fractal and fractional

Article

Mixed Neutral Caputo Fractional Stochastic Evolution Equations with Infinite Delay: Existence, Uniqueness and Averaging Principle

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Abstract: The aim of this article is to consider a class of neutral Caputo fractional stochastic evolution equations with infinite delay (INFSEEs) driven by fractional Brownian motion (fBm) and Poisson jumps in Hilbert space. First, we establish the local and global existence and uniqueness theorems of mild solutions for the aforementioned neutral fractional stochastic system under local and global Carathéodory conditions by using the successive approximations, stochastic analysis, fractional calculus, and stopping time techniques. The obtained existence result in this article is new in the sense that it generalizes some of the existing results in the literature. Furthermore, we discuss the averaging principle for the proposed neutral fractional stochastic system in view of the convergence in mean square between the solution of the standard INFSEEs and that of the simplified equation. Finally, the obtained averaging theory is validated with an example.

Keywords: fractional neutral system; successive approximations; fractional Brownian motion; Poisson jumps; fractional calculus; averaging principle

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1. Introduction

The subject of fractional calculus has received considerable critical attention due to its applications in widespread areas of engineering and science. It is significant and successful in describing systems which have long-term memory and long-range interaction [1–3]. The theory of fractional differential Equations (FDEs) is a major area of interest within the field of fractional calculus. Evidence suggests that different kinds of FDEs appear frequently as the mathematical modeling of systems in various engineering and scientific disciplines, such as solid mechanics [4], physics [5,6], finance [7], chemistry [8], physiology [9] and electromechanics [10]. Mostly, the FDEs models seem to be more regular with the real events compared with the integer-order models, because the fractional integrals and derivatives allow the explanation of the hereditary and memory properties inherent in various processes and materials [11,12]. There are a large number of published studies (e.g., [13–17], and their cited references) that describe the existence of solutions for FDEs.

In light of recent events in the theory of dynamical systems, it is becoming extremely difficult to ignore the existence of random fluctuations. Recently, fractional stochastic differential equations (FSDEs) have been attracting considerable interest due to their successful and potential applications [18–20]. Many results have been investigated on the existence and uniqueness problem for various kinds of fractional stochastic systems, see [21–34] and the references therein. Most studies of dynamical systems depend not only on current and past states but also involve derivatives with delays, as well as the function itself. This motivates people to focus on the theory of neutral functional differential systems, see [35,36]. However, along with this growth in chemical engineering models, as well as the aeroelasticity theory, there is increasing concern over the neutral fractional stochastic differential systems. For example, Cui and Yan [37] proved the existence result for fractional neutral stochastic integro-differential equations with infinite delay. Alnafisah and Ahmed [38] proved the existence and uniqueness of mild solutions for neutral delay Hilfer fractional integro-differential equations perturbed with fBm. Rajivanthi et al. [39] derived the existence and optimal control for delay neutral fractional stochastic differential equations (NFSDEs) driven with Poisson jumps by using successive approximations under non-Lipschitz condition. Dineshkumar et al. [40,41] derived the approximate controllability for Hilfer fractional neutral stochastic delay integro-differential equations driven by Brownian motion.

However, the averaging method is a powerful tool for studying various kinds of nonlinear dynamical systems, since it permits the simplified averaged autonomous system to replace the original complex time-varying system, thus giving a reasonable way for reduction in complexity. The key to establishing an averaging principle is to ascertain the conditions under which the solution for averaged system can approximate the solution for original system. Starting with the work of Khasminskii [42], averaging principles for SDEs and FSDEs have been developed and applied widely [43–54]. On the other hand, the averaging principle for NFSDEs still in its infancy. For example, Liu and Xu [55] derived the averaging principle for impulsive NFSDEs driven by Brownian motion under non-Lipschitz condition. Shen et al. [56] proved the averaging principle for NFSDEs with variable delays driven by Lévy noise. Xu and Xu [57] obtained the averaging theory for Lipschitz NFSDEs driven by Poisson jumps.

Motivated by the above studies, we consider the following INFSEEs driven by fBm and Poisson jumps in Hilbert space:

$$\begin{cases} D_t^{\beta}[y(t)-f(t,y_t)] = A[y(t)-f(t,y_t)] + \Gamma_t^{1-\beta}\Big[g(s,y_s)\frac{dw^H(t)}{dt} \\ \qquad\qquad\qquad\qquad + \int_Z h(t,y_t,\eta)\tilde{N}(dt,d\eta)\Big], t\in[0,T], \\ y(t)=\varrho(t)\in\wp, \quad t\leqslant 0, \end{cases} \tag{1}$$

where D_t^{β}, $0<\beta<1$ is the Caputo fractional derivative of β-order. $\Gamma_t^{1-\beta}(.)$ denotes the $1-\beta$ order fractional integral. Let $A:\mathbb{D}(A)\subset\mathcal{X}\longrightarrow\mathcal{X}$ is the infinitesimal generator of a solution operator, $\{T_\beta(t)\}_{t\geqslant 0}$, defined on a Hilbert space, $\mathcal{X}$, endowed by inner product, $\langle .,.\rangle$, and norm, $\|.\|_{\mathcal{X}}$. $w^H(t)$ is a fBm with Hurst parameter, $1/2<H<1$, defined on a real separable Hilbert space, $\mathcal{Y}$, endowed with inner product, $\langle .,.\rangle$, and norm, $\|.\|_{\mathcal{Y}}$. $\tilde{N}(dt,d\eta)=N(dt,d\eta)-\tilde{\lambda}(d\eta)dt$ represents the compensated Poisson random measure that is independent of w^H. Assume that $f:[0,T]\times\wp\longrightarrow\mathcal{X}$, $g:[0,T]\times\wp\longrightarrow\mathcal{L}_2^0(\mathcal{Y},\mathcal{X})$, $h:[0,T]\times\wp\times Z\longrightarrow\mathcal{X}$ are nonlinear mappings. Let $\wp=\wp((-\infty,0];L^2(\Omega,\mathcal{X}))$ denote the family of all $\mathcal{F}_0$-measurable bounded continuous functions, $\varrho:(-\infty,0]\longrightarrow L^2(\Omega,\mathcal{X})$, equipped with the norm, $\|\varrho\|_t^2=\sup_{-\infty<\theta<0}\|\varrho(\theta)\|^2$. Let $\wp^B_{\mathcal{F}_0}((-\infty,0];\mathcal{X})$ denote the family of all almost surely bounded, $\mathcal{F}_0$-measurable, $\wp$-valued random variables. Let $\mathcal{B}_t$ be a Banach space of all $\mathcal{F}_t$-adapted processes, $\varrho(t,\omega)$, which are almost surely continuous in t for fixed $\omega\in\Omega$ with the following norm:

$$\|\varrho\|_{\mathcal{B}_t}=\big(\sup_{0\leqslant t\leqslant T}\|\varrho\|_t^2\big)^{\frac{1}{2}}.$$

Suppose $y_t = y(t+\theta)$, $-\infty < \theta \leqslant 0$ can be regarded as a $\wp$-valued stochastic process. The initial value $\varrho = \{\varrho(\theta) : -\infty < \theta \leqslant 0\}$ is an $\mathcal{F}_0$-measurable, valued random variable independent of w^H and Poisson process, $\tilde{N}$, with finite second moment.

Recently, Ramkumar et al. [58] utilized the existence and optimal control for NFSEEs (1) with finite delay driven by fBm and Poisson jumps under non-Lipchitz conditions. We observe that the non-Lipchitz conditions do not conclude Carathéodory conditions in general. However, the Carathéodory conditions may derive Lipschitz and non-Lipschitz conditions. Generally speaking, this is an irreversible process. If the existence and uniqueness problem of INFSEEs (1) under Carathéodory conditions is proved, then we may obtain a much bigger degree of freedom for choosing drift and diffusion coefficients in applications. So, existence and uniqueness problem of INFSEEs driven by fBm and Poisson jumps under Carathéodory conditions is desired. However, local and global existence and uniqueness theorems, as well as averaging principle for solutions of INFSEEs under local and global Carathéodory type conditions, have not been considered so far.

The main contributions of this paper are summarized as follows:

- In view of the research gaps and pressing needs, the infinite delays are taken into consideration, which makes the underlying model and the obtained results more general and applicable.
- The local and global existence and uniqueness results for Equation (1), under local and global Carathéodory conditions by means of successive approximation and stopping time techniques, are rarely available in the literature, which is the key inspiration to our research work in this article and seems to be new to our knowledge.
- By using stochastic analysis techniques, we analyzed the averaging results under global Carathéodory conditions for the proposed model (1).

A brief outline of this article is arranged as follows. Section 2 introduces some needed notions and preliminaries about fBm and fractional calculus. In Section 3, we discuss the local and global existence and uniqueness results for system (1) under local and global Carathéodory conditions. In Section 4, we extend the averaging principle for (1) under global Carathéodory conditions. Finally, Section 5 shows an example to illustrate the utility of our obtained averaging theoretical results.

2. Preliminaries

In the present section, some elementary notations and preliminaries are reviewed. Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is a filtered probability space with $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. The fBm $w^H = \{w^H(t)\}_{0\leqslant t\leqslant T}$, $H \in (\frac{1}{2}, 1)$ is a centered Gaussian process with the following variance–covariance function:

$$K_H(u,v) = \mathbb{E}(w^H(u)w^H(v)) = \frac{1}{2}(u^{2H} + v^{2H} - |u-v|^{2H}), \quad u,v \in (-\infty, \infty)$$

and the following second partial derivative [59]:

$$\frac{\partial K_H}{\partial u \partial v} = (2H-2)H|u-v|^{2H-2}, \quad H > \frac{1}{2}$$

so, we can write the following :

$$K_H(u,v) = (2H-2)H \int_0^u \int_0^v |u_1 - v_1|^{2H-2} du_1 dv_1.$$

For any real and separable Hilbert spaces, $\mathcal{X}$ and $\mathcal{Y}$, assume $\mathcal{L}(\mathcal{Y}, \mathcal{X})$ is the space of all bounded linear operators from $\mathcal{Y}$ to $\mathcal{X}$. Let $\mathbb{Q} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ be the operator defined by $\mathbb{Q}e_n = \lambda_n e_n$ with finite trace $tr\mathbb{Q} = \sum_{n=1}^{\infty} \lambda_n < \infty$; for $\lambda_n \geqslant 0$ $(n = 1, 2, \ldots)$ are non-negative real numbers and $\{e_n\}$ is a complete orthonormal basis in $\mathcal{Y}$. The infinite dimensional fBm on $\mathcal{Y}$ is defined as follows:

$$w^H(t) = w^H_{\mathbb{Q}}(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n w^H_n(t),$$

with real independent fBm's w^H_n. Construct the space $\mathcal{L}^0_2 := \mathcal{L}^0_2(\mathcal{Y}, \mathcal{X})$ of all $\mathbb{Q}$-Hilbert Schmidt operators, $\zeta : \mathcal{Y} \longrightarrow \mathcal{X}$, equipped with the inner product $\langle \varphi, \zeta \rangle_{\mathcal{L}^0_2} = \sum_{n=1}^{\infty} \langle \varphi e_n, \zeta e_n \rangle$ and the following norm:

$$\|\zeta\|^2_{\mathcal{L}^0_2} := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \zeta e_n\|^2 < \infty.$$

For any $\phi(s) \in \mathcal{L}^0_2(\mathcal{Y}, \mathcal{X})$, $s \in [0, T]$, such that $\sum_{n=1}^{\infty} \|R^* \phi \mathbb{Q}^{\frac{1}{2}} e_n\|^2_{\mathcal{L}^0_2} < \infty$, the Weiner integral of ϕ, with respect to w^H, is defined by the following:

$$\int_0^t \phi(s) dw^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \phi(s) e_n dw^H_n(s). \tag{2}$$

Lemma 1. *Ref. [60] For any $\phi : [0, T] \longrightarrow \mathcal{L}^0_2(\mathcal{Y}, \mathcal{X})$ with $\int_0^T \|\phi(s)\|^2_{\mathcal{L}^0_2} ds < \infty$, satisfying Equation (2), what follows is satisfied:*

$$\mathbb{E} \left\| \int_0^t \phi(s) dw^H(s) \right\|^2 \leqslant 2Ht^{2H-1} \int_0^t \|\phi(s)\|^2_{\mathcal{L}^0_2} ds.$$

We refer to [61–65] for more details on the stochastic integral with respect fBm.

Definition 1. *Ref. [66,67] The β-order fractional integral of Riemann–Liouville sense for $g : [0, T] \longrightarrow \mathcal{X}$ is expressed by the following:*

$$J^\beta_t g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s) ds, \quad \beta > 0.$$

Definition 2. *Refs. [68] The Caputo β-order derivative with 0 lower bound for $g : [0, T] \longrightarrow \mathcal{X}$ is expressed as follows:*

$$D^\beta_t g(t) = \frac{1}{\Gamma(k-\beta)} \int_0^t \frac{g^k(s)}{(t-s)^{\beta+1-k}} ds = J^{k-\beta}_t g^k(t), \quad \beta > 0,\ t \geqslant 0.$$

For further discussion on the fractional Riemann–Liouville and Caputo derivatives, refer to [66–69].

Next, a two parameter Mittag–Leffler function is defined by the following series expansion:

$$E_{\beta,\alpha}(w) = \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(\beta k + \alpha)} = \frac{1}{2\pi i} \int_c \frac{\lambda^{\beta-\alpha} e^\lambda}{\lambda^\beta - w} d\lambda, \quad \alpha, \beta > 0,\ w \in \mathbb{C}$$

where $\mathbb{C}$ is a contour that starts and ends with $-\infty$ and encircles the disk $|\lambda| \leqslant |w|^{\frac{1}{2}}$ counter clockwise.

We construct the following definition for the mild solution of Equation (1):

Definition 3. *A stochastic process, $\{y(t), t \in (-\infty, T]\}$, $(0 < T < \infty)$, is called a mild solution of Equation (1) if*

(i) $y(t)$ is $\mathcal{F}_t$-adapted;
(ii) for arbitrary $t \in [0, T]$, $y(t)$ satisfies the following integral form:

$$\begin{cases} y(t) = T_\beta(t)[\varrho(0) - f(0,\varrho)] + f(t,y_t) + \int_0^t T_\beta(t-s)g(\tau,y_\tau)dw^H(s) \\ \qquad + \int_0^t \int_Z T_\beta(t-s)h(s,y_s,\eta)\tilde{N}(ds,d\eta), \\ y_0 = \varrho \in \wp. \end{cases}$$

where $T_\beta(t)$ is the solution operator generated by A and given by the following:

$$T_\beta(t) = E_{\beta,1}(At^\beta) = \frac{1}{2\pi i}\int_{\hat{B}_r} e^{\mu t}\frac{\mu^{\beta-1}}{\mu^\beta - A}d\mu.$$

The coming assumptions on the coefficients of (1) are prepared for achieving the main results.

Condition 1. If $A : \mathbb{D} \subset \mathcal{X} \longrightarrow \mathcal{X}$ is the infinitesimal generator of a strong and continuous semigroup of bounded and linear operator $T_\beta(t)$, then there exists some constant $B > 0$ obeying, as follows:

$$\|T_\beta(t)\|^2 \leqslant B, \quad for\ all\ t \in [0,T].$$

Condition 2. There exists a positive constant $c_f \in (0,1)$ such that for all $x_t, y_t \in \wp$, we have the following:

$$\|f(t,x_t) - f(t,y_t)\| \leqslant c_f\|x-y\|_t, \quad and \quad f(t,0) = 0,\ t \geqslant 0.$$

Condition 3. (a) There exists a function, $\mathcal{K}(t,v) : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$, such that $\mathcal{K}(t,v)$ is locally integrable in t for any fixed $v \geqslant 0$ and is continuous, non-decreasing, and concave in v for each fixed $t \in [0,T]$. Furthermore, for any $t \in [0,T]$, $y_t \in \wp$, the following holds:

$$\mathbb{E}\int_0^t \|g(s,y_s)\|^2 ds \vee \mathbb{E}\int_0^t\int_Z \|h(s,y_s,\eta)\|^2\hat{\lambda}(\eta)ds$$
$$\vee\mathbb{E}\left(\int_0^t\int_Z \|h(s,y_s,\eta)\|^4\hat{\lambda}(\eta)ds\right)^{\frac{1}{2}} \leqslant \int_0^t \mathcal{K}(s,\mathbb{E}\|y\|_s^2)ds.$$

(b) For any $C > 0$, the following differential equation:

$$\frac{dv}{dt} = C\mathcal{K}(t,v)$$

has a global solution for any initial value v_0.

Condition 4. (Global conditions) (a) There exists a function, $\mathcal{A}(t,v) : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$, such that $\mathcal{A}(t,v)$ is locally integrable in t for any fixed $v \geqslant 0$ and is continuous, non-decreasing, and concave in v for each fixed $t \in [0,T]$, $\mathcal{A}(t,0) = 0$ and for any fixed $0 \leqslant t \leqslant T$, $\int_{0^+}\frac{1}{\mathcal{A}(t,v)}dv = \infty$. Furthermore, for any $t \in [0,T]$, $x_t, y_t \in \wp$, the following holds:

$$\mathbb{E}\int_0^t \|g(s,x_s) - g(s,y_s)\|^2 ds \vee \mathbb{E}\int_0^t\int_Z \|h(s,x_s,\eta) - h(s,y_s,\eta)\|^2\hat{\lambda}(\eta)ds$$
$$\vee\mathbb{E}\left(\int_0^t\int_Z \|h(s,x_s,\eta) - h(s,y_s,\eta)\|^4\hat{\lambda}(\eta)ds\right)^{\frac{1}{2}} \leqslant \int_0^t \mathcal{A}(s,\mathbb{E}\|x-y\|_s^2)ds.$$

(b) For any constant $K > 0$, if a non-negative function $v(t)$ satisfies the following:

$$v(t) \leqslant K\int_0^t \mathcal{A}(s,v(s))ds, \quad t \in [0,T],$$

then $v(t) \equiv 0$ for any $t \in [0,T]$.

Condition 5. (Local conditions) (a) For any integer, $N > 0$, there exists a function, $\mathcal{A}_N(t,v) : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$, such that $\mathcal{A}_N(t,v)$ is locally integrable in t for any fixed $v \geqslant 0$ and is continuous, non-decreasing and concave in v for each fixed $t \in [0,T]$, $\mathcal{A}_N(t,0) = 0$ for any $t \in [0,T]$. Furthermore, for any $t \in [0,T]$, $x_t, y_t \in \wp$ with $\|x\|_t \leqslant N$, $\|y\|_t \leqslant N$, the following holds:

$$\mathbb{E}\int_0^t \|g(s,x_s) - g(s,y_s)\|^2 ds \vee \mathbb{E}\int_0^t \int_Z \|h(s,x_s,\eta) - h(s,y_s,\eta)\|^2 \hat{\lambda}(\eta) ds$$
$$\vee \mathbb{E}\left(\int_0^t \int_Z \|h(s,x_s,\eta) - h(s,y_s,\eta)\|^4 \hat{\lambda}(\eta) ds\right)^{\frac{1}{2}} \leqslant \int_0^t \mathcal{A}_N(s, \mathbb{E}\|x-y\|_s^2) ds.$$

(b) For any positive constant K, if a non-negative function $v(t)$ satisfies the following:

$$v(t) \leqslant K\int_0^t \mathcal{A}_N(s, v(s)) ds, \quad t \in [0,T],$$

then for all $t \in [0,T]$, $v(t) \equiv 0$.

3. Existence and Uniqueness

In this section, we provide the problem of existence and uniqueness of mild solution to Equation (1) under global and local Carathéodory conditions.

Theorem 1. *If Conditions 1–4 hold, then the system (1) admits a unique mild solution, $y(t) \in \mathcal{B}_{\mathcal{T}}$.*

Proof. In order to prove the existence of the solution to Equation (1), let us introduce the sequence of successive approximations, as follows: $y^0(t) = T_\beta(t)\varrho(0)$, $t \in [0,T]$ and $y^n(t) = \varrho(t)$, $n \geqslant 1$ for $t \in (-\infty, 0]$, and for $t \in [0,T]$ and $n \geqslant 1$, we have the following:

$$\begin{aligned} y^n(t) &= T_\beta(t)[\varrho(0) - f(0,\varrho)] + f(t, y_t^n) + \int_0^t T_\beta(t-s) g(\tau, y_\tau^{n-1}) dw^H(s) \\ &\quad + \int_0^t \int_Z T_\beta(t-s) h(s, y_s^{n-1}, \eta) \tilde{N}(ds, d\eta), \quad n = 1,2,\ldots \end{aligned} \tag{3}$$

The proof will be split into three parts.

Part 1. For all $t \in (-\infty, T]$, we claim that the sequence $y^n \in \mathcal{B}_{\mathcal{T}}$, $n \geqslant 0$ is bounded. It is obvious that $y^0(t) \in \mathcal{B}_{\mathcal{T}}$. From (3), for $0 \leqslant t \leqslant T$, we have by elementary inequality, Lemma 1, Burkholder–Davis–Gundy (B–D–G) inequality for pure jump stochastic integral [70] in $\mathcal{X}$ and Conditions 1–3, as follows:

$$\begin{aligned} \mathbb{E}\|y^n(t)\|^2 &\leqslant 8B[\mathbb{E}\|\varrho(0)\|^2 + c_f^2 \mathbb{E}\|\varrho\|_0^2] + 4c_f^2 \mathbb{E}\|y^n\|_t^2 \\ &\quad + 8BHT^{2H-1}\mathbb{E}\int_0^t \|g(\tau, y_\tau^{n-1})\|^2 ds \\ &\quad + 4BC_T \mathbb{E}\int_0^t \int_Z \|h(s, y_s^{n-1}, \eta)\|^2 \tilde{\lambda}(d\eta) ds \\ &\quad + 4BC_T \mathbb{E}\left(\int_0^t \int_Z \|h(s, y_s^{n-1}, \eta)\|^4 \tilde{\lambda}(d\eta) ds\right)^{\frac{1}{2}} \\ &\leqslant 8B[\mathbb{E}\|\varrho(0)\|^2 + c_f^2 \mathbb{E}\|\varrho\|_0^2] + 4c_f^2 \mathbb{E}\|y^n\|_t^2 \\ &\quad + 8B(C_T + HT^{2H-1})\int_0^t \mathcal{K}(s, \mathbb{E}\|y^{n-1}\|_s^2) ds. \end{aligned}$$

Thus,

$$\mathbb{E}\|y^n\|_t^2 \leqslant \frac{8B[\mathbb{E}\|\varrho(0)\|^2 + c_f^2 \mathbb{E}\|\varrho\|_0^2]}{1 - 4c_f^2} + \frac{8B(C_T + Ht^{2H-1})}{1 - 4c_f^2}\int_0^t \mathcal{K}(s, \mathbb{E}\|y^{n-1}\|_s^2) ds,$$

Then, Condition 3(b) implies that there is a solution, u_t, that satisfies the following:

$$v_t \leqslant C_1 + C_2 \int_0^t \mathcal{K}(s, v_s) ds,$$

where $C_1 = \frac{8B[\mathbb{E}\|\varrho(0)\|^2 + c_f^2 \mathbb{E}\|\varrho\|_0^2]}{1-4c_f^2}$ and $C_2 = \frac{8B(C_T + Ht^{2H-1})}{1-4c_f^2}$.

Since $\mathbb{E}\|y^0(t)\|^2 \leqslant B\mathbb{E}\|\varrho(0)\|^2 < \infty$, we have $\mathbb{E}\|y^n\|_t^2 \leqslant v_t \leqslant v_T < \infty$, which indicates the boundedness of $\{y^n(t)\}_{n\geqslant 0}$ in $\mathcal{B}_{\mathcal{T}}$.

Part 2. We claim that $\{y^n(t), n \geqslant 0\}$ is a Cauchy sequence.

For all $n, m \geqslant 0$ and $t \in [0, T]$, from (3), we have by Lemma 1, B–D–G inequality and Conditions 1, 2 and 4, as follows:

$$\begin{aligned}
&\mathbb{E}\|y^{n+1}(t) - y^{m+1}(t)\|^2 \\
&\leqslant 3\mathbb{E}\|f(t, y_t^{n+1}) - f(t, y_t^{m+1})\|^2 \\
&\quad + 3\mathbb{E}\left\| \int_0^t T_\beta(t-s)[g(\tau, y_\tau^{n+1}) - g(\tau, y_\tau^{m+1})] dw^H(s) \right\|^2 \\
&\quad + 3\mathbb{E}\left\| \int_0^t \int_Z T_\beta(t-s)[h(s, y_s^{n+1}, \eta) - h(s, y_s^{m+1}, \eta)] \tilde{N}(ds, d\eta) \right\|^2 \\
&\leqslant 3c_f^2 \mathbb{E}\|y^{n+1} - y^{m+1}\|_t^2 + 6BT^{2H-1} \int_0^t \mathbb{E}\|g(\tau, y_\tau^{n+1}) - g(\tau, y_\tau^{m+1})\|^2 ds \\
&\quad + 3BC_T \mathbb{E} \int_0^t \int_Z \|h(s, y_s^{n+1}, \eta) - h(s, y_s^{m+1}, \eta)\|^2 \tilde{\lambda}(d\eta) ds \\
&\quad + 3BC_T \mathbb{E}\left(\int_0^t \int_Z \|h(s, y_s^{n+1}, \eta) - h(s, y_s^{m+1}, \eta)\|^4 \tilde{\lambda}(d\eta) ds \right)^{\frac{1}{2}} \\
&\leqslant 3c_f^2 \mathbb{E}\|y^{n+1} - y^{m+1}\|_t^2 + 6B(C_T + HT^{2H-1}) \int_0^t \mathcal{A}(s, \mathbb{E}\|y^n - y^m\|_s^2) ds.
\end{aligned}$$

Thus,

$$\mathbb{E}\|y^{n+1} - y^{m+1}\|_t^2 = \sup_{-\infty \leqslant s \leqslant t} \mathbb{E}\|y^{n+1}(s) - y^{m+1}(s)\|^2 \leqslant C_3 \int_0^t \mathcal{A}(s, \mathbb{E}\|y^n - y^m\|_s^2) ds, \tag{4}$$

where $C_3 = \frac{6B(C_T + HT^{2H-1})}{1-3c_f^2}$.

By Equation (4), Fatou's lemma and Condition 4(b), we obtain the following:

$$\begin{aligned}
&\limsup_{n,m \longrightarrow \infty} \big(\sup_{0 \leqslant s \leqslant t} \mathbb{E}\|y^{n+1}(s) - y^{m+1}(s)\|^2 \big) \\
&\qquad \leqslant C_3 \int_0^t \mathcal{A}\big(s, \limsup_{n,m \longrightarrow \infty} \big(\sup_{0 \leqslant \theta \leqslant s} \mathbb{E}\|y^{n+1}(\theta) - y^{m+1}(\theta)\|^2 \big)\big) ds,
\end{aligned}$$

which gives the following:

$$\lim_{n,m \longrightarrow \infty} \sup_{0 \leqslant s \leqslant T} \mathbb{E}\|y^{n+1}(s) - y^{m+1}(s)\|^2 = 0,$$

and implies that $\{y^n\}_{n\geqslant 0}$ is Cauchy sequence in $\mathcal{B}_{\mathcal{T}}$.

Part 3. We claim the existence and uniqueness of the mild solution to Equation (1). The completeness of $\mathcal{B}_{\mathcal{T}}$ guarantees the existence of a process $y \in \mathcal{B}_{\mathcal{T}}$, such that we obtain the following:

$$\lim_{n\longrightarrow\infty} \sup_{-\infty\leqslant s\leqslant T} \mathbb{E}\|y^n(s) - y(s)\|^2 = 0.$$

Hence, letting $n \longrightarrow \infty$ and taking limits on both sides of (3), we obtain that $y(t)$ is a mild solution to Equation (1). This shows the existence proof. Furthermore, the uniqueness of the solutions could be obtained by the same procedure as Part 2. This completes the proof of Theorem 1. □

Remark 1. *Let $\mathcal{A}(t,v) = L(t)\bar{\mathcal{A}}(v)$, $t \in [0,T]$, where $L(t) \geqslant 0$ is locally integrable and $\bar{\mathcal{A}}(v)$ is a concave non-decreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$, such that $\bar{\mathcal{A}}(0) = 0$, $\bar{\mathcal{A}}(v) > 0$ for $v > 0$ and $\int_{0^+} \frac{1}{\bar{\mathcal{A}}(v)}dv = \infty$. Then, by the comparison theorem of differential equations we know that Condition (4b) holds.*

Remark 2. *If $L(t) = 1$ in Remark 1, the condition is considered in [58]. Therefore, some previous results in [58] are improved and enhanced.*

Next, we will prove the existence and uniqueness of solutions to Equation (1) under local Carathéodory conditions.

Theorem 2. *If Conditions 1–3 and 5 hold, the system (1) admits a unique local mild solution, as follows: $y(t) \in \mathcal{B}_{\mathcal{T}}$.*

Proof. Let N be a positive integer and $T_0 \in (0,T)$. We introduce the sequence of functions $g^N(t,v_t)$ and $h^N(t,v_t,\eta)$, $(t,v_t) \in [0,T] \times \mathcal{B}$, as follows:

$$g^N(t,v_t) = \begin{cases} g(t,v_t), & \|v\|_t \leqslant N; \\ g(t, \frac{Nv}{\|v\|_t}), & \|v\|_t > N, \end{cases}$$

$$h^N(t,v_t,\eta) = \begin{cases} h(t,v_t,\eta), & \|v\|_t \leqslant N; \\ h(t, \frac{Nv}{\|v\|_t},\eta), & \|v\|_t > N. \end{cases}$$

Then, the functions $\{g^N(t,v_t)\}$ and $h^N(t,v_t,\eta)$ satisfy Condition 3, and for any $x, y \in \mathcal{B}$, $t \in [0,T]$, the following inequality holds:

$$\begin{aligned} &\mathbb{E}\int_0^t \|g^N(s,x_s) - g^N(s,y_s)\|^2 ds \vee \mathbb{E}\int_0^t \int_Z \|h^N(s,x_s,\eta) - h^N(s,y_s,\eta)\|^2 \hat{\lambda}(\eta) ds \\ &\quad \vee \mathbb{E}\left(\int_0^t \int_Z \|h^N(s,x_s,\eta) - h^N(s,y_s,\eta)\|^4 \hat{\lambda}(\eta) ds\right)^{\frac{1}{2}} \leqslant \int_0^t \mathcal{A}_N(s, \mathbb{E}\|x-y\|_s^2) ds. \end{aligned}$$

As a consequence of Theorem 1, there exists a unique mild solutions $y^N(t)$ and $y^{N+1}(t)$, respectively, to the following integral equations:

$$\begin{cases} y^N(t) = T_\beta(t)[\varrho(0) - f(0,\varrho)] + f(t,y_t^N) + \int_0^t T_\beta(t-s) g^N(\tau, y_\tau^N) dw^H(s) \\ \qquad\qquad + \int_0^t \int_Z T_\beta(t-s) h^N(s, y_s^N, \eta) \tilde{N}(ds,d\eta), \ t \in [0,T], \\ y^N(t) = \varrho(t), \quad t \leqslant 0, \end{cases} \tag{5}$$

$$\begin{cases} y^{N+1}(t) = T_\beta(t)[\varrho(0) - f(0,\varrho)] + f(t,y_t^{N+1}) + \int_0^t T_\beta(t-s) g^{N+1}(\tau, y_\tau^{N+1}) dw^H(s) \\ \qquad\qquad + \int_0^t \int_Z T_\beta(t-s) h^{N+1}(s, y_s^{N+1}, \eta) \tilde{N}(ds,d\eta), \ t \in [0,T], \\ y^{N+1}(t) = \varrho(t), \quad t \leqslant 0. \end{cases} \tag{6}$$

Define the stopping times, as follows:

$$\delta_N := T_0 \wedge \inf\{t \in [0,T] : \|y^N\|_t \geqslant N\},$$

$$\delta_{N+1} := T_0 \wedge \inf\{t \in [0,T] : \|y^{N+1}\|_t \geqslant N+1\},$$

$$\tau_N := \delta_N \wedge \delta_{N+1}.$$

In view of (5) and (6), we obtain the following:

$$\begin{aligned}
&\mathbb{E}\|y^{N+1}(t) - y^N(t)\|^2 \\
&\quad \leqslant 3\mathbb{E}\|f(t, y_t^{N+1}) - f(t, y_t^N)\|^2 \\
&\qquad + 3\mathbb{E}\left\| \int_0^t T_\beta(t-s)[g^{N+1}(\tau, y_\tau^{N+1}) - g^N(\tau, y_\tau^N)]dw^H(s) \right\|^2 \\
&\qquad + 3\mathbb{E}\left\| \int_0^t \int_Z T_\beta(t-s)[h^{N+1}(s, y_s^{N+1}, \eta) - h^N(s, y_s^N, \eta)]\tilde{N}(ds, d\eta) \right\|^2.
\end{aligned}$$

By using the technique of plus and minus, Lemma 1, B–D–G inequality and Conditions 1 and 2, we obtain the following:

$$\begin{aligned}
&\mathbb{E}\|y^{N+1}(t) - y^N(t)\|^2 \\
&\quad \leqslant 3\mathbb{E}\|f(t, y_t^{N+1}) - f(t, y_t^N)\|^2 \\
&\qquad + 6\mathbb{E}\left\| \int_0^t T_\beta(t-s)[g^{N+1}(\tau, y_\tau^{N+1}) - g^{N+1}(\tau, y_\tau^N)]dw^H(s) \right\|^2 \\
&\qquad + 6\mathbb{E}\left\| \int_0^t \int_Z T_\beta(t-s)[h^{N+1}(s, y_s^{N+1}, \eta) - h^{N+1}(s, y_s^N, \eta)]\tilde{N}(ds, d\eta) \right\|^2 \\
&\quad \leqslant 3c_f^2\mathbb{E}\|y^{N+1} - y^N\|_t^2 \\
&\qquad + 12BHT^{2H-1}\mathbb{E}\int_0^t \|g^{N+1}(\tau, y_\tau^{N+1}) - g^{N+1}(\tau, y_\tau^N)\|^2 ds \\
&\qquad + 6BC_T\mathbb{E}\int_0^t \int_Z \|h^{N+1}(s, y_s^{N+1}, \eta) - h^{N+1}(s, y_s^N, \eta)\|^2 \tilde{\lambda}(d\eta)ds \\
&\qquad + 6BC_T\mathbb{E}\left(\int_0^t \int_Z \|h^{N+1}(s, y_s^{N+1}, \eta) - h^{N+1}(s, y_s^N, \eta)\|^4 \tilde{\lambda}(d\eta)ds \right)^{\frac{1}{2}},
\end{aligned}$$

where we have used that for $0 < v < \tau_N$, as follows:

$$g^{N+1}(v, y_v^N) = g^N(v, y_v^N), \quad h^{N+1}(v, y_v^N, \eta) = h^N(v, y_v^N, \eta).$$

Employing Condition 5, we obtain the following:

$$\begin{aligned}
\mathbb{E}\|y^{N+1}(t) - y^N(t)\|^2 \quad &\leqslant \quad 3c_f^2\mathbb{E}\|y^{N+1} - y^N\|_t^2 \\
&\qquad + 12B(C_T + HT^{2H-1})\int_0^t \mathcal{A}_{N+1}(s, \mathbb{E}\|y^{N+1}y^N\|_s^2)ds.
\end{aligned}$$

Therefore, for all $t \in [0, T_0]$, we obtain the following:

$$\begin{aligned}
\sup_{-\infty < s \leqslant t\wedge\tau_N} \mathbb{E}\|y^{N+1}(s) - y^N(s)\|^2 \quad &\leqslant \quad \frac{12B(C_T + HT^{2H-1})}{1 - 3c_f^2} \\
&\qquad \times \int_0^{t\wedge\tau_N} \mathcal{A}_{N+1}(s, \sup_{-\infty < u \leqslant s\wedge\tau_N} \mathbb{E}\|y^{N+1}y^N\|_u^2)ds.
\end{aligned}$$

Thus, Condition 5 indicates the following:

$$\sup_{-\infty<s\leqslant t\wedge\tau_N}\mathbb{E}\|y^{N+1}(s)-y^N(s)\|^2=0.$$

Then, for a.e. ω, as follows:

$$y^{N+1}(t)=y^N(t),\quad for\;\; 0\leqslant t\leqslant T_0\wedge\tau_N.$$

Note that for each $\omega\in\Omega$, there exists an $N_0(\omega)>0$, such that $0\leqslant T_0\leqslant\tau_{N_0}$. Define $y(t)$ by the following:

$$y(t)=y^{N_0}(t),\quad for\;\; t\in[0,T_0].$$

Since $y(t\wedge\tau_N)=y^N(t\wedge\tau_N)$, the following holds:

$$\begin{aligned}
y(t\wedge\tau_N) &= T_\beta(t\wedge\tau_N)[\varrho(0)-f(0,\varrho)]+f(t\wedge\tau_N,y^N_{t\wedge\tau_N})\\
&\quad+\int_0^{t\wedge\tau_N}T_\beta(t\wedge\tau_N-s)g^N(s,y^N_s)dw^H(s)\\
&\quad+\int_0^{t\wedge\tau_N}\int_Z T_\beta(t\wedge\tau_N-s)h^N(s,y^N_s,\eta)\tilde{N}(ds,d\eta)\\
&= T_\beta(t\wedge\tau_N)[\varrho(0)-f(0,\varrho)]+f(t\wedge\tau_N,y_{t\wedge\tau_N})\\
&\quad+\int_0^{t\wedge\tau_N}T_\beta(t\wedge\tau_N-s)g(s,y_s)dw^H(s)\\
&\quad+\int_0^{t\wedge\tau_N}\int_Z T_\beta(t\wedge\tau_N-s)h(s,y_s,\eta)\tilde{N}(ds,d\eta).
\end{aligned}$$

Allowing $N\longrightarrow\infty$, we obtain the following:

$$\begin{aligned}
y(t) &= T_\beta(t)[\varrho(0)-f(0,\varrho)]+f(t,y_t)+\int_0^t T_\beta(t-s)g(s,y_s)dw^H(s)\\
&\quad+\int_0^t\int_Z T_\beta(t-s)h^N(s,y_s,\eta)\tilde{N}(ds,d\eta),
\end{aligned}$$

which completes the proof of existence. The uniqueness proof can be justified by stopping our process $y(t)$. This completes the proof of Theorem 2. □

Remark 3. *As Condition 4 implies Condition 5, Theorem 2 conditions are weaker than those of Theorem 1. Hence, Theorem 2 generalizes Theorem 1.*

Remark 4. *Let $\mathcal{A}_N(t,v)=\bar{\mathcal{A}}_N(v)$, where $\bar{\mathcal{A}}_N(v)$ is a concave, non-decreasing and continuous function, such that $\bar{\mathcal{A}}_N(0)=0$, and $\int_{0^+}\frac{dv}{\bar{\mathcal{A}}_N(v)}=\infty$. Evidently, our obtained result in Theorem 2 is still new and applicable. Moreover, the results of [58] are improved and extended.*

4. Averaging Principle

This section is devoted to the establishment of an averaging principle for INFSEEs driven by fBm and Poisson jumps. Hence an interesting theoretical result to simplify the systems is presented.

The standard INFSEEs driven by fBm and Poisson jumps is defined as follows:

$$\begin{aligned}
y_\varepsilon(t) &= T_\beta(t)[\varrho(0)-f(0,\varrho)]+f(t,y_{\varepsilon,t})+\varepsilon^H\int_0^t T_\beta(t-s)g(\tau,y_{\varepsilon,\tau})dw^H(s)\\
&\quad+\sqrt{\varepsilon}\int_0^t\int_Z T_\beta(t-s)h(s,y_{\varepsilon,s},\eta)\tilde{N}(ds,d\eta),
\end{aligned}\tag{7}$$

where $t \in [0,T]$, $\varepsilon \in (0,\varepsilon_1]$ $(0 < \varepsilon_1 << 1)$ is a small parameter, the coefficients satisfy the assumptions of Theorem 1.

In what follows, we deduce that when the time scale, ε, approaches 0, the standard mild solution, $y_\varepsilon(t)$, tends to the averaged mild solution, $z_\varepsilon(t)$, of the simplified system, as follows:

$$\begin{aligned} z_\varepsilon(t) &= T_\beta(t)[\varrho(0) - f(0,\varrho)] + f(t, z_{\varepsilon,t}) + \varepsilon^H \int_0^t T_\beta(t-s)\bar{g}(z_{\varepsilon,\tau})dw^H(s) \\ &\quad + \sqrt{\varepsilon}\int_0^t \int_Z T_\beta(t-s)\bar{h}(z_{\varepsilon,s},\eta)\tilde{N}(ds,d\eta), \end{aligned} \tag{8}$$

the coefficients $\bar{g}(.)$, $\bar{h}(.,\eta) \in \mathcal{X}$ are all measurable functions satisfying Conditions 3 and 4 and the following assumptions:

(A1)

$$\frac{1}{T_1}\int_0^{T_1} \|g(s,y) - \bar{g}(y)\|^2 ds \leqslant \rho_1(T_1)\gamma(\|y\|^2),$$

(A2)

$$\begin{aligned} &\frac{1}{T_1}\left[\int_0^{T_1}\int_Z \|h(s,y,\eta) - \bar{h}(y,\eta)\|^2 \tilde{\lambda}(d\eta)ds \right. \\ &\quad \left. \vee \left(\int_0^{T_1}\int_Z \|h(s,y,\eta) - \bar{h}(y,\eta)\|^4 \tilde{\lambda}(d\eta)ds\right)^{\frac{1}{2}}\right] \leqslant \rho_2(T_1)\gamma(\|y\|^2), \end{aligned}$$

where $0 \leqslant T_1 \leqslant T$, functions $\rho_i(T_1)$ are positive bounded obeying $\rho_i(T_1) \longrightarrow 0 (i = 1,2)$ as $T_1 \longrightarrow \infty$, and the function $\gamma : [0,\infty) \longrightarrow [0,\infty)$ is non-decreasing, continuous and concave.

Theorem 3. *Assume Conditions 1–4 and Assumptions (A1) and (A2) can be fulfilled. For any given small number, $\theta_0 > 0$, there exists $P > 0$, $\varepsilon_2 \in (0,\varepsilon_1]$, such that for all $\varepsilon \in (0,\varepsilon_2]$, $t \in [0, P\varepsilon^{-1}] \subseteq [0,T]$, as follows:*

$$\mathbb{E}\|y_\varepsilon(t) - z_\varepsilon(t)\|^2 \leqslant \theta_0.$$

Proof. Owing to Condition 2, for any $u \in [0,T]$, it follows from Equations (7) and (8) that:

$$\sup_{0\leqslant t\leqslant u} \mathbb{E}\|y_\varepsilon(t) - z_\varepsilon(t)\|^2 \leqslant \frac{\sup_{0\leqslant t\leqslant u} \mathbb{E}\|y_\varepsilon(t) - z_\varepsilon(t) - (f(t,y_{\varepsilon,t}) - f(t,z_{\varepsilon,t}))\|^2}{(1-c_f)^2}, \tag{9}$$

where, by elementary inequality, obtain the following:

$$\begin{aligned} &\sup_{0\leqslant t\leqslant u} \mathbb{E}\|y_\varepsilon(t) - z_\varepsilon(t) - (f(t,y_{\varepsilon,t}) - f(t,z_{\varepsilon,t}))\|^2 \\ &\leqslant 2\varepsilon^{2H}\mathbb{E}\left\|\int_0^t T_\beta(t-s)[g(\tau,y_{\varepsilon,\tau}) - \bar{g}(z_{\varepsilon,\tau})dw^H(s)\right\|^2 \\ &\quad + 2\varepsilon\mathbb{E}\left\|\int_0^t\int_Z T_\beta(t-s)[h(s,y_{\varepsilon,s},\eta) - \bar{h}(z_{\varepsilon,s},\eta)\tilde{N}(ds,d\eta)\right\|^2 \\ &:= I_1 + I_2. \end{aligned} \tag{10}$$

We now deal with I_1, as follows:

$$\begin{aligned}
I_1 &= 2\varepsilon^{2H}\mathbb{E}\left\| \int_0^t T_\beta(t-s)[g(\tau, y_{\varepsilon,\tau}) - \bar{g}(z_{\varepsilon,\tau})]dw^H(s)\right\|^2 \\
&\leqslant 4\varepsilon^{2H}\mathbb{E}\left\| \int_0^t T_\beta(t-s)[g(\tau, y_{\varepsilon,\tau}) - g(\tau, z_{\varepsilon,\tau})]dw^H(s)\right\|^2 \\
&\quad +4\varepsilon^{2H}\mathbb{E}\left\| \int_0^t T_\beta(t-s)[g(\tau, z_{\varepsilon,\tau}) - \bar{g}(z_{\varepsilon,\tau})]dw^H(s)\right\|^2 \\
&:= I_{11} + I_{12}.
\end{aligned}$$

By Lemma 1, Conditions 1 and 4, Assumption (A1) and Jensen inequality, we obtain the following:

$$\begin{aligned}
I_{11} &= 4\varepsilon^{2H}\mathbb{E}\left\| \int_0^t T_\beta(t-s)[g(\tau, y_{\varepsilon,\tau}) - g(\tau, z_{\varepsilon,\tau})]dw^H(s)\right\|^2 \\
&\leqslant 8BH\varepsilon^{2H}u^{2H-1}\int_0^t \mathbb{E}\|g(\tau, y_{\varepsilon,\tau}) - g(\tau, z_{\varepsilon,\tau})\|^2 ds \\
&\leqslant 8BH\varepsilon^{2H}u^{2H-1}\int_0^t \mathcal{A}(s, \mathbb{E}\|y_\varepsilon - z_\varepsilon\|_s^2)ds, \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
I_{12} &= 4\varepsilon^{2H}\mathbb{E}\left\| \int_0^t T_\beta(t-s)[g(\tau, z_{\varepsilon,\tau}) - \bar{g}(z_{\varepsilon,\tau})]dw^H(s)\right\|^2 \\
&\leqslant 8BH\varepsilon^{2H}u^{2H-1}\int_0^t \mathbb{E}\|g(\tau, z_{\varepsilon,\tau}) - \bar{g}(z_{\varepsilon,\tau})\|^2 ds \\
&\leqslant 8BH(\varepsilon u)^{2H}\left(\sup_{0\leqslant t\leqslant u}\rho_1(t)\right)\gamma(\sup_{0\leqslant t\leqslant u}\mathbb{E}\|z_{\varepsilon,t}\|^2). \qquad (12)
\end{aligned}$$

Consequently, we obtain the following:

$$\begin{aligned}
I_1 &\leqslant 8BH\varepsilon^{2H}u^{2H-1}\int_0^t \mathcal{A}(s, \mathbb{E}\|y_\varepsilon - z_\varepsilon\|_s^2)ds \\
&\quad +8BH(\varepsilon u)^{2H}\left(\sup_{0\leqslant t\leqslant u}\rho_1(t)\right)\gamma(\sup_{0\leqslant t\leqslant u}\mathbb{E}\|z_{\varepsilon,t}\|^2). \qquad (13)
\end{aligned}$$

Then, by the elementary inequality, we obtain the following:

$$\begin{aligned}
I_2 &= 2\varepsilon\mathbb{E}\left\| \int_0^t \int_Z T_\beta(t-s)[h(s, y_{\varepsilon,s}, \eta) - \bar{h}(z_{\varepsilon,s}, \eta)]\tilde{N}(ds, d\eta)\right\|^2 \\
&\leqslant 4\varepsilon\mathbb{E}\left\| \int_0^t \int_Z T_\beta(t-s)[h(s, y_{\varepsilon,s}, \eta) - h(s, z_{\varepsilon,s}, \eta)]\tilde{N}(ds, d\eta)\right\|^2 \\
&\quad +4\varepsilon\mathbb{E}\left\| \int_0^t \int_Z T_\beta(t-s)[h(s, z_{\varepsilon,s}, \eta) - \bar{h}(z_{\varepsilon,s}, \eta)]\tilde{N}(ds, d\eta)\right\|^2 \\
&:= I_{21} + I_{22}.
\end{aligned}$$

By B–D–G inequality, Conditions 1 and 4, Assumption (A2) and Jensen inequality, we obtain the following:

$$\begin{aligned} I_{21} &= 4\varepsilon\mathbb{E}\left\| \int_0^t \int_Z T_\beta(t-s)[h(s,y_{\varepsilon,s},\eta)-h(s,z_{\varepsilon,s},\eta)]\tilde{N}(ds,d\eta)\right\|^2 \\ &\leqslant 4\varepsilon BC_T\mathbb{E}\int_0^t\int_Z \|h(s,y_{\varepsilon,s},\eta)-h(s,z_{\varepsilon,s},\eta)\|^2\tilde{\lambda}(d\eta)ds \\ &\quad +4\varepsilon BC_T\mathbb{E}\left(\int_0^t\int_Z \|h(s,y_{\varepsilon,s},\eta)-h(s,z_{\varepsilon,s},\eta)\|^4\tilde{\lambda}(d\eta)ds\right)^{\frac{1}{2}} \\ &\leqslant 8\varepsilon BC_T\int_0^t \mathcal{A}(s,\mathbb{E}\|y_\varepsilon-z_\varepsilon\|_s^2)ds, \end{aligned} \tag{14}$$

and

$$\begin{aligned} I_{22} &= 4\varepsilon\mathbb{E}\left\| \int_0^t \int_Z T_\beta(t-s)[h(s,z_{\varepsilon,s},\eta)-\bar{h}(z_{\varepsilon,s},\eta)]\tilde{N}(ds,d\eta)\right\|^2 \\ &\leqslant 4\varepsilon BC_T\mathbb{E}\int_0^t\int_Z \|h(s,z_{\varepsilon,s},\eta)-\bar{h}(z_{\varepsilon,s},\eta)\|^2\tilde{\lambda}(d\eta)ds \\ &\quad +4\varepsilon BC_T\mathbb{E}\left(\int_0^t\int_Z \|h(s,z_{\varepsilon,s},\eta)-\bar{h}(z_{\varepsilon,s},\eta)\|^4\tilde{\lambda}(d\eta)ds\right)^{\frac{1}{2}} \\ &\leqslant 8\varepsilon BC_T u\left(\sup_{0\leqslant t\leqslant u}\rho_2(t)\right)\gamma(\sup_{0\leqslant t\leqslant u}\mathbb{E}\|z_{\varepsilon,t}\|^2) \end{aligned} \tag{15}$$

Consequently, we obtain the following:

$$\begin{aligned} I_2 &\leqslant 8\varepsilon BC_T\int_0^t \mathcal{A}(s,\mathbb{E}\|y_\varepsilon-z_\varepsilon\|_t^2)ds \\ &\quad +8\varepsilon BC_T u\left(\sup_{0\leqslant t\leqslant u}\rho_2(t)\right)\gamma(\sup_{0\leqslant t\leqslant u}\mathbb{E}\|z_{\varepsilon,t}\|^2). \end{aligned} \tag{16}$$

Now, combining Equations (10), (13) and (16), we obtain the following:

$$\begin{aligned} &\sup_{0\leqslant t\leqslant u}\mathbb{E}\|y_\varepsilon(t)-z_\varepsilon(t)-(f(t,y_{\varepsilon,t})-f(t,z_{\varepsilon,t}))\|^2 \\ &\quad\leqslant 8B(\varepsilon C_T+H\varepsilon^{2H}u^{2H-1})\int_0^u \mathcal{A}(s,\mathbb{E}\|y_\varepsilon-z_\varepsilon\|_t^2)dt \\ &\quad\quad +8BH\varepsilon^{2H}u^{2H}\left(\sup_{0\leqslant t\leqslant u}\rho_1(t)\right)\gamma(\sup_{0\leqslant t\leqslant u}\mathbb{E}\|z_{\varepsilon,t}\|^2) \\ &\quad\quad +8\varepsilon BC_T u\left(\sup_{0\leqslant t\leqslant u}\rho_2(t)\right)\gamma(\sup_{0\leqslant t\leqslant u}\mathbb{E}\|z_{\varepsilon,t}\|^2). \end{aligned}$$

If $\mathcal{A}(t,.)$ is a concave function, then there exist $p(t)\geqslant 0$ and $q(t)\geqslant 0$, such that we obtain the following:

$$\mathcal{A}(t,y)\leqslant p(t)+q(t)y,\quad \int_0^u p(t)dt<\infty,\quad \int_0^u q(t)dt<\infty.$$

Then, we obtain the following:

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant u} \mathbb{E}\|y_\varepsilon(t) - z_\varepsilon(t) - (f(t, y_{\varepsilon,t}) - f(t, z_{\varepsilon,t}))\|^2 \\
&\quad\leqslant 8B(\varepsilon C_T u + H\varepsilon^{2H} u^{2H})(\sup_{0\leqslant t\leqslant u} p(t)) \\
&\quad\quad + 8B(\varepsilon C_T + H\varepsilon^{2H} u^{2H-1})(\sup_{0\leqslant t\leqslant u} q(t)) \int_0^u \mathbb{E}\|y_\varepsilon - z_\varepsilon\|_t^2 dt \\
&\quad\quad + 8BH\varepsilon^{2H} u^{2H}\left(\sup_{0\leqslant t\leqslant u} \rho_1(t)\right)\gamma(\sup_{0\leqslant t\leqslant u} \mathbb{E}\|z_{\varepsilon,t}\|^2) \\
&\quad\quad + 8B\varepsilon C_T u\left(\sup_{0\leqslant t\leqslant u} \rho_2(t)\right)\gamma(\sup_{0\leqslant t\leqslant u} \mathbb{E}\|z_{\varepsilon,t}\|^2) \\
&\quad\leqslant (\varepsilon C_T u + H(\varepsilon u)^{2H})\Lambda_1 + (\varepsilon C_T + H\varepsilon^{2H} u^{2H-1})\Lambda_2 \int_0^u \mathbb{E}\|y_\varepsilon - z_\varepsilon\|_t^2 dt \\
&\quad\quad + H(\varepsilon u)^{2H}\Lambda_3 + \varepsilon C_T u \Lambda_4,
\end{aligned} \tag{17}$$

where $\Lambda_1 = 8B(\sup_{0\leqslant t\leqslant u} p(t))$, $\Lambda_2 = 8B(\sup_{0\leqslant t\leqslant u} q(t))$, $\Lambda_3 = 8B\left(\sup_{0\leqslant t\leqslant u} \rho_1(t)\right)$ $\gamma(\sup_{0\leqslant t\leqslant u} \mathbb{E}\|z_{\varepsilon,t}\|^2)$ and $\Lambda_4 = 8B(\sup_{0\leqslant t\leqslant u} \rho_2(t))\gamma(\sup_{0\leqslant t\leqslant u} \mathbb{E}\|z_{\varepsilon,t}\|^2)$.

Therefore, combining Equations (9) and (17), Gronwall's inequality reads as follows:

$$\sup_{0\leqslant t\leqslant u} \mathbb{E}\|y_\varepsilon(t) - z_\varepsilon(t)\|^2 = \sup_{0\leqslant t\leqslant u} \mathbb{E}\|y_\varepsilon - z_\varepsilon\|_t^2 \leqslant \varepsilon Q_1 e^{\varepsilon Q_2 u},$$

where $Q_1 = \frac{[(C_T u + H\varepsilon^{2H-1} u^{2H})\Lambda_1 + \varepsilon^{2H-1} H\Lambda_3 + C_T u \Lambda_4]}{(1-c_f)^2}$, $Q_2 = \frac{(C_T + H(\varepsilon u)^{2H-1})\Lambda_2}{(1-c_f)^2}$.

Choose $P > 0$ such that for all $u \in [0, P\varepsilon^{-1}] \subseteq [0, T]$, we obtain the following:

$$\sup_{0\leqslant t\leqslant u} \mathbb{E}\|y_\varepsilon(t) - z_\varepsilon(t)\|^2 \leqslant \varepsilon Q_1 e^{Q_2 P} = M\varepsilon,$$

where $M = Q_1 e^{Q_2 P}$.

Hence, given any number $\theta_0 > 0$, we may obtain that there exist $\varepsilon_2 \in (0, \varepsilon_1)$, such that for every $\varepsilon \in (0, \varepsilon_2]$, $\forall\ t \in [0, P\varepsilon^{-1}] \subseteq [0, T]$, we obtain the following:

$$\sup_{0\leqslant t\leqslant u} \mathbb{E}\|y_\varepsilon(t) - z_\varepsilon(t)\|^2 \leqslant \theta_0.$$

Hence, consider the proof complete. □

Remark 5. *If $\mathcal{A}(t, v) = L(t)\bar{A}(v)$, $t \in [0, T]$, where $L(t) \geqslant 0$ is locally integrable and $\bar{A}(v)$ is a concave non-decreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$, such that $\bar{A}(0) = 0$, $\bar{A}(v) > 0$ for $v > 0$ and $\int_{0^+} \frac{1}{\bar{A}(v)} dv = \infty$, then our obtained averaging result is still new and applicable.*

5. Example

In this section, we validate the obtained averaging result with an example. Consider the following neutral Caputo fractional stochastic partial functional differential equations, and its nonlinear and stochastic terms depend on a small parameter, ε, as follows:

$$\begin{cases}
D_t^{\frac{1}{2}}[y_\varepsilon(t,x) - a_1(t)y_{\varepsilon,t}(x)] = \frac{\partial^2}{\partial x^2}[y_\varepsilon(t,x) - a_1(t)y_{\varepsilon,t}(x)] + \Gamma_t^{\frac{1}{2}}\Big[2\varepsilon^H \alpha \sin^2(s) y_{\varepsilon,s}(x)\frac{dw^H(t)}{dt} \\
\qquad\qquad + \sqrt{\varepsilon}\int_Z \eta(\cos^2(t) y_{\varepsilon,t}(x))\tilde{N}(dt, d\eta)\Big], \\
y_\varepsilon(t, 0) = y_\varepsilon(t, \pi) = 0, \quad t \in [0, \pi], \\
y_0(t, x) = \varrho(t, x), \quad \varrho(., x) \in \wp, \quad \varrho(t, .) \in \mathcal{L}_2([0, \pi]),\ t \in (-\infty, 0],
\end{cases} \tag{18}$$

with $D_t^{\frac{1}{2}}$ is a Caputo fractional derivative of order $\frac{1}{2}$, and w^H, $\frac{1}{2} < H < 1$ is a fractional Brownian motion. Let $\mathcal{X} = \mathcal{L}_2([0,\pi])$ and the operator $A = \frac{\partial^2}{\partial x^2} : \mathbb{D}(A) \subset \mathcal{X} \longrightarrow \mathcal{X}$, with domain $\mathbb{D}(A) = \{y \in \mathcal{X}; y, \acute{y}$ are absolutely continuous, $\acute{\acute{y}} \in \mathcal{X},\ y(0) = y(\pi) = 0\}$. Then, A generates an analytic compact semigroup of bounded linear operator, $(T(t))_{t\geqslant 0}$, on a separable Hilbert space, $\mathcal{X}$ which is given by the following:

$$T(t)w = \sum_{n=1}^{\infty}(w_n, e_n)e_n, \quad w \in \mathcal{D}(A)$$

where $e_n(x) = \sqrt{\frac{2}{n}}\sin(nx),\ \ 0 \leqslant x \leqslant \pi,\ n \in \mathbb{N}$.

The subordination principle of solution operator implies that A is the infinitesimal generator of a solution operator, $(T_\beta(t))_{t\geqslant 0}$. Because of the strong continuity of $T_\beta(t)$ on $[0,\infty)$ by a uniformly bounded theorem, there exist a constant $B > 0$, such that $\|T_\beta(t)\|^2 \leqslant B$, for $t \in [0,T]$. Then the system (18) can be rewritten in the abstract form of neutral Caputo fractional stochastic evolution equation, as follows:

$$\begin{cases} D_t^{\frac{1}{2}}[y_\varepsilon(t) - a_1(t)y_{\varepsilon,t}] = A[y_\varepsilon(t) - a_1(t)y_{\varepsilon,t}] + \Gamma_t^{\frac{1}{2}}\Big[2\varepsilon^H\alpha\sin^2(s)y_{\varepsilon,s}\frac{dw^H(t)}{dt} \\ \qquad\qquad + \sqrt{\varepsilon}\int_Z \eta(\cos^2(t)y_{\varepsilon,t})\tilde{N}(dt,d\eta)\Big], \\ y_0(t) = \varrho(t), \quad t \in (-\infty,0]. \end{cases} \tag{19}$$

Let

$$\bar{g}(z_{\varepsilon,t}) = \varepsilon^H\alpha z_{\varepsilon,t}, \quad \bar{h}(z_{\varepsilon,t}) = \frac{\sqrt{\varepsilon}}{2}z_{\varepsilon,t} \quad \int_Z \eta^2\tilde{\lambda}(d\eta) < \infty.$$

Then, its easy to verify that all Conditions 1–4 and Hypotheses A1 and A2 in Theorem 1 are satisfied. Hence, the averaged system for Equation (18) is given by

$$\begin{cases} D_t^{\frac{1}{2}}[z_\varepsilon(t) - a_1(t)z_{\varepsilon,t}] = A[z_\varepsilon(t) - a_1(t)z_{\varepsilon,t}] + \Gamma_t^{\frac{1}{2}}\Big[\varepsilon^H\alpha z_{\varepsilon,s}\frac{dw^H(t)}{dt} + \frac{\sqrt{\varepsilon}}{2}\int_Z \eta z_{\varepsilon,t}\tilde{N}(dt,d\eta)\Big], \\ z_0(t) = \varrho(t), \quad t \in (-\infty,0]. \end{cases} \tag{20}$$

Evidently, the simplified system (20) is much simpler than the standard one (18). Better yet, Theorem 3 guarantees that only a small error is generated in the process of substitution.

6. Conclusions

In this paper, we established the local and global existence and uniqueness problems for neutral Caputo fractional stochastic evolution equations with infinite delay driven by fBm and Poisson jumps. The results are proved by fractional calculus, successive approximations and stopping time techniques under local and global Carathéodory conditions with non-Lipschitz condition as a special case. Finally, the averaging principle for INFSEEs is studied. Furthermore, we proved that the mild solution to the simplified system converges to the mild solution to the original system in the mean square sense. Our future work will be studying the existence and averaging principle for impulsive INFSEEs driven by fBm and Poisson jumps.

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