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Mixed Caputo Fractional Neutral Stochastic Differential Equations with Impulses and Variable Delay

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Abstract: In this manuscript, a new class of impulsive fractional Caputo neutral stochastic differential equations with variable delay (IFNSDEs, in short) perturbed by fractional Brownain motion (fBm) and Poisson jumps was studied. We utilized the Carathéodory approximation approach and stochastic calculus to present the existence and uniqueness theorem of the stochastic system under Carathéodory-type conditions with Lipschitz and non-Lipschitz conditions as special cases. Some existing results are generalized and enhanced. Finally, an application is offered to illustrate the obtained theoretical results.

Keywords: fractional differential system of neutral type; fractional brownian motion; fractional calculus; Poisson jump; Carathéodory approximation

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1. Introduction

The theory of fractional differential equations is an important component in the fractional calculus and plays a key role in helping researchers to explore the hidden properties of the dynamics of complex systems in viscoelasticity, electromagnetism, diffusion, mechanics, control, signal processing, physics, and many other fields [1–5]. The most important advantage of utilizing systems of fractional order in the applications is their non-local property. Recently, random fluctuations have appeared commonly in various natural and synthetic systems, and fractional stochastic differential equations (FSDEs) with random perturbations have been applied as the mathematical models of many practical systems. This motivates researchers to move from fractional deterministic models to fractional stochastic models to guarantee the model performance. The theory of FSDEs is of interest because of the applications in many fields of engineering and science such as mechanics, control, physics, economics and many other areas [6–8]. A considerable amount of literature has been published on the existence and uniqueness of solutions for FSDEs. One can see [9–17] and the references therein.

On the other hand, in many mathematical models, the claims often display long-range memories, possibly due to extreme weather or natural disasters, and in some cases, many dynamical systems depend not only on present and past states but also contain the derivatives with delays. Neutral functional differential equations are often used to describe such systems [18]. In recent decades, the existence and uniqueness of mild solutions for fractional neutral SDEs have attracted much attention. For instance, Lakhel and Mckibben [19] established the existence of mild solutions for a class of FNSDEs driven by fBm with infinite delay; Dhanalakshmi and Balasubramaniam [20] showed the stability result of higher-order FNSDEs with infinite delay driven by Poisson jumps and the Rosenblatt process; Ramkumar et al. [21] obtained the existence and optimal control for a class of Caputo FNSDEs driven by fBm and Poisson jumps; Alnafisah and Ahmed [22] proved the existence and controllability for neutral delay Hilfer fractional integrodifferential equations driven with fBm by means of the fixed point theorem and semigroup theory.

The theory of impulsive differential equations is developing as an active area of investigation due to the applications in engineering, biology, physics, and many other areas [23–25]. The systems with impulses are utilized for studying the dynamics of processes subject to abrupt changes at discrete moments. Very recently, impulsive FSDEs (IFSDEs) arising in a very natural way as mathematical models are often applied to describe the case where deterministic changes with impulses are interwoven with noisy fluctuations. For more details on the existence and uniqueness for IFSDEs, see [26–30] and the references cited therein. Dhanalaksmi and Balasubramaniam [31] utilized the existence and exponential stability of mild solutions for impulsive fractional neutral SDEs driven by fBm in Hilbert space. Muthukumar and Thiagu [32] derived the existence and approximate controllability of solutions to fractional neutral impulsive SDEs of order $1 < q \leqslant 2$ with infinite delay and Poisson jumps.

Based on the above and to the best of our knowledge, there is no manuscript considering the solvability of an impulsive Caputo fractional neutral stochastic system driven by both fBm and Poisson jumps. In order to fill this gap, we considered the following IFNSDEs with variable delay driven by fBm and Poisson jumps in Hilbert space:

$$\begin{cases} D_t^{\beta}[y(t) - f(t, y(t-\rho(t)))] = A[y(t) - f(t, y(t-\rho(t)))] + \Gamma_t^{1-\beta}\Big[g(s, y(s), y(s-\rho(s)))\frac{dw^H(t)}{dt} \\ \qquad\qquad + \int_Z h(t, y(t), y(t-\rho(t)), \eta))\tilde{N}(dt, d\eta)\Big], t \in [0,b], t \neq t_k, \\ \triangle y(t_k) = y(t_k^+) - y(t_k^-) = I_k(y(t_k)), \quad t = t_k \quad k = 1, 2, \ ..., q, \ q \in \mathbb{N} \\ y(t) = \varrho(t), \quad -\tau \leqslant t \leqslant 0, \end{cases} \tag{1}$$

where D_t^{β} is the Caputo fractional derivative of order β, $0 < \beta < 1$. $\Gamma_t^{1-\beta}(.)$ denotes the $1-\beta$ order fractional integral. Let $A : \mathbb{D}(A) \subset \mathcal{X} \longrightarrow \mathcal{X}$ be an infinitesimal generator of a solution operator $\{S_\beta(t)\}_{t\geqslant 0}$ defined on a Hilbert space $\mathcal{X}$ endowed by the inner product $\langle .,. \rangle$ and norm $\|.\|_{\mathcal{X}}$. w^H is a fBm with Hurst parameter $1/2 < H < 1$ defined on a real separable Hilbert space $\mathcal{Y}$. Let $f : [0,b] \times \mathcal{X} \longrightarrow \mathcal{X}$, $g : [0,b] \times \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{L}_2^0(\mathcal{Y}, \mathcal{X})$ and $h : [0,b] \times \mathcal{X} \times \mathcal{X} \times Z \longrightarrow \mathcal{X}$ be nonlinear functions. Let $\wp = \wp([-\tau, 0]; \mathcal{X})$ be the Banach space of all continuous functions $\varrho : [-\tau, 0] \longrightarrow \mathcal{X}$ endowed by the norm

$$\|\varrho\| = \sup\{\|\varrho(\theta)\| : -\tau \leqslant \theta \leqslant 0\},$$

and the initial data $y(0) = \varrho = \{\varrho(\theta) : -\tau \leqslant \theta \leqslant 0\}$ is an $\mathcal{F}_0$-measurable $\wp([-\tau, 0], \mathcal{X})$-valued random variable such that $\mathbb{E}\|\varrho\|^2 < \infty$. $\tilde{N}(dt, d\eta) = N(dt, d\eta) - \tilde{\lambda}(d\eta)dt$ is the Poisson counting measure. Let $\mathcal{M}_2([-\tau, b], \mathcal{X})$ be the space of all $\mathcal{X}$-valued $\mathcal{F}_t$-adapted processes $\{y(t), -\tau \leqslant t \leqslant b\}$ endowed with the norm $\mathbb{E}\|y\|^2_{\mathcal{M}_2} = \mathbb{E}\|\varphi\|^2 + \mathbb{E}\int_0^b \|y(t)\|^2 dt < \infty$. Here, $I_k : \mathcal{X} \longrightarrow \mathcal{X}, k = 1, 2, \ ..., q$ are bounded functions with the fixed times t_k satisfying $0 = t_0 < t_1 < t_2 < \ ... \ < t_q < b$, and $y(t_k^+)$ and $y(t_k^-)$ represent the right and left limits of $y(t)$ at time t_k, respectively. Further, $\triangle y(t_j) = y(t_k^+) - y(t_k^-)$ determines the jump in the state y at time t_k, where I_k is the jump size.

It is noted that when $\rho(t) = I_k = 0$, our model (1) reduces to the [21] model, which has been studied by means of successive approximation under non-Lipschitz conditions. In contrast, we used the Carathéodory approximation approach to obtain the existence of the unique mild solution for Equation (1) under Carathéodory conditions with the non-Lipschitz condition used in [21] as a special case. Therefore, some results in [21] are generalized and enhanced. We highlight the contributions of this article as follows.

- This article model's IFNSDEs are more general than the [21] model as it takes the variable delays described by the term $\rho(t)$ and possible jumps shown as impulses into consideration.
- It is noted that the proofs of Theorem 3.3 in [21] and Theorem 3 in [33] are proved by means of successive approximation. However, in our case here, Theorem 3.1 is proved by means of the Carathéodory approximation approach, which is more complicated.
- Our Carathéodory conditions are more general than the non-Lipschitz condition used in [21] and contain it as special case. Hence, some results in [21] are generalized and extended.

This article is arranged as follows. In Section 2, we review some preliminary notions and notations about stochastic integral with respect to fBm as well as fractional calculus and state our assumptions. Section 3 is devoted to proving our main theorem of the existence and uniqueness of mild solutions to Equation (1). Then, an application to validate our research is discussed in Section 4. Finally, the conclusion is given in Section 5.

2. Preliminaries

Through the present section, we collect some notions and notations needed to establish our main result. Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is a filtered probability space with $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets. The fBm with $H \in (\frac{1}{2}, 1)$ is a centered process of Gaussian type $w^H = \{w^H(t)\}_{0 \leqslant t \leqslant T}$ with the following variance–covariance function:

$$K_H(u,v) = \mathbb{E}(w^H(u)w^H(v)) = \frac{1}{2}(u^{2H} + v^{2H} - |u-v|^{2H}), \quad u,v \in (-\infty, \infty)$$

and second partial derivative [19]:

$$\frac{\partial K_H}{\partial u \partial v} = (2H-2)H|u-v|^{2H-2}, \quad H > \frac{1}{2}$$

So, we can write:

$$K_H(u,v) = (2H-2)H \int_0^u \int_0^v |u_1 - v_1|^{2H-2} du_1 dv_1.$$

For any real and separable Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$, assume $\mathcal{L}(\mathcal{Y}, \mathcal{X})$ is the space of all bounded linear operators from $\mathcal{Y}$ to $\mathcal{X}$. Let $\mathbb{Q} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ be the operator defined by $\mathbb{Q}e_n = \lambda_n e_n$ with finite trace $tr\mathbb{Q} = \sum_{n=1}^{\infty} \lambda_n < \infty$, for $\lambda_n \geqslant 0$ $(n = 1, 2, \ldots)$, which are non-negative real numbers and $\{e_n\}$, which is a complete orthonormal basis in $\mathcal{Y}$. The infinite dimensional fBm on $\mathcal{Y}$ is defined as

$$w^H(t) = w^H_{\mathbb{Q}}(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}.e_n.w^H_n(t),$$

with real independent fBm's w^H_n. Construct the space $\mathcal{L}^0_2 := \mathcal{L}^0_2(\mathcal{Y}, \mathcal{X})$ of all Hilbert–Schmidt operators $\zeta : \mathcal{Y} \longrightarrow \mathcal{X}$, equipped with the inner product $\langle \varphi, \zeta \rangle_{\mathcal{L}^0_2} = \sum_{n=1}^{\infty} \langle \varphi e_n, \zeta e_n \rangle$ and norm:

$$\|\zeta\|^2_{\mathcal{L}^0_2} := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\zeta e_n\|^2 < \infty.$$

For any $\phi(s) \in \mathcal{L}^0_2(\mathcal{Y}, \mathcal{X})$, $s \in [0, T]$ such that $\sum_{n=1}^{\infty} \|R^* \phi \mathbb{Q}^{\frac{1}{2}} e_n\|^2_{\mathcal{L}^0_2} < \infty$, the Weiner integral of ϕ with respect to w^H is defined by:

$$\int_0^t \phi(s) dw^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \phi(s) e_n dw^H_n(s). \tag{2}$$

Lemma 1 ([20]). *For any $\phi : [0,b] \longrightarrow \mathcal{L}_2^0(\mathcal{Y},\mathcal{X})$ with $\int_0^b \|\phi(s)\|^2_{\mathcal{L}_2^0} ds < \infty$, satisfying Equation (2), what follows is satisfied:*

$$\mathbb{E}\left\|\int_0^t \phi(s)dw^H(s)\right\|^2 \leqslant 2Ht^{2H-1}\int_0^t \|\phi(s)\|^2_{\mathcal{L}_2^0} ds.$$

We refer to [34–38] for more details on the stochastic integral with respect to fBm.

Definition 1 ([3,5]). *The β-order fractional integral of Riemann–Lioville sense for $g : [0,b] \longrightarrow \mathcal{X}$ is expressed by:*

$$J_t^\beta g(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s)ds, \quad \beta > 0.$$

Definition 2 ([39]). *The Caputo β-order derivative with a 0 lower bound for $g : [0,b] \longrightarrow \mathcal{X}$ is expressed as:*

$$D_t^\beta g(t) = \frac{1}{\Gamma(k-\beta)}\int_0^t \frac{g^{(k)}(s)}{(t-s)^{\beta+1-k}} ds = J_t^{k-\beta} g^{(k)}(t), \quad 0 < k-1 < \beta < k,\ t \geqslant 0.$$

For further discussion on the fractional Riemann–Liouville and Caputo derivatives, one can refer to [3,5,39].

Next, a two-parameter Mittag–Leffler function is defined by the series expansion:

$$E_{\beta,\alpha}(w) = \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(\beta k+\alpha)} = \frac{1}{2\pi i}\int_c \frac{\lambda^{\beta-\alpha}e^\lambda}{\lambda^\beta - w} d\lambda, \quad \alpha,\beta > 0,\ w \in \mathbb{C}$$

where $\mathbb{C}$ is a contour that starts and ends with $-\infty$ and encircles the disk $|\lambda| \leqslant |w|^{\frac{1}{2}}$ counter-clockwise.

Following Definition 2.9 of [21], we constructed the definition of the mild solution for Equation (1) as:

Definition 3. *The mild solution $y : [-\tau,b] \longrightarrow \mathcal{X}$ of (1) is a stochastic process satisfying:*

(i) $y(t)$ *is Cadlag and* $\mathcal{F}_t$*-adapted.*
(ii) $\int_0^b \mathbb{E}\|y(s)\|^2 ds < \infty$*, a.s.*
(iii) *The coming integral equation is true.*

$$y(t) = \varrho(t), \quad t \in [-\tau, 0],$$

$$\begin{aligned} y(t) \;=\; & S_\beta(t)[\varrho(0) - f(0,\varrho)] + f(t, y(t-\rho(t))) \\ & + \int_0^t S_\beta(t-s) g(s, y(s), y(s-\rho(s))) dw^H(s) \\ & + \int_0^t \int_Z S_\beta(t-s) h(s, y(s), y(s-\rho(s)), \eta)\tilde{N}(ds, d\eta) \\ & + \sum_{0<t_k<t} S_\beta(t-t_k) I_k(y(t_k)), \quad t \in [0,b], \end{aligned} \tag{3}$$

where $S_\beta(t)$ is the solution operator generated by A and given by:

$$S_\beta(t) = E_{\beta,1}(At^\beta) = \frac{1}{2\pi i}\int_{\hat{B}_r} e^{\mu t}\frac{\mu^{\beta-1}}{\mu^\beta - A} d\mu.$$

(iv) $P\{y(t) = z(t), \forall\, 0 \leqslant t \leqslant b\} = 1$ *if $z(t)$ is another solution to (1).*

The coming assumptions on the coefficients of (1) were prepared for studying the existence and uniqueness of the mild solution.

(A1). The infinitesimal generator $A : \mathbb{D} \subset \mathcal{X} \longrightarrow \mathcal{X}$ of a strong and continuous semigroup of bounded and linear operator $S_\beta(t)$, satisfying $S_\beta(0) = \mathbb{I}$ (the identity operator on $\mathcal{X}$); there exists some constant $M > 0$ obeying:

$$\|S_\beta(t)\| \leqslant M, \quad for\ all\ t \in [0, b],$$

(A2). There exists a function $\mathcal{K}(t, v) : [0, b] \times [0, \infty) \longrightarrow [0, \infty)$, such that:

(a) $\mathcal{K}(t, v)$ is local and integrable in t for all fixed $v \in [0, \infty)$ and is continuous, monotone and nondecreasing in v for all fixed $t \in [0, b]$.

(b) Furthermore, for all fixed $t \in [0, b]$ and $x, y \in \wp$, this inequality is true:

$$\mathbb{E}\int_0^t \|g(t, x, y)\|^2 ds \vee \mathbb{E}\int_0^t \int_Z \|h(t, x, y, \eta)\|^2 \hat{\lambda}(\eta) ds \vee \mathbb{E}\left(\int_0^t \int_Z \|h(t, x, y, \eta)\|^4 \hat{\lambda}(\eta) ds\right)^{\frac{1}{2}} \leqslant \int_0^t \mathcal{K}(s, \mathbb{E}\|x\|^2 + \mathbb{E}\|y\|^2) ds$$

(c) For any positive constant γ, the deterministic equation

$$\frac{d\vartheta}{dt} = \gamma\mathcal{K}(t, \vartheta), \ \ t \in [0, b]$$

has a global solution for some initial value $\vartheta_0 \geqslant 0$.

(A3). There exists a function $\mathcal{M}(t, v) : [0, b] \times [0, \infty) \longrightarrow [0, \infty)$, such that:

(a) $\mathcal{M}(t, v)$ is local and integrable in t for any fixed $v \in [0, \infty)$ and is continuous, monotone, nondecreasing, and concave in v for any fixed $t \geqslant 0$ such that $\mathcal{M}(t, 0) = 0$ and $\int_{0+} \frac{1}{\mathcal{M}(t,v)} dv = +\infty$.

(b) Furthermore, for any fixed $t \in [0, b]$ and $x_1, x_2, y_1, y_2 \in \wp$, this inequality holds:

$$\begin{aligned} &\mathbb{E}\int_0^t \|g(t, x_1, x_2) - g(t, y_1, y_2)\|^2 ds \\ &\vee \mathbb{E}\int_0^t \int_Z \|h(t, x_1, x_2, \eta) - h(t, y_1, y_2, \eta)\|^2 \hat{\lambda}(\eta) ds \\ &\vee \mathbb{E}\left(\int_0^t \int_Z \|h(t, x_1, x_2, \eta) - h(t, y_1, y_2, \eta)\|^4 \hat{\lambda}(\eta) ds\right)^{\frac{1}{2}} \\ &\leqslant \int_0^t \mathcal{M}(s, \mathbb{E}\|x_1 - x_2\|^2 + \mathbb{E}\|y_1 - y_2\|^2) ds \end{aligned}$$

(c) If a non-negative continuous function $Y(t)$, $t \in [0, b]$ satisfies

$$\begin{cases} Y(t) \leqslant \mathcal{N} \int_0^t \mathcal{M}(s, Y(s)) ds, & 0 \leqslant t \leqslant b, \\ Y(0) = 0, \end{cases}$$

we have $Y(0) \equiv 0$ for all positive constant $\mathcal{N}$ and $0 \leqslant t \leqslant b$.

(A4). For some positive constant C_f and $y_1, y_2 \in \wp$,

$$\|f(t, y_1) - f(t, y_2)\|^2 \leqslant C_f \|y_1 - y_2\|^2 \ and \ \ f(t, 0) = 0,$$

for all $t \geqslant 0$.

(A5). There exists a constants $c_k > 0$ such that for every $k = 1, 2, \ ..., q$,

$$\|I_k(y_1) - I_k(y_2)\|^2 \leqslant c_k \|y_1 - y_2\|^2 \ and \ \|I_k(0)\|^2 = 0,$$

for all $y_1, y_2 \in \wp$.

Remark 1. *Let* $\mathcal{M}(t,v) = B(t)\overline{\mathcal{M}}(v)$, $t \in [0,b]$, *where* $B(t) \geqslant 0$ *is locally integrable and* $\overline{\mathcal{M}}(v)$ *is a concave nondecreasing function from* $[0,\infty[$ *to* $[0,\infty[$ *such that* $\overline{\mathcal{M}}(0) = 0$, $\overline{\mathcal{M}}(v) > 0$ *for* $v > 0$ *and* $\int_{0^+} \frac{1}{\overline{\mathcal{M}}(v)} dv = \infty$. *Then, by the comparison theorem of differential equations, we know that condition (A3-c) holds.*

Now, let us give some concrete examples of the function $\overline{\mathcal{M}}(.)$. *Let* $\epsilon > 0$ *and let* $\kappa \in (0,1)$ *be sufficiently small. Define*

$$\overline{\mathcal{M}}_1(v) = \epsilon v, \qquad v \geqslant 0,$$

$$\overline{\mathcal{M}}_2(v) = \begin{cases} v \log(v^{-1}), & 0 \leqslant v \leqslant \kappa, \\ \kappa \log(\kappa^{-1}) + \acute{\overline{\mathcal{M}}}_2(\kappa-)(v-\kappa), & v > \kappa, \end{cases}$$

where $\acute{\overline{\mathcal{M}}}_2$ *denotes the derivative of function* $\overline{\mathcal{M}}_2$. *They are all concave nondecreasing functions satisfying* $\int_{0^+} \frac{1}{\overline{\mathcal{M}}_i(v)} dv = \infty$ $(i = 1,2)$. *In particular, we see that the non-Lipschitz condition in [21] is a special case of our proposed condition.*

3. Existence and Uniqueness

With the help of assumptions (A1)–(A5), we, through this section, develop the existence of the unique mild solution concerning Equation (1). Assume the Carathéodory approximation $y_n(t)$ is defined for all $b > 0$ and any integer $n \geqslant 2/\tau$ as follows:

$$y_n(t) = \varrho(t), \quad -\tau \leqslant t \leqslant 0,$$

$$\begin{aligned} y_n(t) \;=\; & S_\beta(t)[\varrho(0) - f(0,\varrho)] + f(t, y_n(t-\rho(t))) \\ & + \int_0^t 1_{D_n^c}(s) S_\beta(t-s) g\Big(s, y_n(s-\tfrac{1}{n}), y_n(s-\rho(s))\Big) dw^H(s) \\ & + \int_0^t 1_{D_n^c}(s) S_\beta(t-s) h\Big(s, y_n(s-\tfrac{1}{n}), y_n(s-\rho(s)), \eta\Big) \tilde{N}(ds, d\eta) \\ & + \int_0^t 1_{D_n}(s) S_\beta(t-s) g\Big(s, y_n(s-\tfrac{1}{n}), y_n(s-\rho(s)-\tfrac{1}{n})\Big) dw^H(s) \\ & + \int_0^t 1_{D_n}(s) S_\beta(t-s) h\Big(s, y_n(s-\tfrac{1}{n}), y_n(s-\rho(s)-\tfrac{1}{n}), \eta\Big) \tilde{N}(ds, d\eta) \\ & + \sum_{0<t_k<t} S_\beta(t-t_k) I_k\Big(y_n(t_k - \tfrac{1}{n})\Big), \quad t \in [0,b], \end{aligned} \tag{4}$$

with the indicator functions 1_{D_n} and $1_{D_n^c}$ of $D_n = \{t : \rho(t) < \frac{1}{n},\ 0 \leqslant t \leqslant b\}$ and $D_n^c = [0,b] - D_n$, respectively.

Theorem 1. *Assume assumptions (A1)–(A5) are fulfilled. Suppose* ϱ *is independent of the Poisson counting measure* $\tilde{N}(dt, d\eta)$ *and fBm* $w^H(t)$. *Then, provided* $7C_f + 7qM^2 \sum_{k=1}^{q} c_k < 1$, *Equation (1) has a unique mild solution* $y(t)$ *on* $\mathcal{M}_2([-\tau, b], \mathcal{X})$.

Proof. For the convenience of the readers, the proof is divided into the following steps.

Step 1. This sequence $\{y_n(t)\}_{n \geqslant 2/\tau}$ is claimed to be bounded.

By elementary inequality to (4), it is seen

$$\begin{aligned}
&\mathbb{E}(\sup_{0\leqslant s\leqslant t}\|y_n(s)\|^2)\\
&\leqslant 7\mathbb{E}\sup_{0\leqslant s\leqslant t}\|S_\beta(t)[\varrho(0)-f(0,\varrho)]\|^2+7\mathbb{E}\sup_{0\leqslant s\leqslant t}\|f(s,y_n(s-\rho(s)))\|^2\\
&+7\mathbb{E}\sup_{0\leqslant s\leqslant t}\left\|\int_0^s 1_{D_n^c}(r)S_\beta(t-r)g(r,y_n(r-\tfrac{1}{n}),y_n(r-\rho(r)))dw^H(r)\right\|^2\\
&+7\mathbb{E}\sup_{0\leqslant s\leqslant t}\left\|\int_0^s\int_Z 1_{D_n^c}(r)S_\beta(t-r)h(r,y_n(r-\tfrac{1}{n}),y_n(r-\rho(r)),\eta)\tilde{N}(dr,d\eta)\right\|^2\\
&+7\mathbb{E}\sup_{0\leqslant s\leqslant t}\left\|\int_0^s 1_{D_n}(r)S_\beta(t-r)g(r,y_n(r-\tfrac{1}{n}),y_n(r-\rho(r)-\tfrac{1}{n}))dw^H(r)\right\|^2\\
&+7\mathbb{E}\sup_{0\leqslant s\leqslant t}\left\|\int_0^s\int_Z 1_{D_n}(r)S_\beta(t-r)h(r,y_n(r-\tfrac{1}{n}),y_n(r-\rho(r)-\tfrac{1}{n}),\eta)\tilde{N}(dr,d\eta)\right\|^2\\
&+7\mathbb{E}\sup_{0\leqslant s\leqslant t}\left\|\sum_{0<t_k<s}S_\beta(t-t_k)I_k(y_n(t_k-\tfrac{1}{n}))\right\|^2\\
&\leqslant \sum_{j=1}^7 F_j
\end{aligned}\tag{5}$$

Thus, we have by conditions (A1) and (A4)

$$F_1\leqslant 14M^2[\mathbb{E}\|\varrho\|^2+C_f\mathbb{E}\|\varrho\|^2]\leqslant 14M^2(1+C_f)\mathbb{E}\|\varrho\|^2,\tag{6}$$

and

$$F_2\leqslant 7\mathbb{E}\|f(t,y_n(t-\rho(t)))\|^2\leqslant 7C_f\mathbb{E}\sup_{0\leqslant s\leqslant t}\|y_n(s)\|^2.\tag{7}$$

By Lemma 1 , Burkholder's inequality for pure jump stochastic integrals in $\mathcal{X}$ [40] and condition (A2), we obtain

$$\begin{aligned}
F_3+F_4 &\leqslant 14M^2Ht^{2H-1}\mathbb{E}\int_0^t 1_{D_n^c}(s)\|g(s,y_n(s-\tfrac{1}{n}),y_n(s-\rho(s)))\|^2ds\\
&\quad+7M^2C_b\Big[\mathbb{E}\int_0^t\int_Z 1_{D_n^c}(s)\|h(s,y_n(s-\tfrac{1}{n}),y_n(s-\rho(s)),\eta)\|^2\tilde{\lambda}(d\eta)ds\\
&\quad+\mathbb{E}\Big(\int_0^t\int_Z 1_{D_n^c}(s)\|h(s,y_n(s-\tfrac{1}{n}),y_n(s-\rho(s)),\eta)\|^4\tilde{\lambda}(d\eta)ds\Big)^{\frac{1}{2}}\Big]\\
&\leqslant 7M^2(C_b+2Ht^{2H-1})\int_0^t 1_{D_n^c}(s)\mathcal{K}\Big(s,\mathbb{E}\|y_n(s-\tfrac{1}{n})\|^2+\mathbb{E}\|y_n(s-\rho(s))\|^2\Big)ds.
\end{aligned}\tag{8}$$

Similarly,

$$\begin{aligned}
&F_5+F_6\\
&\leqslant 7M^2(C_b+2Ht^{2H-1})\int_0^t 1_{D_n}(s)\mathcal{K}\Big(s,\mathbb{E}\|y_n(s-\tfrac{1}{n})\|^2+\mathbb{E}\|y_n(s-\rho(s)-\tfrac{1}{n})\|^2\Big)ds.
\end{aligned}\tag{9}$$

For F_7, we have by conditions (A1) and (A5)

$$\begin{aligned}
F_7 &\leqslant 7qM^2\sum_{k=1}^q c_k\mathbb{E}\|y_n(t_k-\tfrac{1}{n})\|^2\\
&\leqslant \Big(7qM^2\sum_{k=1}^q c_k\Big)\mathbb{E}\sup_{0\leqslant s\leqslant t}\|y_n(s)\|^2.
\end{aligned}\tag{10}$$

Combining Equations (5)–(10), it concludes

$$\begin{aligned}
&\mathbb{E}(\sup_{0\leqslant s\leqslant t}\|y_n(s)\|^2)\\
&\leqslant \frac{14M^2(1+C_f)\mathbb{E}\|\varrho\|^2}{1-7C_f-7qM^2\sum_{k=1}^q c_k}+\frac{7M^2(C_b+2Ht^{2H-1})}{1-7C_f-7qM^2\sum_{k=1}^q c_k}
\end{aligned}$$

$$\times \int_0^t 1_{D_n^c}(s)\mathcal{K}\left(s, 2\mathbb{E}\|\varrho\|^2 + 2\mathbb{E}\sup_{0\leqslant r\leqslant s}\|y_n(r)\|^2\right)ds$$
$$+\frac{7M^2(C_b+2Ht^{2H-1})}{1-7C_f-7qM^2\sum_{k=1}^q c_k}\int_0^t 1_{D_n}(s)\mathcal{K}\left(s, 2\mathbb{E}\|\varrho\|^2 + 2\mathbb{E}\sup_{0\leqslant r\leqslant s}\|y_n(r)\|^2\right)ds$$
$$\leqslant \frac{14M^2(1+C_f)\mathbb{E}\|\varrho\|^2}{1-7C_f-7qM^2\sum_{k=1}^q c_k} + \frac{7M^2(C_b+2Ht^{2H-1})}{1-7C_f-7qM^2\sum_{k=1}^q c_k}$$
$$\times \int_0^t \mathcal{K}\left(s, 2\mathbb{E}\|\varrho\|^2 + 2\mathbb{E}\sup_{0\leqslant r\leqslant s}\|y_n(r)\|^2\right)ds.$$

Hence, we have

$$2\mathbb{E}\|\varrho\|^2 + 2\mathbb{E}(\sup_{0\leqslant s\leqslant t}\|y_n(s)\|^2)$$
$$\leqslant \frac{2[1+14M^2(1+C_f)-7C_f-7qM^2\sum_{k=1}^q c_k]}{1-7C_f-7qM^2\sum_{k=1}^q c_k}\mathbb{E}\|\varrho\|^2$$
$$+\frac{14M^2(C_b+2Ht^{2H-1})}{1-7C_f-7qM^2\sum_{k=1}^q c_k}\int_0^t \mathcal{K}\left(s, 2\mathbb{E}\|\varrho\|^2 + 2\mathbb{E}\sup_{0\leqslant r\leqslant s}\|y_n(r)\|^2\right)ds.$$

Then, for any solution ϑ_t, condition (A2-c) gives

$$\vartheta_t \leqslant \frac{2[1+14M^2(1+C_f)-7C_f-7qM^2\sum_{k=1}^q c_k]}{1-7C_f-7qM^2\sum_{k=1}^q c_k}\mathbb{E}\|\varrho\|^2 + \frac{14M^2(C_b+2Ht^{2H-1})}{1-7C_f-7qM^2\sum_{k=1}^q c_k}\int_0^t \mathcal{K}(s,\vartheta_s)ds.$$

Since $\mathbb{E}\|\varrho\|^2 \leqslant \infty$, it reads

$$\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}\|y_n(s)\|^2\right) \leqslant \vartheta_t \leqslant \vartheta_T < \infty, \quad n \geqslant \frac{2}{n}.$$

Therefore, $\{y_n(t)\}_{n\geqslant 2/\tau}$ is uniformly bounded, and Step 1 is then fulfilled.

Step 2. For $s,t \in [0,b],\ s<t$ and $n \geqslant 2/\tau$, it reads

$$\mathbb{E}\|y_n(t)-y_n(s)\|^2 \leqslant C_1\|S_\beta(t-s)-\mathbb{I}\|^2 + C_2(t-s) + C_3\sum_{s<t_k<t} c_k,$$

where C_1, C_2, and C_3 are defined through the proof.

Note that:

$$\begin{aligned}&\mathbb{E}\|y_n(t)-y_n(s)\|^2\\ &\leqslant 2\mathbb{E}\|f(t,y_n(t-\rho(t)))-f(s,y_n(s-\rho(s)))\|^2 + 2\mathbb{E}\|J_n(t,s)\|^2\\ &\leqslant 2C_f\mathbb{E}\|y_n(t)-y_n(s)\|^2 + 2\mathbb{E}\|J_n(t,s)\|^2\\ &\leqslant \tfrac{2}{1-2C_f}\mathbb{E}\|J_n(t,s)\|^2,\end{aligned} \tag{11}$$

where:

$$\mathbb{E}\|J_n(t,s)\|^2$$
$$\leqslant 5\mathbb{E}\left\|\int_0^s 1_{D_n^c}(v)[S_\beta(t-v)-S_\beta(s-v)]g(v,y_n(v-\frac{1}{n}),y_n(v-\rho(u)))dw^H(v)\right.$$

$$
\begin{aligned}
&+\int_s^t 1_{D_n^c}(v)S_\beta(t-v)g(v,y_n(v-\frac{1}{n}),y_n(v-\rho(v)))dw^H(v)\Big\|^2\\
&+5\mathbb{E}\Big\|\int_0^s\int_Z 1_{D_n^c}(v)[S_\beta(t-v)-S_\beta(s-v)]h(v,y_n(v-\frac{1}{n}),y_n(v-\rho(v)),\eta)\tilde{N}(dv,d\eta)\\
&+\int_s^t\int_Z 1_{D_n^c}(v)S_\beta(t-v)h(v,y_n(v-\frac{1}{n}),y_n(v-\rho(v)),\eta)\tilde{N}(dv,d\eta)\Big\|^2\\
&+5\mathbb{E}\Big\|\int_0^s 1_{D_n}(v)[S_\beta(t-v)-S_\beta(s-v)]g(v,y_n(v-\frac{1}{n}),y_n(v-\rho(v)-\frac{1}{n}))dw^H(v)\\
&+\int_s^t 1_{D_n}(v)S_\beta(t-v)g(v,y_n(v-\frac{1}{n}),y_n(v-\rho(v)-\frac{1}{n}))dw^H(v)\Big\|^2\\
&+5\mathbb{E}\Big\|\int_0^s\int_Z 1_{D_n}(v)\\
&\times[S_\beta(t-v)-S_\beta(s-v)]h(v,y_n(v-\frac{1}{n}),y_n(v-\rho(v)-\frac{1}{n}),\eta)\tilde{N}(dv,d\eta)\\
&+\int_s^t\int_Z 1_{D_n}(v)S_\beta(t-v)h(v,y_n(v-\frac{1}{n}),y_n(v-\rho(v)-\frac{1}{n}),\eta)\tilde{N}(dv,d\eta)\Big\|^2\\
&+5\mathbb{E}\Big\|\sum_{0<t_k<s}[S_\beta(t-t_k)-S_\beta(s-t_k)]I_k(y_n(t_k-\frac{1}{n}))\\
&+\sum_{s<t_k<t}S_\beta(t-t_k)I_k(y_n(t_k-\frac{1}{n}))\Big\|^2:=\sum_{i=1}^5 J_i. \qquad (12)
\end{aligned}
$$

Now for J_1, we have by Lemma 1 and conditions (A1) and (A2):

$$
\begin{aligned}
J_1 \leqslant\ & 10\mathbb{E}\Big\|\int_0^s 1_{D_n^c}(v)\big[S_\beta(t-v)-S_\beta(s-v)\big]g(v,y_n(v-\frac{1}{n}),y_n(v-\rho(v)))dw^H(v)\Big\|^2\\
&+10\mathbb{E}\Big\|\int_s^t 1_{D_n^c}(v)S_\beta(t-v)g(v,y_n(v-\frac{1}{n}),y_n(v-\rho(v)))dw^H(v)\Big\|^2\\
\leqslant\ & 20Hs^{2H-1}\|S_\beta(t-v)-S_\beta(s-v)\|^2\int_0^s 1_{D_n^c}(v)\\
&\times\mathcal{K}(v,\mathbb{E}\|y_n(v-\frac{1}{n})\|^2+\mathbb{E}\|y_n(v-\rho(v))\|^2)du\\
&+20H(t-s)^{2H-1}\|S_\beta(t-v)\|^2\int_s^t 1_{D_n^c}(v)\\
&\times\mathcal{K}(v,\mathbb{E}\|y_n(v-\frac{1}{n})\|^2+\mathbb{E}\|y_n(v-\rho(v))\|^2)dv\\
\leqslant\ & 20Hs^{2H-1}M^2\|S_\beta(t-s)-\mathbb{I}\|^2\int_0^s 1_{D_n^c}(v)\mathcal{K}(v,2\mathbb{E}\sup_{0\leqslant r\leqslant v}\|y_n(r)\|^2)dv\\
&+20H(t-s)^{2H-1}M^2\int_s^t 1_{D_n^c}(v)\mathcal{K}(v,2\mathbb{E}\sup_{0\leqslant r\leqslant v}\|y_n(r)\|^2)dv. \qquad (13)
\end{aligned}
$$

For J_2, we have by Burkholder's inequality and conditions (A1) and (A2):

$$
\begin{aligned}
J_2 \leqslant\ & 10\mathbb{E}\Big\|\int_0^s\int_Z 1_{D_n^c}(v)\\
&\times\big[S_\beta(t-v)-S_\beta(s-v)\big]h(v,y_n(v-\frac{1}{n}),y_n(v-\rho(v)),\eta)\tilde{N}(dv,d\eta)\Big\|^2\\
&+10\mathbb{E}\Big\|\int_s^t\int_Z 1_{D_n^c}(v)S_\beta(t-v)h(v,y_n(v-\frac{1}{n}),y_n(v-\rho(v)),\eta)\tilde{N}(dv,d\eta)\Big\|^2
\end{aligned}
$$

$$\begin{aligned}
&\leqslant 10C_bM^2\|S_\beta(t-s)-\mathbb{I}\|^2\mathbb{E}\Big[\int_0^s\int_Z 1_{D_n^c}(v)\|h(v,y_n(v-\frac{1}{n}),y_n(v-\rho(v)),\eta)\|^2\tilde{\lambda}(d\eta)dv \\
&+\Big(\int_0^s\int_Z 1_{D_n^c}(v)\|h(v,y_n(v-\frac{1}{n}),y_n(v-\rho(v)),\eta)\|^4\tilde{\lambda}(d\eta)dv\Big)^{\frac{1}{2}}\Big] \\
&+10C_bM^2\mathbb{E}\Big[\int_s^t\int_Z 1_{D_n^c}(v)\|h(v,y_n(v-\frac{1}{n}),y_n(v-\rho(v)),\eta)\|^2\tilde{\lambda}(d\eta)dv \\
&+\Big(\int_s^t\int_Z 1_{D_n^c}(v)\|h(v,y_n(v-\frac{1}{n}),y_n(v-\rho(v)),\eta)\|^4\tilde{\lambda}(d\eta)dv\Big)^{\frac{1}{2}}\Big] \\
&\leqslant 10C_bM^2\|S_\beta(t-s)-\mathbb{I}\|^2\int_0^s 1_{D_n^c}(v)\mathcal{K}(v,2\mathbb{E}\sup_{0\leqslant r\leqslant v}\|y_n(r)\|^2)dv \\
&+10C_bM^2\int_s^t 1_{D_n^c}(v)\mathcal{K}(v,2\mathbb{E}\sup_{0\leqslant r\leqslant v}\|y_n(r)\|^2)dv. \qquad (14)
\end{aligned}$$

Similarly, for J_3 and J_4, we have:

$$\begin{aligned}
J_3+J_4 &\leqslant 10M^2\|S_\beta(t-s)-\mathbb{I}\|^2[C_b+2Hs^{2H-1}]\int_0^s 1_{D_n}(v)\mathcal{K}(v,2\mathbb{E}\sup_{0\leqslant r\leqslant v}\|y_n(r)\|^2)dv \\
&+10M^2[C_b+2H(t-s)^{2H-1}]\int_s^t 1_{D_n}(v)\mathcal{K}(v,2\mathbb{E}\sup_{0\leqslant r\leqslant v}\|y_n(r)\|^2)dv. \qquad (15)
\end{aligned}$$

For J_5, we have by conditions (A1) and (A5) and Hölder's inequality:

$$\begin{aligned}
J_5 &\leqslant 10\mathbb{E}\Big\|\sum_{0<t_k<s}(S_\beta(t-t_k)-S_\beta(s-t_k))I_k(y_n(t_k-\frac{1}{n}))\Big\|^2 \\
&+10\mathbb{E}\Big\|\sum_{s<t_k<t}S_\beta(t-t_k)I_k(y_n(t_k-\frac{1}{n}))\Big\|^2 \\
&\leqslant 10qM^2\|S_\beta(t-s)-\mathbb{I}\|^2\sum_{k=1}^q c_k\mathbb{E}\|y_n(t_k-\frac{1}{n})\|^2 \\
&+10qM^2\sum_{s<t_k<t}c_k\mathbb{E}\|y_n(t_k-\frac{1}{n})\|^2 \\
&\leqslant 10qM^2\|S_\beta(t-s)-\mathbb{I}\|^2\sum_{k=1}^q c_k\mathbb{E}\sup_{0\leqslant s\leqslant t}\|y_n(s)\|^2 \\
&+10qM^2\sum_{s<t_k<t}c_k\mathbb{E}\sup_{0\leqslant s\leqslant t}\|y_n(s)\|^2. \qquad (16)
\end{aligned}$$

Combining Equations (12)–(16) and using Step 1, we have:

$$\begin{aligned}
\mathbb{E}\|J_n(t,s)\|^2 &\leqslant 10M^2\|S_\beta(t-s)-\mathbb{I}\|^2[C_b+2Hs^{2H-1}]\int_0^s 1_{D_n^c}(v)\mathcal{K}(v,2\mathbb{E}\sup_{0\leqslant r\leqslant v}\|y_n(r)\|^2)dv \\
&+10M^2[C_b+2H(t-s)^{2H-1}]\int_s^t 1_{D_n^c}(v)\mathcal{K}(v,2\mathbb{E}\sup_{0\leqslant r\leqslant v}\|y_n(r)\|^2)dv \\
&+10M^2\|S_\beta(t-s)-\mathbb{I}\|^2[C_b+2Hs^{2H-1}]\int_0^s 1_{D_n}(v)\mathcal{K}(v,2\mathbb{E}\sup_{0\leqslant r\leqslant v}\|y_n(r)\|^2)dv \\
&+10M^2[C_b+2H(t-s)^{2H-1}]\int_s^t 1_{D_n}(v)\mathcal{K}(v,2\mathbb{E}\sup_{0\leqslant r\leqslant v}\|y_n(r)\|^2)dv \\
&+10qM^2\|S_\beta(t-s)-\mathbb{I}\|^2\sum_{k=1}^q c_k\mathbb{E}\sup_{0\leqslant s\leqslant t}\|y_n(s)\|^2
\end{aligned}$$

$$
\begin{aligned}
&+10qM^2\sum_{s<t_k<t}c_k\mathbb{E}\sup_{0\leqslant s\leqslant t}\|y_n(s)\|^2\\
\leqslant\ &10M^2\|S_\beta(t-s)-\mathbb{I}\|^2[C_b+2Hs^{2H-1}]\int_0^s\mathcal{K}(v,2\mathbb{E}\sup_{0\leqslant r\leqslant v}\|y_n(r)\|^2)dv\\
&+10M^2[C_b+2H(t-s)^{2H-1}]\int_s^t\mathcal{K}(v,2\mathbb{E}\sup_{0\leqslant r\leqslant v}\|y_n(r)\|^2)dv\\
&+10qM^2\|S_\beta(t-s)-\mathbb{I}\|^2\sum_{k=1}^{q}c_k\mathbb{E}\sup_{0\leqslant s\leqslant t}\|y_n(s)\|^2\\
&+10qM^2\sum_{s<t_k<t}c_k\mathbb{E}\sup_{0\leqslant s\leqslant t}\|y_n(s)\|^2\\
\leqslant\ &10M^2\left[s(C_b+2Hs^{2H-1})\sup_{0\leqslant v\leqslant s}\mathcal{K}(v,2C)+qC\sum_{k=1}^{q}c_k\right]\|S_\beta(t-s)-\mathbb{I}\|^2\\
&+10M^2[C_b+2H(t-s)^{2H-1}]\sup_{0\leqslant v\leqslant t}\mathcal{K}(v,2C)(t-s)+10qCM^2\sum_{s<t_k<t}c_k,
\end{aligned}
\tag{17}
$$

where the constant C comes from Step 1. Inserting Equation (17) in Equation (11), the required result is obtained with constants:

$$
C_1=\frac{20M^2\left[s(C_b+2Hs^{2H-1})\sup_{0\leqslant v\leqslant s}\mathcal{K}(v,2C)+qC\sum_{k=1}^{q}c_k\right]}{1-2M_f},
$$

$$
C_2=\frac{20M^2[C_b+2H(t-s)^{2H-1}]\sup_{0\leqslant v\leqslant t}\mathcal{K}(v,2C)}{1-2M_f},\quad and\ \ C_3=\frac{20qCM^2}{1-2C_f}.
$$

Step 3. It is proved that $\{y_n(t)\}_{n\geqslant 2/\tau}$ is Cauchy in $\mathcal{M}_2([-\tau,b],\mathbb{X})$. Using Equation (4), for $m>n\geqslant 2/\tau$, it is easy to obtain:

$$
\mathbb{E}(\sup_{0\leqslant s\leqslant t}\|y_m(s)-y_n(s)\|^2)\leqslant\frac{2}{1-2C_f}\mathbb{E}(\sup_{0\leqslant s\leqslant t}\|\Upsilon_n(s)\|^2),
\tag{18}
$$

where:

$$
\begin{aligned}
\Upsilon_n(t)\ =\ &\int_0^t\Big\{1_{D_m^c}(s)S_\beta(t-s)g(s,y_m(s-\frac{1}{m}),y_m(s-\rho(s)))\\
&-1_{D_n^c}(s)S_\beta(t-s)g(s,y_n(s-\frac{1}{n}),y_n(s-\rho(s)))\Big\}dw^H(s)\\
&+\int_0^t\int_Z\Big\{1_{D_m^c}(s)S_\beta(t-s)h(s,y_m(s-\frac{1}{m}),y_m(s-\rho(s)),\eta)\\
&-1_{D_n^c}(s)S_\beta(t-s)h(s,y_n(s-\frac{1}{n}),y_n(s-\rho(s)),\eta)\Big\}\tilde{N}(ds,d\eta)\\
&+\int_0^t\Big\{1_{D_m}(s)S_\beta(t-s)g(s,y_m(s-\frac{1}{m}),y_m(s-\rho(s)-\frac{1}{m}))\\
&-1_{D_n}(s)S_\beta(t-s)g(s,y_n(s-\frac{1}{n}),y_n(s-\rho(s)-\frac{1}{n}))\Big\}dw^H(s)\\
&+\int_0^t\int_Z\Big\{1_{D_m}(s)S_\beta(t-s)h(s,y_m(s-\frac{1}{m}),y_m(s-\rho(s)-\frac{1}{m}),\eta)\\
&-1_{D_n}(s)S_\beta(t-s)h(s,y_n(s-\frac{1}{n}),y_n(s-\rho(s)-\frac{1}{n}),\eta)\Big\}\tilde{N}(ds,d\eta)
\end{aligned}
$$

$$+\sum_{0<t_k<t} S_\beta(t-s)\big[I_k(y_m(t_k-\frac{1}{m}))-I_k(y_n(t_k-\frac{1}{n}))\big]=\sum_{i=1}^{5}\Theta_i. \tag{19}$$

The integrals above are computed by the technique of plus and minus:

$$\begin{aligned}
\Theta_1 \;=\;& \int_0^t 1_{D_m^c}(s)S_\beta(t-s)g(s,y_m(s-\frac{1}{m}),y_m(s-\rho(s)))dw^H(s)\\
&-\int_0^t 1_{D_n^c}(s)S_\beta(t-s)g(s,y_n(s-\frac{1}{n}),y_n(s-\rho(s)))dw^H(s)\\
&+\int_0^t 1_{D_n^c}(s)S_\beta(t-s)g(s,y_m(s-\frac{1}{m}),y_m(s-\rho(s)))dw^H(s)\\
&-\int_0^t 1_{D_n^c}(s)S_\beta(t-s)g(s,y_m(s-\frac{1}{m}),y_m(s-\rho(s)))dw^H(s)\\
&+\int_0^t 1_{D_n^c}(s)S_\beta(t-s)g(s,y_n(s-\frac{1}{m}),y_n(s-\rho(s)))dw^H(s)\\
&-\int_0^t 1_{D_n^c}(s)S_\beta(t-s)g(s,y_n(s-\frac{1}{m}),y_n(s-\rho(s)))dw^H(s)\\
=\;& \int_0^t 1_{D_n^c}(s)S_\beta(t-s)\Big\{g(s,y_m(s-\frac{1}{m}),y_m(s-\rho(s)))\\
&-g(s,y_n(s-\frac{1}{m}),y_n(s-\rho(s)))\Big\}dw^H(s)\\
&+\int_0^t 1_{D_n^c}(s)S_\beta(t-s)\Big\{g(s,y_n(s-\frac{1}{m}),y_n(s-\rho(s)))\\
&-g(s,y_n(s-\frac{1}{n}),y_n(s-\rho(s)))\Big\}dw^H(s)\\
&+\int_0^t 1_{D_m^c-D_n^c}(s)S_\beta(t-s)g(s,y_m(s-\frac{1}{m}),y_m(s-\rho(s)))dw^H(s).
\end{aligned}$$

Taking expectation, elementary inequality, Lemma 1, conditions (A1)–(A3), and Step 1, we have:

$$\begin{aligned}
\mathbb{E}\sup_{0\leqslant s\leqslant t}\|\Theta_1\|^2 \;\leqslant\;& 6Ht^{2H-1}M^2\int_0^t 1_{D_n^c}(s)\mathcal{M}\Big(s,\mathbb{E}\|y_m(s-\frac{1}{m})-y_n(s-\frac{1}{m})\|^2\\
&+\mathbb{E}\|y_m(s-\rho(s))-y_n(s-\rho(s))\|^2\Big)ds\\
&+6Ht^{2H-1}M^2\int_0^t 1_{D_n^c}(s)\mathcal{M}\Big(s,\mathbb{E}\|y_n(s-\frac{1}{m})-y_n(s-\frac{1}{n})\|^2\Big)ds\\
&+6Ht^{2H-1}M^2\int_0^t 1_{D_m^c-D_n^c}(s)\mathcal{K}\Big(s,\mathbb{E}\|y_m(s-\frac{1}{m})\|^2\\
&+\mathbb{E}\|y_m(s-\rho(s))\|^2\Big)ds\\
\leqslant\;& 6Ht^{2H-1}M^2\int_0^t 1_{D_n^c}(s)\mathcal{M}\Big(s,2\mathbb{E}\sup_{0\leqslant v\leqslant s}\|y_m(v)-y_n(v)\|^2\Big)ds\\
&+6Ht^{2H-1}M^2\int_0^b 1_{D_n^c}(s)\mathcal{M}\Big(s,\mathbb{E}\|y_n(s-\frac{1}{m})-y_n(s-\frac{1}{n})\|^2\Big)ds\\
&+6Ht^{2H}M^2\Big(\sup_{0\leqslant s\leqslant t}\mathcal{K}(s,2C)\Big)\mu(D_m^c-D_n^c).
\end{aligned} \tag{20}$$

Similarly, we have for Θ_2 by Burkholder's inequality, conditions (A1)–(A3) and Step 1:

$$
\begin{aligned}
\mathbb{E}\sup_{0\leqslant s\leqslant t}\|\Theta_2\|^2 \leqslant\ & 3\mathbb{E}\sup_{0<s<t}\Big\|\int_0^s\int_Z 1_{D_n^c}(v)S_\beta(s-v)\Big\{h(v,y_m(v-\frac{1}{m}),y_m(v-\rho(v)),\eta)\\
&-h(v,y_n(v-\frac{1}{m}),y_n(v-\rho(v)),\eta)\Big\}\tilde{N}(dv,d\eta)\Big\|^2\\
&+3\mathbb{E}\sup_{0<s<t}\Big\|\int_0^s\int_Z 1_{D_n^c}(v)S_\beta(s-v)\Big\{h(v,y_n(v-\frac{1}{m}),y_n(v-\rho(v)),\eta)\\
&-h(v,y_n(v-\frac{1}{n}),y_n(v-\rho(v)),\eta)\Big\}\tilde{N}(dv,d\eta)\Big\|^2\\
&+3\mathbb{E}\sup_{0<s<t}\Big\|\int_0^s\int_Z 1_{D_m^c-D_n^c}(v)S_\beta(s-v)\\
&\times h(v,y_m(v-\frac{1}{m}),y_m(v-\rho(v)),\eta)\tilde{N}(dv,d\eta)\Big\|^2\\
\leqslant\ & 3C_bM^2\int_0^t 1_{D_n^c}(s)\mathcal{M}\Big(s,2\mathbb{E}\sup_{0\leqslant v\leqslant s}\|y_m(v)-y_n(v)\|^2\Big)ds\\
&+3C_bM^2\int_0^b 1_{D_n^c}(s)\mathcal{M}\Big(s,\mathbb{E}\|y_n(s-\frac{1}{m})-y_n(s-\frac{1}{n})\|^2\Big)ds\\
&+3C_bM^2\Big(\sup_{0\leqslant s\leqslant b}\mathcal{K}(s,2C)\Big)\mu(D_m^c-D_n^c).
\end{aligned}
\tag{21}
$$

By similar analysis for Θ_3 and Θ_4, we obtain:

$$
\begin{aligned}
&\mathbb{E}\sup_{0\leqslant s\leqslant t}(\|\Theta_3\|^2+\|\Theta_4\|^2)\\
&\leqslant 3M^2(C_b+2Ht^{2H-1})\int_0^t 1_{D_m}(s)\mathcal{M}\Big(s,2\mathbb{E}\sup_{0\leqslant v\leqslant s}\|y_m(v)-y_n(v)\|^2\Big)ds\\
&\quad+3M^2(C_b+2Ht^{2H-1})\int_0^b 1_{D_m}(s)\mathcal{M}\Big(s,\mathbb{E}\|y_n(s-\frac{1}{m})-y_n(s-\frac{1}{n})\|^2\\
&\quad+\mathbb{E}\|y_n(s-\rho(s)-\frac{1}{m})-y_n(s-\rho(s)-\frac{1}{n})\|^2\Big)ds\\
&\quad+3M^2(C_b+2Ht^{2H})\Big(\sup_{0\leqslant s\leqslant b}\mathcal{K}(s,2C)\Big)\mu(D_n-D_m).
\end{aligned}
\tag{22}
$$

For Θ_5, we have by the technique of plus and minus, conditions (A1) and (A5):

$$
\begin{aligned}
\mathbb{E}\sup_{0\leqslant s\leqslant t}\|\Theta_5\|^2 \leqslant\ & 2qM^2\sum_{k=1}^q c_k\mathbb{E}\|y_m(t_k-\frac{1}{m})-y_n(t_k-\frac{1}{m})\|^2\\
&+2qM^2\sum_{k=1}^q c_k\mathbb{E}\|y_n(t_k-\frac{1}{m})-y_n(t_k-\frac{1}{n})\|^2\\
\leqslant\ & 2qM^2\sum_{k=1}^q c_k\mathbb{E}\sup_{0\leqslant s\leqslant t}\|y_m(s)-y_n(s)\|^2\\
&+2qM^2\sum_{k=1}^q c_k\mathbb{E}\|y_n(t_k-\frac{1}{m})-y_n(t_k-\frac{1}{n})\|^2.
\end{aligned}
\tag{23}
$$

Collecting (18)–(23), we find:

$$
\mathbb{E}(\sup_{0\leqslant s\leqslant t}|y_m(s)-y_n(s)|^2) \leqslant C_4\int_0^t\mathcal{M}\Big(s,2\mathbb{E}\sup_{0\leqslant v\leqslant s}\|y_m(v)-y_n(v)\|^2\Big)ds
$$

$$
\begin{aligned}
&+C_4\int_0^b 1_{D_n^c}(s)\mathcal{M}\Big(s,\mathbb{E}\|y_n(s-\frac{1}{m})-y_n(s-\frac{1}{n})\|^2\Big)ds\\
&+C_4\int_0^b 1_{D_m}(s)\mathcal{M}\Big(s,\mathbb{E}\|y_n(s-\frac{1}{m})-y_n(s-\frac{1}{n})\|^2\\
&+\mathbb{E}\|y_n(s-\rho(s)-\frac{1}{m})-y_n(s-\rho(s)-\frac{1}{n})\|^2\Big)ds\\
&+C_6\mathbb{E}\|y_n(t_k-\frac{1}{m})-y_n(t_k-\frac{1}{n})\|^2\\
&+C_5\Big(\sup_{0\leqslant s\leqslant b}\mathcal{K}(s,2C)\Big)\mu(D_n-D_m), \qquad (24)
\end{aligned}
$$

where $C_4=\frac{30M^2(C_b+2Ht^{2H-1})}{1-2C_f-24qM^2\sum_{k=1}^{q}c_k}$, $C_5=\frac{60M^2(C_b+2Ht^{2H})}{1-2C_f-24qM^2\sum_{k=1}^{q}c_k}$ and $C_6=\frac{24qM^2\sum_{k=1}^{q}c_k}{1-2C_f-24qM^2\sum_{k=1}^{q}c_k}$.

In the lines that follow, we can estimate:

$$
\begin{aligned}
&\int_0^b 1_{D_n^c}(s)\mathcal{M}\Big(s,\mathbb{E}\|y_n(s-\frac{1}{m})-y_n(s-\frac{1}{n})\|^2\Big)ds\\
&=\int_0^{\frac{1}{n}} 1_{D_n^c}(s)\mathcal{M}\Big(s,\mathbb{E}\|y_n(s-\frac{1}{m})-y_n(s-\frac{1}{n})\|^2\Big)ds\\
&\quad+\int_{\frac{1}{n}}^b 1_{D_n^c}(s)\mathcal{M}\Big(s,\mathbb{E}\|y_n(s-\frac{1}{m})-y_n(s-\frac{1}{n})\|^2\Big)ds\\
&\leqslant\int_0^{\frac{1}{n}} 1_{D_n^c}(s)\mathcal{M}\Big(s,2\mathbb{E}\|y_n(s-\frac{1}{m})\|^2+2\mathbb{E}\|y_n(s-\frac{1}{n})\|^2\Big)ds\\
&\quad+b\mathcal{M}\Bigg(s,C_1\|S_\beta(\frac{1}{n}-\frac{1}{m})-\mathbb{I}\|^2+C_2(\frac{1}{n}-\frac{1}{m})+C_3\sum_{s-\frac{1}{n}<t_k<s-\frac{1}{m}}c_k\Bigg)\\
&\leqslant\mathcal{M}\Big(s,4\mathbb{E}\|\varrho\|^2+4C\Big)\frac{1}{n}\\
&\quad+b\mathcal{M}\Bigg(s,C_1\|S_\beta(\frac{1}{n}-\frac{1}{m})-\mathbb{I}\|^2+C_2(\frac{1}{n}-\frac{1}{m})+C_3\sum_{s-\frac{1}{n}<t_k<s-\frac{1}{m}}c_k\Bigg), \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^b 1_{D_m}(s)\mathcal{M}\Big(s,\mathbb{E}\|y_n(s-\frac{1}{m})-y_n(s-\frac{1}{n})\|^2\\
&\quad+\mathbb{E}\|y_n(s-\rho(s)-\frac{1}{m})-y_n(s-\rho(s)-\frac{1}{n})\|^2\Big)ds\\
&\leqslant\mathcal{M}\Big(s,8\mathbb{E}\|\varrho\|^2+8C\Big)(\frac{1}{n}+\frac{1}{m})\\
&\quad+b\mathcal{M}\Bigg(s,2C_1\|S_\beta(\frac{1}{n}-\frac{1}{m})-\mathbb{I}\|^2+2C_2(\frac{1}{n}-\frac{1}{m})+2C_3\sum_{s-\frac{1}{n}<t_k<s-\frac{1}{m}}c_k\Bigg), \qquad (26)
\end{aligned}
$$

and:

$$
\begin{aligned}
&\mathbb{E}\|y_n(t_k-\frac{1}{m})-y_n(t_k-\frac{1}{n})\|^2\\
&\leqslant C_1\|S_\beta(\frac{1}{n}-\frac{1}{m})-\mathbb{I}\|^2+C_2(\frac{1}{n}-\frac{1}{m})+C_3\sum_{s-\frac{1}{n}<t_k<s-\frac{1}{m}}c_k. \qquad (27)
\end{aligned}
$$

Let

$$Y(t) = \limsup_{m,n\longrightarrow\infty} 2\mathbb{E}(\sup_{0\leqslant s\leqslant t} \|y_m(s) - y_n(s)\|^2). \tag{28}$$

Since $\mu(D_n - D_m) \longrightarrow 0$ and $\mathcal{M}(s,.) = 0$ as $n, m \longrightarrow \infty$, and employing (25)–(27), Equations (24) and (28) beside Fatou's lemma yield:

$$Y(t) \leqslant 2C_4 \int_0^t \mathcal{M}(s, Y(s))ds. \tag{29}$$

Lastly, through Equation (29) and condition (A3), the following is obtained:

$$Y(t) = \limsup_{m,n\longrightarrow\infty} 2\mathbb{E}(\sup_{0\leqslant s\leqslant t} \|y_m(s) - y_n(s)\|^2) = 0,$$

which yields:

$$\limsup_{m,n\longrightarrow\infty} \mathbb{E}(\sup_{0\leqslant s\leqslant t} \|y_m(s) - y_n(s)\|^2) = 0,$$

which shows that $\{y_n(t)\}_{n\geqslant 2/\tau}$ is Cauchy sequence on $\mathcal{M}_2([-\tau, b]; \mathcal{X})$. The Borel–Cantelli lemma shows that, as $n \longrightarrow \infty$, $y_n(t) \longrightarrow y(t)$ holds uniformly for every $t \in [0, b]$. Consequently, by the limit on both sides of (4), $y(t)$ defines the mild solution of (1) with property:

$$\mathbb{E}(\sup_{0\leqslant s\leqslant t} \|y(s)\|^2) = 0, \quad 0 \leqslant t \leqslant b.$$

Lastly, the proof of existence is complete. The uniqueness proof is presented in the Appendix A. Hence, the Theorem 1 proof is completed. □

If $g(t,.,.) \equiv g(t, y_t)$, $h(t,.,.,\eta) \equiv h(t, y_t, \eta)$ and $f(t,.) \equiv f(t, y_t)$, Equation (1) reduces to the following equation:

$$\begin{cases} D_t^{\beta}[y(t) - f(t, y_t)] = A[y(t) - f(t, y_t)] + \Gamma_t^{1-\beta}\Big[f(s, y_s)\frac{dw^H(t)}{dt} \\ \qquad\qquad + \int_Z f(t, y_t, \eta)\tilde{N}(dt, d\eta)\Big], t \in [0, b], t \neq t_k, \\ \triangle y(t_k) = y(t_k^+) - y(t_k^-) = I_k(y(t_k)), \quad t = t_k \;\; k = 1, 2, ..., q, \; q \in \mathbb{N} \\ y(t) = \varrho(t), \quad -\tau \leqslant t \leqslant 0, \end{cases} \tag{30}$$

Corollary 1. *Assume assumptions (A1)–(A5) are fulfilled. Suppose ϱ is independent of the Poisson counting measure $\tilde{N}(dt, d\eta)$ and fBm $w^H(t)$. Then, provided $7C_f + 7qM^2\sum_{k=1}^{q} c_k < 1$, Equation (30) has a unique mild solution $y(t)$ on $\mathcal{M}_2([-\tau, b], \mathcal{X})$.*

Remark 2. *If $I_k(.) \equiv 0 (k = 1, 2, ..., q)$ in Equation (30), Corollary 1 consists with Theorem 3.3 in Ramkumar et al. [21], where the authors applied the successive approximation to utilize existence and uniqueness problem under non-Lipschitz conditions. However, here, the results were utilized by using Carathéodory approximation under Carathéodory conditions with reference to [21] non-Lipschitz conditions as special case. So, Corollary 1 generalizes some of the results in [21].*

4. Application

Consider the following IFNSPDEs driven by fBm and Poisson jumps:

$$\begin{cases} D_t^{\beta}\Big[z(t,y) - \int_{-\tau}^{0}\alpha_1(r)\sin z(t+r,y)dr\Big] = \frac{\partial^2}{\partial y^2}\Big[z(t,y) - \int_{-\tau}^{0}\alpha_1(r)\sin z(t+r,y)dr\Big] \\ \qquad +\frac{1}{\Gamma(1-\beta)})\int_0^t (t-r)^{-\beta}\Big[\int_0^t\int_{-\tau}^{t} e^{4(\tau-t)}z(r,y)drdw^H(r) \\ \qquad +\int_Z \eta\Big(\int_{-\tau}^{t}\alpha_2(r-t)z(r,y)dr\Big)\tilde{N}(dt,d\eta)\Big], \\ \triangle z(t_k,y) := z(t_k^+,y) - z(t_k^-,y) = \frac{\alpha_3}{2^k}z(t_k,y), \quad k\in\mathbb{N} \\ y\in\mathcal{D} = (0,\pi], \quad t\in[0,b] \\ z(t,0) = z(t,\pi) = 0, \quad t\in[0,T], \\ z(t,y) = \varphi(t,y), \quad -\tau\leqslant t\leqslant 0, \quad y\in\mathbb{D}, \end{cases} \tag{31}$$

with D_t^{β}, which is a fractional Caputo derivative of order $0<\beta<1$, and w^H, $\frac{1}{2}<H<1$ is fBm. Let $\mathcal{X} = \mathcal{L}_2([0,\pi])$, and define the operator $A:\mathbb{D}(A)\subset\mathcal{X}\longrightarrow\mathcal{X}$ by $Az = \acute{\acute{z}}$ with domain $\mathbb{D}(A) = \{z\in\mathcal{X}; z,\acute{z}$ are absolutely continuous, $\acute{\acute{z}}\in\mathcal{X}, z(0)=z(\pi)=0\}$. Then, $A_z = \sum_{n=1}^{\infty} n^2(z,z_n)z_n,\ z\in\mathbb{D}(A)$, where $z_n(y) = \sqrt{\frac{2}{\pi}}\sin(ny),\ n\in\mathbb{N}$, which is a set of orthogonal eigenvectors of A. Define the fBm in $\mathcal{Y}$ by:

$$w^H(t) = \sum_{n=1}^{\infty}\sqrt{\lambda_n}w_n^H(t)e_n,$$

where $\{w_n^H(t), n\in\mathbb{N}\}$ are standard mutually independent fBms. Thanks to the subordination principle of solution operator, A is the infinitesimal generator of a solution operator $\{S_\beta(t), t\geqslant 0\}$. Since $S_\beta(t)$ is strongly continuous on $[0,\infty)$ by a uniformly bounded theorem, there exists a constant $M>0$ such that $\|S_\beta(t)\|^2\leqslant M^2$, for $t\in[0,b]$. Define the nonlinear functions $f:[0,b]\times\wp\longrightarrow\mathcal{X}$, $g:[0,b]\times\wp\times\wp\longrightarrow\mathcal{L}_2^0(\mathcal{Y},\mathcal{X})$ and $h:[0,b]\times\wp\times\wp\times Z\longrightarrow\mathcal{X}$ by:

$$f(t,\varrho)(y) = \int_{-\tau}^{0}\alpha_1(\zeta)\sin(\varrho(\zeta)(y))d\zeta, \quad \zeta\in[-\tau,0],\ y\in\mathbb{D},$$

$$g(t,\psi,\varrho)(y) = \int_{-\tau}^{0} e^{-4\zeta}\varrho(\zeta)(y)d\zeta,$$

$$h(t,\psi,\varrho)(y) = \int_{-\tau}^{0}\alpha_2(\zeta)\varrho(\zeta)(y)d\zeta,$$

$$I_k(y(t_k)) = \frac{\alpha_3}{2^k}z(t_k,y), \quad k\in\mathbb{N}$$

and assuming that $\int_Z \eta^2\tilde{\lambda}(d\eta)<\infty$. Finally, system (31) takes the abstract form of model (1) and assumes that assumptions (A1)–(A5) are satisfied. Then, Theorem 1 guarantee that model (31) has a mild solution, which is unique.

5. Conclusions

Through this study, the existence of the unique mild solution for a class of IFNSDEs with variable delay driven by fBm and Poisson jumps was investigated in Hilbert space. Our model is more general than [21,33] models as it takes the variable delays described by the term $\rho(t)$ and possible jumps shown as impulses into consideration. First, a new class of sufficient conditions for the existence of mild solutions of the aforementioned class of equations was established, which is more general than the non-Lipschitz condition used in [21] and contains it as special case. The existence results were formulated and proved by using a solution operator, fractional calculus, the Carathéodory approximation approach, and stochastic analysis techniques. An application was provided to validate the obtained theoretical results. Our results improve and enhance some results in [21]. Our future work is to consider the problem of an averaging principle for IFNSDEs with variable delay under fBm and Poisson-jump perturbations.

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Appendix A

Proof of the Uniqueness: Let $x(t)$, $y(t)$ be two solutions of Equation (1). Then, the uniqueness is obvious on the interval $[-\tau, 0]$, and for $t \in [0, b]$, by similar analysis of Equation (18), elementary inequality, Lemma 1, Burkholder's inequality, and conditions (A1), (A3), and (A5), it is easy to obtain:

$$
\begin{aligned}
&\mathbb{E}(\sup_{0\leqslant s\leqslant t} \|x(s)-y(s)\|^2) \\
&\leqslant \frac{12M^2Ht^{2H-1}}{1-2c_f}\int_0^t \mathbb{E}\|g(s,x(s),x(s-\rho(s)))-g(s,y(s),y(s-\rho(s)))\|^2 ds \\
&+\frac{6M^2C_b}{1-2c_f}\Big[\mathbb{E}\int_0^t\int_Z \|h(s,x(s),x(s-\rho(s)),\eta)-h(s,y(s),y(s-\rho(s)),\eta)\|^2\tilde{\lambda}(d\eta)ds \\
&+\mathbb{E}\Big(\int_0^t\int_Z \|h(s,x(s),x(s-\rho(s)),\eta)-h(s,y(s),y(s-\rho(s)),\eta)\|^4\tilde{\lambda}(d\eta)ds\Big)^{\frac{1}{2}}\Big] \\
&+\frac{6qM^2}{1-2c_f}\sum_{k=1}^{q}\mathbb{E}\|I_k(x(t_k))-I_k(y(t_k))\|^2 \\
&\leqslant \frac{6M^2(C_b+2Ht^{2H-1})}{1-2C_f-6qM^2\sum_{k=1}^q c_k}\int_0^t \mathcal{M}(s, 2\mathbb{E}\sup_{0\leqslant u\leqslant s}\|x(u)-y(u)\|^2)ds.
\end{aligned}
$$

Then:

$$
2\mathbb{E}(\sup_{0\leqslant s\leqslant t}\|x(s)-y(s)\|^2) \leqslant \frac{24M^2(C_b+Ht^{2H-1})}{1-2C_f-6qM^2\sum_{k=1}^q c_k}\int_0^t \mathcal{M}(s, 2\mathbb{E}\sup_{0\leqslant u\leqslant s}\|x(u)-y(u)\|^2)ds,
$$

which, with the aid of condition (A3.c), gives:

$$
\mathbb{E}(\sup_{0\leqslant s\leqslant t}\|x(s)-y(s)\|^2) = 0, \quad 0\leqslant t\leqslant b
$$

Therefore, $x(t) = y(t)$ for all $0 \leqslant t \leqslant b$. Hence, the uniqueness is proved. □

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