Adaptive Output Tracking Control of Piecewise Affine Systems With Prescribed Performance

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***Abstract*—In this article, we investigate the adaptive output tracking control for multiinput–multioutput piecewise affine systems with prescribed performance. Both direct and indirect adaptation approaches are studied. Given a desired trajectory, both control approaches ensure the output tracking error to be confined within a performance bound, which prescribes the steady-state tracking error as well as the transient behavior, such as decaying rate and overshoot. We establish novel common Lyapunov functions without solving the conventional Lyapunov equations. Based on these common Lyapunov functions, the stability of the closed-loop system under arbitrary switching is established. Furthermore, the parameter convergence for both direct and indirect approaches is proved under the persistently exciting condition of the input signals. The dynamic gain adjustment technique is incorporated to counter the singularity problem in the indirect adaptation case. Finally, the numerical simulation validates the effectiveness and correctness of the proposed approaches in both direct and indirect adaptation cases.**

***Index Terms*—Adaptive control, hybrid systems, piecewise affine (PWA) systems, prescribed performance.**

I. Introduction

DURING recent years, the analysis and controller design of hybrid systems have attracted a lot of interest in the research community. Piecewise affine (PWA) systems [1] are proposed to model hybrid systems and to simplify the analysis. The state space of a PWA system is divided into several convex regions. In each region, the PWA system is governed by an associated linear subsystem dynamics. In practice, PWA systems have been used to model switching circuits such as various dc–dc converters [2], [3]. Another favorable application field is mechanical systems with piecewise linear (PWL) characteristics, such as friction [4], backlash [5], and saturation [6].

Manuscript received March 13, 2021; revised June 18, 2021 and September 10, 2021; accepted October 14, 2021. This article was recommended by Associate Editor K. G. Vamvoudakis. *(Corresponding author: Yufeng Gao.)*

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This article has supplementary material provided by the authors and color versions of one or more figures available at https://doi.org/10.1109/TSMC.2021.3122982.

Digital Object Identifier 10.1109/TSMC.2021.3122982

Considering the model uncertainties, changing environments and external disturbances in the real world, the controller with pretuned and static gains may not suffice to stabilize the closed-loop systems. The adaptive control approach is introduced such that the controller gains are adapted in real time and the desired system behavior can be maintained.

In the literature, various adaptive control algorithms for PWA systems or PWL systems, which serve as the modified version of PWA systems, are explored. The direct model reference adaptive control (MRAC) approaches of PWL systems for state tracking and output tracking are reported in [7] and [8], respectively. For PWA systems, a hybrid MRAC approach based on minimal control synthesis is proposed for the continuous case [9] and discrete case [10]. By assuming the existence of a common Lyapunov function, the stability of the controlled PWA system in control canonical form without sliding mode is guaranteed. This approach is extended in [11] such that the stability is ensured even when the closed-loop system exhibits sliding mode. The work in [12] generalizes the MRAC approach to multivariable PWA systems. In particular, the indirect MRAC approach, which is rarely studied for PWA systems before, is also discussed. Given a persistently exciting (PE) reference signal, both the tracking task and the estimation of subsystem parameters of the PWA systems can be achieved.

The aforementioned MRAC approaches for PWA systems ensure asymptotic tracking, namely, zero steady-state tracking error. However, the transient behavior of the closed-loop systems is not guaranteed and can only be improved by manually tuning the adaptation gains or imposing additional PE conditions, which is not always feasible. The analysis and improvement of the transient behavior are an essential issue in adaptive control [13], because an aggressive transient response may result in saturation, oscillation, or even damage to the physical plants in real applications. In this article, we would like to explore the adaptive control of PWA systems with the performance guarantee of transient behavior and steady tracking error.

Prescribed performance control, proposed in [14] and [15], is a popular tool to guarantee the elementwise performance of adaptive systems. With this approach, the steady-state tracking error and the transient response, such as decaying rate as well as overshoot, are confined within a predefined bound. This approach has been incorporated into different areas, such as multiagent systems [16]–[19], helicopter/satellite attitude control [20], [21], underwater vehicles [22], and robot manipulators [23]. Besides, it has also been introduced to the field of switched systems [24], [25].

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.

Despite the reviewed advances, the problem of designing the adaptive control of uncertain PWA systems with a prescribed performance guarantee is still challenging. Most of the existing MRAC approaches of PWA systems proposed in [9]–[12] achieve closed-loop stability under arbitrary switching by constructing the common Lyapunov functions, whose existence relies on the solution of a set of Lyapunov equations associated with the PWL error dynamics. This is not applicable to the systems with prescribed performance technique due to its nonlinear error transform. Besides, the parameter convergence, a topic of major interest in the area of adaptive control [26], has not been fully explored in the area of prescribed performance control [14]–[25]. The nonlinear error transform of the prescribed performance technique introduces extra nonlinearity into the adaptive systems, which makes the classical theorems of parameter convergence for linear and PWA systems [27] not applicable. Furthermore, the loss of controllability issue needs to be carefully treated for PWA systems with uncertain input matrices. The classical solution by using dynamic gain adjustment [12] needs the knowledge of reference systems, which are not available in the context of prescribed performance control.

Our main contribution lies in tackling the direct and indirect adaptive output tracking control problem of uncertain PWA systems with prescribed performance. Specifically, we cast the dynamics of the transformed error metric into linear form, where the nonlinearity and switching are captured as its exogenous input. Based on that, we construct novel common Lyapunov functions, which do not rely on the solution of the conventional Lyapunov equations shown in [9]–[12], and prove the closed-loop stability under arbitrary switching. We further prove that the estimated controller and system parameters converge to their nominal values under PE conditions. Moreover, we propose a novel dynamic gain adjustment technique and solve the loss of controllability issue in the indirect adaptation case.

The remainder of this article is structured as follows. In Section II, the PWA system we study is defined and the prescribed performance is revisited. The design of nominal control goes in Section III, which is followed by the direct adaptive control in Section IV and indirect adaptive control in Section V. The approaches are validated through numerical examples in Section VI. Finally, we give the discussion and conclusion in Section VII.

II. Preliminaries and Problem Formulation

A. System Description

Consider the multi-input multi-output (MIMO) PWA system with *strict relative degree* $r \in \mathbb{N}$ and $s \in \mathbb{N}$ subsystems described by

$$\begin{aligned} x_1^{(r)} &= a_{1i}^T x + b_{1i}^T u + f_{1i} \\ &\vdots \\ x_p^{(r)} &= a_{pi}^T x + b_{pi}^T u + f_{pi}, \quad i = 1, \dots, s \\ y &= [x_1, x_2, \dots, x_p]^T \end{aligned} \tag{1}$$

where

$$x_j^{(r)} = \frac{d^r x_j}{dt^r} \tag{2}$$

and $x = [x_1, \dots, x_1^{(r-1)}, \dots, x_p, \dots, x_p^{(r-1)}]^T \in \mathbb{R}^n$ denotes the overall state vector with $n = pr$. $u, y \in \mathbb{R}^p$ represent the control input and system output, respectively. The output y and its derivatives up to order $r-1$ constitute the state vector x. They are available for the control design. $a_{ji} \in \mathbb{R}^n, b_{ji} \in \mathbb{R}^p, f_{ji} \in \mathbb{R}, j = 1, \dots, p$ denote the system parameters of the ith subsystem. We write system (1) into compact form and obtain

$$\begin{aligned} \dot{x} &= A_i x + B_i u + f_i, \quad i = 1, \dots, s \\ y &= Cx \end{aligned} \tag{3}$$

where $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{p\times n}$, and $f_i \in \mathbb{R}^n$ denote the system parameters of the ith subsystem. $a_{ji}, b_{ji}, f_{ji}, j = 1, \dots, p, i = 1, \dots, s$ are contained in A_i, B_i, and f_i in the corresponding positions, respectively, and thus, A_i, B_i, C, and f_i are in the control canonical form. Since a large class of physical systems can be modeled [28], [29] and transformed [30] into canonical form, its control design is essential and attracts a lot of interests, such as [9]–[11]. In this article, we focus on the prescribed performance adaptive control of MIMO PWA systems in control canonical form.

Since the system has *strict relative degree* r, we have

$$\begin{aligned} CB_i = CA_iB_i = \cdots = CA_i^{r-2}B_i = 0, &\quad CA_i^{r-1}B_i \neq 0 \\ Cf_i = CA_if_i = \cdots = CA_i^{r-2}f_i = 0, &\quad CA_i^{r-1}f_i \neq 0 \end{aligned} \tag{4}$$

for $i = 1, \dots, s$, which leads to

$$\begin{aligned} y &= Cx \\ \dot{y} &= CA_i x \\ &\cdots \\ y^{(r)} &= CA_i^r x + CA_i^{r-1}B_i u + CA_i^{r-1} f_i. \end{aligned} \tag{5}$$

In this article, the system input and output have the same dimension p and the system is a square system. Nevertheless, this will not necessarily restrict our approach, since square systems cover broad applications [31]. Some nonsquare systems can also be transformed into square systems [32], [33].

In PWA systems, the state space $x \in \mathbb{R}^n$ is partitioned by switching hyperplanes into s polyhedral regions $\{\Omega_i\}$ with $i = 1, \dots, s$. Among the regions, there is no overlap, i.e., $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. We use the indicator function to indicate in which region the state locates, or equivalently, which subsystem is activated

$$\chi_i(t) = \begin{cases} 1, & \text{if } x(t) \in \Omega_i \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

and $\sum_{i=1}^s \chi_i = 1$. With the indicator functions, we can rewrite the PWA system as

$$\begin{aligned} \dot{x} &= Ax + Bu + f \\ y &= Cx \end{aligned} \tag{7}$$

where $A = \sum_{i=1}^s \chi_i A_i$, $B = \sum_{i=1}^s \chi_i B_i$, and $f = \sum_{i=1}^s \chi_i f_i$.

Remark 1: The state x in the PWA system (1) is continuous, also on the switching hyperplanes. This leads to the continuity of $y, \dot{y}, \ldots, y^{(r-1)}$ according to the definition of y [see (1)]. This, in turn, implies $CA_i = CA_j, \ldots, CA_i^{r-1} = CA_j^{r-1}$ for $\forall i, j = 1, \ldots, s$. If the PWA system is not in control canonical form, this property does not hold and the output derivative may exhibit jump behavior on the switching hyperplanes.

B. Prescribed Performance Technique

In this article, we investigate the output tracking of the PWA systems. We assume that the reference signals $y_d \in \mathbb{R}^p$ and its derivatives $\dot{y}_d, \ldots, y_d^{(r)} \in \mathbb{R}^p$ are bounded and continuous. To study the output tracking with prescribed performance, we first introduce the definition performance function and study its properties in control systems.

Definition 1 (Performance Function [14]): A smooth positive function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as the performance function if it is decreasing and satisfies $\lim_{t\to\infty} \rho(t) = \rho_\infty > 0$.

A commonly used performance function is

$$\rho(t) = (\rho_0 - \rho_\infty)e^{-lt} + \rho_\infty \tag{8}$$

with $\rho_0, \rho_\infty, l \in \mathbb{R}^+$ and $\rho_0 > \rho_\infty$. We see that $\rho(t)$ is decreasing with $\rho(t=0) = \rho_0$ and $\rho(t \to \infty) = \rho_\infty$.

Given the reference output y_d and a vector performance function $\rho(t) \in \mathbb{R}^p$, let $e = [e_1, e_2, \ldots, e_p]^T \in \mathbb{R}^p$ be the output tracking error $y - y_d$, $\rho_j(t)$ be the performance function of the jth component of ρ, the control objective that the tracking error is confined within a prescribed performance bound can be expressed by the following inequalities:

$$\begin{aligned} -\delta_j\rho_j(t) < e_j(t) < \rho_j(t), &\quad \text{if } e_j(0) > 0 \\ -\rho_j(t) < e_j(t) < \delta_j\rho_j(t), &\quad \text{if } e_j(0) < 0 \end{aligned} \tag{9}$$

for $j = 1, \ldots, p$, for $\delta_j \in [0, 1]$ and $\forall t > 0$. δ_j is a design parameter. With smaller δ_j, the overshoot of the jth component of the tracking error can be reduced. This overshoot becomes 0 if $\delta_j = 0$.

The concept of prescribed performance control is to transform the constrained error (9) into an unconstrained one, and thus, the classical stability theory can be applied to design the controller for the unconstrained transformed error. Let σ_j be the transformed error and define $e_j = \rho_j(t)G_j(\sigma_j)$, where $G_j(\sigma_j)$ is a smooth and strictly increasing function of transformed error σ_j. Note that inequalities in (9) are equivalent to

$$\begin{aligned} -\delta_j < G_j(\sigma_j) < 1, &\quad \text{if } e_j(0) > 0 \\ -1 < G_j(\sigma_j) < \delta_j, &\quad \text{if } e_j(0) < 0 \end{aligned} \tag{10}$$

so the strictly increasing function $G_j(\sigma_j)$ needs to be designed such that (10) holds for $\sigma_j \in (-\infty, +\infty)$. We choose the following function as the most references suggested:

$$\begin{aligned} G_j(\sigma_j) &= \frac{\exp(\sigma_j) - \delta_j\exp(-\sigma_j)}{\exp(\sigma_j) + \exp(-\sigma_j)}, \quad \text{if } e_j(0) > 0 \\ G_j(\sigma_j) &= \frac{\delta_j\exp(\sigma_j) - \exp(-\sigma_j)}{\exp(\sigma_j) + \exp(-\sigma_j)}, \quad \text{if } e_j(0) < 0. \end{aligned} \tag{11}$$

The transformed error σ_j can thus be solved by

$$\sigma_j = \begin{cases} G_j^{-1}\left(\frac{e_j(t)}{\rho_j(t)}\right) = \frac{1}{2}\ln\frac{\delta_j+G_j}{1-G_j}, & \text{if } e_j(0) > 0 \\ G_j^{-1}\left(\frac{e_j(t)}{\rho_j(t)}\right) = \frac{1}{2}\ln\frac{1+G_j}{\delta_j-G_j}, & \text{if } e_j(0) < 0 \end{cases} \tag{12}$$

from which we can see, if σ_j is bounded, then (10) holds, which further implies that (9) holds. To relate the transformed error σ_j with the tracking error e_j, we take for instance the time derivative of σ_j for $e_j(0) > 0$ and it yields

$$\dot{\sigma}_j = q_{0,j}^1 e_j + q_{1,j}^1 \dot{e}_j \tag{13}$$

with

$$\begin{aligned} q_{0,j}^1 &= -\frac{\dot{\rho}_j}{2\rho_j^2}\frac{\delta_j + 1}{\left(1 - \frac{e_j}{\rho_j}\right)\left(\delta_j + \frac{e_j}{\rho_j}\right)} \\ q_{1,j}^1 &= \frac{1}{2\rho_j}\frac{\delta_j + 1}{\left(1 - \frac{e_j}{\rho_j}\right)\left(\delta_j + \frac{e_j}{\rho_j}\right)} \end{aligned}$$

and similarly, the kth derivative of σ_j is

$$\sigma_j^{(k)} = \sum_{l=0}^{k-1} q_{l,j}^k\left(\rho_j, \ldots, \rho_j^{k-l}\right)e_i^{(l)} + \frac{\partial G_j^{-1}}{\partial\left(\frac{e_j}{\rho_j}\right)}\frac{1}{\rho_j}e_j^{(k)} \tag{14}$$

where $q_{l,j}^k(\rho_j, \ldots, \rho_j^{k-l})$ represents a term that depends on $\rho_j, \ldots, \rho_j^{(k-l)}$ for some given k and $l = 1, 2, \ldots, k-1$. Define the error metric E_j

$$E_j = \sigma_j + \sum_{k=1}^{r-1} \lambda_k \sigma_j^{(k)} \tag{15}$$

where $\lambda_k \in \mathbb{R}^+$ are parameters to be chosen, and $\sigma_j^{(k)}$ is the kth derivative of σ_j. E_j is utilized to describe the dynamics of the transformed error system. The derivative of E_j follows:

$$\dot{E}_j = \sum_{k=0}^{r-1}\sum_{l=k-1}^{r-1} \lambda_l q_{k,j}^{l+1} e_j^{(k)} + \lambda_{r-1} q_{r,j}^r e_j^{(r)} \tag{16}$$

with $\lambda_{-1} = 0$ and $\lambda_0 = 1$. We can write the vector form

$$\dot{E} = \sum_{k=0}^{r-1}\sum_{l=k-1}^{r-1} \lambda_l R_k^{l+1} e^{(k)} + \lambda_{r-1} R_r^r e^{(r)} \tag{17}$$

with $E = [E_1, \ldots, E_p]^T \in \mathbb{R}^p$ and

$$R_k^l = \begin{bmatrix} q_{k,1}^l & & 0 \\ & \ddots & \\ 0 & & q_{k,p}^l \end{bmatrix}. \tag{18}$$

Since $\rho(t)$ and y_d are known, each component of their derivative up to the rth order can be calculated. The system state x is assumed to be available and thus $y, \dot{y}, \ldots, y^{(r-1)}$ are also available. Substituting $e^{(r)}$ in (17) with $y^{(r)} - y_d^{(r)}$ and inserting (5) yield

$$\begin{aligned} \dot{E} = &\underbrace{\sum_{k=0}^{r-1}\sum_{l=k-1}^{r-1} \lambda_l R_k^{l+1} e^{(k)} - \lambda_{r-1} R_r^r y_d^{(r)} + \lambda_{r-1} R_r^r CA^r x}_{:=K} \\ &+ \lambda_{r-1} R_r^r CA^{r-1} Bu + \lambda_{r-1} R_r^r CA^{r-1} f. \end{aligned} \tag{19}$$

This step associates the system input u with error metric E. If the control input u is designed such that E is bounded, then the boundedness of $\sigma_j^{(k)}$ is ensured for $k = 1, \ldots, r-1, j = 1, \ldots, p$. This further implies the achievement of prescribed performance described by the inequalities in (9).

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For the purpose of clarity, we replace R_r^r with R and λ_{r-1} with λ in the rest of this article and express $\dot{E}$ as

$$\dot{E} = K + \lambda RCA^r x + \lambda RCA^{r-1}Bu + \lambda RCA^{r-1}f. \tag{20}$$

C. Signal Properties

We revisit some signal properties, which are essential for the analysis in this article.

Definition 2 (Persistence of Excitation (PE) [34]): A piecewise continuous signal vector $z : \mathbb{R}^+ \to \mathbb{R}^n$ is PE with a level of excitation α_0 if there exist constants $\alpha_1, T_0 > 0$ such that

$$\alpha_1 I \geq \frac{1}{T_0}\int_t^{t+T_0} z(\tau)z^T(\tau)\mathrm{d}\tau \geq \alpha_0 I \quad \forall t \geq 0.$$

The idea behind PE property is that some internal signals should contain rich frequency components. A closely related property is sufficiently rich property [34, Def. 5.2.1], namely, a signal $u : \mathbb{R}^+ \to \mathbb{R}$ is called sufficiently rich of order $2n$, if it contains at least n distinct frequencies.

D. Problem Formulation

In multivariable adaptive control, a common assumption is the prior knowledge of a matrix $S_i \in \mathbb{R}^{p\times p}$ such that $K_{ri}^* S_i$ is symmetric and positive definite [26], where $K_{ri}^* = -(CA_i^{r-1}B_i)^{-1}$ is the nominal control gain for the ith subsystem (details will be given in Section III). The meaning of S_i becomes more intuitive for a scalar K_{ri}^*, where S_i becomes the sign of the nominal control gain sign$[K_{ri}^*]$. A known S_i describes that the control direction is known but the control effectiveness (or magnitude) is unknown. This is quite often the case in applications, such as vehicles and aircraft [32], [35]. Interested readers may refer to [26, Sec. 4.2.6] for some relaxation techniques of this assumption. The problem to be addressed in this article is formulated as follows.

Problem 1: Given a PWA system (3) with known subsystem partitions Ω_i, unknown subsystem parameters A_i, B_i, and f_i and known S_i, design an adaptive control law $u(t)$ to enforce the output of the system $y(t)$ to track the given reference signal $y_d(t)$ with prescribed error performance (9). Besides, explore the conditions, under which the estimated gains or estimated parameters converge to their nominal or real values.

III. Nominal Control

We start with the nominal control design, where the subsystem parameters and switching hyperplanes are known exactly.

The following control law, which is suggested by the Lyapunov stability analysis (will be shown in Theorem 1), is proposed:

$$u = K_x^* x + K_r^* \xi + K_f^* \tag{21}$$

where

$$\xi = \frac{1}{\lambda}R^{-1}E + \frac{1}{\lambda}R^{-1}K \tag{22}$$

and

$$\begin{aligned} K_x^* &= \sum_{i=1}^{s}\chi_i K_{xi}^* = -\sum_{i=1}^{s}\chi_i (C\Psi_i)^{-1}C\Phi_i \\ K_r^* &= \sum_{i=1}^{s}\chi_i K_{ri}^* = -\sum_{i=1}^{s}\chi_i (C\Psi_i)^{-1} \\ K_f^* &= \sum_{i=1}^{s}\chi_i K_{fi}^* = -\sum_{i=1}^{s}\chi_i (C\Psi_i)^{-1}C\Upsilon_i \end{aligned} \tag{23}$$

are nominal controller gains with

$$\Phi_i = A_i^r, \quad \Psi_i = A_i^{r-1}B_i, \quad \Upsilon_i = A_i^{r-1}f_i. \tag{24}$$

Note that $C\Psi_i$ is assumed to be invertible for $i = 1, \ldots, s$. The controller structure (21), the definition of ξ (22), as well as the nominal controllers (23) are determined by the Lyapunov-based stability analysis. The performance analysis of the proposed nominal control law and the closed-loop stability is summarized in the following theorem.

Theorem 1: Given the reference signal y_d and predefined performance function $\rho(t)$, let the PWA system (3) with known partition regions Ω_i and known subsystem parameters A_i, B_i, and f_i be controlled by the feedback controller (21). Let ρ be designed such that the inequality (9) holds at initial time instant $t = 0$. The closed-loop system is stable and the output tracking error satisfies the prescribed performance (9).

Proof: Substituting u in (20) with (21) and inserting (23), we obtain

$$\dot{E} = K + \lambda RC\Phi x + \lambda RC\Psi u + \lambda RC\Upsilon = -E \tag{25}$$

where $\Phi = \sum_{i=1}^{s}\chi_i\Phi_i$, $\Psi = \sum_{i=1}^{s}\chi_i\Psi_i$, $\Upsilon = \sum_{i=1}^{s}\chi_i\Upsilon_i$ capture the switching effect. This means that the closed-loop dynamics of E can be described by the homogeneous system $\dot{E} = -E$ by applying the nominal controller (21). Define the following Lyapunov function:

$$V = \frac{1}{2}E^T E \tag{26}$$

taking the derivative along the trajectory (20) yields

$$\dot{V} = -E^T E \leq 0. \tag{27}$$

From (27), it follows $E \in \mathcal{L}_\infty$ and $E \to 0$ as $t \to \infty$. This further implies the boundedness of $\sigma_j, \sigma_j^{(k)}$ with $\sigma_j, \sigma_j^{(k)} \to 0$ as $t \to \infty$ $\forall j = 1, \ldots, p$, which leads to $y, y^{(k)} \in \mathcal{L}_\infty, k = 1, \ldots, r-1$, and thus, $x \in \mathcal{L}_\infty$. From the definition of K and (22), we also have $K \in \mathcal{L}_\infty$ and $\xi \in \mathcal{L}_\infty$. From (12) and the boundedness of σ_j, we can conclude that the tracking error is within the performance bound, i.e., (9) holds. ■

Remark 2: The asymptotic tracking can be achieved under certain conditions. From Theorem 1, we have $\sigma_j \to 0$. Given certain δ_j, $\lim_{t\to\infty} G_j$ can be obtained by solving $\lim_{t\to\infty}\sigma_j = 0$ according to (12). If $\delta_j = 1$, then we obtain $G_j \to 0$ for $t \to \infty$. Since $G_j = e_j/\rho_j$ and $\rho_j \neq 0$, the jth component of the tracking error $e_j \to 0$ as $t \to \infty$.

Remark 3: The controller (21) shares the common structure as the controller of MRAC, i.e., $u = K_x x + K_r r + K_f$ (see [12, eq. (11)]). The difference is that the reference signal r of the MRAC is replaced by ξ in this context. Unlike the reference signal r, which is given as an external signal in MRAC, ξ also contains internal signals. As shown by (22), ξ contains the error metric E and output tracking errors, as well as their higher order derivatives (captured by K). Therefore, its boundedness needs to be specially checked, as shown in the proof of Theorem 1.

Remark 4: According to (15), we have that E depends on σ_j and its derivatives, which in turn relates to the tracking error e and its derivatives. Since y_d and its derivatives, as well as $y, \dot{y}, \ldots, y^{(r-1)}$ are continuous (see Remark 1), E is also continuous even on switching hyperplanes. Therefore, the Lyapunov function (26) is shared by all the subsystems and it decreases independent of which subsystem is activated. This implies that the Lyapunov function (26) is a *common Lypaunov function* and the closed-loop stability can be concluded even under arbitrary switching. As stated in Remark 1, $y, \dot{y}, \ldots, y^{(r-1)}$ may exhibit jump behavior for generalized PWA systems, and this leads to discontinuous E and may ruin the stability for fast switching. A possible way to extend the approach to the generalized PWA systems might be defining the error metric in terms of the state tracking error [35] to avoid output derivatives with jumps.

IV. Direct Adaptation Case

In this section, we study the direct prescribed performance adaptive control for PWA systems with known state space partition and unknown subsystem parameters.

A. Controller Design

The controller takes the same structure as in (21) but with the estimated parameters

$$u = K_x x + K_r \xi + K_f \tag{28}$$

where

$$K_x = \sum_{i=1}^{s} \chi_i K_{xi}, \quad K_r = \sum_{i=1}^{s} \chi_i K_{ri}, \quad K_f = \sum_{i=1}^{s} \chi_i K_{fi}$$

are estimated controller gains. We propose the following adaptation law to update the estimated controller gains:

$$\begin{aligned} \dot{K}_{xi} &= \chi_i \Gamma_{xi} S_i^T R^T E x^T \\ \dot{K}_{ri} &= \chi_i \Gamma_{ri} S_i^T R^T E \xi^T \\ \dot{K}_{fi} &= \chi_i \Gamma_{fi} S_i^T R^T E \end{aligned} \tag{29}$$

where $\Gamma_{xi}, \Gamma_{ri}, \Gamma_{fi} \in \mathbb{R}^+$ are positive scaling factors.

We define the estimation errors of the controller gains as

$$\tilde{K}_{xi} = K_{xi} - K_{xi}^*, \tilde{K}_{ri} = K_{ri} - K_{ri}^*, \tilde{K}_{fi} = K_{fi} - K_{fi}^*. \tag{30}$$

We insert (28) in (20) and obtain

$$\begin{aligned} \dot{E} &= K + \sum_{i=1}^{s} \chi_i (\lambda R C \Phi_i x + \lambda R C \Psi_i u + \lambda R C \Upsilon_i) \\ &= K + \sum_{i=1}^{s} \chi_i \Big(\lambda R C \Phi_i x + \lambda R C \Psi_i K_{xi}^* x \\ &\quad + \lambda R C \Psi_i \tilde{K}_{xi} x + \lambda R C \Psi_i K_{ri}^* \xi + \lambda R C \Psi_i \tilde{K}_{ri} \xi \\ &\quad + \lambda R C \Psi_i K_{fi}^* + \lambda R C \Psi_i \tilde{K}_{fi} + \lambda R C \Upsilon_i \Big). \end{aligned} \tag{31}$$

Inserting the nominal controller gains (23) yields

$$\dot{E} = -E + \lambda R \sum_{i=1}^{s} \chi_i C \Psi_i \big(\tilde{K}_{xi} x + \tilde{K}_{ri} \xi + \tilde{K}_{fi} \big). \tag{32}$$

This equation describes the dynamics of the error metric E when the adaptive controller (28) is utilized. The estimation errors of controller gains $\tilde{K}_x$, $\tilde{K}_r$, and $\tilde{K}_f$ constitute the external inputs of the dynamics. The state transition matrix of E is $-I$ and thus is not affected by switching.

B. Stability Analysis

We study the stability and the tracking performance of the closed-loop system. The result is summarized in the following theorem.

Theorem 2: Given the reference signal y_d and predefined performance function $\rho(t)$, let the PWA system (3) with known partition regions Ω_i and unknown subsystem parameters be controlled by the feedback controller (28) with the update law (29). Let ρ be designed such that the inequality (9) holds at initial time instant $t = 0$. The closed-loop system is stable and the output tracking error satisfies the prescribed performance (9).

Proof: We define the following Lyapunov-like function:

$$\begin{aligned} V = \frac{E^T E}{2\lambda} + \frac{1}{2} \sum_{i=1}^{s} \Big(& \Gamma_{xi}^{-1} \mathrm{tr}\big(\tilde{K}_{xi}^T M_i \tilde{K}_{xi} \big) \\ & + \Gamma_{ri}^{-1} \mathrm{tr}\big(\tilde{K}_{ri}^T M_i \tilde{K}_{ri} \big) + \Gamma_{fi}^{-1} \mathrm{tr}\Big(\tilde{K}_{fi}^T M_i \tilde{K}_{fi} \Big) \Big) \end{aligned} \tag{33}$$

where $M_i = (K_{ri}^* S_i)^{-1} \in \mathbb{R}^{p \times p}$. Taking the time derivative of V, inserting (32) and (29) and doing some simplifications (details can be seen in the supplementary material) yield

$$\dot{V} = -\frac{1}{\lambda} E^T E \leq 0. \tag{34}$$

The negative semidefiniteness of $\dot{V}$ confirms the stability of the closed-loop adaptive system. More precisely, $E, \tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi} \in \mathcal{L}_\infty$. Considering $E \in \mathcal{L}_\infty$, (12), and (15), we have $\sigma, \sigma^{(k)}, e, e^{(k)} \in \mathcal{L}_\infty$, which further indicates $y, y^{(k)} \in \mathcal{L}_\infty, k = 1, \ldots, r-1$, and thus $x \in \mathcal{L}_\infty$.

The boundedness of $e^{(k)}$ leads to $R_k^l \in \mathcal{L}_\infty$ with $k = 0, 1, \ldots, r, l = 1, 2, \ldots, r$, from which we can obtain $K, \xi \in \mathcal{L}_\infty$ and hence, $\dot{K}_{xi}, \dot{K}_{ri}, \dot{K}_{fi} \in \mathcal{L}_\infty$. The boundedness of $\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi}, x$, and ξ gives $u \in \mathcal{L}_\infty$ and $\dot{E} \in \mathcal{L}_\infty$. Equation (34) also implies that $E \in \mathcal{L}_2$, which together with $E, \dot{E} \in \mathcal{L}_\infty$ gives $\lim_{t \to \infty} E \to 0$. This together with the boundedness of σ_j implies that the tracking error e is confined within the prescribed performance bound, i.e., (9) holds.

An essential issue in analyzing the stability of switched systems is that the closed-loop system may enter a sliding mode. Namely, both the vector fields of two neighboring subsystems point toward the switching hyperplane and the trajectory of the system cannot move across the regions. To analyze the stability in sliding mode, we follow the concept in [11] and [12] and observe the derivative of V along the sliding mode solutions, which can be achieved by replacing the indicator function $\chi_i \in \{0, 1\}$ with $\bar{\chi}_i \in [0, 1]$, where $\sum_{i=1}^{s} \bar{\chi}_i = 1$. Specifically, the transformed error dynamics (32) is convexified as

$$\dot{E} = -E + \lambda R \sum_{i=1}^{s} \bar{\chi}_i C\Psi_i\big(\tilde{K}_{xi}x + \tilde{K}_{ri}\xi + \tilde{K}_{fi}\big). \tag{35}$$

Equation (35) holds due to the synchronous switching of the plant and the controller. As a part of the closed-loop dynamics, the adaptation gains during the sliding motion are

$$\begin{aligned}\dot{K}_{xi} &= \bar{\chi}_i \Gamma_{xi} S_i^T R^T E x^T\\ \dot{K}_{ri} &= \bar{\chi}_i \Gamma_{ri} S_i^T R^T E \xi^T\\ \dot{K}_{fi} &= \bar{\chi}_i \Gamma_{fi} S_i^T R^T E.\end{aligned} \tag{36}$$

Inserting (35) and (36) into $\dot{V}$, we still obtain the same expression as in (34), which implies the stability of the controlled system also in sliding mode. ■

Remark 5: Theorem 2 shows that the tracking error stays within the prescribed performance bound. Note that $E, \sigma, \sigma^{(k)} \to 0, k = 1, \ldots, r-1$ as $t \to \infty$, the time limit of tracking error can thus be calculated by solving (12). For $\delta_j = 1, j \in \{1, \ldots, p\}$, we have the solution $\lim_{t\to\infty} e_j(t) = 0$.

Remark 6: Benefitting from the property that the state transition matrix of E is independent of the switching [as shown in (32)], the Lyapunov function (33) is a common Lypaunov function. It ensures the closed-loop stability under arbitrary switching. A similar concept to construct the common Lyapunov function can be found in the adaptive control for switched systems in Brunovsky form [36], where an error metric is constructed based on the tracking error and its derivatives (see [36, eq. (11)]). When comparing to the approach in [36], the distinctive feature of our approach is that the error metric E is expressed in terms of the transformed error σ_j and thus, the transient behavior evolves within the prescribed performance bound if E is bounded.

Remark 7: The stability analysis of classical MRAC of PWA systems in [9] and [12] also relies on the common Lyapunov function. It requires the existence and the knowledge of a common Lyapunov matrix P such that the Lyapunov equation $A_{mi}^T P + P A_{mi} < 0$ holds for all the state matrices A_{mi} of the reference PWA system. Differing from this requirement, the construction of the common Lyapunov function in our work only requires the continuity of the reference signal and its derivatives, which is less restrictive.

Remark 8: Theorem 2 shows that the tracking error e satisfies the prescribed performance condition, i.e., (9) holds. If the performance function is chosen as (8), the tracking error e decays exponentially. In the classical direct MRAC of PWA systems, the PE condition of the reference signals must be introduced to ensure the exponential decaying of tracking errors (see [7, Th. 2] and [12, Th. 2]). Besides, the decaying rate depends on the excitation level of the reference signals. Expressing it explicitly is not straightforward (see [12, eqs. (26) and (27)]). In contrast, the exponential decaying of the tracking error in our approach does not require PE conditions and the decaying rate can be specified directly in the performance function (8) by choosing the value of l.

C. Parameter Convergence

Theorem 2 shows the boundedness of the controller gains K_{xi}, K_{ri}, and K_{fi}. In this section, we discuss if the adaptive controller gains converge to the nominal gains under the classical PE conditions. First, we explore if the signal vector $z = [x^T, \xi^T, 1]^T$ is PE given a sufficiently rich reference signal y_d. This is summarized in the following lemma.

Lemma 1: Let the system (3) be controlled by the controller (28). If the closed-loop system has $E \in \mathcal{L}_\infty$, $E \to 0$ for $t \to \infty$, if the reference signal y_d is sufficiently rich of order $r+1$, and if $\delta_j = 1, j = 1, \ldots, p$, then the vector $z = [x^T, \xi^T, 1]^T$ is PE.

The proof can be seen in the supplementary material.

Remark 9: For the case, where the adaptive systems have to fulfill the desired tracking task y_d, which does not contain a sufficient amount of frequencies, the sufficiently rich condition can be fulfilled by superposing some periodic signals with the required amount of frequencies and small enough amplitudes upon the desired trajectory. By doing so, the sufficiently rich condition can be fulfilled without significantly disturbing the primary tracking task. Further analysis of parameter convergence relies on the PE condition of the closed-loop system signal vector z.

Since a PWA system has multiple subsystems, the controller gains of all the subsystems need to be estimated. To this end, we require that the reference signal y_d to be sufficiently rich and repeatedly activate all the subsystems as also suggested in other works of PWA systems [12], [27]. The conclusion is depicted by the following theorem.

Theorem 3: Let the PWA system (3) with known partition regions Ω_i and unknown subsystem parameters be controlled by the feedback controller (28) with the update law (29). Let ρ be designed such that the inequality (9) holds at initial time instant $t = 0$. Let the reference signals y_d be sufficiently rich of order $r+1$ and cause repeated activation of all subsystems. If the matrices $C\Psi_i$ are invertible, and $\delta_j = 1$ for $j = 1, \ldots, p$, then $\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi} \to 0$ for $t \to \infty$.

Proof: According to Theorem 2, the closed-loop system is stable under arbitrary switching. For clarity, we first study a single subsystem and suppose the ith subsystem to be activated during some time interval, i.e., $\chi_i(t) = 1$. We rewrite $\dot{E}$ as

$$\dot{E} = -E + \lambda R C\Psi_i\big(\tilde{K}_{xi}x + \tilde{K}_{ri}\xi + \tilde{K}_{fi}\big) \tag{37}$$

which can be further simplified by using the Kronecker product

$$\dot{E} = -E + \lambda R \Xi^T \tilde{\theta}_i \tag{38}$$

with

$$\Xi = \begin{bmatrix} x \\ \xi \\ 1 \end{bmatrix} \otimes I_p, \quad \tilde{\theta}_i = \mathrm{vec}\big(C\Psi_i\big[\tilde{K}_{xi} \quad \tilde{K}_{ri} \quad \tilde{K}_{fi}\big]\big) \tag{39}$$

where $\otimes$ denotes the Kronecker product, $I_p \in \mathbb{R}^{p\times p}$ is an identity matrix, and the operator vec$(\cdot)$ represents the vectorization of a matrix.

Note that

$$\begin{aligned}\dot{\tilde{\theta}}_i &= \mathrm{vec}\Big(C\Psi_i\Big[\dot{\tilde{K}}_{xi} \quad \dot{\tilde{K}}_{ri} \quad \dot{\tilde{K}}_{fi}\Big]\Big)\\ &= \mathrm{vec}\big(C\Psi_i S^T R^T E\big[x^T \quad \xi^T \quad 1\big]\big)\\ &= \Xi \cdot \mathrm{vec}\big(C\Psi_i S^T R^T E\big)\\ &= -\Xi \cdot W_i R^T E\end{aligned} \tag{40}$$

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where $W_i = C\Psi_i M_i^{-1}(C\Psi_i)^T$. We write E and $\tilde{\theta}_i$ in a form of a new dynamical system and obtain

$$\begin{bmatrix} \dot{E} \\ \dot{\tilde{\theta}}_i \end{bmatrix} = \begin{bmatrix} -I_p & \lambda R \Xi^T \\ -\Xi W_i R^T & 0 \end{bmatrix} \begin{bmatrix} E \\ \tilde{\theta}_i \end{bmatrix}. \tag{41}$$

From Theorem 2 and $\delta_j = 1, j = 1, \ldots, p$, we have $e_j(t) \to 0$. This leads to $R \to R^*$ as $t \to \infty$, where $R^* \in \mathbb{R}^{p\times p}$ is some constant diagonal matrix. R^* can be calculated by going through the derivation shown in Section II-B. Let r_j^* denote the jth diagonal element of R^* and we have

$$r_j^* = \frac{1}{2\rho_j(t)} \frac{\delta_j + 1}{\left(1 - \frac{e_j(t)}{\rho_j(t)}\right)\left(\delta_j + \frac{e_j(t)}{\rho_j(t)}\right)}\Bigg|_{t\to\infty} = \frac{1}{\rho_{\infty j}} \tag{42}$$

with $\rho_{\infty j} = \rho_j(t \to \infty)$ being the predefined static bound of the jth error component. For $R = R^*$, we have the dynamical system

$$\begin{bmatrix} \dot{E} \\ \dot{\tilde{\theta}}_i \end{bmatrix} = \begin{bmatrix} -I_p & \lambda R^* \Xi^T \\ -\Xi W_i R^{*T} & 0 \end{bmatrix} \begin{bmatrix} E \\ \tilde{\theta}_i \end{bmatrix} \tag{43}$$

which has the same structure as the one of [34, Lemma 5.6.3]. Applying this lemma with the PE property of z (obtained by invoking Lemma 1), we have that $E \to 0$ and $\tilde{\theta}_i \to 0$ exponentially for system (43), which together with $R \to R^*$ implies that $E \to 0$ and $\tilde{\theta}_i \to 0$ as $t \to \infty$ for (41). Note that the exponential convergence property of $[E, \tilde{\theta}_i]$ in (43) is not retained in (41) due to the time varying R. So $[E, \tilde{\theta}_i]$ converges toward zero asymptotically during the inverval, when the ith subsystem is activated. Since all the subsystems are activated repeatedly, we have $\tilde{\theta}_i \to 0$ $\forall i \in \{1, \ldots, s\}$ as $t \to \infty$.

The convergence of $\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi}$ cannot be directly concluded from the convergence of $\tilde{\theta}_i$. Further steps of analysis are needed. Note that

$$\begin{aligned} \tilde{\theta}_i &= \mathrm{vec}\Big(C\Psi_i\Big[K_{xi} - K_{xi}^* \quad K_{ri} - K_{ri}^* \quad K_{fi} - K_{fi}^*\Big]\Big) \\ &= \mathrm{vec}\Big(\Big[C\Psi_i K_{xi} - C\Phi_i \quad C\Psi_i K_{ri} - I \quad C\Psi_i K_{fi} - C\Upsilon_i\Big]\Big) \end{aligned}$$

$\tilde{\theta}_i \to 0$ implies $K_{xi} \to (C\Psi_i)^{-1}C\Phi_i = K_{xi}^*$, $K_{ri} \to (C\Psi_i)^{-1} = K_{ri}^*$, and $K_{fi} \to (C\Psi_i)^{-1}C\Upsilon_i = K_{fi}^*$, because the matrices $C\Psi_i, i = 1, \ldots, s$ are invertible. Hence, $\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi} \to 0$ as $t \to \infty$. ■

V. Indirect Adaptation Case

If the estimation of the system parameters is also a part of the control objective, the indirect adaptation can be applied.

A. Controller Design

The indirect adaptive control uses the same control structure as (28). The concept of indirect adaptation suggests the following update law:

$$\begin{aligned} K_{xi} &= -\left(C\hat{\Psi}_i\right)^{-1} C\hat{\Phi}_i \\ K_{ri} &= -\left(C\hat{\Psi}_i\right)^{-1} \\ K_{fi} &= -\left(C\hat{\Psi}_i\right)^{-1} C\hat{\Upsilon}_i \end{aligned} \tag{44}$$

where $\hat{\Phi}_i$, $\hat{\Psi}_i$, and $\hat{\Upsilon}_i$ denote the estimated ith subsystem parameters. The main difficulty by using this method is the singularity of $(C\hat{\Psi}_i)^{-1}$, which is also known as loss of controllability issue. Since $\hat{\Psi}_i$ is updated by some adaptation law, it cannot be ruled out that the smallest singular value of $C\hat{\Psi}_i$ may go across zero or become some small value around zero, which leads to unbounded controller gains.

To solve this singularity problem, we use the dynamic gain adjustment technique in this work. This concept is originally introduced by [37] and extended to MRAC of PWA systems in [12]. We extend this method to the context of adaptive control of PWA systems with prescribed performance. Specifically, the dynamic gain adjustment in MRAC starts with defining the *closed-loop estimation errors*, which capture the matching errors between the reference system and the controlled closed-loop system with estimated parameters. Unlike the MRAC, there exists no reference system in our context and thus, we propose the following novel closed-loop estimation errors:

$$\begin{aligned} \varepsilon_{\Phi i} &= C\hat{\Phi}_i + C\hat{\Psi}_i K_{xi} \\ \varepsilon_{\Psi i} &= C\hat{\Psi}_i K_{ri} + I \\ \varepsilon_{fi} &= C\hat{\Upsilon}_i + C\hat{\Psi}_i K_{fi}. \end{aligned} \tag{45}$$

These closed-loop estimation errors are obtained by multiplying both sides of (44) with $C\hat{\Psi}_i$ and taking the difference between the left- and right-hand sides. The controller gains are updated by using the closed-loop estimation errors

$$\begin{aligned} \dot{K}_{xi} &= \chi_i \Gamma_{xi} S_i^T R^T E x^T + \Gamma_{xi} S_i^T \varepsilon_{\Phi i} \\ \dot{K}_{ri} &= \chi_i \Gamma_{ri} S_i^T R^T E \xi^T + \Gamma_{ri} S_i^T \varepsilon_{\Psi i} \\ \dot{K}_{fi} &= \chi_i \Gamma_{fi} S_i^T R^T E + \Gamma_{fi} S_i^T \varepsilon_{fi} \end{aligned} \tag{46}$$

and the estimated system parameters are updated by

$$\begin{aligned} \dot{\hat{\Phi}}_i &= -\Gamma_{\Phi i} C^T \varepsilon_{\Phi i} \\ \dot{\hat{\Psi}}_i &= -\Gamma_{\Psi i}\Big(C^T \varepsilon_{\Phi i} K_{xi}^T + C^T \varepsilon_{\Psi i} K_{ri}^T + C^T \varepsilon_{fi} K_{fi}^T\Big) \\ \dot{\hat{\Upsilon}}_i &= -\Gamma_{\Upsilon i} C^T \varepsilon_{fi} \end{aligned} \tag{47}$$

with $\Gamma_{\Phi i}, \Gamma_{\Psi i}, \Gamma_{\Upsilon i} \in \mathbb{R}^+$ being positive scaling factors. The update laws (46) and (47) are derived based on the stability analysis. We can see from (46) and (47) that the inverse calculation shown in (44) is avoided through the utilization of closed-loop estimation errors.

B. Stability Analysis

The stability of the closed-loop system by using the indirect adaptive laws is characterized by the following theorem.

Theorem 4: Given the reference signal y_d and predefined performance function $\rho(t)$, let the PWA system (3) with known partition regions Ω_i and unknown subsystem parameters be controlled by the feedback controller (28) with the update laws (45)–(47). Let ρ be designed such that the inequality (9) holds at initial time instant $t = 0$. The closed-loop system is stable and the output tracking error satisfies the prescribed performance (9).

Proof: For clarity and without loss of generality, we let the scaling factors in (45) and (46) be 1 and propose the

Lyapunov-like function

$$V = \frac{E^T E}{2\lambda} + \frac{1}{2}\sum_{i=1}^{s}\Big(\mathrm{tr}\Big(\tilde{\Phi}_i^T\tilde{\Phi}_i\Big) + \mathrm{tr}\Big(\tilde{\Psi}_i^T\tilde{\Psi}_i\Big) + \mathrm{tr}\Big(\tilde{\Upsilon}_i^T\tilde{\Upsilon}_i\Big)\Big) + \mathrm{tr}\Big(\tilde{K}_{xi}^T M_i\tilde{K}_{xi}\Big) + \mathrm{tr}\Big(\tilde{K}_{ri}^T M_i\tilde{K}_{ri}\Big) + \mathrm{tr}\Big(\tilde{K}_{fi}^T M_i\tilde{K}_{fi}\Big)\Big) \quad (48)$$

where $\tilde{\Phi}_i = \hat{\Phi}_i - \Phi_i$, $\tilde{\Psi}_i = \hat{\Psi}_i - \Psi_i$, $\tilde{\Upsilon}_i = \hat{\Upsilon}_i - \Upsilon_i$. Taking the derivative and inserting (45)–(47) yield

$$\dot{V} = -\frac{E^T E}{\lambda} - \sum_{i=1}^{s}\mathrm{tr}\Big(\varepsilon_{\Phi i}^T\varepsilon_{\Phi i} + \varepsilon_{\Psi i}^T\varepsilon_{\Psi i} + \varepsilon_{fi}^T\varepsilon_{fi}\Big) \le 0. \quad (49)$$

Detailed derivations of this step can be seen in the supplementary material. From the negative semidefiniteness of $\dot{V}$, it follows that $E, \hat{\Phi}_i, \hat{\Psi}_i, \hat{\Upsilon}_i, K_{xi}, K_{ri}, K_{fi} \in \mathcal{L}_\infty$, which together with (45) implies $\varepsilon_{\Phi i}, \varepsilon_{\Psi i}, \varepsilon_{fi} \in \mathcal{L}_\infty$. Thus, we have $\dot{\Phi}_i, \dot{\Psi}_i, \dot{\Upsilon}_i \in \mathcal{L}_\infty$. Moreover, (49) also indicates $E, \varepsilon_{\Phi i}, \varepsilon_{\Psi i}, \varepsilon_{fi} \in \mathcal{L}_2$. Following the same analysis as in the direct adaptation case, one can conclude that $\sigma, \sigma^{(k)}, e, e^{(k)} \in \mathcal{L}_\infty$, which further results in $y, y^{(k)} \in \mathcal{L}_\infty, k = 1, \dots, r-1$, and hence, $x, \xi, K \in \mathcal{L}_\infty$. This, in turn, implies $\dot{K}_{xi}, \dot{K}_{ri}, \dot{K}_{fi} \in \mathcal{L}_\infty$. The boundedness of $u, \dot{E}$ can be concluded from the boundedness of $K_{xi}, K_{ri}, K_{fi}, x, \xi$. Furthermore, $\dot{E} \in \mathcal{L}_\infty$ as well as $E \in \mathcal{L}_\infty \cap \mathcal{L}_2$ results in $\lim_{t\to\infty} E \to 0$ and thus, $\sigma_j, \sigma_j^{(k)} \to 0$ as $t \to \infty$ $\forall j = 1, \dots, p, k = 1, \dots, r-1$. Therefore, we conclude that the tracking error e stays within the performance bound, i.e., inequalities in (9) hold.

Observe that the same expression as (49) can be obtained by replacing χ_i with $\bar{\chi}_i$ in the transformed error dynamics (32) and in adaptation laws (46), and we thus can conclude the closed-loop stability even when the closed-loop system enters sliding mode. ■

Remark 10: Two other methods used to avoid singularity (or loss of controllability) problem can be found in [14] and [15], respectively. While calculating the inverse of a matrix F_G using the formula $F_G^{-1} = \mathrm{adj}(F_G)/\det(F_G)$, the method in [14] adds a positive desgin number $\delta_D \in \mathbb{R}^+$ to the denominator to prevent the division by zero (see [14, eq. (12)]). The method in [15] replaces the denominator with a positive constant if its norm is smaller than a threshold (see [15, eq. (12)]). With these two methods, the transformed tracking error and the parameter estimation error converge only to a bounded set. Differing from these results, one key feature of our approach is that the convergence of the tracking error $e_j \to 0$ is achieved by specifying $\delta_j = 1$. Furthermore, the parameter estimation errors, as will be shown later, also converge to 0 under PE conditions. Nevertheless, more prior knowledge (S_i matrix and the system structure) is required compared to [15].

Remark 11: In the classical indirect MRAC of PWA systems shown in [12], the utilization of the dynamic gain adjustment technique has the disadvantage that the tracking error is not exponentially convergent. This problem is not overcome even when the PE condition of the reference signals is imposed. This issue, however, could be bypassed in our approach by choosing an exponentially decreasing performance function [such as the performance function (8)].

C. Parameter Convergence

Theorem 4 shows the boundedness of the parameter estimation error $\tilde{\Phi}_i$, $\tilde{\Psi}_i$, and $\tilde{\Upsilon}_i$. If one of the control objectives is the estimation of the real system parameters, the PE property of the reference signal y_d should be added to ensure the convergence of the estimated parameter to their real values. This is summarized as follows.

Theorem 5: Let the PWA system (3) with known partition regions Ω_i and unknown subsystem parameters be controlled by the feedback controller (28) with the update laws (45)–(47). Let ρ be designed such that the inequality (9) holds at initial time instant $t = 0$. Let the reference signals in y_d be sufficiently rich of order $r+1$ and cause repeated activation of all subsystems. If the matrices $C\Psi_i$ are invertible, and $\delta_j = 1$ for $j = 1, \dots, p$, then $\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi} \to 0$ and $\tilde{A}_i, \tilde{B}_i, \tilde{f}_i \to 0$ as $t \to \infty$.

Proof: Let $\tilde{\theta}_i = \mathrm{vec}(C\Psi_i[\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi}])$. From (46), we have

$$\begin{aligned}\dot{\tilde{\theta}}_i &= \mathrm{vec}\Big(C\Psi_i\Big[\dot{\tilde{K}}_{xi} \quad \dot{\tilde{K}}_{ri} \quad \dot{\tilde{K}}_{fi}\Big]\Big)\\ &= \mathrm{vec}\big(C\Psi_i S^T\big(R^T E\big[x^T \quad \xi^T \quad 1\big] + \big[\varepsilon_{\Phi i} \quad \varepsilon_{\Psi i} \quad \varepsilon_{fi}\big]\big)\big)\\ &= \Xi \cdot \mathrm{vec}\big(C\Psi_i S^T R^T E\big) + \mathrm{vec}\big(C\Psi_i S^T\big[\varepsilon_{\Phi i} \quad \varepsilon_{\Psi i} \quad \varepsilon_{fi}\big]\big)\\ &= -\Xi W_i R^T E + \mathrm{vec}\big(C\Psi_i S^T\big[\varepsilon_{\Phi i} \quad \varepsilon_{\Psi i} \quad \varepsilon_{fi}\big]\big). \end{aligned} \quad (50)$$

Combining it with (38), we have the dynamical systems with the state $[E, \tilde{\theta}_i]^T$

$$\begin{bmatrix}\dot{E}\\ \dot{\tilde{\theta}}_i\end{bmatrix} = \begin{bmatrix}-I_p & \lambda R\Xi^T\\ -\Xi W_i R^T & 0\end{bmatrix}\begin{bmatrix}E\\ \tilde{\theta}_i\end{bmatrix} + \begin{bmatrix}0\\ \epsilon_i\end{bmatrix} \quad (51)$$

with $\epsilon_i = \mathrm{vec}(C\Psi_i S^T[\varepsilon_{\Phi i}, \varepsilon_{\Psi i}, \varepsilon_{fi}])$. Considering (45) and the property $\dot{\Phi}_i, \dot{\Psi}_i, \dot{\Upsilon}_i, \dot{K}_{xi}, \dot{K}_{ri}, \dot{K}_{fi} \in \mathcal{L}_\infty$, we have $\dot{\varepsilon}_{\Phi i}, \dot{\varepsilon}_{\Psi i}, \dot{\varepsilon}_{fi} \in \mathcal{L}_\infty$, which together with $\varepsilon_{\Phi i}, \varepsilon_{\Psi i}, \varepsilon_{fi} \in \mathcal{L}_\infty \cap \mathcal{L}_2$ leads to $\varepsilon_{\Phi i}, \varepsilon_{\Psi i}, \varepsilon_{fi} \to 0$ as $t \to \infty$. Therefore, the convergence property can be shown through the homogeneous part of (51).

It has already been shown in Theorem 3 that $E, \tilde{\theta}_i \to 0$ asymptotically if $\epsilon_i = 0$ and all subsystems are activated repeatedly, from which one can conclude that $\tilde{K}_{xi} \to 0, \tilde{K}_{ri} \to 0, \tilde{K}_{fi} \to 0$ as $t \to \infty$, namely, the adaptive controller gains converge to the nominal gains $K_{xi} \to K_{xi}^*, K_{ri} \to K_{ri}^*, K_{fi} \to K_{fi}^*$ as $t \to \infty$. Considering $\varepsilon_{\Psi i} \to 0$ and the expression of $\varepsilon_{\Psi i}$ in (45), it follows $C\hat{\Psi}_i \to -(K_{ri}^*)^{-1} = C\Psi_i$. Taking this into the expression of $\varepsilon_{\Phi i}$ and ε_{fi} in (45), we have $C\hat{\Phi}_i \to C\Phi_i$ and $C\hat{\Upsilon}_i \to C\Upsilon_i$ as $t \to \infty$.

Note that

$$\begin{aligned} C\hat{\Phi}_i &= \big[\hat{a}_{1i} \quad \hat{a}_{2i} \quad \cdots \quad \hat{a}_{pi}\big]^T\\ C\hat{\Psi}_i &= \big[\hat{b}_{1i} \quad \hat{b}_{2i} \quad \cdots \quad \hat{b}_{pi}\big]^T\\ C\hat{\Upsilon}_i &= \big[\hat{f}_{1i} \quad \hat{f}_{2i} \quad \cdots \quad \hat{f}_{pi}\big]^T \end{aligned} \quad (52)$$

where $\hat{a}_{ji}, \hat{b}_{ji}$, and $\hat{f}_{ji}$ represent the estimated values of a_{ji}, b_{ji}, and f_{ji} in (1) for $j = 1, \dots, p, i = 1, \dots, s$. The convergence of $C\hat{\Phi}_i, C\hat{\Psi}_i, C\hat{\Upsilon}_i$ implies $\hat{a}_{ji} \to a_{ji}$, $\hat{b}_{ji} \to b_{ji}$, and $\hat{f}_{ji} \to f_{ji}$. Considering that the system is in control canonical form, it follows from the convergence of $\hat{a}_{ji}, \hat{b}_{ji}$, and $\hat{f}_{ji}$ that $\hat{A}_i \to A_i$, $\hat{B}_i \to B_i$, and $\hat{f}_i \to f_i$ as $t \to \infty$. ■

The advantage of our indirect adaptive controller over the direct adaptive controller is that the indirect adaptive controller exhibits the capability to identify the subsystem parameters.

This is, however, achieved at the expense of imposing more complexity into the closed-loop system. Specifically, the update law of the controller gains of the indirect adaptive controller (46) is obtained by fusing the closed-loop estimation errors to the update law of the direct adaptive controller (29). Meanwhile, the subsystem parameters are updated through the information of closed-loop estimation errors and the estimated controller gains. Therefore, more computational costs must be tolerable when applying the indirect adaptive controller.

Remark 12 (Parameter Tuning Guidelines): Larger adaptation gains $\Gamma_{xi}, \Gamma_{ri}, \Gamma_{fi}$ and $\Gamma_{\Phi i}, \Gamma_{\Psi i}, \Gamma_{\Upsilon i}$ speed up the parameter adaptation while too large adaptation gains may lead to numerical instability and high control effort. λ is the coefficient of $\sigma_j^{(r-1)}$ and serves as the input gain of the dynamics of E [see (32)]. A larger λ amplifies the sensibility introduced by the higher order derivative and results in aggressive response of E, whereas a too small λ leads to "stiff" descent of the Lyapunov function [see (34) and (49)], which is numerically difficult to solve.

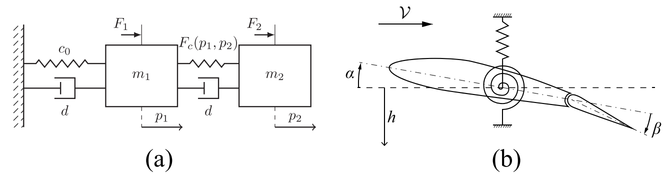
Remark 13: When a PWA system is used to approximate a nonlinear system, there exist approximation errors. If a rigorous robustness analysis is desirable for this case, one can impose robust modifications (such as projection) into adaptation laws and add an auxiliary term $v = \sum_{i=1}^{s} \chi_i S_i^T R^T E$ to the controller (28). This will lead to an inequality of the Lyapunov function in the form of $\dot{V} \leq -2V + \mathcal{B}$ with $\mathcal{B}$ being a bounded term related to the maximal norm of approximation errors, from which the stability can be concluded and the prescribed performance is satisfied.

The concept to convert a constrained error into an unconstrained one to satisfy a prescribed performance requirement has been studied for hybrid systems and switching systems in [24], [25], [38], and [39]. These approaches are based on backstepping design and require either input gains to be completely known [24], [25] or the control direction as well as lower bounds of input gains to be known [38], [39]. Compared to these approaches, only the control direction is assumed to be known in our article. Another feature that differentiates our article from these approaches is that the convergence of gain and parameter estimation errors is achieved under PE conditions.

In prescribed performance control, there also exist approximation-free control methods [20], [40], [41], where no adaptation mechanism is introduced. Such approaches have low controller complexity and computational costs. Compared to these approximation-free methods, our approaches are based on adaptations and can achieve unknown parameter estimation in addition to the tracking task. This is especially useful for monitoring systems with parameter drifts and component aging as well as for joint control and identification tasks.

VI. Numerical Validation

In this section, the proposed adaptive approaches of PWA systems with prescribed performance are validated through two numerical examples.



(a) (b)

Fig. 1. (a) Mass-spring-damper system. (b) Aeroelastic model of aircraft wings [29].

A. Mass-Spring-Damper System

The mass-spring-damper system of interest, taken from [12], is shown in Fig. 1(a). The two masses with $m_1 = 5\,\text{kg}$ and $m_2 = 1\,\text{kg}$ are connected with each other by a damper with $d = 1\,\text{N s/m}$ and a spring with PWA stiffness $F_c(p_1, p_2)$. Let F_1 and F_2 denote the forces acting on the two masses and p_1 and p_2 represent the displacement of the two masses, respectively. The PWA stiffness $F_c(p_1, p_2)$, which is determined by the displacements of the two springs, is given by

$$F_c(p_1,p_2) = \begin{cases} c_1 = 10\,\text{N/m}, & \text{if } |p_2 - p_1| \leq 1\,\text{m} \\ c_2 = 1\,\text{N/m}, & \text{if } p_2 - p_1 > 1\,\text{m} \\ c_3 = 100\,\text{N/m}, & \text{if } p_2 - p_1 < -1\,\text{m}. \end{cases} \quad (53)$$

The left mass is connected with the static environment by the spring with $c_0 = 1\,\text{N/m}$ and the damper with $d = 1\,\text{N s/m}$. Given the state vector $x = [p_1, \dot{p}_1, p_2, \dot{p}_2]^T$, the output vector $y = [p_1, p_2]^T$, and the input vector $u = [F_1, F_2]^T$, the system dynamics can be written as a PWA system

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{c_0+c_i}{m_1} & -\frac{2d}{m_1} & \frac{c_i}{m_1} & \frac{d}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{c_i}{m_2} & \frac{d}{m_2} & -\frac{c_i}{m_2} & -\frac{d}{m_2} \end{bmatrix}}_{A_i} x + \underbrace{\begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}}_{B_i} u + f_i \quad (54)$$

with the affine terms f_i, $i = \{1, 2, 3\}$ being

$$f_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 \\ \frac{c_1-c_2}{m_1} \\ 0 \\ \frac{c_2-c_1}{m_2} \end{bmatrix}, \quad f_3 = \begin{bmatrix} 0 \\ \frac{c_3-c_1}{m_1} \\ 0 \\ \frac{c_1-c_3}{m_2} \end{bmatrix}. \quad (55)$$

In the following simulation, the region partitions are assumed to be known and the subsystem parameters are unknown. Both direct and indirect adaptation cases are analyzed as follows.

1) Direct Adaptation: Now, we test the tracking performance of the direct prescribed performance adaptive control approach, abbreviated as PPAC. To compare this performance with the one of MRAC [12], we let the desired trajectory y_d be the output of the reference system $y_d = W_m(s)r$, where $W_m(s) = \text{diag}\{[1/(0.2s+1)^2], [1/(0.2s+1)^2]\}$ denotes the transfer matrix of the reference system (see [12, Sec. V]), the input signal r is chosen as $r = [2\sin(0.2t), 2\sin(0.5t)]^T$. We define the performance bounds by specifying $\rho_0 = [10, 10]^T$ and $\rho_\infty = [0.1, 0.1]^T$ with the decaying rates $l = [l_1, l_2]^T = [1, 1]^T$. The error bounds in (9) are chosen to be symmetric by letting $\delta_1 = \delta_2 = 1$. λ is selected to be 0.04. Besides, we use unit scaling factors for controller gains

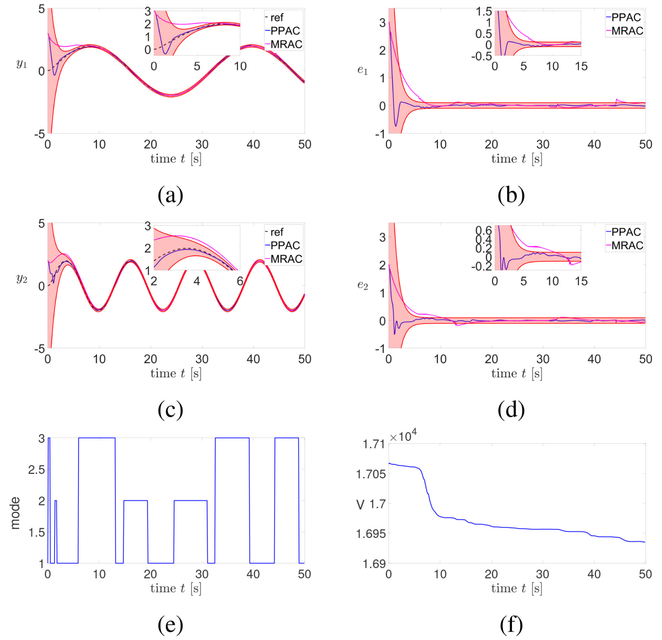


Fig. 2. Output tracking performance of direct adaptation case. (a) Output y_1. (b) Tracking error e_1. (c) Output y_2. (d) Tracking error e_2. (e) Switching signal. (d) Lyapunov function V.

adaptation, $\Gamma_{xi} = \Gamma_{ri} = \Gamma_{fi} = 1\ \forall i = 1, 2, 3$ and we specify $S_i = -I_2\ \forall i = 1, 2, 3$.

The output tracking performance of PPAC and MRAC is shown in Fig. 2. In Fig. 2(a) and (c), the red regions represent the prescribed performance bounds of the output. Blue solid lines indicate the real system output of PPAC and the black dashed lines depict the desired output. In Fig. 2(b) and (d), the tracking errors as well as the performance bound of errors are displayed in blue lines and red regions, respectively. Besides, the mode information is given in Fig. 2(e) and the common Lyapunov function in Fig. 2(f). The Lyapunov function is continuous at each switching instant and strictly decreasing. It can be seen from the figures that both components of the output tracking error of the controlled system stay within the prescribed performance bounds. For comparison purpose, the tracking performance of the MRAC approach is displayed with magenta lines. We observe that the transients of MRAC converge slower than the one of PPAC and violate the prescribed performance constraints.

To validate the convergence of the controller gains under PE conditions, the desired output signal is chosen as $y_d = [2\sin(0.2t) - 0.2\sin(3t), 2\sin(0.5t) - 0.2\sin(7t)]$. The relative degree of the system is $r = 2$. According to Theorem 2, y_d should be sufficiently rich of order 3 to guarantee the convergence of the controller gains to their nominal values. Since each component of y_d contains two distinct frequencies, the sufficiently rich condition is satisfied. Besides, the chosen desired output signal ensures that all the subsystems are activated repeatedly. The scaling factors are chosen as $\Gamma_{xi} = \Gamma_{ri} = \Gamma_{fi} = 5\ \forall i = 1, 2, 3$ and λ is specified as 0.01. The performance bounds are specified by $\rho_0 = [10, 10]^T$ and $\rho_\infty = [0.15, 0.15]^T$ with the decaying rates $l = [l_1, l_2]^T = [0.5, 0.5]^T$.

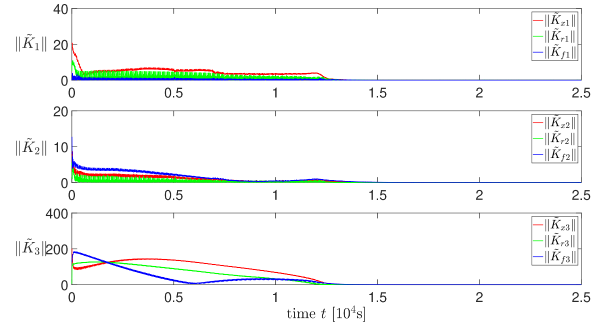


Fig. 3. Convergence of estimation errors of controller gains of direct adaptation case.

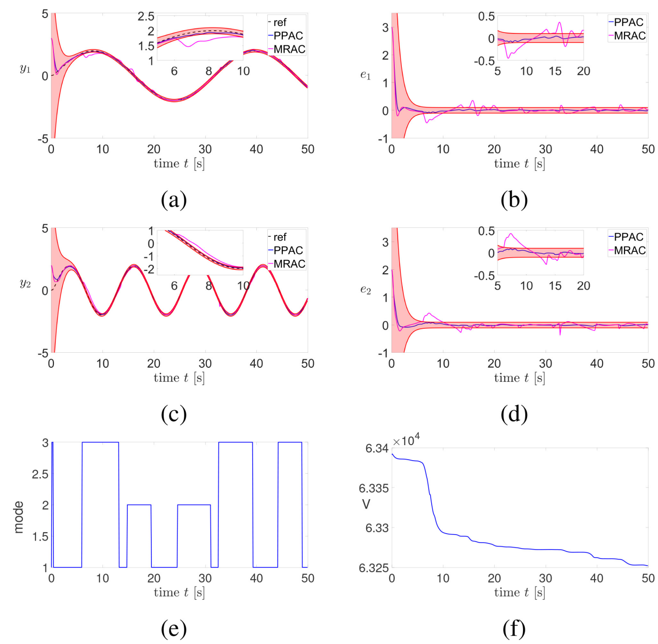


Fig. 4. Output tracking performance of indirect adaptation case. (a) Output y_1. (b) Tracking error e_1. (c) Output y_2. (d) Tracking error e_2. (e) Switching signal. (d) Lyapunov function V.

Fig. 3 shows the convergence of the errors between estimated controller gains and nominal controller gains. We use $\tilde{K}_i$ on the vertical axis to represent the set of estimation errors of the controller gains for the ith subsystem, i.e., $\tilde{K}_i = \{\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi}\}$. As we can conclude from the figure, the estimated controller gains of all the subsystems converge to their nominal values. This validates the theoretical results of Theorem 3.

2) Indirect Adaptation: The tracking performance of the indirect adaptation case is tested with the same parameters as in the direct adaptation case. Fig. 4(a) and (c) displays the desired output in black dashed lines, the real output of PPAC in blue solid lines, as well as the performance bound of output in red lines. The tracking errors, as well as the performance bound of the errors, are presented in Fig. 4(b) and (d) with blue and red colors, respectively. The switches are displayed in Fig. 4(e) and the common Lyapunov function in Fig. 4(f), which is continuous at each switching instant and strictly decreasing. As we can see, the output of the controlled system is enclosed by the performance bound and the prescribed transient performance is satisfied. In comparison to this, the

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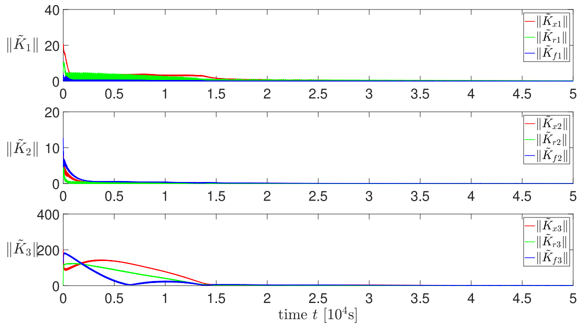

Fig. 5. Convergence of estimation errors of controller gains of indirect adaptation case.

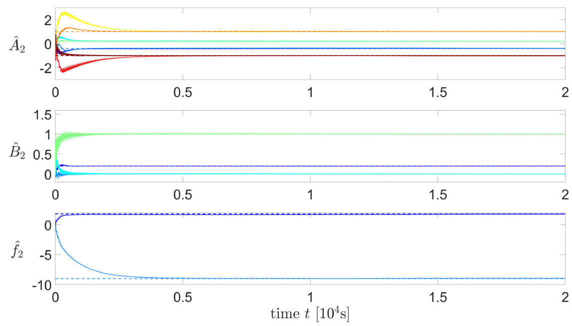

Fig. 6. Convergence of estimated parameters of indirect adaptation case.

tracking performance of the MRAC approach, displayed with magenta lines, violates the prescribed performance constraints.

The convergence of the controller gains and the estimated parameters is tested by applying the same PE input signal with the same setting of parameters as in the direct case. In addition, $\Gamma_{\Phi i}, \Gamma_{\Psi i}, \Gamma_{\Upsilon i} = 1 \; \forall i = 1, 2, 3$. As Fig. 5 shows, the estimation error of the controller gains $\tilde{K}_{xi}$, $\tilde{K}_{ri}$, and $\tilde{K}_{fi}$ converges to zero. The parameter estimation of subsystem 2 is displayed in Fig. 6. Note that only the to be estimated components rather than all the components in the parameter matrices are displayed [see (52)]. The dashed lines represent the real values and the solid lines depict the estimated values. As can be seen from the figure, the estimated system parameters converge to the real values. It can also be seen from Figs. 3 and 5 that the estimated controller gains converge after 10 000 s. That is because the parameter convergence is asymptotic instead of exponential (see the analysis in Theorem 3). In practice, properly choosing larger adaptation gains may be one possible way to improve the rapidity of the parameter convergence.

B. Aeroelastic Model

In this section, the proposed approaches are tested with an engineering application example, the aeroelastic model of aircraft wings [29], [42]. The wing fluctuation is simplified as the dynamics of an airfoil with linear and torsional spring, which is illustrated in Fig. 1(b). The airfoil has two degrees of freedom: 1) plunging and 2) pitching. h denotes the plunging deflection and α represents the pitch angle about the elastic axis. $\beta = [\beta_1, \beta_2]^T$ serves as the input signal and denotes the left and right flap deflection angles, which are not distinguished from each other in Fig. 1(b) due to the side view. $\mathcal{V}$ denotes the constant airspeed. Let $y = [h, \alpha]^T$ be the system output. The motion of the aeroelastic model can be described by the equation

$$\mathcal{M}\ddot{y} + \mathcal{C}\dot{y} + \mathcal{K}y + \mathcal{W}_q = \mathcal{B}_\mu \beta \tag{56}$$

where $\mathcal{M}$ denotes the mass and inertia matrix, and $\mathcal{B}_\mu$ represents the control gain. The structural damping effect, stiffness, aerodynamic lift, and moment effect are included in matrices $\mathcal{C}$ and $\mathcal{K}$. Their values are known and detailed derivations can be seen in [42]. $\mathcal{W}_q = [0, \bar{K}\alpha]^T$ constitutes the source of uncertainties with $\bar{K}$ being the nonlinear torsional stiffness

$$\bar{K} = 2.82 - 62.322\alpha + 3709.71\alpha^2 - 24195.6\alpha^3 + 48756.954\alpha^4.$$

TABLE I
PWL APPROXIMATION OF $\bar{K}\alpha$

mode	1	2	3	4
$\bar{a}_i$	10044	5992	2482.1	19.141
$\bar{b}_i$	2732.9	1377.8	422.78	2.8463
Region	[-0.38,-0.33]	[-0.33,-0.27]	[-0.27,-0.17]	[-0.17, 0.38]

The characteristics of the nonlinear term $\bar{K}\alpha$ in the interval $\alpha \in [-0.38, 0.38]$ can be divided into four regions and its PWL approximation in form of $\bar{a}_i\alpha + \bar{b}_i, i = 1, \ldots, 4$ is given in Table I. Let the state be $x = [h, \alpha, \dot{h}, \dot{\alpha}]^T$. The dynamics (56) can be approximated by the PWA system in form of (3) with

$$A_i = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -293.27 & -100.59 + 0.66\bar{a}_i & -5.9027 & -0.40542 \\ 1885.9 & 743.79 - 19.65\bar{a}_i & 34.728 & 2.4687 \end{bmatrix}$$

$$B_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -7606.8 & -7642.6 \\ 14250 & 9021.9 \end{bmatrix}, f_i = \begin{bmatrix} 0 \\ 0 \\ 0.66\bar{b}_i \\ -19.65\bar{b}_i \end{bmatrix}, i = 1, \ldots, 4. \tag{57}$$

Now, we test the tracking performance of both direct and indirect PPAC approaches on the nonlinear system (56), which is equivalent to the PWA system (57) with approximation errors as external disturbances. Gaussian noise with zero mean and 0.001 variance is added to the state measurements. We define the performance bounds by specifying $\rho_0 = [5, \pi/6]^T$ and $\rho_\infty = [0.1, 0.04]^T$ with the decaying rates $l = [0.2, 0.2]^T$. The error bounds are symmetric with $\delta_1 = \delta_2 = 1$. λ is selected to be 0.01. The adaptation gains are $\Gamma_{xi} = \Gamma_{fi} = 1, \Gamma_{ri} = 0.001 \; \forall i = 1, \ldots, 4$ and we specify the reference signal as $y_d = [0, -0.4e^{-0.03t}\sin(0.5t + \pi/2)]^T$. The initial state of the system reads $x(0) = [1, -0.35, 0, 0]^T$. The initial guess of the parameters for each subsystem is specified by letting $\bar{a}_i = \bar{b}_i = 0$ in (57). The following S_i matrices are applied:

$$S_i = \begin{bmatrix} 0.7607 & 0.7643 \\ -1.4250 & -0.9022 \end{bmatrix} \quad \forall i = 1, \ldots, 4. \tag{58}$$

The output tracking performance of direct and indirect PPAC is shown in Fig. 7. In Fig. 7(a) and (b), the blue lines and magenta lines depict the output tracking errors of direct and indirect approaches. The mode switches by using direct and indirect PPAC are shown in Fig. 7(c) and (d), respectively. It can be seen

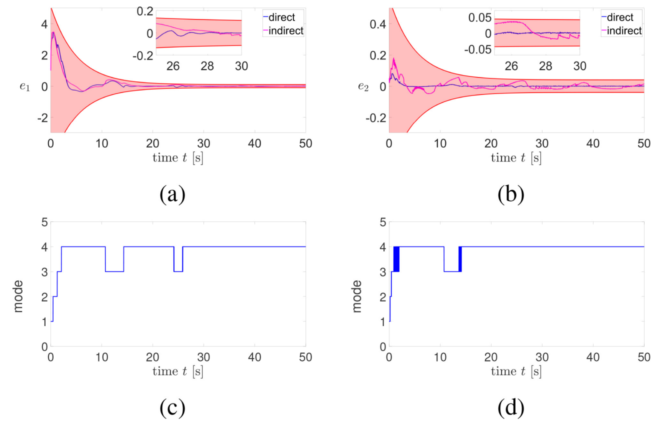

Fig. 7. Output tracking performance of direct and indirect adaptation cases. (a) Tracking error e_1. (b) Tracking error e_2. (c) Switching signal (direct case). (d) Switching signal (indirect case).

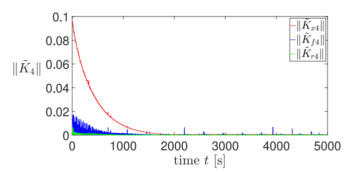

Fig. 8. Convergence of estimated controller gains of indirect adaptation case.

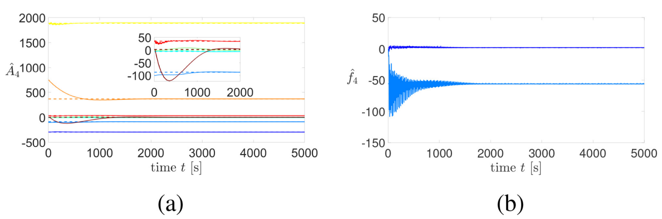

Fig. 9. Convergence of estimated parameters of indirect adaptation case. (a) Convergence of $\hat{A}_4$. (b) Convergence of $\hat{f}_4$.

from the figures that the output tracking errors of both direct and indirect approaches stay within the prescribed performance bounds. This also suggests some degree of robustness of our approaches against noise and disturbances.

The parameter convergence property is tested on the PWA system (57) with an indirect adaptation approach. The reference signal is $y_d = [0.5\sin(0.2t)+0.05\sin(0.9t), 0.2\sin(0.5t)+0.05\sin(1.2t)]$ without Gaussian noise. The adaptation gains, the performance bound, and the initial guess of parameters are chosen the same as those of the tracking case. Besides, we specify $\lambda = 0.04$.

Fig. 8 shows the convergence of the estimation errors of the controller gains of subsystem 4. The red line, green line, and blue line represent the estimation errors $\|\tilde{K}_{x4}\|$, $\|\tilde{K}_{r4}\|$, $\|\tilde{K}_{f4}\|$, respectively. The figure validates the convergence of the estimated controller gains to the nominal ones.

Similarly, the componentwise convergence of the estimated parameters of subsystem 4 by using the indirect PPAC approach is shown in Fig. 9. As can be seen from the figure, the estimated system parameters, displayed by solid lines, converge to the real values (dashed lines).

VII. Conclusion

In this article, we have investigated the adaptive control approaches for MIMO PWA systems with prescribed performance in terms of both direct and indirect adaptations, respectively. For both control approaches, we have shown that the output tracking errors stay within the prescribed performance bounds. Based on novel common Lyapunov functions, which do not rely on the solution of conventional Lyapunov equations, closed-loop stability is achieved under arbitrary switching. The controller gains and estimated subsystem parameters are proved to converge to their nominal and real values if the desired trajectory is PE. The incorporation of the dynamic gain adjustment technique prevents the singularity in indirect adaptation. One limitation of our approaches is that the PWA system with state jump or jump in the derivatives of the state/output cannot be handled. Extending our approaches to such cases can be our future work. While our methods ensure the closed-loop stability under sliding mode (chattering Zeno), the study of genuinely Zeno behavior (infinite switching events within a finite time interval) in adaptive PWA systems remains an open topic for future work.

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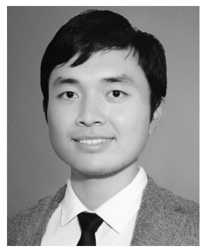

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