
ROBUST LEARNING IN HETEROGENEOUS CONTEXTS

A PREPRINT

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February 18, 2022

ABSTRACT

We consider the problem of learning from training data obtained in different contexts, where the underlying context distribution is unknown and is estimated empirically. We develop a robust method that takes into account the uncertainty of the context distribution. Unlike the conventional and overly conservative minimax approach, we focus on excess risks and construct distribution sets with statistical coverage to achieve an appropriate trade-off between performance and robustness. The proposed method is computationally scalable and shown to interpolate between empirical risk minimization and minimax regret objectives. Using both real and synthetic data, we demonstrate its ability to provide robustness in worst-case scenarios without harming performance in the nominal scenario.

1 Introduction

Machine-learning methods often leverage large amounts of training data that is collected from several sources in varying contexts. For instance, image classification data is labeled by several people and the data itself may be captured in different environments with different backgrounds or levels of illumination [1]. More generally, data could be collected in contexts with different covariate distributions [21, 15] or different interventions [4, 13].

In this paper, we consider a finite set of contexts, indexed by an integer $c \in \mathcal{C}$. In given context c , training data is drawn independently and identically as

$$\mathbf{z}_i \sim p(\mathbf{z}|c), \quad i = 1, \dots, n_c, \quad (1)$$

where the training distribution is unknown. Let $\boldsymbol{\theta} \in \Theta$ be a decision parameter in a specific task and $\ell_{\boldsymbol{\theta}}(\mathbf{z})$ denote the loss at $\boldsymbol{\theta}$ incurred for test point \mathbf{z} . Then each context has a conditional risk denoted

$$R_c(\boldsymbol{\theta}) = \mathbb{E}[\ell_{\boldsymbol{\theta}}(\mathbf{z})|c]$$

Using n_c training samples the conditional risk can be approximated by an empirical average (or a regularized form of it).

Example 1 (Stock control). *Suppose we must decide on the quantity θ to keep in stock for sale. Let x denote the purchasing cost per unit and y the total demand so that the loss of a specific stock level $\theta \in [0, \theta_{\max}]$ is*

$$\ell_{\boldsymbol{\theta}}(\mathbf{z}) = \theta x - r \min(\theta, y)$$

where r is a given price per unit and $\mathbf{z} = (x, y)$. Figure 1a illustrates past data observed in two different contexts, for instance sunny or rainy weather. Since they are heterogeneous, their conditional risks $R_c(\boldsymbol{\theta})$ at stock level θ differ, as seen in Figure 1b.

When future contexts are unknown, as in the example above, we treat c as a random variable with a distribution $p(c)$ and consider the parameter $\boldsymbol{\theta}$ that minimizes the (overall) risk

$$R(\boldsymbol{\theta}; p) \equiv \sum_{c \in \mathcal{C}} R_c(\boldsymbol{\theta}) p(c), \quad (2)$$

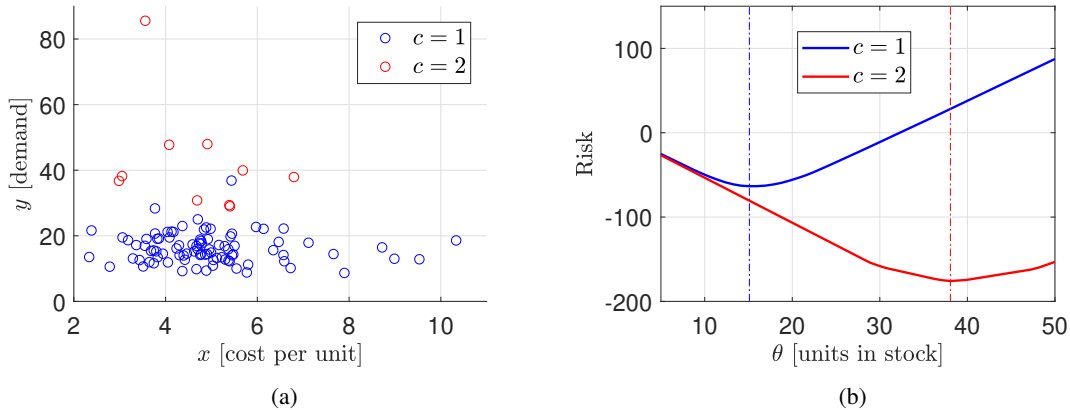


Figure 1: Stock control example. (a) Training data for cost per unit, x , and total demand, y , in two different contexts $c = 1$ and $c = 2$. The number of samples are $n_1 = 90$ and $n_2 = 10$, respectively. (b) Conditional risk $R_c(\theta)$ in each context (solid lines) as a function of stock level θ . Vertical lines indicate the corresponding optimal stock levels. The risk has no closed-form solution as was computed numerically using Monte Carlo simulations.

Empirical risk minimization (ERM) [23, 20] is a standard learning approach that forms an empirical approximation of (2) with an *unbiased* estimate of $p(c)$. When the number of contexts $|\mathcal{C}|$ is relatively large, such an unbiased estimate can readily underemphasize more challenging contexts leading to poor worst-context performance as a result.

The classical robust approach, dating back to [24], is the minimax method which focuses on the worst-case risk alone. This approach was soon recognized to be overly conservative and can result in a poor trade-off between performance and robustness [8]. [17] recognized that what is the ‘worst-case’ context must be put in relation to the minimum risk achievable in that context. He therefore proposed focusing on the worst-case *excess* risk (aka. regret) instead. [18], on the other hand, recognized that considering a worst-case *distribution* with a specified set of distributions would avoid an overemphasis on very improbable scenarios. This approach was extended to distribution sets for $p(z)$ with a size controlled by a user parameter. These sets have been constructed using a variety of measures, including moment-based constraints, Wasserstein distances, f-divergences and minimum mean discrepancy measures [7, 19, 14, 2, 22].

In this paper, we focus on the uncertainty of $p(c)$ using confidence-based distribution sets building upon principles congruent with those recently advanced in the decision-theoretic literature on imprecise probabilities [10, 3]. We combine the insights from both Savage’s focus on excess risks and Scarf’s focus on distributional uncertainty to develop a method for learning from training data obtained in heterogeneous contexts. The method has the following properties:

- it is robust in the worst-case contexts,
- computationally efficient,
- uses distribution sets with a desired level of statistical confidence,
- and interpolates adaptively between the empirical risk minimization and minimax regret methods.

We demonstrate the methodology in different tasks using real and synthetic data.

2 Problem Formulation

Using a total of n samples, we seek to learn a parameter $\hat{\theta}$ that approximates a minimizer of the risk $R(\theta; p)$, which may depend heavily on the distribution of contexts p .

Example 2 (Stock control, cont’d). Consider two different context distributions, $p' = \{0.90, 0.10\}$ and $p'' = \{0.20, 0.80\}$. Figure 2a shows how the risk $R(\theta; p)$ differs under two context distributions, along with the resulting optimal stock levels θ .

The empirical risk minimization approach (ERM) solves

$$\min_{\theta} \sum_{c \in \mathcal{C}} \hat{R}_c(\theta) \hat{p}(c), \quad (3)$$

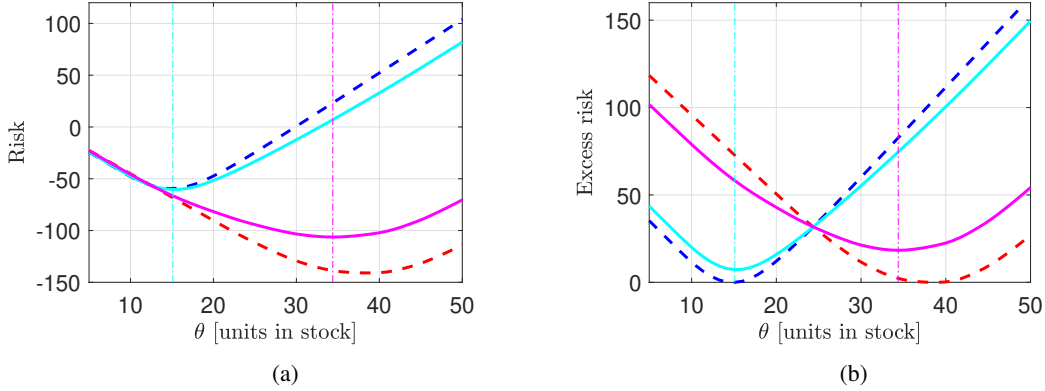


Figure 2: Stock control problem under two different context distributions, p' and p'' . (a) Risk $R(\theta; p)$ under p' (solid cyan) and p'' (solid magenta). Dashed lines show the conditional risks $R_c(\theta)$, see Figure 1b. (b) Excess risk $\Delta(\theta; p)$ under p' (solid cyan) and p'' (solid magenta). Dashed lines show the conditional excess risk in each context. Vertical lines show the optimal stock levels for p' and p'' , respectively.

where $\hat{p}(c) = \frac{n_c}{n}$ is an unbiased estimate of $p(c)$ and $\hat{R}_c(\theta) = \frac{1}{n_c} \sum_i \ell_\theta(z_i)$ is an empirical estimate. The unbiased estimates can lead to an underemphasis on contexts with risks that are sensitive to θ . Wald's robust approach focuses on the worst-case context and the (empirical) minimax risk method solves

$$\min_{\theta} \max_{c \in \mathcal{C}} \hat{R}_c(\theta) \quad (4)$$

It has two main drawbacks: it can overemphasize highly improbable contexts and also ignore how the risk can be reduced in alternative contexts. For instance, in a classification task the minimax approach could focus on a context where the error rate is high but insensitive with respect to θ , while ignoring contexts with low minimum error rates that are sensitive to θ . Moreover, while conceptually simple, finding a solution to (4) is often a computationally challenging problem.

We now consider an alternative approach to achieve an appropriate trade-off between performance and robustness against worst-case contexts.

3 Robust method

The minimum achievable risk, i.e., $\min_{\theta} R_c(\theta)$, may differ across heterogeneous contexts. Following [17], we focus only on the part of the risk that can be reduced by the parameter θ , unlike the overly conservative nature of the approach in (4). Let us now consider the (overall) excess risk that θ incurs,

$$\Delta(\theta; p) \equiv \sum_{c \in \mathcal{C}} \left[R_c(\theta) - \min_{\theta'} R_c(\theta') \right] p(c), \quad (5)$$

for a given context distribution p .

Example 3 (Stock control, cont'd). *The excess risk $\Delta(\theta; p)$ under two different context distributions p' and p'' is shown in Figure 2b.*

The excess risk at stake for any pair (θ, p) is given by $\Delta(\theta; p)$. For a given p , a parameter θ that minimizes excess risk (5) also minimizes the overall risk (2), and vice versa. In lieu of p , we consider a distribution p^* that yield worst-case excess risk in a *confidence set* of distributions, following [18].

3.1 Confidence-based learning

Let \mathcal{P} denote the set of all distributions p . Given n training data points, we seek a subset $\mathcal{P}_\beta^n \subseteq \mathcal{P}$ that covers the unknown p at a specified confidence level β :

$$\mathbb{P}\{p \in \mathcal{P}_\beta^n\} \geq \beta \quad (6)$$

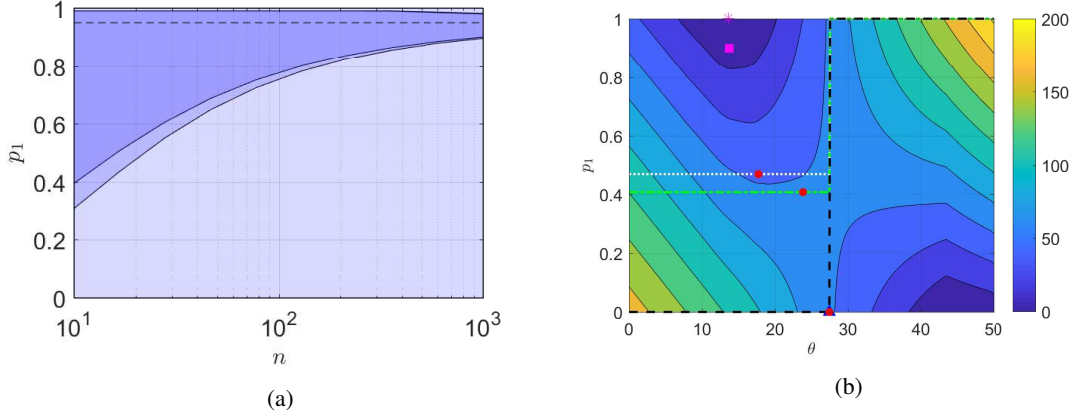


Figure 3: Stock control problem with increasing amounts of training data n and unknown context distribution $p = \{p_1, 1 - p_1\}$. (a) Confidence sets \mathcal{P}_β^n with levels 90%, 99% and 100% (shaded), which are nested according to (7). Note that at $\beta = 100\%$, the set contains all distributions. Dashed line shows the unknown probability p_1 . (b) Contour plot of empirical excess risk $\widehat{\Delta}(\theta; p)$ when $n = 20$. The lines show the least favourable p_1 for each θ , obtained from the inner problem in (10), using three confidence levels β : 90% (dotted white), 99% (dashed cyan) and 100% (solid black). The red dots on the lines denote the corresponding minimizers $\widehat{\theta}_\beta$ of (10). For comparison, we include ERM (3) (square) and minimax risk θ (4) (star).

such that

$$\beta < \beta' \Leftrightarrow \mathcal{P}_\beta^n \subset \mathcal{P}_{\beta'}^n, \quad \text{and} \quad \mathcal{P}_1^n \equiv \mathcal{P} \quad (7)$$

Thus the confidence level controls the size of the distribution set \mathcal{P}_β^n at any given n .

Result 1 (Confidence set). *Let*

$$\mathcal{P}_\beta^n = \{p' : D(\widehat{p} \| p') \leq \varepsilon_\beta\}, \quad (8)$$

where $D(\widehat{p} \| p') = \widehat{\mathbb{E}}[\log_2(\widehat{p}(c)/p'(c))]$ denotes the Kullback-Leibler divergence from $\widehat{p}(c) = n_c/n$. By setting

$$\varepsilon_\beta = \frac{1}{n} \left(|C| \log_2(n+1) - \log_2(1-\beta) \right), \quad (9)$$

the set (8) satisfies the coverage property (6) and the nested property (7).

Proof. We have that $\{p : D(\widehat{p} \| p) \leq \varepsilon\} \subset \{p : D(\widehat{p} \| p) \leq \varepsilon'\} \Leftrightarrow \varepsilon < \varepsilon'$, and from (9) we have $\varepsilon < \varepsilon' \Leftrightarrow \beta < \beta'$ so that (7) holds. Using [5, thm. 11.2.1], it follows that $\mathbb{P}\{p \notin \mathcal{P}_\beta^n\} \leq 2^{|C| \log_2(n+1) - n\varepsilon} \equiv 1 - \beta$. Thus solving for ε yields (9) such that (6) is satisfied. \square

The tasks for which the stakes of choosing θ are high, require a confidence level β closer to 100% and thus a larger set \mathcal{P}_β^n . Conversely, for a given confidence level the evidence accumulated as n increases should exclude certain context distributions. This principle has been advanced more recently in the decision-theoretic literature on imprecise probabilities, see, e.g., [10, 3].

Example 4 (Stock control, cont'd). *Consider a context distribution $p = \{p_1, 1 - p_1\}$. Figure 3a illustrates nested confidence sets \mathcal{P}_β^n at levels 90%, 99% and 100%. As the amount of training data n increases, the confidence set narrows down when $\beta < 100\%$. Here the underlying context distribution is $p = \{0.95, 0.05\}$.*

We now consider θ that minimizes worst-case excess risk over all distributions in the confidence set \mathcal{P}_β^n to overcome the two main drawbacks of the minimax risk approach (4). That is,

$$\widehat{\theta}_\beta \in \arg \min_{\theta} \max_{p \in \mathcal{P}_\beta^n} \widehat{\Delta}(\theta; p) \quad (10)$$

using estimates the conditional risks in (5). By increasing the level β , we are increasingly confident that we cover the unknown p in (6).

Example 5 (Stock control, cont'd). *Figure 3b shows the empirical excess risk $\widehat{\Delta}(\theta; p)$ for different stock levels θ and context distributions $q = \{p_1, 1 - p_1\}$. ERM and the minimax risk methods give overwhelming weight to context $c = 1$. By contrast, as the confidence level β increases, the robust approach takes the excess risk of θ in context $c = 2$ increasingly into account so that (10) approaches $\theta \approx 25$ at which the excess risks for each context are equal. Note that for θ lower (or greater) than this level, the least-favorable distribution p^* gives maximum weight to $c = 2$ (or, alternatively $c = 1$).*

We now show that the approach in (10) interpolates adaptively between empirical risk minimization and minimax regret depending on the confidence level β .

Result 2. *Let p^* be the least-favorable distribution in (10) and*

$$\widehat{\Delta}_c(\theta) = \widehat{R}_c(\theta) - \min_{\theta'} \widehat{R}_c(\theta') \geq 0$$

denote the empirical excess risk for context c . Then the cost function in (10) is equivalent to a weighted combination of empirical risk and excess risk:

$$\widehat{\Delta}(\theta; p^*) = \widehat{R}(\theta; \widehat{p}) + \sum_c \widehat{p}(c) w_c^\beta(\theta) \widehat{\Delta}_c(\theta) + K, \quad (11)$$

where K is a constant and the weights $w_c^\beta(\theta) \in [-1, 1]$. Specifically,

$$w_c^\beta(\theta) = \left(\sum_{c'} \widehat{p}(c') \frac{\nu^* - \widehat{\Delta}_c(\theta)}{\nu^* - \widehat{\Delta}_{c'}(\theta)} \right)^{-1} - 1 \quad (12)$$

where $\nu^* > \max_c \widehat{\Delta}_c(\theta)$ is the root of the following equation

$$\sum_{c \in \mathcal{C}} \widehat{p}(c) \log_2 \left(\nu^* - \widehat{\Delta}_c(\theta) \right) + \log_2 \left[\sum_{c \in \mathcal{C}} \frac{\widehat{p}(c)}{\nu^* - \widehat{\Delta}_c(\theta)} \right] - \varepsilon_\beta = 0. \quad (13)$$

Furthermore, we have that

1. as $n \rightarrow \infty$ for a given β :

$$\widehat{\Delta}(\theta; p^*) \rightarrow \widehat{R}(\theta; \widehat{p}) + K$$

2. as $\beta \rightarrow 1$, for a given n :

$$\widehat{\Delta}(\theta; p^*) \rightarrow \max_c \widehat{\Delta}_c(\theta)$$

That is, in the limiting cases (10) approaches ERM as n become large and it approaches an empirical minimax regret method with increasing confidence level β .

Proof. The Lagrangian of the inner maximization problem of (10) is,

$$L(p, \boldsymbol{\lambda}, \nu) = - \sum_{c \in \mathcal{C}} \widehat{\Delta}_c(\theta) p_c + \lambda_0 [D(\widehat{p}|p) - \varepsilon_\beta] + \sum_{c \in \mathcal{C}} \lambda_c (-p_c) + \nu \left(\sum_{c \in \mathcal{C}} p_c - 1 \right). \quad (14)$$

We have $\widehat{p}_c = n_c/n > 0$ for all c . When $p \equiv \widehat{p}$, the inequality constraints are inactive and Slater's condition is satisfied, which implies that the primal-dual optimal solutions $(p^*, \boldsymbol{\lambda}^*, \nu^*)$ can be obtained by solving the following Karush-Kuhn-Tucker (KKT) conditions:

$$\begin{aligned} \partial_{p_c} L(p^*, \boldsymbol{\lambda}^*, \nu^*) &= 0, & \lambda_c^* p_c^* &= 0, & p_c^* &\geq 0 \text{ for } c \in \mathcal{C}, \\ \lambda_0^* [D(\widehat{p}|p^*) - \varepsilon_\beta] &= 0, & \sum_c p_c^* &= 1, & D(\widehat{p}|p^*) &\leq \varepsilon_\beta \end{aligned} \quad (15)$$

From the derivative condition on the Lagrangian, we obtain

$$p_c^* = \frac{\lambda_0^* \widehat{p}_c}{\nu^* - \widehat{\Delta}_c(\theta)} \text{ for } c \in \mathcal{C}. \quad (16)$$

Since $p_c^* > 0$ it follows that $\lambda_c^* = 0$. Moreover, $\lambda_0^* > 0$ since it is a common scale factor and $\lambda_0^* = (\sum_c \frac{\hat{p}_c}{\nu^* - \hat{\Delta}_c(\theta)})^{-1}$ is obtained from $\sum_c p_c^* = 1$. It follows that $D(\hat{p}||p^*) - \varepsilon_\beta = 0$, for which ν^* is a root.

Now let $\hat{R}(\theta)$ be the empirical risk in (3), $\hat{r}_c = \min_{\theta} \hat{R}_c(\theta)$ be the minimum risk in context c and $\hat{\Delta}_c(\theta) = \hat{R}_c(\theta) - \hat{r}_c$, then

$$\hat{\Delta}(\theta; p^*) - \hat{R}(\theta) = \sum_c p^*(c) \hat{\Delta}_c(\theta) - \sum_c \hat{p}(c) \hat{R}_c(\theta). \quad (17)$$

Plugging $p^*(c)$ from (16) gives

$$\hat{\Delta}(\theta; p^*) - \hat{R}(\theta) = \sum_c \hat{p}(c) \left(\hat{\Delta}_c(\theta) \frac{\lambda_0^*}{\nu^* - \hat{\Delta}_c(\theta)} - \hat{R}_c(\theta) \right). \quad (18)$$

Adding and subtracting $\sum_c \hat{p}(c) \hat{r}_c$ on the right hand side we get,

$$\hat{\Delta}(\theta; p^*) - \hat{R}(\theta) = \sum_c \hat{p}(c) \hat{\Delta}_c(\theta) \left(\frac{\lambda_0^*}{\nu^* - \hat{\Delta}_c(\theta)} - 1 \right) + K. \quad (19)$$

Let

$$w_c^\beta(\theta) = \frac{\lambda_0^*}{\nu^* - \hat{\Delta}_c(\theta)} - 1. \quad (20)$$

Plugging in λ_0^* in the above equation and simplifying gives (12).

In the first case, $\varepsilon_\beta \rightarrow 0$ as $n \rightarrow \infty$ using (9). Then $\nu^* \rightarrow \infty$ (13) and from (12), it can be seen that the fractions in the denominator cancel in the limit and $w_c^\beta(\theta) \rightarrow (\hat{p}(c) + \sum_{c' \neq c} \hat{p}(c'))^{-1} - 1 = 0 \forall c$. Hence, $\hat{\Delta}(\theta; p^*) \rightarrow \hat{R}(\theta) + K$ in (11). In the second case when $\beta \rightarrow 1$, $\varepsilon_\beta \rightarrow \infty$ and ν^* approaches $\max_c \hat{\Delta}_c(\theta)$. Then from (12), $w_c^\beta(\theta) \rightarrow -1 \forall c$ except the context with maximum regret for which it approaches $\frac{1 - \hat{p}(c)}{\hat{p}(c)}$. Plugging these in (11) gives $\hat{\Delta}(\theta; p^*) \rightarrow \max_c \hat{\Delta}_c(\theta)$. □

3.2 Numerical search method

Learning problems typically require using numerical search methods. Here we propose using gradient-based approaches to solve the problem (10). Let

$$\tilde{\Delta}(\theta) = \max_{p \in \mathcal{P}_\beta^n} \hat{\Delta}(\theta; p) \quad (21)$$

In case of loss functions for which the expected losses are convex and differentiable in θ , e.g., squared-error loss or cross-entropy loss for models that are linear in the parameters, we follow the approach in [11]. Then the gradient of (21) can be obtained using Danskin's theorem [6]: $\partial_\theta \tilde{\Delta}(\theta)|_{\theta'}$ $\equiv \partial_\theta \hat{\Delta}(\theta; p^*)|_{\theta'}$, where p^* maximizes $\hat{\Delta}(\theta', p)$ and be computed efficiently by (16) evaluated at θ' . We can therefore readily use gradient-based algorithms to compute $\hat{\theta}_\beta$, such as Algorithm 1.

Algorithm 1 Gradient descent algorithm

- 1: Input: data $\{(z_i, c_i)\}_{i=1}^n$, confidence level: β , initial estimate: $\theta' = \hat{\theta}_{\text{erm}}$, step size : η
 - 2: **repeat**
 - 3: Find solution ν^* of (13)
 - 4: Compute p^* given by (16)
 - 5: Compute gradient $\partial_\theta \tilde{\Delta}(\theta)|_{\theta'} = \sum_c \frac{p_c^*}{n_c} \sum_{i=1}^{n_c} [\partial_\theta \ell_\theta(z_i)]|_{\theta'}$
 - 6: Update $\theta' = \theta' - \eta \partial_\theta \tilde{\Delta}(\theta)|_{\theta'}$
 - 7: **until convergence**
 - 8: Output: $\hat{\theta}_\beta = \theta'$
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For cases where the loss function $\ell_\theta(z)$ is not convex in θ , an alternative method is to use the two-time scale gradient descent-ascent algorithm, which converges to a local Nash equilibrium [9].

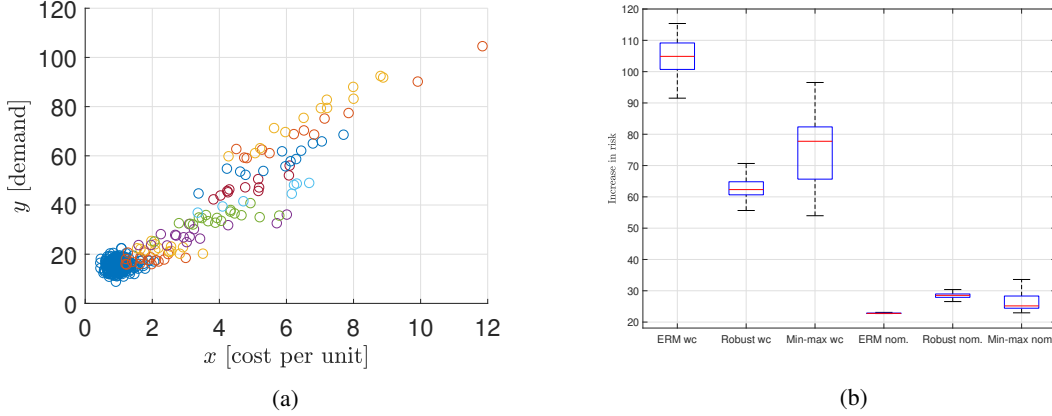


Figure 4: Stock control task: (a) Training data obtained from 10 different contexts (indicated using different colours). Data for context 1 is most prevalent while data for other contexts are scarce. (b) Box plot of excess risk for ERM, minimax and robust methods under worst-case (wc) and nominal (nom) test scenarios, using training data from 50 Monte Carlo runs.

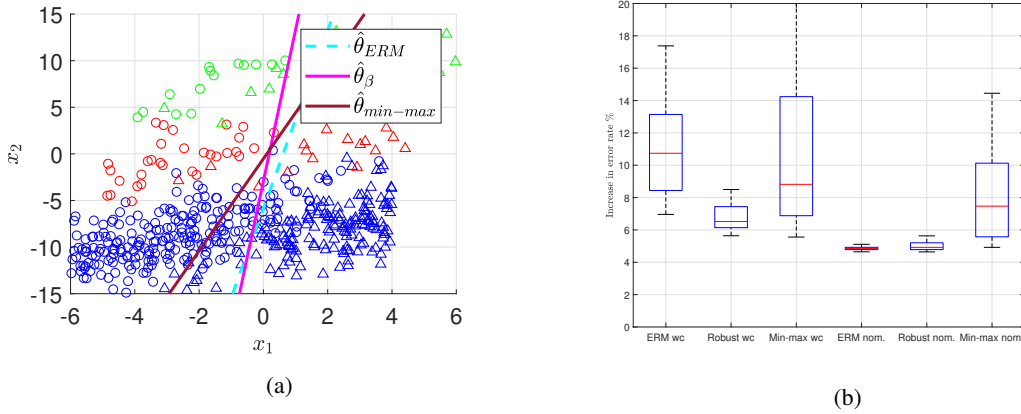


Figure 5: Classification task: (a) Training data from 3 different contexts (different colours) with label $y = 0$ (circle) and $y = 1$ (triangle). Plot also shows learned decision-boundaries parameterized by θ . (b) Box plot of excess error rates under worst-case (wc) and nominal (nom) test conditions, using training data from 50 Monte Carlo runs.

4 Experiments

We illustrate the proposed robust method for two different tasks – stock control and classification – and evaluate the excess risk it incurs across contexts. The data is drawn as (1) with a context distribution $p(c)$. We consider two extremes: (i) *nominal case* such that p will be covered by \mathcal{P}_β^n with probability β and (ii) *worst case* in which p is the least-favourable distribution in the entire set \mathcal{P} . This corresponds to focusing on the context that yields the worst excess risk. Throughout all experiments we use a confidence level of $\beta = 99\%$.

4.1 Stock control task

We consider the stock control problem again but with $|\mathcal{C}| = 10$ different contexts. We obtain $n = 400$ samples on the price per unit x , total demand y and context c , drawn as

$$\begin{aligned} c &\sim \text{Cat}(p_1, \dots, p_{|C|}), & x|c &\sim \text{Log-Normal}(\mu_c, 0.25), \\ y|x, c &\sim \text{Normal}(a_c x + b_c, 4), \end{aligned} \quad (22)$$

where the probability of the first context is $p_1 = 0.70$ and the remaining contexts have equal probabilities. The training data is illustrated in Figure 4a, using coefficients

$$\begin{aligned} \mu_c &= \frac{6}{|\mathcal{C}| - 1}(c - 1) + 1, & a_c &= \frac{6.9}{|\mathcal{C}| - 1}(c - 1) + 0.1, \\ b_c &= \frac{15}{|\mathcal{C}| - 1}(c - 1) + 15 \end{aligned} \quad (23)$$

We compare the proposed method with the ERM and minimax risk methods (3) and (4). The increased risk that $\hat{\theta}_{\text{erm}}$, $\hat{\theta}_{\text{min-max}}$ and $\hat{\theta}_\beta$ incur over all contexts is shown in Figure 4b using 50 different draws of training data. We see that in the worst case, the robust method has an excess risk around 60, as compared to 105 for ERM and 80 for minimax. This robustness is gained at the minor expense of raising the excess risk in the nominal case to about 28 as compared to 23 for ERM and 25 for minimax.

4.2 Classification task

We consider a binary classification problem with two dimensional covariates \mathbf{x} observable in $|\mathcal{C}| = 3$ contexts. For the training data, the context probabilities are $p = \{0.8, 0.1, 0.1\}$ so that context $c = 1$ is most prevalent.

We obtain $n = 1000$ samples, drawn as

$$\begin{aligned} c &\sim \text{Cat}(p_1, \dots, p_{|\mathcal{C}|}), & x_1|c &\sim \text{Unif}([-5 + \mu_c, 5 + \mu_c]), \\ y|x_1, c &\sim \text{Binomial}(\sigma(x_1)), \\ x_2|x_1, y, c &\sim \mathcal{N}(x_1 + a_c + 2(\mathbf{1}\{y = 0\} - \mathbf{1}\{y = 1\}), 4), \end{aligned} \quad (24)$$

where $\sigma(\cdot)$ is the logistic function. The training data is illustrated in Figure 5a when the coefficients are $\mu_c \in \{-1, 0, 1\}$ and $a_c \in \{-8, 0, 8\}$ corresponding to $c = 1, 2, 3$. Note that given x_1 , the covariate x_2 provides no additional information about the outcome y . For the classification task, we use logistic regression model with cross-entropy loss function, i.e.,

$$\ell_\theta(\mathbf{x}, y) = -y \ln(\sigma(\mathbf{x}^\top \boldsymbol{\theta})) - (1 - y) \ln(1 - \sigma(\mathbf{x}^\top \boldsymbol{\theta})),$$

The minimum error rates for the three contexts were 6.86%, 7.42% and 6.70% respectively.

We compare our method with ERM and minimax risk methods in (3) and (4) respectively. For the minimax method we use the algorithm proposed in [16] which updates p and $\boldsymbol{\theta}$ iteratively and also has a convergence guarantee. No explicit guideline is provided in [16] for choosing the step sizes and number of iteration parameters. We used a large number of iterations $T = 2 \times 10^4$ iterations and step sizes of 0.1. Figure 5a shows the learned decision boundaries parameterized by $\boldsymbol{\theta}$. We see that $\hat{\boldsymbol{\theta}}_\beta$ accounts for excess risks in less frequent contexts by a more vertical decision boundary than $\hat{\boldsymbol{\theta}}_{\text{erm}}$ and $\hat{\boldsymbol{\theta}}_{\text{min-max}}$. Since the cross entropy risk is taken as a proxy for test error, we evaluate the learning methods with respect to excess test error rate across contexts in Figure 5b, using 50 Monte Carlo runs. In comparison with ERM and minimax risk parameters, the robust method significantly reduces the excess error rate in the worst case, with a minor increase in the nominal case. In Algorithm 1, we used a step size $\eta = 0.05$ to obtain $\hat{\boldsymbol{\theta}}_\beta$.

4.3 Colored MNIST classification task

Finally, we consider a binary classification task with real data of handwritten digits [12] inspired by [1]. The task is to classify images of handwritten digits as `low` (range 0–4) and `high` (range 5–9), with a corresponding label $y \in \{0, 1\}$. For the images we use an autoencoder to extract features \mathbf{x} from grey scale images of size 28×28 . In addition, we consider the images to be (synthetically) colored as `green` or `red`, which is indicated by a covariate $x^{\text{col}} \in \{0, 1\}$, see Figure 6a. We consider the association between image color and the digits to vary across $|\mathcal{C}| = 5$ contexts, i.e.

$$p(x^{\text{col}} = 1|y = 0, c) = p(x^{\text{col}} = 0|y = 1, c)$$

$\in \{0.95, 0.84, 0.72, 0.61, 0.50\}$ corresponding to $c = 1, \dots, 5$. Note that in context $c = 5$, the color is entirely uninformative of the digit. We draw $n = 900$ samples randomly using the MNIST training data with context distribution $p = \{0.6, 0.1, 0.1, 0.1, 0.1\}$. For simplicity, we use a four-dimensional feature vector \mathbf{x} since it provides sufficient accuracy in this problem and use a logistic regression model with cross-entropy loss function as in the previous example. The minimum error rates for contexts 1 to 5 were estimated as 5.03%, 14.74%, 24%, 31.90% and 34.31%, respectively, using 10^4 data points drawn from the MNIST test data for each context.

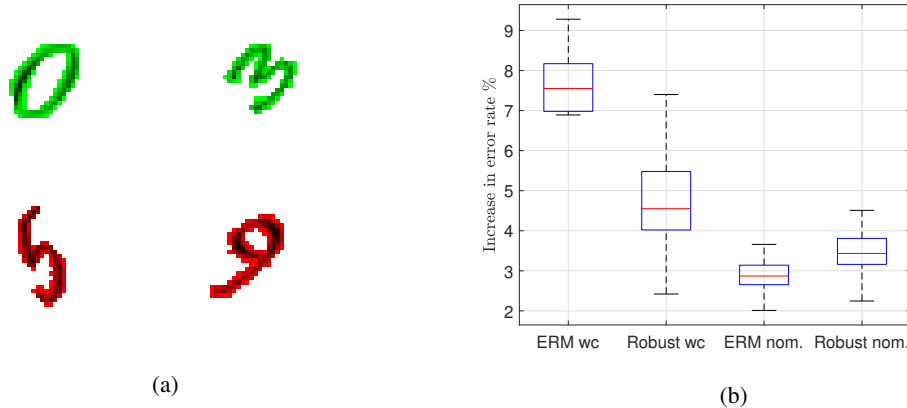


Figure 6: Colored MNIST classification task: (a) Hand-written and colored digits. (b) Box plot of excess error rates under worst-case (wc) and nominal (nominal test conditions, using training data from 50 Monte Carlo simulations).

We compare our method with only ERM here because the numerical solver for minimax proposed by [16] does not provide a reasonable solution with the parameters chosen above. Figure 6b shows the results using 50 Monte Carlo simulations. Compared to ERM, the robust method once again significantly reduces the excess error rate in the worst case, and results in a minor increase in the nominal case. To obtain $\hat{\theta}_\beta$, we used a step size $\eta = 0.01$ in Algorithm 1.

5 Discussion

We have considered the problem of learning from training data obtained in different contexts. The underlying context distribution is unknown and is estimated empirically, which can underemphasize challenging contexts. Building on the insights from [17] and [18], we consider all distributions in a *confidence set* and seek the decision parameter that is robust against the distribution that yields the worst *excess* risk. This approach is consistent with the confidence-based decision-theoretic principles that have recently been proposed to tackle imprecise probabilities, cf. [10, 3].

We showed that this approach interpolates between empirical risk minimization and minimax regret objectives based on the chosen confidence level. In this way, it seeks to achieve a statistically motivated trade-off between performance and robustness that is better balanced than the conventional minimax risk approach. Using both real and synthetic data, we demonstrate its ability to provide robustness in worst-case scenarios without harming performance in the nominal scenario. Further work may investigate specific applications in which regularized learning is particularly relevant.

Acknowledgement

This research was partially supported by the Swedish Research Council (contract no.: 2018-05040) and the *Wallenberg AI, Autonomous Systems and Software Program* (WASP) funded by Knut and Alice Wallenberg Foundation.

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