

A Generic Complementary Sequence Construction and Associated Encoder/Decoder Design

Alphan Şahin, *Member, IEEE*, and Rui Yang, *Member, IEEE*

Abstract—In this study, we propose a flexible construction of complementary sequences (CSs) that can contain zero-valued elements. To derive the construction, we use Boolean functions to represent a polynomial generated with a recursion. By applying this representation to recursive CS constructions, we show the impact of construction parameters such as sign, amplitude, phase rotation used in the recursion on the elements of the synthesized CS. As a result, we extend Davis and Jedwab’s CS construction by obtaining independent functions for the amplitude and phase of each element of the CS, and the seed sequence positions in the CS. The proposed construction shows that a set of distinct CSs compatible with non-contiguous resource allocations for orthogonal frequency-division multiplexing (OFDM) and various constellations can be synthesized systematically. It also leads to a low peak-to-mean-envelope-power ratio (PMEPR) multiple accessing scheme in the uplink and a low-complexity recursive decoder. We demonstrate the performance of the proposed encoder and decoder through comprehensive simulations.

Index Terms—Complementary sequences, Gray code, peak-to-average-power ratio, pseudo-Boolean function, Reed-Muller code, non-contiguous resource allocation

I. INTRODUCTION

Orthogonal frequency-division multiplexing (OFDM) enables a radio to transmit multiple modulation symbols on orthogonal channels separated in the frequency domain. One drawback of OFDM transmission is that it causes high peak-to-mean-envelope-power ratio (PMEPR) for arbitrary modulation symbols. In the literature, there are numerous approaches to reduce the PMEPR [1], [2]. One of the trends is to design codes such that the PMEPR of the resulting signal is bounded (see [3] and the references therein). In this direction, complementary sequences (CSs) from the cosets of Reed-Muller (RM) codes can limit the PMEPR of an OFDM symbol to be less than or equal to 3 dB while still providing coding gain [4]. However, these sequences have to be contiguous with phase shift keying (PSK) alphabet and limited choice of lengths. In this study, we address the issue of constructing CSs compatible with a set of flexible resource allocation in the frequency domain for OFDM while achieving expressions for not only phase but also amplitude of the elements of the CSs.

CSs were introduced by M. Golay in [5]. He established a framework to synthesize Golay complementary pairs (GCPs) with binary alphabet. His constructions were later extended by R. J. Turyn [6] and Sivasamy [7]. In [8], Budišin introduced the permutations of shifted phase-rotated sequences instead of simple concatenation seed CSs to obtain more CSs. In

[9], he proposed another construction that explains how the scaling factors can be taken into account in a recursive method to generate multi-level GCPs. In [10] and [11], recursive constructions with phase factors and scaling factors were combined in a single recursion. Nevertheless, these earlier studies did not propose direct constructions for CSs or investigate CSs with certain alphabets.

Appealing applications of CSs on communication systems were first demonstrated in [4], [12]–[15]. By generalizing an earlier work of Boyd [12], Popovic showed that CSs were beneficial to control the peak instantaneous power of OFDM signals [13]. Davis and Jedwab proved that a large class of CSs over \mathbb{Z}_{2^h} of length 2^m can be obtained through generalized Boolean functions related to RM codes [4], [14] for $h \in \mathbb{Z}^+$. Davis and Jedwab’s result was also generalized to a larger class of CSs over \mathbb{Z}_H of length 2^m where H is even positive integer in [15]. In the literature, the CSs generated through Davis and Jedwab’s construction is sometimes referred to as *Golay-Davis-Jedwab (GDJ)* or *standard* sequences.

The connection between CSs and RM codes has been a cornerstone of various research directions. The first major challenge has been the low code rate of the encoder proposed by Davis and Jedwab. Since the sequence length increases much faster than the number of distinct CSs for the encoder, the corresponding code rate decreases for a larger number of uncoded bits. To address this issue, one major direction has been exploring the CSs which cannot be generated through method in [4], i.e., *nonstandard* sequences. Until Li and Chu found 1024 quaternary nonstandard CSs of length 16 via a computer search in 2005 [16], there were no evidence on the existence of nonstandard CSs of length 2^m . In [17], Li and Kao showed that these sequences contain third-order monomials and can be synthesized by concatenating or interleaving of quaternary GCP of length 8. In [18], Fiedler and Jedwab provided another explanation how these sequences arise by showing that some of the standard CSs in different cosets have identical aperiodic auto-correlation function (AACF). In [19], they also provided a construction leading to the nonstandard CSs in [16] by expressing CSs as a projection of a multi-dimensional array.

To increase the number of distinct CSs for a given sequence length, another direction has been the CS construction with larger alphabets such as M -quadrature amplitude modulation (QAM) constellations. In [20], Rößing and V. Tarokh showed that 16-QAM sequences which result in low PMEPR can be constructed through a weighted sum of two CSs with the alphabet of quadrature phase-shift keying (QPSK) constellation. In [21], [22], Chong et al. showed that a 16-QAM CS can

Alphan Şahin and Rui Yang are with University of South Carolina, Columbia, SC and InterDigital, NY, respectively. E-mail: asahin@mailbox.sc.edu, rui.yang@interdigital.com

be formed by introducing the "offset" sequences based on the Boolean functions to a "base" quaternary standard sequence and there exist at least $(14 + 12m)(m!/2)4^{m+1}$ distinct CSs of length 2^m with the alphabet of 16-QAM constellation. In [23], Li revised Chong et al. and Lee and Golomb's theorems [24] and introduced new offset pairs leading to 64-QAM CSs proposed in [25]. In [26], Li showed that there exist at least $[(m+1)4^{2(q-1)} - (m+1)4^{(q-1)} + 2^{q-1}](m!/2)4^{(m+1)}$ CSs for 4^q -QAM sequences. Although the offset method is helpful for synthesizing a large number of distinct QAM CSs, it typically does not give concise constructions as Davis and Jedwab's method for the standard sequences [22], [23]. This issue is one of our motivations to synthesize CSs by separating the functions for its amplitude and the phase terms. From this aspect, our study is in line with [27] and [28]. However, our work differs from [27] and [28] as we focus on a framework relying on linear operators, which enables us to generate non-contiguous CSs, i.e., CS with zero-valued elements, the CSs of length non-power-of-two, and the CS with various constellations. They also pave the way for developing an encoder and a decoder.

Another noteworthy direction for increasing the number of distinct sequences is to extend CSs to complementary sets or complete complementary codes at the expense of a larger PMEPR [29]–[31]. For these extensions, paraunitary matrices with larger sizes are often utilized. Since we focus on CSs in this study, we refer the reader to [29] and [30] and the references therein for further details.

A. Contributions and Organization

A framework based on linear operators: The first contribution is Lemma 1 that re-expresses the result of a recursion evolved with two linear operators at each step with Boolean functions. It describes how the operators applied to the input polynomials at the n th recursion step are distributed to the coefficients of the final polynomials.

A novel CS construction: By applying Lemma 1 to recursive CS constructions [8]–[11], [32], we show the impact of each parameter in the recursion on the synthesized CS. By removing the redundant parameters, we obtain a concise construction given in Theorem 2 as the main contribution of this study. It extends James and Davis's result [4] by providing independent pseudo-Boolean functions for the amplitude, phase, seed sequences, and the support of the CS.

Distinct CSs with a wide range of constellations: We introduce a constellation family consisting of QAM, PSK, DVB-16APSK, DVB-32APSK, and a modified version IEEE 802.11ay 64-NUC, called *equiangular sub-symmetric constellation (ESC)*. We show that distinct CSs can be enumerated systematically for a given ESC. We also discuss how to generate non-contiguous CSs where the non-zero-valued elements are in ESC.

Encoder/decoder for CSs with an ESC: We develop a generic encoder and maximum-likelihood (ML)-based decoder for CSs with ESC. We demonstrate its applicability for a practical wireless communication system through a low-PMEPR multiple accessing scheme in the uplink with non-contiguous resource allocation for OFDM.

The rest of the paper is organized as follows. In Section II, preliminaries are provided. In Section III, we establish Lemma 1. In Section IV, Theorem 2 is discussed. In Section V, we enumerate CSs for an arbitrary ESC. In Section VI, we discuss the encoder and the decoder. In Section VII, we provide numeral results. We conclude the paper in Section VIII.

Notation: The sets of complex numbers, real numbers, non-negative real numbers, integers, non-negative integers, positive integers, and integers modulo H are denoted by \mathbb{C} , \mathbb{R} , \mathbb{R}_0^+ , \mathbb{Z} , \mathbb{Z}_0^+ , \mathbb{Z}^+ , and \mathbb{Z}_H , respectively. The set of m -dimensional integers where each element is in \mathbb{Z}_H is denoted by \mathbb{Z}_H^m . The transpose, Hermitian, the complex conjugation, the modulo 2, and the assignments are denoted by $(\cdot)^T$, $(\cdot)^H$, $(\cdot)^*$, $(\cdot)_2$, and \leftarrow , respectively. The constant j is $\sqrt{-1}$.

II. PRELIMINARIES AND DEFINITIONS

A. Sequences, OFDM, and PMEPR

Let $\mathbf{a} = (a_i)_{i=0}^{N-1} \triangleq (a_0, a_1, \dots, a_{N-1})$ be a sequence of length N , where $a_i \in \mathbb{C}$ and $a_{N-1} \neq 0$. We associate the sequence \mathbf{a} with the polynomial $A(z) = a_{N-1}z^{N-1} + a_{N-2}z^{N-2} + \dots + a_0$ in indeterminate z . When it is clear from the context, $A(z)$ is a polynomial function.

Let $s_a(t) = \sum_{i=0}^{N-1} a_i e^{j2\pi i \frac{t}{T_s}}$ for $t \in [0, T_s)$ be the continuous-time baseband OFDM symbol generated from the sequence \mathbf{a} with the symbol duration T_s . Since $s_a(t) = A(e^{j\frac{2\pi t}{T_s}})$, the instantaneous envelope power of the OFDM symbol can be examined by evaluating the polynomial $A(z)$ at $|z| = 1$. When z is restricted to be on the unit circle in the complex plane, $|A(z)|^2$ corresponds to $\sum_{k=-N+1}^{N-1} \rho_a(k)z^k$, where $\rho_a(k)$ is the AACF of the sequence \mathbf{a} given by

$$\rho_a(k) \triangleq \begin{cases} \sum_{i=0}^{N-k-1} a_i^* a_{i+k}, & 0 \leq k \leq N-1 \\ \sum_{i=0}^{N+k-1} a_i a_{i-k}^*, & -N+1 \leq k < 0 \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

Hence, an upper bound for the instantaneous power can be obtained as $\max_{t \in [0, T_s)} |s_a(t)|^2 = \max_{|z|=1} |A(z)|^2 \leq \rho_a(0) + 2 \sum_{k=1}^{N-1} |\rho_a(k)|$, which indicates that a sequence with a smaller $|\rho_a(k)|$ for $k \neq 0$ also provides less power fluctuations when it is used for OFDM transmission.

Let \mathcal{C} denote a code of length N . Let $\mathbf{c} \in \mathbb{C}^N$ be an admissible sequence, i.e., a codeword, in \mathcal{C} . In this study, we define the PMEPR of the codeword \mathbf{c} as $\max_{t \in [0, T_s)} |s_c(t)|^2 / P_{av}$, where $P_{av} = \mathbb{E}_{\mathbf{c}}[\rho_c(0)]$ is a constant that depends on the code.

B. Sequence Synthesis

A generalized Boolean function is a function f that maps from $\mathbb{Z}_2^m = \{(x_1, x_2, \dots, x_m) | \forall x_j \in \mathbb{Z}_2\}$ to \mathbb{Z}_H as $f: \mathbb{Z}_2^m \rightarrow \mathbb{Z}_H$, where H is an integer. It can be uniquely expressed as a linear combination of the monomials over \mathbb{Z}_H , i.e.,

$$f(\mathbf{x}) = \sum_{k=0}^{2^m-1} c_k \prod_{j=1}^m x_j^{k_j} = c_0 1 + \dots + c_{2^m-1} x_1 x_2 \dots x_m, \quad (2)$$

where $\mathbf{x} \triangleq (x_1, x_2, \dots, x_m)$ and $c_k \in \mathbb{Z}_H$ for $k = \sum_{j=1}^m k_j 2^{m-j}$ and $k_j \in \mathbb{Z}_2$ (i.e., the coefficient of $(k+1)$ th

monomial $x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}$ belongs to \mathbb{Z}_H). The expression given in (2) is called algebraic normal form (ANF) of $f(\mathbf{x})$ [33].

The co-domain of the function f can be extended to \mathbb{R} and \mathbb{Z}_0^+ . In the former case, $f: \mathbb{Z}_2^m \rightarrow \mathbb{R}$ is a pseudo-Boolean function and the sequences generated through the monomials construct a basis in \mathbb{R}^{2^m} and $c_k \in \mathbb{R}$. In the latter case, i.e., $f: \mathbb{Z}_2^m \rightarrow \mathbb{Z}_0^+$, the monomial coefficients are in \mathbb{Z}_0^+ . In this study, we use $\check{\cdot}$ on top of a function to denote a function that maps the non-negative integers to the co-domain of f as $\check{f}(i) \triangleq f \circ e^{-1}(i)$, where $i = e(\mathbf{x}) \triangleq \sum_{j=1}^m x_j 2^{m-j}$, i.e., a decimal representation of the binary number constructed using all elements in the sequence \mathbf{x} , where the most significant bit is x_1 .

Let $f_r: \mathbb{Z}_2^m \rightarrow \mathbb{R}$, $f_i: \mathbb{Z}_2^m \rightarrow \mathbb{R}$, and $f_s: \mathbb{Z}_2^m \rightarrow \mathbb{Z}_0^+$. In this study, we generate a complex sequence $\mathbf{a} = (a_i)_{i=0}^{2^m-1}$ by listing $a_i = \xi^{\check{f}_r(i)+j\check{f}_i(i)}$ for $\xi \triangleq e^{\frac{2\pi}{H}}$. We utilize the function $f_s(\mathbf{x})$ for generating complex sequences with zero-valued elements by modifying the associated polynomial of $\mathbf{a} = (a_i)_{i=0}^{2^m-1}$ as

$$C(z) = \sum_{i=0}^{2^m-1} a_i z^{\check{f}_s(i)+i}.$$

Therefore, the sequence \mathbf{c} can be characterized with $f_r(\mathbf{x})$, $f_i(\mathbf{x})$, and $f_s(\mathbf{x})$.

Let $f: \mathbb{Z}_2^m \rightarrow \mathbb{Z}_2$ be a Boolean function and \mathbf{a} and \mathbf{b} be seed sequences of length N . The polynomial $S(\mathbf{x}, z) = A(z)(1+f(\mathbf{x}))_2 + B(z)f(\mathbf{x})$ is then equal to $A(z)$ for $f(\mathbf{x}) = 0$ and $B(z)$ for $f(\mathbf{x}) = 1$. For $U \in \mathbb{Z}_0^+$, consider the polynomial given by

$$C(z) = \sum_{i=0}^{2^m-1} \check{S}(i, z) a_i z^{\check{f}_s(i)+U i}.$$

For $U = N$ and $\check{f}_s(i) = 0$ for all i , the polynomial indicates that the 2^m sequences (i.e., $A(z)\xi^{\check{f}_r(i)+j\check{f}_i(i)}$ or $B(z)\xi^{\check{f}_r(i)+j\check{f}_i(i)}$ based on $\check{f}(i)$) are concatenated. While the seed sequences are separated by $U - N$ zero-valued elements for $U > N$, the $N - U$ elements of the two adjacent seed sequences are overlapped for $U < N$.

C. Constellations

Let $p_u \in \mathbb{C}$ for $u \in \{1, 2, \dots, S\}$ and $S \in \mathbb{Z}^+$. Let r_u and ψ_u denote the amplitude and the phase of p_u subtracted by $\pi/4$ radian, respectively. We define \mathbb{S}_H as a set of distinct S complex numbers such that $-jp_u^* \in \mathbb{S}_H$ if $|\psi_u| < \pi/H$ for all $p_u \in \mathbb{S}_H$ and $-\pi/H < \psi_u \leq \pi/H$. In other words, the points in \mathbb{S}_H are symmetric with respect to the xy -diagonal of the complex plane (i.e., sub-symmetry), except the points on the line $y = \tan(\pi/H + \pi/4)x$. By rotating all the points in \mathbb{S}_H by an integer multiple of $2\pi/H$ (i.e., equiangularity), we define ESC as follows:

Definition 1. *Equiangular sub-symmetric constellation is $\mathbb{S}_{\text{esc}} = \{p_u \times e^{j2\pi k/H} \mid \forall p_u \in \mathbb{S}_H \text{ and } \forall k \in \mathbb{Z}_H\}$.*

QAM and PSK are ESCs. For example, let $4s^2$ -QAM constellation be a set of points given by $\{(2u-1)+j(2v-1) \mid u, v \in \mathbb{Z}, -s < u \leq s, -s < v \leq s\}$ for $s \in \mathbb{Z}^+$. A $4s^2$ -QAM

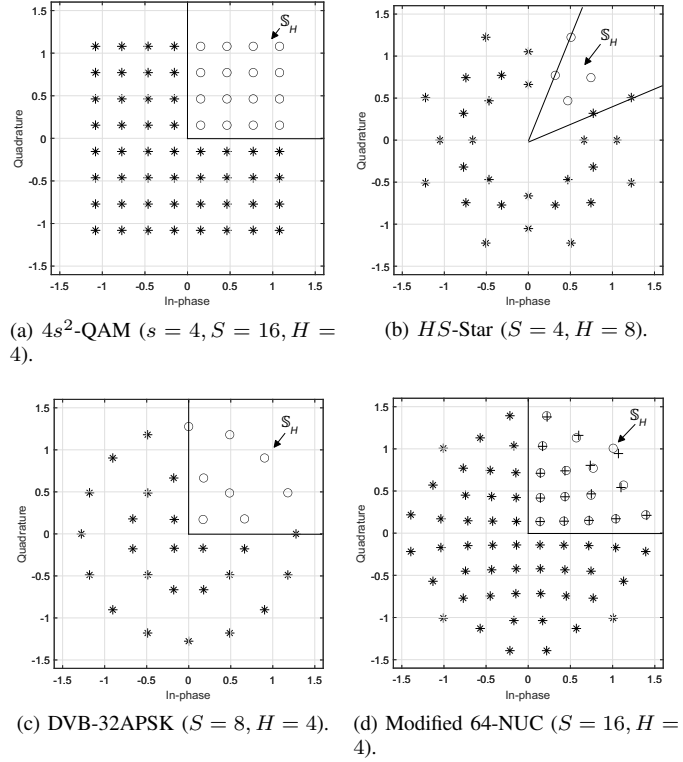


Figure 1. Example ESCs (*: points in \mathbb{S}_{esc} , o: points in \mathbb{S}_H , +: original 64-NUC constellation on the first quadrant).

constellation is an ESC for $H = 4$ and the corresponding \mathbb{S}_H is the set of $S = s^2$ points on the first quadrant as shown in Figure 1(a) for $s = 4$. HS -Star is another ESC defined as the points in \mathbb{S}_H satisfying $r_u < r_{u+1}$ and $\psi_u = \pi/H \times ((u-1) \bmod 2)$ for $u = 1, 2, \dots, S$ as exemplified in Figure 1(b). DVB-16/32APSK defined in [34] are also ESCs for $H = 4$. DVB-32APSK is shown in Figure 1(c). IEEE 802.11ay 64-NUC constellation [35] is not an ESC. However, it can be modified to be an ESC for $H = 4$ by switching the real and imaginary components and calculating the mean of two closest points in 2-D complex plane. The modified IEEE 802.11ay 64-NUC is provided in Figure 1(d).

In this study, we synthesize complex sequences where their non-zero-valued elements in an ESC via $\xi^{\check{f}_r(\mathbf{x})+j\check{f}_i(\mathbf{x})}$ for $f_r: \mathbb{Z}_2^m \rightarrow \mathbb{R}$ and $f_i: \mathbb{Z}_2^m \rightarrow \mathbb{R}$. The fundamental strategy that we use is to offset the monomial coefficients of $f_r(\mathbf{x})$ and $f_i(\mathbf{x})$ such that the elements are in the targeted ESC. For instance, let $f_i(\mathbf{x}) = k' + k_1 x_1$ and $f_r(\mathbf{x}) = e_1 x_1$ for $m = 2$, $k' = \kappa' + \Delta_1$, $k_1 \in \mathbb{Z}_2$, $\kappa' \in \mathbb{Z}_2$, $e_1 \in \mathbb{R}$, and $H = 2$. For $\{\Delta_1 = 0, e_1 = 0\}$, $\{\Delta_1 = \pi/2, e_1 = 0\}$, $\{\Delta_1 = 0, e_1 = \ln(3)/\pi\}$, and $\{\Delta_1 = \pi/2, e_1 = \ln(3)/\pi\}$, the elements of the complex sequence are in $\{-1, 1\}$, $\{-j, j\}$, $\{-3, -1, 1, 3\}$, and $\{-3j, -1j, 1j, 3j\}$, respectively, which forms an ESC of $S = 8$ and $H = 2$.

D. Golay Complementary Pair and Complementary Sequence

The sequence pair (\mathbf{a}, \mathbf{b}) of length N is a GCP if $\rho_a(k) + \rho_b(k) = 0$ for $k \neq 0$. The sequences \mathbf{a} and \mathbf{b} are referred to as CSs. By using the definition of GCP, a GCP (\mathbf{a}, \mathbf{b}) satisfies $|A(z)|^2 + |B(z)|^2 = \rho_a(0) + \rho_b(0)$ for

$|z| = 1$. The instantaneous peak power of the corresponding OFDM signal generated through a CS \mathbf{a} is bounded since $\max_{t \in [0, T_s)} |s_{\mathbf{a}}(t)|^2 \leq \rho_{\mathbf{a}}(0) + \rho_{\mathbf{b}}(0)$.

A general well-known recursive GCP construction can be given as follows:

Theorem 1 ([8]–[11], [32]). *Let (\mathbf{a}, \mathbf{b}) be a GCP of length N . For $\alpha_n, \beta_n \in \mathbb{R}_0^+$, $\tau_n, U \in \mathbb{Z}_0^+$, $\omega_n, \gamma_n, \eta_n \in \{u : u \in \mathbb{C}, |u| = 1\}$ for $n = 1, 2, \dots, m$, and the sequence $\psi = (\psi_n)_{i=1}^m$ defined by a permutation of $\{0, 1, \dots, m-1\}$, assume that the following operations occur at the n th step of a recursion:*

$$\begin{aligned} A^{(n)}(z) &= \gamma_n \alpha_n A^{(n-1)}(z) + \eta_n \beta_n \omega_n B^{(n-1)}(z) z^{\tau_n + U 2^{\psi_n}}, \\ B^{(n)}(z) &= \eta_n^* \beta_n A^{(n-1)}(z) - \gamma_n^* \alpha_n \omega_n B^{(n-1)}(z) z^{\tau_n + U 2^{\psi_n}}, \end{aligned}$$

where $A^{(0)}(z) = A(z)$, $B^{(0)}(z) = B(z)$. The sequences $\mathbf{a}^{(m)}$ and $\mathbf{b}^{(m)}$ associated with the polynomials $A^{(m)}(z)$ and $B^{(m)}(z)$, respectively, construct a GCP for $m \geq 1$.

Theorem 1 shows the Budišin's recursive constructions in [8, Eq. (4)] and [9, Eqs. (3-4)], the García's constructions in [10, Eq. (2)] and [11, Eq. (5)], a special case of Golay's concatenation in [32, Eq. (10)]. The expression in Theorem 1 can also be written in a matrix form as

$$\begin{bmatrix} A^{(n)}(z) \\ B^{(n)}(z) \end{bmatrix} = \begin{bmatrix} \gamma_n \alpha_n & \eta_n \beta_n \omega_n z^{\tau_n + U 2^{\psi_n}} \\ \eta_n^* \beta_n & -\gamma_n^* \alpha_n \omega_n z^{\tau_n + U 2^{\psi_n}} \end{bmatrix} \begin{bmatrix} A^{(n-1)}(z) \\ B^{(n-1)}(z) \end{bmatrix},$$

which shows that CS construction can involve 2-by-2 paraunitary matrices. The special cases of the paraunitary matrices are investigated in the literature. For example, [27] and [29] consider the case $\{\mathbf{a} = \mathbf{b} = 1, \tau_n = 0, U = 1$ for $n = 1, 2, \dots, m\}$. Both [27] and [29] show that Davis and Jedwab's construction can also be derived as a special case of paraunitary matrices, which implies that Theorem 1 is general enough to construct standard CSs.

To utilize CSs in communications, their distinctness must be ensured. However, the recursion in Theorem 1 (and the corresponding paraunitary matrix representation) by itself does not guarantee distinct CSs for different parameters [27, Section IV] after m iterations:

Example 1. Assume that $\mathbf{a} = \mathbf{b} = 1$, $U = 1$, $m = 3$, $\psi = (1, 2, 3)$, $\tau_n = 0$, $\gamma_n = 0$, and $\eta_n = 0$ for $n = 1, 2, 3$. Consider two different configurations: $\{(\alpha_1, \alpha_2, \alpha_3) = (1, 2, 3), (\beta_1, \beta_2, \beta_3) = (1/2, 2, 3)\}$ and $\{(\alpha_1, \alpha_2, \alpha_3) = (3, 1, 2), (\beta_1, \beta_2, \beta_3) = (3/2, 1, 2)\}$. Both configurations result in the same CS $\mathbf{a}^{(3)} = (6, 3, 3, -6, 6, 6, 3, -3, 6)$.

To resolve this issue, in [27], the distinct CSs are generated by using the properties of Gaussian integers for QAM alphabet. In this study, we eliminate the redundancy in Theorem 1 by using pseudo-Boolean functions, which leads to a generalization of Davis and Jedwab's construction.

E. Operators

An operator O transforms one polynomial to another polynomial. If $O\{aA(z) + bB(z)\} = aO\{A(z)\} + bO\{B(z)\}$ for all $A(z)$ and $B(z)$ and $a, b \in \mathbb{C}$, the operator O is linear. Several examples of linear operators are $O\{R(z)\} = z^d R(z)$,

$O\{R(z)\} = -R(z)$, $O\{R(z)\} = \xi^a R(z)$, and $O\{R(z)\} = \xi^{ib} R(z)$ for $d \in \mathbb{Z}_0^+$ and $a, b \in \mathbb{R}$, where $R(z)$ is an arbitrary polynomial.

Let O_0 and O_1 be two linear operators and $f : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_2$ be a Boolean function. Consider the polynomial given by $P^{(m)}(z) = \sum_{i=0}^{2^m-1} O_{\tilde{f}(i)}\{R(z)\} z^i$. The Boolean function $\tilde{f}(i)$ determines which one of the two operators affects the coefficient of z^i . Thus, one can simply define the polynomial $P^{(m)}(z)$ with the operators O_0 and O_1 and the Boolean function f . We use this notation to define the polynomials as shown in Table I.

III. REPRESENTATION OF A RECURSION

Let $P^{(n)}(z)$ and $Q^{(n)}(z)$ for $n = 1, 2, \dots, m$ be the polynomials generated by

$$\begin{aligned} P^{(n)}(z) &= O_{11}^{(n)}\{P^{(n-1)}(z)\} + O_{12}^{(n)}\{Q^{(n-1)}(z)\} z^{2^{\psi_n}}, \\ Q^{(n)}(z) &= O_{21}^{(n)}\{P^{(n-1)}(z)\} + O_{22}^{(n)}\{Q^{(n-1)}(z)\} z^{2^{\psi_n}}, \end{aligned} \quad (3)$$

where ψ_n is the n th element of the sequence $\psi \triangleq (\psi_n)_{i=1}^m$ defined by a permutation of $\{0, 1, \dots, m-1\}$, $O_{ij}^{(n)} \in \{O_0^{(n)}, O_1^{(n)}\}$ is a linear operator, and $P^{(0)}(z) = Q^{(0)}(z) = R(z)$. Since ψ includes all non-negative integers less than m and $O_0^{(n)}$ and $O_1^{(n)}$ are linear operators, $P^{(m)}(z)$ and $Q^{(m)}(z)$ can be obtained as

$$P^{(m)}(z) = \sum_{i=0}^{2^m-1} \overbrace{O_{\tilde{f}_m(i)}^{(m)} \dots O_{\tilde{f}_2(i)}^{(2)} O_{\tilde{f}_1(i)}^{(1)}}^{\mathcal{F}_i\{R(z)\}} \{R(z)\} z^i, \quad (4)$$

and

$$Q^{(m)}(z) = \sum_{i=0}^{2^m-1} \overbrace{O_{\tilde{g}_m(i)}^{(m)} \dots O_{\tilde{g}_2(i)}^{(2)} O_{\tilde{g}_1(i)}^{(1)}}^{\mathcal{G}_i\{R(z)\}} \{R(z)\} z^i, \quad (5)$$

where the Boolean functions $f_n : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_2$ and $g_n : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_2$ indicate which of the two operators, i.e., $O_0^{(n)}$ and $O_1^{(n)}$, are involved in the constructions of the composite operators $\mathcal{F}_i\{R(z)\}$ and $\mathcal{G}_i\{R(z)\}$ by setting the subscripts of $O_{\tilde{f}_n(i)}^{(n)}$ and $O_{\tilde{g}_n(i)}^{(n)}$, respectively. The Boolean functions can be obtained in closed-form as follows:

Lemma 1. *Let \mathbf{b}_n be a configuration vector defined by $\mathbf{b}_n^T \triangleq [b_{11}^{(n)} \ b_{12}^{(n)} \ b_{21}^{(n)} \ b_{22}^{(n)}] \in \mathbb{Z}_2^4$, where $b_{ij}^{(n)} = 0$ if $O_{ij}^{(n)} = O_0^{(n)}$ and $b_{ij}^{(n)} = 1$ if $O_{ij}^{(n)} = O_1^{(n)}$. Then, $f_n(\mathbf{x})$ and $g_n(\mathbf{x})$ are the Boolean functions given by*

$$f_n(\mathbf{x}) = \begin{cases} b_{11}^{(n)}(1 - x_{\pi_n}) + b_{12}^{(n)} x_{\pi_n}, & n = m \\ b_{11}^{(n)}(1 - x_{\pi_n})(1 - x_{\pi_{n+1}}) \\ \quad + b_{12}^{(n)} x_{\pi_n}(1 - x_{\pi_{n+1}}) \\ \quad + b_{21}^{(n)}(1 - x_{\pi_n})x_{\pi_{n+1}} \\ \quad + b_{22}^{(n)} x_{\pi_n}x_{\pi_{n+1}}, & n < m \end{cases}, \quad (6)$$

$$g_n(\mathbf{x}) = \begin{cases} b_{21}^{(n)}(1 - x_{\pi_n}) + b_{22}^{(n)} x_{\pi_n}, & n = m \\ b_{11}^{(n)}(1 - x_{\pi_n})(1 - x_{\pi_{n+1}}) \\ \quad + b_{12}^{(n)} x_{\pi_n}(1 - x_{\pi_{n+1}}) \\ \quad + b_{21}^{(n)}(1 - x_{\pi_n})x_{\pi_{n+1}} \\ \quad + b_{22}^{(n)} x_{\pi_n}x_{\pi_{n+1}}, & n < m \end{cases}, \quad (7)$$

for $n = 1, 2, \dots, m$, where $\pi_n = m - \psi_n$ is the n th element of the sequence $\boldsymbol{\pi} \triangleq (\pi_n)_{n=1}^m$.

The proof for Lemma 1 is given in Appendix A. Since there exist 16 possible options for the n th configuration vector \mathbf{b}_n , there also exist 16 different Boolean functions for each step in (3).

Example 2. Assume the operator configuration given by $\mathcal{O}_{11}^{(n)} = \mathcal{O}_{12}^{(n)} = \mathcal{O}_{21}^{(n)} = \mathcal{O}_0^{(n)}$ and $\mathcal{O}_{22}^{(n)} = \mathcal{O}_1^{(n)}$ for all iterations, $m = 2$, and $\boldsymbol{\psi} = (0, 1)$. When $n = 1$, $P^{(1)}(z) = \mathcal{O}_0^{(1)}\{R(z)\} + \mathcal{O}_0^{(1)}\{R(z)\}z$ and $Q^{(1)}(z) = \mathcal{O}_0^{(1)}\{R(z)\} + \mathcal{O}_1^{(1)}\{R(z)\}z$. When $n = 2$, $P^{(2)}(z) = \mathcal{O}_0^{(2)}\mathcal{O}_0^{(1)}\{R(z)\} + \mathcal{O}_0^{(2)}\mathcal{O}_0^{(1)}\{R(z)\}z + \mathcal{O}_0^{(2)}\mathcal{O}_1^{(1)}\{R(z)\}z^2 + \mathcal{O}_0^{(2)}\mathcal{O}_1^{(1)}\{R(z)\}z^3$ and $Q^{(2)}(z) = \mathcal{O}_0^{(2)}\mathcal{O}_0^{(1)}\{R(z)\} + \mathcal{O}_0^{(2)}\mathcal{O}_1^{(1)}\{R(z)\}z + \mathcal{O}_1^{(2)}\mathcal{O}_0^{(1)}\{R(z)\}z^2 + \mathcal{O}_1^{(2)}\mathcal{O}_1^{(1)}\{R(z)\}z^3$. By investigating the distribution of the operators at the final polynomial, the Boolean functions related to the first and the second steps should produce the sequences $\mathbf{f}_1 = (0, 0, 0, 1)$, $\mathbf{f}_2 = (0, 0, 0, 0)$, $\mathbf{g}_1 = (0, 0, 0, 1)$, and $\mathbf{g}_2 = (0, 0, 1, 1)$.

Example 3. Consider Example 2. Since the recursion is configured as $\mathcal{O}_{11}^{(n)} = \mathcal{O}_{12}^{(n)} = \mathcal{O}_{21}^{(n)} = \mathcal{O}_0^{(n)}$ and $\mathcal{O}_{22}^{(n)} = \mathcal{O}_1^{(n)}$ for all iterations, the configuration vector is obtained as $\mathbf{b}_n^T = [0 \ 0 \ 0 \ 1]$ for $n = \{1, 2\}$. Hence, by using (6) and (7), $f_n(\mathbf{x})$ and $g_n(\mathbf{x})$ are obtained as

$$f_n(\mathbf{x}) = \begin{cases} 0, & \text{if } n = m \\ x_{\pi_n} x_{\pi_{n+1}}, & \text{if } n < m \end{cases},$$

and

$$g_n(\mathbf{x}) = \begin{cases} x_{\pi_m}, & \text{if } n = m \\ x_{\pi_n} x_{\pi_{n+1}}, & \text{if } n < m \end{cases},$$

respectively. Since $\pi_n = m - \psi_n$ and $\boldsymbol{\psi} = (0, 1)$, the Boolean functions that lead to the sequences $\mathbf{f}_1, \mathbf{f}_2, \mathbf{g}_1$, and \mathbf{f}_2 are obtained as $f_1(\mathbf{x}) = x_{\pi_1} x_{\pi_2} = x_2 x_1$, $f_2(\mathbf{x}) = 0$, $g_1(\mathbf{x}) = x_{\pi_1} x_{\pi_2} = x_2 x_1$, and $f_2(\mathbf{x}) = x_{\pi_2} = x_1$, respectively, which result in the same sequences obtained in Example 2. Now assume that $\mathcal{O}_0^{(n)}\{R(z)\} = R(z)$ and $\mathcal{O}_1^{(n)}\{R(z)\} = -R(z)$. Since $\xi^{jH/2} = -1$, we can then express the composite operator as $\mathcal{F}_i\{R(z)\} = \xi^{j\tilde{f}(i)} \times R(z)$, where the underlying function determining the sign is $f(\mathbf{x}) = \frac{H}{2} \sum_{n=1}^{m-1} x_{\pi_n} x_{\pi_{n+1}}$.

Lemma 1 can be used for any recursion in the form given in (3) to identify underlying code. For example, the underlying code for m -bit Gray code can be obtained as follows:

Example 4 (Gray Code). Assume the operator configuration given by $\mathcal{O}_{11}^{(n)} = \mathcal{O}_{22}^{(n)} = \mathcal{O}_0^{(n)}$ and $\mathcal{O}_{12}^{(n)} = \mathcal{O}_{21}^{(n)} = \mathcal{O}_1^{(n)}$ for all iterations and $\boldsymbol{\psi} = (0, 1, \dots, m-1)$. In this case, the configuration vector is $\mathbf{b}_n^T = [0 \ 1 \ 1 \ 0]$ for $n = 1, 2, \dots, m$. Hence, by using (6), $f_n(\mathbf{x})$ is obtained as

$$f_n(\mathbf{x}) = \begin{cases} x_{\pi_m}, & \text{if } n = m \\ (x_{\pi_n} + x_{\pi_{n+1}})_2, & \text{if } n < m \end{cases}, \quad (8)$$

where $\boldsymbol{\pi} = (m, m-1, \dots, 1)$. As the recursion combines two polynomials after applying $\mathcal{O}_1^{(n)}$ to the second polynomial and the positions of $(n-1)$ th operators are reflected in $P^{(n-1)}(z)$

and $Q^{(n-1)}(z)$, $f_n(\mathbf{x})$ for $n = 1, 2, \dots, m$ form the basis vectors of m -bit Gray code, e.g., $\mathbf{f}_1 = (0, 1, 1, 0, 0, 1, 1, 0)$, $\mathbf{f}_2 = (0, 0, 1, 1, 1, 1, 0, 0)$, and $\mathbf{f}_3 = (0, 0, 0, 0, 1, 1, 1, 1)$ for $m = 3$.

IV. A GENERIC CS CONSTRUCTION

In this section, we first utilize Lemma 1 to develop a direct GCP construction by re-expressing the recursion given in Theorem 1. We then discuss how to generate distinct CS without an auxiliary method and compare it with other constructions in the literature through examples.

Theorem 2 (Main Theorem). Let (\mathbf{a}, \mathbf{b}) be a GCP of length $N \geq 1$ and $\boldsymbol{\pi} = (\pi_n)_{n=1}^m$ be a sequence defined by a permutation of $\{1, 2, \dots, m\}$. For any $H \in \mathbb{Z}^+$, $e_n, e' \in \mathbb{R}$, and $k_n, k', k'' \in [0, H]$ for $n = 1, 2, \dots, m$ and $\xi = e^{\frac{2\pi}{H}}$, let

$$f_r(\mathbf{x}) = e_m x_{\pi_m} + \sum_{n=1}^{m-1} e_n (x_{\pi_n} + x_{\pi_{n+1}})_2 + e', \quad (9)$$

$$g_r(\mathbf{x}) = e_m (1 + x_{\pi_m})_2 + \sum_{n=1}^{m-1} e_n (x_{\pi_n} + x_{\pi_{n+1}})_2 + e', \quad (10)$$

$$f_i(\mathbf{x}) = \frac{H}{2} \sum_{n=1}^{m-1} x_{\pi_n} x_{\pi_{n+1}} + \sum_{n=1}^m k_n x_{\pi_n} + k', \quad (11)$$

$$g_i(\mathbf{x}) = \frac{H}{2} \left(x_{\pi_m} + \sum_{n=1}^{m-1} x_{\pi_n} x_{\pi_{n+1}} \right)_2 + \sum_{n=1}^m k_n x_{\pi_n} + k'', \quad (12)$$

$$f_s(\mathbf{x}) = \sum_{n=1}^m d_n x_{\pi_n}, \quad (13)$$

$$S(\mathbf{x}, z) = A(z)(1 + x_{\pi_1})_2 + B(z)x_{\pi_1}. \quad (14)$$

Then, the sequences \mathbf{c} and \mathbf{d} associated with the polynomials given by

$$C(z) = \sum_{i=0}^{2^m-1} \check{S}(i, z) \xi^{\tilde{f}_r(i) + j\tilde{f}_i(i)} z^{\tilde{f}_s(i) + U i}, \quad (15)$$

$$D(z) = \sum_{i=0}^{2^m-1} \check{S}(i, z) \xi^{\tilde{g}_r(i) + j\tilde{g}_i(i)} z^{\tilde{f}_s(i) + U i}, \quad (16)$$

construct a GCP.

The proof is given in Appendix B in detail.

Theorem 2 shows how each element of the generated CSs is formed explicitly as compared to Theorem 1. As described in Section II, while $f_r(\mathbf{x})$ alters the amplitude of $\xi^{f_r(\mathbf{x}) + j\tilde{f}_i(\mathbf{x})}$, $f_i(\mathbf{x})$ changes its phase. The function $f_s(\mathbf{x})$ determines the position of the seed sequences in the constructed CS by offsetting the order of the polynomial $S(\mathbf{x}, z)$. The seed sequences are also scaled by different complex values, i.e., $\xi^{f_r(\mathbf{x}) + j\tilde{f}_i(\mathbf{x})}$ and $\xi^{g_r(\mathbf{x}) + j\tilde{g}_i(\mathbf{x})}$.

The proof of Theorem 2 results in the following conclusions:

Corollary 1. Let $\alpha_n = \xi^{a_n}$, $\beta_n = \xi^{b_n}$, $\gamma_n = \xi^{j\tilde{c}_n}$, $\eta_n = \xi^{j\tilde{c}_n}$, and $\omega_n = \xi^{j\tilde{c}_n}$. Theorem 1 and Theorem 2 lead to identical GCP if $\pi_n = m - \psi_n$, $d_{|n|} = \tau_n$, $e_{|n|} = b_n - a_n$ for $n =$

$1, 2, \dots, m$, $e' = \sum_{n=1}^m a_n$, $k_1 = c_1 + \check{c}_1 - \dot{c}_1$, $k_{n|n=2,3,\dots,m} = c_n + \check{c}_n - \dot{c}_n - \check{c}_{n-1} - \dot{c}_{n-1}$, $k' = \sum_{n=1}^m \dot{c}_n$, and $k'' = -\check{c}_m + \sum_{n=1}^{m-1} \dot{c}_n$.

Corollary 2. *The length of the constructed CSs in (15) and (16) are $U(2^m - 1) + N + \sum_{n=1}^m d_n$.*

Proof. The length is equal to $\max_i U_i + N + \check{f}_s(i)$, which gives $U(2^m - 1) + N + \sum_{n=1}^m d_n$ since the function $\check{f}_s(i)$ gets its maximum and minimum values when i is maximum, i.e., $2^m - 1$, and minimum, i.e., 0, respectively. \square

In the rest of the study, we provide our discussions based on $C(z)$ as $C(z)$ and $D(z)$ share the same properties.

A. Distinct CSs with a Fixed Support

1) *Avoiding Overlapping:* For a given i , the order of the polynomial $\check{S}(i, z)z^{U_i + \check{f}_s(i)}$ is $U_i + \check{f}_s(i) + N - 1$. Hence, the seed sequences are placed side-by-side for $U = N$ and $\check{f}_s(i) = 0$ for all i . On the other hand, for different values of i , the resulting polynomials can consist of the same monomials for $\check{f}_s(i) \neq 0$ or $U \neq N$. Thus, $f_s(\mathbf{x})$ can change the phase and amplitude of the elements through the superposition of the seed sequences. This issue can be avoided if

$$d_{k|\pi_k=a} \geq \sum_{\ell \in \{l|\pi_l > a\}} d_\ell, \quad U \geq N, \quad (17)$$

for $1 \leq a \leq m - 1$. For example, for $\pi = (m, m - 1, \dots, 1)$, several conditions can be listed as $d_m \geq d_{m-1} + d_{m-2} + \dots + d_1$ for $a = 1$, $d_{m-1} \geq d_{m-2} + d_{m-3} + \dots + 1$ for $a = 2$, and $d_2 \geq d_1$ for $a = m - 1$. The conditions in (17) can be derived by evaluating the Boolean functions in (13). For instance, x_1 in (13) shifts the last 2^{m-1} seed sequences by $d_{k|\pi_k=1}$ as the last 2^{m-1} elements of the corresponding sequence for x_1 is 1. Therefore, it gives a room of size $d_{k|\pi_k=1}$ for other shift parameters for $U = N$ as expressed in (17) for $a = 1$.

2) *Fixed Support:* Let $f_s(\mathbf{x})$ be a fixed function for all π by using a fixed set of d_{π_n} for $n = 1, 2, \dots, m$. In this case, the supports of the seed sequences in the CS do not change for different π values, which is instrumental to generate CSs with a certain support.

3) *Distinct Sequences:* Let $f_s(\mathbf{x})$ be a fixed function where its coefficients satisfy (17). Then, the superpositions of the seed sequences are avoided and the supports of the seed sequences are fixed. Therefore, the functionality of $f_s(\mathbf{x})$ in (15) is completely separated the ones for $f_i(\mathbf{x})$, $f_r(\mathbf{x})$, and $S(\mathbf{x}, z)$. Hence, for distinct $f_r(\mathbf{x})$, $f_i(\mathbf{x})$, or $S(\mathbf{x}, z)$, the CS \mathbf{c} is distinct.

The functions $f_r(\mathbf{x})$, $f_i(\mathbf{x})$, or $S(\mathbf{x}, z)$ lead to sets of distinct sequences for a given π . On the other hand, for different π , their combined impact result in distinct CSs under certain conditions:

- For $N = 1$, when the elements of π are reversed, $\sum_{n=1}^{m-1} x_{\pi_n} x_{\pi_{n+1}}$ does not change. Therefore, π and the reversed π do not change the set of sequences generated by $f_i(\mathbf{x})$ if $k_n, k' \in \mathbb{Z}_H$. However, if one of $\{k_n\}$ is offset by a non-integer value for even m , π and the reversed π lead to different set of sequences. For odd m , π and

the reversed π can be used if one of $\{k_{n|n \neq \lfloor m/2 \rfloor}\}$ is offset by a non-integer value. Also, all permutations generate a set of distinct CSs when $e_m \neq e_{n \neq m}$ since $f_r(\mathbf{x})$ yields different functions for π and reversed π even $f_i(\mathbf{x})$ remains unchanged.

- For $N > 1$ ($\mathbf{a} \neq k\mathbf{b}$ for $k \in \mathbb{C}$), each permutation lead to a distinct set of CSs. This is because the polynomial $S(\mathbf{x}, z)$ is a function of x_{π_1} . When π is reversed, the placements of the seed sequences in the CS \mathbf{c} changes.

B. Comparisons with Other Constructions and Examples

Theorem 2 under the condition (17) provides further insight into CSs as follows:

1) *Independent functions for amplitude and phase:* Theorem 2 extends the Davis and Jedwab's construction [4] by providing independent functions that manipulate the amplitude and phase of the elements of the generated CSs, where the one for phase coincides with the result in [4]. The parameters e_n for $n = 1, 2, \dots, m$ and e' change the real part of the exponent of ξ , which is beneficial to synthesize CSs with various amplitude levels without affecting the phase of the elements. The offset method, e.g., [20]–[26], [36]–[39], and para-unitary matrix-based construction, e.g., [27], do not yield concise expressions related to the phase, amplitude, or seed sequences for synthesizing CSs. As we shown in Section V and Section VI, the independent pseudo-Boolean functions for amplitude and phase can be exploited to synthesize CSs with various constellation and develop algorithms for receiver.

Example 5. Let $m = 3$, $U = 1$, $\pi = (3, 2, 1)$, $e_1 = \frac{H}{2\pi} \ln(3)$, $e_{n|n=2,3} = 0$, $e' = 0$, $H = 4$, $k_{n|n=1,2,3} = 0$, $k' = 0$, $d_{n|n=1,2,3} = 0$, $\mathbf{a} = \mathbf{b} = (1)$. Based on Theorem 2, the basis vectors for $f_r(\mathbf{x})$ can be listed as $x_{\pi_3} = (0, 0, 0, 0, 1, 1, 1, 1)$, $x_{\pi_2} + x_{\pi_3} = (0, 0, 1, 1, 1, 1, 0, 0)$, $x_{\pi_1} + x_{\pi_2} = (0, 1, 1, 0, 0, 1, 1, 0)$, $\mathbf{1} = (1, 1, 1, 1, 1, 1, 1, 1)$. The function $f_r(\mathbf{x})$ leads to the sequence $(0, e_1, e_1, 0, 0, e_1, e_1, 0)$ while the function $f_i(\mathbf{x})$ gives the sequence $(0, 0, 0, 2, 0, 0, 2, 0)$ independently. The function $S(\mathbf{x}, z) = A(z)(1 - x_{\pi_1})_2 + B(z)x_{\pi_1} = A(z)(1 - x_3)_2 + B(z)x_3 = 1$ results in $(1, 1, 1, 1, 1, 1, 1, 1)$. As $d_{n|n=1,2,3} = 0$, $f_s(\mathbf{x}) = 0$. Finally, combining all functions with (15), the sequence \mathbf{c} is obtained as $(1, 3, 3, -1, 1, 3, -3, 1)$.

Note that for $\pi = (m, m - 1, \dots, 1)$, the amplitude of the elements of the CS are determined by the linear combinations of the columns of the m -bit Gray code (see (8) and (9)) over \mathbb{R} since the variables that change the amplitudes in Theorem 1 alternate as in Example 4.

2) *Enumeration under the presence of a seed GCP:* Theorem 2 gives contiguous CSs of length $N \cdot 2^m$ when a seed GCP (\mathbf{a}, \mathbf{b}) of length $N > 1$ is utilized for $U = N$ and $f_s(\mathbf{x}) = 0$. It reduces to the method in [4] for $f_r(\mathbf{x}) = 0$, $k_n, k', k'' \in \mathbb{Z}_H$, and $\mathbf{a} = \mathbf{b} = 1$. However, under the same configuration for $N > 1$, it determines $m! \cdot H^{m+1}$ CSs, i.e., doubles the number in [4], as all permutations are valid for generating distinct CSs in the presence of the seed sequences.

Example 6. Consider the parameters in Example 5 and replace \mathbf{a} and \mathbf{b} as $\mathbf{a} = (1, j, 1, 1, 1, -1)$, and $\mathbf{b} = (1, j, 1, -1, -1, 1)$

[40]. For $\pi = (3, 2, 1)$ and $U = 6$, the function $S(\mathbf{x}, z) = A(z)(1-x_{\pi_1})_2 + B(z)x_{\pi_1} = A(z)(1-x_3)_2 + B(z)x_3$ results in $(A(z), B(z), A(z), B(z), A(z), B(z), A(z), B(z))$. Therefore, the resulting sequence \mathbf{c} is length of 48 and can be expressed as $(\mathbf{a}, 3 \cdot \mathbf{b}, 3 \cdot \mathbf{a}, -\mathbf{b}, \mathbf{a}, 3 \cdot \mathbf{b}, -3 \cdot \mathbf{a}, \mathbf{b})$ from (15). Now assume the order of the elements in π are reversed, $e_{n|n=1,3} = 0$, and $e_2 = \frac{2}{\pi} \ln(3)$. The function $S(\mathbf{x}, z) = A(z)(1-x_1)_2 + B(z)x_1$ then gives $(A(z), A(z), A(z), A(z), B(z), B(z), B(z), B(z))$, while $f_r(\mathbf{x})$ and $f_i(\mathbf{x})$ remain the same. The final sequence is obtained as $(\mathbf{a}, 3 \cdot \mathbf{a}, 3 \cdot \mathbf{a}, -\mathbf{a}, \mathbf{b}, 3 \cdot \mathbf{b}, -3 \cdot \mathbf{b}, \mathbf{b})$, which is different from the original one as the positions of the seed sequences changes.

In [19], the multi-dimensional arrays were exploited for generating CSs by projecting the arrays to a lower dimension, where the projection operation re-orders the elements of the multi-dimensional arrays. As compared to [19], we assume that the seed CSs are given in this study. Our construction can be inferred as a projection from two-dimensional array to a single dimension in [19] for $U = N$. However, for $U \neq N$, our construction extends the definition of projection by allowing zero-valued elements or summation of the elements of the seed CSs. We also identify the seed sequence positions for a given π , which is useful for multiplexing as done in Section VII.

3) *Flexible support*: In the literature, the constructions mainly focus on contiguous CSs, i.e., CSs with no zero-valued elements [20]–[27], [36]–[39]. In contrast, Theorem 2 explains how to generate distinct non-contiguous CSs under the condition given in (17). In [30], various resource allocations for complementary sets were considered for preamble design. However, the maximum PMEPR of the resulting waveform reaches 4 dB. In [41], several correlation bounds under non-contiguous resource allocation and several constructions were discussed. However, systematic construction of the zero-valued elements for CSs with (13) was not investigated in these studies.

Example 7. Consider the parameters given in Example 5. Let $\mathbf{a} = (1, j, 1)$, $\mathbf{b} = (1, 1, -1)$ [40] and $U = 3$. The final sequence \mathbf{c} is $(\mathbf{a}, 3 \cdot \mathbf{b}, 3 \cdot \mathbf{a}, -\mathbf{b}, \mathbf{a}, 3 \cdot \mathbf{b}, -3 \cdot \mathbf{a}, \mathbf{b})$. Now assume that $d_3 = 60$ and $d_{n|n=1,2} = 0$. The function $f_s(\mathbf{x})$ leads to $(0, 0, 0, 0, 60, 60, 60, 60)$. Hence, from (15), the support of the last four parts of \mathbf{c} are shifted by 60. The resulting sequence can be expressed as

$$\mathbf{c} = (\mathbf{a}, 3 \cdot \mathbf{b}, 3 \cdot \mathbf{a}, -\mathbf{b}, \overbrace{0, \dots, 0}^{d_1=60}, \mathbf{a}, 3 \cdot \mathbf{b}, -3 \cdot \mathbf{a}, \mathbf{b}).$$

In other words, a non-contiguous CS with two clusters of length of 12 separated by $d_2 = 60$ zero-valued elements is obtained where the alphabet on the clusters remains the same. Under the same condition, assume that $\pi = (1, 2, 3)$. Let $d_1 = 60$ and $d_{n|n=2,3} = 0$. Since $f_s(\mathbf{x})$ still gives $(0, 0, 0, 0, 60, 60, 60, 60)$ the resulting sequence keeps the original support as

$$\mathbf{c} = (\mathbf{a}, 3 \cdot \mathbf{a}, 3 \cdot \mathbf{a}, -\mathbf{a}, \overbrace{0, \dots, 0}^{d_3=60}, \mathbf{b}, 3 \cdot \mathbf{b}, -3 \cdot \mathbf{b}, \mathbf{b})$$

while the position of the seed sequence change.

4) *Continuous parameters*: Theorem 2 provides a framework for synthesizing CSs with various alphabets through the continuous parameters e_n and e' , and k_n and k' , as opposed to existing work on CSs [15], [26], [27]. For example, since k_n and k' are in $[0, H)$, i.e., a range rather than \mathbb{Z}_H , it offers a flexibility in phase. Hence, it is possible to generate CSs with various alphabets as discussed in Section V in detail.

V. COMPLEMENTARY SEQUENCES WITH EQUIANGULAR SUB-SYMMETRIC CONSTELLATIONS

Consider two points p_u and p_v in \mathbb{S}_H indexed by $u, v \in \{1, 2, \dots, S\}$, where H is an even number¹. By choosing $e' = \frac{H}{2\pi} \ln r_u$ and $e_l = \frac{H}{2\pi} \ln \frac{r_v}{r_u}$ while $e_{n|n \neq l} = 0$ for $n \in \{1, 2, \dots, m\}$ in (9) for $l \in \{1, \dots, m\}$, we equate the amplitudes of the elements of the CSs to either r_u or r_v . Since our construction also admits real values for (11), we set $k' = \kappa' + \frac{H}{2\pi} \Delta'$ and $k_n = \kappa_n + \frac{H}{2\pi} \Delta_n$ for $\Delta', \Delta_n \in \mathbb{R}_0^+$ and $\kappa', \kappa_n \in \mathbb{Z}_H$ for $n \in \{1, 2, \dots, m\}$. While l determines the parameters to be offset, u and v identify where to offset.

Consider the case $u = v$. Due to the equiangularity property, an ESC consists of the points in \mathbb{S}_H rotating with an integer multiple of $2\pi/H$ radian. Hence, by offsetting κ' by $\Delta' = \psi_u$, the rotated and scaled H -PSK GDJ sequences on an ESC alphabet can be obtained. Since there are S elements in \mathbb{S}_H , only S valid permutations exist for (u, v) , where each permutation generates $G \triangleq \frac{m!}{2} H^{m+1}$ sequences for $N = 1$ and $2G$ sequences for $N > 1$, based on Section IV-A3.

Now, consider the case $u \neq v$. Assume that $k_1 = q_1$, $k_n = q_n \pm q_{n-1}$ for $n = 2, \dots, m$, and $k' = q'$. We then write $\sum_{n=1}^m k_n x_{\pi_n} + k' = q_m x_{\pi_m} + \sum_{n=1}^{m-1} q_n ((x_{\pi_n})_2 \pm (x_{\pi_{n+1}})_2) + q'$, which allows us to express the phase and amplitude functions in similar forms. For example, if $k_n = q_n - q_{n-1}$,

$$(x_{\pi_n})_2 - (x_{\pi_{n+1}})_2 \in \begin{cases} \{0\}, & \text{if } (x_{\pi_n} + x_{\pi_{n+1}})_2 = 0 \\ \{-1, 1\}, & \text{if } (x_{\pi_n} + x_{\pi_{n+1}})_2 = 1 \end{cases}. \quad (18)$$

Therefore, q_l and e_l alter the same elements of the sequence. Since the non-zero values of $(x_{\pi_l})_2 - (x_{\pi_{l+1}})_2$ are either -1 or 1 , the phase function rotates the elements scaled with e_l in the counterclockwise and the clockwise directions equally on the unit circle. For $q_l = \psi_v$, the amount of the phase rotation is ψ_v . If $k_n = q_n + q_{n-1}$,

$$1 - (x_{\pi_n})_2 - (x_{\pi_{n+1}})_2 \in \begin{cases} \{0\}, & \text{if } (x_{\pi_n} + x_{\pi_{n+1}})_2 = 1 \\ \{-1, 1\}, & \text{if } (x_{\pi_n} + x_{\pi_{n+1}})_2 = 0 \end{cases}, \quad (19)$$

which implies that the amount of phase rotation in the counterclockwise and the clockwise directions for the elements that are *not* scaled by e_l is ψ_u for $q' = \psi_u$ and $q_l = -\psi_u$.

The rotation in the counterclockwise and the clockwise directions equally is exactly aligned with the sub-symmetry property of an ESC. Equations (18) and (19) allow us to alter the different elements of the sequence, for given l, u , and v .

¹As the coset term in (11) can flip the sign of the elements of CSs, we ensure that p_u and $-p_u$ are in \mathbb{S}_{esc} with an even H as in [15].

Their combined impacts on the phase offset parameters can be calculated as $\Delta' = \psi_u$, $\Delta_l = \psi_v - \psi_u$, $\Delta_{l+1} = -\psi_v - \psi_u$ if $l < m$, and $\Delta_{n|n \neq l, l+1} = 0$ for $l \in \{1, 2, \dots, m\}$.

For $N > 1$, the total number of (u, v) permutations where $u \neq v$ is $S^2 - S$ and $l \in \{1, 2, \dots, m\}$. This implies that the total number CSs with an ESC generated through the sub-symmetry is $2((S^2 - S)m) \cdot G$. For $N = 1$, π and the reversed π give different CSs when $r_u \neq r_v$ for $l = m$ based on the discussions in Section IV-A3. In addition, π and the reversed π yield different CSs for $l = m$ if $\Delta_m = \psi_v - \psi_u \neq 0$ and $\psi_u + \psi_v \neq 0$, which implies that $|\psi_u| \neq |\psi_v|$ should hold. Since these special cases generate $2G$ different CSs for $l = m$ while others lead to G different CSs, the total number of CSs can be calculated as $((S^2 - S)m + A + B) \cdot G$, where A and B are the number of (u, v) permutations for $r_u \neq r_v$ and the number of (u, v) permutations for $r_u = r_v$ and $|\psi_u| \neq |\psi_v|$, respectively.

We summarize how the phase and amplitude parameters are modified for given u, v , and l for an arbitrary ESC in Algorithm 1. The number of distinct CSs are calculated as follows:

Corollary 3. *The numbers of distinct CSs with an ESC of $H \cdot S$ points are at least $((S^2 - S)m + A + B + S) \cdot G$ and $2((S^2 - S)m + S) \cdot G$ for $N = 1$ and $N > 1$, respectively.*

Note that non-contiguous CSs with a certain support can be generated by fixing $f_s(\mathbf{x})$ as in Example 7. Hence, for a given support, the number of distinct non-contiguous CSs is equal to the number of distinct contiguous CSs.

A. Spectral Efficiency and Minimum Distance

Based on Corollary 3, the spectral efficiency (SE) ρ with CSs with an arbitrary ESC can be as high as $\lfloor \log_2((S^2 - S)m + A + B + S) \rfloor \cdot G / 2^m$ bit/s/Hz and $\lfloor \log_2(2((S^2 - S)m + S) \cdot G) \rfloor / (2^m \cdot N)$ bit/s/Hz for $N = 1$ and $N > 1$, respectively. To infer the minimum Euclidean distance, denoted by d_{\min} , let $r_{\min} \triangleq \min_{p_u \in \mathbb{S}_H} |p_u|$ and $\delta_{\min} \triangleq \min_{p_u, p_v \in \mathbb{S}_H, u \neq v} |p_u - p_v|$. In [4, Theorem 10], it was shown that the minimum Lee distances for the phase function $f_i(\mathbf{x})$ are 2^{m-1} and 2^{m-2} for $H > 2$ and $H = 2$, respectively. Therefore, it can be shown that the minimum Euclidean distance for all sequences generated with $r_{\min} \zeta^{j f_i(\mathbf{x})}$ is $2^{(m+1)/2} r_{\min} \sin(\pi/H)$ for $H > 2$ and $2^{m/2} r_{\min} \sin(\pi/H)$ for $H = 2$. Also, the amplitude of 2^{m-1} of the elements of CSs are set to either r_u and r_v in this study. Therefore, for a given $f_i(\mathbf{x})$, the minimum distance is less than or equal to $2^{(m-1)/2} \delta_{\min}$. By combining these results, for $N = 1$ and $\mathbf{a} = \mathbf{b} = (1)$,

$$d_{\min} \leq \begin{cases} \min\{2^{\frac{m+1}{2}} \sin(\frac{\pi}{H}) r_{\min}, 2^{\frac{m-1}{2}} \delta_{\min}\}, & H > 2 \\ \min\{2^{\frac{m}{2}} \sin(\frac{\pi}{H}) r_{\min}, 2^{\frac{m-1}{2}} \delta_{\min}\}, & H = 2 \end{cases} \quad (20)$$

When the seed sequences exist, d_{\min} is a function of the seed sequences in general.

Our numerical analysis on d_{\min} shows that (20) is exact in certain scenarios, e.g., $4s^2$ -QAM for $(s, m) = \{(2, 2), (2, 3), (3, 2), (3, 3)\}$, HS -STAR for $(s, m) = \{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$ for

Algorithm 1: Offsetting amplitude and phase parameters for an arbitrary ESC

Input: $\mathbb{S}_H, H, N, m, u, v, l$, and $\kappa', \kappa_n \in \mathbb{Z}_H$ for $n = 1, 2, \dots, m$

Output: e', e_n, k', k_n for $n = 1, 2, \dots, m$, isReversedPiValid

Function calculateParametersWithOffsets

```

 $e' \leftarrow \frac{H}{2\pi} \ln r_u, e_l \leftarrow \frac{H}{2\pi} \ln \frac{r_u}{r_v}$ 
 $e_{n|n \neq l} \leftarrow 0$  for  $n = 1, 2, \dots, m$ 
if  $u = v$  then
   $k' \leftarrow \kappa' + \frac{H}{2\pi} \psi_u$ 
else
   $k' \leftarrow \kappa' + \frac{H}{2\pi} \psi_u$ 
   $k_l \leftarrow \kappa_l + \frac{H}{2\pi} \psi_v - \frac{H}{2\pi} \psi_u$ 
   $k_{l+1} \leftarrow \kappa_{l+1} - \frac{H}{2\pi} \psi_v - \frac{H}{2\pi} \psi_u$  (if  $l+1 \leq m$ )
if  $N > 1$  then
   $\text{isReversedPiValid} \leftarrow 1$ 
else
  if  $l = m$  and  $(r_u \neq r_v$  or  $|\psi_u| \neq |\psi_v|)$  then
     $\text{isReversedPiValid} \leftarrow 1$ 
  else
     $\text{isReversedPiValid} \leftarrow 0$ 

```

$H = 4$). However, we do not have a proof that shows the tightness of (20) in this study.

B. Examples

1) $4s^2$ -QAM: The number of (u, v) permutations for $r_u = r_v$ and $|\psi_u| = |\psi_v|$ can be calculated $s + 4s(s-1)/2 = 2s^2 - s$. Hence, $A + B$ must be equal to $s^4 - (2s^2 - s) = s^4 - 2s^2 + s$. Thus, the total number $4s^2$ -QAM CSs can be calculated as $((s^4 - s^2)(m+1) + s) \cdot G$ and $2((s^4 - s^2)m + s^2) \cdot G$ for $N = 1$ and $N > 1$, respectively. For $4s^2$ -QAM, $r_{\min} = \sqrt{3/(4s^2 - 1)}$ and $\delta_{\min} = \sqrt{6/(4s^2 - 1)}$. Therefore, $d_{\min} \leq 2^{m/2} \sqrt{3/(4s^2 - 1)}$ for $N = 1$.

Our result extends the enumerations in [26] via offset method and Case I-II results in [27] via para-unitary matrices. In [26], for $N = 1$, Li showed that there exist at least $((m+1)4^{2(q-1)} - (m+1)4^{(q-1)} + 2^{q-1}) \cdot G$ CSs for 4^q -QAM alphabet. For $s = 2^{q-1}$, we obtain the same result when it is plugged into $(s^4 - s^2)(m+1) + s$ without using an auxiliary method such as Gaussian integers. As a new result, for $N > 1$, we show that the total number of sequences is $2(m4^{2(q-1)} - m4^{(q-1)} + 2^{2q-2}) \cdot G$ for 4^q -QAM alphabet. Note that our framework also enumerates the CSs for non-square constellations, e.g., $4s^2 = 36$ points in \mathbb{S}_{esc} for $s = 3$.

In this study, $f_r(\mathbf{x})$ is utilized to generate CSs with two distinct amplitude levels. By exploiting the grid nature of QAM alphabet, it is possible to obtain the phase and amplitude parameters with a computer search for CSs with more than two amplitude levels, as done in [27] with para-unitary matrix construction. However, we do not have a systematic rule to enumerate QAM-CSs with more than two amplitude levels. Since these sequences also require a large q [27], [28], it can decrease d_{\min} drastically without a particular CS selection.

2) HS -Star: For this case, $(A, B) = (S(S-1), 0)$. Thus, there are at least $((S^2 - S)(m+1) + S) \cdot G$ and $2((S^2 - S)m + S) \cdot G$ CSs with an HS -Star alphabet for $N = 1$ and $N > 1$, respectively.

3) $DVB-16/32APSK$: For $DVB-16APSK$ and $DVB-32APSK$, $(A, B) = (6, 4)$ and $(A, B) = (38, 14)$, respectively. Thus, the numbers of CSs with $DVB-16APSK$ constellation

are at least $(240m + 26) \cdot G$ and $(480m + 32) \cdot G$ for $N = 1$ and $N > 1$. For DVB-32APSK constellation, there are at least $(992m + 84) \cdot G$ and $(1984m + 64) \cdot G$ CSs for $N = 1$ and $N > 1$, respectively.

4) *Modified IEEE 802.11ay 64-NUC*: For modified IEEE 802.11ay 64-NUC constellation, $(A, B) = (228, 0)$ and the numbers of CSs are at least $(4032m + 292) \cdot G$ and $(8064m + 128) \cdot G$ for $N = 1$ and $N > 1$, respectively.

C. Constant-Modulus CSs with an ESC

Using a constellation different from PSK for CSs can increase the mean power of the OFDM symbol. This issue demotes the main motivation of using CSs with a larger constellation for a practical system since the PMEPR can be higher than 3 dB [26]. On the other hand, our framework provides a way to control the mean power of OFDM symbol through $\{(u, v), l\}$ for a constellation different from PSK. The choice of $e' = \frac{H}{2\pi} \ln r_u$, $e_l = \frac{H}{2\pi} \ln \frac{r_u}{r_v}$, and $e_n |n \neq l = 0$ in Theorem 2 scales halves of the elements to r_u and r_v . Hence, the OFDM symbol power can be fixed through constant-modulus CSs if $(r_u^2 + r_v^2)/2 = 1$ holds. To retain the PMEPR benefit of CSs, we consider only $\{(u, v), l\}$ combinations that satisfies $(r_u^2 + r_v^2)/2 = 1$. Under this constraint, the number of distinct CSs reduces, but it can still be high if the constellation is designed well. For example, if $S = 4$ for *HS-Star* constellation, $(r_1^2, r_2^2, r_3^2, r_4^2)$ can be chosen as $(2 - b^2, 2 - a^2, a^2, b^2)$ for any $1 < a < b < \sqrt{2}$. For QAM, there exist (u, v) permutations that satisfy $(r_u^2 + r_v^2)/2 = 1$ and the valid permutations can be easily identified through a computer search. It is also worth noting that ensuring a larger d_{\min} with a constellation optimization or a constant-modulus CS selection algorithm is not considered in this work.

VI. ENCODER AND DECODER

A. Encoder

The proposed encoder encodes the information bits with $\{(u, v), l\}$ combinations, $\kappa', \kappa_n \in \mathbb{Z}_H$ for $n \in \{1, 2, \dots, m\}$, and the permutations of π , where $k' = \kappa' + \frac{H}{2\pi} \Delta'$, $k_n = \kappa_n + \frac{H}{2\pi} \Delta_n$ for $n \in \{1, 2, \dots, m\}$ and Δ' , Δ_n , e' , and e_n are functions of $\{(u, v), l\}$ based on Algorithm 1, and H is assumed to be power-of-two. We utilize orthogonal seed GCPs to support orthogonal multiple access in the uplink by using the properties of modulated unimodular sequences [42] and use a fixed π_1 to keep the locations of the seed sequences at a cost of maximum $\lceil \log_2 m \rceil$ bits. Therefore, the same subcarriers can be shared by N users for a seed GCP of length N .

Based on Corollary 3, we transmit $\lceil \log_2 (N_{u,v,l}(m-1)!) \rceil + (m+1) \log_2 H$ information bits over $N2^m$ subcarriers, where $N_{u,v,l}$ is the number of $\{(u, v), l\}$ combinations and equal to $(S^2 - S)m + S$. To utilizes all available constellation points, $K_{u,v,l} = \lceil \log_2 (N_{u,v,l}(m-1)!) \rceil$ information bits is first converted to an integer $i_{u,v,l,\pi} \in \{0, 1, \dots, 2^{K_{u,v,l}} - 1\}$. It is then decomposed as $i_{u,v,l,\pi} = i_\pi N_{u,v,l} + i_{u,v,l}$, where $i_{u,v,l} \in \{0, 1, \dots, N_{u,v,l} - 1\}$. The indices $i_{u,v,l}$ and i_π are utilized to identify a combination of $\{(u, v), l\}$ and

a permutation of π , respectively. We use factoradic via Lehmer code to construct a bijective mapping from integers to permutations.

B. Decoder

At the receiver side, we first separate the users by using the orthogonality of the seed sequences and exploiting the fixed $S(\mathbf{x}, z)$. We then concatenate the matched filter outputs, i.e., the sequence length reduces to 2^m for each user. The i th element of the sequence of a user can be expressed as $r_i = c_i \xi^{\tilde{f}_r(i) + j\tilde{f}_i(i)} + n_i$, where c_i and n_i are the complex channel and the noise coefficients, respectively. Assuming that c_i is available at the receiver, an ML decoder is equivalent to a minimum distance decoder for additive white Gaussian noise (AWGN), i.e.,

$$\begin{aligned} \{\tilde{\theta}_r, \tilde{\theta}_i\} &= \arg \min_{\theta_r, \theta_i} \sum_{i=0}^{2^m-1} |c_i \xi^{\tilde{f}_r(i; \theta_r) + j\tilde{f}_i(i; \theta_i)} - r_i|^2 \\ &= \arg \max_{\theta_r, \theta_i} \Re \left\{ \sum_{i=0}^{2^m-1} \xi^{\tilde{f}_r(i; \theta_r) - j\tilde{f}_i(i; \theta_i)} s_i - \xi^{2\tilde{f}_r(i; \theta_r)} h_i \right\}, \end{aligned} \quad (21)$$

where $\theta_r \cup \theta_i = \{(u, v), l, \pi, k', k_1, \dots, k_m\}$, $s_i = c_i^* r_i$, and $h_i = |c_i|^2/2$. However, solving (21) is not easy task and the complexity with a brute-force search can be very high.

1) *Principle*: To achieve a low-complexity ML detector, we decompose $f_r(\mathbf{x}; \theta_r)$ and $f_i(\mathbf{x}; \theta_i)$ as $f_r(\mathbf{x}; \theta_r) = g_r^\ell(\mathbf{x}^\ell; \psi_r) + x_\ell f_r^\ell(\mathbf{x}^\ell; \phi_r)$ and $f_i(\mathbf{x}; \theta_i) = g_i^\ell(\mathbf{x}^\ell; \psi_i) + x_\ell f_i^\ell(\mathbf{x}^\ell; \phi_i)$ for $\ell = \{1, 2, \dots, m\}$, respectively, where $\mathbf{x}^\ell \triangleq (x_1, x_2, \dots, x_{\ell-1}, x_{\ell+1}, \dots, x_m)$. While the order of $g^\ell(\mathbf{x}^\ell)$ is r , $f^\ell(\mathbf{x}^\ell)$ is a polynomial with the order of $r - 1$ and can be calculated as $f^\ell(\mathbf{x}^\ell) = \partial f(\mathbf{x}) / \partial x_\ell$. Both $f^\ell(\mathbf{x}^\ell)$ and $g^\ell(\mathbf{x}^\ell)$ lead to the sequences of length 2^{m-1} . By using these decompositions, the maximization in (21) can be re-expressed as

$$\max_{\psi_r, \phi_r, \psi_i, \phi_i} \Re \left\{ \sum_{i=0}^{2^{m-1}-1} \xi^{\tilde{g}_r^\ell(i; \psi_r) - j\tilde{g}_i^\ell(i; \psi_i)} s_i^\ell - \xi^{2\tilde{g}_r^\ell(i; \psi_r)} h_i^\ell \right\}, \quad (22)$$

where

$$s_i^\ell = s_{I_\ell(i)} + s_{I_\ell(i) + 2^{m-\ell}} \xi^{\tilde{f}_r^\ell(i; \phi_r) - j\tilde{f}_i^\ell(i; \phi_i)}, \quad (23)$$

$$h_i^\ell = h_{I_\ell(i)} + h_{I_\ell(i) + 2^{m-\ell}} \xi^{2\tilde{f}_r^\ell(i; \phi_r)}, \quad (24)$$

and $I_\ell(i)$ maps the integers from 0 to $2^{m-1} - 1$ to the values of $\sum_{j=1}^m x_j 2^{m-j}$ in ascending order for $x_\ell = 0$. The variables s_i^ℓ and h_i^ℓ can be considered as the element of a new received sequence and the half of absolute square of channel gain as in the original expression in (21), where the original sequence length is halved.

Theorem 2 provides $f_r(\mathbf{x})$ and $f_i(\mathbf{x})$. By using (27), $f_r(\mathbf{x})$ in (9) can be re-written as

$$f_r(\mathbf{x}) = e' + e_1 x_{\pi_1} + \sum_{n=2}^m (e_n + e_{n-1}) x_{\pi_n} - 2 \sum_{n=1}^{m-1} e_n x_{\pi_n} x_{\pi_{n+1}}.$$

Algorithm 2: ML Decoder for Fading Channel & Pruning

Input: $\pi_1, m, H, (c_i)_{i=0}^{2^m-1}, (r_i)_{i=0}^{2^m-1}$
Output: $\hat{i}_{u,v,l,\pi}, (\hat{\kappa}_1, \dots, \hat{\kappa}_m, \hat{\kappa}')$
Function main
 Prepare S_m, H_m, L_m, O_m for $N_{u,v,l}$ sequences
 Run $(\Sigma_m, \Pi_m, \Theta_m) = \text{dec}(S_m, H_m, L_m, O_m)$
 Calculate $\hat{i}_{u,v,l,\pi}$ and $(\hat{\kappa}_1, \dots, \hat{\kappa}_m, \hat{\kappa}')$ from Π_m and Θ_m for the index that maximizes Σ_m
return $\hat{i}_{u,v,l,\pi}, (\hat{\kappa}_1, \dots, \hat{\kappa}_m, \hat{\kappa}')$

Function $(\Sigma_m, \Pi_m, \Theta_m) = \text{dec}(S_m, H_m, L_m, O_m)$
 Enumerate $S_{m-1}, H_{m-1}, L_{m-1}, O_{m-1}$ based on (23) and (24)
if $m = 1$ **then**
 Obtain $\hat{\kappa}', \hat{\kappa}_1$ for each of L sequences based in (22)
 Populate $\Pi_m | i \leftarrow 1, \Theta_m | i \leftarrow (\hat{\kappa}_1, \hat{\kappa}'), \Sigma_m | i \leftarrow (22)$
else
 Populate the indices N_{best} best sequences in N_m
 Prune $S_{m-1}, H_{m-1}, L_{m-1}, O_{m-1}$ based on N_m
 Run $(\Sigma_{m-1}, \Pi_{m-1}, \Theta_{m-1}) = \text{dec}(S_{m-1}, H_{m-1}, L_{m-1}, O_{m-1})$
 Calculate $\Sigma_m, \Pi_m, \Theta_m$ for each sequence indexed in N_m
 Set the scores in Σ_m for the pruned sequences to $-\infty$
return $\Sigma_m, \Pi_m, \Theta_m$

Therefore, $f_r^\ell(\mathbf{x}^\ell)$ can be calculated as

$$f_r^\ell(\mathbf{x}^\ell) = \begin{cases} e_{m-1} - 2e_{m-1}x_{\pi_{m-1}} + e_m, & \pi_m = \ell \\ e_{n-1} - 2e_{n-1}x_{\pi_{n-1}} + e_n - 2e_nx_{\pi_{n+1}}, & \pi_n = \ell \\ e_1 - 2e_1x_{\pi_2}, & \pi_1 = \ell \end{cases}.$$

Similarly, we can express $f_i(\mathbf{x}) = g_i^\ell(\mathbf{x}^\ell) + x_\ell f_i^\ell(\mathbf{x}^\ell)$, where

$$f_i^\ell(\mathbf{x}^\ell) = \begin{cases} k_m + \frac{H}{2}x_{\pi_{m-1}}, & \pi_m = \ell \\ k_n + \frac{H}{2}x_{\pi_{n-1}} + \frac{H}{2}x_{\pi_{n+1}}, & \pi_n = \ell \\ k_1 + \frac{H}{2}x_{\pi_2}, & \pi_1 = \ell \end{cases}.$$

As π_1 is fixed and available at the receiver, $f_r^\ell(\mathbf{x}^\ell) = e_1 - 2e_1x_{\pi_2}$ and $f_i^\ell(\mathbf{x}^\ell) = k_1 + \frac{H}{2}x_{\pi_2}$ for $\ell = \pi_1$. Hence, $\phi_l = \{k_1, \pi_2\}$ and $\phi_r = \{e_1, \pi_2\}$, where $k_1 = \kappa_1 + \Delta_1$ and e_1 depend on $i_{u,v,l}$. Hence, for a hypothesized $i_{u,v,l}$, we can calculate (23) and (24) for different κ_1 and π_2 . This procedure can be done continuously, where the sequence length is halved for each step. By backtracking the branch that maximizes (22), a recursive ML detector can be achieved. We adopt this principle and address the exponential growth by terminating unpromising branches early.

2) *Algorithm:* The proposed algorithm has four different phases given as follows:

a) *Preparation:* We calculate $N_{u,v,l}$ offset parameters, i.e., $\Delta', \Delta_1, \dots, \Delta_m, e', e_1, \dots, e_m$ based on the ESC for all $\{(u, v), l\}$ combinations. Since e' and Δ' alter the amplitude and phase of all elements in the sequence, we combine them with the channel coefficients as $c_i \leftarrow c_i \xi^{e' + j\Delta'}$. We then calculate $s_i \leftarrow c_i^* r_i$ and $h_i \leftarrow |c_i|^2 / 2$ for each offset and store $N_{u,v,l}$ resulting sequences in S_m and H_m , respectively. The parameter ℓ (i.e., the fixed π_1 for the preparation) and $i_{u,v,l}$ are stored in L_m and O_m , respectively, for identifying the offsets in the following steps.

b) *Enumeration:* There are $m - 1$ and H options for π_2 and k_1 for each sequence in S_m , respectively. By taking e_1 and Δ_1 into account, we calculate (23) and (24) for all combinations of π_2 and κ_1 . We populate $(m - 1)HL$ resulting sequences in S_{m-1} and H_{m-1} , respectively, where L is the

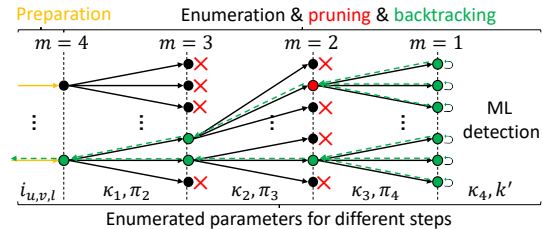


Figure 2. An example for preparation, enumeration, pruning, backtracking and the parameters for $m = 4$. While e' and Δ' are used for the preparation, Δ_n and e_n are used only at the n th step for calculating (23) and (24).

number of sequences given to the algorithm. We populate the corresponding $i_{u,v,l}$ and the information related to ℓ for the next recursion (i.e., the enumerated values for π_2) in O_{m-1} and L_{m-1} , respectively.

c) *Pruning:* To address the exponential growth of the enumerated sequences, we keep only N_{best} enumerated sequences for the next step. By using the high signal-to-noise ratio (SNR) approximations of $\xi^{f_r(\mathbf{x};\theta_r) + jf_i(\mathbf{x};\theta_i)} \approx s_i / 2h_i$ and $\xi^{2f_r(\mathbf{x};\theta_r)} \approx |s_i|^2 / 4h_i^2$ in (21), we calculate an estimate of (21) and choose N_{best} sequences based on the resulting metric, i.e., $\sum_{i=1}^{2^m-1} |s_i|^2 / 4h_i$. The indices of the best sequences are populated in N_m for backtracking.

d) *Backtracking:* We recursively run the enumeration and pruning phases till $m = 1$. For $m = 1$, we perform an ML detection based on (22). The detected parameters, i.e., $(\hat{\kappa}_1, \hat{\kappa}')$, $\pi = (1)$, and the value of (22) for all sequences are stored in Π_m , Θ_m , and Σ_m , where $\hat{\kappa}_1$ and $\hat{\kappa}'$ correspond to the detected κ_m and κ' , respectively. The algorithm then returns Π_m , Θ_m , and Σ_m for the step above. For $m \geq 2$, the detected parameters from the previous finalized step are utilized to calculate (22) for each enumerated sequence. The best $\hat{\kappa}_1$ and $\hat{\pi}_1$ for the sequences indicated in N_m are obtained by using O_m and L_m and the results are combined with the parameters from the previous step in Π_m and Θ_m . The scores with (22) for different sequences are also populated in Σ_m .

After the recursions are finalized, the sequence index that returns the maximum (22) among $N_{u,v,l}$ sequences is assigned to $\hat{i}_{u,v,l}$. The parameters $\hat{\pi}$ and $(\hat{\kappa}_1, \dots, \hat{\kappa}_m, \hat{\kappa}')$ are detected from Π_m and Θ_m . After mapping $\hat{\pi}$ to an integer \hat{i}_π , $\hat{i}_{u,v,l,\pi} = \hat{i}_\pi N_{u,v,l} + \hat{i}_{u,v,l}$ is calculated. The binary representation of $\hat{i}_{u,v,l,\pi}$ corresponds to the received bits over π and $\{(u, v), l\}$. The pseudocode and an example for the phases are given in Algorithm 2 and Figure VI-B2c, respectively.

3) *Complexity Reduction with Pruning:* With pruning, the remaining complexity is due to the generation of $\max(m - i, 1)H \min(L, N_{\text{best}})$ sequences of length 2^{m-i} at the i th step, the process of choosing N_{best} sequences, and the calculation of (22) for N_{best} sequences. For instance, for $N_{\text{best}} = 400$, maximum $400H(i - 1)$ sequences of length 2^{m-i} are enumerated and 400 of them survive for the $(i + 1)$ th step, which corresponds to a substantially reduced search space.

4) *Comparisons with Other Decoders for CSs:* Although there are many studies that focus on the enumeration of CSs, the literature is *not* rich with decoding algorithms for the codes related to CSs. Particularly, there are only few studies that consider the coset term changing based on π . In

[4], a supercode decoder that subtracts each possible coset representative and decodes the remaining sequences with a fast Hadamard transformation (FHT) were proposed. In [43], FHT was extended for decoding the first-order generalized RM codes. In [44], for a given coset term, a recursive decoding which can reduce the complexity of generalized FHT further is proposed. In [45], $f^\ell(\mathbf{x}^\ell)$ is defined as $f(\mathbf{x}|x_\ell = 1) - f(\mathbf{x}|x_\ell = 0)$ and utilized for developing efficient decoding algorithms. Although Algorithm 8 in [45] considers the coset term, it consists of a step (i.e., Step 2) that multiplies two noisy terms, which would cause noise enhancement.

The proposed decoder can be considered as an extension of the decoders in [44] and [45]. The recursion steps in our decoder follow the same rationale discussed in [44] and exploit the order reduction proposed in [45]. However, it differs from [44] as we consider the coset term. As opposed to the decoder in [45], we utilize $f^\ell(\mathbf{x}^\ell)$ such that the decoder does not cause noise enhancement. We also use the derivatives of both phase and amplitude functions in Theorem 2, which allow us to develop the decoder for different constellations and selective fading channels.

VII. NUMERICAL RESULTS

In this section, we evaluate the proposed encoder and decoder for an uplink scenario numerically. For comparison, we also generate results with a polar code and binary phase-shift keying (BPSK) under similar SE conditions. We assume that multiple users access the communication medium with a single OFDM symbol synchronously over a non-contiguous resource allocation in the frequency domain to exploit the frequency diversity. Considering the compatibility with 3GPP Fifth Generation (5G) New Radio (NR) waveform parameters, we assume that each user utilizes 384 subcarriers distributed over 8 clusters in the frequency domain. Each cluster consists of 48 subcarriers. We assume that 132 subcarriers at the center of the band are allocated for random access channel and left unoccupied as shown in Figure 3(a). The spacing between the clusters on the left and right portions of the band is set to 96 subcarriers. Hence, three interlaced groups are constructed. We support multiple access on each group by exploiting orthogonal seed sequences of length $N = 3$. Therefore, 9 users can be multiplexed in the uplink within a single OFDM symbol. To align with the resource allocation, we set $m = 7$ for the proposed encoder and polar code. We choose $\pi_1 = 7$. For ESCs, we consider HS -Star for $H = \{4, 8\}$ and $S = \{1, 2, 4\}$ and $4s^2$ -QAM for $s = \{2, 4\}$. For HS -Star, (r_1^2, r_2^2) and $(r_1^2, r_2^2, r_3^2, r_4^2)$ are chosen as $(2 - a^2, a^2)$ where $a = 1.256$ for $S = 2$ and $(2 - b^2, 2 - a^2, a^2, b^2)$ where $a = 1.1$ and $b = 1.33$ for $S = 4$, respectively. For the proposed decoder, we set $N_{\text{best}} = 400$. For polar decoder, we consider soft decoding with successive interference cancellation (SIC) and the equalizer is minimum mean square error (MMSE) frequency-domain equalization (FDE). The design SNR for the polar code is set to 3 dB. The demodulation and equalization are combined in the proposed decoder for the CSs (see Algorithm 2). The subcarrier spacing is set to 30 kHz.

In Figure 3(b), we compare PMEPR distributions of different schemes. While PMEPR of OFDM symbols with the polar

code reaches to 12 dB at the rate of $1e-2$, the maximum PMEPR is 6 dB for all cases of CSs. When the OFDM symbol power is fixed by limiting the (u, v) combinations, the maximum PMEPRs of the schemes based on CSs reduce to 3 dB, which leaves a 9 dB extra room for transmit power as compared to OFDM with the polar code. It is worth noting that the PMEPR of the OFDM symbols generated with the CSs is not affected by the non-contiguous resource allocation as the support can be achieved via $f_s(\mathbf{x})$ given in (13), i.e., $d_{\pi_n=1} = 288$, $d_{\pi_n=2} = 96$, $d_{\pi_n=3} = 48$, and $d_{\pi_n \neq 1,2,3} = 0$. The SEs of the schemes are provided in Figure 3(b) for the fixed OFDM symbol power. While the number of bits sacrificed for fixed OFDM symbol power is only 1 bit for 16-QAM and HS -Star for $S = 4$ and $H = \{4, 8\}$, it is 3 bits for 64-QAM. No bit is sacrificed with HS -Star for $S = 2$ and $H = \{4, 8\}$.

In Figure 3(c) and Figure 3(d), we provide bit-error rate (BER) and block-error rate (BLER) curves in AWGN and fading channel, respectively. We consider ITU Vehicular A for the multi-path channel model without mobility. We perform the evaluations with constant-modulus CSs by restricting (u, v) permutations. For $S = 1$, HS -Star is equivalent to H -PSK constellation and our encoder reduces to the one in [4]. The BERs are at $1e-3$ when $E_b/N_0 = 2$ dB and $E_b/N_0 = 4$ dB for $H = 4$ and $H = 8$ in AWGN channel as shown in Figure 3(c). In the fading channel, $1e-2$ BLER is achieved when SNRs are -7 dB and -3 dB for $H = 4$ and $H = 8$, respectively, as in Figure 3(d). For $S = \{2, 4\}$, the SE increases gradually as more distinct CSs are synthesized without sacrificing 3 dB PMEPR. For $H = 4$, $1e-2$ BLER is achieved at -5.6 dB and -4.5 dB when $S = 2$ and $S = 4$, respectively. For $H = 8$, the number of information bits reach up to 37 and 38 bits per user for $S = 2$ and $S = 4$, respectively, where $1e-2$ BLER is achieved at -2 dB and 0 dB. The results indicate that HS -Star performs better than QAM in average BLER. While both HS -Star for $H = 4$ and $S = 4$ and 16-QAM result in the same SE under PMEPR constraint, HS -Star provides 1 dB improvement at $1e-2$ BLER. The SNR gap between HS -Star for $H = 1$ and $S = 8$ and 64-QAM reaches to 3 dB. The OFDM with polar code provides similar BER versus E_b/N_0 curves for both $\rho = 25/384$ bit/s/Hz and $\rho = 38/384$ bit/s/Hz (i.e., the code rates are 25/128 and 38/128, respectively) as shown in Figure 3(c). It performs 2 dB worse than the QPSK CSs as the decoder for polar code relies on SIC. The polar code is more reliable than the CSs with high-order constellations in AWGN. It outperforms CSs with HS -Star for $S = \{2, 4\}$ for $H = 8$ with 2 dB and 4 dB, respectively. Nevertheless, the proposed encoder under a similar SE provides a 9 dB room for transmit power and can offset the degradation due to the SNR loss in practice.

VIII. CONCLUSION

In this study, we introduce a method that allows one to re-express a polynomial generated with a recursion consisting of linear operators by indicating how the operators are distributed to coefficients of the polynomial via Boolean functions. By

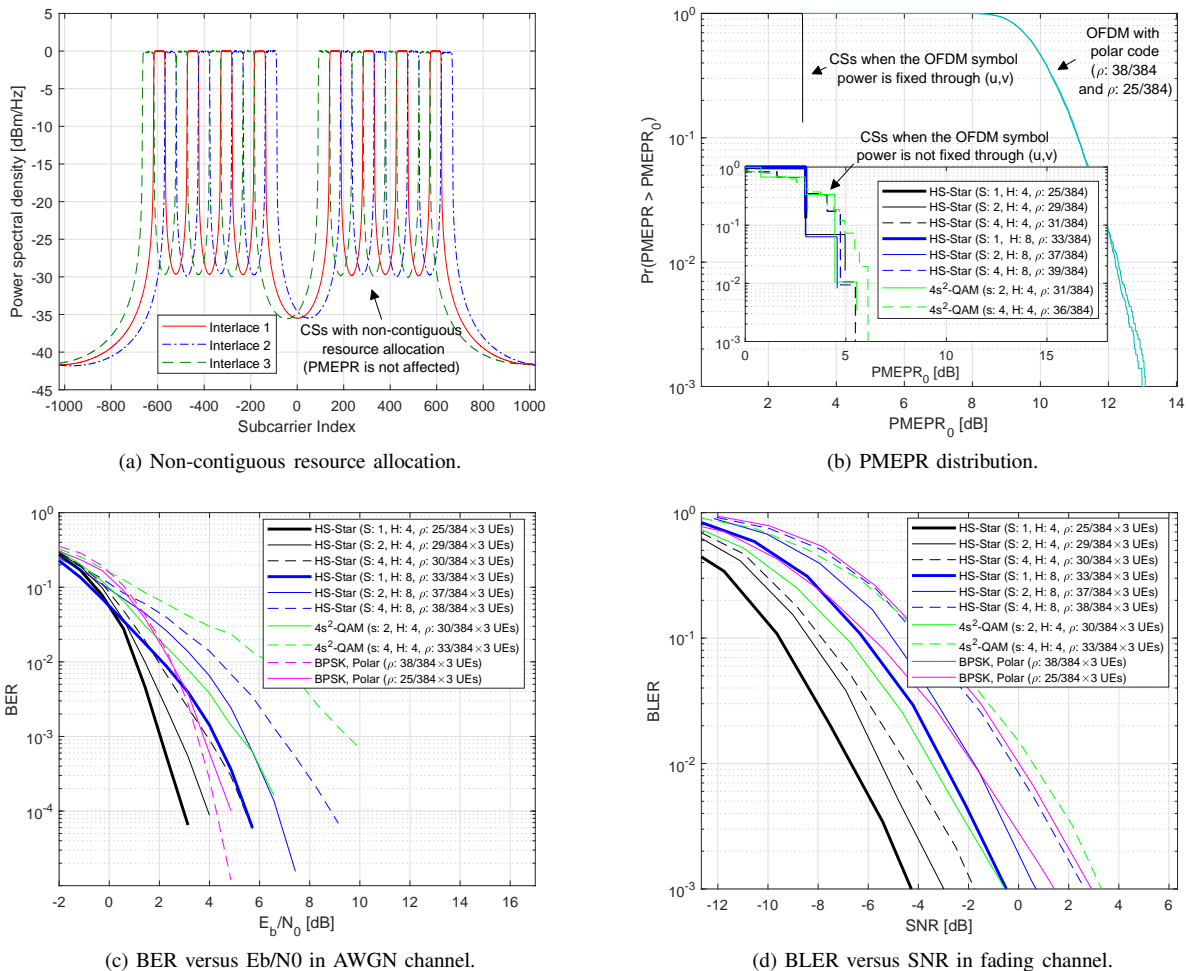


Figure 3. Resource allocation, PMEPR distribution, BER, and BLER for different schemes ($N_{\text{best}} = 400$, $m = 7$, $N = 3$).

applying this method to a recursive GCP constructions, we show how the seed sequences, phase rotations, signs, real scalars, and the shifting factors applied at each step of the recursion alter the elements of the sequences in the GCP. As a result, we obtain a generic CS construction that extends David and Jedwab's construction in [4] by deriving separate pseudo-Boolean functions for the amplitude, the phase, the seed sequences, and the support of the CS. We show that distinct CSs with a wide range of constellations can be enumerated systematically. We also present an encoder and an ML decoder. We show that the proposed encoder can achieve low-PMEPR OFDM symbols without utilizing any optimization technique while supporting multiple users, different constellations, and non-contiguous resource allocation. The proposed decoder also decodes the information bits on the second-order coset in the phase function while being compatible with different constellations.

We demonstrate the performance of the encoder through an uplink simulation. The error rate is a strong function of the constellation. Under a maximum 3 dB PMEPR constraint, the encoder with *HS-Star* outperforms the one with QAM in terms of average BLER. The polar code is superior to the proposed encoder with high-order constellations in AWGN while the performance gap diminishes in a fading channel.

Also, the proposed encoder offers a 9 dB room for increasing the transmit power without any clipping, which can offset the performance difference in practice.

APPENDIX A PROOF OF LEMMA 1

Proof. The sequence $\mathbf{1} \triangleq (1)_{i=0}^{2^m-1}$ can be decomposed as

$$\begin{aligned}
 \mathbf{1}(z) &= \prod_{n=1}^m (1 + z^{2^{\psi_n}}) \\
 &= \underbrace{\prod_{\substack{n=1 \\ n \neq l}}^m (1 + z^{2^{\psi_n}})}_{H_1(z)} + z^{2^{\psi_l}} \underbrace{\prod_{\substack{n=1 \\ n \neq l}}^m (1 + z^{2^{\psi_n}})}_{H_2(z)} \\
 &= \underbrace{\prod_{\substack{n=1 \\ n \neq l, l+1}}^m (1 + z^{2^{\psi_n}})}_{H_{11}(z)} + z^{2^{\psi_{l+1}}} \underbrace{\prod_{\substack{n=1 \\ n \neq l, l+1}}^m (1 + z^{2^{\psi_n}})}_{H_{21}(z)} \\
 &\quad + z^{2^{\psi_l}} \underbrace{\prod_{\substack{n=1 \\ n \neq l, l+1}}^m (1 + z^{2^{\psi_n}})}_{H_{12}(z)} + z^{2^{\psi_l} + 2^{\psi_{l+1}}} \underbrace{\prod_{\substack{n=1 \\ n \neq l, l+1}}^m (1 + z^{2^{\psi_n}})}_{H_{22}(z)}.
 \end{aligned}$$

By the definitions, $H_1(z)$ and $H_2(z)$ should have no common monomials, i.e., $\mathbf{h}_1 + \mathbf{h}_2 = \mathbf{1}$. The polynomial $H_2(z)$ corresponds to a polynomial where the degree of each monomial in $H_1(z)$ is increased by 2^{ψ_l} by their definitions. Hence, the sequence \mathbf{h}_2 should be the shifted version of the sequence \mathbf{h}_1 by 2^{ψ_l} . Since $\mathbf{h}_1 + \mathbf{h}_2 = \mathbf{1}$ and \mathbf{h}_2 is the shifted version of \mathbf{h}_1 by 2^{ψ_l} , the u th element of the sequence \mathbf{h}_1 should be 1 for $u = s2^{\psi_l} + (1, 2, \dots, 2^{\psi_l})$ for even s and 0 for odd s where $0 \leq s < 2^{m-\psi_l}$. Thus, the Boolean functions generating \mathbf{h}_1 and \mathbf{h}_2 can be obtained as x_{π_l} and $1 - x_{\pi_l}$, respectively. The polynomials $H_1(z)$ and $H_2(z)$ can also be decomposed as $H_{11}(z) + H_{21}(z)$ and $H_{12}(z) + H_{22}(z)$, respectively, i.e., $\mathbf{h}_{11} + \mathbf{h}_{21} = \mathbf{h}_1$ and $\mathbf{h}_{12} + \mathbf{h}_{22} = \mathbf{h}_2$. The sequences \mathbf{h}_{21} and \mathbf{h}_{22} are the shifted version of the sequence \mathbf{h}_{11} and \mathbf{h}_{12} by $2^{\psi_{l+1}}$, respectively. Thus, the Boolean functions generating \mathbf{h}_{11} , \mathbf{h}_{21} , \mathbf{h}_{12} , and \mathbf{h}_{22} can then be expressed as $(1 - \pi_l)(1 - \pi_{l+1})$, $(1 - \pi_l)\pi_{l+1}$, $\pi_l(1 - \pi_{l+1})$, and $\pi_l\pi_{l+1}$, respectively.

Now, consider a new recursion given by $P_l^{(n)}(z)$ is $b_{11}^{(l)}P_l^{(n-1)}(z) + b_{12}^{(l)}Q_l^{(n-1)}(z)z^{2^{\psi_n}}$ for $n = l$, otherwise $P_l^{(n-1)}(z) + Q_l^{(n-1)}(z)z^{2^{\psi_n}}$ and $Q_l^{(n)}(z)$ is $b_{21}^{(l)}P_l^{(n-1)}(z) + b_{22}^{(l)}Q_l^{(n-1)}(z)z^{2^{\psi_n}}$ for $n = l$, otherwise $P_l^{(n-1)}(z) + Q_l^{(n-1)}(z)z^{2^{\psi_n}}$, where $P_l^{(0)}(z) = Q_l^{(0)}(z) = 1$. By the definition, $P_l^{(m)}(z)$ and $Q_l^{(m)}(z)$ correspond to the associated polynomials for the sequences \mathbf{f}_l and \mathbf{g}_l , respectively. By evaluating the new recursion, $P_l^{(m)}(z)$ and $Q_l^{(m)}(z)$ can be expressed as

$$F_l(z) = P_l^{(m)}(z) = \begin{cases} b_{11}^{(l)}H_1(z) + b_{12}^{(l)}H_2(z), & l = m \\ b_{11}^{(l)}H_{11}(z) + b_{12}^{(l)}H_{12}(z) \\ + b_{21}^{(l)}H_{21}(z) + b_{22}^{(l)}H_{22}(z), & l < m \end{cases},$$

$$G_l(z) = Q_l^{(m)}(z) = \begin{cases} b_{21}^{(l)}H_1(z) + b_{22}^{(l)}H_2(z), & l = m \\ b_{11}^{(l)}H_{11}(z) + b_{12}^{(l)}H_{12}(z) \\ + b_{21}^{(l)}H_{21}(z) + b_{22}^{(l)}H_{22}(z), & l < m \end{cases}.$$

APPENDIX B PROOF OF THEOREM 2

Proof. We re-write $A^{(n)}(z)$ and $B^{(n)}(z)$ in Theorem 1 as

$$P^{(n)}(z) = \dot{\Omega}_1^{(n)} \dot{\Upsilon}_0^{(n)} \ddot{\Omega}_0^{(n)} \ddot{\Upsilon}_0^{(n)} \\ S_0^{(n)} A_1^{(n)} B_0^{(n)} \Omega_0^{(n)} \Delta_0^{(n)} \dot{\Omega}_1^{(n)} \ddot{\Omega}_0^{(n)} \{P^{(n-1)}(z)\} \\ + \dot{\Omega}_0^{(n)} \dot{\Upsilon}_0^{(n)} \ddot{\Omega}_1^{(n)} \ddot{\Upsilon}_0^{(n)} \\ S_0^{(n)} A_0^{(n)} B_1^{(n)} \Omega_1^{(n)} \Delta_0^{(n)} \dot{\Omega}_0^{(n)} \ddot{\Omega}_1^{(n)} \{Q^{(n-1)}(z)\} z^{2^{\psi_n}},$$

$$Q^{(n)}(z) = \dot{\Omega}_0^{(n)} \dot{\Upsilon}_0^{(n)} \ddot{\Omega}_0^{(n)} \ddot{\Upsilon}_1^{(n)} \\ S_0^{(n)} A_0^{(n)} B_1^{(n)} \Omega_0^{(n)} \Delta_1^{(n)} \dot{\Omega}_1^{(n)} \ddot{\Omega}_0^{(n)} \{P^{(n-1)}(z)\} \\ + \dot{\Omega}_0^{(n)} \dot{\Upsilon}_1^{(n)} \ddot{\Omega}_0^{(n)} \ddot{\Upsilon}_0^{(n)} \\ S_1^{(n)} A_1^{(n)} B_0^{(n)} \Omega_1^{(n)} \Delta_1^{(n)} \dot{\Omega}_0^{(n)} \ddot{\Omega}_1^{(n)} \{Q^{(n-1)}(z)\} z^{2^{\psi_n}},$$

by using the operators defined in Table I. The operators related to the scalars, i.e., α_n and β_n , and the phase rotations, i.e., γ_n , γ_n^* , η_n , η_n^* , and ω_n , are denoted by $A_{\{0,1\}}^{(n)}$, $\dot{\Omega}_{\{0,1\}}^{(n)}$, $\dot{\Upsilon}_{\{0,1\}}^{(n)}$, $\ddot{\Omega}_{\{0,1\}}^{(n)}$, $\ddot{\Upsilon}_{\{0,1\}}^{(n)}$, and $\Omega_{\{0,1\}}^{(n)}$ respectively. The signs for

$A^{(n-1)}(z)$ and $B^{(n-1)}(z)$ in Theorem 1 are generated with the operators $S_{\{0,1\}}^{(n)}$. While the operators $\Delta_{\{0,1\}}^{(n)}$ apply a shift to their arguments, the operators $\dot{\Omega}_{\{0,1\}}^{(n)}$ and $\ddot{\Omega}_{\{0,1\}}^{(n)}$ generate the polynomials $A(z)$ and $B(z)$ for $n = 1$, respectively. For the sake of unifying the operator formats, we express the operators by using ξ as $\alpha_n \triangleq \xi^{a_n}$, $\beta_n \triangleq \xi^{b_n}$, $\gamma_n \triangleq \xi^{j\dot{c}_n}$, $\eta_n \triangleq \xi^{j\ddot{c}_n}$, $\omega_n \triangleq \xi^{j\dot{c}_n}$, and $\xi^{j\tau_n} \triangleq \xi^{j\dot{d}_n}$, and exploit the identity $\xi^{j\frac{H}{2}} = -1$ in Table I. The configuration vectors \mathbf{b}_n for the sub-recursions are also given in Table I. By using (6) and (7), $f_n(\mathbf{x})$ and $g_n(\mathbf{x})$ are obtained for each sub-recursion as in Table I. Finally, by composing all operators based on $f_n(\mathbf{x})$ and $g_n(\mathbf{x})$, the polynomials $P^{(m)}(z)$ and $Q^{(m)}(z)$ can be calculated as

$$P^{(m)}(z) = \sum_{i=0}^{2^m-1} \check{S}(i, z) \xi^{\check{f}_r(i) + j\check{f}_s(i)} z^{\check{f}_s(i) + iU}, \quad (25)$$

$$Q^{(m)}(z) = \sum_{i=0}^{2^m-1} \check{S}(i, z) \xi^{\check{g}_r(i) + j\check{g}_s(i)} z^{\check{f}_s(i) + iU}, \quad (26)$$

where $i = \sum_{j=1}^m x_j 2^{m-j}$, $f_s(\mathbf{x}) = \sum_{n=1}^m d_n x_{\pi_n}$, $S(\mathbf{x}, z) = A(z)(1 - x_{\pi_1})_2 + B(z)x_{\pi_1}$, $f_r(\mathbf{x}) = c_r(\mathbf{x}) + a_m(1 - x_{\pi_m})_2 + b_m x_{\pi_m}$, $g_r(\mathbf{x}) = c_r(\mathbf{x}) + b_m(1 - x_{\pi_m})_2 + a_m x_{\pi_m}$, $f_i(\mathbf{x}) = c_i(\mathbf{x}) + \dot{c}_m(1 - x_{\pi_m})_2 + \ddot{c}_m x_{\pi_m}$, $g_i(\mathbf{x}) = c_i(\mathbf{x}) - \dot{c}_m x_{\pi_m} - \ddot{c}_m(1 - x_{\pi_m})_2 + \frac{H}{2} x_{\pi_m}$, where $c_r(\mathbf{x}) \triangleq \sum_{n=1}^{m-1} a_n(1 - x_{\pi_n} - x_{\pi_{n+1}})_2 + b_n(x_{\pi_n} + x_{\pi_{n+1}})_2$ and

$$c_i(\mathbf{x}) \triangleq \frac{H}{2} \sum_{n=1}^{m-1} x_{\pi_n} x_{\pi_{n+1}} + \sum_{n=1}^m c_n x_{\pi_n} \\ + \sum_{n=1}^{m-1} \dot{c}_n((1 - x_{\pi_n})(1 - x_{\pi_{n+1}}))_2 - \dot{c}_n(x_{\pi_n} x_{\pi_{n+1}})_2 \\ + \sum_{n=1}^{m-1} \ddot{c}_n(x_{\pi_n}(1 - x_{\pi_{n+1}}))_2 - \ddot{c}_n((1 - x_{\pi_n})x_{\pi_{n+1}})_2,$$

Based on (27) in Lemma 2, $a_n(1 - x_{\pi_n} - x_{\pi_{n+1}})_2 = a_n - a_n(x_{\pi_n} + x_{\pi_{n+1}})_2$, $a_m(1 - x_{\pi_m})_2 = a_m - a_m x_{\pi_m}$, and $b_m(1 - x_{\pi_m})_2 = b_m - b_m x_{\pi_m}$. Hence, $f_r(\mathbf{x})$ and $g_r(\mathbf{x})$ can be re-written as in (9) and (10), respectively, where $e_n = (b_n - a_n)$ for $n = 1, 2, \dots, m$, and $e' = \sum_{n=1}^m a_n$. By using the identities given in (27) and (28), $\dot{c}_m(1 - x_{\pi_m})_2 + \ddot{c}_m x_{\pi_m} = \dot{c}_m + (\ddot{c}_m - \dot{c}_m)x_{\pi_m}$, $\dot{c}_m x_{\pi_m} + \ddot{c}_m(1 - x_{\pi_m})_2 = \ddot{c}_m + (\dot{c}_m - \ddot{c}_m)x_{\pi_m}$, $\dot{c}_n((1 - x_{\pi_n})(1 - x_{\pi_{n+1}}))_2 - \dot{c}_n(x_{\pi_n} x_{\pi_{n+1}})_2 = \dot{c}_n - \dot{c}_n x_{\pi_n} - \dot{c}_n x_{\pi_{n+1}}$, and $\ddot{c}_n(x_{\pi_n}(1 - x_{\pi_{n+1}}))_2 - \ddot{c}_n((1 - x_{\pi_n})x_{\pi_{n+1}})_2 = \ddot{c}_n x_{\pi_n} - \ddot{c}_n x_{\pi_{n+1}}$. Hence, $f_i(\mathbf{x})$ and $g_i(\mathbf{x})$ can be re-expressed as in (11) and (12), respectively, where $k_1 = c_1 + \dot{c}_1 - \dot{c}_1$, $k_{n|n=2,3,\dots,m} = c_n + \ddot{c}_n - \dot{c}_n - \ddot{c}_{n-1} - \dot{c}_{n-1}$, $k' = \sum_{n=1}^m \dot{c}_n$, and $k'' = -\ddot{c}_m + \sum_{n=1}^{m-1} \dot{c}_n$, respectively. \square

APPENDIX C IDENTITIES

Lemma 2. For $f_1 : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_2$, $f_2 : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_2$ and $k_1 \in \mathbb{R}$,

$$k_1(f_1(\mathbf{x}) + f_2(\mathbf{x}))_2 \\ = k_1 f_1(\mathbf{x}) + k_1 f_2(\mathbf{x}) - 2k_1(f_1(\mathbf{x})f_2(\mathbf{x}))_2, \quad (27)$$

$$k_1(f_1(\mathbf{x})(1 + f_2(\mathbf{x})))_2 - k_1((1 + f_1(\mathbf{x}))f_2(\mathbf{x}))_2 \\ = k_1 f_1(\mathbf{x}) - k_1 f_2(\mathbf{x}). \quad (28)$$

Table I
THE OPERATOR DEFINITIONS AND THE BOOLEAN FUNCTIONS.

	$O_0^{(n)}\{R(z)\}$	$O_1^{(n)}\{R(z)\}$	\mathbf{b}_n^T	$f_n(\mathbf{x})$	$g_n(\mathbf{x})$
Scalar α_n	$A_0^{(n)}\{R(z)\} \triangleq \xi^0 R(z)$	$A_1^{(n)}\{R(z)\} \triangleq \xi^{a_n} R(z)$	[1 0 0 1]	$\begin{cases} 1 - x_{\pi_m}, & n = m \\ 1 - x_{\pi_n} - x_{\pi_{n+1}}, & n < m \end{cases}$	$\begin{cases} x_{\pi_m}, & n = m \\ 1 - x_{\pi_n} - x_{\pi_{n+1}}, & n < m \end{cases}$
Scalar β_n	$B_0^{(n)}\{R(z)\} \triangleq \xi^0 R(z)$	$B_1^{(n)}\{R(z)\} \triangleq \xi^{b_n} R(z)$	[0 1 1 0]	$\begin{cases} x_{\pi_m}, & n = m \\ x_{\pi_n} + x_{\pi_{n+1}}, & n < m \end{cases}$	$\begin{cases} 1 - x_{\pi_m}, & n = m \\ x_{\pi_n} + x_{\pi_{n+1}}, & n < m \end{cases}$
Phase γ_n	$\dot{O}_0^{(n)}\{R(z)\} \triangleq \xi^0 R(z)$	$\dot{O}_1^{(n)}\{R(z)\} \triangleq \xi^{j c_n} R(z)$	[1 0 0 0]	$\begin{cases} 1 - x_{\pi_m}, & n = m \\ (1 - x_{\pi_n})(1 - x_{\pi_{n+1}}), & n < m \end{cases}$	$\begin{cases} 0, & n = m \\ (1 - x_{\pi_n})(1 - x_{\pi_{n+1}}), & n < m \end{cases}$
Phase γ_n^*	$\dot{O}_0^{(n)}\{R(z)\} \triangleq \xi^0 R(z)$	$\dot{O}_1^{(n)}\{R(z)\} \triangleq \xi^{-j c_n} R(z)$	[0 0 0 1]	$\begin{cases} 0, & n = m \\ x_{\pi_n} x_{\pi_{n+1}}, & n < m \end{cases}$	$\begin{cases} x_{\pi_m}, & n = m \\ x_{\pi_n} x_{\pi_{n+1}}, & n < m \end{cases}$
Phase η_n	$\ddot{O}_0^{(n)}\{R(z)\} \triangleq \xi^0 R(z)$	$\ddot{O}_1^{(n)}\{R(z)\} \triangleq \xi^{j e_n} R(z)$	[0 1 0 0]	$\begin{cases} x_{\pi_m}, & n = m \\ x_{\pi_n}(1 - x_{\pi_{n+1}}), & n < m \end{cases}$	$\begin{cases} 0, & n = m \\ x_{\pi_n}(1 - x_{\pi_{n+1}}), & n < m \end{cases}$
Phase η_n^*	$\ddot{O}_0^{(n)}\{R(z)\} \triangleq \xi^0 R(z)$	$\ddot{O}_1^{(n)}\{R(z)\} \triangleq \xi^{-j e_n} R(z)$	[0 0 1 0]	$\begin{cases} 0, & n = m \\ (1 - x_{\pi_n})x_{\pi_{n+1}}, & n < m \end{cases}$	$\begin{cases} 1 - x_{\pi_m}, & n = m \\ (1 - x_{\pi_n})x_{\pi_{n+1}}, & n < m \end{cases}$
Sign	$S_0^{(n)}\{R(z)\} \triangleq \xi^0 R(z)$	$S_1^{(n)}\{R(z)\} \triangleq \xi^{j \frac{d}{2}} R(z)$	[0 0 0 1]	$\begin{cases} 0, & n = m \\ x_{\pi_n} x_{\pi_{n+1}}, & n < m \end{cases}$	$\begin{cases} x_{\pi_m}, & n = m \\ x_{\pi_n} x_{\pi_{n+1}}, & n < m \end{cases}$
Phase ω_n	$\Omega_0^{(n)}\{R(z)\} \triangleq \xi^0 R(z)$	$\Omega_1^{(n)}\{R(z)\} \triangleq \xi^{j c_n} R(z)$	[0 1 0 1]	x_{π_n}	x_{π_n}
Shift τ_n	$\Delta_0^{(n)}\{R(z)\} \triangleq z^0 R(z)$	$\Delta_1^{(n)}\{R(z)\} \triangleq z^{d_n} R(z)$	[0 1 0 1]	x_{π_n}	x_{π_n}
$A(z)$	$\dot{O}_0^{(n)}\{R(z)\} \triangleq \xi^0 R(z)$	$\dot{O}_1^{(n)}\{R(z)\} \triangleq \begin{cases} \xi^0 R(z), & n \neq 1 \\ A(z), & n = 1 \end{cases}$	[1 0 1 0]	$1 - x_{\pi_n}$	$1 - x_{\pi_n}$
$B(z)$	$\ddot{O}_0^{(n)}\{R(z)\} \triangleq \xi^0 R(z)$	$\ddot{O}_1^{(n)}\{R(z)\} \triangleq \begin{cases} \xi^0 R(z), & n \neq 1 \\ B(z), & n = 1 \end{cases}$	[0 1 0 1]	x_{π_n}	x_{π_n}

Proof. The right hand side of (27) is 0 for $f_1(\mathbf{x}) = f_2(\mathbf{x})$ and k_1 for $f_1(\mathbf{x}) \neq f_2(\mathbf{x})$. For (28), after the arguments of $(\cdot)_2$ are arranged as the summations of two Boolean functions, (27) is applied. \square

REFERENCES

- [1] S. H. Han and J. H. Lee, "An overview of peak-to-average power ratio reduction techniques for multicarrier transmission," *IEEE Wireless Commun.*, vol. 12, no. 2, pp. 56–65, Apr. 2005.
- [2] Y. Rahmatallah and S. Mohan, "Peak-to-average power ratio reduction in OFDM systems: A survey and taxonomy," *IEEE Commun. Surveys Tut.*, vol. 15, no. 4, pp. 1567–1592, Fourth 2013.
- [3] G. Wunder, R. F. H. Fischer, H. Boche, S. Litsyn, and J. No, "The PAPR problem in OFDM transmission: New directions for a long-lasting problem," *IEEE Signal Processing Magazine*, vol. 30, no. 6, pp. 130–144, Nov. 2013.
- [4] J. A. Davis and J. Jedwab, "Peak-to-mean power control in OFDM, Golay complementary sequences, and Reed-Muller codes," *IEEE Trans. Inf. Theory*, vol. 45, no. 7, pp. 2397–2417, Nov. 1999.
- [5] M. Golay, "Complementary series," *IRE Trans. Inf. Theory*, vol. 7, no. 2, pp. 82–87, Apr. 1961.
- [6] R. Turyn, "Hadamard matrices, Baumert-Hall units, four-symbol sequences, pulse compression, and surface wave encodings," *Journal of Combinatorial Theory, Series A*, vol. 16, no. 3, pp. 313–333, 1974.
- [7] R. Sivaswamy, "Multiphase complementary codes," *IEEE Trans. Inf. Theory*, vol. 24, no. 5, pp. 546–552, Sep. 1978.
- [8] S. Z. Budisin, "New complementary pairs of sequences," *Electronics Letters*, vol. 26, no. 13, pp. 881–883, Jun. 1990.
- [9] —, "New multilevel complementary pairs of sequences," *Electronics Letters*, vol. 26, no. 22, pp. 1861–1863, Oct. 1990.
- [10] E. Garcia, J. J. Garcia, J. U. A. M. C. Perez, and A. Hernandez, "Generation algorithm for multilevel LS codes," *Electronics Letters*, vol. 46, no. 21, pp. 1465–1467, Oct. 2010.
- [11] E. Garcia, J. Ureña, and J. J. Garcia, "Generation and correlation architectures of multilevel complementary sets of sequences," *IEEE Trans. Signal Process.*, vol. 61, no. 24, pp. 6333–6343, Dec. 2013.
- [12] S. Boyd, "Multitone signals with low crest factor," *IEEE Trans. Circuits Syst.*, vol. 33, no. 10, pp. 1018–1022, Oct. 1986.
- [13] B. M. Popovic, "Synthesis of power efficient multitone signals with flat amplitude spectrum," *IEEE Trans. Commun.*, vol. 39, no. 7, pp. 1031–1033, Jul. 1991.
- [14] J. A. Davis and J. Jedwab, "Peak-to-mean power control and error correction for OFDM transmission using Golay sequences and Reed-Muller codes," *Electronics Letters*, vol. 33, no. 4, pp. 267–268, Feb. 1997.
- [15] K. G. Paterson, "Generalized Reed-Muller codes and power control in OFDM modulation," *IEEE Trans. Inf. Theory*, vol. 46, no. 1, pp. 104–120, Jan. 2000.
- [16] Y. Li and W. B. Chu, "More Golay sequences," *IEEE Trans. Inf. Theory*, vol. 51, no. 3, pp. 1141–1145, Mar. 2005.
- [17] Y. Li and Y.-C. Kao, "Structures of non-GDJ golay sequences," in *Proc. International Symposium on Information Theory (ISIT)*, Sep. 2005, pp. 378–381.
- [18] F. Fiedler and J. Jedwab, "How do more Golay sequences arise?" *IEEE Trans. Inf. Theory*, vol. 52, no. 9, pp. 4261–4266, Sep. 2006.
- [19] F. Fiedler, J. Jedwab, and M. G. Parker, "A multi-dimensional approach to the construction and enumeration of Golay complementary sequences," *Journal of Combinatorial Theory, Series A*, vol. 115, no. 5, pp. 753–776, 2008.
- [20] C. Rößing and V. Tarokh, "A construction of OFDM 16-QAM sequences having low peak powers," *IEEE Trans. Inf. Theory*, vol. 47, no. 5, pp. 2091–2094, Jul. 2001.
- [21] C. V. Chong and V. Tarokh, "Two constructions of 16-QAM Golay complementary sequences," in *Proc. IEEE International Symposium on Information Theory (ISIT)*, Jun. 2002, p. 240.
- [22] C. V. Chong, R. Venkataramani, and V. Tarokh, "A new construction of 16-QAM Golay complementary sequences," *IEEE Trans. Inf. Theory*, vol. 49, no. 11, pp. 2953–2959, Nov. 2003.
- [23] Y. Li, "Comments on 'a new construction of 16-QAM Golay complementary sequences' and extension for 64-QAM Golay sequences," *IEEE Trans. Inf. Theory*, vol. 54, no. 7, pp. 3246–3251, Jul. 2008.
- [24] H. Lee and S. W. Golomb, "A new construction of 64-QAM Golay complementary sequences," *IEEE Trans. Inf. Theory*, vol. 52, no. 4, pp. 1663–1670, Apr. 2006.
- [25] C. Chang, Y. Li, and J. Hirata, "New 64-QAM Golay complementary sequences," *IEEE Trans. Inf. Theory*, vol. 56, no. 5, pp. 2479–2485, May 2010.
- [26] Y. Li, "A construction of general QAM Golay complementary sequences," *IEEE Trans. Inf. Theory*, vol. 56, no. 11, pp. 5765–5771, Nov. 2010.
- [27] S. Z. Budišin and P. Spasojević, "Paraunitary-based Boolean generator for QAM complementary sequences of length 2^K ," *IEEE Trans. Inf. Theory*, vol. 64, no. 8, pp. 5938–5956, Aug. 2018.
- [28] Z. Wang, E. Xue, and G. Gong, "New constructions of complementary sequence pairs over 4^q -QAM," 2020.

- [29] Z. Wang, D. Ma, G. Gong, and E. Xue, "New construction of complementary sequence (or array) sets and complete complementary codes," 2020.
- [30] Y. Zhou, Z. Zhou, Z. Liu, P. Fan, and Y. L. Guan, "Low-PMEPR preamble sequence design for dynamic spectrum allocation in OFDMA systems," *IEEE Trans. Commun.*, vol. 68, no. 5, pp. 2922–2933, Feb. 2020.
- [31] C. Chen, "Complementary sets of non-power-of-two length for peak-to-average power ratio reduction in OFDM," *IEEE Trans. Inf. Theory*, vol. 62, no. 12, pp. 7538–7545, Dec. 2016.
- [32] M. G. Parker, K. G. Paterson, and C. Tellambura, "Golay complementary sequences," in *Wiley Encyclopedia of Telecommunications*, 2003.
- [33] F. Fiedler, J. Jedwab, and M. G. Parker, "A framework for the construction of Golay sequences," *IEEE Trans. Inf. Theory*, vol. 54, no. 7, pp. 3114–3129, Jul. 2008.
- [34] ETSI, "Digital Video Broadcasting (DVB); Second generation framing structure, channel coding and modulation systems for broadcasting, interactive services, news gathering and other broadband satellite applications; Part 1: DVB-S2," EN 302 307-1 V1.4.1, Nov. 2014.
- [35] IEEE, "P802.11ay/D5.0 IEEE standard for information technology Part 11: Wireless LAN medium access control (MAC) and physical layer (PHY) specifications - Amendment 2: Enhanced throughput for operation in license-exempt bands above 45 GHz," *IEEE P802.11ay/D5.0*, pp. 1–790, Oct. 2019.
- [36] F. Zeng, "A sufficient condition producing 16-QAM Golay complementary sequences," *IEEE Commun. Lett.*, vol. 18, no. 11, pp. 1875–1878, Nov. 2014.
- [37] T. Jiang, C. Ni, and Y. Xu, "Novel 16-QAM and 64-QAM near-complementary sequences with low PMEPR in OFDM systems," *IEEE Trans. Commun.*, vol. 64, no. 10, pp. 4320–4330, Oct. 2016.
- [38] B. Tarokh and H. R. Sadjadpour, "Construction of OFDM M-QAM sequences with low peak-to-average power ratio," *IEEE Trans. Inf. Theory*, vol. 51, no. 1, pp. 25–28, Jan. 2003.
- [39] Z. Liu, Y. Li, and Y. L. Guan, "New constructions of general QAM Golay complementary sequences," *IEEE Trans. Inf. Theory*, vol. 59, no. 11, pp. 7684–7692, Nov. 2013.
- [40] W. H. Holzmann and H. Kharaghani, "A computer search for complex Golay sequences," *Aust. Journ. Comb.*, vol. 10, pp. 251–258, Apr. 1994.
- [41] Z. Liu, Y. L. Guan, U. Paramalli, and S. Hu, "Spectrally-constrained sequences: Bounds and constructions," *IEEE Trans. Inf. Theory*, vol. 64, no. 4, pp. 2571–2582, Apr. 2018.
- [42] J. J. Benedetto, I. Konstantinidis, and M. Rangaswamy, "Phase-coded waveforms and their design," *IEEE Signal Processing Magazine*, vol. 26, no. 1, pp. 22–31, Jan. 2009.
- [43] A. J. Grant and R. D. Van Nee, "Efficient maximum-likelihood decoding of Q-ary modulated Reed-Muller codes," *IEEE Communications Letters*, vol. 2, no. 5, pp. 134–136, 1998.
- [44] K. Schmidt and A. Finger, "Simple maximum-likelihood decoding of generalized first-order Reed-Muller codes," *IEEE Communications Letters*, vol. 9, no. 10, pp. 912–914, Oct. 2005.
- [45] K. G. Paterson and A. E. Jones, "Efficient decoding algorithms for generalized Reed-Muller codes," *IEEE Trans. Commun.*, vol. 48, no. 8, pp. 1272–1285, 2000.