mk, qk) where m is prime. The first type yields SGB and the second type, together with the first, implies SSGB.

□

Note. The structure S_g reveals that a principle known as *emergence* lies beneath the Goldbach statement: For a given nk, n ≥ 4, k ≥ 1, the existence of two odd primes p, q such that nk is the arithmetic mean of pk and qk becomes visible only when we consider all odd primes and all k simultaneously. The triple form (pk, mk, qk) for all multiples nk, n ≥ 3, of a fixed k ≥ 1 is an effect that emerges from the interaction of all such triples when k runs through $\mathbb{N}$. See also the Remark 5.3.

3. EXAMPLES FOR SGB AND SSGB

In the previous chapter we have seen that the multiples of the numbers k in $\mathbb{N}_3$ are strictly set by our structure S_g. Let us call these multiples the occurrences of k within the structure.

In the proof it was essential to understand that the representation of a nk, where n > 5 is composite, as nk = n'k', $n' \in \mathbb{P}_3$ or n' = 4, k' ≠ k, constitutes two distinct occurrences, i.e. one of the number k and another of the number k'. The occurrences of both, k and k', are ruled by the triples separately.

3.1. n = 14 and k = 3:
Let us assume that n is not the arithmetic mean of two primes. For nk = 42, we find for example (pk', mk', qk') = (3·6, 5·6, **7·6**), which is part of the occurrence of 6 in the structure. But there is no triple (p·3, m·3, q·3) that contains n·3. Thus, n·3 violates the occurrence of 3 in the structure.

This contradiction can be resolved only if n = m, that is, n must be the arithmetic mean of, for example, 11 and 17.

3.2. n = 9 and k = 3:
Let us assume that n is not the arithmetic mean of two primes. For nk = 27, we only find (pk', mk', qk') = (**3·9**, m·9, q·9), which is part of the occurrence of 9 in the structure. But there is no triple (p·3, m·3, q·3) that contains n·3. Thus, n·3 violates the occurrence of 3 in the structure.

This contradiction can be resolved only if n = m, that is, n must be the arithmetic mean of, for example, 7 and 11.

3.3. n = 19 and k = 3:
Let us assume that n is not the arithmetic mean of two primes. For nk = 57, we find for example (p'k, m'k, q'k) = (17·3, 18·3, **19·3**), which is part of the occurrence of 3 in the structure. But there is no triple (p·3, m·3, q·3) with p < 19 < q that contains n·3. Thus, n·3 violates the occurrence of 3 in the structure.

This contradiction can be resolved only if n = m, that is, n must be the arithmetic mean of 7 and 31.

4. GENERALIZATION AND FURTHER RESULTS

First, we will embed the structure used in the proof of SSGB in a general concept. We give the following definitions:

Definition 4.1. *Let T be a non-empty subset of $\mathbb{N}_3 \times \mathbb{N}_3 \times \mathbb{N}_3$. A triple structure, or simply,* ***structure*** *S in $\mathbb{N}_3$ is a set defined by $S := \{ (t_1 \cdot k, t_2 \cdot k, t_3 \cdot k) \mid (t_1, t_2, t_3) \in T; k \in \mathbb{N} \}$.*

Definition 4.2. *Let S be a structure in $\mathbb{N}_3$, given by the triples (s_1, s_2, s_3). Then, a set $N \subseteq \mathbb{N}_3$ is* ***covered*** *by the structure S if every $n \in N$ can be represented by at least one s_i, $1 \le i \le 3$; that is, $\forall n \in N \; \exists s_i$, $1 \le i \le 3$, such that $n = s_i$. We say that the structure S provides a* ***covering*** *of N.*

Based on these definitions, we can make the following elementary statement:

Lemma 4.3. *Let S be a structure based on the set T. Then, $\mathbb{N}_3$ is covered by S if and only if* $\mathbb{P}_3 \cup \{ 4 \} \subseteq \bigcup_{1 \le i \le 3} \pi_i(T)$.

Proof. Let the union of the sets $\pi_i(T)$, $1 \le i \le 3$, contain all odd primes and the number 4.

Then, every prime number in $\mathbb{N}_3$ is represented by a component $t_i \cdot k$ with $t_i \in \mathbb{P}_3$ and $k = 1$.

Furthermore, every composite number n, different from the powers of 2, has a prime decomposition $n = pk$ with $p \in \mathbb{P}_3$ and $k \in \mathbb{N}$, and as such, is represented by a triple component $t_i \cdot k$ of S.

The powers of 2 are represented by $t_i \cdot k$ with $t_i = 4$ and k = 1, 2, 4, 8, 16, … .

So, the whole range of $\mathbb{N}_3$ is covered by S. On the other hand, if any odd prime or the number 4 is missing in the union of the sets $\pi_i(T)$, $1 \le i \le 3$, at least one of the representations described above is no longer possible.

□

For the structure S_g, used in chapter 2., that covers $\mathbb{N}_3$ we have $\pi_1(T_g) = \mathbb{P}_3$ and $4 \in \pi_2(T_g)$. Based on the above definitions, we can generalize S_g in the following manner:

Let P be a subset of the set of all odd numbers in $\mathbb{N}_3$ with at least two elements. For a subset $T_P \subseteq P \times \mathbb{N}_3 \times P$, where $p < m < q$ for all $(p, m, q) \in T_P$, we then define the structure $S_P := \{ (pk, mk, qk) \mid k \in \mathbb{N}; (p, m, q) \in T_P \}$. We call the structure S_P maximal if all pairs $(p, q) \in P \times P$ with $p < q$ are used in T_P. Furthermore, we call the structure S_P distance-preserving if for all $(p, m, q) \in T_P : (q – m) – (m – p) = c$ with a constant c. Specifically, we call S_P equidistant if $c = 0$. We note that in a distance-preserving S_P the component mk is uniquely determined by pk, qk. In the case of an equidistant S_P, we obtain the arithmetic mean for m.

In order to get a covering of $\mathbb{N}_3 \setminus \{$ powers of 2 $\}$ through the components pk, qk of S_P, P must contain all odd primes. In this case, due to the construction of S_P, a maximal and distance-preserving S_P covers $\mathbb{N}_3$ and is equidistant because the triples (3k, 4k, 5k) are contained. We then obtain the structure S_g by setting $P = \mathbb{P}_3$ and we realize that $\mathbb{P}_3$ is the smallest subset of odd numbers in $\mathbb{N}_3$ that enables such a complete covering of $\mathbb{N}_3$ through the triples (pk, mk, qk) with $p, q \in \mathbb{P}_3$.

Now, for a generalization in terms of the numbers m we consider functions $f : \mathbb{P}_3 \times \mathbb{P}_3 \rightarrow \mathbb{Z}$. To achieve useful results, we define the following restrictions on f:

First, we restrict f with the condition (f1): For all pairs $(p, q) \in \mathbb{P}_3 \times \mathbb{P}_3$ with $p < q$ the triples (p, q, f(p,q)) have the same numerical ordering and the difference between the two distances of successive components remains constant. I.e., the resulting triples (t_1, t_2, t_3) satisfy: $t_1 < t_2 < t_3$ and $(t_3 – t_2) – (t_2 – t_1) = c$ with a constant c.

Additionally, we set the condition (f2): $\exists$ (p, q) $\in \mathbb{P}_3$ x $\mathbb{P}_3$, p < q, with f(p,q) = 4. So, the powers of 2 are contained when we consider the triples (pk, qk, f(p,q)k) for all k ≥ 1. Therefore, according to Lemma 4.3, the whole range of $\mathbb{N}_3$ is covered by the components of these triples.

For the function f with the conditions (f1), (f2) we then define a f-specific, distance-preserving, structure which covers $\mathbb{N}_3$ by

$$S_f := \{ (pk, qk, f(p,q)k) \mid k \in \mathbb{N};\ p, q \in \mathbb{P}_3,\ p < q;\ f(p,q) \in \mathbb{N}_3 \}$$

and call it the f-structure. Also here we use the maximality considering all pairs (p, q) of odd primes with p < q. And again the pairs (p, q) are the exclusive parameters in the triples (pk, qk, f(p,q)k) for each k. Moreover, for each fixed k the triples (pk, qk, f(p,q)k) distribute their components uniformly in accordance with (f1).

From this we obtain the following:

Any function f as above generates exactly one of three possible classes of numbers: only even numbers or only odd numbers or both. We call this the f-class. In case of odd numbers, f cannot satisfy (f2) so that the f-structure would not yield a complete covering of $\mathbb{N}_3$. So, f is restricted to be a function which generates either only even numbers or both even and odd numbers. Furthermore, we notice that a f-structure provides a distribution exclusively for f(p,q)k by means of the triples (pk, qk, f(p,q)k) for each k. If, for example, f produces only even numbers, only such even multiples of k are being distributed through the structure S_f. In this case, we would have no information regarding the odd multiples which are not prime.

A few observations on the special case when f is the arithmetic mean:

In the proof of SSGB we used the function f = g that determines the arithmetic mean g(p,q) = (p + q) ⁄ 2 = m. g generates even and odd numbers and satisfies the conditions (f1) and (f2) for building the g-structure on $\mathbb{N}_3$. In this case, c = 0 so that distance-preserving means equidistance. The pairs (pk, qk) are expanded into triples (pk, mk, qk) including the powers of 2 through (3k, 4k, 5k). As is easily verified, the arithmetic mean is the only function fulfilling (f1) and (f2) that has its values in the middle component of the ordered triples and that generates even and odd numbers.

For the definition of S_f we replaced the arithmetic mean m used in the structure S_g by numbers f(p,q) $\in \mathbb{N}_3$ determined by a function f with the conditions (f1), (f2). Now, we apply the proof of SSGB to the triples (pk, qk, f(p,q)k) replacing the argument of m being constantly the arithmetic mean by the condition (f1), where the parameters n, x $\in \mathbb{N}_4$ used in that proof are now of the f-class. Then, based on the f-structure S_f, we obtain the following property as a generalization of SSGB:

(F) *For each fixed k ≥ 1 the triples (pk, qk, f(p,q)k) form a distribution of all nk, n ≥ 4, n of the f-class, with respect to pk, qk that is determined by (f1).*

Let us now consider other functions f which satisfy the conditions (f1) and (f2) and build a f-structure on $\mathbb{N}_3$, with the outcome that f(p,q) represents all even integers greater than 2. Due to (f2), in this case f(p,q) is always the first component in the ordered triples.

For example, we can state

Corollary 4.4. *All even positive integers are of the form 2p – q + 1 with odd primes p < q.*

Proof. For the number 2 we have: 2 = 2·3 – 5 + 1. For all even numbers in $\mathbb{N}_3$ we apply our concept of the f-structure.

As is easily verified, f(p,q) = 2p – q + 1 satisfies the conditions (f1) and (f2) for building a f-structure on $\mathbb{N}_3$. We consider only those f(p,q) which lie in $\mathbb{N}_3$ and we note that by the Bertrand-Chebyshev theorem: $\forall p \in \mathbb{P}_3$, p > 3, $\exists q \in \mathbb{P}_3$, q > p, such that f(p,q) $\in \mathbb{N}_3$.

We now assume that there is an even integer n > 2 which is not of the form n = 2p – q + 1 with two odd primes p, q. We then consider the multiple nk for any k ≥ 1 and note that nk belongs to none of the triples ((2p – q + 1)k, pk, qk). This causes a contradiction to (F) and proves the corollary.

□

Note. If we interchange the primes p, q and consider f ' (p,q) = 2q – p + 1, then f ' also satisfies the condition (f1) for building a f-structure. But for a complete covering of $\mathbb{N}_3$ the number 4 is missing, and we can easily verify that there are other even numbers in $\mathbb{N}_3$ which cannot be represented by f '(p,q).

Another interesting example is f(p,q) = 2p – q – 3 versus f ' (p,q) = 2q – p – 3. f satisfies all conditions, including the covering, and therefore represents all even numbers, whereas f ' satisfies the covering because of f ' (3,5) = 4, but it violates numerical ordering and distance-preserving. There are even numbers in $\mathbb{N}_3$ which cannot be represented by f ' (p,q).

5. REMARKS

5.1. Due to the unpredictable way that the primes are distributed, all studies on the representation of natural numbers as the sum of primes are problematic when they use approaches based on the distribution of the primes.

Despite tremendous efforts over the centuries, the best result so far was five summands. I was always convinced that the solution must lie in the constructive characteristics of the prime numbers and not in their distribution.

5.2. The statement in the binary Goldbach conjecture actually is nothing more than the symmetric structure (pk, mk, qk) used in the proof. As we have shown, it is in fact a specific case of a general distribution principle within the natural numbers. Furthermore, we note that the property of the prime numbers and their infinitude are merely used to guarantee the complete covering of $\mathbb{N}_3$ through the structure.

In order to discard the usual interpretation of the conjecture that focuses on the sums of primes and thus opposes their multiplicative character, we have tackled the problem differently after shifting to the triple form: Instead of searching for primes which determine the needed arithmetic mean equal to a given n, we have approached the issue from the opposite direction. Based on the multiplicative prime decomposition, we identify nk as the component of a structure, in this case determined by the arithmetic mean.

A key point in the proof is the dual role of the numbers k: As multiplier they generate composite numbers while their own multiples in $\mathbb{N}_3$ are strictly set by the used structure.

5.3. In other subject areas, the effect of the formation of new properties after the transition from single items to a whole system is called *emergence* (‘*The whole is more than the sum of its parts.*’). The structure S_g reveals that such principle lies beneath the Goldbach statement: For a given nk, n ≥ 4, k ≥ 1, the existence of two odd primes p, q such that nk is the arithmetic mean of pk and qk becomes visible only when we consider all odd primes and all k simultaneously. There is a remarkable aspect of this *emergence*: The two primes which form the so-called Goldbach partition of a given even number 2n are located before 2n, however, the reason for the existence of that partition also involves the primes beyond 2n.

It can be expected that also other questions in number theory own a solution based on this underlying principle.

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