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REVIEW OF SOLUTION METHODS FOR THE FRACTIONAL OPTIMAL CONTROL

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Abstract. *Optimal control was introduced in 1954 by Pontryagin and Bellman. The fractional calculus of variation gains the momentum in last three decades. The present review investigates the progress in the field of fractional calculus of variation and henceforth the development of fractional optimal control that is first introduced by Agrawal in 2002. It covers the formulation of optimal control problem, approximation of fractional optimal control problem and solution methodologies for the fractional optimal control problem.*

Keywords: Fractional calculus, FOCP, Fractional E-L, LQR, Orthogonal polynomials

1. **Introduction.** The fractional calculus originated in 1695 when Guillaume de l'Hôpital sent a letter to Gottfried Wilhelm Leibniz where he mentioned about the fractional derivative. The basic need for the formulation of any optimal control problem is to form and solve the Euler-Lagrange equations. The fractional E-L equations were introduced first time in [2]. Since then the fractional calculus of variation and henceforth the fractional optimal control gain the momentum.

The fractional variational principle and fractional calculus has been mentioned in much literature [3-5]. Since fractional order calculus theory is now established, various researchers have used it for the applications of system modelling and control purposes. Some of the recent control applications include DC motor control [6], brushless DC motor control [7], linear motor control [8], contactless actuation system [9], renewable energy integration system that includes pump and solar panel [10], vibration control in seismic structures [11] and control of satellite system [12]. The main motivation for this review paper is to provide detailed information about the recent development in fractional order optimal control part to a suitable researcher. Using this information one can design the control system in fractional order domain in optimal manner. This paper is mainly concerned with fractional order optimal control solution. Since fractional order optimal control requires the solution of differential equations and solution of fractional integration, some numerical methods are mentioned related to the solution of fractional differential equations and fractional order optimal control. The solution of fractional optimal control requires the study of different pseudo spectral methods including different types of wavelets, orthogonal functions, numerical integration techniques, collocation points, etc. Since almost all the methods for the solution of fractional optimal control problem are based upon one or another form of numerical techniques, for the sake of simplicity we classify them in to five different classes.

The paper is organized as follows. Section 2 covers the basic introduction to fractional calculus and various definitions of fractional integration and differentiation. Fractional

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optimal control system is introduced in Section 3. Based upon the pioneering work carried out by the various researchers, the authors have classified five different areas for solution of fractional optimal control problems namely 1) polynomial approach, 2) basic numerical method approach, 3) various wavelet method based approach, 4) neural network based approach and 5) LQR based approach. These methods are introduced in Section 4. At last in Section 5, conclusion regarding fractional optimal controller is given followed by references. Now in next section brief review for the fractional calculus and some of the commonly used definitions will be illustrated.

2. **Fractional Calculus.** The following definitions [13] are widely used in fractional integration and differentiation. The left and right Riemann and Liouville derivatives are given by

$$\left(D_{a+}^{\alpha} y\right)(x) = \frac{1}{\Gamma(n-\alpha)} \cdot \left(\frac{d}{dx}\right)^{n} \int_{a}^{x} (x-t)^{n-\alpha-1} y(t)dt \quad \alpha \in [n-1, n) \tag{1}$$

and

$$\left(D_{b-}^{\alpha} y\right)(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^{n} \int_{x}^{b} (t-x)^{n-\alpha-1} y(t)dt \quad \alpha \in [n-1, n) \tag{2}$$

respectively.

The left and right Caputo derivatives are given by

$$\left(D_{b-}^{\alpha} y\right)(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^{n} \int_{x}^{b} (t-x)^{n-\alpha-1} y(t)dt \quad \alpha \in [n-1, n) \tag{3}$$

and

$$\left({}^{C}D_{b-}^{\alpha} y\right)(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)} \int_{x}^{b} (t-x)^{n-\alpha-1} y^{(n)}(t)dt \quad \alpha \in (n-1, n] \tag{4}$$

respectively.

The left and right G-L definitions are given by

$$\left({}^{GL}D_{a+}^{\alpha} y\right)(x) = \lim_{\substack{h \to 0 \\ h>0}} \frac{\sum_{k=0}^{N_a} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} \cdot y(x-kh)}{h^{\alpha}} \tag{5}$$

and

$$\left({}^{GL}D_{b-}^{\alpha} y\right)(x) = \lim_{\substack{h \to 0 \\ h>0}} \frac{\sum_{k=0}^{N_b} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} \cdot y(x+kh)}{h^{\alpha}} \tag{6}$$

respectively.

The Caputo definition is most popular for the fractional differentiation and for the fractional integration.

The relationship between the RL and Caputo definition is given by

$$_{RL}D_{a}^{\alpha} f(x) = {}_{C}D_{a}^{\alpha} f(x) + \sum_{k=0}^{n-1} \frac{f^{k}(a^{+})}{\Gamma(1+k-\alpha)} (x-a)^{k-\alpha} \tag{7}$$

The fractional E-L equations are given in [14] by Equation (8):

$$\frac{\partial F}{\partial y} + {}_{x}D_{b}^{\alpha} \frac{\partial F}{\partial_{a} D_{x}^{\alpha} y} + {}_{a}D_{x}^{\beta} \frac{\partial F}{\partial_{x} D_{b}^{\beta} y} = 0 \tag{8}$$

The fractional Hamilton [15,16] is formed and related work is carried out in [17-22]. In [23] the Caputo definition is used again to relate the fractional Euler-Lagrange to fractional Hamilton equations. The generalized fractional Euler-Lagrange equations are derived in [24] where the fixed boundary conditions are used. Some efforts have been made to solve the fractional E-L equations for the multipoint boundary but no generalization has been made in [25]. In [26] a generalized fractional operator is introduced for formulation of generalized Euler-Lagrange equation. The generalized fractional order optimal control problem is usually stated as follows. The performance function is given by

$$J = \int_0^T (f(x,t), u(t), t)dt \tag{9}$$

subject to system dynamics

$$X^\alpha = f(x,t) + u(t) \tag{10}$$

where α is the fractional order operator. Like integer order optimal control problem formulation, the fractional order optimal problem may be formulated using input constraint and state constraint. The solution of the fractional optimal control problems using various numerical methods will be mentioned in the next section.

3. **Solution of Fractional Order Problem in Integer Order Domain.** The optimal control of integer order system is really a wide topic and there is much literature available. Interested readers can refer [27-30]. Almost all the methods for the solution of integer order optimal control system are utilized for the solution of fractional order optimal control system as well. Methods are available to approximate the fractional order operator and hence fractional order system to integer order system [31-35]. In [35] the authors have considered the variable order fractional Captuto operator and problem is transferred to integer order derivative. So once the problem is transformed to the integer order domain one can easily implement the available optimal control methods.

4. **Solution of Fractional Optimal Control Problems Using Different Methods.**

4.1. **Polynomial approach.** In this approach various polynomials like Lagurre, Legendre, Shifted Legendre, Jacobi, Chebyshev, Bernstein, and Genocchi, are used to approximate the fractional order derivative and fractional order integration. The pioneering work has been done by A. H. Bhrawy. The basic idea in this is to form the operation matrix for the solution of fractional order differentiation and integration [36]. For example, in [37] the Shifted Legendre polynomial based operation matrix is formed to solve the fractional optimal control problem. The Shifted Legendre polynomial based matrix is used again in [38] with numerical integration based method named as Gaussian quadrature method. In [37-39] the single dimensional fractional order system is used. Whereas in [40] FOCP is solved for the multidimensional system, further the quadratic performance index is used in the performance function. Legendre polynomials are also used for solving the Volterra integrals [41]. Further Legendre polynomials are used for the solution of linear PDE [42] and fractional order PDE in [43]. In [44] the Chebyshev spectral method based on the Chebyshev polynomials is used for the fractional order differentiation. Further the second kind of Chebyshev polynomials based operation matrix is used for solution of fractional differential equation in [45]. In [46] Jacobi polynomial is used whereas in [47-51] Lagurre polynomial based operational matrix is used. The same kind of either Shifted Legendre or Chebyshev polynomial operational matrix based approaches are mentioned for various types of fractional optimal control problems in [52-56]. The Bernoulli [57] polynomials are used for the first time for solution of optimal control problem. The Bernstein polynomial based operational matrix is proposed in [58]. The Bernstein polynomial based fractional optimal control is proposed for the multidimensional fractional order system in [59] without delay and with delay in [60]. Also along with Ritz method, the Bernstein operational matrix is used in [61]. The Genocchi polynomials based operation matrix is introduced in [62] and for the FOCP solution in it used in [63]. The solution of fractional order differentiation using various polynomials can be used for direct evaluation of fractional optimal control problems. Further in [64] SVD based rational approximation scheme is proposed to solve the fractional optimal control problem. Here again linear approximation of fractional operator is used to approximate it to integer order domain. The pseudo state space approach is used in [65] also. The hat basis function is used for the solution of the fractional variational problem solution in [66].

The following procedure [40] illustrates the basic procedure for the evaluation of fractional optimal control problem using polynomial approach.

Suppose a performance function is given by

$$\text{Min. } J = \frac{1}{2}\int_0^t \left(\sum_{i=1}^{n}\left[b_i(t)x_i^2(t)\right] + a_0(t)u^2(t)\right) \tag{11}$$

subject to dynamic constraints

$$D^\alpha x_j(t) = \sum_{i=1}^{n}\left[b_{jn+i}(t)x_i(t)\right] + a_j(t)u(t) \tag{12}$$

$j = 1, 2, \ldots, n$, $x_j(0) = x_j$.

Step 1: Approximate fractional derivative using given polynomial (i.e., SLP).

$$D^\alpha x(t) \cong C_j^T \Delta_N(t)$$

where

$$\Delta_N(t) = \begin{pmatrix} P_0^*(t) \\ P_1^*(t) \\ \vdots \\ P_N^*(t) \end{pmatrix} \tag{13}$$

The shifted orthogonal Legendre polynomials [56] $P_k^*(t)$ mentioned in the above matrix are given by equation

$$P_{k+1}^*(t) = \frac{2k+1}{k+1}(2t-1)P_k^*(t) - \frac{k}{k+1}P_{k-1}^*(t) \tag{14}$$

Step 2: Similarly one can approximate coefficients, initial conditions and input $u(t)$ in terms of orthogonal polynomials.

Step 3: Use of operational matrix

$$I^\alpha D^\alpha x_j(t) \cong C_j^T I^{(\alpha)}\Delta_N(t) \tag{15}$$

where $I^{(\alpha)}$ is an operational matrix.

Step 4: Similarly one can convert the dynamic constraint to the system of algebraic linear equations.

Step 5: Using the Lagrange multiplier method, the solution is obtained in terms of coefficients.

The coefficients value found out from the above procedure is used again to find the state and co-state vectors through iterative process.

4.2. **Numerical method based approach.** For the fractional variational problem a method is proposed based upon the indefinite integral in [67]. A hybrid function based numerical method is proposed in [68] for the solution of the fractional order optimal control problem. Mostly the methods are restricted to one dimensional problem solution without state or input constraints. One of the simple numerical methods is given in [69], and author shows the solution of fractional optimal control using proposed numerical method. The Caputo definition is used to formulate the optimal control problem. Similarly the finite element method is used in [70] and in [71] the central difference based numerical scheme is used. In [72] an approximate method is used to solve the fractional optimal control problem using integer order state space formulation. In this method authors have approximated the fractional order term in to pseudo state space form. In [73] a simple numerical approach is used using Caputo definition to solve the fractional order optimal control problems.

There are some methods where fractional order operator is approximated to time domain based series. The way this numerical method is different from other method is that this method is especially designed for particular class of fractional order optimal control problem. In this method [74-77] the following approximation is used and optimal variational principle is used along with the fractional optimal control.

$$ {}_{t_0}^{RL}D_t^{\alpha}x(t) = A(\alpha)(t-t_0)^{-\alpha}x(t) + B(\alpha)(t-t_0)^{1-\alpha}\dot{x}(t) - \sum_{p=2}^{s} C(\alpha,p)(t-t_0)^{1-p-\alpha}x_p(t) \quad (16) $$

where

$$ A(\alpha) = \frac{1}{\Gamma(1-\alpha)}\left(1 + \sum_{p=2}^{s}\left(\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha)(p-1)!}\right)\right) \quad (17) $$

$$ B(\alpha) = \frac{1}{\Gamma(2-\alpha)}\left(1 + \sum_{p=2}^{s}\left(\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)(p)!}\right)\right) \quad (18) $$

$$ \text{and} \qquad C(\alpha) = \frac{\Gamma(p-1+\alpha)}{\Gamma(2-\alpha)\Gamma(\alpha-1)(p-1)!} \quad (19) $$

In the above methods as shown in Equations (17)-(19), time series based formulas are used for the approximation. The s is an integer value which is used to limit the infinite time series to finite series. In [78] the discrete direct method based approach is used to solve the fractional variational problems. The system dynamics used for simulation in the above mentioned are given by (10) and (20). The system dynamics in (20) is proposed for the novelty purpose only.

$$ X^{\alpha} + \dot{X} = f(x,t) + u(t) \quad (20) $$

Since most of the wavelets are made from polynomials like Legendre, Shifted Legendre, and Chebyshev, there is very narrow difference between wavelet and polynomial based approach.

4.3. **Wavelet based approach.** The wavelet method is a proven method for the solution of ODE, PDE and FOPDE and also for the variational problem solution [79]. As for the example in [80] the Legendre wavelet is used for the solution linear PDE, solution of fractional order nonlinear PDE in [81]. Some of the related applications of the Legendre wavelet are mentioned in [82-84]. Chebyshev wavelet is used for the fractional PDE in [85]. The Haar wavelet also finds its application in fractional order system [86-88]. In [88] the Haar wavelet based approach is used to solve the fractional optimal control problem. The Legendre wavelet based approach is addressed in [83]. The Chebyshev wavelet is proposed in [89]. In [90] again Legendre wavelet based method is used for the solution of fractional optimal control problems. The variable order fractional optimal control problem is solved using the Legendre wavelet method in [91].

4.4. **Neural network based approach.** In this approach an indirect method is used for the optimal control of the fractional order system. This method is described in [92]. The Hamiltonian function is formed as given by Equation (21) and necessary conditions for the fractional optimal control are established as per Equation (22)

$$ H(x,u,\lambda,t) = f(x,u,t) + \lambda g(x,u,t) \quad (21) $$

$$
\begin{aligned}
&{}_aD_t^\alpha(x(t)) = \frac{\partial H}{\partial \lambda}, \quad x(a) = x_a \\
&{}_tD_b^\alpha(\lambda(t)) = \frac{\partial H}{\partial x}, \quad \lambda(b) = \lambda_b \\
&\frac{\partial H}{\partial u} = 0
\end{aligned} \tag{22}
$$

After forming the Volterra integral equations, Equation (23) is proposed to approximate the FOCP problem in terms of neural network.

$$
\begin{aligned}
x_N(t, \psi_x) &= A(t) + B(t)N(t, \psi_x) \\
u_N(t, \psi_u) &= C(t) + D(t)N(t, \psi_u) \\
\lambda_N(t, \psi_\lambda) &= F(t) + G(t)N(t, \psi_\lambda)
\end{aligned} \tag{23}
$$

where $A(t)$, $B(t)$, $C(t)$, $D(t)$, $F(t)$ and $G(t)$ are real single variable functions. ψ_x, ψ_u, ψ_λ are weight vectors containing weights of input, output and bias of $x(t)$, $u(t)$ and $\lambda(t)$ respectively. In this method authors have used Matlab 7 and the building in optimization tool box is used. However, in [93] the dynamic optimization scheme is proposed to solve the nonlinear fractional order control system using neural network.

4.5. **Fractional LQR based approach.** The fractional LQR problem for the single state continuous time system is solved analytically in [94]. For the discrete time case it is mentioned in [95]. Recently a practical approach is mentioned in which first the integer order LQR is designed and after that fractional order system LQR is designed [96].

The following steps illustrate the fractional LQR design.

Step 1: Formulate the E-L equation

$$u = -R^{-1}B^T\lambda \ \text{ and } \ {}_0D_{T_f}^\alpha = -A^T\lambda - Qx \tag{24}$$

Step 2: Formulate Ω matrix and find $\mathrm{W}^{-1}\Omega\mathrm{W}$

Here

$$\Omega = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \tag{25}$$

And $\mathrm{W}^{-1}\Omega\mathrm{W} = \begin{bmatrix} -M & 0 \\ 0 & -M \end{bmatrix}$ where M is diagonal matrix with right half eigen values.

Step 3: Find $T(t)$ and $\lambda(t)$ from the following equations

$$T(t) = E_\alpha(-M(T_f - t)^\alpha)W_{21}W_{11}^{-1}E_\alpha^{-1}(M(T_f - t)^\alpha) \tag{26}$$

and

$$
\begin{aligned}
\lambda(t) = {} & [W_{22} - T(t)W_{12}]^{-1}[T(t)W_{11} - W_{21}]x(t) \\
& + [W_{22} - T(t)W_{12}]^{-1}[W_{22}E_\alpha(-M(T_f - t)^\alpha) - T(t)W_{12}E_\alpha(-M(T_f - t)^\alpha) \\
& + E_\alpha(-M(T_f - t)^\alpha) \cdot [W_{22} - W_{11}^{-1}W_{12}]\lambda(T_f)] + T_f^\alpha
\end{aligned} \tag{27}
$$

Step 4: Find the $u(t)$ and $x(t)$ by using Equations (26) and (27).

5. **Conclusion.** In this paper, solution methods for the fractional optimal control and fractional variational problem are addressed. In integer order based classical control system, variational principle leads to optimality condition and henceforth formulation and solution of optimal control problem are done easily using Riccati equations especially for the LQR problem. However, in fractional order optimal control problem formulation and solution, there is no connectivity between FOCP, LQR and Riccati equations and hence solution of FOCP required the use of various numerical method. Further most of the methods for the solution of optimal control problem in fractional domain are based upon the numerical methods. Since most of the numerical methods are already used for the integer order domain, after some modifications, these methods are also used for the

fractional order optimal control problem solution. Apart from that, advances in fractional optimal control problem solution lead to introduction of few new numerical methods. Most of the available methods are used for the single state or single dimension fractional order system. Also most of the methods are applicable to commensurate order fractional order systems. In this paper neural network based fractional optimal control and formulation of fractional LQR problem is also summarized. From this review it can be concluded that very few attempts are made for the implementation of fractional optimal control for the real-time system.

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