ON THE DEPTH OF GÖDEL'S INCOMPLETENESS THEOREM

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ABSTRACT. In this paper, we use Gödel's incompleteness theorem as a case study for investigating mathematical depth. We take for granted the widespread judgment by mathematical logicians that Gödel's incompleteness theorem is deep, and focus on the philosophical question of what its depth consists in. We focus on the methodological study of the depth of Gödel's incompleteness theorem, and propose three criteria to account for its depth: influence, fruitfulness, and unity. Finally, we give some explanations for our account of the depth of Gödel's incompleteness theorem.

1. INTRODUCTION

"Mathematical depth" is an often used notion when mathematicians assess and evaluate the work of their fellows. Mathematics is full of disagreements over what is deep work. And even mathematicians who do agree in judging one thing or another deep, are likely to disagree sharply on what makes it deep. The notion of mathematical depth is not well-defined, and there is no single widely accepted account of mathematical depth. There is a lot of discussion of mathematical depth from the literature (see [Ernst, Heis, Maddy, McNulty and Weatherall, 2015a, 2015b][Arana, 2015] [Gray, 2015] [Lange, 2015][Stillwell, 2015][Urquhart, 2015]). The following four types of questions have been widely discussed in the current philosophical investigation (see [Ernst, Heis, Maddy, McNulty and Weatherall, 2015a, pp.155-162]): (1) is there agreement that the cited examples are deep or not deep? (2) Are there commonalities in the kinds of features cited in defense of depth and non-depth assessments in the various examples? (3) Is depth the same as or different from such notions as fruitfulness, surprisingness, importance, elegance, difficulty, fundamentalness, explanatoriness, beauty, etc.? (4) Is depth an objective feature or something essentially tied to our interests, abilities, and so on?

In this paper, the bearer of depth is mathematical theorems. There may be different ways of being deep, and different theorems may have distinct criteria for their depth. It is hard for us to give a universal account of mathematical depth. In this paper, we do not attempt to give a universal account of mathematical depth. Instead, our strategy in this paper is to use Gödel's incompleteness theorem as a case study for investigating mathematical depth. In this paper, we take for granted the widespread judgment by mathematical logicians that Gödel's incompleteness theorem is deep, and focus on the philosophical question of what its depth consists in. In particular, we focus on the methodological study of the depth of Gödel's incompleteness theorem: we attempt to find some fundamental criteria to account for the depth of the incompleteness theorem based on the current research on incompleteness from the literature. After briefly introducing Gödel's incompleteness theorem, we account for the depth of the incompleteness theorem according to the following three criteria: influence, fruitfulness, and unity. In this paper, we only cover the most important mathematical evidences (as far as we know) of the three criteria. We make no attempt at completeness here. There are many more mathematical evidences of the depth of Gödel's incompleteness theorem from the literature than we can cover here. This paper is a modest attempt to bring some

coherence to philosophical understandings of the depth of Gödel's incompleteness theorem. This work may help us to get a sense of the viability of a methodological inquiry into what counts as deep mathematics and why. We hope this work will advance our philosophical understanding of mathematical depth, perhaps making way for more unified accounts of mathematical depth to follow.

This paper is written in the same spirit of Arana's [2015]. After introducing Szemerédi's Theorem and gesturing at its proofs, Arana [2015] articulates four different accounts of the depth of Szemerédi's Theorem (genetic views, evidentialist views, consequentialist views, and cosmological views), and indicates ways in which each is apt and inapt for characterizing depth as it occurs in mathematical practice. The genetic view of depth identifies a deep theorem as one proved by sufficiently talented mathematicians. The evidentialist view of depth links the depth of a theorem with some quality of its proof. The consequentialist view of depth measures the depth of a theorem by some quality of its consequences, or of the consequences of its proofs. The cosmological view of depth measures the depth of a theorem by measuring the order the theorem has established and the unexpected structure it has revealed. However, Arana's four views of depth have features that many philosophers would consider problematic: either by falling into vagueness, by failing to pick out theorems that obviously count as deep, or by making depth subjective (see [Ernst, Heis, Maddy, McNulty, and Weatherall, 2015a]). Our three criteria of the depth of Gödel's incompleteness theorem are isolated from Arana's evidentialist, consequentialist, and cosmological views of depth. We basically agree with Arana's analysis of these views of depth. The focus of this paper is the justification of the depth of Gödel's incompleteness theorem from the three criteria we propose.

In this paper, G1 stands for Gödel's first incompleteness theorem, and G2 stands for Gödel's second incompleteness theorem. This paper is structured as follows. In Section 1, we introduce the motivation and main content of this paper. In Section 2, we give a brief overview of Gödel's incompleteness theorem and a sketch of main ideas of the proof of G1 and G2. In Section 3, we account for the depth of Gödel's incompleteness theorem based on the following three criteria: influence, fruitfulness and unity. In Section 4, we give some explanations for our account of the depth of Gödel's incompleteness theorem.

2. Gödel's incompleteness theorem

In this section, we give an overview of Gödel's incompleteness theorem. For textbooks on Gödel's incompleteness theorem, we refer to [Enderton 2001], [Murawski, 1999], [Lindström, 1997], [Smith, 2007], [Boolos, 1993]. For survey papers on Gödel's incompleteness theorem, we refer to [Smoryński, 1977], [Beklemishev, 2010], [Kotlarski, 2004], [Visser, 2016], [Cheng, 2019c].

We first review some basic notions used in this paper. We focus on first order theories with a countable language. For a given theory T, let $L(T)$ denote the language of T. In this paper, arithmetization refers to the method in mathematical logic that replaces reasonings on the expressions of first order language by reasonings on natural numbers. For this purpose, the replacement is constructed by some one-to-one mapping of the set of all expressions (in the alphabet of the language under consideration) into the natural number sequence. Relations and operations defined on expressions are transformed by this mapping into relations and operations on natural numbers. If unless stated otherwise, we always assume the arithmetization of the base theory with a recursive set of non-logical constants. For more details about arithmetization, we refer to [Murawski, 1999]. Under arithmetization, any formula (or finite sequence of formulas) can be coded by a natural number (called *Gödel's number*). We use $\ulcorner \phi \urcorner$ to denote the corresponding numeral

of the Gödel number of a formula ϕ. Given a theory T, we say a sentence ϕ in $L(T)$ is independent of T if $T \nvdash \phi$ and $T \nvdash \neg\phi$. A theory T is *incomplete* if there is a sentence ϕ in $L(T)$ such that ϕ is independent of T; otherwise, T is *complete*. A theory T is *recursively axiomatizable* if it has a recursive set of axioms, i.e. the set of Gödel numbers of axioms of T is recursive. A n-ary relation $R(x_1, \cdots, x_n)$ on $\mathbb{N}^n$ is *representable* in a theory T if there is a formula $\phi(x_1, \cdots, x_n)$ such that if $R(m_1, \cdots, m_n)$ holds, then $T \vdash \phi(\overline{m_1}, \cdots, \overline{m_n})$;[1] and if $R(m_1, \cdots, m_n)$ does not hold, then $T \vdash \neg\phi(\overline{m_1}, \cdots, \overline{m_n})$.[2]

Robinson Arithmetic $\mathbf{Q}$ is introduced in [Tarski, Mostowski and Robinson, 1953] as a base axiomatic theory for investigating incompleteness and undecidability.

Definition 2.1. Robinson Arithmetic $\mathbf{Q}$ is defined in the language $\{\mathbf{0}, \mathbf{S}, +, \times\}$ with the following axioms:

> $\mathbf{Q}_1$: $\forall x \forall y (\mathbf{S}x = \mathbf{S}y \to x = y)$;
> $\mathbf{Q}_2$: $\forall x (\mathbf{S}x \neq \mathbf{0})$;
> $\mathbf{Q}_3$: $\forall x (x \neq \mathbf{0} \to \exists y(x = \mathbf{S}y))$;
> $\mathbf{Q}_4$: $\forall x \forall y (x + \mathbf{0} = x)$;
> $\mathbf{Q}_5$: $\forall x \forall y (x + \mathbf{S}y = \mathbf{S}(x + y))$;
> $\mathbf{Q}_6$: $\forall x (x \times \mathbf{0} = \mathbf{0})$;
> $\mathbf{Q}_7$: $\forall x \forall y (x \times \mathbf{S}y = x \times y + x)$.

The theory $\mathbf{PA}$ consists of axioms $\mathbf{Q}_1$-$\mathbf{Q}_2$, $\mathbf{Q}_4$-$\mathbf{Q}_7$ in Definition 2.1 and the axiom scheme of induction $(\phi(\mathbf{0}) \wedge \forall x(\phi(x) \to \phi(\mathbf{S}x))) \to \forall x\phi(x)$, where ϕ is a formula with at least one free variable x. Let $\mathfrak{N} = \langle \mathbb{N}, +, \times \rangle$ denote the standard model of arithmetic. We say $\phi \in L(\mathbf{PA})$ is a true sentence of arithmetic if $\mathfrak{N} \models \phi$.

We introduce a hierarchy of $L(\mathbf{PA})$-formulas called the *arithmetical hierarchy* (see [P. Hájek and P. Pudlák, 1993]). *Bounded formulas* (Σ_0^0, or Π_0^0, or Δ_0^0 formula) are built from atomic formulas using only propositional connectives and bounded quantifiers (in the form $\forall x \leq y$ or $\exists x \leq y$). A formula is Σ_{n+1}^0 if it has the form $\exists x\phi$ where ϕ is Π_n^0. A formula is Π_{n+1}^0 if it has the form $\forall x\phi$ where ϕ is Σ_n^0. Thus, a Σ_n^0-formula has a block of n alternating quantifiers, the first one being existential, and this block is followed by a bounded formula. Similarly for Π_n^0-formulas. A formula is Δ_n^0 if it is equivalent to both a Σ_n^0 formula and a Π_n^0 formula in $\mathbf{PA}$.

A theory T is said to be ω-*consistent* if there is no formula $\varphi(x)$ such that $T \vdash \exists x\varphi(x)$ and for any $n \in \mathbb{N}$, $T \vdash \neg\varphi(\bar{n})$; T is 1-*consistent* if there is no such a Δ_1^0 formula $\varphi(x)$. We say a theory T is Σ_n^0-*definable* if there is a Σ_n^0 formula $\alpha(x)$ such that n is the Gödel number of some sentence of T if and only if $\mathfrak{N} \models \alpha(\overline{n})$. A theory T is Σ_n^0-*sound* if for all Σ_n^0 sentences ϕ, $T \vdash \phi$ implies $\mathfrak{N} \models \phi$. A theory T is Π_n^0-*decisive* if for all Π_n^0 sentences ϕ, either $T \vdash \phi$ or $T \vdash \neg\phi$ holds.

The notion of interpretation provides us with a method to measure and compare the strength of different theories in different languages. Informally, an interpretation of a theory T in a theory S is a mapping from formulas of T to formulas of S that maps all axioms of T to sentences provable in S. For the precise definition of interpretation, we refer to [Visser, 2011] for more details. Let $T \trianglelefteq S$ denote that T is interpretable in S, and let $T \lhd S$ denote that $T \trianglelefteq S$ but $S \trianglelefteq T$ does not hold. In this paper, we say that T is weaker than S w.r.t. interpretation if $T \lhd S$.

Gödel proves his incompleteness theorem in [Gödel, 1931] for a certain formal system $\mathbf{P}$ related to Russell-Whitehead's Principia Mathematica and based on the simple theory of types over the natural number series and the Dedekind-Peano axioms (see [Beklemishev, 2010, p.3]). Gödel's original first incompleteness theorem ([Gödel, 1931]) says that for any formal theory T formulated in the language of $\mathbf{P}$

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[1] For $n \in \mathbb{N}$, $\bar{n}$ denotes the corresponding numeral in $L(T)$ for n.

[2] In this case, we say the formula $\phi(x_1, \cdots, x_n)$ represents the relation $R(x_1, \cdots, x_n)$.

and obtained by adding a primitive recursive set of axioms to the system $\mathbf{P}$, if T is ω-consistent, then T is incomplete. The following is a modern reformulation of Gödel's first incompleteness theorem.

Theorem 2.2 (Gödel's first incompleteness theorem (G1)). *Let T be a recursively axiomatized extension of* $\mathbf{PA}$. *Then there exists a Gödel's sentence* $\mathbf{G}$ *such that:*

- *if T is consistent, then $T \nvdash \mathbf{G}$;*
- *if T is ω-consistent, then $T \nvdash \neg\mathbf{G}$.*

From Theorem 2.2, if T is ω-consistent, then Gödel's sentence $\mathbf{G}$ is independent of T and hence T is incomplete.

Now, we give a sketch of the main idea of Gödel's proof of G1. In the rest of this section, we assume that T is a recursively axiomatized consistent extension of $\mathbf{PA}$ in $L(\mathbf{PA})$. Gödel's proof of the incompleteness theorem depends on a long chain of ideas, each involving a significant insight. The three main ideas in Gödel's proof of G1 are arithmetization of the syntax of T, representability of recursive functions in $\mathbf{PA}$ and self-reference construction of Gödel's sentence.[3] Under the arithmetization, we could establish the one-to-one correspondence between expressions of $L(T)$ and natural numbers. Under this correspondence, we can translate metamathematical statements about the formal theory T into statements about natural numbers. Moreover, fundamental metamathematical relations can be translated in this way into certain recursive relations, hence into relations representable in the theory T. Consequently, one can speak about a formal system of arithmetic and about its properties as a theory in the system itself (see [Murawski, 1999])! This is the essence of Gödel's idea of arithmetization.[4]

Now, we can define some relations on $\mathbb{N}$ which express some metamathematical properties of T. For example, we can define a binary relation on $\mathbb{N}^2$ as follows: $Proof_T(m, n)$ iff n is the Gödel's number of a proof in T of the formula with Gödel number m. Moreover, we can prove that the relation $Proof_T(m, n)$ is recursive. Gödel proves that every recursive relation is representable in $\mathbf{PA}$. Let $\mathbf{Proof}_T(x, y)$ be the formula which represents $Proof_T(m, n)$ in $\mathbf{PA}$.[5] From the formula $\mathbf{Proof}_T(x, y)$, we can define the provability predicate $\mathbf{Prov}_T(x)$ as $\exists y \mathbf{Proof}_T(x, y)$. Finally, Gödel effectively constructs Gödel's sentence $\mathbf{G}$ which asserts its own unprovability in T (i.e. $T \vdash \mathbf{G} \leftrightarrow \neg\mathbf{Prov}_T(\ulcorner \mathbf{G} \urcorner)$). Gödel shows that if T is consistent, then $T \nvdash \mathbf{G}$; and if T is ω-consistent, then $T \nvdash \neg\mathbf{G}$.

Since we will discuss general provability predicates based on proof predicates, now we give a general definition of proof predicate which is a generalization of properties of Gödel's proof predicate $Proof_T(x, y)$. We say a formula $\mathbf{Prf}_T(x, y)$ is a *proof predicate* of T if it satisfies the following conditions:

(1) $\mathbf{Prf}_T(x, y)$ is Δ_1^0;
(2) $\mathbf{PA} \vdash \forall x(\mathbf{Prov}_T(x) \leftrightarrow \exists y \mathbf{Prf}_T(x, y))$;
(3) for any $n \in \omega$ and formula ϕ, $\mathbb{N} \models \mathbf{Proof}_T(\ulcorner \phi \urcorner, \overline{n}) \leftrightarrow \mathbf{Prf}_T(\ulcorner \phi \urcorner, \overline{n})$;
(4) $\mathbf{PA} \vdash \forall x \forall x' \forall y(\mathbf{Prf}_T(x, y) \wedge \mathbf{Prf}_T(x', y) \rightarrow x = x')$.

Note that each proof predicate represents the relation "y is the code of a proof in T of a formula with Gödel number x". We define the provability predicate $\mathbf{Pr}_T(x)$ from a proof predicate $\mathbf{Prf}_T(x, y)$ by $\exists y \mathbf{Prf}_T(x, y)$, and the consistency statement $\mathbf{Con}(T)$ from a provability predicate $\mathbf{Pr}_T(x)$ by $\neg\mathbf{Pr}_T(\ulcorner \mathbf{0} \neq \mathbf{0} \urcorner)$.

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[3]Before Gödel, Emil Post independently discovered a statement undecidable within Principia Mathematica whose truth could nevertheless be established by metamathematical considerations. But he never published the result. See [Murawski, 1999, p. 203].

[4]For more details on arithmetization, we refer to [Murawski, 1999].

[5]One can speak about the property of T in $\mathbf{PA}$ itself via arithmetization and representability!

The following conditions **D1**-**D3** are called drivability conditions of provability predicate $\mathbf{Pr}_T(x)$.

D1: If $T \vdash \varphi$, then $T \vdash \mathbf{Pr}_T(\ulcorner\varphi\urcorner)$;

D2: $T \vdash \mathbf{Pr}_T(\ulcorner\varphi\urcorner) \rightarrow (\mathbf{Pr}_T(\ulcorner\varphi \rightarrow \psi\urcorner) \rightarrow \mathbf{Pr}_T(\ulcorner\psi\urcorner))$;

D3: $T \vdash \mathbf{Pr}_T(\ulcorner\varphi\urcorner) \rightarrow \mathbf{Pr}_T(\ulcorner\mathbf{Pr}_T(\ulcorner\varphi\urcorner)\urcorner)$.

We say a provability predicate is standard if it satisfies conditions **D1**-**D3**. One important non-standard provability predicate is Rosser provability predicate $\mathbf{Pr}_T^R(x)$ introduced by Rosser [1936] to improve Gödel's first incompleteness theorem. The Rosser provability predicate $\mathbf{Pr}_T^R(x)$ is defined as the formula $\exists y(\mathbf{Prf}_T(x,y) \wedge \forall z \leq y \neg \mathbf{Prf}_T(\dot{\neg}(x),z))$, where $\dot{\neg}$ is a function symbol expressing a primitive recursive function calculating the code of $\neg\phi$ from the code of ϕ.

In this paper, unless stated otherwise, we assume that the provability predicate $\mathbf{Pr}_T(x)$ is standard, and $\mathbf{Con}(T)$ defined as $\neg\mathbf{Pr}_T(\ulcorner\mathbf{0} \neq \mathbf{0}\urcorner)$ is the canonical consistence statement of T formulated via a standard provability predicate $\mathbf{Pr}_T(x)$. The importance of standard provability predicate and canonical consistence statement lies in that $\mathsf{G2}$ holds for the canonical consistence statement formulated via the standard provability predicate as we will show. However, $\mathsf{G2}$ may fail for non-standard provability predicates and non-canonical consistence statements. There are a lot of research on non-standard provability predicates and non-canonical consistence statements from the literature (see [Feferman, 1960] and [Visser, 2011]).

Gödel announces the second incompleteness theorem in an abstract published in October 1930: no consistency proof of systems such as Principia, Zermelo-Fraenkel set theory, or the systems investigated by Ackermann and von Neumann is possible by methods that can be formulated in these systems (see [Zach, 2007, p.431]). In the modern formulation, the second incompleteness theorem states that if T is consistent, then the arithmetical formula $\mathbf{Con}(T)$ that expresses the consistency of T is not provable in T.[6] For the proof of $\mathsf{G2}$, we use the key fact that the provability predicate $\mathbf{Prov}_T(x)$ is standard and satisfies conditions **D1**-**D3**. Based on this fact, we can show that $T \vdash \mathbf{Con}(T) \leftrightarrow \mathbf{G}$. Thus, $\mathsf{G2}$ holds: if T is consistent, then $T \nvdash \mathbf{Con}(T)$.

For Gödel's proof of $\mathsf{G1}$, only assuming that T is consistent is not enough to show that Gödel's sentence is independent of T. In fact, the optimal condition to show that Gödel's sentence is independent of T is that $T + \mathbf{Con}(T)$ is consistent (see [Isaacson, 2011, Theorems 35-36]).[7] Only assuming that T is consistent is not sufficient to show that $T \nvdash \neg\mathbf{Con}(T)$. But we can prove that $\mathbf{Con}(T)$ is independent of T by assuming that T is 1-consistent which is stronger than "T is consistent". For more details of proofs of $\mathsf{G1}$ and $\mathsf{G2}$, we refer to Chapter 2 in [Murawski, 1999].

3. On the depth of Gödel's incompleteness theorem

After introducing Gödel's incompleteness theorem, now we turn to the analysis of the depth of Gödel's theorem. In this section, we account for the depth of Gödel's incompleteness theorem from the following three criteria: influence, fruitfulness, and unity.

3.1. **Influence.** In this section, we justify for the influence of Gödel's incompleteness theorem from its impact on foundations of mathematics, philosophy, mathematics, and theoretic computer science, that is revealed by the research practice on incompleteness after Gödel. We make no attempt to exhaustively discuss the

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[6]Even if $\mathbf{Con}(T)$ is not provable in T, T is reflective: for each finite sub-theory S of T, $T \vdash \mathbf{Con}(S)$ (see [Murawski, 1999, Theorem 2.6.12]).

[7]This optimal condition is much weaker than ω-consistency.

full impact of Gödel's incompleteness theorem and all of the ongoing important research programs that it suggests.

Gödel's incompleteness theorem is one of the most remarkable and profound discoveries in the 20th century, an important milestone in the history of modern logic. Gödel's incompleteness theorem has had wide and profound influence on the development of logic, philosophy, mathematics, theoretical computer science and other fields, substantially shaping foundations of mathematics after 1931. On the impact of Gödel's incompleteness theorems, Feferman said: "their relevance to mathematical logic (and its offspring in the theory of computation) is paramount; further, their philosophical relevance is significant, but in just what way is far from settled; and finally, their mathematical relevance outside of logic is very much unsubstantiated but is the object of ongoing, tantalizing efforts" (see [Feferman, 2006, p.434]).

The influence of Gödel's incompleteness theorem on foundations of mathematics is reflected in the following five aspects: (1) Gödel's incompleteness theorem reveals the independence phenomenon which is common in mathematics and logic; (2) Gödel's incompleteness theorem shows certain weaknesses and the essential limitation of one given formal system (or the limit of proof and computation); (3) Gödel's incompleteness theorem reveals the essential difference between the notion of "provability in **PA**" and the notion of "truth in the standard model of arithmetic";[8] (4) Gödel's incompleteness theorem is a blow to Whitehead-Russell's program for proving that all mathematics (or at least quite a lot of it) could be derived solely from logic in their three-volume Principia Mathematica; (5) Gödel's incompleteness theorem has profound influence on the development of Hilbert's program.

There is extensive literature about the development of Hilbert's program after Gödel's incompleteness theorem, and its effect on mathematical logic (especially proof theory) and philosophy of mathematics (see [Feferman, 1988], [Franks, 2009], [Murawski, 1999], [Simpson, 1988], [Zach, 2007]). The above aspects (2)-(5) are well known, and we only give some explanations of the independence phenomenon in mathematics and logic.

Nowadays, independence is ubiquitous in logic. The independence phenomenon reveals the big gap between mathematical truth and provability in formal systems. Gödel's incompleteness theorem firstly reveals the independence phenomenon of formal systems at the level of arithmetic, and shows the essential limitation of any formal system containing "enough" information of arithmetic. After Gödel, people have found many arithmetic sentences from classic mathematics that are independent of **PA**. The sequent research after Gödel reveals the independence phenomenon of stronger formal systems such as higher order arithmetic and **ZFC**. For example, Gödel and Cohen show that Continuum Hypothesis (**CH**) is independent of **ZFC**, that means **ZFC** is essentially incomplete to capture all set-theoretic truth.[9] Moreover, after Gödel people have found many examples of statements that are independent of **ZFC** from varied fields of mathematics such as analysis, algebra, topology and mathematical logic.

Now, we give a brief account of the impact of Gödel's incompleteness theorem in mathematics. The incompleteness theorem and its proofs are strikingly original mathematics. It has been often thought that Gödel's proof of G1 uses pure logical method and has no relevance with mathematics: Gödel's sentence constructed via

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[8]I.e. there is a true sentence of arithmetic which is independent of **PA**.

[9]Gödel proposes the research program to find new axioms of set theory to capture all set-theoretic truth, that is one of the central motivations of current research in set theory.

the meta-mathematical method is artificial (not natural), and has no real mathematical content. As Isaacson [1987] points out, Gödel's sentence is constructed not by reflecting about arithmetical properties of natural numbers, but by reflecting about an axiomatic system in which those properties are formalized. A natural question after Gödel is: can we find examples of natural independent sentences with real mathematical contents? We call the incompleteness phenomenon revealed by natural independent sentences with real mathematical contents as concrete incompleteness.

In fact, the incompleteness phenomenon prevails in classic mathematics. The research program of concrete incompleteness seeks for natural independent sentences with real mathematical contents. After Gödel, many natural independent arithmetic sentences with real mathematical contents have been found. These independent sentences have a clear mathematical flavor, and do not refer to the arithmetization of syntax and provability predicate. Paris and Harrington [1977] proposes a mathematically natural true statement unprovable in **PA**: *Paris-Harrington Principle* (PH). Following PH, many other mathematically natural statements independent of **PA** with combinatorial or number-theoretic contents were formulated: the Kanamori-McAloon principle [Kanamori and McAloon, 1987], the Kirby-Paris sentence [Kirby and Paris, 1982], the Hercules-Hydra game [Kirby and Paris, 1982], the Worm principle [Beklemishev, 2003][Hamano and Okada, 1997], the flipping principle [Kirby, 1982], the arboreal statement [Mills, 1980], P.Pudlák's Principle [Pudlák, 1979][Hájek and Paris, 1986], the kiralic and regal principles [Clote and McAloon, 1983] (see [Bovykin, 2006, p. 40]). All these statements are thought of as much more genuinely and purely mathematical than Gödel's sentence, and reveal the concrete incompleteness of first order arithmetic. All these concrete independent arithmetic sentences from mathematics are provable in fragments of second order arithmetic, and are more complex than Gödel's sentence: Gödel's sentence is equivalent to **Con(PA)** in **PA**; but all these arithmetic sentences are not only independent of **PA** but also independent of **PA + Con(PA)** (see [Beklemishev, 2010, p. 36] and [Murawski, 1999, p. 301]).

Harvey Friedman is a leading researcher in the field of concrete incompleteness. Friedman's work extends the research on concrete incompleteness from first order arithmetic to higher order arithmetic. Friedman's book "Boolean Relation Theory and Incompleteness" is a comprehensive monograph on concrete incompleteness in mathematics, and provides many examples of concrete mathematical theorems not provable in subsystems of second-order arithmetic stronger than **PA**, and a number of concrete mathematical statements provable in third-order arithmetic but not provable in second-order arithmetic (see [Friedman, forthcoming]). Cheng [2015, 2019a] finds a concrete mathematical theorem "Harrington's principle implies the existence of zero sharp", and shows that this theorem is expressible in second-order arithmetic, not provable in second-order or third-order arithmetic, but provable in fourth-order arithmetic. For more examples of concrete mathematical incompleteness and discussions of this subject, we refer to [Simpson, 1987], [Simpson, 1985], [Pacholski, 1980], [Berline, McAloon and Ressayre, 1981], [Cheng, 2019a], [Bovykin, 2006] and [Friedman, forthcoming].

The impact of Gödel's incompleteness theorem is not confined to the community of mathematicians and logicians, and it has been very popular and widely used outside mathematics and logic. Gödel's incompleteness theorem has significant philosophical meaning, and raises a number of philosophical questions concerning the nature of mind and machine, as well as the limit of proof and computation. Gödel succeeds with his proof of the incompleteness theorem because he recognizes the central importance of distinguishing theory from metatheory, logic from meta-logic,

signs from their referents (see [Baaz, Papadimitriou, Putnam, Scott and Harper, 2014]). In the literature, there are a lot of discussions about the philosophical meaning of Gödel's incompleteness theorem (see [Gaifman, 2000][Resnik, 1974] [Auerbach, 1985][Detlefsen, 1979][Detlefsen, 1980][Franks, 2009][Pudlák, 1999]). For a popular book about the use and misuse of Gödel's incompleteness theorem in and outside mathematics and logic for a wider audience, we refer to [Franzen, 2005]. In the following, we only give an overview of the philosophical influence of Gödel's incompleteness theorem on the Anti-Mechanism Argument and Gödel's Disjunctive Thesis.

There are a lot of discussions in the literature about the influence of Gödel's incompleteness theorem on the philosophical question of whether the mind can be mechanized (see [Penrose, 1989] [Chalmers, 1995] [Lucas, 1996] [Lindström, 2006] [Feferman, 2009] [Shapiro, 1998] [Shapiro, 2003] [Koellner, 2016] [Koellner, 2018a][Koellner, 2018b] [Cheng, 2020]). The Anti-Mechanism Argument claims that the mind cannot be mechanized in the sense that the mathematical outputs of the idealized human mind outstrip the mathematical outputs of any Turing machine.[10] A popular interpretation of G1 is that G1 implies that the Anti-Mechanism Argument holds. Gödel did not argue that his incompleteness theorem implies that the mind cannot be mechanized. For Gödel, the human mind cannot be mechanized and human mind is sufficiently powerful to capture all mathematical truths. Gödel believes that the distinctiveness of the human mind when compared to a Turing machine is evident in its ability to come up with new axioms and develop new mathematical theories. Based on his rationalistic optimism, Gödel believes that we are arithmetically omniscient. However, Gödel admits that he cannot give a convincing argument for either the thesis "the human mind cannot be mechanized" or the thesis "there are absolutely undecidable statements". Gödel thinks that the most he could claim from his incompleteness theorem is a weaker conclusion, Gödel's Disjunctive Thesis (GD),[11] which claims that if the human mind can be mechanized, then there are absolutely undecidable statements in the sense that there are mathematical truths that cannot be proved by the idealized human mind.[12] We refer to [Horsten and Welch, 2016] for more discussions of GD.

For Gödel, GD is a mathematically established fact of great philosophical interest which follows from his incompleteness theorem, and it is entirely independent from the standpoint taken toward the foundation of mathematics (see [Gödel, 1951]).[13] For more detailed discussions of Gödel's Disjunctive Thesis and the relationship between Gödel's incompleteness theorem and the Anti-Mechanism Argument, we refer to [Cheng, 2020], [Horsten and Welch, 2016] and Koellner's recent nice papers [Koellner, 2016, 2018a, 2018b].

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[10]We will not consider the performance of actual human minds, with their limitations and defects; but only consider the idealized human mind and look at what it can do in principle. See [Koellner, 2018a].

[11]The original version of GD was introduce by Gödel in [Gödel, 1995], p. 310: "So the following disjunctive conclusion is inevitable: either mathematics is incompletable in this sense, that its evident axioms can never be comprised in a finite rule, that is to say, the human mind (even within the realm of pure mathematics) infinitely surpasses the powers of any finite machine, or else there exist absolutely unsolvable diophantine problems of the type specified (where the case that both terms of the disjunction are true is not excluded, so that there are, strictly speaking, three alternatives)".

[12]Gödel's Disjunctive Thesis concerns the limit of mathematical knowledge and the possibility of the existence of mathematical truths that are inaccessible to the idealized human mind.

[13]In the literature, there is a consensus that Gödel's argument for GD is definitive, but until now we have no compelling evidence for or against any of the two disjuncts (see [Horsten and Welch, 2016]).

Now, we give a brief account of the impact of Gödel's incompleteness theorem on theoretic computer science. Theoretical computer science is about the power and limitation of computation. Gödel's incompleteness theorem contains several technical ideas that can be recognized as computational. In Gödel's proof of the incompleteness theorem, Gödel uses primitive recursive function and arithmetization which are important tools in theoretic computer science. The technique of arithmetization which represents syntactic elements, such as logical terms, formulas, and proofs, as numbers, has been used crucially in theoretical computer science (see [Papadimitriou, 2014]). Negative results constitute an important and distinguishing tradition in theoretical computer science. One typical example of negative results in theoretic computer science is the undecidability of the halting problem (the problem of telling whether a given program will eventually terminate) proved by Alan Turing. Gödel's incompleteness theorem is an ideal archetype, and Turing's halting problem can be seen as a sharpening of Gödel's theorem. For more discussions of the influence of Gödel's incompleteness theorem on theoretic computer science, we refer to [Baaz, Papadimitriou, Putnam, Scott and Harper, 2014].

3.2. **Fruitfulness.** In this section, we discuss another criteria of the depth of Gödel's incompleteness theorem: fruitfulness. The fruitfulness of a theorem measures the degree to which a theorem (or a proof of a theorem) leads to yet further theorems and proofs (see [Arana, 2015]). In this section, we account for the fruitfulness of Gödel's incompleteness theorem from the following three indicators: different proofs of the theorem, generalizations of the theorem (how and in what degree the theorem can be generalized or extended), and the boundary (or the limit) of the theorem (i.e. under what conditions the theorem holds and under what conditions the theorem fails).

The first indicator of the fruitfulness of Gödel's incompleteness theorem is the diversity of its proof methods. After Gödel, people have found many different proofs of Gödel's incompleteness theorem. We first give some definitions. We say a proof of G1 is constructive if it explicitly constructs the independent sentence from the base theory algorithmically. A non-constructive proof of G1 proves the mere existence of the independent sentence, and does not show its existence algorithmically. We say that a proof of G1 has the Rosser property if the proof only assumes that the base theory is consistent instead of assuming that the base theory is ω-consistent or 1-consistent.

We could classify different proofs of Gödel's incompleteness theorem from the literature based on the following features: (1) proof-theoretic proof; (2) recursion-theoretic proof;[14] (3) model-theoretic proof;[15] (4) proof via arithmetization; (5) proof via the Diagonalization Lemma;[16] (6) proof based on logical paradox;[17] (7) constructive proof; (8) proof with the Rosser property; (9) proof via Kolmogorov complexity;[18] (10) concrete incompleteness (i.e. proof via an independent sentence

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[14]For example, Kleene gives a simple proof of G1 via recursion theory: for any consistent recursive enumerable theory T that contains **Q**, there exists some $t \in \omega$ such that $\varphi_t(t) \uparrow$ holds but $T \nvdash$ "$\varphi_t(t) \uparrow$" (see [Salehi and Seraji, 2018, Theorem 2.2]).

[15]Arithmetic completeness theorem ([Lindström, 1997]) is an important tool in the model-theoretic proof of the incompleteness theorem.

[16]Let T be a consistent r.e. extension of **Q**. The Diagonalization Lemma says that for any formula $\phi(x)$ with exactly one free variable, there exists a sentence θ such that $T \vdash \theta \leftrightarrow \phi(\ulcorner\theta\urcorner)$.

[17]Many paradoxes have been used to give new proofs of incompleteness theorems: e.g. the Liar Paradox, Berry's Paradox, Grelling-Nelson's Paradox and Yablo's Paradox.

[18]Kolmogorov complexity is a measure of the quantity of information in finite objects. Chaitin [1974] gives information-theoretic formulation of G1, and proves a weaker version of G1 in terms of Kolmogorov complexity. Kikuchi [1997] proves the formalized version of G1 via Kolmogorov complexity.

with real mathematical contents). Gödel's proof of G1 has the following features: (1) uses proof-theoretic method with arithmetization; (2) does not directly use the Diagonalization Lemma; (3) the proof formalizes the liar paradox; (4) the proof is constructive;[19] (5) Gödel's proof does not have the Rosser property; (6) Gödel's sentence is constructed via metamathematical method, and has no real mathematical content.[20]

We give some comments about these features of proofs of Gödel's incompleteness theorem. Firstly, these features are not exclusive: a proof of Gödel's theorem may have several above features. Secondly, each of the above features of Gödel's incompleteness theorem is not a necessary condition to prove Gödel's theorem. We have examples of proofs of G1 with the above features and examples of proofs of G1 without the above features from the literature. For example, for the proof of G1, we also have examples of proofs which are non-constructive,[21] proofs having the Rosser property, and proofs without the use of arithmetization.[22] Thirdly, these different proofs of Gödel's incompleteness theorem establish the connection among different fields: proof theory, recursion theory, logical paradox, model theory, Kolmogorov complexity, ect.

The second indicator of the fruitfulness of Gödel's incompleteness theorem is the great variety of its generalizations. From the literature, G1 and G2 can be generalized to both extensions of **PA** and weaker theories than **PA** w.r.t. interpretation. These generalizations show the applicability and explanatory power of Gödel's incompleteness theorem. In the following, we give some typical examples to explain this.

We first discuss generalizations of G1. The first example is Rosser's improvement of G1. Gödel's proof of G1 assumes that the base theory is ω-consistent. Rosser proves G1 only assuming that the base theory is consistent: Rosser constructs a Rosser sentence (Π_1^0 sentence), and shows that if T is a recursively axiomatized consistent extension of **Q**, then the Rosser sentence is independent of T. Note that ω-consistency implies consistency. But the converse does not hold, and the notion of ω-consistency is stronger than consistency since we can find examples of theories that are consistent but not ω-consistent.[23]

The second example is the generalization of G1 to arithmetically definable theories. From G1, if a theory T is a Σ_1^0-definable and consistent extension of **PA**, then T is not Π_1^0-decisive. Kikuchi-Kurahashi and Salehi-Seraji generalize G1 to arithmetically definable theories (see [Kikuchi and Kurahashi, 2017] and [Salehi and Seraji, 2017]), and show that if T is a Σ_{n+1}^0-definable and Σ_n^0-sound extension of **Q**, then T is not Π_{n+1}^0-decisive.

Thirdly, G1 can also be generalized via the notion of interpretation. We define that G1 holds for a theory T iff for any recursively axiomatizable consistent theory S, if T is interpretable in S, then S is incomplete ([Cheng, 2019b]). In fact, G1 also

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[19]I.e. given a consistent r.e. extension T of **PA**, one can effectively find a true Π_1^0 sentence G_T of arithmetic such that G_T is independent of T. Gödel calls this the "incompletability or inexhaustability of mathematics".

[20]I.e. Gödel's sentence is a pure logical construction (via the arithmetization of syntax and provability predicate) and has no relevance with classic mathematics (without any combinatorial or number-theoretic content). On the contrary, Paris-Harrington Principle is an independent arithmetic sentence from classic mathematics with combinatorial contents.

[21]A non-constructive proof of G1 proves the mere existence of the independent sentence and does not show its existence algorithmically.

[22]All proofs of G1 we have discussed use arithmetization. However, A. Grzegorczyk proposes the theory **TC** in [Grzegorczyk, 2005] as a possible alternative theory for studying incompleteness and undecidability, and proves that **TC** is incomplete without the use of arithmetization.

[23]For example, assuming **PA** is consistent, then **PA** + ¬**Con**(**PA**) is consistent, but not ω-consistent.

holds for many weaker theories than **PA** w.r.t. interpretation. Let **R** be the theory consisting of schemes Ax1-Ax5 with $L(\mathbf{R}) = \{\mathbf{0}, \mathbf{S}, +, \times, \leq\}$ where $\leq$ is a primitive binary relation symbol, and $\overline{n} = \mathbf{S}^n\mathbf{0}$ for $n \in \mathbb{N}$:

Ax1: $\overline{m} + \overline{n} = \overline{m+n}$;
Ax2: $\overline{m} \times \overline{n} = \overline{m \cdot n}$;
Ax3: $\overline{m} \neq \overline{n}$, if $m \neq n$;
Ax4: $\forall x(x \leq \overline{n} \rightarrow x = \overline{0} \vee \cdots \vee x = \overline{n})$;
Ax5: $\forall x(x \leq \overline{n} \vee \overline{n} \leq x)$.

It is well known that G1 holds for **Q** and **R** (see [Vaught, 1962]). For more examples of weaker theories than **PA** w.r.t. interpretation for which G1 holds, we refer to [Cheng, 2019b] for more discussions.

Now, we discuss generalizations of G2. Let T be a recursively axiomatizable consistent extension of **Q**. Recall that $\mathbf{Con}(T)$ is the canonical arithmetic sentence expressing the consistency of T. In fact, G2 can also be generalized in different ways. Here, we only give two typical examples we think important. Firstly, G2 can be generalized via the notion of interpretation: there is no r.e. theory T such that $\mathbf{Q} + \mathbf{Con}(T)$ is interpretable in T, i.e. $\mathbf{Q} + \mathbf{Con}(T) \ntrianglelefteq T$ (see [Visser, 2011]). As a corollary, G2 holds for any consistent r.e. theory interpreting **Q**. Secondly, Löb's theorem is an important generalization of G2. Löb shows that for any standard provability predicate $\mathbf{Pr}_T(x)$ and any formula ϕ, if $T \vdash \mathbf{Pr}_T(\phi) \rightarrow \phi$, then $T \vdash \phi$. As a corollary, we have $T \nvdash \mathbf{Con}(T)$.

The third indicator of the fruitiness of Gödel's incompleteness theorem is the boundary (or the limit) of the theorem. The research on the boundary of Gödel's incompleteness theorem reveals the limit of the applicability of Gödel's incompleteness theorem, greatly deepens our understanding of the scope of Gödel's incompleteness theorem, and contributes to new mathematical evidences of the fruitfulness of Gödel's incompleteness theorem.

We first give a brief account of the boundary (or the limit) of G1. There are many consistent formal theories which are complete.[24] Whether a theory about arithmetic is complete depends on the language of the theory. The theory **PA** is incomplete in the language $L(\mathbf{0}, \mathbf{S}, +, \times)$. There are respectively recursively axiomatized complete arithmetic theories in the language of $L(\mathbf{0}, \mathbf{S})$, $L(\mathbf{0}, \mathbf{S}, <)$ and $L(\mathbf{0}, \mathbf{S}, <, +)$ (see Section 3.1-3.2 in [Enderton, 2001]). Firstly, containing enough information of arithmetic is essential for the proof of G1.[25] Secondly, containing the information about the arithmetic of multiplication is essential for the proof of G1. If the theory contains only the information about the arithmetic of addition without multiplication, then it could be complete.[26] Finally, containing the arithmetic of multiplication is not a sufficient condition for a theory to be incomplete.[27]

Recall that G1 holds for some arithmetically definable extensions of **Q**, but it is not true that any arithmetically definable extension of **Q** is incomplete.[28] It was

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[24]For example, the following theories are complete: the theory of dense linear orderings without endpoints (**DLO**), the theory of ordered divisible groups (**ODG**), the theory of algebraically closed fields of given characteristic (**ACF$_\mathbf{p}$**), and the theory of real closed fields (**RCF**), etc (see [Epstein, 2011] for details of these theories).

[25]For example, the Euclidean geometry is not about arithmetic but only about points, circles and lines in general; but the Euclidean geometry is complete as Tarski has proved.

[26]For example, Presburger arithmetic is the theory of arithmetic of addition, and its language only contains non-logical symbols $\mathbf{0}, \mathbf{S}$ and $+$; but Presburger arithmetic is complete (see [Murawski, 1999, Theorem 3.2.2]).

[27]For example, there exists a complete recursively axiomatized theory in the language $L(\mathbf{0}, \times)$ (see [Murawski, 1999, p.230]).

[28]For example, there exists a Σ_{n+1}^0-definable, Σ_{n-1}^0-sound ($n \geq 1$) theory that is a complete extension of **Q** (see [Salehi and Seraji, 2017, Theorem 2.6]).

often thought that $\mathbf{R}$ is the weakest theory w.r.t. interpretation for which G1 holds. In fact, we can find many theories S weaker than $\mathbf{R}$ such that G1 holds for S.[29] We conjecture that there is no minimal r.e. theory w.r.t. interpretation for which G1 holds.

Now, we give a brief account of the boundary (or the limit) of G2. Both mathematically and philosophically, G2 is more problematic than G1. The difference between G1 and G2 is that, in the case of G1, we are mainly interested in the fact that it shows that *some* sentence is independent of the base theory. We make no claim to the effect that that sentence "really" expresses what we would express by saying "$\mathbf{PA}$ cannot prove this sentence". But in the case of G2, we are also interested in the content of the statement. In the following, we give a brief overview of the intensionality of G2 (we refer to [Cheng, 2019c] for more details).

For a consistent theory T, we say that G2 holds for T if the consistency statement of T is not provable in T. However, this definition is vague, and whether G2 holds for T depends on how we formulate the consistency statement. We refer to this phenomenon as the intensionality of G2. The status of G2 is essentially different from G1 due to the intensionality of G2. We can say that G1 is extensional in the sense that we can construct a concrete independent mathematical statement without referring to arithmetization and provability predicate. However, G2 is intensional, and "whether G2 holds for T" depends on varied factors as we will discuss.

The intensionality of G2 has been widely discussed from the literature (e.g. [Halbach and Visser, 2014a], [Halbach and Visser, 2014b], [Visser, 2011]). Visser [2011] locates three sources of indeterminacy in the formalisation of a consistency statement for a theory T: (I) the choice of a proof system; (II) the choice of a way of numbering; (III) the choice of a specific formula numerating the axiom set of T.

In this section, unless stated otherwise, we make the following assumptions:

- The theory T is a recursively axiomatized consistent extension of $\mathbf{Q}$;
- The canonical arithmetic formula to express the consistency of the base theory T is $\mathbf{Con}(T) \triangleq \neg \mathbf{Pr}_T(\mathbf{0 \neq 0})$;
- The canonical numbering we use is Gödel's numbering;
- The provability predicate we use is standard;
- The formula representing the set of axioms is Σ_1^0.

Based on the current research on incompleteness from the literature, we argue that "whether G2 holds for T" depends on the following factors:

(1) the choice of the base theory T;
(2) the choice of the method to express consistency;
(3) the choice of a provability predicate;
(4) the choice of a numbering;
(5) the choice of a specific formula numerating the axiom set of T.

These factors are not independent, and a choice made at an earlier stage may have effects on the choices made at a later stage. In the following, when we discuss how G2 depends on one factor, we always assume that other factors are fixed as in the above default assumptions, and only the factor we are discussing is varied. For example, Visser [2011] rests on fixed choices for (1)-(2) and (4)-(5) but varies the choice of (3); Grabmayr [2019] rests on fixed choices for (1) and (3)-(5) but varies the choice of (2); Feferman [1960] rests on fixed choices for (1)-(4) but varies the choice of (5). In the following, we give a brief discussion of how G2 depends on

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[29]For example, Cheng [2019b] shows that for any recursively inseparable pair $\langle A, B \rangle$, there is a theory $U_{\langle A,B \rangle}$ such that G1 holds for $U_{\langle A,B \rangle}$ and $U_{\langle A,B \rangle} \lhd \mathbf{R}$.

the above five factors. For more detailed discussions of these factors, we refer to [Cheng, 2019c].

Firstly, "Whether G2 holds for T" depends on the choice of the base theory. A foundational question about G2 is: how much of information about arithmetic is required for the proof of G2. If the base theory does not contain enough information of arithmetic, then G2 may fail in the sense that the consistency statement is provable in the base theory.[30] Pakhomov [2019] defines a theory $H_{<\omega}$, and shows that it proves its own canonical consistency. Thus, G2 fails for the theory $H_{<\omega}$.[31]

Secondly, "Whether G2 holds for T" depends on the choice of the method to express consistency. From the philosophical point of view, one can ask: what a consistency statement of a theory is? when can we reasonably say that the arithmetic sentence $\mathbf{Con}(T)$ does really express the consistency of T? (see [Visser, 2011, p. 545]). These questions are difficult to answer, and have been investigated by many logicians, among them Resnik [1974], Detlefsen [1980], Visser [2016, 2011], Feferman [1960], Auerbach [1985] and Franks [2009]. In the literature, we usually use an arithmetic formula in the language of T to express the consistency of T. Artemov [2019] argues that in Hilbert's consistency program, the original formulation of consistency "no sequence of formulas is a derivation of a contradiction" is about finite sequences of formulas, not about arithmetization, proof codes, and internalized quantifiers. Artemov concludes that G2 does not actually exclude finitary consistency proofs of the original formulation of consistency. Artemov shows that the original formulation of consistency admits a direct proof in informal arithmetic, and this proof is formalizable in $\mathbf{PA}$ (see [Artemov, 2019]).[32]

In the following, we use a single arithmetic sentence to express the consistency statement. Even among consistency statements defined via a single arithmetic sentence, we still have different ways to express the consistency of T. For example, another way to express the consistency of T is $\mathbf{Con}^0(T) \triangleq \forall x(\mathbf{Fml}(x) \land \mathbf{Pr}_T(x) \to \neg\mathbf{Pr}_T(\dot{\neg}x))$.[33] Kurahashi [2019] constructs a Rosser provability predicate such that G2 holds for the consistency statement formulated via $\mathbf{Con}^0(T)$,[34] but G2 fails for the consistency statement formulated via $\mathbf{Con}(T)$ (i.e. the consistency statement formulated via $\mathbf{Con}(T)$ and the Rosser provability predicate is provable in T).

Thirdly, "whether G2 holds for T" depends on the choice of the provability predicate. Visser [2016] argues that, being a consistency statement is not an absolute concept but a role w.r.t. a choice of the provability predicate (see [Visser, 2016]). Recall that G2 holds for standard provability predicates.[35] However, G2 may fail for non-standard provability predicates. Define the consistency statement $\mathbf{Con}^R(T)$ via Rosser provability predicate as $\neg\mathbf{Pr}_T^R(\ulcorner\mathbf{0} \neq \mathbf{0}\urcorner)$. Then G2 fails for Rosser provability predicate in the sense that $T \vdash \mathbf{Con}^R(T)$.

Fourthly, "Whether G2 holds for T" depends on the choice of numberings. Any injective function γ from a set of $L(\mathbf{PA})$-expressions to ω qualifies as a numbering. Gödel's numbering is a special kind of numberings under which the Gödel number

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[30]Willard [2006] explores the generality and boundary-case exceptions of G2 under some base theories. Willard constructs examples of recursively enumerable arithmetical theories that couldn't prove the totality of successor function but could prove their own canonical consistency (see [Willard, 2001], [Willard, 2006]).

[31]Unlike Willard's theories, $H_{<\omega}$ isn't an arithmetical theory but a theory formulated in the language of set theory with an additional unary function.

[32]Informal arithmetic is the theory of informal elementary number theory containing recursive identities of addition and multiplication as well as the induction principle. The formal arithmetic $\mathbf{PA}$ is just the conventional formalization of the informal arithmetic (see [Artemov, 2019]).

[33]$\mathbf{Fml}(x)$ is the formula which represents the relation that x is a code of a formula.

[34]I.e. the consistency statement formulated via $\mathbf{Con}^0(T)$ and the Rosser provability predicate is not provable in T.

[35]If a provability predicate $\mathbf{Pr}_T(x)$ is standard, then $T \nvdash \neg\mathbf{Pr}_T(\ulcorner\mathbf{0} \neq \mathbf{0}\urcorner)$.

of the set of axioms of **PA** is recursive. Grabmayr [2019] shows that G2 holds for acceptable numberings.[36] But G2 fails for some non-acceptable numberings.

Finally, "Whether G2 holds for T" depends on the numeration of T. We say that a formula $\alpha(x)$ is a numeration of T if for any n, we have **PA** $\vdash \alpha(\overline{n})$ iff n is the Gödel number of some $\phi \in T$. As a generalization, G2 holds for any Σ_1^0 numeration of T.[37] However, G2 fails for some Π_1^0 numerations of T. For example, Feferman [1960] constructs a Π_1^0 numeration $\tau(u)$ of T such that G2 fails under this numeration (i.e. $T \vdash \mathbf{Con}_\tau(T)$).

3.3. **Unity.** In this section, we discuss the third criteria of the depth of Gödel's incompleteness theorem we propose: unity. The unity of Gödel's incompleteness theorem means that it ties together apparently disparate fields and draws interconnections between these fields. In this section, we give a brief account of the unity of Gödel's incompleteness theorem from the following aspects:

- finding the order in chaos between mathematics and meta-mathematics;
- the close relationship with the theory of undecidability;
- the close relationship with logical paradox;
- the close relationship with provability logic;
- the close relationship with the formal theory of truth.

Gödel's proof of the incompleteness theorem uses methods from both mathematics and logic. For example, in Gödel's proof, he uses the Chinese Remainder Theorem and the unique factorization in number theory, as well as some meta-mathematical methods in logic such as arithmetization, representability, and self-reference construction. In Section 3.1, we give some examples of natural independent sentences with real mathematical contents.[38] Section 3.1 emphasizes the difference between meta-mathematical independent sentences constructed via pure logic and concrete independent sentences with real mathematical contents. An interesting and amazing fact is that all the mathematically natural independent sentences with combinatorial or number-theoretic contents we list in Section 3.1 are in fact provably equivalent in **PA** to a certain meta-mathematical sentence. Consider the following reflection principle for Σ_1^0 sentences: for any Σ_1^0 sentence ϕ in $L(\mathbf{PA})$, if ϕ is provable in **PA**, then ϕ is true. Using the arithmetization of syntax, one can write this principle as a sentence of $L(\mathbf{PA})$ and denote it by $\mathsf{Rfn}_{\Sigma_1^0}(\mathbf{PA})$ (see [Murawski, 1999, p.301]). McAloon has shown in **PA** that the Paris-Harrington Principle is equivalent to $\mathsf{Rfn}_{\Sigma_1^0}(\mathbf{PA})$ (see [Murawski, 1999, p. 301]). In fact, similar equivalences can be established for all the natural independent sentences with combinatorial or number-theoretic contents we list in Section 3.1 (see [Beklemishev, 2010, p. 36], [Beklemishev, 2003, p. 3] and [Murawski, 1999, p. 301]). Isaacson [1987] argues that this result reveals something of the implicit (hidden) higher-order content of the mathematically natural independent sentences we list in Section 3.1. This phenomenon shows that the difference between mathematical and meta-mathematical statements is not as huge as we might have expected (see [Dean, 2015] for more discussions of this claim).

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[36]For the definition of acceptable numberings, we refer to [Grabmayr, 2019].

[37]Given a formula $\alpha(x)$ in $L(T)$, define the formula $\mathbf{Prf}_\alpha(x, y)$ saying "y is the Gödel number of a proof of the formula with Gödel number x from the set of all sentences satisfying $\alpha(x)$"; define the provability predicate $\mathbf{Pr}_\alpha(x)$ of $\alpha(x)$ as $\exists y \mathbf{Prf}_\alpha(x, y)$ and the consistency statement $\mathbf{Con}_\alpha(T)$ as $\neg\mathbf{Pr}_\alpha(\ulcorner \mathbf{0} \neq \mathbf{0} \urcorner)$. This generalization says that if $\alpha(x)$ is a Σ_1^0 numeration of T, then $T \nvdash \mathbf{Con}_\alpha(T)$.

[38]For example, the Paris-Harrington Principle, the Kanamori-McAloon principle, the Kirby-Paris sentence, the Hercules-Hydra game, the Worm principle, the flipping principle, the arboreal statement, the P.Pudlák's Principle, the kiralic and regal principles.

In the rest of this section, we give a brief account of the close relationship among Gödel's incompleteness theorem, the theory of undecidability, logical paradox, provability logic, and the formal theory of truth.

Gödel's work has many crucial connections to the theory of computation and undecidability. Gödel's proof contains the germs of such influential computational ideas as arithmetization and primitive recursion. The method of arithmetization plays a major role in the growth of recursion theory. We say a theory T is *essentially undecidable* if any recursively axiomatizable consistent extension of T in the same language is undecidable; and T is *essentially incomplete* if any recursively axiomatizable consistent extension of T is incomplete. Since a theory T is essentially undecidable if and only if T is essentially incomplete,[39] the theory of completeness/incompleteness is closely related to the theory of decidability/undecidability. Recall that we have defined the notion "G1 holds for a theory T". From [Cheng, 2019b], G1 holds for T if and only if T is essentially undecidable. It is well known that we can prove G1 and G2 in terms of the undecidability of the halting problem. All of these show the close relationship between the theory of incompleteness and the theory of undecidability.

The current research practice on Gödel's incompleteness theorem reveals that G1 is closely related to logical paradox. Gödel comments in his famous paper that "any epistemological antinomy could be used for a similar proof of the existence of undecidable propositions" (see [Feferman, 1995a]). In Gödel's proof of G1, we can view Gödel's sentence as the formalization of the Liar Paradox. Gödel's sentence concerns the notion of provability, but the liar sentence in the Liar Paradox concerns the notion of truth in the standard model of arithmetic. Except for the Liar Paradox, many other paradoxes have been properly formalized to give new proofs of the incompleteness theorem: for example, Berry's Paradox in [Boolos, 1989][Chaitin, 1974][Kikuchi, 1994] [Kikuchi, Kurahashi and Sakai, 2012] [Kikuchi and Tanaka, 1994][Vopenka, 1966], Grelling-Nelson's Paradox in [Cieśliński, 2002], the Unexpected Examination Paradox in [Fitch, 1964] [Kritchman and Raz, 2010], and Yablo's Paradox in [Cieśliński and Urbaniak, 2013] [Kurahashi, 2014][Kurahashi, 2014][Priest, 1997].

One important consequence of Gödel's incompleteness theorem is Tarski's undefinability theorem of truth as an application of the Diagonalization Lemma. Define **Prov** $= \{\phi \in L(\mathbf{PA}) : \mathbf{PA} \vdash \phi\}$ and **Truth** $= \{\phi \in L(\mathbf{PA}) : \mathfrak{N} \models \phi\}$. From Tarski's theorem, **Truth** (the set of true sentences of arithmetic) is not definable in the standard model of arithmetic; as a corollary, **Truth** is not arithmetic and not representable in **PA**. But **Prov** (the set of sentences provable in **PA**) is definable in the standard model of arithmetic and recursive enumerable, even if it is not recursive (for details of properties of **Truth** and **Prov**, we refer to [Murawski, 1999] [Tarski, Mostowski and Robinson, 1953]). Current research practice reveals the relationship between Gödel's incompleteness theorem and Tarski's undefinability theorem of truth. For example, Visser [2019] gives a self-reference-free proof of Gödel's second incompleteness theorem from Tarski's undefinability theorem of truth.

Provability logic is an important tool for the study of incompleteness and metamathematics of arithmetic. The origins of provability logic (e.g. Henkin's problem, the isolation of derivability conditions, Löb's theorem) are all closely tied to the incompleteness theorem historically. In this sense, we can say that Gödel's incompleteness theorem plays a unifying role between first order arithmetic and modal

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[39]This follows from the following two facts: (1) every consistent recursively axiomatizable complete theory is decidable; (2) every incomplete decidable theory has a consistent, decidable complete extension in the same language (see [Murawski, 1999, p. 214-215]).

logic. The notion of arithmetical interpretation provides us with an important tool to establish the relationship between provability logic and meta-mathematics of arithmetic.[40] Surprisingly, Solovay's Arithmetical Completeness Theorems for **GL** and **GLS**[41] characterize the difference between **Prov** and **Truth** via provability logic.[42]

Provability logic is the logic of properties of provability predicates. Note that the proof of Gödel's incompleteness theorem depends on the property of provability predicates. Provability logic provides us with a new perspective and an important tool to understand incompleteness. Provability logics based on different provability predicates reveal the intensionability of provability predicates which is one source of the intensionability of G2, and provide us with a new route to examine the intensionability of provability predicates. Under different numerations of the base theory, the provability predicate may have different properties, and hence may correspond to different provability logics (i.e. different modal principles under arithmetic interpretations). For more discussions about the relationship between incompleteness and provability predicate, we refer to [Kurahashi, 2019a] [Kurahashi, 2019b].

4. Some explanations

In this section, we give some explanations for our account of the depth of Gödel's incompleteness theorem.

In the literature, some criteria for mathematical depth have been proposed. From [Ernst, Heis, Maddy, McNulty, and Weatherall, 2015b], the following five candidate criteria for mathematical depth have gained the widest support:

(1) ties together apparently disparate fields;
(2) involves impurity (definitions that reach into higher types, proofs that appeal to concepts other than those in the statement proved);
(3) finds order in chaos;
(4) exhibits organizational or explanatory power;
(5) transforms a field or opens a new one.

No examples of mathematical depth that failed on all above five criteria were proposed. From our account of the depth of Gödel's incompleteness theorem, Gödel's incompleteness theorem satisfies all of the above criteria. Compare our three criteria with the above five criteria, our three criteria are more general than the above five criteria. Each one of our three criteria is not a sufficient condition for a theorem to be deep. We can not say that if a theorem is influential (or fruitful, or exhibiting unity), then this theorem is deep. For example, a theorem may be fruitful but not widely considered as deep. It is controversial whether each one of our three criteria is a necessary condition for a theorem to be deep (i.e. is it true that if a theorem is not influential (or not fruitful, or not exhibiting unity), then this theorem is not deep). We do not know whether there might be an example of depth without fruitfulness.

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[40]Let T be a consistent r.e. extension of **Q**. A mapping from the set of all modal propositional variables to the set of $L(T)$-sentences is called an *arithmetical interpretation*. Every arithmetical interpretation f is uniquely extended to the mapping f^* from the set of all modal formulas to the set of $L(T)$-sentences so that f^* satisfies the following conditions: (1) $f^*(p) = f(p)$ for each propositional variable p; (2) f^* commutes with every propositional connective; (3) $f^*(\Box A)$ is $\mathbf{Pr}_T(\ulcorner f^*(A) \urcorner)$ for every modal formula A. We equate an arithmetical interpretation f with its unique extension f^* defined on the set of all modal formulas.

[41]The definition of **GL** and **GLS** is standard, and we refer to [Boolos, 1993].

[42]Solovay's Arithmetical Completeness Theorem for **GL** says that if T is a Σ_1^0-sound r.e. extension of **Q**, then for any modal formula ϕ in $L(\mathbf{GL})$, $\mathbf{GL} \vdash \phi$ iff $T \vdash \phi^f$ for every arithmetic interpretation f. Solovay's Arithmetical Completeness Theorem for **GLS** says that for any modal formula ϕ, $\mathbf{GLS} \vdash \phi$ iff $\mathfrak{N} \models \phi^f$ for every arithmetic interpretation f.

The depth of a theorem is not an essential intrinsic property of the theorem but a property of mathematical practice of this theorem. Depth is historically located or contextual. What people once thought deep could turn out not to be. For example, the theorem on the irrationality of the square root of two was regarded as deep by the ancients, but perhaps today it appears too simple to be deep (see [Ernst, Heis, Maddy, McNulty, and Weatherall, 2015a]). A theorem may not seem deep immediately after its first publication, but may be widely considered as deep during the mathematical practice. Deep theorems are generally the work of several generations of mathematicians. For example, for the influence, fruitfulness and unity of Gödel's incompleteness theorem, Gödel may not even realize that his theorem is so influential on foundations of mathematics, philosophy, mathematics and theoretic computer science; has so many different proofs and generalizations; establishes so many connections among varied fields; and whether G2 holds depends on so many factors. Nowadays, Gödel's incompleteness theorem and its proof are standard materials of logic textbooks for advanced undergraduates. The current research practice on incompleteness (such as concrete incompleteness) is rather complex or even more technical than Gödel's original proof. Thus, if the depth of Gödel's incompleteness theorem is only linked to the original proof by Gödel, and is not related to the research practice of this theorem after Gödel, then we may no longer view Gödel's incompleteness theorem as deep since the later research practice on incompleteness has greatly deepen our understanding of Gödel's incompleteness theorem. Finally, influence, fruitfulness and unity of a theorem all depend on the level of research practice of this theorem. But there is no limit of research practice, and as research practice goes on, people may find more and more evidences of influence, fruitfulness and unity of this theorem.

A natural question is: is depth an objective property (independent of our interests and abilities) or a subjective property of the theorem (something essentially tied to our interests, abilities, and so on)? It is not our goal to decide on the larger question of whether depth is objective and what objectivity would consist in. However, it is an interesting question whether our account of the depth of Gödel's incompleteness theorem is objective (and if so, in what sense), and whether it can deliver a notion of depth that is not essentially dependent on our contingent interests and abilities. For us, this depends on how we view the objectivity of our account of the depth of Gödel's incompleteness theorem. Assuming we view our account as objective if our justifications are based on mathematical evidences of the theorem, then our account of the depth of Gödel's incompleteness theorem is objective since our account is based on mathematical evidences from the research practice of Gödel's theorem, not based on individual preferences, interests and abilities (even if we have limited knowledge about the current research of Gödel's incompleteness theorem and the mathematical evidences we give here are limited). The evaluation of the depth of a mathematical theorem may be person-dependent: for the same theorem, some may think it as interesting and deep, but others may not. For mathematicians interested in foundations of mathematics, they may view Gödel's incompleteness theorem as influential; but for mathematicians without any interest on foundations of mathematics, they may not view Gödel's incompleteness theorem as influential. But we may have an objective account of mathematical depth of a theorem from some academic community. The depth of a mathematical theorem is more than a fact about the theorem but an evaluation of this theorem from the specific academic community. For example, the depth of Gödel's incompleteness theorem should not be judged by the academic community from topology; instead, it should be judged by the academic community from mathematical logic. We can describe a possible practical procedure of judging whether a given mathematical

theorem is deep. Given a mathematical theorem A in some field X, whether theorem A is deep can be judged by a group of academic committees which consist of top scholars around the world in the field X. Following the commonly accepted criteria of mathematical depth by this group (such as our Influence-Fruitfulness-Unity criteria), this group of academic committees can rank the depth of theorem A according to the commonly accepted criteria.

Another natural question is: are there salient differences between the depth of Gödel's incompleteness theorem and the depth of a pure mathematical theorem such as Szemerédi's Theorem? Based on Arana's work in [2015], we can argue that Szemerédi's Theorem also satisfies our Influence-Fruitfulness-Unity criteria.[43] Thus, according to our Influence-Fruitfulness-Unity criteria, both Szemerédi's Theorem and Gödel's incompleteness theorem are deep. However, even if both Szemerédi's Theorem and Gödel's incompleteness theorem satisfy our Influence-Fruitfulness-Unity criteria, but they have different justifications for the three criteria. For example, even if both Szemerédi's Theorem and Gödel's incompleteness theorem are influential, their influence cover different fields. The impact of Szemerédi's Theorem is more on pure mathematics and especially number theory. But the impact of Gödel's incompleteness theorem is more on logic. We do not know any general criterion of mathematical theorems which Szemerédi's Theorem satisfies but Gödel's incompleteness theorem does not satisfy.

In summary, in this paper, we put forward Gödel's incompleteness theorem as a case for studying mathematical depth. We propose three criteria (influence, fruitfulness, and unity) to account for the depth of Gödel's incompleteness theorem based on the current research practice, and justify that Gödel's theorem satisfies our Influence-Fruitfulness-Unity criteria. Many points discussed in this paper are worth further exploration. For example, the uniform criteria of mathematical depth, the objectivity of mathematical depth, the difference between the depth of theorems and the depth of proofs, the method to compare the depth of different mathematical theorems, and the difference between the depth of pure logical theorems and the depth of pure mathematical theorems are all worthy of further study. Deeper research about these topics might bring more insights of mathematical depth to light. In this paper, we focus on the methodological study of what the depth of Gödel's incompleteness theorem consists in? We hope our account of the depth of Gödel's incompleteness theorem sheds a little light on mathematical depth as a notion with many faces.

Reference

Arana, Andrew [2015]: 'On the Depth of Szemerédi's Theorem', *Philosophia Mathematica*, Volume 23, Issue 2, Pages 163-176.

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[43]This is an observation based on Arana's work in [2015]. Due to the space limit, we can not provide the details of the argument for this claim.

Artemov, Sergei [2019]: 'The Provability of Consistency', reprint, see arXiv:1902.07404v5, 2019.

Auerbach, D.David [1985]: 'Intensionality and the Gödel Theorems', *Philosophical Studies: An International Journal for Philosophy in the Analytic Tradition*, Vol. 48, No. 3, pp. 337-351.

Avigad, Jeremy [2005]: 'Incompleteness via the halting problem', 2005.

Baaz, Matthias, Christos H. Papadimitriou, Hilary W. Putnam, Dana S. Scott, and Charles L. Harper Jr (Edited)[2014]: 'Kurt Gödel and the Foundations of Mathematics: Horizons of Truth', Cambridge University Press.

Beklemishev, D. Lev [2003]: 'The Worm principle', Logic Group Preprint Series, 219, Utrecht Univ. March.

Beklemishev, D. Lev [2010]: 'Gödel incompleteness theorems and the limits of their applicability I', *Russian Math Surveys.*

Berline, C., McAloon, K., Ressayre, J.P. (editors) [1981]: *'Model Theory and Arithmetic'*, Lecture Notes in Mathematics, vol. 890, Springer, Berlin.

Boolos, George [1989]: 'A new proof of the Gödel incompleteness theorem', *Notices Amer. Math. Soc.*36, 388-390.

Boolos, George [1993]: *'The Logic of Provability'*, Cambridge University Press.

Bovykin, Andrey [2006]: 'Brief introduction to unprovability', Logic Colloquium, Lecture Notes in Logic 32.

Chaitin, J. Gregory [1974]: 'Information-theoretic limitations of formal systems', *Journal of the Association for Computing Machinery*, 21:403-424.

Chalmers, David J [1995]: 'Minds, machines, and mathematics: A review of Shadows of the mind by Roger Penrose', *Journal Psyche*, 2.

Chao, Conden and Seraji Payam [2018]: 'Gödel's second incompleteness theorem for Σ_n-definable theories', *Logic Journal of the IGPL*, Volume 26, Issue 2, Pages 255-257.

Cheng, Yong [2019a]: *'Incompleteness for Higher-Order Arithmetic: An Example Based on Harrington's Principle'*, Springer series: Springerbrief in Mathematics, Springer.

Cheng, Yong [2019b]: 'Finding the limit of incompleteness I', accepted and to appear in *The Bulletin of Symbolic Logic*, DOI: 10.1017/bsl.2020.09.

Cheng, Yong [2019c]: 'Current research on Gödel's incompleteness theorem', accepted and to appear in The Bulletin of Symbolic Logic, DOI: 10.1017/bsl.2020.44.

Cheng, Yong and Schindler Ralf [2015]: 'Harrington's Principle in higher order arithmetic', *The Journal of Symbolic Logic,* Volume 80,Issue 02, pp 477-489.

Cieśliński, Cezary [2002]: 'Heterologicality and incompleteness', *Mathematical Logic Quarterly*, 48(1), 105-110.

Cieśliński, Cezary and Urbaniak, R [2013]: 'Godelizing the Yablo sequence', *Journal of Philosophical Logic*, 42(5), 679-695.

Clote, P. and McAloon, K [1983]: 'Two further combinatorial theorems equivalent to the 1-consistency of Peano arithmetic', *J. Symb. Log.* vol.48, no.4, pp. 1090-1104.

Dean, Walter [2019]: 'Incompleteness via paradox and completeness', to appear in *Review of symbolic logic.*

Detlefsen, Michael [1979]: 'On Interpreting Gödel's Second Theorem', *Journal of Philosophical Logic* 8, 297-313.

Detlefsen, Michael [1980]: 'On a theorem of Feferman', *Philosophical Studies: An International Journal for Philosophy in the Analytic Tradition*, Vol. 38, No. 2, pp. 129-140.

Enderton B. Herbert [2001]: *'A mathematical introduction to logic'* (2nd ed.), Boston, MA: Academic Press.

Epstein L. Richard [2011] (with contributions by Lesław W.Szczerba): *'Classical mathematical logic: The semantic foundations of logic'*, Princeton University Press, 2011.

Ernst, Michael, Heis Jeremy , Maddy Penelope, McNulty Bennett Michael, Weatherall Owen James [2015a]: 'Foreword to Special Issue on Mathematical Depth', *Philosophia Mathematica*, Volume 23, Issue 2, Pages 155-162.

Ernst, Michael, Heis Jeremy , Maddy Penelope, McNulty Bennett Michael, Weatherall Owen James [2015b]: 'Afterword to Special Issue on Mathematical Depth', *Philosophia Mathematica*, Volume 23, Issue 2, Pages 242-254.

Feferman, Solomon [1960]: 'Arithmetization of mathematics in a general setting', *Fundamenta Mathematicae*, vol. 49, pp. 35-92.

Feferman, Solomon [1988]: 'Hilbert's Program Relativized: Proof-Theoretical and Foundational Reductions', *The Journal of Symbolic Logic*, Vol. 53, No. 2, pp. 364-384.

Feferman, Solomon [2006]: 'The Impact of the Incompleteness Theorems on Mathematics', *Notices of the AMS*, Volume 53, Number 4, p.434-439.

Feferman, Solomon [2009]: 'Gödel, Nagel, minds, and machines', *The Journal of Philosophy*, CVI(4):201-219.

Feferman, Solomon, Dawson, Jr. W. John, Goldfarb Warren, Parsons Charles, and Solovay M. Robert (Edited) [1995a]: 'Kurt Gödel's Collected Works (vol. 1, pp. 145-195)', Oxford University Press, New York and Oxford.

Feferman, Solomon, Dawson, Jr. W. John, Goldfarb Warren, Parsons Charles, and Solovay M. Robert (Edited) [1995b]: 'Collected Works, Volume III: Unpublished Essays and Lectures', Oxford University Press, New York and Oxford.

Fitch, F.B [1964]: 'A goedelized formulation of the prediction paradox', *American Philosophical Quarterly*, 1, 161-164.

Franks, Curtis [2009]: *'The Autonomy of Mathematical Knowledge: Hilbert's Program Revisited'*, Cambridge University Press.

Franzen, Torkel [2005]: *'Gödel's Theorem: an incomplete guide to its use and abuse'*, A.K.Peters.

Friedman, M. Harvey [formthcoming]: *'Boolean Relation Theory and Incompleteness'*, Lecture Notes in Logic, Association for Symbolic Logic.

Gaifman, Haim [2000]: 'What Gödel's incompletness result does and does not show', *The Journal of Philosophy*, XCVII(8):462-470.

Gödel, Kurt [1931]: 'Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I', *Monatsh. Math. Phys.* 38:1, 173-198.

Gödel, Kurt [1995]: 'Some basic theorems on the foundations of mathematics and their implications', in 'Collected Works, Volume III: Unpublished Essays and Lectures', pp. 304-323, Oxford University Press.

Grabmayr, Balthasar [2019]: 'On the Invariance of Gödel's Second Theorem with regard to Numberings', Reprint.

Gray, Jeremy [2015]: 'Depth-A Gaussian Tradition in Mathematics', *Philosophia Mathematica*, Volume 23, Issue 2, Pages 177-195.

Grzegorczyk, Andrzej [2005]: 'Undecidability without arithmetization', *Studia Logica*, 79(2):163-230.

Hájek, Petr and Paris J [1986]: 'Combinatorial principles concerning approximations of functions', *Archive Math. Logic*, vol.26, no.1-2, pp.13-28.

Hájek, Petr and Pudlák Pavel [1993]: *'Metamathematics of First-Order Arithmetic'*, Springer-Verlag, Berlin-Heidelberg-New York.

Halbach, Volker and Visser Albert [2014a]: 'Self-reference in arithmetic I', *Review of Symbolic Logic* 7(4), 671-691.

Halbach, Volker and Visser Albert [2014b]: 'Self-Reference in Arithmetic II', *Review of Symbolic Logic* 7(4), 692-712.

Hamano, M and M Okada [1997]: 'A relationship among Gentzen's proof-reduction, Kirby-Paris' hydra game, and Buchholz's hydra game', *Math. Logic Quart*, 43:1, 103-120.

Horsten, Leon and Philip Welch [2016]: *'Gödel's Disjunction: The scope and limits of mathematical knowledge'*, Oxford University Press.

Isaacson, D. [2011]: 'Necessary and sufficient conditions for undecidability of the Gödel sentence and its truth', In: D. DeVidi, etal. (Eds.), Logic, Mathematics, Philosophy: Vintage Enthusiasms, Springer, ISBN 9789400702134, pp. 135-152.

Kanamori, A and McAloon K.[1987]: 'On Gödel's incompleteness and finite combinatorics', *Ann. Pure Appl. Logic*, 33:1, 23-41.

Kikuchi, Makoto [1994]: 'A note on Boolos' proof of the incompleteness theorem', *Math. Log. Quart.* 40, 528-532.

Kikuchi, Makoto [1997]: 'Kolmogorov complexity and the second incompleteness theorem', *Archive for Mathematical Logic*, 36(6):437-443.

Kikuchi, Makoto and Kurahashi Taishi [2017]: 'Generalizations of Gödel's incompleteness theorems for Σ_n-definable theories of arithmetic', *Rew. Symb. Logic*, 10(4): 603-616.

Kikuchi, Makoto, Kurahashi Taishi and Sakai, H [2012]: 'On proofs of the incompleteness theorems based on Berry's paradox by Vopěnka, Chaitin, and Boolos', *Mathematical Logic Quarterly*, 58(4-5), 307-316.

Kikuchi, Makoto and Tanaka Kazuyuki [1994]: 'On formalization of model-theoretic proofs of Gödel's theorems', *Notre Dame Journal of Formal Logic*, 35(3):403-412.

Kirby, L [1982]: 'Flipping properties in arithmetic', *J. Symb. Log.* vol.47, no.2, pp. 416-422.

Kirby, L. A. S. and Paris J. [1982]: 'Accessible independence results for Peano arithmetic', *Bull. London Math. Soc*, 14:4, 285-293.

Koellner, Peter [2016]: 'Gödel's Disjunction. Charpter in Gödel's Disjunction: The scope and limits of mathematical knowledge', edited by Leon Horsten and Philip Welch, Oxford University Press.

Koellner, Peter [2018]: 'On the Question of Whether the Mind Can Be Mechanized Part I: From Gödel to Penrose', *Journal of Philosophy*, Volume 115, Issue 7, Pages 337-360.

Koellner, Peter [2018]: 'On the Question of Whether the Mind can be Mechanized Part II: Penrose's New Argument', *Journal of Philosophy*, Volume 115, Issue 9, Pages 453-484.

Kotlarski, Henryk [2004]: 'The incompleteness theorems after 70 years', *Annals of Pure and Applied Logic* 126, 125-138, 2004.

Kritchman, S. and Raz, R. [2010]: 'The surprise examination paradox and the second incompleteness theorem', *Notices of the American Mathematical Society*, 57(11), 1454-1458.

Kurahashi, Taishi [2014]: 'Rosser-type undecidable sentences based on Yablo's paradox', *Journal of Philosophical Logic*, 43(5), 999-1017.

Kurahashi, Taishi [2019a]: 'A note on derivability conditions', Reprint.

Kurahashi, Taishi [2019b]: 'Rosser provability and the second incompleteness theorem', Reprint.

Lange, Marc [2015]: 'Depth and Explanation in Mathematics', *Philosophia Mathematica*, Volume 23, Issue 2, Pages 196-214.

Lindström, Per [1997]: *'Aspects of Incompleteness'*, Lecture Notes in Logic v. 10.

Lindström, Per [2006]: 'Remarks on Penrose's new argument', *Journal of Philosophical Logic*, 35:231-237.

Lucas, R. J. [1996]: 'Minds, machines, and Gödel: A retrospect', in 'Machines and thought: The legacy of Alan Turing, Volume 1' (P. J. R. Millican and A. Clark, editors), Oxford University Press, Oxford.

Mills, G. [1980]: 'A tree analysis of unprovable combinatorial statements', *Model Theory of Algebra and Arithmetic*, Lecture Notes in Mathematics, vol.834, Springer, Berlin pp.248-311.

Murawski, Roman [1999]: *'Recursive Functions and Metamathematics: Problems of Completeness and Decidability, Gödel's Theorems'*, Springer Netherlands.

Pacholski, L. and Wierzejewski,J. [1980]: *'Model Theory of Algebra and Arithmetic'*, Lecture Notes in Mathematics, vol.834, Springer, Berlin.

Pakhomov, Fedor [2019]: 'A weak set theory that proves its own consistency', Reprint.

Papadimitriou, H. Christos [2014]: 'Computation and Intractability: Echoes of Kurt Godel', in 'Kurt Gödel and the Foundations of Mathematics: Horizons of Truth'.

Paris, J. and Harrington, L. [1977]: 'A mathematical incompleteness in Peano arithmetic', Handbook of mathematical logic (J. Barwise, ed.), Stud. Logic Found. Math., vol. 90, North-Holland, Amsterdam-New York-Oxford, pp. 1133-1142.

Pudlak, Pavel [1979]: 'Another combinatorial principle independent of Peano's axioms', Unpublished.

Pudlak, Pavel [1999]: 'A note on applicability of the incompleteness theorem to human mind', *Annals of Pure and Applied Logic* 96, 335-342.

Penrose, Roger [1989]: *'The Emperor's New Mind: Concerning Computeres, Minds, and the Laws of Physics'*, Oxford University Press.

Priest, G [1997]: 'Yablo's paradox', *Analysis*, 57(4), 236-242.

Resnik, D. M. [1974]: 'On the Philosophical Significance of Consistency Proofs', *Journal of Philosophical Logic* 3, 133-147.

Rosser, Barkley John [1936]: 'Extensions of some theorems of Gödel and Church', *The Journal of Symbolic Logic*, 1(3):87-91.

Salehi, Saeed and Seraji Payam [2017]: 'Gödel-Rosser's Incompleteness Theorem, generalized and optimized for definable theories', *Journal of Logic and Computation*, Volume 27, Issue 5, Pages 1391-1397.

Salehi, Saeed and Seraji Payam [2018]: 'On constructivity and the Rosser property: a closer look at some Gödelean proofs', *Annals of Pure and Applied Logic* 169 (2018) 971-980.

Shapiro, Stewart [1998]: 'Incompleteness, Mechanism, and Optimism', *The Bulletin of Symbolic Logic*, Vol. 4, No. 3, pp. 273-302.

Shapiro, Stewart [2003]: 'Mechanism, truth, and Penrose's new argument', *Journal of Philosophical Logic*, XXXII(1):19-42.

Solovay, M. R.[1976]: 'Provability interpretations of modal logic', *Israel Journal of Mathematics* 25, pp. 287-304.

Stephen, G.Simpson [1985]: *'Harvey Friedman's Research on the Foundations of Mathematics'*, Studies in Logic and the Foundations of Mathematics, vol. 117, North-Holland Publishing, Amsterdam.

Stephen, G.Simpson (editor) [1987]: 'Logic and Combinatorics', in *Contemporary Mathematics*, vol. 65, AMS, Providence, RI.

Stephen, G.Simpson [1988]: 'Partial Realizations of Hilbert's Program', *The Journal of Symbolic Logic*, Vol. 53, No. 2, pp. 349-363.

Smith, Peter [2007]: *'An Introduction to Gödel's Theorems'*, Cambridge University Press.

Smoryński, C. [1977]: 'The Incompleteness Theorems', in: J. Barwise (Ed.), Handbook of Mathematical Logic, North-Holland, Amsterdam, pp. 821-865.

Stillwell, John [2015]: 'What Does "Depth" Mean in Mathematics'? *Philosophia Mathematica*, Volume 23, Issue 2, Pages 215-232.

Tarski, A., Mostowski, A. and Robinson M. R. [1953]: *'Undecidable Theories'*, North-Holland.

Urquhart, Alasdair [2015]: 'Mathematical Depth', *Philosophia Mathematica*, Volume 23, Issue 2, Pages 233-241.

Vaught L. R. [1962]: 'On a theorem of Cobham concerning undecidable theories', In: Nagel, E., Suppes, P., Tarksi, A. (eds.): Logic, Methodology, and Philosophy of Science, p. 18. Proceedings of the 1960 International Congress. Stanford, CA: Stanford University Press.

Visser, Albert [2011]: 'Can we make the second incompleteness theorem coordinate free'? *Journal of Logic and Computation* 21(4), 543-560.

Visser, Albert [2016]: 'The Second Incompleteness Theorem: Reflections and Ruminations', Chapter in Gödel's Disjunction: The scope and limits of mathematical knowledge, edited by Leon Horsten and Philip Welch, Oxford University Press.

Visser, Albert [2019]: 'From Tarski to Gödel: or, how to derive the second incompleteness theorem from the undefinability of truth without self-reference', Reprint.

Vopěnka, Petr: 'A new proof of Gödel's result on non-provability of consistency', *Bulletin del'Académie Polonaise des Sciences. Série des Sciences Mathématiques. Astronomiques et Physiques*, 14, 111-116.

Willard, E. D.[2001]: 'Self-verifying axiom systems, the incompleteness theorem and related reflection principles', *Journal of Symbolic Logic*, 66(2):536-596.

Willard, E. D.[2006]: 'A generalization of the second incompleteness theorem and some exceptions to it', *Ann. Pure Appl. Logic*, 141(3):472-496.

Zach, Richard [2007]: 'Hilbert's Program Then and Now', Philosophy of Logic, Handbook of the Philosophy of Science, Pages 411-447.

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