IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received December 11, 2020, accepted December 30, 2020, date of publication January 8, 2021, date of current version January 20, 2021.

Digital Object Identifier 10.1109/ACCESS.2021.3049919

Finite-Time Bounded Tracking Control for Fractional-Order Systems

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This work was supported in part by the Oriented Award Foundation for Science and Technological Innovation, Inner Mongolia Autonomous Region, China, under Grant 2012, and in part by the National Key Research and Development Program of China under Grant 2017YFF0207401.

ABSTRACT This article solves the finite-time bounded tracking control problem for fractional-order systems. Firstly, by taking the fractional derivative on state equations and error signals, a fractional-order error system is constructed, and the error signal is taken as the output vector of the error system. Secondly, a state feedback controller is introduced into the error system, and the fractional derivative of disturbance signals and desired tracking signals are combined as the disturbance signal of the error system. Thus, the original problem is converted into the input-output finite time stability problem of the closed-loop error systems. Thirdly, based on the linear matrix inequalities (LMIs), the sufficient conditions which ensure the finite-time bounded tracking for the desired tracking signals are derived. Therefore, the finite-time bounded tracking controller of the original system is obtained. Finally, simulation results elucidate the effectiveness of the controller.

INDEX TERMS Finite-time bounded tracking, fractional-order systems, error systems, linear matrix inequalities.

I. INTRODUCTION

The systems, represented by fractional calculus equations, are called fractional-order systems. Many practical systems are described by fractional differential equations, because it can better represent the essential characteristics and dynamic behaviors of practical systems [1]–[3]. For example, by adopting the fractional-order, the memory phenomenon, in the mechanical system with viscous damping structure, is particularly easy to be showed [2]. The genetic characteristics in microbial fermentation process can be better depicted by employing fractional-order system [3]. In recent years, the theory of fractional-order control systems has captured many scholars' attention and achieved fruitful results [4]–[6]. Sakthivel *et al.* solved the robust fault estimation-based synchronization problem for a class of fractional-order multi-weighted complex dynamic networks subject to external disturbances [4]. Sakthivel *et al.* considered the output tracking control problem and disturbance rejection performance for a class of fractional-order T-S fuzzy systems with time-varying delay and external disturbances [5]. More interesting results in this field can be found in [6]. It is well known that stability is usually the first problem to be considered and solved in the analysis and design of a system. Therefore, many scholars have investigated the stability theory of fractional-order systems. Li *et al.* researched the stability for a type of fractional-order nonlinear systems based on Lyapunov direct method [7]. N'Doye *et al.* discussed the problem of robust stabilization for uncertain descriptor fractional-order systems [8]. HosseinNia *et al.* explored the stability of fractional-order switched systems [9]. Zhao *et al.* gave the stability criterion of fractional-order positive switched systems by using the fractional-order Lyapunov function [10].

It is worth pointing out that most of the researches analyzing the stability of fractional-order systems are mainly about Lyapunov stability which exposes the behaviors of systems in the infinite time interval. However, in some practical problems, engineers pour more attention to the dynamic behaviors of systems in a fixed time interval. Meanwhile, excessive state value is not allowed. For instance, the circuit would be damaged, if the voltage is too high in a boost circuit system [11].

The associate editor coordinating the review of this manuscript and approving it for publication was Chao Shen.

Hence, Dorato *et al.* proposed the finite time stability (FTS) to reflect the characteristic that the state of the system does not exceed a given range in a finite time [12],[13]. Furthermore, if there are external disturbances in the system, FTS can be extended to finite-time bounded (FTB) [14]. In order to discuss the input-output behavior of system over a finite time interval, Amato *et al.* presented the input-output finite time stability (IO-FTS) in 2010 [15]. Currently, the research on FTS, FTB, and IO-FTS has been spread from ordinary integer-order systems to fractional-order systems [16]–[19]. By utilizing the generalized Gronwall inequality, Lazarević and Spasić investigated the FTS problem of fractional-order delay systems, and gave the sufficient conditions for the system to be FTS [16]. Ma *et al.* contributed the definition of FTS and FTB for fractional-order linear systems [17]. The IO-FTS problems of normal and singular fractional-order linear systems were solved, and the design methods of state feedback controller were presented in [18]. Subsequently, Liang *et al.* investigate the problem of IO-FTS for fractional-order positive switched systems [19].

In practical engineering applications, there are plenty of tracking problems. Consequently, tracking control is always one of the research hotspots in the control field. At present, there are numerous important research results in tracking control, for example, optimal tracking control [20], [21], adaptive tracking control [22], [23], tracking control based on iterative learning [24], etc. In addition, Kohler *et al.* proposed a nonlinear model predictive control scheme for tracking of dynamic target signals by utilizing reference generic offline computations [25]. In order to keep track of a time-varying steady state target, an output feedback model predictive control for fuzzy systems was presented in [26]. Li *et.al* investigated event-triggered tracking control for a class of nonlinear systems with disturbances [27]. In some tracking problems, scholars sometimes desire that the output of system can always remain within the specified neighborhood of the desired tracking signal in a finite time. For instance, the robot is expected to move along the planned path in a given period of time [28]. In view of this, the concept of finite-time bounded tracking, which reflects the characteristic that the output within a given threshold of the desired tracking signal in a finite time, is proposed in [28], [29]. However, to our knowledge, the research on the finite-time bounded tracking still stays in the ordinary integer-order system. None of the study, about the finite-time bounded tracking of fractional-order system, has achieved so far. The problem of the finite-time bounded tracking for fractional-order systems is very challenging. One reason is that fractional-order systems have more complex dynamic behaviors than integral-order systems. The other reason is that the existing methods and conclusions about finite-time tracking of integer-order systems cannot be directly applied to fractional-order systems. Therefore, this article proposes and solves the finite-time bounded tracking problem for a class of fractional-order systems. The contributions of this research are summarized as follows: 1) The finite-time bounded tracking control is extended to the fractional-order systems for the first time; 2) a finite-time bounded tracking controller is designed for a type of fractional-order systems; 3) the conception, method and conclusion of this article can be applied to the integer-order systems, directly.

Notations: $A \in R^{m\times n}$ means that A is an $m \times n$ real matrix; I denotes the identity matrix; $Q < 0 (Q > 0)$ represents that Q is a negative (positive) definite matrix; $Q \leq 0$ $(Q \geq 0)$ represents that Q is a negative (positive) semidefinite matrix.

II. PRELIMINARIES

In this section, some basic notions and properties for fractional calculus are reviewed. For further details, please refer to [1].

The left-sided Riemann-Liouville fractional integral with order $\alpha > 0$ of the integrable function $x(t)$ is defined as

$$ {}_{t_0}I_t^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}x(\tau)d\tau, $$

where $\alpha \in R$, ${}_{t_0}I_t^{\alpha}$ is the integral operator of order α on $[t_0, t]$, $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$ [1], p.69.

Because the Caputo derivative is the most frequently used in control engineering, this article adopts the Caputo fractional derivative, which is defined as

$$ {}_{t_0}^{C}D_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-\tau)^{n-\alpha-1}x^{(n)}(\tau)d\tau, $$

where n is a positive integer, $x(t)$ is a differentiable function with the order n, $n-1 < \alpha < n$ [1], p. 92.

Property 1 ([4, Th. 3.16]): Caputo fractional derivative is a linear operator, that is, for any constant λ_1, λ_2,

$$ {}_{t_0}^{C}D_t^{\alpha}[\lambda_1 x_1(t) + \lambda_2 x_2(t)] = \lambda_1 {}_{t_0}^{C}D_t^{\alpha}x_1(t) + \lambda_2 {}_{t_0}^{C}D_t^{\alpha}x_2(t). $$

Property 2 ([1, Lemma 2.22]): Letting $\alpha > 0$, $x(t)$ be an order n differentiable function, the relationship

$$ {}_{t_0}I_t^{\alpha}({}_{t_0}^{C}D_t^{\alpha}x(t)) = x(t) - \sum_{k=0}^{n-1}\frac{(t-t_0)^k}{k!}x^{(k)}(t_0) $$

is obtained, here $x^{(0)}(t) = x(t)$, $n-1 < \alpha < n$.

In addition, this article needs utilize the following lemmas.

Lemma 1 [30]: Let $0 < \alpha < 1$, $x(t) \in R^n$ be a vector of differentiable function. Then, when $t \geq t_0$, there is

$$ {}_{t_0}^{C}D_t^{\alpha}\left[x^T(t)Px(t)\right] \leq x^T(t)P{}_{t_0}^{C}D_t^{\alpha}x(t) + ({}_{t_0}^{C}D_t^{\alpha}x(t))^T Px(t), $$

where $P \in R^{n\times n}$ and $P > 0$.

Lemma 2 ([31, Lemma 2.8]): Consider the matrix

$$ S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} $$

in which both S_{11} and S_{22} are invertible and symmetric matrices. The following conditions are equivalent:

(i) $S < 0$;

(ii) $S_{22} < 0, \quad S_{11} - S_{12}S_{22}^{-1}S_{12}^T < 0$;

(iii) $S_{11} < 0, \quad S_{22} - S_{12}^T S_{11}^{-1}S_{12} < 0$.

III. PROBLEM DESCRIPTION AND ASSUMPTIONS

Consider a fractional-order system as follows:

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha}x(t) = Ax(t) + Ew(t), \quad x(t_0) = 0 \\ y(t) = Cx(t), \end{cases} \tag{1}$$

where $0 < \alpha < 1$; $x(t) \in R^n$, $y(t) \in R^p$, and $w(t) \in R^q$ denote the state vector, the output vector, and the disturbance input, respectively; moreover, $A \in R^{n\times n}$, $C \in R^{p\times n}$, and $E \in R^{n\times q}$ are known constant matrices.

The definition of IO-FTS for system (1) is showed as follows.

Definition 1: The three scalars $c_1 > 0$, $c_2 > 0$, $T > t_0$, and two matrices $Q > 0$, $\Phi > 0$ are given. Under the initial value condition $x(t_0) = 0$, system (1) is referred to as IO-FTS with respect to (c_1, c_2, Q, Φ, T), if

$$\sup_{t\in[t_0,T]} w^T(t)Qw(t) \le c_1 \Rightarrow y^T(t)\Phi y(t) < c_2, \forall t \in [t_0, T].$$

Remark 1: Definition 1 is slightly different from the one in [18]. In fact, two definitions can be transformed with each other if the appropriate parameters are selected.

Then, we extend Definition 1 to discuss the finite-time bounded tracking of system (1). The finite-time bounded tracking means that the output $y(t)$ of (1) always remains in a given neighborhood of the desired tracking signal $y_d(t) \in R^p$ under certain conditions. The difference between the desired tracking signal and the output signal is defined as the error signal $e(t)$, that is

$$e(t) = y(t) - y_d(t). \tag{2}$$

The strict definition of finite-time bounded tracking for System (1) is as follows:

Definition 2: Given three scalars $c_1 > 0$, $c_2 > 0$, $T > t_0$, two matrices $Q > 0$, $\Phi > 0$, and initial condition $x(t_0) = 0$, the outputs of system (1) complete finite-time bounded tracking for $y_d(t)$ with respect to (c_1, c_2, Q, Φ, T), if

$$\sup_{t\in[t_0,T]} w^T(t)Qw(t) \le c_1 \Rightarrow e^T(t)\Phi e(t) < c_2, \forall t \in [t_0, T].$$

Remark 2: Particularly, $y_d(t) \equiv 0$, the finite-time bounded tracking degenerate to the IO-FTS.

Let us consider the fractional-order system

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha}x(t) = Ax(t) + Bu(t) + Ew(t), \quad x(t_0) = 0 \\ y(t) = Cx(t), \end{cases} \tag{3}$$

where $0 < \alpha < 1$; $x(t) \in R^n$ denotes the state vector, $u(t) \in R^m$ is the control input vector, $w(t) \in R^q$ represents the disturbance vector, $y(t) \in R^p$ means the output vector; $A \in R^{n\times n}$, $B \in R^{n\times m}$, $E \in R^{n\times q}$, and $C \in R^{p\times n}$ denote known constant matrices.

The assumptions on the desired tracking signal $y_d(t)$ and disturbance signal $w(t)$ of system (3) are presented as follows.

Assumption 1: $y_d(t)$ is a piecewise continuous differentiable function with $y_d(t_0) = 0$, and satisfies

$$\left(\frac{(T-t_0)^{1-\alpha}}{\Gamma(2-\alpha)}\right)^2 \sup_{t\in[t_0,T]} \dot{y}_d^T(t)Q_1\dot{y}_d(t) \le c_{11},$$

where $Q_1 \in R^{p\times p}$ and c_{11} are given positive matrix and number, respectively.

Assumption 2: $w(t)$ is a piecewise continuous differentiable function and satisfies

$$\left(\frac{(T-t_0)^{1-\alpha}}{\Gamma(2-\alpha)}\right)^2 \sup_{t\in[t_0,T]} \dot{w}^T(t)Q_2\dot{w}(t) \le c_{22},$$

where $Q_2 \in R^{q\times q}$ and c_{22} are given positive matrix and number, respectively.

Remark 3: According to Assumption 1 and 2, it follows that $y_d(t)$ and $w(t)$ are not differentiable at some isolated points. At this juncture, the one-sided derivative of $y_d(t)$ and $w(t)$ are taken.

The objective of this study is to design a controller for system (3) so that the output $y(t)$ of (3) completes finite-time bounded tracking for $y_d(t)$ under certain conditions.

IV. DESIGN OF THE CONTROLLER

The method of the constructing error systems in [32], [33] is implemented to solve the problem. Firstly, a fractional-order error system is constructed so that the error signal is included in the state vector.

Taking the order α Caputo derivative on both sides of the state equation of (3), and utilizing Property 1, the following will be obtained:

$${}_{t_0}^{C}D_t^{\alpha}({}_{t_0}^{C}D_t^{\alpha}x(t)) = A{}_{t_0}^{C}D_t^{\alpha}x(t) + B{}_{t_0}^{C}D_t^{\alpha}u(t) + E{}_{t_0}^{C}D_t^{\alpha}w(t). \tag{4}$$

By applying ${}_{t_0}^{C}D_t^{\alpha}$ to both sides of (2), it follows that

$${}_{t_0}^{C}D_t^{\alpha}e(t) = {}_{t_0}^{C}D_t^{\alpha}y(t) - {}_{t_0}^{C}D_t^{\alpha}y_d(t) = C{}_{t_0}^{C}D_t^{\alpha}x(t) - {}_{t_0}^{C}D_t^{\alpha}y_d(t). \tag{5}$$

Upon combining (4) and (5), one can acquire

$${}_{t_0}^{C}D_t^{\alpha}z(t) = \bar{A}z(t) + \bar{B}{}_{t_0}^{C}D_t^{\alpha}u(t) + \bar{E}{}_{t_0}^{C}D_t^{\alpha}w(t) + \bar{G}{}_{t_0}^{C}D_t^{\alpha}y_d(t), \tag{6}$$

where

$$z(t) = \begin{bmatrix} e(t) \\ {}_{t_0}^{C}D_t^{\alpha}x(t) \end{bmatrix} \in R^{p+n}, \quad \bar{A} = \begin{bmatrix} 0 & C \\ 0 & A \end{bmatrix} \in R^{(p+n)\times(p+n)},$$

$$\bar{B} = \begin{bmatrix} 0 \\ B \end{bmatrix} \in R^{(p+n)\times m}, \quad \bar{E} = \begin{bmatrix} 0 \\ E \end{bmatrix} \in R^{(p+n)\times q},$$

$$\bar{G} = \begin{bmatrix} -I \\ 0 \end{bmatrix} \in R^{(p+n)\times p}.$$

Since the output $y(t)$ and the desired tracking signal $y_d(t)$ are known, $e(t)$ can be taken as the output of (6), that is, the following output equation can be introduced for (6).

$$e(t) = \bar{C}z(t), \tag{7}$$

where $\bar{C} = \begin{bmatrix} I & 0 \end{bmatrix} \in R^{p\times(p+n)}$.

Combining (6) and (7) yields

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha}z(t) = \bar{A}z(t) + \bar{B}{}_{t_0}^{C}D_t^{\alpha}u(t) + \bar{E}{}_{t_0}^{C}D_t^{\alpha}w(t) \\ \qquad + \bar{G}{}_{t_0}^{C}D_t^{\alpha}y_d(t) \\ e(t) = \bar{C}z(t). \end{cases} \tag{8}$$

(8) is the error system of system (3).

It is interesting to note that if a state feedback controller

$$ {}_{t_0}^{C}D_t^{\alpha}u(t) = Kz(t). \tag{9} $$

can make the closed-loop system of (8) IO-FTS, then the output of (3) completes finite-time bounded tracking for $y_d(t)$. The gain matrix K of (9) will be given in the following.

Introducing (9) to (8) results

$$ \begin{cases} {}_{t_0}^{C}D_t^{\alpha}z(t) = (\bar{A} + \bar{B}K)z(t) + \bar{E}{}_{t_0}^{C}D_t^{\alpha}w(t) + \bar{G}{}_{t_0}^{C}D_t^{\alpha}y_d(t) \\ e(t) = \bar{C}z(t), \end{cases} \tag{10} $$

which is the closed-loop system of (8). Note that compared to system (1), there is one more term $\bar{G}{}_{t_0}^{C}D_t^{\alpha}y_d(t)$. Let us treat ${}_{t_0}^{C}D_t^{\alpha}y_d(t)$ as a disturbance, and integrate ${}_{t_0}^{C}D_t^{\alpha}w(t)$ and ${}_{t_0}^{C}D_t^{\alpha}y_d(t)$ together to form a new disturbance vector

$$ \bar{w}(t) = \begin{bmatrix} {}_{t_0}^{C}D_t^{\alpha}y_d(t) \\ {}_{t_0}^{C}D_t^{\alpha}w(t) \end{bmatrix}. $$

In this case, (10) can be written as

$$ \begin{cases} {}_{t_0}^{C}D_t^{\alpha}z(t) = (\bar{A} + \bar{B}K)z(t) + \tilde{E}\bar{w}(t) \\ e(t) = \bar{C}z(t), \end{cases} \tag{11} $$

where $\tilde{E} = \begin{bmatrix} \bar{G} & \bar{E} \end{bmatrix} \in R^{(p+n)\times(q+p)}$.

Remark 4: Treating ${}_{t_0}^{C}D_t^{\alpha}y_d(t)$ as an external disturbance leads to some conservatism in the result. From the point of view of mathematics, this method is reasonable. The problem of finite-time bounded tracking control for a part of desired tracking signals can be solved by adopting this method.

(11) has the same form as (1). Thus, the theory and method of the IO-FTS for fractional-order system can be adopted as a reference. The first critical theorem of this research is given as following.

Theorem 1: For $\forall t \in [t_0, T]$, there is $e^T(t)\Phi e(t) < c_2$, that is, system (11) is input-output finite time stability with respect to (c_1, c_2, Q, Φ, T), if under Assumption 1 and 2, there exists a matrix $P > 0$ such that

$$ \begin{bmatrix} P\bar{A} + \bar{A}^TP + P\bar{B}K + K^T\bar{B}^TP & P\tilde{E} \\ \tilde{E}^TP & -Q \end{bmatrix} < 0 \tag{12} $$

and

$$ \bar{C}^T\Phi\bar{C} - \frac{c_2\Gamma(\alpha+1)}{c_1(T-t_0)^{\alpha}}P < 0, \tag{13} $$

where

$$ c_1 \geq c_{11} + c_{22},\, Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \quad \text{and } {}_{t_0}^{C}D_t^{\alpha}u(t) = Kz(t). $$

Proof: The quadratic form $V = z^TPz$ is constructed based on the positive definite matrix P satisfying (12) and (13). Taking the fractional derivative of V with regard to time t along the trajectory of (11) and considering Lemma 1, one can obtain

$$ \begin{aligned} &{}_{t_0}^{C}D_t^{\alpha}V\,\big|_{(11)} \\ &= {}_{t_0}^{C}D_t^{\alpha}(z^T(t)Pz(t)) \\ &\leq z^T(t)P{}_{t_0}^{C}D_t^{\alpha}z(t) + ({}_{t_0}^{C}D_t^{\alpha}z(t))^TPz(t) \\ &= z^T(t)P\left((\bar{A}+\bar{B}K)z(t) + \tilde{E}\bar{w}(t)\right) \\ &\quad + \left((\bar{A}+\bar{B}K)z(t) + \tilde{E}\bar{w}(t)\right)^T Pz(t) \\ &= z^T(t)(P\bar{A} + \bar{A}^TP + P\bar{B}K + K^T\bar{B}^TP)z(t) \\ &\quad + \bar{w}^T(t)\tilde{E}^TPz(t) + z^T(t)P\tilde{E}\bar{w}(t) \\ &= \begin{bmatrix} z^T(t) & \bar{w}^T(t) \end{bmatrix} \\ &\quad \begin{bmatrix} P\bar{A} + \bar{A}^TP + P\bar{B}K + K^T\bar{B}^TP & P\tilde{E} \\ \tilde{E}^TP & 0 \end{bmatrix} \begin{bmatrix} z(t) \\ \bar{w}(t) \end{bmatrix} \\ &= \begin{bmatrix} z^T(t) & \bar{w}^T(t) \end{bmatrix} \\ &\quad \begin{bmatrix} P\bar{A} + \bar{A}^TP + P\bar{B}K + K^T\bar{B}^TP & P\tilde{E} \\ \tilde{E}^TP & -Q \end{bmatrix} \\ &\quad \begin{bmatrix} z(t) \\ \bar{w}(t) \end{bmatrix} \\ &\quad + \bar{w}^T(t)Q\bar{w}(t) \end{aligned} \tag{14} $$

By using (12), the following will be established

$$ {}_{t_0}^{C}D_t^{\alpha}V\,\big|_{(11)} < \bar{w}^T(t)Q\bar{w}(t). \tag{15} $$

Based on the assumption of zero initial conditions, the following is obtained

$$ V(z(t_0)) = z^T(t_0)Pz(t_0) = 0. $$

Integrating both sides of (15) with respect to order α from to t_0 and taking t Property 2 into account, it can be achieved that

$$ \begin{aligned} V(z(t)) &< {}_{t_0}I_t^{\alpha}(\bar{w}^T(t)Q\bar{w}(t)) \\ &= \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}\bar{w}^T(\tau)Q\bar{w}(\tau)d\tau. \end{aligned} \tag{16} $$

Now let us estimate the upper bound on the right side of (16). Because of

$$ \bar{w}(t) = \begin{bmatrix} {}_{t_0}^{C}D_t^{\alpha}y_d(t) \\ {}_{t_0}^{C}D_t^{\alpha}w(t) \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, $$

there is

$$ \begin{aligned} \bar{w}^T(t)Q\bar{w}(t) &= ({}_{t_0}^{C}D_t^{\alpha}y_d(t))^TQ_1{}_{t_0}^{C}D_t^{\alpha}y_d(t) \\ &\quad + ({}_{t_0}^{C}D_t^{\alpha}w(t))^TQ_2{}_{t_0}^{C}D_t^{\alpha}w(t), \quad \forall t \in [t_0, T] \end{aligned} \tag{17} $$

Consider the two terms in the right of (17). For the first term, it is known from Assumption 1 that $\dot{y}_d(t)$ has only the first kind of discontinuity at most. Thus, $\dot{y}_d(t)$ is a bounded function. Moreover, the relation $(t-\tau)^{-\alpha}$ of τ remains its sign in $[t_0, t]$. Therefore, according to the integral mean value theorem in [34, p. 352], there exists a point $\xi \in [t_0, t]$ such that

$$ \begin{aligned} \int_{t_0}^{t}(t-\tau)^{-\alpha}\dot{y}_d(\tau)d\tau &= \left[\int_{t_0}^{t}(t-\tau)^{-\alpha}d\tau\right]\dot{y}_d(\xi) \\ &= \frac{1}{1-\alpha}(t-t_0)^{1-\alpha}\dot{y}_d(\xi). \end{aligned} $$

Then

$$
\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha} y_d(t) &= \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-\tau)^{-\alpha}\dot{y}_d(\tau)d\tau \\
&= \frac{1}{(1-\alpha)\Gamma(1-\alpha)}(t-t_0)^{1-\alpha}\dot{y}_d(\xi) \\
&= \frac{1}{\Gamma(2-\alpha)}(t-t_0)^{1-\alpha}\dot{y}_d(\xi).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&({}_{t_0}^{C}D_t^{\alpha} y_d(t))^T Q_1 {}_{t_0}^{C}D_t^{\alpha} y_d(t) \\
&\quad = \left(\frac{1}{\Gamma(2-\alpha)}(t-t_0)^{1-\alpha}\dot{y}_d(\xi)\right)^T \\
&\qquad \times Q_1\left(\frac{1}{\Gamma(2-\alpha)}(t-t_0)^{1-\alpha}\dot{y}_d(\xi)\right) \\
&\quad \le \left(\frac{(t-t_0)^{1-\alpha}}{\Gamma(2-\alpha)}\right)^2 \sup_{\xi\in[t_0,t]} \dot{y}_d^T(\xi)Q_1\dot{y}_d(\xi) \\
&\quad \le \left(\frac{(T-t_0)^{1-\alpha}}{\Gamma(2-\alpha)}\right)^2 \sup_{t\in[t_0,T]} \dot{y}_d^T(t)Q_1\dot{y}_d(t).
\end{aligned}
$$

Based on Assumption 1, we have

$$
({}_{t_0}^{C}D_t^{\alpha} y_d(t))^T Q_1 {}_{t_0}^{C}D_t^{\alpha} y_d(t) \le c_{11}. \tag{18}
$$

Similarly, if Assumption 2 holds, it is known that the second term on the right side of (17) satisfies

$$
({}_{t_0}^{C}D_t^{\alpha} w(t))^T Q_2 {}_{t_0}^{C}D_t^{\alpha} w(t) \le c_{22}. \tag{19}
$$

Merging (17), (18), and (19), the following can be achieved

$$
\bar{w}^T(t)Q\bar{w}(t) \le c_{11} + c_{22} \le c_1. \tag{20}
$$

By putting (20) into (16), one can obtain the upper bound of (16) which is the upper bound of $V(z(t))$, as follows:

$$
\begin{aligned}
V(z(t)) &< \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}\bar{w}^T(\tau)Q\bar{w}(\tau)d\tau \\
&\le \frac{c_1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}d\tau = \frac{c_1(t-t_0)^{\alpha}}{\Gamma(\alpha+1)} \\
&\le \frac{c_1(T-t_0)^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned} \tag{21}
$$

Because of $e(t) = \bar{C}z(t)$, considering (13) and (21), when $t \in [t_0, T]$, one can get

$$
\begin{aligned}
e^T(t)\Phi e(t) &= z^T(t)\bar{C}^T\Phi\bar{C}z(t) \\
&< \frac{c_2\Gamma(\alpha+1)}{c_1(T-t_0)^{\alpha}}z^T(t)Pz(t) = \frac{c_2\Gamma(\alpha+1)}{c_1(T-t_0)^{\alpha}}V(z(t)) \\
&< c_2.
\end{aligned}
$$

The proof is completed.

Nevertheless, the inequality (12) of Theorem 1 is not LMI, so (12) cannot be tackled by LMI toolbox in MATLAB. By converting (12) into LMI, one acquires the second theorem of this article.

Theorem 2: If under Assumption 1 and 2, there exists matrices $L > 0$ and Y satisfying

$$
\begin{bmatrix} \bar{A}L + L\bar{A}^T + \bar{B}Y + Y^T\bar{B}^T & \tilde{E} \\ \tilde{E}^T & -Q \end{bmatrix} < 0, \tag{22}
$$

and

$$
\begin{bmatrix} -\dfrac{c_2\Gamma(\alpha+1)}{c_1(T-t_0)^{\alpha}}L & L\bar{C}^T \\ \bar{C}L & -\Phi^{-1} \end{bmatrix} < 0, \tag{23}
$$

then system (3) achieves the finite-time bounded tracking for $y_d(t)$ with respect to (c_1, c_2, Q, Φ, T). Besides, the gain matrix $K = YL^{-1}$ and ${}_{t_0}^{C}D_t^{\alpha}u(t) = Kz(t)$.

Proof: we just need to prove that if the conditions of this theorem are true, the conditions of Theorem 1 are also true. The congruent transformation is implemented on the matrix on the left of (12) by pre-multiplying an invertible matrix $diag(P^{-1}, I)$ and post-multiplying the transpose of this matrix. Due to the fact that the congruent transformation remains the positive definiteness of a matrix, (12) is equivalent to

$$
\begin{bmatrix} \bar{A}P^{-1} + P^{-1}\bar{A}^T + \bar{B}KP^{-1} + P^{-1}K^T\bar{B}^T & \tilde{E} \\ \tilde{E}^T & -Q \end{bmatrix} < 0. \tag{24}
$$

Denoting $L = P^{-1}$, $K = YL^{-1}$, it can be seen that (24) becomes (22), which implies that (12) is satisfied if and only if (22) hold.

Pre- and post-multiply the left side of (13) by an invertible matrix L and its transpose (namely, L), respectively. Similarly, for the reason that congruent transformation remains the positive definiteness of a matrix, (13) is equivalent to

$$
L\bar{C}^T\Phi\bar{C}L - \frac{c_2\Gamma(\alpha+1)}{c_1(T-t_0)^{\alpha}}LPL < 0.
$$

Because of $L = P^{-1}$, $\Phi > 0$, the inequality above can be rewritten as

$$
-\frac{c_2\Gamma(\alpha+1)}{c_1(T-t_0)^{\alpha}}L - L\bar{C}^T(-\Phi^{-1})^{-1}\bar{C}L < 0. \tag{25}
$$

In view of $-\Phi^{-1} < 0$, according to Lemma 2, (25) and (23) are equivalent. Consequently, (13) is true if and only if (23) is established. Moreover, $L > 0$ if and only if $P > 0$. Thus, Theorem 2 is proved.

Further, we consider the control input of (3). The third crucial theorem of this article is as follows:

Theorem 3: If there exist matrices $L > 0$ and Y satisfying (22) and (23), besides, Assumption 1 and 2 are true, i.e., the conditions of Theorem 2 are satisfied, the control input of (3) can be taken as

$$
\begin{aligned}
u(t) &= K_e {}_{t_0}I_t^{\alpha}e(t) + K_x x(t) + u(t_0) \\
&= \frac{K_e}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}e(\tau)d\tau + K_x x(t) + u(t_0),
\end{aligned} \tag{26}
$$

where $K = YL^{-1}$, L and Y are determined by (22), (23) and $L > 0$, $K = \begin{bmatrix} K_e & K_x \end{bmatrix}$. Under control law (26), the output $y(t)$ of (3) can achieves finite-time bounded tracking for $y_d(t)$ with respect to (c_1, c_2, Q, Φ, T).

Proof: If the conditions of Theorem 2 are true, the controller of (8) is ${}_{t_0}^{C}D_t^{\alpha}u(t) = Kz(t)$, which is also a control input of system (3). Divide K into $\begin{bmatrix} K_e & K_x \end{bmatrix}$, where $K_e \in R^{m\times p}$, $K_x \in R^{m\times n}$. At this time, ${}_{t_0}^{C}D_t^{\alpha}u(t) = Kz(t)$ can be written as

$$ {}_{t_0}^{C}D_t^{\alpha}u(t) = K_e e(t) + K_x {}_{t_0}^{C}D_t^{\alpha}x(t). $$

Integrating both sides of the above equation with α from t_0 to t and employing Property 2, the following will be established

$$ u(t) - u(t_0) = K_e {}_{t_0}I_t^{\alpha}e(t) + K_x(x(t) - x(t_0)). \tag{27} $$

Shifting $u(t_0)$ to the right side of the equal sign in (27) and considering the initial condition $x(t_0) = 0$, (26) can be derived. This proof completes.

Remark 5: In (26), $\frac{K_e}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}e(\tau)d\tau$ represents the integrator which is fractional; $K_x x(t)$ means a state feedback; $u(t_0)$ is the initial value of the control input. The proper selection of $u(t_0)$ can accelerate the tracking speed of the output signal to $y_d(t)$. Generally, $u(t_0) = 0$.

Remark 6: In fact, all the concepts, methods and conclusions in this article can be directly applied to ordinary integer-order systems. It is only necessary to rewrite system (3) into ordinary integer-order system and take $\alpha = 1$ in derivation and conclusion. Therefore, the ordinary integer-order system can be treated as a special case of this article.

V. NUMERICAL SIMULATION

In this section, two examples are presented to illustrate the effectiveness of the proposed controller design.

Example 1: Consider the following numerical academic example

$$ \begin{cases} {}_{0}^{C}D_t^{\alpha}x(t) = \underbrace{\begin{bmatrix} -1 & -1 & 0 \\ 1.5 & 0.3 & 0.6 \\ 0.5 & 1 & -1 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} -1 \\ 1.5 \\ 3 \end{bmatrix}}_{B} u(t) \\ \qquad + \underbrace{\begin{bmatrix} 0.5 \\ 3 \\ -4 \end{bmatrix}}_{E} w(t), x(0) = 0 \\ y(t) = \underbrace{\begin{bmatrix} 4 & 0 & 0 \end{bmatrix}}_{C} x(t) \end{cases} \tag{28} $$

Take $\alpha = 0.8$, $\Phi = I$, $c_1 = 1$, $c_2 = 5$, $T = 10$. The desired tracking signal is taken as

$$ y_d(t) = \begin{cases} 0, & t < 2 \\ 0.5(t-2), & 2 \le t < 4 \\ 1, & t \ge 4 \end{cases} \tag{29} $$

By letting the weight matrix $Q_1 = 0.3$, simple calculation gives that

$$ \left(\frac{(T-t_0)^{1-\alpha}}{\Gamma(2-\alpha)}\right)^2 \sup_{t\in[t_0,T]} \dot{y}_d^T(t)Q_1\dot{y}_d(t) \approx 0.2235 < 0.65 \overset{def}{=} c_{11}. $$

The disturbance signal is

$$ w(t) = 0.1\cos(2t) + 0.3. \tag{30} $$

Assume that the weight matrix is $Q_2 = 1$, then the following can be obtained

$$ \left(\frac{(T-t_0)^{1-\alpha}}{\Gamma(2-\alpha)}\right)^2 \sup_{t\in[t_0,T]} \dot{w}^T(t)Q_2\dot{w}(t) \approx 0.1192 < 0.35 \overset{def}{=} c_{22}. $$

At this time,

$$ Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} = \begin{bmatrix} 0.3 & 0 \\ 0 & 1 \end{bmatrix}, c_{11} + c_{22} = 1 \le c_1. $$

Considering Theorem 3 and employing the LMI toolbox in MATLAB, the following will be established

$$ L = \begin{bmatrix} 0.6092 & -0.9340 & 0.5863 & 1.0742 \\ -0.9340 & 3.2686 & -1.9925 & -4.9718 \\ 0.5863 & -1.9925 & 30.2468 & 11.7315 \\ 1.0742 & -4.9718 & 11.7315 & 47.8437 \end{bmatrix}, $$
$$ Y = \begin{bmatrix} 7.2440 & 5.2602 & -17.4902 & 8.1969 \end{bmatrix}. $$

Furthermore,

$$ \begin{aligned} K &= YL^{-1} = \begin{bmatrix} 26.3911 & 9.8987 & -0.7441 & 0.7899 \end{bmatrix}, \\ K_e &= 26.3911, \\ K_x &= \begin{bmatrix} 9.8987 & -0.7441 & 0.7899 \end{bmatrix}. \end{aligned} $$

Figures 1 and 2 depicts the output response and the trajectory of $e^T(t)\Phi e(t)$ of system (28), respectively. It can be observed from Fig. 2 that in the time interval [0, 10], always $e^T(t)\Phi e(t) < 5$, which illustrates that under the action of the designed controller, system (28) realizes finite-time bounded tracking for $y_d(t)$ with respect to $(1, 5, Q, I, 10)$.

Furthermore, in order to compare the performance of the designed controller on different order, we change the parameter α while keeping the other parameters. Take $\alpha = 0.6$, $\alpha = 0.75$, α=1 (when α=1, (28) is an ordinary integer-order system), then it can be verified that the above three values satisfy the condition of Theorem 3. The output responses of different order systems and the corresponding trajectory $e^T(t)\Phi e(t)$ are showed in Fig. 3 and 4 respectively.

It can be seen from Figures 3 and 4 that the closed-loop system achieves finite-time bounded tracking for $y_d(t)$ with respect to $(1, 5, Q, I, 10)$ under different fractional orders. Moreover, the tracking performance of the system with higher order is better than that of low order.

Example 2: Let us consider the viscoelastic system, which can be described by the following fractional differential equations [35]

$$ \begin{cases} m\ddot{x}(t) + \delta {}_{t_0}^{C}D_t^{1/2}x(t) + \gamma x(t) = u(t) + \eta w(t) \\ x(0) = a_1, \dot{x}(0) = a_2 \end{cases} \tag{31} $$

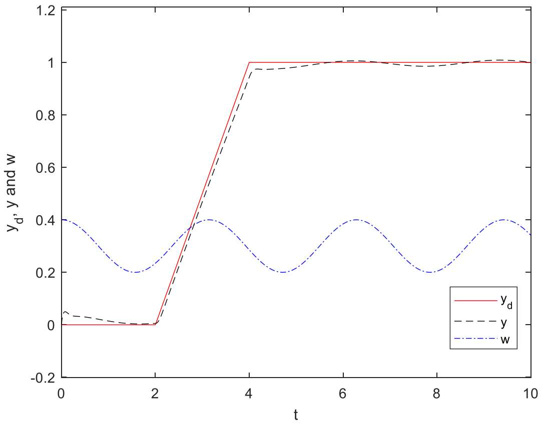

FIGURE 1. The output response of (28) with disturbance signal (30).

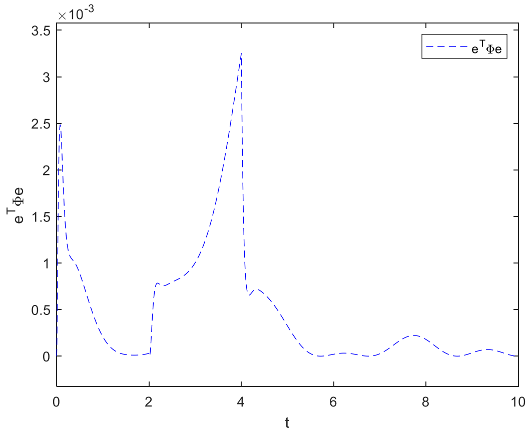

FIGURE 2. The trajectory of $e^T(t)\Phi e(t)$.

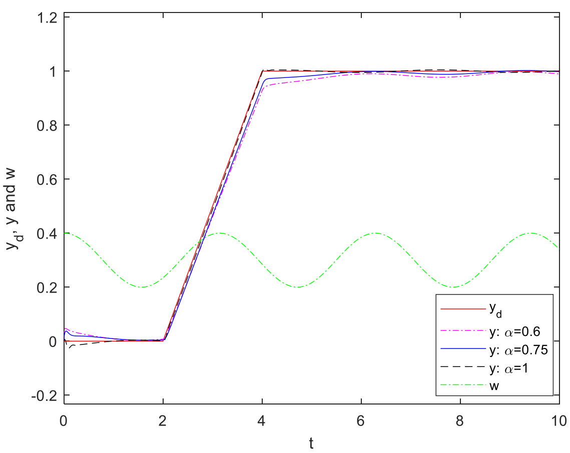

FIGURE 3. The output response with different orders.

where m, δ, γ, and η represent mass, damping coefficient, elastic coefficient, and disturbance coefficient, respectively; a_1 and a_2 are constants; $x(t)$ is the displacement function; $u(t)$ denotes the control input; $w(t)$ is the disturbance input.

Selecting a set of state variables

$$x_1(t) = x(t), x_2(t) = {}_0^C D_t^{1/2} x(t), x_3(t) = \dot{x}(t),$$
$$x_4(t) = {}_0^C D_t^{3/2} x(t),$$

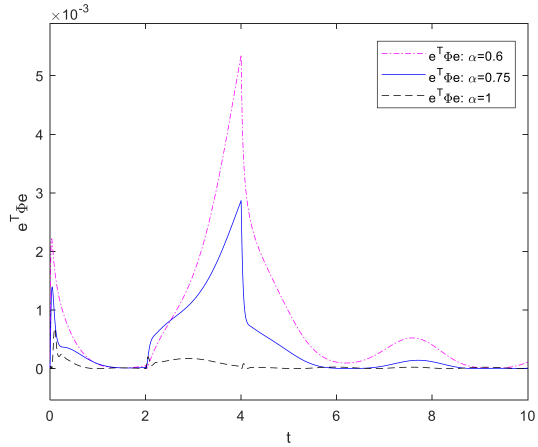

FIGURE 4. The trajectory of $e^T(t)\Phi e(t)$ with different orders.

one can get

$$ {}_0^C D_t^{1/2} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\gamma}{m} & -\frac{\delta}{m} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m} \end{bmatrix} \times u(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\eta}{m} \end{bmatrix} w(t)$$

The output of system (31) is $x(t)$. Letting $m = 0.25$, $\gamma = 0.5$, $\delta = 0.25$, $\eta = -2.5$, $a_1 = a_2 = 0$, we have

$$\begin{cases} {}_0^C D_t^{1/2} \tilde{x}(t) = A\tilde{x}(t) + Bu(t) + Ew(t) \\ y(t) = C\tilde{x}(t), \end{cases} \tag{32}$$

where

$$\tilde{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix},$$
$$E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -10 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

The initial state is $\tilde{x}(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$.

Let $\Phi = I$, $c_1 = 1$, $c_2 = 1$, $T = 10$. The desired tracking signal is chosen as

$$y_d(t) = 0.25 \sin(t) \tag{33}$$

Select the weight matrix $Q_1 = 1$. We have

$$\left(\frac{(T - t_0)^{1-\alpha}}{\Gamma(2-\alpha)} \right)^2 \sup_{t \in [t_0, T]} \dot{y}_d^T(t) Q_1 \dot{y}_d(t)$$
$$\approx 0.7958 < 0.9 \overset{def}{=} c_{11}$$

The disturbance signal is

$$w(t) = 0.15. \tag{34}$$

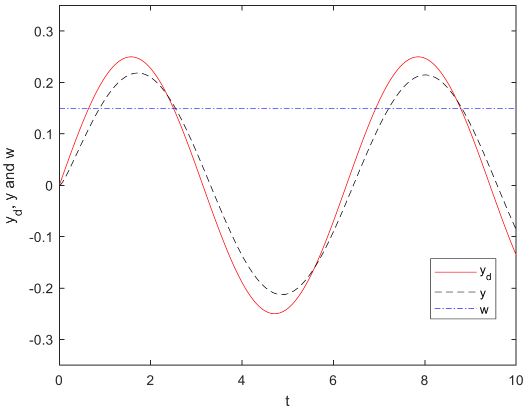

FIGURE 5. The output response of (32) with disturbance signal (34).

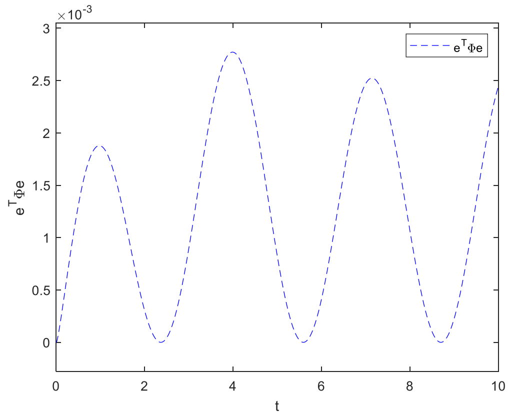

FIGURE 6. The trajectory of $e^T(t)\Phi e(t)$.

Take the weight matrix $Q_2 = 1$. By calculating, it follows that

$$\left(\frac{(T-t_0)^{1-\alpha}}{\Gamma(2-\alpha)}\right)^2 \sup_{t\in[t_0,T]} \dot{w}^T(t)Q_2\dot{w}(t) = 0 < 0.1 \overset{def}{=} c_{22}$$

Note that $Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $c_{11} + c_{22} = 1 = c_1$.

In light of Theorem 3, applying the LMI toolbox in MATLAB, the matrices L and Y are obtained.

On this basis, the gain matrix K is calculated:

Figure 5 shows the closed-loop output curve of system (32). Figure 6 is the trajectory of $e^T(t)\Phi e(t)$.

As it can be seen from Figure 5, under the action of the designed controller, the output signal of system (32) is always within the neighborhood of the desired tracking signal in a given time interval. Meanwhile, it can be observed from Fig. 6, in the time interval [0, 10], $e^T(t)\Phi e(t)$ remains in the specific threshold. This indicates that system (32) realizes finite-time bounded tracking for $y_d(t)$ with respect to $(1, 1, Q, I, 10)$.

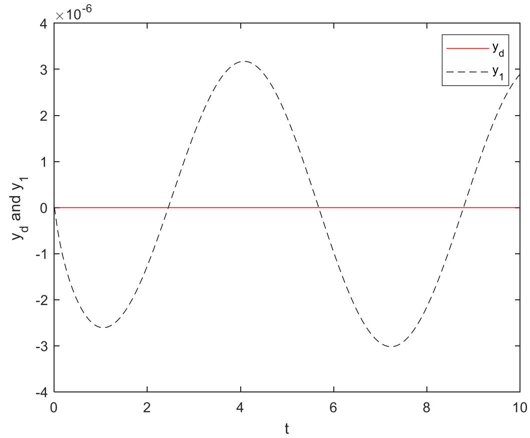

FIGURE 7. The output obtained by the control method in this article.

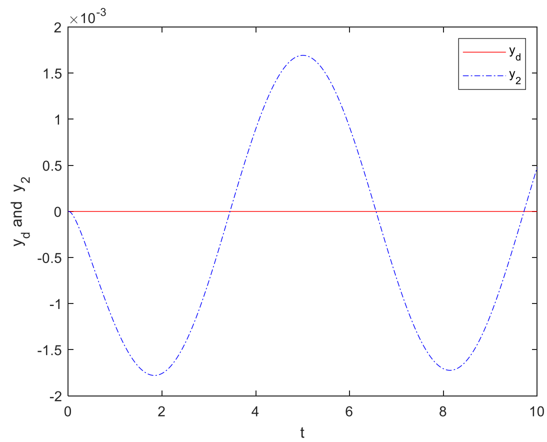

FIGURE 8. The output obtained by the control method in [18].

This article studies the finite-time bounded tracking of fractional-order systems, while previous literature studied the input-output finite time stability. In fact, if output of system (3) tracks the zero vector, in other words, let $y_d(t) \equiv 0$, then result of this article is input-output finite time stability of the fractional-order system. The coefficient matrix and parameters in Example 2 are still adopted. The disturbance signal is taken as $w(t) = 0.2\sin(t)$. We compare the result of

$$L = \begin{bmatrix} 0.2705 & -1.0472 & -3.9490 & 25.3661 & 266.3576 \\ -1.0472 & 24.3566 & -224.8731 & 646.6073 & 3703.5708 \\ -3.9490 & -224.8731 & 4095.5383 & -26707.7103 & -46612.4840 \\ 25.3661 & 646.6073 & -26707.7103 & 328081.8163 & -903495.6571 \\ 266.3576 & 3703.5708 & -46612.4840 & -903495.6571 & 27377626.9659 \end{bmatrix}$$

$$Y = \begin{bmatrix} -927.4067 & 11609.1630 & 226784.3353 & -6850753.6251 & -590813.1649 \end{bmatrix}$$

$$K = \begin{bmatrix} -139556.0037 & -30524.9445 & -3661.1341 & -275.0819 & -9.8459 \end{bmatrix}$$

this article with those in [18]. The output of the closed-loop system in this article is denoted as $y_1(t)$ and the output of the closed-loop system in [18] is denoted as $y_2(t)$. Figure 7 is the output curve obtained by using the controller designed in this article. Figure 8 shows the output response obtained by utilizing the method in [18]. By comparing Figures 7 and 8, it can be observed in the order of magnitude of the vertical axis that the control effect of this article is better than that of [18].

VI. CONCLUSION

This article designs a finite-time bounded tracking controller for a type of fractional-order system. By means of the method of constructing the error system in the preview control theory, the original problem is transformed into an IO-FTS problem. Then, the finite-time bounded tracking controller is acquired by utilizing the LMI. Theoretical results and numerical simulation demonstrate that under the action of the designed controller, the output of the original system realizes finite-time bounded tracking for desired tracking signal under certain conditions. Due to the aging of components and the delay of measurement, the system model often has the characteristics of uncertainty and delay. Therefore, the proposed finite-time bounded tracking control approach can be extended to other models, such as uncertain systems and delay systems and so on, which can be a good topic for further investigation.

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