

Multi-Source Causal Inference Using Control Variates

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Abstract

While many areas of machine learning have benefited from the increasing availability of large and varied datasets, the benefit to causal inference has been limited given the strong assumptions needed to ensure identifiability of causal effects; these are often not satisfied in real-world datasets. For example, many large observational datasets (e.g., case-control studies in epidemiology, click-through data in recommender systems) suffer from selection bias on the outcome, which makes the average treatment effect (ATE) unidentifiable. We propose a general algorithm to estimate causal effects from *multiple* data sources, where the ATE may be identifiable only in some datasets but not others. The key idea is to construct control variates using the datasets in which the ATE is not identifiable. We show theoretically that this reduces the variance of the ATE estimate. We apply this framework to inference from observational data under an outcome selection bias, assuming access to an auxiliary small dataset from which we can obtain a consistent estimate of the ATE. We construct a control variate by taking the difference of the odds ratio estimates from the two datasets. Across simulations and two case studies with real data, we show that this control variate can significantly reduce the variance of the ATE estimate.

1 Introduction

The ongoing rapid growth in the scale and scope of data sources has challenged many research communities, including optimization, machine learning, and causal inference [Hand, 2007, Agarwal and Duchi, 2012, Bottou et al., 2018, Shiffrin, 2016]. In particular, in causal inference, there has been a surge of interest in developing tools to draw causal conclusions from large-scale observational data [Imbens and Rubin, 2015, Pearl, 2009, Maathuis et al., 2010, Kleinberg, 2013, Maslove and Leisman, 2019, Wachinger et al., 2019]. Compared to randomized trial designs, these observational data sources can often offer longitudinal data, fine-grained measurements, and much larger sample sizes. For example, electronic health data, including electronic health records (EHR) used for clinical care, may contain extensive details that include the timing, intensity, and quality of the interventions received by individuals. Moreover, randomized clinical trials are sometimes not feasible due to logistical, economic or ethical reasons, and

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even when feasible can be seriously limited in terms of sample size [Stuart et al., 2013]. Thus, large scale observational datasets hold open the promise of a much greater impact for causal inference methodology.

However, conceptual problems arise in the observational data setting which can make causal effects unidentifiable or hard to estimate. The problems include unmeasured confounding, noisy measurements, inconsistency, and selection bias [Rosenbaum and Rubin, 1983, Angrist et al., 1996, Nalatore et al., 2007, Hernán et al., 2004]. These problems have generally been studied in the setting of a single observational data source, and in such a setting it is natural to view them through an all-or-nothing lens—either selection bias is present or it is not, either confounding is present or it is not, etc. In such cases causal inference is possible only when the data source satisfies certain delicate assumptions. These assumptions are often invalid; in particular, selection bias is notorious for being difficult to assume away—for example, in case-control datasets in epidemiological studies, cases are much more likely to be reported than non-cases; in observational data in recommender systems, certain items are more likely to receive clicks and ratings [Rothman et al., 2008, Robins et al., 2000, Robins, 2001, Hernán et al., 2004, Wang et al., 2016, Schnabel et al., 2016, Wang et al., 2020]. Under the existence of such selection bias, causal effects are in general unidentifiable.

In this paper, we take a different route and consider estimating causal effects from *multiple* data sources. In particular, we ask: Can we combine large, possibly biased datasets with smaller, unbiased datasets to develop efficient estimators of causal effects? Our work is motivated by the observation that, in practice, we might be able to obtain a small dataset where the causal effects are identifiable; for example, from small-scale randomized trials. Causal inference may not be efficient in such small datasets alone, due to limited sample size. The hope is that large observational datasets, while not permitting causal inference by themselves, may be useful in improving the efficiency of the causal effects estimators from the small, unbiased dataset.

Contributions. We present an affirmative answer to this question in this paper, proposing a general framework to estimate causal effects from multiple data sources. Specifically, we show how to perform inference when the average treatment effect (ATE) may be identifiable only in some datasets but not others. The key is to use certain features that are robust and transportable across datasets to construct meaningful control variates. We show that such control variates allow us to design new ATE estimators which enjoy variance reduction, and we theoretically quantify the reduction in variance. To demonstrate the effectiveness of this general approach, we apply it to case-control studies with outcome selection bias. In particular, we construct control variates by taking the difference of the odds ratio estimates across datasets. We also present experiments, using synthetic data and two real data case studies, to show that this control variate can significantly reduce the variance of ATE estimates using a variety of estimators.

1.1 Related work

The problem of combining multiple datasets to estimate causal effects has attracted much recent research interest given the strong practical incentives, especially combining datasets from observational and experimental sources [Colnet et al., 2020, Rosenfeld et al., 2017, Rosenman et al., 2018, 2020, Kallus et al., 2018], where the observational data may suffer from hidden confounders and certain missing data. Cannings and Fan [2019] propose new estimators by combining datasets with complete cases and further observations with missing values, in order to improve on the performance of the complete-case estimator; Yang and Ding [2020] propose estimators by combining a main dataset with unmeasured

confounders and a smaller validation dataset with supplementary information on these confounders. Their technique is based on constructing multiple error-prone estimators that are transportable across the main and validation datasets. In fact, their “error-prone estimators” are used to design one particular choice of control variates in our setting. However, their estimators rely on the identifiability of the ATE in observational data, and therefore do not handle selection bias.

Selection bias is induced by preferential selection of data points, and it is often governed by unknown factors that can interact with treatments, outcomes and their consequences. Operationally, selection bias cannot be easily eliminated by random sampling. There have been extensive studies on methods that deal with mitigating certain selection biases in observational studies. [Bareinboim and Pearl \[2012\]](#) discuss graphical and algebraic methods, and derive a general condition together with a procedure for recovering the odds ratio under selection bias. They also propose using instrumental variables for the removal of selection bias in the presence of confounding bias. [Zhang \[2008\]](#) studies special cases in which selection bias can be detected even from the observations, as captured by a non-chordal undirected graphical component. [Robins et al. \[2000\]](#) and [Hernán et al. \[2004\]](#) propose epidemiological methods that assume knowledge of the probability of selection given treatment, which can be estimated from data in certain cases.

Control variates technique serves as a natural choice to reduce the variance of the estimate, and there have been applications of it to causal effect estimation [[Tan, 2006](#)]. More generally, the key idea in our approach is to identify an estimand that is transportable between the observational data, i.e. selection biased dataset, to the experimental data. And if that has sufficient correlation with the target estimand one is interested in, it could be potentially used to construct control variates. To this end, our work relates to the literature on transportability in causal inference [[Bareinboim and Pearl, 2014](#), [Lee et al., 2020](#), [Bareinboim and Pearl, 2016](#)]. These works investigate what causal quantities are identifiable, and allow us to construct control variates accordingly.

2 Preliminaries

In this section, we present the basic setup for causal inference with multiple data sources. We also formalize our assumptions on the identification of causal effects.

Potential outcomes and ATE estimation. We use the potential outcomes framework to define causal effects [[Neyman, 1923](#), [Rubin, 1974](#)]. Let Z denote a binary treatment random variable, with 0 and 1 being the labels for control and active treatments, respectively. For each realization of the level of treatment $z \in \{0, 1\}$, we assume that there exists a potential outcome $Y(z)$ representing the outcome had the subject been given treatment z (possibly contrary to fact). Then, the observed outcome is $Y = Y(Z) = ZY(1) + (1 - Z)Y(0)$. Further, we denote a vector of observed pretreatment covariates as $X \in \mathbb{R}^d$. We focus on estimating the ATE:

$$\tau = E[Y(1) - Y(0)]. \tag{1}$$

Data sources. We consider a main data source that consists of observations $\mathcal{O}_1 = \{(Z_i, X_i, Y_i) : i \in \mathcal{S}_1\}$, with sample size $n_1 = |\mathcal{S}_1|$, and a validation data source with observations $\mathcal{O}_2 = \{(Z_j, X_j, Y_j) : j \in \mathcal{S}_2\}$ and sample size $n_2 = |\mathcal{S}_2|$. We assume that the ATE is identifiable from the validation data source but not necessarily the main data source. For simplicity, we consider two data sources here. It is straightforward to generalize the same framework to multiple data sources.

A fundamental problem in causal inference is that the counterfactuals are not observable. Therefore, to allow for the identification of ATE, we make the following ignorability assumption [Rosenbaum and Rubin, 1983] with respect to the validation data \mathcal{O}_2 .

Assumption 2.1 (Ignorability). $Y(z) \perp\!\!\!\perp Z \mid X$ for $z = 0$ and $z = 1$.

Under Assumption 2.1, many methods for estimating the ATE from a single observational dataset exist in the causal inference literature [see, e.g., Rosenbaum, 2002, Imbens, 2004, Rubin, 2006].

3 Methodology

In this section, we describe our proposed methodology for utilizing a main dataset \mathcal{O}_1 to improve the efficiency of the ATE estimates from the validation dataset \mathcal{O}_2 . Such a strategy allows us to design efficient ATE estimators using all the data, without requiring the ATE to be identifiable in all individual data sources.

3.1 A general strategy with control variates

We present a general strategy for efficient estimation of the ATE by utilizing both the main and validation data. Informally, we want to identify and use certain features that are robust and transportable across datasets to construct a consistent estimator. With such features, we exploit information across the datasets and improve the efficiency of the estimator using all datasets. This strategy is reminiscent of a control-variate methodology for general variance reduction in Monte Carlo simulations Owen [2013].

Let ψ be an estimand for which there exist consistent estimators obtainable from datasets \mathcal{O}_1 and \mathcal{O}_2 (with consistent estimators denoted by $\hat{\psi}_1$ and $\hat{\psi}_2$, respectively). Let $\hat{\tau}_2$ denote a consistent estimator of the true ATE τ (see Definition in Eq (1)) that we obtain using dataset \mathcal{O}_2 . In particular, we consider a class of estimators satisfying

$$n_2^{1/2} \begin{pmatrix} \hat{\tau}_2 - \tau \\ \hat{\psi}_2 - \hat{\psi}_1 \end{pmatrix} \rightarrow \mathcal{N} \left\{ 0, \begin{pmatrix} v_2 & \Gamma^\top \\ \Gamma & V \end{pmatrix} \right\}, \quad (2)$$

for some V and Γ . If Eq. (2) holds exactly rather than asymptotically, by multivariate normal theory, we have the following the conditional distribution:

$$\begin{aligned} & n_2^{1/2}(\hat{\tau}_2 - \tau) \mid n_2^{1/2}(\hat{\psi}_2 - \hat{\psi}_1) \\ & \sim \mathcal{N} \left\{ n_2^{1/2} \Gamma^\top V^{-1} (\hat{\psi}_2 - \hat{\psi}_1), v_2 - \Gamma^\top V^{-1} \Gamma \right\}. \end{aligned}$$

Then, we apply the control variates method and use the estimators for τ and ψ jointly [Owen, 2013] to build a new estimator of τ which has a lower variance than $\hat{\tau}_2$. Specifically, we construct a new estimator for ATE using control variates as follows: $\hat{\tau}_{CV}(\beta) = \hat{\tau}_2 - \beta^\top (\hat{\psi}_2 - \hat{\psi}_1)$.

Solving for the optimal β , we obtain the new estimator

$$\hat{\tau}_{CV} = \hat{\tau}_2 - \Gamma^\top V^{-1} (\hat{\psi}_2 - \hat{\psi}_1), \quad (3)$$

where $V = \text{Var}(\hat{\psi}_2 - \hat{\psi}_1)^{-1}$ and $\Gamma = \text{Cov}(\hat{\psi}_2 - \hat{\psi}_1, \hat{\tau}_2)$.

Theorem 3.1. (Proof in Appendix C) Denote the asymptotic variance of $\widehat{\tau}_2$ as v_2 . Under Assumption 2.1, if Equation (2) holds, then $\widehat{\tau}_{CV}$ is consistent for τ , and we have:

$$n_2^{1/2}(\widehat{\tau}_{CV} - \tau) \rightarrow \mathcal{N}(0, v_2 - \Gamma^\top V^{-1} \Gamma),$$

in distribution as $n_2 \rightarrow \infty$. Given a nonzero Γ , the asymptotic variance, $v_2 - \Gamma^\top V^{-1} \Gamma$, is smaller than v_2 .

Empirically, to estimate this optimal value of β , we can use estimators \widehat{V} and $\widehat{\Gamma}$ for the variance and covariance in Eq. (3). These estimators \widehat{V} and $\widehat{\Gamma}$ can be obtained by bootstrap sampling from the given data with B bootstrap replicates. We present the details of the bootstrap sampling procedure in Appendix B.

4 Control Variates for Selection Bias

We apply this general strategy to datasets with selection bias, which are common in observational studies. Specifically, we instantiate the main data \mathcal{O}_1 as observational data that suffers from selection bias on the outcome, from which the ATE is unidentifiable. Such selection bias on the outcome occurs in practice in case control studies in epidemiology Rothman et al. [2008], Robins [2001], Robins et al. [2000], and in recommender systems as a problem with *implicit* feedback Wang et al. [2016], Schnabel et al. [2016], Wang et al. [2020]. Under selection bias on the outcome, we show that we are able to obtain consistent odds ratio estimates in various models, which can then be used to construct control variates across all datasets.

4.1 Outcome selection bias

Outcome selection bias is induced by preferential selection of units based on the outcome. To illuminate the nature of this bias, consider the model of Figure 1, in which $S \in \{0, 1\}$ represents the selection mechanism: $S = 1$ means presence in the sample, and $S = 0$ means absence. Recall that X represents the pre-treatment covariates, Z represents a binary treatment, and Y represents a binary outcome.

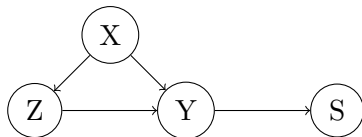


Figure 1: Causal graph for dataset \mathcal{O}_1 where there is a selection bias that depends on the outcome.

Under the existence of such bias on Y , the ATE is in general unidentifiable, even if we assume that there are no unobserved confounders linking Z and Y . Thus, we consider the main data \mathcal{O}_1 to be a dataset suffering from outcome selection bias, and the validation data \mathcal{O}_2 to be unbiased (e.g., obtained from small-scale randomized trials).

4.2 Variance reduction with the odds ratios

Eq. (3) applies to any estimand ψ , as long as it is correlated with ATE and we are able to obtain consistent estimators of it from both datasets. We show that when \mathcal{O}_1 suffers from outcome selection

bias, we can use the odds ratios as the choice of ψ . This is based on two observations: First, the conditional odds ratios and the ATE are correlated; second, the conditional odds ratios are robust to outcome selection bias under a variety of data generation models.

Definition 4.1. *The conditional odds ratio between a binary treatment Z and a binary outcome Y conditioned on covariates X is (for brevity, x denotes $X = x$):*

$$\text{OR}(x) = \frac{P(Y(1) = 1|x)P(Y(0) = 0|x)}{P(Y(1) = 0|x)P(Y(0) = 1|x)}.$$

Under Assumption 2.1, we have $P(Y(1) = 1|x) = P(Y(1) = 1|Z = 1, x) = P(Y = 1|Z = 1, x)$. Therefore, we can rewrite $\text{OR}(x)$ in Definition (4.1) as:

$$\text{OR}(x) = \frac{P(Y = 1|Z = 1, x)P(Y = 0|Z = 0, x)}{P(Y = 0|Z = 1, x)P(Y = 1|Z = 0, x)}$$

Lemma 4.1 further allows us to directly estimate the conditional odds ratios empirically with finite samples, even under selection bias.

Lemma 4.1. *(Proof in Appendix C) If the selection S depends solely on Y (as in Figure 1), then the conditional odds ratio is transportable and given by:*

$$\begin{aligned} \text{OR}(x) &= \frac{P(Y = 1|S = 1, Z = 1, x)P(Y = 0|S = 1, Z = 0, x)}{P(Y = 0|S = 1, Z = 1, x)P(Y = 1|S = 1, Z = 0, x)}. \end{aligned}$$

Lemma 4.1 guarantees that the identifiability does not rely on any modeling assumption; see also Didelez et al. [2010] and Jiang and Ding [2017]. Therefore, to construct control variates, it is sufficient to derive consistent estimators of the odds ratio from the datasets \mathcal{O}_1 and \mathcal{O}_2 . Let $\widehat{\text{OR}}_1(x)$ and $\widehat{\text{OR}}_2(x)$ denote consistent estimators for $\text{OR}(x)$ obtained from the datasets \mathcal{O}_1 and \mathcal{O}_2 , respectively. For a set of covariate values $\{x_1, \dots, x_k\}$, one possible control variate construction is to take $\psi = (\text{OR}(x_1), \dots, \text{OR}(x_k))^\top$. Then $\widehat{\psi}_1 = (\widehat{\text{OR}}_1(x_1), \dots, \widehat{\text{OR}}_1(x_k))^\top$, $\widehat{\psi}_2 = (\widehat{\text{OR}}_2(x_1), \dots, \widehat{\text{OR}}_2(x_k))^\top$. Substituting these into Eq. (3) gives the new ATE estimator with control variates.

When X is continuous or there are too many discrete values of X , we can reduce the large number of conditional odds ratios $\text{OR}(x)$ down to a manageable control variate by integrating $\text{OR}(x)$ over a common distribution $F(x)$: let $\psi = \int \text{OR}(x)F(dx)$. Then $\widehat{\psi}_1 = \int \widehat{\text{OR}}_1(x)F(dx)$, and $\widehat{\psi}_2 = \int \widehat{\text{OR}}_2(x)F(dx)$.

4.3 Estimating the odds ratios with selection bias

The main difficulty in applying Eq. (3) is to find consistent estimators for constructing the control variates. We demonstrate that this can be achieved by estimating the conditional odds ratios parametrically using a logistic model with varying coefficients, or non-parametrically using kernel smoothing. For the consistency analysis of estimators for ATE and the odds ratios, we assume that the data points in \mathcal{O}_1 and \mathcal{O}_2 before selection bias are IID samples from a same underlying population.

4.3.1 Logistic outcome model

One approach to estimating the odds ratio $\text{OR}(x)$ uses a logistic model with varying coefficients Cleveland [1991] to parameterize the outcome distribution:

$$P(Y = 1|Z = z, x) = \frac{e^{\beta_0^x + \beta_1^x z}}{1 + e^{\beta_0^x + \beta_1^x z}}, \quad (4)$$

Here, β_0^x, β_1^x are coefficients that depend on the covariates x . If X is discrete and finite, then there would be a discrete and finite number of parameters β_0^x, β_1^x . Otherwise, β_0^x and β_1^x can be viewed as functions of x .

If the data is truly generated by the outcome model defined in Eq. (4), then Theorem 4.2 below shows that selection bias on the outcome will not change the coefficient β_1^x across \mathcal{O}_1 and \mathcal{O}_2 . Furthermore, β_1^x is the only parameter needed to compute the conditional odds ratio $\text{OR}(x)$. Thus, any consistent estimates of β_1^x for both \mathcal{O}_1 and \mathcal{O}_2 would provide consistent estimates of the conditional odds ratio that are robust to selection bias.

Theorem 4.2. (Proof in Appendix C) *If the selection S depends solely on Y (as in Figure 1) and $P(Y = 1|Z = z, X = x)$ follows the logistic model in (4), then $P(Y = 1|Z = z, X = x, S = 1)$ also follows a logistic model, with the same coefficient β_1^x on Z as the logistic model for $P(Y = 1|Z = z, X = x)$ for each covariate value x . Furthermore, the conditional odds ratio $\text{OR}(x) = e^{\beta_1^x}$.*

Theorem 4.2 extends Prentice and Pyke [1979]; see also Agresti [2015]. By Theorem 4.2, we need only compute consistent estimators $\hat{\beta}_{1, \mathcal{O}_1}^x$ and $\hat{\beta}_{1, \mathcal{O}_2}^x$ of β_1^x from \mathcal{O}_1 and \mathcal{O}_2 to produce consistent estimators of the true underlying conditional odds ratio $\text{OR}(x)$, $\widehat{\text{OR}}_1(x) = e^{\hat{\beta}_{1, \mathcal{O}_1}^x}$, $\widehat{\text{OR}}_2(x) = e^{\hat{\beta}_{1, \mathcal{O}_2}^x}$.

When X is discrete, we can obtain such consistent estimators $\hat{\beta}_{1, \mathcal{O}_1}^x$ and $\hat{\beta}_{1, \mathcal{O}_2}^x$ by stratifying the data on X and performing logistic regression within each stratum. Let $\hat{\beta}_{1, \mathcal{O}_2}^x$ be the maximum likelihood estimator for $\beta_{1, \mathcal{O}_2}^x$ in the stratum (or subset of data) with $X = x$ from \mathcal{O}_2 (and $\hat{\beta}_{1, \mathcal{O}_1}^x$ be the same for \mathcal{O}_1). Since the maximum likelihood estimator is consistent, these are consistent estimators for the true β_1^x .

For continuous X , producing a theoretically consistent estimator for β_0^x, β_1^x is more challenging. One technique is to assume parametric models for the functions $\beta_0^x = f_0(x, \theta_0)$, $\beta_1^x = f_1(x, \theta_1)$. For example, if these functions are linear (i.e., $f_0(x, \theta_0) = \theta_0^\top x$, $f_1(x, \theta_1) = \theta_1^\top x$), then the problem of estimating β_1^x reduces to maximum likelihood estimation of θ_1 over a logistic model: $P(Y = 1|Z = z, x) = \frac{e^{\theta_0^\top x + \theta_1^\top x z}}{1 + e^{\theta_0^\top x + \theta_1^\top x z}}$. We may also allow $f_0(x, \theta_0), f_1(x, \theta_1)$ to take more general functional forms, such as neural networks. Depending on the complexity of the functions, it becomes more challenging to theoretically guarantee asymptotic consistency and a rate of convergence to the true β_1^x . However, such methods may still work well in practice, and we explore this in experiments on real data in Section 6.

4.3.2 Kernel smoothing

When X is continuous, we can also estimate the odds ratio using kernel smoothing without making any parametric assumptions on the exact outcome model or functional form of β_1^x . First, notice that $\text{OR}(x) = \frac{\mathbb{E}[YZ|x] \cdot \mathbb{E}[(1-Y)(1-Z)|x]}{\mathbb{E}[Y(1-Z)|x] \cdot \mathbb{E}[(1-Y)Z|x]}$.

Further, by Lemma 4.1 this is similar under selection bias:

$$\text{OR}(x) = \frac{\mathbb{E}[YZ|S = 1, x] \cdot \mathbb{E}[(1-Y)(1-Z)|S = 1, x]}{\mathbb{E}[Y(1-Z)|S = 1, x] \cdot \mathbb{E}[(1-Y)Z|S = 1, x]}.$$

Therefore, estimating $\text{OR}(x)$ is equivalent to estimating $\mathbb{E}[W|x]$ and $\mathbb{E}[W|S = 1, x]$ from \mathcal{O}_1 and \mathcal{O}_2 , respectively, where $W \in \{YZ, (1-Y)(1-Z), Y(1-Z), Z(1-Y)\}$. Choose a kernel function $K(\cdot)$ and the bandwidth λ . Given a dataset with n data points $(X_i, Y_i, Z_i)_{i=1}^n$, for a random variable $W \in \{YZ, (1-Y)(1-Z), Y(1-Z), Z(1-Y)\}$, $\hat{\mathbb{E}}[W|x] = \frac{\sum_{i=1}^N K(\frac{x-X_i}{\lambda})W_i}{\sum_{i=1}^N K(\frac{x-X_i}{\lambda})}$. Therefore, the kernel estimator is:

$$\begin{aligned} \widehat{\text{OR}}(x) &= \frac{\sum_{i=1}^N K(\frac{x-X_i}{\lambda})Y_iZ_i \sum_{i=1}^N K(\frac{x-X_i}{\lambda})(1-Y_i)(1-Z_i)}{\sum_{i=1}^N K(\frac{x-X_i}{\lambda})Y_i(1-Z_i)K(\frac{x-X_i}{\lambda})(1-Y_i)Z_i}. \end{aligned}$$

This estimator is consistent under selection bias on the outcome as shown by which Lemma 4.1. However, unlike the parametric estimators using the MLE, we note that the asymptotic convergence of the kernel estimator depends on the bandwidth λ and the dimensionality d . We provide further analysis of this convergence for the odds ratio in Appendix A.

5 Simulation Experiments

We first demonstrate the finite-sample performance of estimators with and without the proposed control variates in a simulation study. We simulate an observational dataset with confounding from X using a logistic model adapted from Zhang [2009].

5.1 Data generation and estimators

We generate the dataset \mathcal{O}_2 by sampling n_2 samples using the following data-generating process. Let $X \in \mathbb{R}^2$ have two components X_1, X_2 , which are i.i.d. Bernoulli($p = 0.5$). Given X , the treatment assignment Z is distributed as $P(Z = 1|X = x) = \frac{e^{a_0 + a_1^T x}}{1 + e^{a_0 + a_1^T x}}$.

As specific parameters, we set $a_1 = [-1, 1]$, and $a_0 = -E[a_1^T X]$, which implies that $P(Z = 1) = 0.5$. Setting $a_1 = [0, 0]$ would correspond to a randomized study, whereas we set $a_1 = [-1, 1]$ to simulate an observational study with confounding. The potential outcomes are distributed as

$$\begin{aligned} P(Y(0) = 1|x) &= \frac{e^{b_{0,0} + b_{0,1}^T x}}{1 + e^{b_{0,0} + b_{0,1}^T x}}, \\ P(Y(1) = 1|x) &= \frac{e^{b_{1,0} + b_{1,1}^T x}}{1 + e^{b_{1,0} + b_{1,1}^T x}}. \end{aligned} \tag{5}$$

Eq. (5) builds in ignorability in Assumption 2.1, which is also equivalent to generating the outcome Y from

$$P(Y = 1|Z = z, x) = \frac{e^{\beta_0 + \beta_1 z + \beta_2^T x + \beta_3^T xz}}{1 + e^{\beta_0 + \beta_1 z + \beta_2^T x + \beta_3^T xz}},$$

where $\beta_0 = b_{0,0}$, $\beta_1 = b_{1,0} - b_{0,1}$, $\beta_2 = b_{0,1}$, and $\beta_3 = b_{1,1} - b_{0,1}$. When $\beta_3 = 0$ and $b_{1,1} = b_{0,1}$, then there is no interaction term between X and Z and the conditional odds ratio is simply e^{β_1} . We set $b_{0,1} = [-1, 1]$ and $b_{1,1} = [1, -1]$ ($\beta_3 \neq 0$) so that the conditional odds ratio varies as a function of x . As done by Zhang [2009], the intercept terms are determined by $b_{0,0} = -0.5 - E[b_{0,1}^T X]$ and $b_{1,0} = 0.5 - E[b_{1,1}^T X]$.

To generate \mathcal{O}_1 , we add an additional selection bias component. After drawing $(Z_i, X_i, Y_i)_{i=1}^N$ samples from the same data-generating process as \mathcal{O}_2 , we include each sample (Z_i, X_i, Y_i) in \mathcal{O}_1 with probability $P(S_i = 1|Y_i = 1) = 0.9$ and probability $P(S_i = 1|Y_i = 0) = 0.1$. This simulates selection bias in favor of positive outcomes as happens in case-control studies in practice.

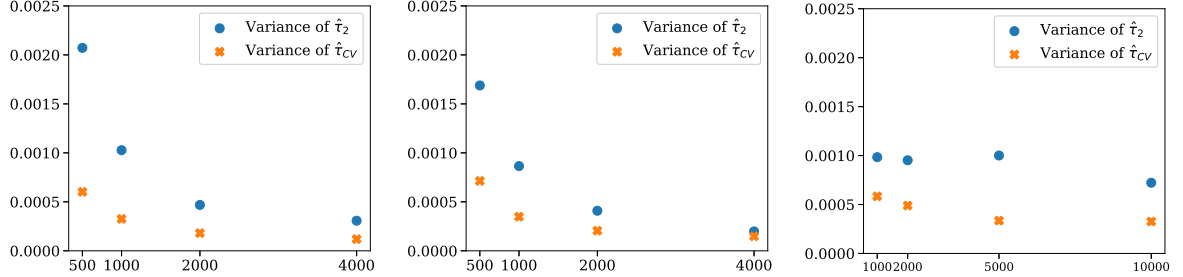


Figure 2: **Scenario 1 (left):** Comparisons of variance as the size of the observational dataset n_2 increases under the general logistic model. The ratio n_2/n_1 is kept constant at 1/10. **Scenario 2 (middle):** Comparisons of variance as the size of the observational dataset n_2 increases under the general logistic model. The size of the selection bias dataset is fixed at $n_1 = 10000$. **Scenario 3 (right):** Comparisons of variance as the size of the selection bias dataset n_1 increases under the general logistic model. The size of the observational dataset is fixed at $n_2 = 1000$. *Lower is better.*

5.1.1 Estimating the ATE

To obtain an estimate $\hat{\tau}_2$ of the ATE from \mathcal{O}_2 , we use a parametric imputation estimator as follows. Denote the coefficient and intercept resulting from logistic regression of Y on Z for stratum $X = x$ as $\hat{\beta}_1^x$ and $\hat{\beta}_0^x$. The regression imputation estimator of the ATE is given by:

$$\hat{\tau}_2 = n_2^{-1} \sum_{i=1}^{n_2} \left\{ \frac{e^{\hat{\beta}_0^{X_i} + \hat{\beta}_1^{X_i}}}{1 + e^{\hat{\beta}_0^{X_i} + \hat{\beta}_1^{X_i}}} - \frac{e^{\hat{\beta}_0^{X_i}}}{1 + e^{\hat{\beta}_0^{X_i}}} \right\}. \quad (6)$$

Since this reflects the true data-generating model, this estimator is not subject to model misspecification.

5.1.2 Estimating the control variate

To estimate the conditional odds ratio, we perform logistic regression of Y on Z on each stratum with $X = x$ to obtain estimates of β_1^x from both \mathcal{O}_1 and \mathcal{O}_2 , which produces estimates $\widehat{OR}_1(x), \widehat{OR}_2(x)$ as described in Section 4.3.1. To compute the proposed control variates estimator $\hat{\tau}_{CV}$ (Eq. (3)), we ran $B = 100$ bootstrap replicates to estimate the covariances Γ and V , which we use estimate the optimal coefficient $\hat{\Gamma}^T \hat{V}^{-1}$ for the control variate.

5.2 Results

We consider three scenarios to analyze the finite-sample performance of the proposed estimators. For all three scenarios, we compare the variance for the ATE estimator $\hat{\tau}_2$ with the variance of the ATE estimator with control variates $\hat{\tau}_{CV}$. To compare these, we ran $B = 100$ bootstrap replicates to measure the variance of the ATE estimator with and without the odds ratio control variates. (Details on the bootstrap algorithm can be found in Appendix B).

Scenario 1: We vary the size of the observational dataset, n_2 , while keeping a constant ratio for the size of the observational dataset relative to the size of the selection biased dataset: $n_2/n_1 = 1/10$. This illustrates the simple asymptotic performance of the estimators as the sample sizes increase without changing the proportional sizes of the two datasets relative to each other.

Scenario 2: We vary the size of the observational dataset, n_2 , while keeping the size of the selection biased dataset constant and relatively large: $n_1 = 10000$. This illustrates the scenario when a practitioner has access to a large fixed amount of case-control data with selection bias (\mathcal{O}_1), and must decide how much observational or experimental data to collect with identifiable ATE (\mathcal{O}_2).

Scenario 3: We vary the size of the selection bias dataset, n_1 , while keeping the size of the observational dataset constant and relatively small: $n_2 = 1000$. This illustrates the relative utility of including more selection biased samples to estimate the control variate.

Throughout all three scenarios, the estimator with control variates $\hat{\tau}_{CV}$ had significantly reduced variance compared to $\hat{\tau}_2$ alone. However, the impact of increasing n_1 and n_2 varies, with n_2 mattering much more for improving the variance. Figure 2 shows that the variance of $\hat{\tau}_2$ and $\hat{\tau}_{CV}$ both decrease significantly as n_2 increases, even if the ratio n_2/n_1 is not necessarily fixed. Consistent with these findings, in Figure 2, we also observe that when there is a limited fixed amount of observational data n_2 , increasing the amount of selection biased data does not seem to significantly improve the variance of the estimator with control variates, $\hat{\tau}_{CV}$.

We further report the bias of each estimator over the bootstrap replicates in Appendix D (Figures 7, 8, and 9). In general, the bias decreases as n_2 increases, and is not significantly different with or without control variates.

6 Real Data Case Studies

In addition to the full simulation, we evaluated the performance of the proposed control variates on two case studies with real public datasets. All estimators and data generation code will be made publicly available.

Case study 1: flu shot encouragement with selection bias from case-control studies:

Our first case study uses a flu shot encouragement experiment dataset that has been repeatedly studied in the causal inference literature [McDonald et al., 1992, Hirano et al., 2000, Ding and Lu, 2017]. As done in prior work, we use the data from 1980 which contains data for 2,861 patients collected from an encouragement experiment in which participating physicians were assigned treatments Z uniformly at random, where $Z = 1$ indicates that the patient’s physician was sent a letter encouraging them to vaccinate their patients, and $Z = 0$ indicates that the patient’s physician was not sent a letter. The outcome of interest Y is the binary outcome of whether the patient was hospitalized for flu-related reasons the following winter. The dataset contains eight additional covariates X , one of which is a continuous *age* variable, and seven of which are binary indicators of prior medical conditions of the patient (e.g., history of heart disease). There is an additional intermediate variable indicating whether or not the patient received a flu shot, but we do not consider this variable in our study.

We set \mathcal{O}_2 to be this original dataset. For the second dataset \mathcal{O}_1 , we consider a realistic scenario where an additional observational dataset exists consisting mostly of patients who have already been hospitalized for flu-related reasons. Observational studies consisting mostly of positive outcomes are common in epidemiology as case-control studies. Such a dataset may be easier to collect than the original encouragement experiment, since such data may already be available from hospitals without setting up an explicit controlled experiment. However, the ATE is not identifiable from such a dataset,

since the data is subject to selection bias on the outcome of hospitalization. We simulate \mathcal{O}_1 by training the following logistic model on the original dataset \mathcal{O}_2 ,

$$P(Y = 1|Z = z, x) = \frac{e^{\beta_0 + \beta_1 z + \beta_2 x + \beta_3 xz}}{1 + e^{\beta_0 + \beta_1 z + \beta_2 x + \beta_3 xz}} \quad (7)$$

and generating samples according the fitted distribution (details in Appendix E). We assume that the “true” ATE is given by this model.

Case study 2: spam email detection with selection bias from implicit feedback: For a second case study, we use a dataset constructed for the Atlantic Causal Inference Conference (ACIC) 2019 Data Challenge based on the Spambase dataset for spam email detection from UCI [Gruber et al., 2019, Dua and Graff, 2017]. The dataset consists of emails with outcome of interest Y being whether or a user marked the email as spam. The treatment Z is whether or not the email contains more than a given threshold of capital letters (the threshold is computed by a mean over the full dataset). There are 22 continuous covariates X which are word frequencies given as percentages.

The ACIC competition does not use the original data from UCI directly, but instead generates modified versions using pre-specified data generating processes with known true ATE. We specifically use their “modification 1” out of four different modifications of the spam email dataset. We detail the specific data generating process in Appendix E, for which code is also available on the ACIC 2019 website. We generate \mathcal{O}_2 directly using ACIC’s data generating process.

For \mathcal{O}_1 , we generate data from the same data generating process and apply selection bias $P(S = 1|Y = 1) = 0.9$ and $P(S = 1|Y = 0) = 0.1$ to produce $n_1 = 30,000$ examples. This simulates a practical scenario where a user marking an email as spam constitutes explicit feedback, but a user *not* taking action to mark an email as spam constitutes *implicit* feedback. This implicit feedback is unreliable, since if a user does not mark an email as spam, there is no guarantee that the user actually even read the email in full. This implicit feedback problem and resulting selection bias has been repeatedly identified as a fundamental challenge in learning-to-rank systems and recommender systems [Wang et al., 2016, Schnabel et al., 2016, Wang et al., 2020]. Disregarding the unreliable implicit feedback, the resulting dataset containing only explicit feedback is subject to selection bias on the outcome $Y = 1$ of being marked as spam, rendering the ATE non-identifiable. In our experiment, \mathcal{O}_2 is assumed to be a small curated dataset without the implicit feedback problem, which may be constructed by, e.g., asking users to explicitly mark emails as “not spam.” We ran two experiments to illustrate the effects of the size of this curated dataset, one with $n_2 = 3,000$ and one with a larger $n_2 = 10,000$.

6.1 Estimators and implementation

To handle the continuous X values, we estimate the conditional odds ratio for a finite set of values \mathcal{X} taken from \mathcal{O}_2 , and set the control variate ψ to be average over all of these conditional odds ratios. Furthermore, for these datasets with continuous covariates X , we found that using the log conditional odds ratio was more effective as a control variate as it had lower variance for extreme values of X . We report results with the control variate estimand $\psi = |\mathcal{X}|^{-1} \sum_{x \in \mathcal{X}} \log \text{OR}(x)$. To estimate the ATE $\hat{\tau}_2$ and the odds ratios for the control variates, we apply three different methods described below.

Logistic model with interaction: We first apply a logistic model with an interaction term between X and Z (Eq. (7)) to estimate both the ATE and conditional odds ratios (details in Appendix E). Since this is the same model used to generate the flu shot encouragement dataset \mathcal{O}_1 , there is no model misspecification when using this estimator for case study 1. We set \mathcal{X} to be the set of all X_i in \mathcal{O}_2 .

Neural network: To allow for more flexibility, we use a neural network to estimate a logistic outcome model with varying coefficients (Eq.(4)), where $\beta_0^x = f_0(x; \theta)$, $\beta_1^x = f_1(x; \theta)$ are outputs of the neural network with parameters θ . The optimization objective is the logistic loss on the final outcome prediction, and we choose the neural network architecture using five-fold cross validation (more details in Appendix E). We estimate both the ATE and the odds ratio using this neural network varying coefficient model. We only report results with the neural network on the spam email dataset since the flu dataset contains a small number of mostly binary covariates, and the additional flexibility from the neural network does not provide much additional benefit. We set \mathcal{X} to be the set of all X_i in \mathcal{O}_2 .

Kernel smoothing: As a third technique, we estimate the ATE using the logistic model in Eq. (7), but apply kernel smoothing to estimate the odds ratios for the control variate (as in Section 4.3.2). This non-parametric estimate gets around any problems of model misspecification when estimating the odds ratio. We set \mathcal{X} to be a random sample of 50 values of X_i from \mathcal{O}_2 , due to computational constraints.

As done in the simulation study, we ran $B = 300$ bootstrap replicates to estimate the variance of the ATE estimate and the covariance between the ATE estimate and the odds ratio control variates, which we use to compute the optimal coefficient $\hat{\Gamma}^\top \hat{V}^{-1}$ for the control variate.

6.2 Results

Tables 1 and 2 report the variances of the ATE estimators with and without the proposed control variates, where the variance is computed over $B = 300$ bootstrap replicates. For both case studies with all estimator combinations, the variance is reduced with the control variates. However, the amount that the variance is reduced varies significantly. For the flu dataset, the variance reduction is significantly higher than for the spam email dataset. This is likely because on the flu dataset, there is no model misspecification, as we generate \mathcal{O}_1 using the same logistic regression model as we use to estimate the ATE and odds ratios.

On both case studies, kernel smoothing performs better than the logistic and neural network estimators. The kernel smoothing estimator performs significantly better than the explicit modeled estimators on the spam email dataset, which we hypothesize is due to model misspecification for the logistic regression model, and overfitting for the neural network model. Interestingly, the variance reduction of the neural network is better than the simpler logistic model when there are more samples n_2 , which suggests possible overfitting for the neural network when n_2 is too small.

Table 1: **Case study 1:** Variances on flu shot encouragement data with $n_1 = 10,000$.

METRIC	LOGISTIC	KERNEL
VAR (% DIFF)	71.160%	77.599%
VAR $\hat{\tau}_2$	1.023e-4	1.072e-4
VAR $\hat{\tau}_{CV}$	2.952e-5	2.400e-5

Variance reduction sometimes came at a cost of higher bias, which was more pronounced on the spam email dataset (Table 4 in the Appendix). This speaks to the care that is needed in managing the bias/variance tradeoff via control variates when there is possible model misspecification.

Table 2: **Case study 2:** Variances on spam email detection data with $n_1 = 30,000$.

METRIC	LOGISTIC	NN	KERNEL
$n_2 = 3,000$			
VAR (% DIFF)	3.522%	0.638%	21.231%
VAR $\hat{\tau}_2$	4.576e-4	1.074e-3	4.309e-4
VAR $\hat{\tau}_{CV}$	4.415e-4	1.067e-3	3.394e-4
$n_2 = 10,000$			
VAR (% DIFF)	0.874%	5.940%	23.134%
VAR $\hat{\tau}_2$	1.443e-4	4.762e-4	1.351e-4
VAR $\hat{\tau}_{CV}$	1.430e-4	4.479e-4	1.038e-4

7 Conclusion

Causal inference can be highly sensitive to challenges in data collection. As a common practical example, selection bias on the outcome of interest renders the ATE unidentifiable. This situation arises in practice in case-control studies and recommender systems, where it can be significantly more costly to obtain data without such selection bias. Our proposed method allows such selection biased data to be used to reduce the variance of ATE estimates when combined with unbiased datasets. More generally, this method builds on the classical methodology of control variates, and is flexible enough to be applied with any estimands that can be estimated consistently across datasets. We showed theoretically that our proposed ATE estimator achieves lower variance than standard ATE estimators which rely only on a single dataset. Across synthetic and semi-synthetic case studies with real data, we also empirically demonstrated this variance reduction in finite samples.

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A Further Analysis of the Kernel Estimator

In this section, we analyze the asymptotic properties of the kernel odds ratio estimator. We work with the log of the odds ratio for an easier analysis of the asymptotic properties.

Theorem A.1. *Given a symmetric kernel function $K(u)$ such that $\int K(u)du = 1$, $\int uK(u)du = 0$, and λ is the bandwidth. Consider the kernel odds ratio estimator as defined in Section 4.3.2. Denote $f(x)$ as the density of X , $X \in \mathbb{R}^d$ and consider $\lambda = o(\frac{1}{N})^{\frac{1}{d+4}}$. Define $\mathbf{w} \triangleq (YZ, (1-Y)(1-Z), Y(1-Z), Z(1-Y))^\top$, $\Sigma(x) \triangleq \text{Cov}(\mathbf{w}|x)$, $g(x) = (g^1(x), g^2(x), g^3(x), g^4(x))^\top \triangleq \mathbb{E}[\mathbf{w}|x]$, and $A(x) \triangleq (\frac{1}{g^1(x)}, \frac{1}{g^2(x)}, \frac{1}{g^3(x)}, \frac{1}{g^4(x)})^\top$. Then,*

$$(n_2\lambda^d)^{1/2}(\log \widehat{\text{OR}}(x) - \log \text{OR}(x)) \rightarrow \mathcal{N}\left(0, A^\top(x)\Sigma(x)A(x)f(x)^{-1} \int K^2(u)du\right),$$

in distribution as $N \rightarrow \infty$.

Proof. By definition,

$$\text{OR}(x) = \frac{g^1(x)g^2(x)}{g^3(x)g^4(x)}. \quad (8)$$

Also by definition, the kernel estimator is

$$\widehat{\text{OR}}(x) = \frac{\hat{g}^1(x)\hat{g}^2(x)}{\hat{g}^3(x)\hat{g}^4(x)}, \quad (9)$$

where $\hat{g}^j(x) = \frac{\sum_{i=1}^N K(\frac{x-X_i}{\lambda})W_i^j}{\sum_{i=1}^N K(\frac{x-X_i}{\lambda})}$, $i = 1 \dots 4$, N as the number of the samples, and $(W^1, W^2, W^3, W^4)^\top = \mathbf{w} = (YZ, (1-Y)(1-Z), Y(1-Z), Z(1-Y))^\top$. Then, $\log \widehat{\text{OR}}(x)$ is simply $\log(\hat{g}^1(x)) + \log(\hat{g}^2(x)) - \log(\hat{g}^3(x)) - \log(\hat{g}^4(x))$.

We first study an asymptotic analysis for $g^j(x)$, $j = 1 \dots 4$. Then we apply the Delta method to obtain the asymptotic consistency of $\log \widehat{\text{OR}}(x)$. Denote that

$$\hat{g}^j(x) = \frac{\frac{1}{N\lambda^d} \sum_{i=1}^N K(\frac{x-X_i}{\lambda})W_i^j}{\frac{1}{N\lambda^d} \sum_{i=1}^N K(\frac{x-X_i}{\lambda})} = \frac{\hat{\tau}^j(x)}{\hat{f}(x)}.$$

Then we have

$$\begin{aligned} \mathbb{E}[\hat{f}(x)] &= \mathbb{E}\left[\frac{1}{N\lambda^d} \sum_{i=1}^N K\left(\frac{x-X_i}{\lambda}\right)\right] \\ &= \mathbb{E}\left[\frac{1}{\lambda^d} K\left(\frac{x-X_i}{\lambda}\right)\right] \\ &= \int \frac{1}{\lambda^d} K\left(\frac{x-z}{\lambda}\right) f(z) dz. \end{aligned}$$

Making the change-of-variables formula for multivariate densities $u = \frac{x-z}{\lambda}$, $du = \lambda^{-d} dz$, then

$$\mathbb{E}[\hat{f}(x)] = \int K(u) f(x - \lambda u) du.$$

When $f(x)$ is continuous, bounded above and $\int K(u)du = 1$, the above converges to $f(x)$ as λ goes to 0 by dominated convergence theorem. To compute the bias, take the second order Taylor expansion of $f(x - \lambda u)$, we have:

$$f(x - \lambda u) = f(x) - \lambda \frac{\partial f(x)}{\partial x'} u + \frac{\lambda^2}{2} \text{tr} \left(\frac{\partial^2 f(x)}{\partial x \partial x'} u u' \right) + o(\lambda^2)$$

Therefore, since the kernel is symmetric, the bias is $\mathcal{O}(\lambda^2)$.

Similarly, we calculate the variance of $\hat{f}(x)$:

$$\begin{aligned} \text{Var}(\hat{f}(x)) &= \text{Var} \left(\frac{1}{N} \sum_{i=1}^N \frac{1}{\lambda^d} K \left(\frac{x - X_i}{\lambda} \right) \right) \\ &= \frac{1}{N} \text{Var} \left(\frac{1}{\lambda^d} K \left(\frac{x - X_i}{\lambda} \right) \right) \\ &= \frac{1}{N} \mathbb{E} \left[\left(\frac{1}{\lambda^d} K \left(\frac{x - X_i}{\lambda} \right) \right)^2 \right] - \frac{1}{N} \left(\mathbb{E} \left[\frac{1}{\lambda^d} K \left(\frac{x - X_i}{\lambda} \right) \right] \right)^2 \\ &= \frac{1}{N} \int \frac{1}{\lambda^{2d}} \left(K \left(\frac{x - z}{\lambda} \right) \right)^2 f(z) dz - \frac{1}{N} \left(\mathbb{E}[\hat{f}(x)] \right)^2 \\ &= \frac{1}{N \lambda^d} \int (K(u))^2 f(x - \lambda u) du - \frac{1}{N} \left(\mathbb{E}[\hat{f}(x)] \right)^2 \\ &= \frac{f(x)}{N \lambda^d} \int (K(u))^2 du + o\left(\frac{1}{N \lambda^d}\right), \end{aligned}$$

where we make the change of variable $u = \frac{x-z}{\lambda}$ again. Therefore, the variance of $\hat{f}(x)$ is $\mathcal{O}(\frac{1}{N \lambda^d})$. Recall its bias is $\mathcal{O}(\lambda^2)$ given that the kernel is symmetric, and the optimal bandwidth equates the rate of convergence of the squared bias and variance, i.e. $\mathcal{O}((\lambda^*)^4) = \mathcal{O}(\frac{1}{N(\lambda^*)^d})$. Therefore, the optimal bandwidth is $\lambda^* = \mathcal{O}(\frac{1}{N})^{\frac{1}{d+4}}$.

Further, we can consider $\hat{f}(x)$ as an average of a triangular array:

$$\hat{f}(x) = \frac{1}{N} \sum_{i=1}^N Z_{in},$$

with $Z_{in} = \frac{1}{\lambda^d} K(\frac{x - X_i}{\lambda})$. By the Lyapunov CLT theorem, when $\lambda \rightarrow 0$, $N \lambda^d \rightarrow \infty$, we have

$$\frac{\hat{f}(x) - \mathbb{E}[\hat{f}(x)]}{\sqrt{\text{Var}(\hat{f}(x))}} \rightarrow N(0, 1),$$

as $N \rightarrow \infty$.

Notice that

$$\frac{\hat{f}(x) - f(x)}{\sqrt{\text{Var}(\hat{f}(x))}} = \frac{\hat{f}(x) - \mathbb{E}[\hat{f}(x)]}{\sqrt{\text{Var}(\hat{f}(x))}} + \frac{\mathbb{E}[\hat{f}(x)] - f(x)}{\sqrt{\text{Var}(\hat{f}(x))}}.$$

Pick $\lambda = o(\frac{1}{N})^{\frac{1}{d+4}}$ ensures that and the bias term vanishes from the asymptotic distribution and is negligible relative to the variance. Therefore, we have

$$\frac{\hat{f}(x) - f(x)}{\sqrt{\text{Var}(\hat{f}(x))}} \rightarrow N(0, 1),$$

in distribution as $N \rightarrow \infty$.

Next we analyze $\hat{\tau}^j(x)$, $j = 1 \cdots 4$. By definition,

$$\begin{aligned} \mathbb{E}[\hat{\tau}^j(x)] &= \mathbb{E}\left[\frac{1}{\lambda^d} K\left(\frac{x - X_i}{\lambda}\right) W_i^j\right] \\ &= \mathbb{E}\left[\frac{1}{\lambda^d} K\left(\frac{x - X_i}{\lambda}\right) g^j(x_i)\right] \\ &= \int \frac{1}{\lambda^d} K\left(\frac{x - X_i}{\lambda}\right) g^j(z) f(z) dz \end{aligned}$$

Similar to the derivation of $\mathbb{E}[\hat{f}(x)]$, we have

$$\begin{aligned} \mathbb{E}[\hat{\tau}^j(x)] &= g^j(x) f(x) + \mathcal{O}(\lambda^d) \\ &\rightarrow g^j(x) f(x), \end{aligned}$$

as $N \rightarrow \infty$, and

$$\begin{aligned} \text{Var}(\hat{\tau}^j(x)) &= \text{Var}\left(\frac{1}{N} \sum_{i=1}^N \frac{1}{\lambda^d} K\left(\frac{x - X_i}{\lambda}\right) W_i^j\right) \\ &= \frac{\mathbb{E}[(W^j)^2|x]}{N\lambda^d} \int K^2(u) du + o\left(\frac{1}{N\lambda^d}\right). \end{aligned}$$

As $N\lambda^d \rightarrow \infty$, $\begin{pmatrix} \hat{\tau}(x) \\ \hat{f}(x) \end{pmatrix}$ are jointly normal. Applying the Delta method, we have

$$(N\lambda^d)^{\frac{1}{2}}(\hat{g}^j(x) - g^j(x)) \rightarrow \mathcal{N}\left(0, \frac{\text{Var}(W_i^j|x_i=x)}{f(x)} \int K^2(u) du\right).$$

Therefore, for $j = 1 \cdots 4$, we have $\text{Var}(\hat{g}^j(x)) = \frac{\text{Var}(W_i^j|x_i=x)}{f(x)} \int K^2(u) du$. Similarly, we have that $\text{Cov}(\hat{g}^j(x), \hat{g}^k(x)) = \frac{\text{Cov}(W_i^j, W_i^k|x_i=x)}{f(x)} \int K^2(u) du$ for all $j, k = 1 \cdots 4$, $j \neq k$ as N goes to ∞ . Thus,

$$(n_2\lambda^d)^{1/2}(\hat{g}(x) - g(x)) \rightarrow \mathcal{N}\left(0, \Sigma(x) f(x)^{-1} \int K^2(u) du\right), \quad (10)$$

Lastly, we use the Delta method again to analyze the asymptotic convergence of $\log \widehat{\text{OR}}(x)$. Recall that $\log \widehat{\text{OR}}(x) = \log(\hat{g}^1(x)) + \log(\hat{g}^2(x)) - \log(\hat{g}^3(x)) - \log(\hat{g}^4(x))$. By definition and Taylor expansion,

$$\begin{aligned} &\log \widehat{\text{OR}}(x) - \log \text{OR}(x) \\ &= \frac{1}{g^1(x)}(\hat{g}^1(x) - g^1(x)) + \frac{1}{g^2(x)}(\hat{g}^2(x) - g^2(x)) \\ &\quad - \frac{1}{g^3(x)}(\hat{g}^3(x) - g^3(x)) - \frac{1}{g^4(x)}(\hat{g}^4(x) - g^4(x)) \end{aligned}$$

Therefore, by Delta method and Eq (10), the asymptotic variance of $\log \widehat{\text{OR}}(x)$ is:

$$A^\top(x)\Sigma(x)A(x)f(x)^{-1} \int K^2(u)du.$$

Therefore, we have

$$(n_2\lambda^d)^{1/2}(\log \widehat{\text{OR}}(x) - \log \text{OR}(x)) \rightarrow \mathcal{N}\left(0, A^\top(x)\Sigma(x)A(x)f(x)^{-1} \int K^2(u)du\right),$$

in distribution. This completes the proof. \square

Let $\log \widehat{\text{OR}}_1(x), \log \widehat{\text{OR}}_2(x)$ denotes the two consistent estimators obtained from datasets \mathcal{O}_1 and \mathcal{O}_2 . Let $\widehat{\tau}_2$ denote a consistent estimator of the true ATE τ (see Definition in Eq (1)) that we obtain using dataset \mathcal{O}_2 . Then

$$\left(\begin{array}{c} n_2^{1/2}(\widehat{\tau}_2 - \tau) \\ (n_2\lambda^d)^{1/2} \left(\log \widehat{\text{OR}}_1(x) - \log \widehat{\text{OR}}_2(x) \right) \end{array} \right) \rightarrow \mathcal{N}\left\{0, \begin{pmatrix} v_2 & \Gamma^\top \\ \Gamma & V \end{pmatrix}\right\}, \quad (11)$$

for some V and Γ . If Eq. (11) holds exactly rather than asymptotically, by multivariate normal theory, we have the following the conditional distribution:

$$\begin{aligned} & n_2^{1/2}(\widehat{\tau}_2 - \tau) \mid (n_2\lambda^d)^{1/2} \left(\log \widehat{\text{OR}}_1(x) - \log \widehat{\text{OR}}_2(x) \right) \\ & \sim \mathcal{N}\left\{(n_2\lambda^d)^{1/2}\Gamma^\top V^{-1}(n_2\lambda^d)^{1/2} \left(\log \widehat{\text{OR}}_1(x) - \log \widehat{\text{OR}}_2(x) \right), v_2 - \Gamma^\top V^{-1}\Gamma\right\}. \end{aligned}$$

Then, we apply the control variates method to build a new estimator of τ which has a lower variance than $\widehat{\tau}_2$. The new bias-corrected estimator for ATE is as follows: $\widehat{\tau}_{\text{CV}}(\beta) = \widehat{\tau}_2 - \beta \left(\log \widehat{\text{OR}}_1(x) - \log \widehat{\text{OR}}_2(x) \right)$.

Solving for the optimal β , we obtain the new estimator

$$\widehat{\tau}_{\text{CV}} = \widehat{\tau}_2 - \sqrt{\lambda^d}\Gamma^\top V^{-1} \left(\log \widehat{\text{OR}}_1(x) - \log \widehat{\text{OR}}_2(x) \right), \quad (12)$$

where $V = \text{Var} \left(\log \widehat{\text{OR}}_1(x) - \log \widehat{\text{OR}}_2(x) \right)^{-1}$, and $\Gamma = \text{Cov}(\log \widehat{\text{OR}}_1(x) - \log \widehat{\text{OR}}_2(x), \widehat{\tau}_2)$, and $\lambda^* = o\left(\frac{1}{N}\right)^{\frac{1}{d+4}}$.

Denote the asymptotic variance of $\widehat{\tau}_2$ as v_2 . Under Assumption 2.1, if Equation (11) holds, then $\widehat{\tau}_{\text{CV}}$ is consistent for τ , and we have:

$$n_2^{1/2}(\widehat{\tau}_{\text{CV}} - \tau) \rightarrow \mathcal{N}(0, v_2 - \Gamma^\top V^{-1}\Gamma),$$

in distribution as $n_2 \rightarrow \infty$. Given a nonzero Γ , the asymptotic variance, $v_2 - \Gamma^\top V^{-1}\Gamma$, is smaller than v_2 .

B Bootstrap Sampling Procedure

Our bootstrap sampling procedure is similar to the one in [Yang and Ding \[2020\]](#). For $b = 1, \dots, B$, we construct bootstrap replicates for the estimators as follows:

Step 1. Sample n_2 units from \mathcal{O}_2 with replacement as $O_2^{*(b)}$, and sample n_1 units from \mathcal{O}_1 with replacement as $O_1^{*(b)}$.

Step 2. Compute the bootstrap replicate $\hat{\tau}_2^{(b)}$ using the dataset $O_2^{*(b)}$, and compute the bootstrap replicates $\hat{\psi}_2^{(b)}$, and $\hat{\psi}_1^{(b)}$ using the dataset $O_1^{*(b)}$.

Based on the bootstrap replicates, we estimate the sample covariance $\hat{\Gamma}$ and \hat{V} by

$$\hat{\Gamma} = \frac{1}{B-1} \sum_{b=1}^B (\hat{\tau}_2^{(b)} - \hat{\tau}_2)(\hat{\psi}_2^{(b)} - \hat{\psi}_1^{(b)} - \hat{\psi}_2 + \hat{\psi}_1),$$

$$\hat{V} = \frac{1}{B-1} \sum_{b=1}^B (\hat{\psi}_2^{(b)} - \hat{\psi}_1^{(b)} - \hat{\psi}_2 + \hat{\psi}_1)(\hat{\psi}_2^{(b)} - \hat{\psi}_1^{(b)} - \hat{\psi}_2 + \hat{\psi}_1)^\top.$$

The bootstrap covariance estimates $\hat{\Gamma}$ and \hat{V} are consistent if the estimators $\hat{\tau}_2$, $\hat{\psi}_1$, and $\hat{\psi}_2$ are regular asymptotically linear (RAL) estimators, as shown by [Efron and Tibshirani \[1986\]](#) and [Shao and Tu \[2012\]](#).

Definition B.1. An estimator $\hat{\tau}$ for a statistic τ estimated from a dataset $\{Z_i, X_i, Y_i\}_{i=1}^n$ is RAL if it can be asymptotically approximated by a sum of IID random vectors with mean 0:

$$\hat{\tau} - \tau \cong \frac{1}{n} \sum_{i=1}^n \phi(Z_i, X_i, Y_i)$$

$\phi(Z, X, Y)$ is also known as the influence function for $\hat{\tau}$.

A common example of a RAL estimator for the ATE is the regression imputation estimator, which we used in experiments.

B.1 Matching estimators

Another common class of ATE estimators is matching estimators. Matching estimators do not have smooth influence functions, so the direct bootstrap procedure above may not be consistent [Abadie and Imbens \[2008\]](#). However, [Yang and Ding \[2020\]](#) and [Abadie and Imbens \[2006\]](#) show that the bias of a matching estimator $\hat{\tau}$ can still be expressed in an asymptotically linear form:

$$\hat{\tau} - \tau \cong \frac{1}{n} \sum_{i=1}^n \phi_i$$

Using these linear terms, a slightly modified bootstrap procedure can be used, which [Yang and Ding \[2020\]](#) show to be consistent for both RAL estimators and matching estimators. This procedure uses a modified version of Step 2 which estimates the asymptotically linear terms. Let $\phi_i^{\tau_2}$ indicate the asymptotically linear term for estimator $\hat{\tau}_2$ (e.g. $\phi(Z_i, X_i, Y_i)$ for RAL $\hat{\tau}_2$), and let $\phi_i^{\psi_1}$, $\phi_i^{\psi_2}$ indicate

the same for estimators $\hat{\psi}_2, \hat{\psi}_1$, respectively. Let $\hat{\phi}_i^{\tau_2}, \hat{\phi}_i^{\psi_1}, \hat{\phi}_i^{\psi_2}$ denote estimates for the population quantities.

Step 2 (modified for matching). Compute the bootstrap replicates using the dataset $O_2^{*(b)}$ as

$$\hat{\tau}_2^{(b)} - \hat{\tau}_2 = \frac{1}{n_2} \sum_{i=1}^{n_2} \hat{\phi}_i^{\tau_2}$$

and compute the bootstrap replicates using the dataset $O_1^{*(b)}$ as

$$\hat{\psi}_1^{(b)} - \hat{\psi}_1 = \frac{1}{n_1} \sum_{i=1}^{n_1} \hat{\phi}_i^{\psi_1}; \quad \hat{\psi}_2^{(b)} - \hat{\psi}_2 = \frac{1}{n_1} \sum_{i=1}^{n_1} \hat{\phi}_i^{\psi_2}.$$

Theorem B.1. (Theorem 3 from [Yang and Ding \[2020\]](#)) If $\hat{\tau}_2, \hat{\psi}_1$, and $\hat{\psi}_2$ are RAL estimators or matching estimators, then under certain regularity conditions, the bootstrap estimates $\hat{\Gamma}, \hat{V}$ under this modified procedure are consistent for Γ, V .

C Proofs and Further Analysis of the Odds Ratio

We provide proofs for the theorems and lemma presented in Sections 3 and 4, as well as further analysis of the odds ratio's effectiveness as a control variate.

C.1 Proofs from Section 3

Theorem 3.1. Denote the asymptotic variance of $\hat{\tau}_2$ as v_2 . Under Assumption 2.1, if Equation (2) holds, then $\hat{\tau}_{CV}$ is consistent for τ , and we have:

$$n_2^{1/2}(\hat{\tau}_{CV} - \tau) \rightarrow \mathcal{N}(0, v_2 - \Gamma^\top V^{-1}\Gamma),$$

in distribution as $n_2 \rightarrow \infty$. Given a nonzero Γ , the asymptotic variance, $v_2 - \Gamma^\top V^{-1}\Gamma$, is smaller than v_2 .

Proof. The theorem statement follows directly from

$$n_2^{1/2} \begin{pmatrix} \hat{\tau}_2 - \tau \\ \hat{\psi}_2 - \hat{\psi}_1 \end{pmatrix} \rightarrow \mathcal{N} \left\{ 0, \begin{pmatrix} v_2 & \Gamma^\top \\ \Gamma & V \end{pmatrix} \right\}.$$

By construction, $\hat{\tau}_{CV} = \hat{\tau}_2 - \Gamma^\top V^{-1}(\hat{\psi}_2 - \hat{\psi}_1)$. To compute the asymptotic variance, notice that

$$\begin{aligned} & \text{Var}(n_2^{1/2}(\hat{\tau}_{CV} - \tau)) \\ &= \text{Var}(n_2^{1/2}(\hat{\tau}_2 - \tau)) \\ &= \text{Var}(n_2^{1/2}(\hat{\tau}_2 - \tau - \Gamma^\top V^{-1}(\hat{\psi}_2 - \hat{\psi}_1))) \\ &= \text{Var}(n_2^{1/2}(\hat{\tau}_2 - \tau)) + \Gamma^\top V^{-1} \text{Var}(n_2^{1/2}(\hat{\psi}_2 - \hat{\psi}_1)) V^{-1} \Gamma - 2 \text{Cov}(n_2^{1/2}(\hat{\tau}_2 - \tau), n_2^{1/2} \Gamma^\top V^{-1}(\hat{\psi}_2 - \hat{\psi}_1)) \\ &= v_2 + \Gamma^\top V^{-1} \Gamma - 2 \Gamma^\top V^{-1} \Gamma \\ &= v_2 - \Gamma^\top V^{-1} \Gamma \end{aligned}$$

Therefore, we have:

$$n_2^{1/2}(\hat{\tau}_{CV} - \tau) \rightarrow \mathcal{N}(0, v_2 - \Gamma^\top V^{-1}\Gamma),$$

in distribution as $n_2 \rightarrow \infty$, which completes the proof. \square

C.2 Proofs from Section 4

Lemma 4.1. *If the selection S depends solely on Y (as in Figure 1), then the conditional odds ratio is transportable and given by:*

$$\text{OR}(x) = \frac{P(Y = 1|S = 1, Z = 1, x)P(Y = 0|S = 1, Z = 0, x)}{P(Y = 0|S = 1, Z = 1, x)P(Y = 1|S = 1, Z = 0, x)}.$$

Proof. By Bayes' theorem,

$$P(Y = y|Z = z, x) = \frac{P(Y = y|S = 1, Z = z, x)P(S = 1|Z = z, x)}{P(S = 1|Y = y, Z = z, x)}.$$

Since S depends solely on Y , S is conditionally independent of X and Z given Y . Therefore, we can rewrite the equation above as

$$P(Y = y|Z = z, x) = \frac{P(Y = y|S = 1, Z = z, x)P(S = 1|Z = z, x)}{P(S = 1|Y = y)}.$$

Substituting this into Definition 4.1 (under Assumption 2.1),

$$\begin{aligned} \text{OR}(x) &= \frac{P(Y = 1|Z = 1, x)P(Y = 0|Z = 0, x)}{P(Y = 0|Z = 1, x)P(Y = 1|Z = 0, x)} \\ &= \frac{\frac{P(Y=1|S=1, Z=1, x)P(S=1|Z=1, x)}{P(S=1|Y=1)} \frac{P(Y=0|S=1, Z=0, x)P(S=1|Z=0, x)}{P(S=1|Y=0)}}{\frac{P(Y=0|S=1, Z=1, x)P(S=1|Z=1, x)}{P(S=1|Y=0)} \frac{P(Y=1|S=1, Z=0, x)P(S=1|Z=0, x)}{P(S=1|Y=1)}} \\ &= \frac{P(Y = 1|S = 1, Z = 1, x)P(Y = 0|S = 1, Z = 0, x)}{P(Y = 0|S = 1, Z = 1, x)P(Y = 1|S = 1, Z = 0, x)} \end{aligned}$$

Therefore, the conditional odds ratio is transportable under selection bias given by S . \square

Theorem 4.2. *If the selection S depends solely on Y (as in Figure 1) and $P(Y = 1|Z = z, X = x)$ follows the logistic model in (4), then $P(Y = 1|Z = z, X = x, S = 1)$ also follows a logistic model, with the same coefficient β_1^x on Z as the logistic model for $P(Y = 1|Z = z, X = x)$ for each covariate value x . Furthermore, the conditional odds ratio $\text{OR}(x) = e^{\beta_1^x}$.*

Proof. Given the assumed outcome model, we have

$$P(Y = 1|Z = z, x) = \frac{e^{\beta_0^x + \beta_1^x z}}{1 + e^{\beta_0^x + \beta_1^x z}}.$$

Let $p_1 = P(S = 1|Y = 1)$ and $p_0 = P(S = 1|Y = 0)$. Since the selection S depends solely on Y , S is conditionally independent of X and Z given Y .

The outcome model under selection bias is given by:

$$\begin{aligned}
& P(Y = 1|Z = z, X = x, S = 1) \\
&= \frac{P(Y = 1|Z = z, X = x)P(S = 1|Z = z, X = x, Y = 1)}{P(S = 1|Z = z, X = x)} \\
&= \frac{P(Y = 1|Z = z, X = x)P(S = 1|Z = z, X = x, Y = 1)}{\sum_{y \in \{0,1\}} P(Y = y|Z = z, X = x)P(S = 1|Z = z, X = x, Y = y)} \\
&= \frac{P(Y = 1|Z = z, X = x)p_1}{P(Y = 1|Z = z, X = x)p_1 + P(Y = 0|Z = z, X = x)p_0} \\
&= \frac{\frac{e^{\beta_0^x + \beta_1^x z}}{1 + e^{\beta_0^x + \beta_1^x z}} p_1}{\frac{e^{\beta_0^x + \beta_1^x z}}{1 + e^{\beta_0^x + \beta_1^x z}} p_1 + \frac{1}{1 + e^{\beta_0^x + \beta_1^x z}} p_0} \\
&= \frac{e^{\beta_0^x + \beta_1^x z} p_1}{e^{\beta_0^x + \beta_1^x z} p_1 + p_0} \\
&= \frac{e^{\beta_0^x + \beta_1^x z} p_1 / p_0}{e^{\beta_0^x + \beta_1^x z} p_1 / p_0 + 1} \\
&= \frac{e^{\delta + \beta_0^x + \beta_1^x z}}{1 + e^{\delta + \beta_0^x + \beta_1^x z}}.
\end{aligned}$$

where $\delta = \log(p_1/p_0)$. Thus, on the selection biased dataset \mathcal{O}_1 , the outcome model $P(Y = 1|Z = z, X = x, S = 1)$ also follows a logistic model, with the same coefficient β_1^x on Z as the logistic model for $P(Y = 1|Z = z, X = x)$ for each covariate value x . Furthermore, a simple calculation shows that the conditional odds ratio is $\text{OR}(x) = e^{\beta_1^x}$:

$$\begin{aligned}
\text{OR}(x) &= \frac{P(Y = 1|Z = 1, x)P(Y = 0|Z = 0, x)}{P(Y = 0|Z = 1, x)P(Y = 1|Z = 0, x)} \\
&= \frac{\frac{e^{\beta_0^x + \beta_1^x}}{1 + e^{\beta_0^x + \beta_1^x}} \frac{1}{1 + e^{\beta_0^x}}}{\frac{1}{1 + e^{\beta_0^x + \beta_1^x}} \frac{e^{\beta_0^x}}{1 + e^{\beta_0^x}}} \\
&= \frac{e^{\beta_0^x + \beta_1^x}}{e^{\beta_0^x}} \\
&= e^{\beta_1^x}.
\end{aligned}$$

□

C.3 Analysis of Nonlinear Relationship Between ATE and OR

As discussed in Section 3, the variance reduction from adding control variates depends on the strength of the correlation between the control variates and the ATE estimator. Since we propose to use the odds ratio for selection biased datasets, here we examine the relationship between the ATE and the odds ratio. Specifically, we derive an explicit expression for the ATE using the marginal odds ratio OR assuming a binary covariate X and the following simple logistic outcome model:

$$P(Y = 1|Z = z, X = x) = \frac{e^{\beta_0 + \beta_1 z + \beta_2 x}}{1 + e^{\beta_0 + \beta_1 z + \beta_2 x}},$$

where the marginal odds ratio OR is defined as

$$\text{OR} = \frac{P(Y(1) = 1)P(Y(0) = 0)}{P(Y(1) = 0)P(Y(0) = 1)}.$$

Under this simple logistic outcome model, $\text{OR} = e^{\beta_1}$.

By Assumption 2.1 and backdoor adjustment,

$$\begin{aligned} \mathbb{E}[Y(1)] &= \mathbb{E}[Y|do(Z = 1)] \\ &= \int \mathbb{E}[Y|Z = 1, X = x]P(X = x)dx \\ &= \int \frac{e^{\beta_0 + \beta_1 + \beta_2 x}}{e^{\beta_0 + \beta_1 + \beta_2 x} + 1} P(X = x)dx \\ &= \frac{e^{\beta_0 + \beta_1 + \beta_2}}{e^{\beta_0 + \beta_1 + \beta_2} + 1} P(X = 1) + \frac{e^{\beta_0 + \beta_1}}{e^{\beta_0 + \beta_1} + 1} (1 - P(X = 1)) \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbb{E}[Y(0)] &= \mathbb{E}[Y|do(Z = 0)] \\ &= \int \mathbb{E}[Y|Z = 0, X = x]P(X = x)dx \\ &= \int \frac{e^{\beta_0 + \beta_2 x}}{e^{\beta_0 + \beta_2 x} + 1} P(X = x)dx \\ &= \frac{e^{\beta_0 + \beta_2}}{e^{\beta_0 + \beta_2} + 1} P(X = 1) + \frac{e^{\beta_0}}{e^{\beta_0} + 1} (1 - P(X = 1)) \end{aligned}$$

Therefore, by some algebra we obtain that

$$\begin{aligned} \tau &= \mathbb{E}[Y(1)] - \mathbb{E}[Y(0)] \\ &= \frac{\gamma ab\psi}{ab\psi + 1} - \frac{\gamma a\psi}{a\psi + 1} + \frac{a\psi - a}{a^2\psi - a\psi + a + 1} - C, \end{aligned}$$

where $\psi = \text{OR}$, $\gamma = P(X = 1)$, $a = e^{\beta_0}$, $b = e^{\beta_2}$, with a constant term $C = \frac{a}{a+1} - \frac{ab}{ab+1}$.

D Additional Experimental Results for Simulation Study

This section provides additional experimental results for the simulation study. All code for running the simulation study is provided with the supplementary materials.

D.1 Simple logistic outcome model without interaction between X and Z

In addition to the data generation setting described in Section 5, we also include results with a simpler data generation setting without interaction between X and Z . We set $b_{1,1} = b_{0,1} = (-1, 1)^\top$, which implies that $\beta_3 = 0$ in Section 5.1. In this simpler model, the conditional odds ratio is constant in X and is given by e^{β_1} .

Figure 3 shows that adding control variates still improves the variance of the ATE estimator under this simpler outcome model without interaction between X and Z .

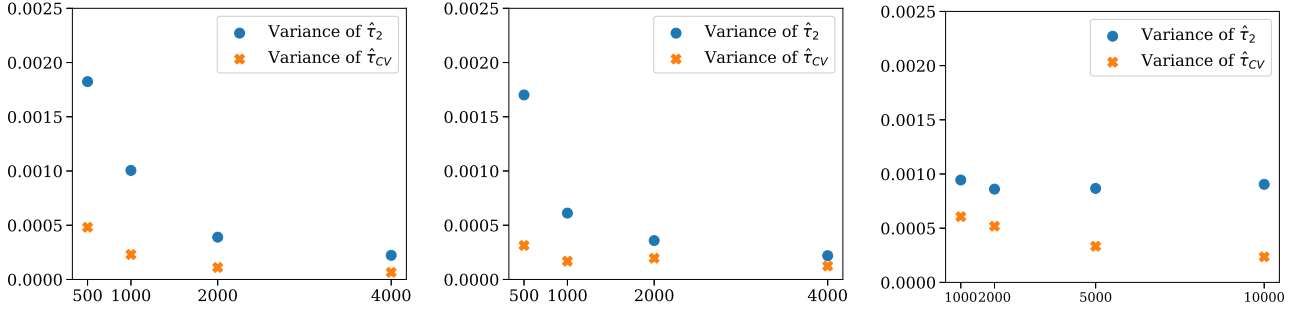


Figure 3: **Scenario 1 (left)**: Comparisons of variance as the size of the observational dataset n_2 increases under the simple logistic model with $\beta_3 = 0$. The ratio n_2/n_1 is kept constant at $1/10$. **Scenario 2 (middle)**: Comparisons of variance as the size of the observational dataset n_2 increases under the simple logistic model with $\beta_3 = 0$. The size of the selection bias dataset is fixed at $n_1 = 10000$. **Scenario 3 (right)**: Comparisons of variance as the size of the selection bias dataset n_1 increases under the simple logistic model with $\beta_3 = 0$. The size of the observational dataset is fixed at $n_2 = 1000$. *Lower is better.*

D.2 Bias results

In this section, we report the bias for the ATE estimators with and without control variates, i.e. $\hat{\tau}_2$ and $\hat{\tau}_{CV}$. The bias is calculated over $B = 100$ bootstrap replicates, and is defined as the difference between the average value of the ATE estimator over $B = 100$ bootstrap replicates and the true ATE.

The bias for the simple logistic outcome model without interaction between X and Z described in Appendix D.1 is given in Figures 4, 5, and 6. The bias for the logistic model with varying coefficients described in Section 5.1 is given in Figures 7, 8, and 9. In general, across all three scenarios, the bias decreases as n_2 increases, and is not significantly different with or without control variates. We report the bias mainly to confirm that adding control variates does not significantly increase the bias of the estimator in finite samples.

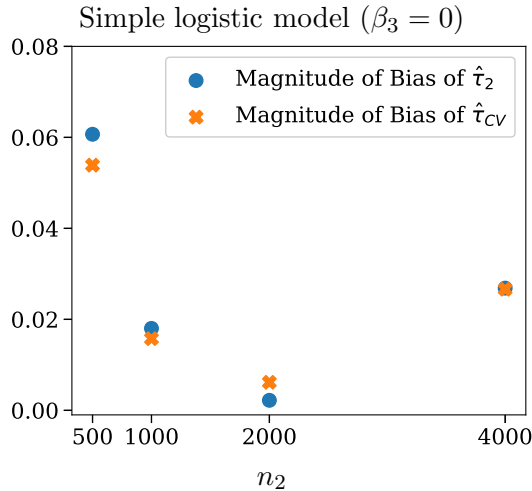


Figure 4: **Scenario 1**: Comparisons of bias as the size of the observational dataset n_2 increases under the simple logistic model. The ratio n_2/n_1 is kept constant at $1/10$.

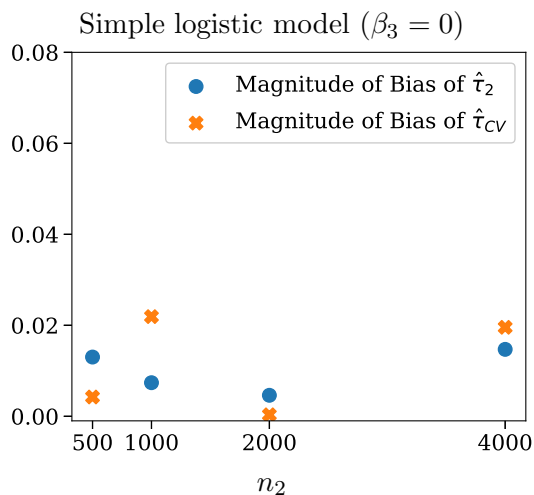


Figure 5: **Scenario 2:** Comparisons of bias as the size of the observational dataset n_2 increases under the simple logistic model. The size of the selection bias dataset is fixed at $n_1 = 10000$.

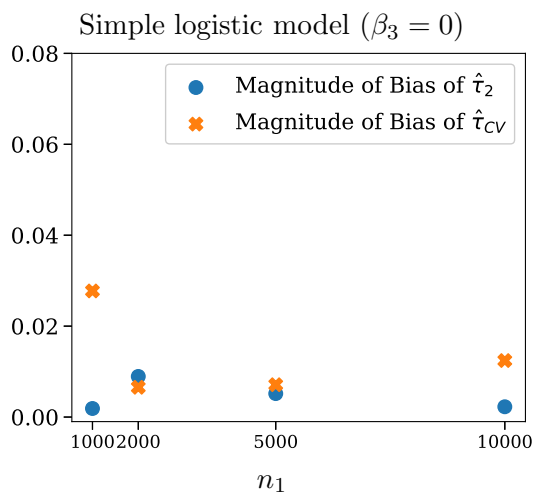


Figure 6: **Scenario 3:** Comparisons of bias as the size of the selection bias dataset n_1 increases under the simple logistic model. The size of the observational dataset is fixed at $n_2 = 1000$.

E Additional Experimental Details and Results for Real Data Case Studies

This section provides additional experimental details and results for the real data case studies. All code for generating data and running the experiments is provided with the supplementary materials.

E.1 Data generation for case study 1: flu shot encouragement

We provide a detailed breakdown for generating the selection biased dataset \mathcal{O}_1 for Case Study 1 on flu shot encouragement below. Code for this is also included with the supplementary materials.

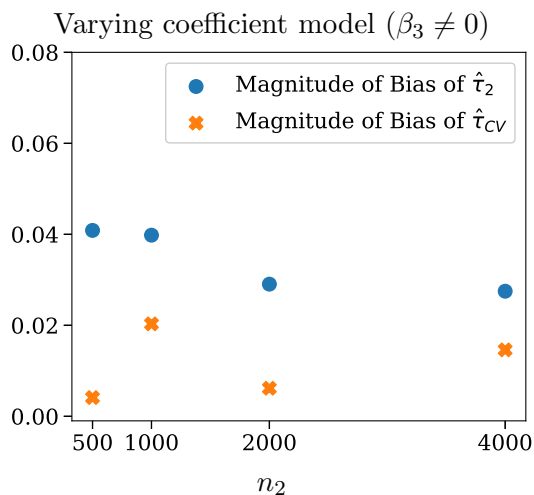


Figure 7: **Scenario 1:** Comparisons of bias as the size of the observational dataset n_2 increases under the logistic model with varying coefficients ($\beta_3 \neq 0$) from Section 5.1. The ratio n_2/n_1 is kept constant at 1/10.

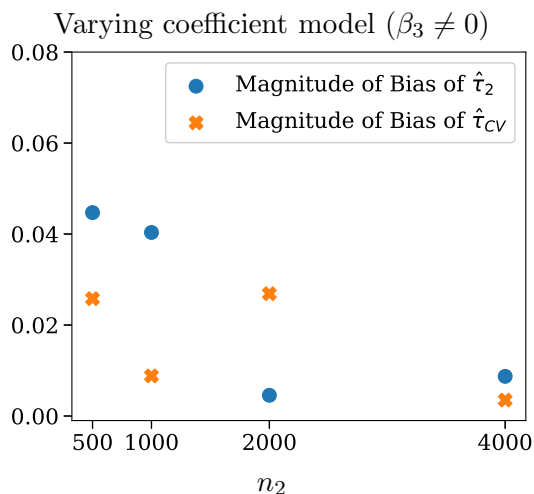


Figure 8: **Scenario 2:** Comparisons of bias as the size of the observational dataset n_2 increases under the logistic model with varying coefficients ($\beta_3 \neq 0$) from Section 5.1. The size of the selection bias dataset is fixed at $n_1 = 10000$.

1. Fit a logistic regression model on the original dataset \mathcal{O}_2 with inputs X, Z and outcome Y according to

$$P(Y = 1|Z = z, X = x) = g(\beta, z, x) = \frac{e^{\beta_0 + \beta_1 z + \beta_2 x + \beta_3 xz}}{1 + e^{\beta_0 + \beta_1 z + \beta_2 x + \beta_3 xz}}. \quad (13)$$

This results in estimated parameters $\hat{\beta}$.

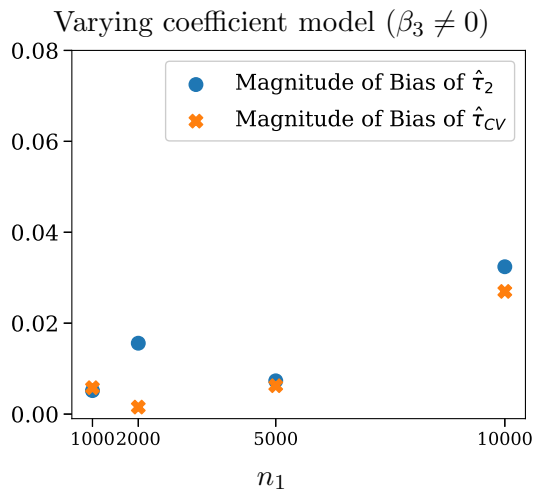


Figure 9: **Scenario 3:** Comparisons of bias as the size of the selection bias dataset n_1 increases under the logistic model with varying coefficients ($\beta_3 \neq 0$) from Section 5.1. The size of the observational dataset is fixed at $n_2 = 1000$.

2. Fit a logistic regression model to estimate the propensity score according to

$$P(Z = 1|X = x) = h(a, x) = \frac{e^{a_0 + a_1^T x}}{1 + e^{a_0 + a_1^T x}},$$

This results in estimated parameters \hat{a} .

3. Sample covariates $\{X_i\}_1^N$ with replacement from \mathcal{O}_2 .
4. Generate Z_i according to the estimated propensity score distribution, $P(Z_i = 1|X = X_i) = h(\hat{a}, X_i)$.
5. Generate Y_i according to the estimated outcome distribution, $P(Y_i = 1|Z = Z_i, X = X_i) = g(\hat{\beta}, Z_i, X_i)$.
6. Apply selection bias on the outcome according to the distribution $P(S_i = 1|Y_i = 1) = 0.9$ to generate the final dataset \mathcal{O}_1 .

E.2 Data generation for case study 2: spam email detection

For case study 2, we directly apply the code provided by the Atlantic Causal Inference Conference (ACIC) Data Challenge to generate both \mathcal{O}_1 and \mathcal{O}_2 . The ACIC data generation code is publicly available at <https://sites.google.com/view/acic2019datachallenge/data-challenge>. The ACIC data generation code modifies existing real datasets in a variety of ways to generate datasets for the ACIC data challenge with known true ATEs. Specifically, we use ACIC’s “modification 1” of the Spambase spam email detection dataset from UCI, which applies a logistic outcome model very close to the logistic regression models fitted to the actual data. For convenience, we include the exact script for “modification 1” in our supplementary materials.

E.3 Further details on the estimators

We provide further details on the estimators for the ATE and odds ratio used in the real data experiments.

Logistic model with interaction: We model the outcome Y using a logistic model with an interaction term between X and Z in Eq. (7), repeated here for convenience:

$$P(Y = 1|Z = z, X = x) = g(\beta, z, x) = \frac{e^{\beta_0 + \beta_1 z + \beta_2 x + \beta_3 xz}}{1 + e^{\beta_0 + \beta_1 z + \beta_2 x + \beta_3 xz}}.$$

To estimate the ATE from the dataset without selection bias \mathcal{O}_2 , we perform logistic regression of Y on X , Z , and the interaction term XZ , to produce estimates $\hat{\beta}_{\mathcal{O}_2}$. The ATE estimator is then given by

$$\hat{\tau}_2 = g(\hat{\beta}_{\mathcal{O}_2}, 1, x) - g(\hat{\beta}_{\mathcal{O}_2}, 0, x).$$

We then estimate the conditional odds ratio from \mathcal{O}_2 as $\widehat{\text{OR}}_2(x) = e^{\hat{\beta}_1 \cdot \mathcal{O}_2 + \hat{\beta}_3 \cdot \mathcal{O}_2}$.

We similarly estimate the conditional odds ratio from \mathcal{O}_1 by following the same logistic regression procedure as above, resulting in $\widehat{\text{OR}}_1(x) = e^{\hat{\beta}_1 \cdot \mathcal{O}_1 + \hat{\beta}_3 \cdot \mathcal{O}_1}$.

Neural network: For a more general outcome model, we model the outcome Y using a logistic model with varying coefficients in Eq. (4), where the functions $\beta_0^x = f_0(x; \theta)$, $\beta_1^x = f_1(x; \theta)$ make up the two-dimensional output of a single neural network with parameters θ :

$$P(Y = 1|Z = z, X = x) = g(\theta, z, x) = \frac{e^{f_0(x; \theta) + f_1(x; \theta)z}}{1 + e^{f_0(x; \theta) + f_1(x; \theta)z}}.$$

The optimization objective for the neural network is the logistic loss on the final outcome prediction, $g(\theta, z, x)$. We optimize the neural network using using ADAM with a default learning rate of 0.001 for 1,000 epochs with batch size n_2 . We choose the neural network architecture using five-fold cross validation over \mathcal{O}_2 . Specifically, we search over $\{4, 8\}$ hidden layers, and hidden layer sizes of $\{4, 8, 16, 32\}$. We use the TensorFlow framework and include all code in the supplementary materials.

Once an architecture has been chosen by the method above, we estimate the ATE from \mathcal{O}_2 by optimizing the neural network parameters θ over the dataset \mathcal{O}_2 to obtain an estimate $\hat{\theta}_{\mathcal{O}_2}$, and the ATE estimator is given by

$$\hat{\tau}_2 = g(\hat{\theta}_{\mathcal{O}_2}, 1, x) - g(\hat{\theta}_{\mathcal{O}_2}, 0, x).$$

We then estimate the conditional odds ratio from \mathcal{O}_2 as $\widehat{\text{OR}}_2(x) = e^{f_1(x; \hat{\theta}_{\mathcal{O}_2})}$.

Finally, using the same architecture as chosen above, we estimate the conditional odds ratio from \mathcal{O}_1 by optimizing the neural network parameters θ over the dataset \mathcal{O}_1 to obtain an estimate $\hat{\theta}_{\mathcal{O}_1}$, resulting in $\widehat{\text{OR}}_1(x) = e^{f_1(x; \hat{\theta}_{\mathcal{O}_1})}$.

Table 3: **Case study 1:** Bias on flu shot encouragement data with $n_1 = 10,000$

METRIC	LOGISTIC	KERNEL
BIAS $\hat{\tau}_2$	4.283e-4	5.384e-4
BIAS $\hat{\tau}_{CV}$	6.431e-4	-4.464e-3

Table 4: **Case study 2:** Bias on spam email detection data with $n_1 = 30,000$.

METRIC	LOGISTIC	NN	KERNEL
$n_2 = 3,000$			
BIAS $\hat{\tau}_2$	6.785e-4	-1.095e-2	-4.250e-3
BIAS $\hat{\tau}_{CV}$	2.132e-3	-1.129e-2	-6.390e-3
$n_2 = 10,000$			
BIAS $\hat{\tau}_2$	7.230e-3	3.427e-4	-2.827e-4
BIAS $\hat{\tau}_{CV}$	3.085e-3	1.361e-3	-1.378e-3

E.4 Bias results

We report the bias for the ATE estimators with and without control variates, $\hat{\tau}_2, \hat{\tau}_{CV}$ for the real data experiments. The bias is calculated over $B = 300$ bootstrap replicates, and is defined as the difference between the average value of the ATE estimator over $B = 300$ bootstrap replicates and the true ATE.

Tables 3 and 4 show the bias of $\hat{\tau}_2, \hat{\tau}_{CV}$ for each of the different estimation methods for case study 1 and case study 2, respectively. While there was not much difference in bias between $\hat{\tau}_2$ and $\hat{\tau}_{CV}$ in the simulation study (Appendix D.2), we often observe higher bias for $\hat{\tau}_{CV}$ in finite samples on the real data. This bias is more pronounced in case study 2, which may be due to a high dimensional continuous covariate vector X .