

# ON THE $P_3$ -HULL NUMBER AND INFECTING TIMES OF GENERALIZED PETERSEN GRAPHS

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ABSTRACT. The  $P_3$ -hull number of a graph is the size of a minimal infecting set of vertices that will eventually infect the entire graph under the rule that uninfected nodes become infected if two or more neighbors are infected. In this paper, we study the  $P_3$ -hull number for Petersen graphs and a number of closely related graphs that arise from surgery or more generalized permutations. In addition, the number of components of the complement of a minimal infecting set is calculated for the Petersen graph and shown to always be 1 or 2. In addition, infecting times for a minimal infecting set are studied. Bounds are given and complete information is given in special cases.

## 1. INTRODUCTION

As introduced in [4] and [7], the  $P_3$ -hull number of a simple connected graph is the minimum cardinality of a set  $U$  of initially uninfected vertices that will eventually infect the entire graph where an uninfected node becomes infected if two or more of its neighbors are infected. There has been much work on formulas for the  $P_3$ -hull numbers of various types of graphs, [5, 6, 8, 10], as well as with the closely related notion of the 2-neighbor bootstrap percolation problem, [3, 12, 13].

Important to this paper is the decycling number. Given a graph  $G$ , its decycling number,  $\nabla(G)$ , is the minimum cardinality of a set  $U$  of vertices such that  $G - U$  is acyclic. In general, it is very hard to compute a graph's decycling number. In fact, it has been shown to be NP-complete [11]. However, results in special cases have been obtained, [1, 2, 9, 14, 15].

In this paper, after initial definitions, we show that for a cubic graph, the  $P_3$ -hull number and the decycling number coincide, Theorem 3.1. By [9], it follows that the  $P_3$ -hull number of the generalized Petersen graph,  $G(n, k)$ , is  $\lceil \frac{n+1}{2} \rceil$ , Corollary 3.2. Furthermore, the complement of the initial minimal infecting set is a forest. In Theorem 3.4, it is shown that this forest always has exactly one or two components.

In addition, we introduce the notion of the infecting time of an infecting set and study it for the Petersen graph. Explicit times are computed for our initial infecting set, Theorem 4.1. Giving explicit formulas for the minimal and maximal infecting times is a very difficult problem. However, we give complete answers for the special case of  $G(n, 1)$ , Theorem 4.3.

Finally, we introduce a number of graphs related to the generalized Petersen graph. For a type of surgery,  $G(n, k) \# G(n, k)$ , the  $P_3$ -hull number is computed in Theorem 5.1. Associated to a permutation  $\sigma$  of  $S_n$ , a generalization of  $G(n, k)$ , called  $GG(n, \sigma)$ , is introduced. General bounds for its  $P_3$ -hull number are given in Theorem 5.4. An exact answer is computed under an additional hypothesis in Theorem 5.5.

## 2. INITIAL DEFINITIONS

Throughout the paper, let  $G = (V, E)$  be a finite, simple, connected graph and let  $S \subseteq V$ . We write  $G[S]$  for the corresponding induced subgraph of  $G$  on  $S$ . We say  $G$  is *cubic* if each vertex of  $G$  has degree 3.

Following [10], the  $P_3$ -interval,  $I[S]$ , is the set  $S$  together with all vertices in  $G$  that have two or more neighbors in  $S$ . If  $I[S] = S$ , then the set  $S$  is called  $P_3$ -convex. The  $P_3$ -convex hull,  $H_C(S)$  of  $S$ , is the smallest  $P_3$ -convex set containing  $S$ . Iteratively, define  $I^0[S] = S$  and  $I^p[S] = I[I^{p-1}[S]]$  for any positive integer  $p$ . Then  $H_C(S)$  is the union of all  $I^p[S]$ .

If  $H_C(S) = V$ , we say that  $S$  is a  $P_3$ -hull set of  $G$ . We also refer to a  $P_3$ -hull set as an *infecting set*. The cardinality,  $h_{P_3}(G)$ , of a minimum  $P_3$ -hull set in  $G$  is called the  $P_3$ -hull number of  $G$ . This will be the main object of our study. For a  $P_3$ -hull set  $S$ , we say that the *infecting time* of  $S$ , denoted  $T_I(S)$ , is the smallest integer  $p$  such that  $I^p[S] = V$ .

Relevant for this paper, we say that  $S$  is a *decycling set* of  $G$  if the induced subgraph  $G[V - S]$  is acyclic, [1, 2]. The minimum cardinality of a decycling set of  $G$  is called the *decycling number* of  $G$  and denoted by  $\nabla(G)$ .

Recall that a *generalized Petersen graph*,  $G(n, k)$  with  $1 \leq k < \frac{n}{2}$ , has vertex set

$$V = \{u_0, u_1, \dots, u_{n-1}, v_0, v_1, \dots, v_{n-1}\}$$

and edge set (interpreting each index modulo  $n$ )

$$E = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : 0 \leq i \leq n-1\}.$$

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### 3. $P_3$ -HULL NUMBERS AND MINIMAL INFECTING SETS FOR $G(n, k)$

**Theorem 3.1.** *Let  $G = (V, E)$  be a cubic graph and  $S \subseteq V$ . Then  $S$  is an infecting set of  $G$  if and only if it is a decycling set of  $G$ . In particular,  $h_{P_3}(G) = \nabla(G)$ .*

*Proof.* We show first that an infecting set is a decycling set via the contrapositive. Assume that  $S$  is not a decycling set. Then there exists some nonempty  $W \subseteq V - S$  so that  $G[W]$  is a cycle. Hence each  $w \in W$  has exactly two neighbors in  $W$  and one neighbor in  $V - W$ .

Suppose now that  $S$  is an infecting set. Since  $S \cap W = \emptyset$ , there is a minimal integer  $n \geq 0$  so that  $I^n[S] \cap W = \emptyset$  and  $I^{n+1}[S] \cap W \neq \emptyset$ . But then for  $w \in I^{n+1}[S] \cap W$ , there must have been two neighbors of  $w$  in  $I^n[S]$ . As  $I^n[S] \cap W = \emptyset$ , these two neighbors must lie in  $V - W$  which is a contradiction.

Next we show that a decycling set is an infecting set. Suppose that  $S$  is a decycling set of  $G$ . If  $S$  is not infecting, then  $H_C(S)$  is a proper subset of  $V$  that is still decycling. As the subgraph induced by  $V - H_C(S)$  is a forest, there exists some  $v \in V - H_C(S)$  with degree 1. This implies two neighbors of  $v$  lie in  $H_C(S)$  which is a contradiction.  $\square$

**Corollary 3.2.**  $h_{P_3}(G(n, k)) = \lceil \frac{n+1}{2} \rceil$ .

*Proof.* This follows from Theorem 3.1 and [9, Theorem 3.1], where it is shown that  $\nabla(G(n, k)) = \lceil \frac{n+1}{2} \rceil$ .  $\square$

As a prelude to infecting time calculations, we present an explicit minimal infecting set for  $G(n, k)$  (examples can be seen in Figure 1). An alternate example may be found in [9, Lemma 3.3].

**Corollary 3.3.** *Let  $c = \gcd(n, k)$ ,  $l = \frac{n}{c}$ , and*

$$\mathcal{S}_v = \{v_{j+ik} : 0 \leq i \leq l-1, 0 \leq j \leq c-1, i \text{ odd}\}.$$

*For  $l$  even, let*

$$\mathcal{S}_u = \{u_0\}$$

*and for  $l$  odd, let*

$$\mathcal{S}_u = \{u_{c-1}\} \cup \{u_j : 0 \leq j \leq c-1, j \text{ even}\}.$$

*Then*

$$\mathcal{S} = \mathcal{S}_v \cup \mathcal{S}_u$$

*is a minimal infecting set for  $G(n, k)$ .*

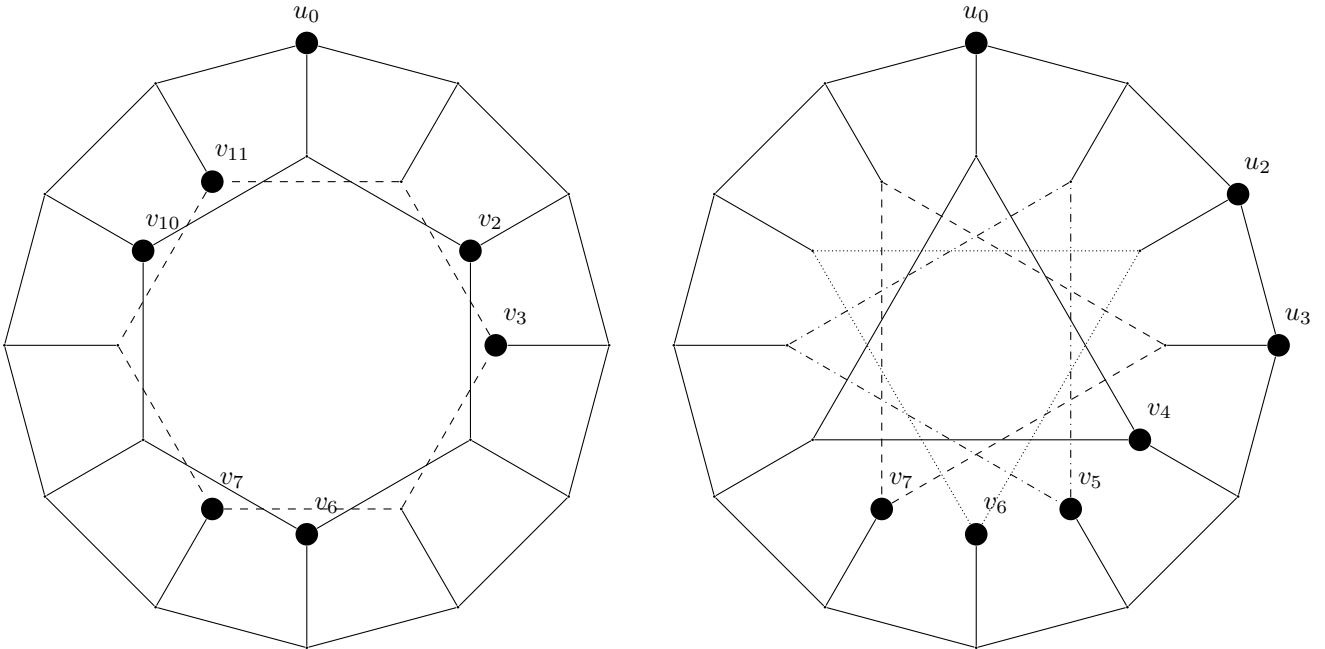


FIGURE 1. Minimal infecting sets for  $G(12, 2)$  and  $G(12, 4)$ , respectively.

*Proof.* Note that the subgraph of  $G(n, k)$  which is induced on the vertex set  $\{v_0, v_1, \dots, v_{n-1}\}$  is the disjoint union of  $c$  cycles of length  $l$ . The corresponding vertex set of each cycle is given by

$$V_j = \{v_{j+ik} : 0 \leq i \leq l-1\}$$

with  $0 \leq j \leq c-1$ .

We will distinguish two cases. The first is when  $l$  is even. Given  $\mathcal{S}$  as an initial set of infected points, the infection will spread to infect every vertex set  $V_j$ ,  $0 \leq j \leq c-1$ . From there with  $u_0$  the infection spreads to all of  $\{u_0, u_1, \dots, u_{n-1}\}$ . Note that  $n = cl$  is even and

$$|\mathcal{S}| = 1 + c \frac{l}{2} = \frac{n}{2} + 1 = \left\lceil \frac{n+1}{2} \right\rceil$$

so that  $\mathcal{S}$  is a minimal infecting set by Corollary 3.2.

Turn now to the case of  $l$  odd. Given  $\mathcal{S}$  as an initial set of infected points, the infection will spread to  $\{u_j : 0 \leq j \leq c-1\}$ . With  $u_j$  and  $v_{j+k}$  infected, also  $v_j$  will be infected, and the infection will spread to infect every vertex set  $V_j$ ,  $0 \leq j \leq c-1$ . From there with  $\{u_j : 0 \leq j \leq c-1\}$  the infection spreads to all of  $\{u_0, u_1, \dots, u_{n-1}\}$ . As

$$|\mathcal{S}| = \left\lceil \frac{c+1}{2} \right\rceil + c \frac{l-1}{2} = \left\lceil c \frac{l-1}{2} + \frac{c+1}{2} \right\rceil = \left\lceil \frac{n+1}{2} \right\rceil,$$

it follows that  $\mathcal{S}$  is a minimal infecting set. □

By Theorem 3.1, the complement of a minimal infecting set of a cubic graph is a forest. The next theorem constrains the number of connected components of this forest for  $G(n, k)$ .

**Theorem 3.4.** *Let  $S$  be a minimal infecting set of the generalized Petersen graph  $G = G(n, k) = (V, E)$ .*

- *If  $n$  is odd, then  $G[V - S]$  is a tree.*
- *If  $n$  is even, then the forest  $G[V - S]$  may have two or one connected components. It will have two connected components if and only if  $S$  has no neighboring points and one connected component if and only if  $S$  has exactly one pair of points that are neighbors.*

*Proof.* Recall for  $G$  that  $|V| = 2n$  and  $|E| = 3n$ . Write  $\nu$  and  $\epsilon$  for the number of vertices and edges, respectively, of  $G[V - S]$ . As  $G[V - S]$  is a forest,  $\nu - \epsilon$  is the number of trees in the forest.

When  $n$  is odd, write  $n = 2k + 1$  so that  $|\mathcal{S}| = \lceil \frac{n+1}{2} \rceil = k + 1$  and  $\nu = 2(2k + 1) - (k + 1) = 3k + 1$ . At most, if  $S$  has no neighboring points, passing from  $G$  to  $G[V - S]$  would remove  $3|\mathcal{S}|$  edges. Therefore,  $\epsilon \geq 3(2k + 1) - 3(k + 1) = 3k$ . It follows that  $\nu - \epsilon \leq 1$ , and  $G[V - S]$  must be a single tree.

When  $n$  is even, write  $n = 2k$  so  $|\mathcal{S}| = \lceil \frac{n+1}{2} \rceil = k + 1$  and  $\nu = 2(2k) - (k + 1) = 3k - 1$ . As in the previous paragraph, we get  $\epsilon \geq 3(2k) - 3(k + 1) = 3k - 3$ . It follows that  $\nu - \epsilon \leq 2$ , and  $G[V - S]$  has either one or two connected components. In addition,  $G[V - S]$  is a tree exactly when  $S$  has no neighboring points, and  $G[V - S]$  has two trees exactly when  $S$  has exactly one pair of neighboring points. □

#### 4. INFECTING TIMES

**Theorem 4.1.** *Let  $c = \gcd(n, k)$ ,  $l = \frac{n}{c}$ , and  $\mathcal{S}$  be the infecting set for  $G(n, k)$  from Corollary 3.3.*

- *When  $l$  is even, the infecting time for  $\mathcal{S}$  is  $\frac{n}{2}$ .*
- *When  $l$  is odd, the infecting time for  $\mathcal{S}$  is  $\frac{n-c}{2} + 1$ .*

*Proof.* Note that  $n = cl$  with  $1 \leq c \leq k < \frac{n}{2}$ . Thus  $l \geq 3$ .

Begin with the case of  $l \geq 4$  even. Obviously,  $\{u_0\} \cup \{v_k : 0 \leq k \leq n-1\} \subseteq I^1[\mathcal{S}]$  by definition. However, also  $u_{n-1} \in I^1[\mathcal{S}]$ . To see this, note that  $v_{n-1} \in V_{c-1}$ , and  $(c-1) + ik \equiv n-1 \pmod{n}$  holds iff  $i \frac{k}{c} \equiv -1 \pmod{l}$ . As  $l$  is even, the last congruency can only hold for  $i$  odd. Thus,  $v_{n-1} \in \mathcal{S}_v \subseteq \mathcal{S}$  and  $u_{n-1} \in I^1[\mathcal{S}]$ .

In addition to  $u_{n-1} \in I^1[\mathcal{S}]$ ,  $u_1 \in I^1[\mathcal{S}]$  happens if and only if  $c = 1$ . In any case, another  $\frac{n}{2} - 1$  steps are necessary for the infection to spread from  $I^1[\mathcal{S}]$  to the rest of the graph.

Turn now to the case of  $l \geq 3$  odd. Obviously,

$$\{u_j : 0 \leq j \leq c-1\} \cup \{v_{j+ik} : 0 < i < l-1, 0 \leq j \leq c-1\} \subseteq I^1[\mathcal{S}]$$

by definition, where some of the  $v_{j+ik}$  for  $i \in \{0, l-1\}$  will still be missing. In addition, exactly one of  $u_c$  and  $u_{n-1}$  will be in  $I^1[\mathcal{S}]$ . To see this, note that  $v_c \in V_0$ , and  $i_1 k \equiv c \pmod{n}$  holds iff  $i_1 \frac{k}{c} \equiv 1 \pmod{l}$ . Similarly,  $v_{n-1} \in V_{c-1}$ , and  $(c-1) + i_2 k \equiv n-1 \pmod{n}$  holds iff  $i_2 \frac{k}{c} \equiv -1 \pmod{l}$ . Thus,  $i_1 \frac{k}{c} + i_2 \frac{k}{c} \equiv 0 \pmod{l}$ , and  $i_1 + i_2$  is a multiple of  $l$ . With  $0 < i_1, i_2 < l$ , we must have  $i_1 + i_2 = l$  odd. Thus, exactly one of  $i_1, i_2$  is odd, exactly one of  $v_c, v_{n-1}$  is in  $\mathcal{S}_v \subseteq \mathcal{S}$ , and exactly one of  $u_c, u_{n-1}$  is in  $I^1[\mathcal{S}]$ . In particular,

$I^1[\mathcal{S}]$  contains exactly  $c + 1$  of the points  $u_j$ . These  $c + 1$  points are the vertices of a path of length  $c$  on the exterior cycle.

Continuing with  $v_c$  and  $v_{n-1}$ , we have  $v_c \in I^1[\mathcal{S}]$  and  $u_c \in I^2[\mathcal{S}]$ . To see this, note that  $v_c \in I^1[\mathcal{S}]$  fails iff  $i_1 = l - 1$ . However,  $i_1 = l - 1$  implies  $\frac{k}{c} \equiv -1 \pmod{l}$  and  $k + c \equiv 0 \pmod{n}$  which is impossible for  $1 \leq c \leq k < \frac{n}{2}$ . A similar argument shows that  $u_{n-1} \in I^2[\mathcal{S}]$  unless  $k = c = \gcd(n, k)$ , i.e., unless  $n$  is a multiple of  $k$ . Thus,  $u_c, u_{n-1} \in I^2[\mathcal{S}]$  for  $k \nmid n$  while a direct inspection shows  $u_c, u_{c+1} \in I^2[\mathcal{S}]$  for  $k \mid n$ . In any case,

$I^2[\mathcal{S}]$  will add at least one new point  $u_j$  to  $I^1[\mathcal{S}]$  but not more than two.

For  $c \in \{1, 2\}$ , note  $\{v_k : 0 \leq k \leq n - 1\} \subseteq I^2[\mathcal{S}]$ . If  $c = 1$ , then  $n = l$  is odd and the infection will need another  $\frac{n+1}{2} - 2$  steps to spread from  $I^2[\mathcal{S}]$  to the rest of the graph. If  $c = 2$ , then  $n = 2l$  is even and the infection will need another  $\frac{n}{2} - 2$  steps to spread from  $I^2[\mathcal{S}]$  to the rest of the graph.

For  $c \geq 3$ , we have  $\{v_k : 0 \leq k \leq n - 1\} \subseteq I^3[\mathcal{S}]$ . If  $k \nmid n$  and  $u_c \in I^1[\mathcal{S}]$ , then  $u_{c+1}, u_{n-1} \in I^2[\mathcal{S}]$ ,  $u_{c+2}, u_{n-2} \in I^3[\mathcal{S}]$ , and the infection will need a total of  $\frac{n-c}{2} + 1$  steps to spread from  $\mathcal{S}$  to the rest of the graph. A similar argument is true for  $k \nmid n$  and  $u_{n-1} \in I^1[\mathcal{S}]$ . If  $k \mid n$ , then  $u_{c+1} \in I^2[\mathcal{S}]$ ,  $u_{c+2}, u_{n-1} \in I^3[\mathcal{S}]$ , and the infection will need again a total of  $\frac{n-c}{2} + 1$  steps to spread from  $\mathcal{S}$  to the rest of the graph.  $\square$

**Lemma 4.2.** *For a cubic graph,  $G = (V, E)$ , and infecting set  $S$ , the infecting time is  $\lceil \frac{d+1}{2} \rceil$  where  $d$  is the largest diameter of a tree of the forest  $G[V - S]$ .*

*Proof.* From Theorem 3.1,  $G[V - S]$  is a forest. The result follows by observing that since  $G$  is cubic, the infection will first infect the leaves of  $G[V - S]$  and progress in a similar fashion iteratively.  $\square$

In general, determining the minimal or maximal infecting time of a minimal infecting set for  $G(n, k)$  is very difficult. We give next full results for the special case of  $G(n, 1)$ , including an exhaustive list of all possible infecting times. By Lemma 4.2, it suffices to calculate the maximal diameter for the connected components of the complement of a minimal infecting set. We give these maximal diameters below as it is finer information.

**Theorem 4.3.** *For  $G(n, 1)$ , the set of possible maximal diameters of connected components for the complement of a minimal infecting set  $S$  is*

$$\left\{ n, n + 2, n + 4, \dots, n + 2 \left\lfloor \frac{n - 3}{4} \right\rfloor \right\}$$

when  $n \geq 3$  is odd and

$$\left\{ d : \left\lfloor \frac{n}{2} \right\rfloor \leq d \leq \frac{3}{2}n - 2 \text{ with } d \neq n - 1 \right\}$$

when  $n \geq 4$  is even.

*Proof.* Begin by viewing  $G = G(n, 1)$  as a discrete annulus as in Figure 2.

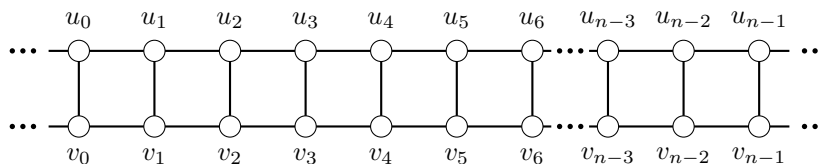


FIGURE 2.

Start with the case where  $n$  is odd ( $n \geq 3$ ) and write  $n = 2k + 1$ . By Corollary 3.2 and Theorem 3.4, every minimal infecting set,  $S$ , contains  $k + 1$  vertices, the remaining vertices constitute a tree,  $T$ , and no two vertices of  $S$  are neighbors. To prevent a cycle from appearing in  $T$ , there must exist a (square) face of the annulus whose top and bottom edges both are not in  $T$ . From this it follows, possibly after turning the annulus over, that there is a face with exactly two vertices of  $S$  configured as in Figure 3, where marked points belong to  $S$ .

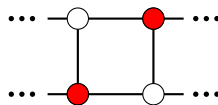


FIGURE 3.

After relabeling, the top left vertex can be labeled as  $u_{n-1}$  and the bottom right as  $v_0$ . Cut the annulus now through the middle of this square and unfold it. The tree  $T$  must connect  $v_0$  to  $u_{2k}$  and is forced to contain  $u_1, v_1, u_{2k-1}$ , and  $v_{2k-1}$ , see

Figure 2. Recalling that  $S$  has no neighboring vertices, to prevent  $T$  from having a cycle around the face with indices 1 and 2,  $S$  must contain exactly one element of  $\{u_2, v_2\}$ . Arguing iteratively, it follows that each set  $\{u_{2i}, v_{2i}\}$ ,  $0 \leq i \leq k$  contains exactly one element of  $S$ .

The shortest diameter for such a tree  $T$  is pictured in Figure 4. It has diameter  $n$  and, by Lemma 4.2, a corresponding infecting time of  $\lceil \frac{n+1}{2} \rceil$ .

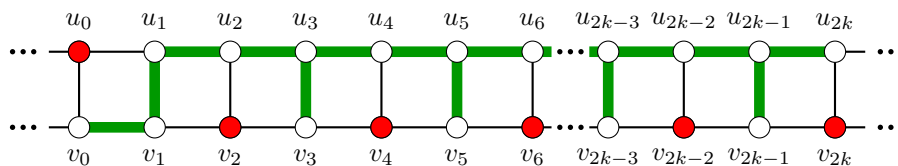


FIGURE 4. Minimal Diameter Infecting Set, I

Larger diameters may be obtained by “weaving in and out.” For example, exchanging  $v_4$  for  $u_4$  in Figure 4 to “weave around”  $u_4$  gives a diameter of  $n + 2$  (see Figure 5). Additional bumps may be added and each increases the diameter of  $T$  by 2, to a maximum diameter of  $n + 2\lfloor \frac{n-3}{4} \rfloor$  (see Figure 6 for  $k$  odd).

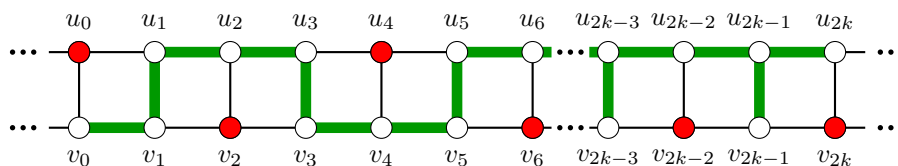


FIGURE 5. Increasing the Diameter by 2

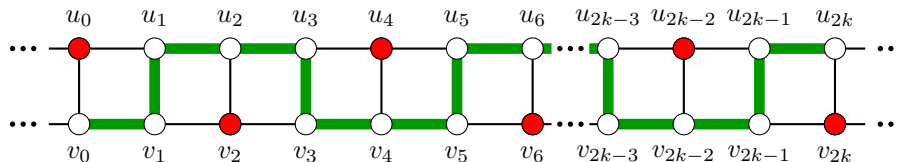


FIGURE 6. Maximal Diameter Infecting Set,  $k$  Odd

Turn now to the case of  $n$  even and write  $n = 2k$  ( $n \geq 4$ ). Again a minimal infecting set  $S$  has  $k + 1$  vertices. This case has several subcases, but as each is handled similarly to the previous case, we only sketch the major steps.

By Theorem 3.4,  $G[V - S]$  may be a forest with one or two connected components. Consider first the case where it is a single tree. In this case there now exists precisely one pair of neighboring vertices in  $S$ . We can still find a square face  $F$  of the annulus with both horizontal edges not in  $T$ . As before, split the annulus at this face. There will be two subcases to examine.

The first subcase is when  $F$  has the same form as Figure 3. The minimal diameter tree will look similar to Figure 4, except that  $T$  will connect  $v_1$  to  $v_2$  before moving up to  $u_2$  and continuing horizontally, and the indices will have a maximal label of  $2k - 1$ . Weaving will increase the diameter by 2 and the possible diameters of  $T$  are  $n, n + 2, \dots, n + 2\lfloor \frac{n-4}{4} \rfloor$ .

The second subcase is when the neighboring vertices of  $F$  are, after relabeling,  $u_0$  and  $v_0$ . In this case, the tree  $T$  with minimal diameter is pictured in Figure 7. Weaving allows incrementing the diameter by 1 and the possible diameters of  $T$  are  $n, n + 1, \dots, \frac{3}{2}n - 2$ .

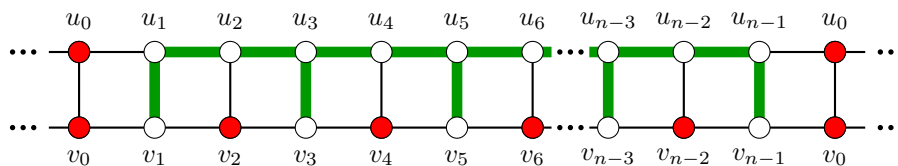


FIGURE 7. Minimal Diameter Infecting Set, II

The final case is when  $G[V - S]$  constitutes 2 connected components and  $S$  has no neighboring vertices. The infecting time is controlled by the diameter of the larger tree. The analysis is similar, but now the annulus may be cut in two places. After relabeling, there are two possibilities. The first is that  $S$  contains  $\{u_0, u_m, v_{m+1}, v_{2k-1}\}$  and the first tree connects  $v_0$  to  $v_m$  and the second tree connects  $u_{m+1}$  to  $u_{2k-1}$ . The other is that  $S$  contains  $\{u_0, v_m, u_{m+1}, v_{2k-1}\}$  and the first tree connects  $v_0$  to  $u_m$  and the second tree connects  $v_{m+1}$  to  $u_{2k-1}$ . A direct inspection shows that the smallest possible diameter for the larger tree is  $\lfloor \frac{n}{2} \rfloor$ . Weaving allows incrementing the diameter by 1 and the possible diameters for the larger tree are  $\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1, \dots, n - 2$ .

We now summarize the possible maximal diameters of connected components for the complement of a minimal infecting set  $S$  for the case of  $n$  even. The diameter  $d$  may be achieved for each  $d$  satisfying  $\lfloor \frac{n}{2} \rfloor \leq d \leq \frac{3}{2}n - 2$  with  $d \neq n - 1$ .  $\square$

## 5. FURTHER RESULTS

In this section, we examine some closely related variants of  $G(n, k)$ .

When working with two copies of  $G(n, k)$ , we will write  $\{u'_k, v'_k : 0 \leq k \leq n - 1\}$  for the second set of vertices. We then write  $G(n, k) \# G(n, k)$  for the graph obtained by surgery on the two copies of  $G(n, k)$  along the edges  $u_0 u_1$  and  $u'_0 u'_1$ . By this we mean that  $G(n, k) \# G(n, k)$  is the union of the two graphs with the edges  $u_0 u_1$  and  $u'_0 u'_1$  replaced by  $u_0 u'_0$  and  $u_1 u'_1$ . The following result calculates the  $P_3$ -hull number of  $G(n, k) \# G(n, k)$ . In fact, the result generalizes to most types of surgery along other edges.

**Theorem 5.1.**  $h_{P_3}(G(n, k) \# G(n, k)) = n + 1$ .

*Proof.* Write  $G = G(n, k) \# G(n, k)$ . Then  $G$  is cubic with  $4n$  vertices and  $6n$  edges, so that, by Theorem 3.1 and [16, Corollary 2], its  $P_3$ -hull number is at least  $n + 1$ . We distinguish two cases. For each recall that  $G(n, k)$  has a minimum infecting set of size  $\lceil \frac{n+1}{2} \rceil$  with an explicit realization in Corollary 3.3. We will use notation from that Corollary.

Consider first the case where  $n$  is odd so that  $l$  (and  $c$ ) are also odd. Apply an index shift to  $G(n, k) \# G(n, k)$ , so that the surgery happens along the edges  $u_{c-1} u_c$  and  $u'_{c-1} u'_c$ . By inspection, the union of our infecting sets for each  $G(n, k)$  infects  $G$ . As the order of this union is  $n + 1$ , we are done.

Consider next the case where  $n$  is even. When  $l$  is also even, a slight modification of  $\mathcal{S}$  provides the infecting set

$$\{u_0\} \cup \{v_{j+ik}, v'_{j+ik} : 0 \leq i \leq l - 1, 0 \leq j \leq c - 1, i \text{ even}\}$$

of size  $n + 1$ . When  $l$  is odd, then  $c$  is even. Apply an index shift to  $G(n, k) \# G(n, k)$ , so that the surgery happens along the edges  $u_{c-1} u_c$  and  $u'_{c-1} u'_c$ . This time, an infecting set is given by the union of the two infecting sets for  $G(n, k)$  minus  $\{u'_{c-1}\}$ . The size of this set is  $(\frac{n}{2} + 1) + \frac{n}{2} = n + 1$ , and we are done.  $\square$

The following is a generalization of  $G(n, k)$ .

**Definition 5.2.** Let  $\sigma$  be a nontrivial permutation in  $S_n$  and write it as a product of disjoint, nontrivial cycles,  $\sigma = \prod_{i=1}^m \sigma_i$ . Assume  $|\sigma_i| \geq 3$  for  $1 \leq i \leq m$ .

The *GGP graph* for  $\sigma$ ,  $GGP(n, \sigma)$  is a graph on  $2n$  vertices with vertex set

$$V = \{u_0, u_1, \dots, u_{n-1}, v_0, v_1, \dots, v_{n-1}\}$$

and edge set

$$E = \{u_i u_{i+1}, u_i v_i, v_i v_{\sigma(i)} : 0 \leq i \leq n - 1\}$$

with indices interpreted modulo  $n$ .

The assumption above that each  $|\sigma_i| \geq 3$  guarantees that  $GGP(n, \sigma)$  is a simple cubic graph. Note that if  $\sigma(i) = i + k$ , then  $GGP(n, \sigma) = G(n, k)$ .

**Example 5.3.** The following is an example of a GGP graph.

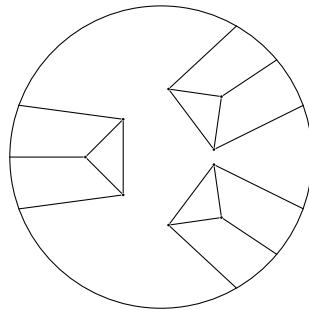


FIGURE 8. Example of  $GGP(9, \sigma)$

Note that the example of  $GGP(9, \sigma)$  in Figure 8 has a decycling number of 6, but that  $\lceil \frac{9+1}{2} \rceil = 5$ . It follows that Corollary 3.2 does not hold for all  $GGP$  graphs. It is expected that a general formula for the  $P_3$ -hull number for all  $GGP$  graphs is difficult. We can, however, provide some bounds.

**Theorem 5.4.** *Let  $\sigma$  be a nontrivial permutation in  $S_n$  written as a product of disjoint, nontrivial cycles,  $\sigma = \prod_{i=1}^m \sigma_i$ , with  $n_i = |\sigma_i| \geq 3$  for  $1 \leq i \leq m$ . Write  $k$  for the number of  $n_i$  that are odd and let  $G = GGP(n, \sigma)$ .*

*Then*

$$h_{P_3}(G) \geq \left\lceil \frac{n+1}{2} \right\rceil.$$

*If each  $n_i$  is even, then*

$$h_{P_3}(G) = \frac{n+2}{2}.$$

*If there are odd  $n_i$ , then*

$$h_{P_3}(G) \leq \frac{n+k}{2}.$$

*Note, in particular,  $h_{P_3}(G) = \frac{n+2}{2}$  for  $k = 0$ ,  $h_{P_3}(G) = \frac{n+1}{2}$  for  $k = 1$ , and  $h_{P_3}(G) = \frac{n+2}{2}$  for  $k = 2$ .*

*Proof.* Since  $G = (V, E)$  is cubic, Theorem 3.1 and [16, Corollary 2] provide the lower bound. In addition, [16, Theorem 3] shows that  $h_{P_3}(G) = \gamma_M(G) + \xi(G)$ , where  $\gamma_M(G)$  is the maximum genus of  $G$  and  $\xi(G)$  is its Betti deficiency. As  $G$  is connected,  $\gamma_M(G) = \frac{\beta(G) - \xi(G)}{2}$ , where  $\beta(G)$  is the cycle rank of  $G$ , which is  $|E| - |V| + 1 = n + 1$  for a cubic connected graph on  $2n$  vertices. Thus  $h_{P_3}(G) = \frac{\beta(G) + \xi(G)}{2} = \frac{n+1+\xi(G)}{2}$ .

Consider first the case of  $k = 0$ . Recall that the Betti deficiency,  $\xi(G)$ , is the minimum over all co-trees of  $G$  of the number of components with an odd number of edges. So, we only need to construct a co-tree of  $G$  with at most 1 odd component. We can construct such a tree  $T$  by taking all exterior edges  $u_i u_{i+1}$  except  $u_0 u_1$ , plus all in-between edges,  $u_i v_i$ . Thus the components of the co-tree corresponding to  $T$  are the interior cycles and  $u_0 u_1$ . There are no odd interior cycles so  $u_0 u_1$  is the only odd component. Thus  $\xi(G) \leq 1$  and  $h_{P_3}(G) \leq \frac{n+2}{2}$ . However, note that  $n = \sum_{i=1}^m n_i \equiv k \pmod{2}$  is even and  $h_{P_3}(G) \geq \lceil \frac{n+1}{2} \rceil = \frac{n+2}{2}$  by our lower bound.

Turn now to the case of  $k \geq 1$ . To show that  $\xi(G) \leq k - 1$ , it suffices to construct a co-tree,  $T$ , of  $G$  with  $k - 1$  odd components. As a visual guide for the construction, see Figure 9.

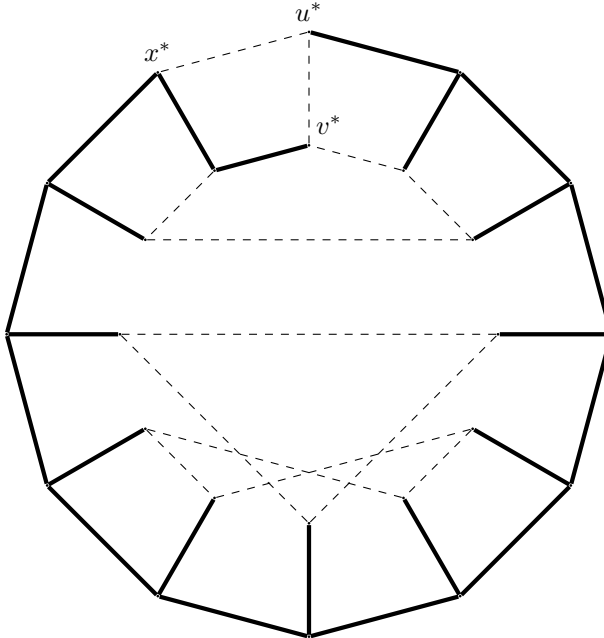


FIGURE 9. A  $GGP$  on 24 Vertices With 2 Odd Inner Cycles

Begin with any interior cycle,  $C$ , of odd length in  $G$ . Pick an edge  $e^* = u^* v^*$  between  $v^* \in C$  and  $u^* \notin C$ . Include all edges  $u_i v_i$ , except  $e^*$ , in  $T$ . Label the neighbors of  $u^*$  on the exterior cycle as  $x^*$  and  $y^*$ . Also include every exterior edge,  $u_i u_{i+1}$  except  $u^* x^*$ , in  $T$  (this includes  $u^* y^*$ ). Finally, add to  $T$  an edge of  $C$  that is incident to  $v^*$ . This results in a spanning tree of  $G$ .

Write  $H$  for the co-tree of  $G$  corresponding to  $T$ . Observe that the remaining edges of  $C$ , together with  $e^*$  and  $u^*x^*$ , form an even component of  $H$ . The only other components of  $H$  are the rest of the interior cycles. Thus the number of odd components of  $H$  is one less than the number of odd interior cycles of  $G$ , so  $\xi(G) \leq k - 1$ .  $\square$

The upper bound in Theorem 5.4 is achieved for certain combinations of  $n$  and  $k$ . For instance, Figure 8 has  $n = 9, k = 3$  and  $h_{P_3}(G) = 6$ . More generally,  $n = 3k$  and  $h_{P_3}(G) = 2k$  holds for the corresponding graph with  $k$  triangles in the interior achieving the upper bound. However, there are quite general cases where the lower bound is achieved. The next theorem gives a sufficient condition based on an interleaving principle.

**Theorem 5.5.** *Let  $\sigma$  be a nontrivial permutation in  $S_n$  written as a product of disjoint, nontrivial cycles,  $\sigma = \prod_{i=1}^m \sigma_i$ , with  $n_i = |\sigma_i| \geq 3$  for  $1 \leq i \leq m$ . Let  $G = GGP(n, \sigma)$ . If there are  $k$  odd  $n_i$ , assume there is a path of  $k$  exterior vertices  $u_j$ , each of which connects to a different odd internal cycle. Then  $h_{P_3}(G) = \lceil \frac{n+1}{2} \rceil$ .*

*Proof.* Define an infecting set by following a similar strategy to Corollary 3.3. Relabel so that the vertices of our path of  $k$  exterior vertices are  $u_0, \dots, u_{k-1}$ . Each odd internal cycle connects uniquely to this path by some edge  $u_i v_i$ ,  $0 \leq i \leq k - 1$ .

To define the initial infecting set,  $\mathcal{S}$ , for each internal cycle,  $C_i$ , add to  $\mathcal{S}$  every other vertex of the cycle as was done with  $\mathcal{S}_v$ , avoiding the points  $v_0, \dots, v_{k-1}$  on the odd internal cycles. If there are only even cycles, add one external vertex for a total of  $\frac{n}{2} + 1 = \lceil \frac{n+1}{2} \rceil$  points. If there are odd cycles, add vertices from our path of  $k$  exterior vertices  $u_0, \dots, u_{k-1}$  as was done with  $\mathcal{S}_u$  in Corollary 3.3 in the case of  $l$  odd. It is straightforward to verify this is an infecting set of size  $\lceil \frac{n+1}{2} \rceil$ .  $\square$

There are numerous generalizations of Theorem 5.5 that can be made and it would be very interesting to determine exactly when the possible minimal and maximal  $P_3$ -hull numbers are obtained for  $GGP$  graphs.

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## 6. STATEMENTS AND DECLARATIONS

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