

The Vertical Logic of Hamiltonian Methods (Part 1)

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(Dated: May 10, 2022)

We discuss the key role that Hamiltonian notions play in physics. Five examples are given that illustrate the versatility and generality of Hamiltonian notions. The given examples concern the interconnection between quantum mechanics, special relativity and electromagnetism. It will be demonstrated that a derivation of these core concepts of modern physics requires little more than an abstract classical analysis of linear Hamiltonian theory.

I. INTRODUCTION

It is a tradition to celebrate that physics is able to provide us with the deepest possible insights into the nature of reality. And it is true, of course – this is exactly what physics does. There is a tendency to think that, for providing these deep insights, physics must be in the possession of the knowledge of deep and profound principles. We shall provide examples that support a different view, namely that not everything, which is considered to be a deep physical insight, requires an explanation that stems from deep and profound principles. Deep and profound are not so much the principles but the conclusions that bright people were and still are able to draw from them.

This article has two parts. In this first part we provide some examples that illustrate, as we think, how essential concepts of modern physics can be developed 'naturally' on the basis of classical Hamiltonian methods. In part two (forthcoming), we shall discuss these findings in more depth and provide an outlook to further possible developments.

Hamiltonian notions are usually introduced to students as a reformulation of Newtonian mechanics. However, as Whittaker reported in his famous treatment on analytical dynamics¹, a theorem due to Lie and Koenigs has demonstrated that “all the differential equations which arise from problems in the Calculus of Variations, with one independent variable, can be expressed in the Hamiltonian form.”

If this result is correct, then Hamiltonian methods are mostly mathematical and do not derive from the axioms of Newtonian mechanics nor from any other metaphysical principle². Instead this theorem suggest that the Hamiltonian form has a general significance for any *thinkable* form of classical mechanics. At least to the degree to which classical mechanics is the physics of “differential equations which arise from problems in the Calculus of Variations, with one independent variable”.

Even if it might perhaps be a historical truth that Hamiltons equations of motion were found as a reformulation of Newtonian mechanics, mediated by Lagrangian

mechanics, the theorem of Lie and Koenigs demonstrates that Hamiltonian methods are more general. They can be applied in all branches of science in which dynamical systems are of interest. There are applications in thermodynamics [3, 4], but also in biology [5, 6] and epidemiology [7]. Strictly speaking, Newtonian mechanics has been falsified by Einstein's special theory of relativity (STR). Though it remains a valid approximation for small velocities, the conceptual basis of special relativity is fundamentally different from that of Newtonian physics. Every student learns “Newton's axioms”, but Newton himself called them *definitiones*. But “[...] the traditional meaning of axiom [is] a self-evident first proposition, which is neither provable nor in need of a proof” [8]. Hence we think that Newton's attitude towards time, space and motion are the real axioms of Newtonian mechanics, because these were the elements of his theory that Newton really considered to be self-evident [9]:

I do not define time, space, place and motion, as being well known to all.

Hamiltonian methods need nothing but the raw ingredients of a general dynamical system: dynamical variables, an independent variable (mostly “time”) and a constant of motion. They can be used to describe classical mechanics, but relativistic mechanics as well.

Newtons “self-evident” understanding of space and time, the *common sense interpretation*, turned out to be wrong. But also his *definitions*, which are presented to students as *axioms* or as *laws*, are not unproblematic. Henri Poincare, for instance, remarked [10]:

The Principle of Inertia – A body under the action of no force can only move uniformly in a straight line. Is this a truth imposed on the mind a priori? If this be so, how is it that the Greeks ignored it? How could they have believed that motion ceases with the cause of motion? or, again, that every body, if there is nothing to prevent it, will move in a circle, the noblest of all forms of motion? If it be said that the velocity of a body cannot change, if there is no reason for it to change, may we not just as legitimately maintain that the position of a body cannot change, or that the curvature of its path cannot change, without the agency of an external cause? Is, then, the principle of inertia, which

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¹ See page 265 in Ref. [1].

² For a discussion of the metaphysical problems related to the *principle of least action*, see for instance Ref. [2].

is not an a priori truth, an experimental fact? Have there ever been experiments on bodies acted on by no forces? and, if so, how did we know that no forces were acting?

The status of Newtons “laws of motion” is far from evident. K.R. Symon described this as follows [11]:

The status of Newton’s first two laws, [...], is often the subject of dispute. We may regard Eqs. (1.9) [$F = ma$] as defining force in terms of mass and acceleration. In this case, Newton’s first two laws are not laws at all but merely definitions of a new concept to be introduced in the theory.

Newton considered absolute space, absolute time and motion of matter in space as fundamental and self-evident notions. Einstein’s STR tells us that Newton was wrong in this respect. While his mechanics is still “valid” as an approximation for small velocities, the “absolute” notions of space and time can hardly be regarded as approximations of the “relative” notions that space and time became with the introduction of STR. One might say that the law of inertia apparently remained fully valid. But this “law” remains mysterious. It seems to claim a cause not for change, but for a resistance against change. Is such a “law” required? And what is its status? Is it a definition, an axiom or an experimental fact about nature? Poincare wrote [10]:

Has this generalised law of inertia been verified by experiment, and can it be so verified? When Newton wrote the Principia, he certainly regarded this truth as experimentally acquired and demonstrated. It was so in his eyes, not only from the anthropomorphic conception to which I shall later refer, but also because of the work of Galileo. It was so proved by the laws of Kepler. According to those laws, in fact, the path of a planet is entirely determined by its initial position and initial velocity; this, indeed, is what our generalised law of inertia requires.

According to Poincare, the essence of the law of inertia is that in mechanics, *two* initial values (per degree of freedom) are required to fully determine the path of objects. This is an intriguing view and a view that enters a new level of abstraction. It is a Lagrangian or Hamiltonian point of view.

Newtonian mechanics, though it introduced a new level of abstraction into the natural sciences, still suffered from a lack of abstraction. The notions of Newtonian mechanics are too abstract to be intuitively clear but in some sense they are not abstract enough. Newtonian mechanics presupposed specific notions like space, time and motion of massive objects as self-evident, as if these notions would somehow be the basis of any *thinkable* physical world.

Hamiltonian mechanics presupposes nothing like that. While Lagrangian mechanics derives from the beautiful but opaque “principle of least action”, Hamiltonian notions can be derived without any *specific* assumption, purely from the distinction between those physical quantities that may vary in time from those quantities that may not. The former ones are called *dynamical variables*, the latter are called constants of motion (COMs), which are the conserved quantities³. In preceding articles we argued on the basis of *pure* Hamiltonian theory, which follows from the assumption that *any* constant physical quantities with ontological content essentially derives from constants of motion [12, 13].

The examples to be discussed will illustrate that the formal constraints which are imposed on dynamical equations by Hamiltonian notions, suffice to derive the core elements of modern physics.

The first example (Sec. II) is taken from accelerator physics and concerns an application of the Hamiltonian method which documents the remarkable fact, that Hamiltonian notions emerging from apparently disconnected levels of physical description fit together seamlessly. We call these interconnections of different levels “vertical” and shall illustrate this in what follows⁴.

In the second example (Sec. III) we summarize and discuss a two-page derivation of Schrödinger’s equation from a simple classical Hamiltonian constraint on the dispersion relation [14]. In the third example (Sec. IV) we shall show that special relativity and the Dirac equation can be obtained from pure Hamiltonian concepts. The last two sections are dedicated to show how the Lorentz transformations (Sec. V) and finally Maxwell’s equations (Sec. VI) fit into (or even follow from) the sketched Hamiltonian (symplectic) framework.

II. FIRST EXAMPLE (SETUP): CYCLOTRON MOTION

Many years ago a (now retired) colleague published a paper titled “Application of the Phase Compression - Phase Expansion Effect for Isochronous Storage Rings” [15]. This is a very specialized topic, but the point we intend to make does not require deep expertise in accelerators. Consider a classical cyclotron (Fig. 1), i.e. particles in almost circular motion in a plane perpendicular to a homogeneous magnetic field B . It is well known that the motion of particles in electromagnetic fields can be derived from the classical Hamiltonian function of a point particle. This does neither exhausts the

³ It is part of the results of this article to show that also invariants like the rest mass can, on a basic level of Hamiltonian description, be described as constants of motion.

⁴ In part two of this article we shall try to expose the idea of vertical connections in part two in more mathematical detail. In this first part we proceed more intuitively.

possibilities of the Hamiltonian method nor does it solve the full problem. The solution of the equations of motion obtained from the Hamiltonian of a relativistic particle in external electromagnetic fields and yields some trajectory $(\vec{x}(t), \vec{v}(t))$, which is then used as a reference.

In order to verify the stability of a beam on the reference trajectory, accelerator physicists use the Hamiltonian techniques again to analyze the *local* behavior for all starting conditions in the *vicinity* of the reference trajectory, i.e. at $\vec{x}(0) + \delta x(t)$ and with velocity $\vec{v}(0) + \delta \vec{v}(0)$. A real particle beam will only show up if a non-vanishing vicinity of starting conditions yields trajectories that stay in the vicinity of the reference trajectory, i.e. if the “betatron motion” is oscillatory⁵. This is analyzed in the local co-moving frame, i.e. by solving the equations of motion for $(\delta \vec{x}(t), \delta \vec{v}(t))$.

The importance of this type of stability analysis in mechanics depends, of course, on the specific problem at hand. For the prediction of planetary orbits of the next century, stability analysis can almost always be omitted. But the reason to allow for this omission does not lie in the stability of planetary orbits, but in the small number of revolutions relevant to us. However, with respect to the long-term stability of a dynamical system, stability analysis is inevitably required. Optical transitions in atoms have frequencies in the range of a few hundred THz. Nonetheless most atoms⁶ are stable. This stability is, from a point-mechanical perspective, difficult to explain.

Coming back to the example: The circulation frequency ω_c of a coasting particle in a cyclotron is given by the ratio of the particle’s velocity to the length of the closed orbit. It can be fine-tuned by the value of the magnetic field at the respective radius. In so-called “isochronous” machines, the field is tuned, for instance by iron shims or by trim-coils, in order to precisely synchronize the phase ϕ between the radio frequency (rf) oscillation of the rf-electrodes, in cyclotrons traditionally called “Dees”, and the particle’s circulation frequency [17]. During the passage of the acceleration gap, the circulating particles may gain or lose energy, depending on the rf-phase ϕ at passage with $\cos \phi$. The maximal energy gain dE/dn for one turn can hence be written as

$$\frac{dE}{dn} = E_G \cos \phi, \quad (1)$$

where E is the particle’s kinetic energy, E_G the maximal energy gain per turn and n is the turn number⁷. Eq. 1 fixes the phase of maximal energy gain to be zero. Note

⁵ The orbits are stable if the “tunes” are real. In circular accelerators, the periodicity of distortions requires that more conditions have to be met to preserve stability, typically that the tunes are not integers and not in some integer relation [16].

⁶ Those with a stable nucleus.

⁷ The use of the (discrete) turn number as a continuous independent variable is called the “smooth acceleration approximation”.

that in cyclotrons, the radius R of the reference orbit rises monotonically with the kinetic energy E so that one may express, for particles in the vicinity of the reference orbit, radius by energy and vice versa. Since the phase is proportional to a time variable $\phi = \Delta\omega t$, it is the Hamiltonian conjugate of energy.

The existence of a canonical pair (E, ϕ) , formally proven by Lie and Koenigs, allows to infer that Eq. 1 is one of two Hamiltonian equations of motion and that a Hamiltonian function $\mathcal{H}(E, \phi)$ should exist such that Hamilton’s equations of motion hold true:

$$\begin{aligned} \frac{dE}{dn} &= \frac{\partial \mathcal{H}}{\partial \phi} \\ \frac{d\phi}{dn} &= -\frac{\partial \mathcal{H}}{\partial E}. \end{aligned} \quad (2)$$

The combination of Eq. 1 and Eq. 2 results in

$$\frac{\partial \mathcal{H}}{\partial \phi} = E_G \cos \phi, \quad (3)$$

the integration of which then yields

$$\mathcal{H} = E_G \sin \phi + F(E), \quad (4)$$

where $F(E)$ is an integration “constant”⁸ and describes the radial phase shift by the radial profile of the (static) magnetic field as we described before. Hence $F(E) = 0$ holds in a perfectly isochronous machine. However, inserting Eq. 4 into the second of Eq. 2 yields another non-zero phase shift

$$\frac{d\phi}{dn} = -\frac{\partial E_G}{\partial E} \sin \phi. \quad (5)$$

which, because of its phase dependency, must be related to the rf-acceleration. The maximum energy gain E_G is equal to the particle’s charge multiplied by the maximal Dee voltage $V(R)$ which may (but does not have to) depend on radius (and hence on energy):

$$E_G = qV(R) \quad (6)$$

Inserting this into Eq. 5 results

$$\frac{d\phi}{dn} = -q \frac{dV}{dR} \frac{dR}{dE} \sin \phi. \quad (7)$$

The term $\frac{dR}{dE}$ is called radius gain with energy and can be derived from $p = RqB$. Since p rises monotonically with energy, $\frac{dR}{dE}$ is a monotonic relationship.

Hence there is – even in a perfectly isochronous magnetic field – the possibility for a non-zero phase shift per turn if the accelerating voltage depends on the radius.

⁸ $F(E)$ does not depend on ϕ and is hence a constant with respect to partial differentiation.

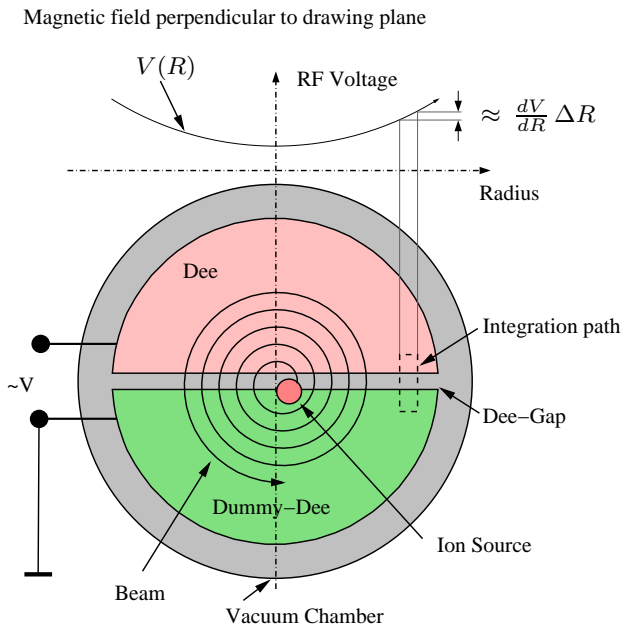


FIG. 1. Classical cyclotron. Particles are accelerated when passing the gap between the Dee and the (grounded) dummy-Dee. An example for a radius dependent gap voltage $V(R)$ is plotted above the sketched cyclotron. The integration path of Eq. 8 is shown as a dashed rectangle.

This dependency can otherwise only be derived using Maxwell's equations [15]:

$$\begin{aligned} \partial_t \int \vec{B} d\vec{A} &= - \int \vec{\nabla} \times \vec{E} d\vec{A} = - \oint \vec{E} d\vec{s} \\ B_{\text{rf}} g &= - \frac{dV}{dR} \frac{\sin(\phi)}{\omega_{\text{rf}}}, \end{aligned} \quad (8)$$

where g is the gap distance. Hence a voltage gradient is necessarily accompanied by a non-zero magnetic high-frequency field. The integration area (integration path) is shown in Fig. 1. The electric field inside the Dees vanishes so that the right side of Eq. 8 is proportional to the voltage difference between the corresponding radial positions when the integration passes the Dee gap. The causal explanation is as follows: according to Maxwell's equations, the gradient of the oscillating rf voltage is accompanied by an oscillating axial magnetic field, the average of which is proportional to $\sin \phi$ as seen by the particle along its orbit. This rf contribution to the magnetic field *causes* a horizontal kick that changes the orbit length and therefore results in a phase shift⁹.

But how is it possible to derive the same result from a Hamiltonian that did not refer in any obvious way to the causal story? One might suspect that we “somehow” smuggled high-level knowledge (some content of

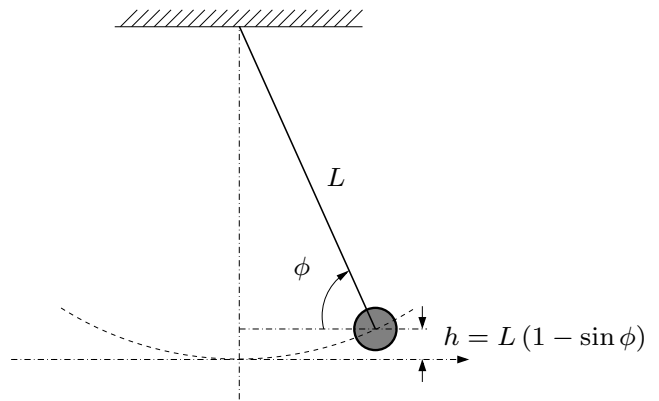


FIG. 2. Classical pendulum. The potential energy is $mgh = mgL(1 - \sin \phi)$.

Maxwell's equations) into EQ. 4. There is no doubt that a cyclotron is a device that can only be constructed with sufficient knowledge of Maxwell's equations. But EQ. 4 is, up to a sign, identical to the momentum change in a classical mechanical pendulum as depicted in Fig. 2. The equation of motion for the momentum is

$$\dot{p} = -mg \cos \phi, \quad (9)$$

which is (up to a sign) formally identical to EQ. 4 for the presumed condition $F(E) = 0$. The only difference lies in the fact that the mass m and the gravitational constant g do (“classically”) not vary with the momentum p . So if ϕ is (proportional to) the canonical conjugate of p , i.e. $x = \varepsilon\phi$, then the presumed Hamiltonian equation

$$\dot{p} = - \frac{\partial \mathcal{H}(p, \phi)}{\partial(\varepsilon\phi)} = -mg \cos \phi, \quad (10)$$

leads to

$$\mathcal{H}(x, p) = -\varepsilon mg \sin \phi + T(p) + C \quad (11)$$

and results in

$$\dot{x} = \frac{\partial \mathcal{H}(p, \phi)}{\partial p} = \frac{dT}{dp}. \quad (12)$$

and eventually with the correct value of the constants $\varepsilon = L$ and $C = Lmg$ to

$$\mathcal{H}(x, p) = mgL(1 - \sin \phi) + T(p) = mgh + T(p) = \text{const} \quad (13)$$

It is hardly possible to defend the idea that an equation of motion of the form of EQ. 4 encodes high-level knowledge about electromagnetism. If it would, why does the same equation describe a simple pendulum where electrodynamics can safely be ignored?

The analytical (and predictive) power of the Hamiltonian method is remarkable and far more versatile and general than Newtonian notions. Nonetheless the physics curriculum, sticking to an historical account of classical

⁹ For further details of the calculation see Ref. [15]. For 3D electromagnetic modeling see also Ref. [18].

mechanics, teaches Hamiltonian mechanics as secondary, as something which somehow derives from Newtonian mechanics.

This has possibly its legitimization in pedagogical reasons, but implies that one derives a correct and powerful method, namely Hamiltonian mechanics, not from basic *mathematical* principles, but instead from a falsified and phenomenological theory, namely Newtonian mechanics. Consequently every physics student has to un-learn (Newtonian based) and re-learn (Hamiltonian based) physics in order to understand the logic of quantum theory and (special) relativity.

Hamiltonian notions are used in various levels of description, often treated separately in separate branches of physics. There is a Hamiltonian for the Dirac particle, which is then sometimes replaced by a relativistic point particle Hamiltonian, or by a Schrödinger Hamiltonian for an orbital electron, then comes the Hamiltonian describing the inter-action that governs ionic binding in crystals, then some kind of classical Hamiltonian that describes the motion of the crystal being a grain of dust in space and so on. In reality all these levels are interconnected, even though they are treated in separate physics textbooks, books on quantum electrodynamics and quantum mechanics (QM), atomic physics, solid state physics and finally the grain of dust by classical mechanics or astrophysics. There is no universally accepted coherent theoretical account known to the author describing the effects which physical constraints resulting from one level of description have on other levels. From a birds eye view, physical reasoning is mostly horizontal, i.e. it stays within one level. Little is known about the general patterns and the “vertical” interconnection between different levels of Hamiltonian description.

The above example illustrates that we might in practice, when solving daily problems, for instance in accelerator physics, take a logical coherence of different levels for granted which is mostly ignored in the usual presentation of physics.

III. SECOND EXAMPLE: SCHRÖDINGER'S EQUATION

The problem is not the interpretation of quantum mechanics. That's getting things just backwards. The problem is the interpretation of classical mechanics.

– *Sidney Coleman* [19]

In a preceding article we gave a short derivation of Schrödinger's equation [14], which we shall briefly summarize and discuss in this section.

Assumed we would, for whatever reason, refuse the notion of the point particle and replace it with a (classical) density distribution $\rho(t, \vec{x})$ in space and time. If the density is supposed to represent a finite amount of matter,

it must of course be normalizable:

$$\int \rho(\vec{x}, t) d^3x = 1. \quad (14)$$

The density must furthermore be positive semi-definite $\rho \geq 0$ ¹⁰ quantity. This latter condition can be automatically fulfilled, if one expresses ρ by (possibly a sum of) squares of some auxiliary function $\psi(\vec{x}, t)$. Then ψ is, due to Eq. 14, square-integrable so that its Fourier transform exists and is well-behaved. The only reason and objective to use ψ is to replace ρ by a positive semi-definite expression, so that we are free to opt for a complex ψ , mostly for reasons of elegance and simplicity of the following Fourier expressions. Hence we write the density as¹¹

$$\rho = \psi^\dagger \psi. \quad (15)$$

Then the Fourier transform can be written as

$$\psi(t, \vec{x}) = \frac{1}{(2\pi)^2} \int \tilde{\psi}(\omega, \vec{k}) \exp[-i(\omega t - \vec{k} \cdot \vec{x})] d^3k d\omega. \quad (16)$$

The translation of normalizability and positive definiteness into square integrability directly leads to the space of L^2 -normalizable functions, i.e. to a Hilbert space. In a Hilbert space it is allowed to freely select a basis which best suits to solve the problem. The use of canonical transformations has the exact same purpose: To find a transformation to new variables in which the solution is possible or - in the best case - trivial. The use of the Fourier transform is hence nothing but the choice of a convenient basis. Since it is a reversible transformation, no information can get lost by its application.

Physics textbooks introduce wave packets as something alien to classical physics which can only be legitimized empirically but not theoretically. It is part of this lore that the interference experiments with particles generate the surprising and counter-intuitive *necessity* to introduce a mechanical theory based on waves. But this is not the only possible view and not the only possible motivation to consider the use of wave packets in order to represent matter densities: any *normalizable* semi-positive density can be represented by the square of a complex wave function, i.e. by a “wave-packet”, no matter if one regards the idea to *use* a wave packet representation as (a consequence of) an experimental discovery or merely as an arbitrary mathematical transformation.

The description of a density distribution by a wave packet is, as we have just argued, little more than a free choice of basis in Hilbert space. We might for instance

¹⁰ These requirements are incidentally the same as one would demand for a probability distribution.

¹¹ This choice obviously introduces a “hidden variable”, namely the phase of the complex function ψ , which can not be obtained by a measurement of ρ .

be motivated by the fact that the notion of the “point particle” is not an uncontroversial element of classical thought. Maybe we just like to spell out the consequences of a purely mathematical game. Or we might be driven by the question, why the transmission of physical matter through space (i.e. “motion”) should be described by different rules of local causality compared to the transmission of energy by waves¹²: Why would nature invent two types of motion? And if it would, why then should they be related by a constant velocity of light, which builds the bridge between relativistic mechanics (particle motion) and electrodynamics (i.e. wave motion)?

It is another math fact that, provided the frequencies ω of the partial waves are related to the wavelength by some relationship $\omega = \omega(\vec{k})$, then the “velocity” of the wave packet is in linear approximation given by the group velocity¹³:

$$\vec{v}_{gr} = \vec{\nabla}_k \omega(\vec{k}) = \left(\frac{\partial \omega}{\partial k_x}, \frac{\partial \omega}{\partial k_y}, \frac{\partial \omega}{\partial k_z} \right)^T. \quad (17)$$

As a matter of fact, the velocity of a classical particle is given by an expression of the exact same mathematical form: Hamilton’s equations of motion (EQOM) for the coordinate velocity is

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}. \quad (18)$$

In vector notation, this reads

$$\vec{v} = \vec{\nabla}_p \mathcal{H}(\vec{p}, \vec{x}). \quad (19)$$

Hence the motion of the matter density ρ can only be consistent with (Hamiltonian) mechanics, if both velocities agree:

$$\vec{\nabla}_{\vec{k}} \omega(\vec{k}) = \vec{\nabla}_{\vec{p}} \mathcal{H}(\vec{p}, \vec{x}), \quad (20)$$

Since both descriptions are of full generality, Eq. 20 must hold in any (Euclidean) space-time of any dimension. The general equivalence of dispersion relations and Hamiltonians is well-known, for instance in geometrical optics [24]. Hence this equivalence is just a math fact applicable to any linear physical wave motion. It follows, by (partial) integration, that

$$\begin{aligned} \vec{k} &\propto \vec{p} + \varepsilon \vec{A}(\vec{x}) \\ \omega(\vec{k}) &\propto \mathcal{H}(\vec{p}, \vec{x}) + \varepsilon \phi(\vec{x}) \end{aligned} \quad (21)$$

where the additive “constants” ϕ and \vec{A} are, in general, functions of the variables which are conjugate to the mechanical momentum, i.e. the spatial position \vec{x} ¹⁴.

Therefore, given we use the conventional units of energy and frequency, some constant conversion factor must be introduced that allows to express energies in units of frequency and momenta in units of wavelength. This factor is usually represented by the symbol \hbar and its value is solely determined by the historical choice of units.

Hermann Weyl explained in 1930 that the significance of the two constants \hbar and c is, from a logical viewpoint, equivalent – they are both scaling constants which can not be obtained theoretically [25]:

The constants c and \hbar , the velocity of light and the quantum of action, have caused some trouble. The insight into the significance of these constants, obtained by the theory of relativity on the one hand and quantum theory on the other, is most forcibly expressed by the fact that they do not occur in the laws of Nature in a thoroughly systematic development of these theories.

The impossibility to obtain the value of these constants theoretically is due to the fact that both are the result of a contingent (historical) choice of units.

The appearance of the action constant is often *interpreted* as if it would provide evidence that energy, by means of some mysterious logical leap, is “chunked in portions”. But as we have just seen, the appearance of an action constant can be derived from a small number of mathematical assumptions: it is mostly the mathematical consequence of a continuity assumption, namely that matter should fill out space continuously, which is quite opposite of “chunking”. Eq. 20 reveals how and why the introduction of an universal action constant at some “fundamental level” leads to a kind of *wave-particle correspondence*: In order to obtain a wave-packet, the center of which moves with a *variable* velocity, we need to equate both, the dispersion relation of the wave and the velocity as defined by the Hamiltonian formalism. At first sight, it does not seem to establish a “duality” of physical pictures.

We admit that the density $\rho = \psi^* \psi$ can not simply be reinterpreted as a charge density in the sense of classical electrodynamics. Firstly, there was no mention of a charge to this point. And secondly, the (self-) energy of a charge distribution would have to be directly related to the (spatial) size of the wave-packet. The above reasoning, however, suggests instead to equate energy to frequency at a fundamental level. In the words of Zeh [26]:

In this formulation of quantum theory by means of wave functions, Planck’s constant is

¹² This latter question receives full significance only in context of Einstein’s theory of relativity, i.e. with the insight that also mass is a form of energy.

¹³ The well-known linear expressions for the group velocity have first been given by Sir W.R. Hamilton (sic!) in Ref. [20]. See also Refs. [21–23].

¹⁴ We shall come back to these quantities in the last example, in Sec. VI.

not used primarily to define discrete quantities ("quanta"), but rather as a scaling parameter, required to replace canonical momenta and energies by wave lengths and frequencies, respectively, – just as time is replaced by length by means of the velocity of light in the theory of relativity.

In any case the above Ansatz directly leads to Schrödinger's equation which is known to give the correct physical results, while the idea of a point particle leads literally to nothing but problems of both, mathematical and meta-physical nature [27, 28].

What would be the "classical" mathematical alternative to Eq. 20? What would we have to presume about the partial waves in order to end up in a distribution of constant shape (DCS), which might be closer to a "classical vision" of a particle as some kind of rigid tiny billiard ball, as suggested by Newton in his treatise on optics [29]? What kind of (linear) wave equation is able to produce such a DCS? It is well known that any function of the form $f(x - v_{ph} t)$ solves the homogeneous wave equation $(\partial_t^2 - v_{ph}^2 \partial_x^2) f = 0$ and produces a DCS, provided that $v_{ph} = \text{const.}$ But this kind of wave equation with constant (phase-) velocity represents a physical situation in which the velocity is a property of the transmitting medium and not of the moving "particle". There was no word of such a medium. The desired dispersion does not derive from the physical properties of some medium: the whole Ansatz derives from little more than math facts. Therefore Schrödinger's theory can be regarded as a *general and abstract* theory of motion for continuous distributions. Apparently it does not represent the motion of a specific kind of matter, but of any *fundamental* kind of matter. Here "fundamental" means little more than "not specific". We presumed that the matter-density "represents" a "particle", i.e. an object which appears to an observer to be fully describable by its mass, position and momentum.

It is part of the conventional lore to refuse or even ridicule the very idea that \hbar could be rationalized (to avoid the term "derived") on classical grounds. The curriculum taught us to accept \hbar as a joke of nature, unexplainable by classical rational reasoning. This narrative suggests that the classical worldview, though it appears to be entirely rational and logical, is 'de facto' wrong and that quantum physics, though 'de facto' correct, is and will always remain deeply irrational. But is this fatalistic conclusion, which must eventually shatter the foundations of enlightenment, namely *reason*, and the scientific enterprise itself, really unavoidable?

There is an oddity related to the often repeated claim that classical mechanics would necessarily presume point particles: the mathematical point-mass is among the most discontinuous ideas to be found in physics, hence in maximal contradiction with a central claim of classical thought, namely that nature does not jump (*natura non facit saltus*). Therefore a fully self-consistent *classical* ontology would have to abandon the gospel of the

point particle and to consider a smooth (differentiable) distribution of matter. This accepted, the above reasoning shows that one can rationalize the action constant by applying the idea of continuity to the notion of a classical "particle": It is little more than a mathematical expression of a smoothly localized substance moving with variable velocity.

Intuitively it seems evident that some law connecting frequency and wave length must indeed exist in nature since otherwise traveling waves, i.e. local and causal signal transport could not exist. But we do not have to refer to intuition alone, since Toll provided a rigorous general proof of the logical equivalence of causality and dispersion[30]:

[...] the logical equivalence of strict causality and a dispersion relation can be expected in any problem in which an "output" function is related to a freely variable "input" by a linear, bounded, time-invariant connection. From the invariance of the connection under time displacement, it follows" that each frequency component is mapped onto itself with only a change in magnitude and phase.

Hence, to the degree that particle motion is a causal process in time, there must be a dispersion relation connected to it. Consequently the so-called wave-particle "duality" is not (only) a mysterious and alienating property of nature but first of all it is an unavoidable consequence of math facts about causality and continuity. There is no need to regard it as forced upon the theory exclusively by experimental results: It is a math fact that such a representation must exist.

It might be surprising and the consequences difficult to understand, but it can be derived from classical assumptions and mathematical logic. Therefore the choice to use (an ensemble of) waves to represent matter density is not quite as weird as usually claimed: not only does it allow to cast spatial causality into mathematical form, it might even be the only way to do so under the presumed conditions. The dispersion relation $\omega(\vec{k})$ is, simply by its mathematical form, a Hamiltonian function [24]:

The key to geometric optics is the dispersion relation, which acts as a hamiltonian for the propagation.

Hence the "ensemble" of partial waves corresponds *by mathematical form* to an ensemble of "points" in an abstract Hamiltonian phase space. Hence here we have the connection of two Hamiltonian descriptions: The Hamiltonian of a classical particle, the velocity of which must agree with another Hamiltonian process, namely the Hamiltonian motion of the real and imaginary wavefunction components [31].

Mara Beller criticized the rhetoric of "inevitability" of the founding fathers[32]. And she had a point insofar as the founding fathers claimed inevitability but did not elaborate on it. Hamiltonian notions provide some

evidence that Schrödinger's equation has indeed an kind of mathematical inevitability. It results from spelling out some simple basic principles mathematically, namely continuity, causality and *extension*. Energy quantization is then not a principle at the foundation, it emerges from the solutions of Schrödinger's equation.

This analysis reveals that the invention of an action constant neither stems necessarily from discontinuity nor does it contradict causality. In the contrary, we used both arguments to demonstrate the “deeper” logic in Schrödinger's equation that presents itself in our view as a theory of motion for smooth “ensembles” (i.e. distributions). There is no “chunking” of energy but simply a unit conversion factor indicating the equivalence of frequency and energy on this supposedly fundamental level. As Hermann Weyl pointed out, it has the same significance as the speed of light which implies that it can, using a natural system of units, it is superfluous. Also John Ralston emphasized that quantum mechanics can be formulated in a way such that \hbar is absent and that this approach has many advantages [33].

The true origin of discreteness is not \hbar , but was hidden in the initial assumption: “Assumed we would, for whatever reason, refuse the notion of the point particle and replace it with a (classical) density distribution $\rho(t, \vec{x})$ in space and time.” We not only presumed a distributed density, but we presumed that this density should represent *one* particle, i.e. we presumed a *normalization to a discrete number, i.e. to one*. From the perspective of classical continuum mechanics, the normalization postulate appears as “non-classical”: Why should the some density represent exactly “one particle” and not some arbitrary amount of matter or energy? On the other hand one can not avoid to admit that Newton did exactly the same when he presumed finite discrete particles in his “Opticks” [29]

[...], it seems probable to me, that God in the Beginning form'd Matter in solid, massy, hard, impenetrable, moveable Particles, of such Sizes and Figures, and with such other Properties, and in such Proportion to Space, as most conduced to the End for which he form'd them; and that these primitive Particles being Solids, are incomparably harder than any porous Bodies compounded of them; even so very hard, as never to wear or break in pieces; no ordinary Power being able to divide what God himself made one in the first Creation.

Hence the postulate of discrete particles is genuinely classical. It goes back, at least, to the atomistic school of the ancient greek philosophers. Hence the invention of “quanta” can be regarded as a mathematically matured version of classical atomistic theory. The discreteness is not a consequence of \hbar . It is more the other way around: The necessity for \hbar is a consequence of the discreteness, which was presumed by identifying *one* group

velocity with the Hamiltonian equation of motion of *one* particle¹⁵.

Precise knowledge of the form of the Hamiltonian function $\omega(\vec{k})$ is, at this point, not required but will be (re-) constructed in the third example. Note that our second example is similar to the first example in an important aspect: In the setup example, only half of Hamilton's equation of motion was given and the mere presumption of the validity of Hamiltonian notions enables to derive results that can otherwise only be obtained from a more general (and apparently distinct) theory, namely Maxwell's electrodynamics. Again one obtains a physically correct result while it remains somewhat unclear why this is so.

From Eq. 21 one obtains, for the field free case ($\phi = 0 = \vec{A}$), the de Broglie relations:

$$\begin{aligned}\mathcal{E} &= \hbar\omega \\ \vec{p} &= \hbar\vec{k}.\end{aligned}\tag{22}$$

This allows to write

$$\psi(t, \vec{x}) \propto \int \tilde{\psi}(\mathcal{E}, \vec{p}) \exp[-i(\mathcal{E}t - \vec{p} \cdot \vec{x})/\hbar] d^3p d\mathcal{E},\tag{23}$$

which then leads to the “canonical” quantization relations

$$\begin{aligned}\mathcal{E} &\rightarrow i\hbar\partial_t \\ \vec{p} &= -i\hbar\vec{\nabla}.\end{aligned}\tag{24}$$

If we apply Newton's energy-momentum-relation (EMR) $\mathcal{E} = \vec{p}^2/(2m)$ for a free particle, then Schrödinger's equation of a free particle pops out:

$$i\hbar\frac{\partial}{\partial t}\psi(t, \vec{x}) = -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi(t, \vec{x}).\tag{25}$$

Combining this with the classical potential energy (density) $\rho(t, \vec{x})V(\vec{x})$ yields Schrödinger's equation of a particle in some external potential $V(x)$:

$$i\hbar\frac{\partial}{\partial t}\psi(t, \vec{x}) = \left(-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{x})\right)\psi(t, \vec{x}).\tag{26}$$

Again, though the Hamiltonian formalism involves a causality requirement, it is difficult to understand why Eq. 20 is in fact more than a “mathematical trick”. Nonetheless only a single physical (Hamiltonian) constraint, Eq. 20, was required to arrive at Schrödinger's equation. All other steps follow (auto-/mathe-)matically¹⁶.

¹⁵ We smuggled another, inherently Hamiltonian method, into the derivation, namely the use of complex numbers. We shall come back to this.

¹⁶ In the established nomenclature of quantum physics, the “opera-

A. 2nd Example, Aftermath

The second example contains little that can not be found in standard textbooks on QM. We only changed the narrative, the presentation of the math. This change did not even concern the interpretation of essential difficulties like the “measurement problem”, but only our attitude towards the mathematics of the classical theory. The major difference has been implemented by the initial sentence “Assumed we would, for whatever reasons, reject the notion of a classical point particle and replace it with a (classical) density distribution [...]”.

Some textbooks on QM, for instance Messiah’s [34] as well as Weinberg’s [35] and Schiff’s [36] refer to the group velocity (Eq. 17) and the corresponding Hamiltonian expression (Eq. 19). But all omit to directly derive Schrödinger’s equation this way. It is interesting to see what they do instead. Messiah first introduces both velocities and writes (page 52): “From the condition $v = v_g$ and from relation (I.2) one obtains the de Broglie relations.” On page 55 he provides another analogy to classical mechanics based on Fermat’s principle. But then, on page 61, one reads the following sentence about the possibility to derive Schrödinger’s equation:

It is quite clear that no deductive reasoning can lead us to that equation. Like all equations of mathematical physics it must be postulated and its only justification lies in the comparison of its predictions with experimental results.

On the same page, Messiah continues to provide three more conditions that the desired equation must obey, namely a) linearity and homogeneity, b) first order in time and c) agreement with classical mechanics. On the same page, he then writes:

All these conditions lead us to the Schrödinger equation in a natural way.

With all due respect¹⁷, but these passages send an inconsistent message: on the one side, we are lead “in a natural way” to Schrödinger’s equation but, on the other side, it can only be postulated, for reasons that are “quite clear”. Whatever Messiah considered to be quite clear,

tor” on the right of Eq. 26 is called “Hamiltonian” and often the word *operator* is dropped. Since the nomenclature of QM dominates contemporary physics, we emphasize that we use “Hamiltonian” here in the classical sense: We refer either to Hamiltonian *functions* or, in Sec. IV, to Hamiltonian *matrices*. The concept of Hamiltonian *operators* emerges from the above formalism, but is not necessarily fundamental. It is required to *apply* quantum mechanics (QM), but not for its derivation.

¹⁷ It is not my intention to criticize Messiah (or Weinberg) specifically. Many “modern” textbooks on QM are *de facto* committed to the view that one can not (and/or should not) make quantum theory plausible at all.

apparently it was not so clear to those authors who presented derivations of Schrödinger’s equation¹⁸ and it apparently was not clear to P.A.M. Dirac either. Dirac even thought it would have been possible that Hamilton himself could have possibly stumbled upon quantum mechanics [63]:

Now Schrödinger’s theory is connected with Hamilton’s second development, the introduction of the families. Schrödinger’s wave function is related to Hamilton’s principal function S . In first approximation This result has great physical significance. It means that an atomic state corresponds, not to an individual solution of the classical equations of motion, but to a family of solutions. The families, which were a mathematical curiosity at the time of Hamilton, are now seen in their true importance. Schrödinger’s form of quantum mechanics may be looked upon as a wave mechanics which is a natural generalisation of Hamilton’s theory of families, in the same way in which wave optics is a generalisation of geometrical optics. If Hamilton had known there was any need to generalise the mechanics of his time, he might have made this step, just by following the optical analogy, and so have discovered quantum mechanics.

Also Budiyono and D. Rohrlich seem to promote a position quite different from that of Messiah [57]:

We cannot fully explain how the theories [quantum and classical mechanics] differ until we can derive them within a single axiomatic framework, allowing an unambiguous account of how one theory is the limit of the other.

Also Weinberg gives a “historical” introduction and mentions both, group velocity and the equivalence with Hamiltonian mechanics on page 14. But also Weinberg does not use these relations to derive Schrödinger’s equation. Though, according to Weinberg (page 21), “Schrödinger showed how the principles of matrix mechanics can be derived from those of wave mechanics.” he favors a different approach and writes (page 23): “The approach that will be adopted when we come to the general principles of quantum mechanics in Chapter 3 will be neither matrix mechanics nor wave mechanics, but a more abstract formulation, that Dirac called transformation theory, from which matrix mechanics and wave mechanics can both be derived.” Again students are left with the impression that Schrödinger’s equation is somehow important but also somehow impotent.

¹⁸ Some examples for such derivations are to be found in Refs. [37–62]. This list is long but not exhaustive.

Schiff's book also refers to Eq. 20 (Ref. [36], page 17), but speaks of the "plausibility" that the de Broglie's relations receive by it and that there is "agreement" found between the group velocity and classical mechanics due to Eq. 20. Again there is no hint that one could reverse the argument and derive Planck's constant, de Broglie's famous relations and Schrödinger's equation altogether from the "assumption" that physical "particles" can't *really* be point-like.

Though physicists are usually solicitous to present their science as a deductive enterprise, the list of authors that are (supposedly) able but (apparently) unwilling to derive Schrödinger's equation does not stop here. Cohen-Tannoudji, Diu and Laloë frankly admitted their lack of interest [64]:

It is possible to introduce it in a very natural way, using the Planck and de Broglie relations. Nevertheless, we have no intention of proving this fundamental equation, which is called Schrödinger Equation. We shall simply assume it.

We have no idea why the authors of a textbook on quantum mechanics could have *no intention* to derive Schrödinger's equation other than that they lost confidence in the value of deductive reasoning.

Then Messiah surprises (page 6) with the assertion that "the desire to unify the various branches of their science has always been one of the most fruitful preoccupations of the physicists", but neither his nor other textbooks on quantum theory provide evidence that this "preoccupation" is more than lip-service when it comes to the question whether one could unify classical and quantum notions. In the contrary, many textbooks express the pre-occupation that these notions can by no means be unified¹⁹.

There are reasons to doubt that "point particles" have *ever* been an uncontroversial ontological element of classical thought. In the contrary, a classical lore would rather describe material objects as *res extensa*, i.e. as objects with extension. Of course, it is true that textbooks on classical analytical mechanics *describe* particles *as if* they had no other properties but a "definite" position, momentum, mass and possibly charge. This approximation was extremely successful and we agree with F. Rohrlich who wrote[66]:

[...] the greatness of these people lay exactly in that fact: that they were not deterred by objections no matter how serious they seem to be. Lesser scientists may not have dared to proceed in this way.

But this does not imply that there was agreement among physicists of the classical era that the point particle

method implies any *ontological* commitment, namely that those physicists would have subscribed to the idea that a particle *is* a mathematical point, i.e. an object without volume. Pre-quantum physicists merely did, what Rohrlich praised so emphatically: They proceeded despite all known difficulties.

One of the classical natural philosophers who supported (if not introduced) the gospel of the material point was Boscovich, a Serbian Jesuit scientist. Boscovich explicitly postulated material objects without extension, likewise called "material points" or "mass points" [67]. But these ideas were far from being accepted mainstream in classical physics. As Glazebrook reported, James Clerk Maxwell was one of the prominent figures who rejected Boscovich's view [68]:

We make no assumption with respect to the nature of the small parts – whether they are all of one magnitude. We do not even assume them to have extension and figure. [...] The simplest theory we could formulate would be that the molecules behaved like elastic spheres, and that the action between any two was a collision following the laws which we know apply to the collision of elastic bodies.

But Maxwell was by far not the only one who refused any ontological claim about the "nature of the small parts".

How about the curriculum? Let us take a look at some arbitrarily selected²⁰ classical textbooks on analytical mechanics. In Peck's *Elementary Treatise on Analytical Mechanics* from 1887 we find on page 9 [69]:

A body is a collection of material particles. A body whose dimensions are exceedingly small is called a material point. In what follows the term point will generally be used in this sense.

Note that Peck wrote *exceedingly small*, but not *point-like*. Twisden explained in 1874 [70]:

A limited portion of matter is called a body e.g. a lump of lead is a body. When a body is so small that for the purposes of any discussion the relative positions of its parts need not be considered, it is spoken of as a material pointy or a heavy point, or simply as a point.

In Love's textbook on Dynamics, from 1906, we can read [71]:

We have said that our object is the description of the motions of bodies. The necessity for a simplification arises from the fact

¹⁹ Sean Carroll even suspected that physicists actually might *not want* to understand quantum theory [65].

²⁰ Most can be found on the internet archive: <https://archive.org/>.

that, in general, all parts of a body have not the same motion, and the simplification we make is to consider the motion of so small a portion of a body that the differences between the motions of its parts are unimportant. How small the portion must be in order that this may be the case we cannot say beforehand, but we avoid the difficulty thus arising by regarding it as a geometrical point. We think then in the first place of the motion of a point.

Bartlett wrote 1873 [72]

A Primary Property is that without which the existence of the body cannot be conceived. There are two of these - Extension and Impenetrability.

In Michie's textbook from 1887 we read on the first page [73]:

Of the ultimate nature of matter we are ignorant; but from close observation of natural laws it has been assumed: (1) That every material substance is composed of one or more *simple substances* or *elements*, so called because they have thus far resisted simplification by subdivision. (2) That each of these simple substances is composed of very minute, but finite and definite, portions, called *atoms*. (3) That in any substance, simple or compound, two or more of these atoms are, in general, so united as to form the smallest portion that can exist by itself and remain the same substance. This combination of atoms is called a *molecule*.

No mention of point particles. Dühring wrote 1873 (page 8)²¹:

First of all it is the concept of the *centre of gravity* that forms the starting point for the theoretical investigation

Dühring refers to a point in space, namely the *center of gravity*. But of course it would be meaningless to speak of the center of gravity of point particles: A point *has* no center, a point *is* a center. The notion of the *center of gravity* makes only sense, if an object has spatial extension: Analytical mechanics refers to the center of gravity because this is a successful *analytical method*, not because classical physicists is "intrinsically" committed to some point-particle-ontology.

²¹ "Vor Allem ist es der Begriff des Schwerpunkts, der den ersten Ausgangspunkt für die rein theoretischen Untersuchungen bildet." [74].

Wilhelm Schell wrote explicitly (1870) that²²:

[...] it must be noted that the objects of mechanics, as they are described here, are creations of *thought* like geometric figures and that we have to carefully consider in each single case of application of mechanics to situation of the physical world to which degree it is legitimate and accurate to approximate the reality of physical bodies by a system of one type or another.

M. Abraham suggested in an article with the title "basic assumptions of electron theory" in 1904[76] to understand an electron as a rigid body of finite size²³:

F. The electron is not able to change form.
G. It is a sphere with homogenous volume or surface charge.

Again there is no mention of an electron as being a "material point".

This list is not exhaustive and there might be counter-examples of physicists who were willing to follow Boscovich. Indeed many textbooks on classical mechanics, notably the most "modern" ones, do not define nor discuss the notion of the material point or point particle *at all*, but omit any discussion about the value and limits of the mathematical point as a representation of "material bodies". If they mention these issue at all, then it is usually asserted that "below a certain radius", somehow, classical reasoning has to end and quantum theory takes over. However, as we have shown, it is mostly the point-particle-approximation that reached it's limit of validity.

Physicists of the nineteenth century however, right before (and also after) Planck's "quantum-hypothesis", apparently were aware of the fact that classical mechanics offered no understanding or explanation of the nature of the presumed "smallest parts"; even Boltzmann's kinetic theory of gases was criticized because it required atoms to exist. The nature of "particles" was simply an unexplored area. But if the known concepts of mechanics did not offer a solution to the particle problem, why should it then be a big surprise if it turns out that the old concepts were indeed incomplete and insufficient? They were known to be incomplete and insufficient at all times.

²² "Im Uebrigen muss bemerkt werden, dass die Gebilde der Mechanik, wie sie hier aufgefasst werden, nur gedachte Dinge sind, wie die geometrischen und dass bei der Anwendung der Mechanik auf Vorgänge der physischen Welt in jedem einzelnen Fall sorgfältig zu prüfen ist, mit welcher Berechtigung und mit welchem Grade der Annäherung an die Wirklichkeit man einen physischen Körper als ein materielles System der einen- oder andern Art ansehen darf." [75]

²³ Original in German: "F. Das Elektron ist einer Formänderung überhaupt nicht fähig. G. Es ist eine Kugel mit gleichförmiger Volum- oder Flächenladung." [76].

Hence there never really was a general ontological commitment of classical analytical mechanics to point particles. Physicists mostly regarded the use of point particle merely as a method to simplify calculus. But many textbooks on QM tell this story differently. The general lamento about the “difficulties” with QM suggests that those difficulties were absent from classical physics, as if classical physics had been in the possession of some satisfactory theory of matter that unexpectedly failed to make the correct predictions. But, as Ralston sub-titled, “there is no classical theory of matter” [77]. Hence the alleged contradiction between classical and quantum mechanics misses the point: it is mostly meaningless. Taken serious, it would mean that an existing and experimentally confirmed theory, namely Schrödinger’s wave mechanics, is in conflict with a theory of material points that actually never existed.

Some texts, specifically those addressing a non-expert readership, use the narrative of the classical point particle to underline the revolutionary new and unexpected content of quantum mechanics – claiming (for instance) that quantum theory requires that point particles are “smeared out” in space [78]. But a particle needs to be smeared out in the theory only if it was imagined to be point-like in the beginning. Another example of this strange narrative has been given by Sean Carroll, as he wrote [79]:

[...] if we think an electron wave function is a diffuse cloud centered on the nucleus, when we actually look at it we don’t see such a cloud, we see a point-like particle at some particular location.

This is a common assertion. However, it is odd that theoretical physicists claim to “see” other things than experimental physicists. While Carroll (and many others) claim to “see” point-like particles (which is somewhat difficult to understand), experimentalists claim to “see” orbitals [80–83]. Of course, since this is in conflict with the (certain readings of the) Copenhagen doctrine, the latter received harsh criticism since they claimed to see something that does, according to the theory, not exist [84]. Even though “orbitals” are entirely a quantum mechanical concept, Scerri wrote [85]:

[...] if these claims [of observed orbitals] were to be sustained it would imply an outright refutation of quantum mechanics.

The experimentalists recanted and admitted that the correct wording would be “electron density” [86]. However, the discussion wasn’t finished. Ten years later Labarca and Lombardi discussed once again “why orbitals do not exist” [87] and suggested a split between quantum mechanics and molecular chemistry:

Therefore, the quantum world has no priority over the world of molecular chemistry: chemical entities do not need the support of

quantum entities to legitimate their objective existence. From this perspective, orbitals exist in the ontology of molecular chemistry, in spite of the fact that they do not exist in the quantum world.

Hence, if quantum theorists start to work in molecular chemistry (or vice versa), apparently they should receive a training in doublethink beforehand.

But not only classical analytical mechanics avoided a commitment to point particles. Sebens wrote recently [88]:

[...] we examined some of the reasons why it is appealing to think of electron charge as spread out in the way Schrödinger proposed. [...] Although quantum chemists regularly treat wave functions as describing spread-out distributions of charge, scholars working on the foundations of quantum mechanics rarely explicitly include such charge densities in the ontologically precise formulations of quantum mechanics that they propose. [...] Here I have argued that their fit with quantum chemistry is a point in their favor. When we move to quantum field theory, I think the case for a spread-out electron charge density is particularly strong as the theory can be viewed as describing quantum superpositions of classical field states where electron charge is spread-out.

And Rangacharyulu, 1997 [28]:

[...] in microscopic physics the discussions of wave-particle duality are not meaningful. This reasoning is based on the observation that waves are a conglomeration of coherent disturbances in a many body system and as such they do not represent individual entities. Newtonian kinematics have no predictive power and they do not offer a physical description of participants in an interaction, except to say that they obey the conservation laws. The point-particle concept is an unnecessary complication in physics.

Or S.N. Lyle [27]:

The point particle approximation has been extraordinarily successful. But [...] we might understand physics better by knowing what can be done with spatially extended particles.

Even the prominent string theorist M. Kaku regards it as an achievement that strings are not point particles [89]:

From a technical point of view, superstring theory seems to be totally free of quantum anomalies and divergences, which riddle all known point-particle theories of gravity and matter. [...] In quantum field

theory, point-particles interact via Feynman graphs, which badly diverge when the graph is “pinched,” that is, when one of the legs of the graph shrinks to zero. When the string moves, it repeatedly splits and reforms, thereby tracing the topology of two-dimensional sheets or Riemann surfaces, such as a doughnut. However, since it is difficult to pinch or stretch a doughnut, one can show that the string graphs are actually ultraviolet finite. Thus, the topology of the string removes the divergences of quantum gravity.

This does not imply that we intend to promote string theory. But it should be noted that the point particle imagery does not enjoy universal support, neither in classical nor in post-classical physics.

Our small literature survey of classical textbooks revealed a variety of suggestions of how particles should be understood, some more pragmatic and some which are more philosophically minded. But there is no support for the narrative that the classical worldview of the nineteenth century is unequivocally based on “material points” in an ontological sense. None of the textbooks on analytical mechanics that we found contained ontological claims about the nature of particles. The author does not claim deep expertise in the history of science, but it seems to us that the insistence on the alleged “classical” point-particle is mostly an invention of the 20th century, maybe with the (somewhat legitimate) intention to underline the revolutionary content of quantum theory.

But if classicality requires particles - but not *point*-particles, then few arguments are left to deny that Schrödinger’s equation is *as such* perfectly classical: it provides a mathematical description of distributed normalized matter density moving in space constrained by a classical Hamiltonian dispersion relation. It provides a continuity equation and hence obeys a local conservation law. It is, taken as such, in any reasonable sense of the word a classical theory. Not only does the presumption that particles can’t be represented by mathematical points allow to “naturally” arrive at Schroedinger’s equation, it has also been shown elsewhere that the classical limit of free wave packets, often suggested to be restored by $\hbar \rightarrow 0$, indeed reproduces classical point mechanics [90]:

In the limit $\hbar \rightarrow 0$, the extension of the Gaussian wave packet considered in Refs. 1 and 4 approaches zero in both coordinate space and momentum space.

It is often taught that free wave-packets are always spreading out in space and that wave-packets are therefore somehow non-classical. However, it has been shown that wave-packets do not *always* spread out [91], but that

their width Δx is actually given by

$$\begin{aligned}\Delta x_t &= \sqrt{\Delta x_0^2 + A t/m + B t^2/m^2} \\ A &= \langle \mathbf{x} \mathbf{p} + \mathbf{p} \mathbf{x} \rangle_0 - 2 \langle \mathbf{x}_0 \rangle \langle \mathbf{p} \rangle \\ B &= \delta p^2\end{aligned}\quad (27)$$

This means that they spread out *eventually* but, since A can be negative, not necessarily at all times: They might be convergent in the beginning. Let us compare this result to a bunch of classical particles with $\mathbf{x}_i = \langle \mathbf{x} \rangle + \delta \mathbf{x}_i$ and $\mathbf{p}_i = \langle \mathbf{p} \rangle + \delta \mathbf{p}_i$. In the force-free case we have (for every direction x, y, z):

$$\begin{aligned}x_i(t) &= x_i(0) + p_i/m t \\ \langle x \rangle(t) &= \langle x \rangle(0) + \langle p \rangle/m t \\ \delta x_i(t) &= x_i(0) + p_i/m t - \langle x \rangle(0) - \langle p \rangle/m t \\ &= \delta x_i(0) + \delta p_i/m t\end{aligned}\quad (28)$$

The square is:

$$\begin{aligned}\delta x_i^2(t) &= (\delta x_i(0) + \delta p_i/m t)^2 \\ &= \delta x_i^2(0) + 2 \delta x_i(0) \delta p_i/m t + \delta p_i^2/m^2 t^2 \\ \langle \delta x^2 \rangle(t) &= \langle \delta x^2 \rangle(0) + 2 \langle \delta x \delta p \rangle(0)/m t + \langle \delta p^2 \rangle/m^2 t^2\end{aligned}\quad (29)$$

with

$$\begin{aligned}A &= 2 \langle \delta x \delta p \rangle(0) \\ &= \langle x p + p x \rangle - 2 \langle x \rangle_0 \langle p \rangle\end{aligned}\quad (30)$$

this expression is identical to the quantum-mechanical result. Hence the wave-packet expands exactly as an ensemble of straight trajectories (a beam of drifting particles) expands – no more and no less.

In any real-world experiment, the precision with which we “prepare” a particle with a specific momentum at a certain position, is always finite. How do we (classically) “prepare” particles in a certain state of motion? Most likely one does not *prepare* at all, but *select*: Typically one would use a beam of particles and select the right particles by two (or more) small-aperture collimators in order to define angle and position within specific ranges. The number of particles able to pass two distant collimators, i.e. the intensity of the transmitted beam, then is usually proportional to the area of the apertures and to the range of spatial angles, because the incoming beam, stemming from a source of finite temperature, will have a thermal statistical distribution. This product of area and momentum range is called the phase space volume and the intensity of a beam of particles is proportional to the phase space volume that the setup of collimators accepts. Hence infinite precision is mathematically possible but implies zero particle flow and is hence *physically meaningless*, both in classical mechanics as well as in quantum theory.

Now quantum mechanics has a bit more to say, namely that particles are “res extensa” in phase space. This means that “a particle” cannot be defined by its finite spatial volume alone: if material particles are *res extensa*, this concerns phase space: spatial volume is *always* just one factor, the other being the volume in momentum-space²⁴. From a mathematical perspective, the problem to squeeze quantum particles through tiny holes, is (disregarding interference) comparable to the problem of squeezing a statistical ensemble of point particles through tiny holes. It can only be effective with a “convergent” wave-function, which is identical to a large negative correlation of position and momentum (see Eq. 27). The exact same Eq. 29 holds for a statistical ensemble of particles: The smaller the region where to concentrate the beam, the steeper the required focusing²⁵. That’s basically a supplement to the content of Liouville’s theorem: we can’t squeeze, even a single particle, into a phase space below a certain volume [92].

The (partial) equivalence (and competition) between particle- and wave-picture was known for long in classical optics. The issue was thought to be solved by Young’s double-slit experiment in 1801: Light shows interference effects and hence “is” a wave. However, the proof was based, as we know today, on an incomplete understanding of matter. Since Young’s double slit experiment has meanwhile been successfully repeated with matter-waves, we have to review it’s original goal and conclusion. It should be clear today that Young’s double slit experiment actually did *not* decide the issue: *There never really was an issue to decide*. The conviction that particles can not interfere is obviously erroneous, because it was derived from a flawed concepts of particles and their motion: Schrödinger’s theory can be regarded as the first causal theory of the motion of (distributed) matter.

Based on *macroscopic* experience with matter classical physicists had wrongly concluded that there is a fundamental difference between waves and *microscopic* particles. But the physical properties of macroscopic solid bodies are emergent; they are (obviously) not equal to the fundamental properties of individual “particles”. The projection of the macroscopic notions down to the dimension of single “particles” created the impression of a dichotomy that only exists in our imagination. This is the straightforward lesson to be drawn from the success of Schrödinger’s theory.

Heisenberg remarked that Eq. 14 could be normalized to any value and not just to the number of particles [93]. This is of course correct but it makes no difference in principle whether one claims that the density must be

normalized to unity because it is one particle that is described or because it is the probability density “to find one particle”. The difference is merely in verbiage, not in (mathematical) essence. In both cases it is the act of normalization that generates the real “quantization” of matter by declaring that some function represents one particle.

Nonetheless it should be emphasized that the “classical” perspective described here neither originates nor requires specific metaphysical presumptions. The mentioned requirements of finality, continuity and causality, no matter the metaphysical interpretation of the density, are sufficient for the mathematical derivation of Schrödinger’s equation.

Hence, up to this point we did not (need to) specify what the density ρ actually represents. It might still be open for philosophers to argue for a “real” matter density or “just” a probability density. However, in view of the mentioned math it makes few sense to describe the squared wave function as a probability to find a “point particle”.

Whatever philosophy will eventually prevail, it should be noted that Max Born, awarded the Nobel prize for his probabilistic interpretation of the wave-function, wrote[94]:

though the wave functions are representing, by their square, probabilities, they have a character of reality. That probability has some kind of reality cannot be denied. How could, otherwise, a prediction based on probability calculus have any application to the real world?

Hence even Born, a virulent defender of the Copenhagen philosophy, could not believe in it’s most obscure anti-realistic gospel: Whatever the wave function represents, it can not fully be abstracted from reality if it is supposed to represent (aspects of) this reality. Maybe there is, after all, no reason to scandalize the appearance of probability densities, nor to scandalize the imagery of electron-waves. Probabilities are known in classical mechanics as well and the “difficulties” to decide between waves and particles are known from classical physics as well. Quantum theory is not a new disease but the cure of a long-standing problem that physics was not aware of: How to rationalize motion of extended matter in space in terms of strict local causality.

Newton’s axiom that force free bodies move rectilinear presumes that “motion” is a self-evident process that needs no further explanation²⁶. Apparently Schrödinger’s equation mostly provides a new and different account of motion²⁷. However, in light of Toll’s

²⁴ It is one of the absurd consequences of the Newtonian curriculum that many post-graduate students are able to play back the Copenhagen gospel but have not been told about (or not absorbed) the concept of phase space.

²⁵ Focusing means exactly this: to generate a large negative correlation of (local) position and momentum.

²⁶ His theory requires, however, the opaque notion of inertia, yet another unsolved riddle of Newtonian mechanics.

²⁷ See also Ref. [95, 96].

proof, the Schrödinger equation is indeed the simplest causal theory of the motion of a distributed portion of matter in space - using a minimal set of specifications.

For those who study the history of quantum mechanics and in view of this fairly simple and straightforward logic, the question may arise, what kind of rationalization the “founding fathers” actually had in mind when they proclaimed that the Copenhagen interpretation clarified everything.

Weizsäcker recalled an interesting conversation (see page 184 in Ref. [97]):

With him [Heisenberg] I existed in a state of tension such as can only arise when one is very close to another person. In Berlin in April 1927, in a taxi, he told me of the uncertainty principle saying, “I think I have refuted the law of causality”; in that moment I decided to study physics to understand this.”

This quote refers to the celebrated “uncertainty paper” and testifies Heisenberg’s enormous ambitions but also raises doubts concerning the strength of his scientific sobriety that might have prevented him from drawing quickly bold conclusions. These doubts are also due to what MacKinnon reported [98]:

When Bohr returned from Norway he read the [draft of the uncertainty] paper and thought that it should be treated the way initial drafts of his own papers were. It should serve as a basis for discussion and be written and rewritten until every detail was correct. Heisenberg ignored all such suggestions and sent the hastily written paper, with all its imperfections, to the *Zeitschrift für Physik*. The indeterminacy principle decisively undercut Schrödinger’s wave picture, which in principle assumed precise specifications of both position and momentum. Heisenberg wanted his paper published as soon as possible.

Heisenberg himself described his motivation to demolish Schrödinger’s competing interpretation quite frankly [99]:

Now Schrödinger’s interpretation – and this was the great novelty – simply denied the existence of these discontinuities. [...] This hypotheses seemed to me too good to be true, and I mustered what arguments I could to show that discontinuities were a fact of life, however inconvenient.

This quote contains a central point that requires critical review: We are not aware of any physical method which would allow to verify true discontinuities as a “fact of life”. As we shall argue in part two, true discontinuities have the intrinsic property that they can neither be

verified nor falsified by experiment. It is therefore questionable whether it is a legitimate scientific hypothesis, after all.

Schroedinger’s theory, though derivable from classical logic, by far exceeds the scope of the classical theory. It generates new features as compared to the conventional classical treatment of motion. His theory introduces a second levels of superposition: The superposition of the “auxiliary” functions ψ and of the densities $\rho = \psi\psi^*$. This implies some “non-classical” features. The (linear) superposition of densities is given by²⁸

$$\rho_1 + \rho_2 = \psi_1\psi_1^* + \psi_2\psi_2^*, \quad (31)$$

while the superposition of wave-functions yields

$$(\psi_1 + \psi_2)(\psi_1 + \psi_2)^* = \rho_1 + \rho_2 + \psi_1\psi_2^* + \psi_2\psi_1^* \quad (32)$$

Both equations can only agree if the wave functions don’t overlap or if the product of the wavefunction is skew-symmetric with respect to an exchange of the “particle’s” index. Hence these wave functions cannot be superimposed arbitrarily²⁹ and if one simply scales the normalization of the wave-function to represent two particles instead of one, one breaks the rules suggested by the dispersion relation: For two particles, two dispersion relations are required in order to obtain two velocities. Hence the “configuration space” of the partial waves must be of a higher dimension. Furthermore, from a classical viewpoint, there is few reason to expect that density-distributions are intrinsically additive unless the represented particles don’t interact *at all*.

A new perspective, even if it is based on classical logic, may have unexpected consequences that go beyond the range of “classical” reasoning. If we prefer the view that all consequences of “classical” thought are by definition classical results, then Schrödinger’s equation is classical – and that was according to all known sources his own initial understanding. But if we prefer to say that quantum theory begins with Planck’s constant, then the wave-particle duality of Eq. 20 can be regarded as the *logical* origin of quantum theory. Everything that follows from it is “quantum”, no matter how we interpret its content. Maybe the distinction between “classical” and “quantum” is rather a matter of philosophy than a matter of mathematical logic [101]:

What is ‘classical physics’? Physicists have typically treated it as a useful and unproblematic category to characterize their discipline from Newton until the advent of ‘modern physics’ in the early twentieth century. But from the historian’s point of view,

²⁸ See also Ref. [100].

²⁹ This feature, when projected backwards to the idea of classical “point particles”, remains a mystery. But it is entirely comprehensible in the wave picture.

over the last three decades several major interpretive difficulties have become apparent, not least the absence of unequivocal criteria for labelling physicists and their work as ‘classical’, whether during the nineteenth century or earlier. Some historians have consequently either treated the term as a retrospectively contrived anachronism (such as Olivier Darrigol), or carefully avoided using it in their analyses (such as Jed Buchwald). Nevertheless, current historiographies have not systematically explored the implications of abandoning ‘classical physics’ as an analytical category. As a result, they arguably overstate the unity of the physics prior to the rise of quantum and relativity theories in the twentieth century. Moreover, many studies into the activities of late nineteenth-century physicists have adopted the perspective of later theoretical developments typically associated with the birth of one type or another of ‘modern physics’, for example the origins of microphysics and, through special relativity, the history of electrodynamics. This focus on theoretical discontinuities, implicit in the classical/modern distinction, has long diverted attention away from important historical continuities in both experimental practice and the applications of physics. We take these reasons as sufficient motivation for rethinking ‘classical physics’.

The main theme of this article is to show the amazing power of Hamiltonian notions in physics. Even if we follow them almost blindly, apparently they can guide us to new valuable insights. We think that Schrödinger’s equation not only provides an excellent example for the fruitful use of Hamiltonian methods, but is also an appropriate and possibly necessary introduction to the next example. Influenced by Whittaker’s *Treatise on the Analytical Dynamics*, it was Dirac’s explicit intention to keep as much as possible of the classical Hamiltonian notions [102]. As we shall argue in the next example, Dirac’s theory can even be regarded as a Hamiltonization of space-time geometry.

IV. THIRD EXAMPLE: DIRAC’S EQUATION

The discovery of this [Dirac’s] equation was the most important advance in the theory of the electron since the Maxwell-Lorentz equations of classical electrodynamics. Bohr’s semiclassical theory and non-relativistic quantum mechanics served only as transitional theories.

– Sokolov, Loskutov and Ternov [103]

Dispersion relations are well known to mathematicians,

engineers and physicists [20–23, 30, 104, 105]. We repeat that Toll proved the “logical equivalence of strict causality and a dispersion relation” [30]. However, from the Hamiltonian perspective, it is disturbing that we had to use Newton’s EMR to arrive at Schrödinger’s equation. We might better argue that the frequency must, for reasons of isotropy, be an even function of the wave-vector and therefore must have a Taylor series expansion $\omega(\vec{k}) = c_0 + c_2 \vec{k}^2/2 + \dots$. This is a strong argument, based on a minimum specification, to establish Newton’s dispersion relation. But we must be willing to follow Newton’s method and to presuppose 3-dim. isotropic absolute space. But there is an alternative to this Newtonian logic. We suggest to follow a purely Hamiltonian way of thought, since this appears to provide reason for the Newtonian approximation and for the need to replace Schrödinger’s by Dirac’s equation.

So the question remains: where does the relativistic dispersion relation (RDR) come from? In the conventional ‘historical’ account, it is not derived from Dirac’s equation, but rather the other way around: Dirac somehow ‘guessed’ his equation in order to reproduce the RDR with a first order (i.e. Hamiltonian) equation. The conventional lore suggests that we have to presume the constancy of the speed of light and the “metric” of Minkowski space-time in order to arrive at the RDR. But is that the only possibility? Do we *have to* speak about “inertial frames” and “clock synchronizations” in the first place³⁰?

The common presentation of special relativity rests on two central pillars, the constancy of the speed of light and the invariance of the laws of physics. The latter principle, however, namely the invariance of Hamilton’s equations of motion under canonical transformations, is already integral part of Hamiltonian notions and the theory of canonical transformations. It might be defensible as a fundamental requirement for a theory that is point-of-view-invariant [107].

But the postulate of the constancy of the speed of light suggests that some *specific* physical wave phenomenon, namely light, serves as a *general* limit that determines the relative scale of spatial vs. temporal coordinates. This implies that this *specific* physical phenomenon has *fundamental* physical significance, i.e. that it is in fact not a *specific* phenomenon but instead a *general* principle, that does not *logically* originate in the transmission of light. But, to our knowledge, the theory of special relativity does not explain *why* electromagnetic phenomena should play this fundamental role since the standard approach regards electromagnetism as one of four fundamental “forces of nature”, i.e. as one among several

³⁰ Einstein himself was not fully satisfied with the notion of the *inertial frame*. In a letter to Jaffe he wrote in 1954: “I see the most essential thing in the overcoming of the inertial system, a thing which acts upon all processes, but undergoes no reaction. The concept is in principle no better than that of the centre of the universe in Aristotelian physics” [106].

others. There is no explanation given why the propagation of light should have any *special* significance in a space-time theory. It is therefore not surprising that many scientists sought for alternative foundations of the Lorentz transformations and relativity [108–128].

The question then is: is it required to raise the mere phenomenological fact of the constancy of the speed of light to the status of a fundamental principle? As it is mere an empirical fact about nature, it should be an outcome of the theory, and not be placed at the very foundation. We shall show in what follows that the relativistic dispersion relation can be indeed be derived from Hamiltonian symmetries. This requires to use some bits of linear algebra, but though many math facts also hold in more general cases, we have to apply them to nothing more demanding than real 4×4 -matrices. The sequence of arguments that allows to derive the Dirac algebra and the relativistic dispersion relation from pure Hamiltonian arguments is long but quite rigorous [12, 13].

Dirac’s idea to implement the RDR by matrices, as ingenious as it was, was yet another ad-hoc approach that pre-supposes the Minkowski metric and leaves physics students with new riddles: what is the meaning of spinors and the Dirac matrices? Why do we need to multiply the Dirac spinor by γ_0 to obtain the “adjunct” spinor? What is the sense and significance of γ_0 anyway? What are the arguments that might lead to a “space-time metric” which is not positive (semi-) definite?

David Hestenes was probably the first who recognized the elegance of an algebraic description of geometry on the basis of the Dirac-Clifford algebra. He reformulated Dirac’s theory and developed the *space-time algebra* (STA), as he called it. This algebra reveals the deeper connection between Dirac’s theory, the geometry of space-time and electrodynamics [129]. We think that this was an essential step forward without which it is impossible to grasp the dynamical emergence of space-time. But to our knowledge Hestenes made no attempt within his theory to explain why this specific algebra (and not some other) should be ideally suited to describe space-time and electrodynamics: why is the Dirac algebra fundamental?

Hence there is still a missing bit, a missing logical reason for why things are as they are and not some other way. There are good arguments supporting the view that this missing bit is the connection of the Dirac equation with “classical” symplectic motion³¹. Though the insight that quantum theory comes along with a symplectic structure is not new, it has long been overlooked that the Dirac algebra itself can be derived from

purely Hamiltonian notions, not only by analogy, but literally [12, 13, 134]. Even this is no new insight but had been understood, for instance by Kim and Noz, already forty years ago [135]:

From a mathematical standpoint, special relativity is the physics of Lorentz transformation, and quantum mechanics is the physics of Fourier transformation. It is easy to see, if not well known, that the Lorentz boost is a symplectic transformation in the plane of longitudinal and timelike coordinates. In Fourier transformation, the width of the momentum distribution is inversely proportional to that of the spatial distribution. This is also a transformation property of the symplectic group. Thus, it is not unreasonable to suspect that the natural language of relativistic quantum mechanics is the symplectic group.

Dirac’s algebra is not only useful to describe the geometry of Minkowski’s space-time and relativistic quantum mechanics. The real Dirac algebra provides a general parameterization of arbitrary real 4×4 -matrices. This parameterization, however, not only matches the requirements of Lorentz covariance, it furthermore allows to derive the Minkowski metric: the real Dirac algebra provides an optimal parameterization of the Lie algebra of the symplectic group $sp(2n = 4, \mathbb{R})$ which generates the full set of possible linear transformations between two canonical pairs [136–139]. In other words: The Dirac algebra pops out of a formal analysis of classical Hamiltonian mechanics and therefore, it is *intrinsically* related to the notion of ensembles in classical Hamiltonian phase space. Hence if the partial waves that have been used in Schrödinger’s equation are indeed points in a Hamiltonian phase space, then the dispersion relation should have its logical origin in Hamiltonian notions.

The wave-function, i.e. the Dirac spinor, corresponds in the momentum representation to ensembles of two classical oscillators, just as Schrödinger’s equation suggests and in perfect agreement with Dirac’s understanding of what the components of his spinor actually represent [140]:

These new degrees of freedom are to be associated here with certain dynamical variables (q_1, p_1) and (q_2, p_2) to be thought of as corresponding to two independent linear harmonic oscillators.

But what can be said about two such oscillators without a specific description of the oscillating system, for instance in terms of masses and spring constants? A different type of analysis is needed, a kind of “contentless deductive theory” [141]:

If the elements of the group are not given any realization, and the group is essentially

³¹ We use quotes in “classical”, because the symplectic group was introduced by Hermann Weyl [130] in 1939 and was therefore unknown when Dirac formulated his relativistic theory of the electron. Though the symplectic group originally seemed to belong to the realm of classical physics, the intrinsic connection to quantum mechanics is known and has been elaborated in some detail; see Ref. [92, 131–133].

defined by its multiplication table, one obtains an abstract group theory, which may have realizations ranging from atomic physics to oriental carpets.

That is, instead of starting from a metaphysical theory of how space, time, matter and fields should be understood, we start with an empty pure and general Hamiltonian form.

One then has to proceed, lacking any phenomenological context and further specification, with a survey of the full space of *possible* solutions of arbitrary general Hamiltonian functions of an arbitrary number of canonical pairs: What are the general features of such systems which might explain how some general kind of dispersion relation emerges? Here we take the classical path and start with a general Hamiltonian function in the form of a Taylor series in the dynamical variables. We then apply some simplifying assumptions. These assumptions should not be misunderstood as metaphysical presuppositions. It is simply the approved method in the history of physics to attack a problem of this dimension and generality by considering the simplest possible cases first. This might help to develop the notions required to elaborate solutions for more complex situations. Hence we start with a second order approximation for a Hamiltonian of n canonical pairs, i.e. with the most general form of linear equations of motion in $2n$ variables. The simplest non-trivial Hamiltonian system is a canonical pair and the simplest system with coupling consists of two canonical pairs. It should provide us with an understanding of the *principle form* of linear Hamiltonian couplings.

Before we enter the details, let us emphasize that it is not as far-fetched as it might appear at first sight to relate linear Hamiltonian couplings to wave mechanics. The difference between an ensemble of non-interacting oscillators and a linear chain – and hence wave motion – lies in the coupling between oscillators: Waves are, in a very general sense, the result of *coupled* oscillations [23]. Hence it is reasonable that the *general* algebraic structure of the coupling determines the *general* characteristics of wave motion, namely of the motion that we expect to generate a physically meaningful purely Hamiltonian dispersion relation. The simplest oscillator is represented in Hamiltonian theory by a single canonical pair, hence the simplest coupling requires two canonical pairs.

Let $\psi = (q_1, p_1, q_2, p_2)^T$ represent Dirac's two classical canonical pairs (two degrees of freedom), then the quadratic terms of a general Hamiltonian function are given by

$$\mathcal{H} = \frac{1}{2} \psi^T \mathbf{A} \psi \quad (33)$$

where \mathbf{A} is a positive definite real symmetric 4×4 matrix. We restrict our considerations to symmetric matrices since skew-symmetric components do not contribute to the Hamiltonian function.

The Hamiltonian function is a constant of motion if

$$\frac{d\mathcal{H}}{d\tau} = \dot{\mathcal{H}} = (\nabla_{\psi} \mathcal{H}) \cdot \dot{\psi} = (\psi^T \mathbf{A}) \cdot \dot{\psi} = 0, \quad (34)$$

which has the general solution

$$\dot{\psi} = \gamma_0 \nabla_{\psi} \mathcal{H} = \gamma_0 \mathbf{A} \psi = \mathbf{F} \psi. \quad (35)$$

where γ_0 is a skew-symmetric matrix, the so-called *symplectic unit matrix* (SUM). In principle this matrix could have an arbitrary skew-symmetric form, but it can (with some mild and reasonable assumptions) be brought into the following form [142]:

$$\gamma_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (36)$$

Hence the SUM γ_0 implements Hamilton's equations of motion in algebraic form, which becomes obvious if one writes Eq. 35 explicitly in components:

$$\begin{pmatrix} \dot{q}_1 \\ \dot{p}_1 \\ \dot{q}_2 \\ \dot{p}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \mathcal{H}}{\partial q_1} \\ \frac{\partial \mathcal{H}}{\partial p_1} \\ \frac{\partial \mathcal{H}}{\partial q_2} \\ \frac{\partial \mathcal{H}}{\partial p_2} \end{pmatrix} \quad (37)$$

Remarkably, the skew-symmetry of γ_0 alone suffices to qualify Eq. 35 as a solution for Eq. 34.

Matrices of the form $\mathbf{F} = \gamma_0 \mathbf{A}$ are called Hamiltonian and they are the starting point of linear Hamiltonian theory. More generally, a matrix \mathbf{F} is said to be *Hamiltonian*, iff it obeys [142]

$$\gamma_0 \mathbf{F} \gamma_0 = \mathbf{F}^T. \quad (38)$$

It is not immediately obvious from Eq. 38, but in combination with $\gamma_0^2 = -\mathbf{1}$ and $\gamma_0 = -\gamma_0^T$, Eq. 38 combines matrix transposition with commutation relations. Two matrices \mathbf{A} and \mathbf{B} are said to commute, if $\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A} = 0$ and to anti-commute, if $\mathbf{A}\mathbf{B} + \mathbf{B}\mathbf{A} = 0$. Eq. 38 allows to construct two matrices $\mathbf{F}_{a,c}$ such that \mathbf{F}_a anti-commutes with γ_0 while \mathbf{F}_c commutes with γ_0 :

$$\begin{aligned} \mathbf{F}_a &= \mathbf{F} + \gamma_0 \mathbf{F} \gamma_0 \\ \mathbf{F}_c &= \mathbf{F} - \gamma_0 \mathbf{F} \gamma_0 \end{aligned} \quad (39)$$

The original matrix is $\mathbf{F} = (\mathbf{F}_a + \mathbf{F}_c)/2$. Inserting Eq. 38 into Eq. 39 results in

$$\begin{aligned} \mathbf{F}_a &= \mathbf{F} + \mathbf{F}^T \\ \mathbf{F}_c &= \mathbf{F} - \mathbf{F}^T \end{aligned} \quad (40)$$

such that \mathbf{F}_a is symmetric and \mathbf{F}_c is skew-symmetric. Hence Hamiltonian matrices that commute with the

SUM γ_0 , are skew-symmetric and those that anti-commute with γ_0 , are symmetric.

The general solution of Eq. 35 for constant \mathbf{F} is given by the matrix exponent $\mathbf{M}(\tau)$

$$\psi(\tau) = \exp(\mathbf{F} \tau) \psi(0) = \mathbf{M}(\tau) \psi(0). \quad (41)$$

It is a math fact that \mathbf{M} is a symplectic matrix, iff \mathbf{F} is Hamiltonian. The evolution in time, generated by some Hamiltonian matrix \mathbf{F} , is a symplectic (canonical) transformation. One can show that symplectic matrices obey [142]:

$$\mathbf{M} \gamma_0 \mathbf{M}^T = \gamma_0. \quad (42)$$

Symplectic matrices form a group which means that any product of symplectic matrices is again a symplectic matrix.

In Hamiltonian theory observables are generators of canonical transformations³². So what are the observables and how do they correspond to generators? In Sec. II we started with the description of a density in space, a volume smoothly filled with “matter”. By the use of the Fourier transform, we switched to an *ensemble* of waves, the “wave-packet”. By introducing the wave-particle-duality (Eq. 20) however, we introduced a new Hamiltonian and by doing so we (implicitly) introduced an ensemble of oscillators in some Hamiltonian phase space by Eq. 17. We did not make that very explicit in Sec. III, but here we explicitly consider (non-interacting) *ensembles* of solutions of Eq. 35.

Classical ensembles of non-interacting (or weakly interacting) systems are subject of classical statistical mechanics, similar to ensembles of particles in accelerator bunches and can be described by a phase space density $\rho(\psi)$. In contrast to ensembles from classical mechanics, where the density is a density of a huge but countable number of “mass points”, the density we presume here is a smooth and continuous distribution in phase space. Distributions can of course be described by various mathematical methods. One possibility is a description based on the moments $\langle q_i^\mu p_j^\nu \rangle$ of the distribution. The most important moments are the second moments,

represented by the (“auto-correlation”) matrix Σ of second moments. The autocorrelation matrix allows to construct the desired correspondence between observables and generators: There are ten independent parameters in the symmetric 4×4 matrix \mathbf{A} (and hence in the Hamiltonian matrix \mathbf{F}) and also ten parameters in the (symmetric) matrix Σ . Let $\Sigma = \langle \psi \psi^T \rangle$ be the matrix of second moments of solutions of Eq. 35, then it follows that

$$\begin{aligned} \dot{\Sigma} &= \langle \dot{\psi} \psi^T \rangle + \langle \psi \dot{\psi}^T \rangle \\ &= \mathbf{F} \langle \psi \psi^T \rangle + \langle \psi \psi^T \rangle \mathbf{F}^T \\ &= \mathbf{F} \Sigma + \Sigma \mathbf{F}^T \end{aligned} \quad (43)$$

Eq. 43 is well-known in accelerator physics and used to describe the development of the second moments of a distribution of particles within a frame co-moving with the bunch. The second moments allow to define the RMS-“size” of the beam by the diagonal elements $\Sigma_{11} = \langle x^2 \rangle$ ³³.

Accelerator physics is mostly satisfied with Eq. 43, but let us take one more step which enables to arrive at a much more transparent framework[144]. This step consists in by a multiplication of both sides of Eq. 43 with γ_0^T from the right and in the definition of a Hamiltonian matrix $\mathbf{S} = \Sigma \gamma_0^T$. Then one obtains the following equation of motion for second moments:

$$\dot{\mathbf{S}} = \mathbf{F} \mathbf{S} - \mathbf{S} \mathbf{F} \quad (44)$$

This equation is the general equation of motion of any Hamiltonian matrix \mathbf{F} . The general transformation law of these matrices is that of a symplectic transformation:

$$\mathbf{F}(\tau) = \mathbf{M}(\tau) \mathbf{F}(0) \mathbf{M}^{-1}(\tau) \quad (45)$$

with a symplectic matrix $\mathbf{M}(\tau)$, which can always be written as a matrix exponent of another Hamiltonian matrix \mathbf{G} :

$$\mathbf{M}(\tau) = \exp(\mathbf{G} \tau/2) \quad (46)$$

and

$$\mathbf{M}^{-1}(\tau) = \exp(-\mathbf{G} \tau/2) = \mathbf{M}(-\tau) \quad (47)$$

Therefore one finds in few steps³⁴:

$$\dot{\mathbf{F}} = \frac{d\mathbf{F}}{d\tau} = \frac{1}{2} (\mathbf{G} \mathbf{F}(\tau) - \mathbf{F}(\tau) \mathbf{G}). \quad (48)$$

³² Andre Heslot[143]:

Current textbooks often emphasize the generator aspect of observables in quantum mechanics, but it is seldom mentioned that this aspect already exists in classical mechanics. As a consequence, notions which already make sense in classical theory are too often considered as purely quantum ones: The spin is a striking example of such a confusion. As in quantum mechanics, the generator aspect of observables in classical mechanics may be dealt with by mathematically sophisticated group theoretical considerations, but we hope we have convinced the reader that this aspect proves relevant even at the elementary level.

³³ Of course, in accelerator physics, the involved matrices are in general of size 6×6 . As mentioned before, median plane symmetry often reduces the size of the problem effectively to 4×4 and 2×2 .

³⁴ Whether or not the right side of Eq. 48 has to be scaled by a factor of $1/2$ depends on the choice of units for τ or \mathbf{G} , respectively.

This is Heisenberg’s equation of motion for operators, an equation that is usually regarded as intrinsically “quantum”. It is automatically identical to the equation of motion of observables, if the observables are second moments in phase space.

The fact that quantum mechanics and (classical) beam dynamics are based on the same mathematical (namely symplectic) structures has been recognized and emphasized by many [92, 145–149], but is mostly ignored in common textbooks on quantum theory. Some physicists believe that it is “misleading” to emphasize the common mathematical basis of quantum and classical theory, namely Hamiltonian mechanics. But how can it be misleading, if not due to the presupposition that classical and quantum mechanics are fundamentally different?

If it is not immediately obvious, that Eq. 44 is literally Heisenberg’s equation of motion, then likely because the quantum “look and feel” requires the use of the unit imaginary and of Planck’s constant \hbar . If we restrict our considerations to non-singular systems, then those eigenvalues of \mathbf{F} , that represent stable oscillation, are purely imaginary [142]. Furthermore they have the unit of a frequency. Hence, the “classical operator” (i.e. the matrix) \mathbf{H} has imaginary eigenvalues, representing the frequencies of oscillation, and hence those of the operator $i\mathbf{H}$ are real. When multiplied by \hbar , the real eigenvalues have the unit of energy

$$\mathbf{H} = \mp i \hbar \mathbf{F}, \quad (49)$$

so that one obtains:

$$\dot{\mathbf{S}} = \pm i/\hbar (\mathbf{H}\mathbf{S} - \mathbf{S}\mathbf{H}). \quad (50)$$

This testifies that much of the differences between equations appearing in QM and those of classical Hamiltonian mechanics are due to the specific notation used in the former, which has been established in the course of the development, i.e. by convention. But a successful description of nature does not depend on notational conventions and no system of equation becomes “quantum” just because we use the unit imaginary explicitly instead of implicitly. But while Bohr’s complementarity and Heisenberg’s uncertainty are supposed to characterize the apparent fundamental difference between classical and quantum notions, Dirac aimed for the opposite and usually emphasized the analogy between Poisson brackets and commutators [63]:

Hamilton made a further development of dynamical theory. He set up a formalism which describes, not an individual solution of the equations of motion, but a whole set of solutions together. [...] At the time when Hamilton set up his theory there was no physical reason why one should be interested in a family of solutions rather than an individual Solution. The latter seemed quite adequate for a description of Nature. Hamilton must have been inspired to know that his

work was important. He showed that there was an analogy between his dynamical theory and geometrical optics, (namely that optics in which one neglects effects arising from the finite wave-length of light.) But this analogy appeared at that time to be just a mathematical curiosity without physical significance. It needed a hundred years of progress in physics to show up the value of Hamilton’s work. Both of Hamilton’s developments of dynamical theory are of the greatest importance for quantum mechanics and are thus needed for a description of Nature on the atomic scale. [...] The concept of the P. b. [Poisson bracket] is the all-important link in the passage from classical to quantum mechanics, and this concept enters into classical mechanics only with Hamilton’s form of the theory.

However Eq. 48 is a direct and *literal* consequence of Hamiltonian methods. Provided the Hamiltonian concerns ensembles of partial waves that inhabit an abstract phase space as suggested by the dispersion relation (Eq. 17), then one has Dirac’s wave equation in momentum representation. Apparently it needs another century to admit that this is more than an “analogy”, more than a “mathematical curiosity without physical significance”.

It follows from Eq. 44 that the second moments are constants of motion when $\dot{\mathbf{S}} = 0$, i.e. if \mathbf{S} and \mathbf{F} commute. It is a math fact of linear algebra that commuting matrices share a system of eigenvectors and that these eigenvectors must be complex, whenever the eigenvalues are imaginary, i.e. in case of stable systems. \mathbf{S} provides the simplest possible (though in most cases incomplete) description of phase space ensembles³⁵. Applying Eq. 41, the autocorrelation matrix $\mathbf{S}(\tau)$ of the phase space ensemble as a function of time is given by

$$\begin{aligned} \Sigma(\tau) &= \langle \psi(\tau)\psi(\tau)^T \rangle \\ &= \mathbf{M}(\tau)\langle \psi(0)\psi(0)^T \rangle \mathbf{M}^T(\tau) \\ &= \mathbf{M}(\tau)\Sigma(0)\mathbf{M}^T(\tau) \end{aligned} \quad (51)$$

This equation, at first sight, seems to suggest that the evolution in time is an orthogonal transformation. However \mathbf{M} is orthogonal only in special cases, but always symplectic (Eq. 42). Again we multiply by γ_0^T from the right and obtain

$$\mathbf{S}(\tau) = \mathbf{M}(\tau)\mathbf{S}(0)\mathbf{M}^{-1}(\tau), \quad (52)$$

where we used Eq. 42 in the last step: The symplectic evolution in time is a similarity transformation, but not

³⁵ Only if an ensemble is Gaussian, the matrix of second moments \mathbf{S} provides a complete description.

necessarily an orthogonal one. Hence the eigenvalues of \mathbf{S} are constants of motion³⁶.

There is a theorem in classical statistical mechanics about ensembles in phase space, which states that the constant phase space density of (thermal) equilibrium is a function of the energy, i.e. the Hamiltonian, or more generally, a function of the constants of motion, hence in our case, of the eigenvalues [151]. If Λ is the (diagonal) matrix of eigenvalues of \mathbf{M} and λ the (diagonal) matrix of eigenvalues of \mathbf{S} , then, applied to the case at hand, this means that, in equilibrium, $\mathbf{S} = f(\mathbf{M})$ can be reduced to $\lambda = f(\Lambda)$. This is the case iff \mathbf{S} and \mathbf{M} have a common system of eigenvectors. Hence thermal equilibrium corresponds to a matched distribution and we can leave the question what exactly determines the form of \mathbf{F} open: both, external fields but also the properties of the considered system itself might be responsible for the precise form of \mathbf{F} .

It is a known, though maybe not well-known, math fact that real 4×4 -matrices can be parameterized by the use of a Clifford algebra. Hestenes elaborated in detail how the Dirac Clifford algebra generates geometrical significance [129]. Insofar our approach is, yet again, close to known presentations. But here we use an approach slightly different from that of Hestenes. We shall derive the Dirac algebra from Hamiltonian notions based on the SUM γ_0 as an essential structure generating element³⁷. In fact *any* real squared matrix of size $2^n \times 2^n$ can be written as a sum

$$\mathbf{M} = \sum_{k=0}^{15} m_k \gamma_k, \quad (53)$$

where γ_k are the unit elements of the Clifford algebra and the index k runs over all unit elements (vectors, bivectors, etc.). But why should it be sensible to apply such a change of variables from profane matrix elements m_{ij} to something fancy like the coefficients of a Clifford algebra? Is this necessary or just ornamental like \hbar and the unit imaginary in QM? Can we make the case from the perspective of Hamiltonian mechanics?

The representation by Clifford algebras charges numbers (m_k) with “structural significance”. The simplest

case of one degree of freedom requires only 2×2 matrices:

$$\mathbf{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}. \quad (54)$$

But since a Hamiltonian matrix $\mathbf{F} = \gamma_0 \mathbf{A}$ is a product of a skew-symmetric and a symmetric matrix, it has a vanishing trace. That is, Hamiltonian matrices have the boundary condition of a vanishing trace, here $m_{11} + m_{22} = 0$. Therefore we define new variables $c = m_{11} - m_{22}$ and $d = m_{11} + m_{22}$ and obtain

$$\mathbf{M} = \begin{pmatrix} (c+d)/2 & m_{12} \\ m_{21} & (d-c)/2 \end{pmatrix}, \quad (55)$$

so that the parameter d is directly proportional to the trace. But as we have shown, also the distinction of symmetric and skew-symmetric elements is of severe importance in linear Hamiltonian theory, so that eventually we write

$$\mathbf{M} = \begin{pmatrix} (c+d)/2 & (a+b)/2 \\ (b-a)/2 & (d-c)/2 \end{pmatrix}, \quad (56)$$

and out pops the representation of a Clifford algebra, namely of the real Pauli algebra $Cl(1,1)$ or $Cl(2,0)$, respectively³⁸:

$$\mathbf{M} = a \eta_0 + b \eta_1 + c \eta_2 + d \mathbf{1}. \quad (57)$$

A symmetric matrix corresponds to $a = 0$, and $d = 0$ implies a matrix with vanishing trace: We thus constructed a scheme in which numbers (quantities) receive structural significance: Quantity and structure become “entangled”, but in a systematic way, so that all coefficients a, b, c, d quantify specific symmetries of the Clifford algebra and hence of the dynamical properties of the system.

It is not directly evident from Eq. 57 that the derived set of four matrices η_k is indeed the representation (rep) of a Clifford algebra (CA), but it becomes evident if we look at the anti-commutators:

$$\eta_i \eta_j + \eta_j \eta_i = \pm 2 \delta_{ij} \quad (58)$$

Hence all Pauli-matrices square to $\pm \mathbf{1}$ and all of them either commute or anti-commute with all others. Then they are a representation (rep) of some CA. There is no need to *postulate* physical significance of CAs. The physical significance of Clifford algebras can be obtained from Hamiltonian notions alone. And indeed, quadratic forms and Clifford algebras are deeply related [152, 153].

The Hamiltonian symmetries introduced for 2×2 -matrices are preserved (and more emerge), if matrices of more complex systems with more degrees of freedom are constructed from the real Pauli algebra by Kronecker

³⁶ They are “emittances”, decorated by a unit imaginary [144]. In accelerator physics, the matrix \mathbf{M} is the so-called “transport matrix”. It is a product of the transport matrices of all involved beam guiding elements (bending magnets, quadrupole magnets, buncher etc., see Ref. [136]) and it is determined by the properties of the beamline elements, i.e. the “outside world”. A beam described by $\mathbf{S}(\tau)$ is called “matched” to a given beamline described by \mathbf{M} , if \mathbf{S} and \mathbf{M} commute. This description is reasonably accurate as long as both, nonlinear terms and self-interaction by space charge or intra-beam-scattering can be neglected. However, if bunches have a non-negligible self-interaction due to space charge, the matrix \mathbf{F} and hence \mathbf{M} also depends on the size of the beam: then \mathbf{F} itself depends on (elements of) \mathbf{S} [150].

³⁷ A very brief intro to Clifford algebras is given in App. A.

³⁸ The usual complex Pauli matrices are a reduction derived from the Dirac algebra and are *therefore* complex.

multiplication. For two degrees of freedom, we have to consider all Kronecker products of the (real) Pauli matrices (Eq. 57) and out pops the real Dirac algebra. More generally it seems that any Clifford algebra that fully conforms with Hamiltonian notions, has a rep that can be obtained from Kronecker products of the real Pauli algebra.

Since the Dirac algebra meets the symmetries of Hamiltonian mechanics, the distinction between Hamiltonian and skew-Hamiltonian elements splits the 16 coefficients m_k into two sets of matrices, 10 of which are Hamiltonian and 6 skew-Hamiltonian. A Hamiltonian 4×4 matrix can then be written as

$$\mathbf{F} = \sum_{k=0}^9 f_k \gamma_k. \quad (59)$$

We have shown that the use of Clifford algebras in Hamiltonian theory can be motivated purely by Hamiltonian symmetries, but one can only make use of the Clifford algebraic approach, if the matrix \mathbf{A} is of size $2^n \times 2^n$, for instance in our case of the coupling of two degrees of freedom.

The Fourier transformation used in Sec. III is a (unitary) transformation to new variables, and the use of the real Dirac algebra is but another transformation to new variables. It is a general phenomenon that most work to solve a (solvable) physical problem is done when we have found the transformation to appropriate variables. Though the use of a Clifford algebra results from the analysis of dynamical symmetries of pure classical phase space, it nonetheless is a new element that was unknown in classical pre-quantum physics. The reason is that this is a method of maximal generality. Before the advent of quantum mechanics, “classical” mechanics was mostly used to describe specific systems with specific forms of \mathbf{F} . And though ensembles in phase space were subject of statistical mechanics, it was mostly understood as the phase space of ensembles of point-particles in 3d-space (and time).

As we shall briefly sketch in the following, the Dirac algebra has the additional and unexpected feature to automatically provide us with a unique interpretation in the sense, that the commutation properties of the algebra alone suffice to determine the transformation properties of all Hamiltonian coefficients [12, 13]. This automatically and inevitably generates a system of 10 variables and their behavior under symplectic transformations (generated by the same 10 quantities) which is able to represent the well-known set of physical quantities which are relevant for the description of a relativistic charged particle in an external electromagnetic field³⁹, namely energy \mathcal{E} and the cartesian components of momentum \vec{P} , electric and magnetic field \vec{E} and \vec{B} , respectively.

The analysis of the elements of the Clifford algebra that is represented by real 4×4 matrices naturally begins with the distinction between Hamiltonian and skew-Hamiltonian matrix elements. It follows from Eq. 38 that γ_0 itself is Hamiltonian. It is therefore the first of 10 parameters necessary to specify the linear Hamiltonian system of two canonical pairs. If γ_0 is the first *basis* element of the Clifford algebra, then any other *basis* element γ_a must anti-commute with γ_0 . This follows from the definition of Clifford algebras. If one furthermore demands that all basis elements γ_a must be Hamiltonian, then all other basis elements, except γ_0 , must be symmetric (see Eq. 40) and therefore square to $+1$. We call a real Clifford algebra (CA) with purely Hamiltonian basis a *Hamiltonian* Clifford algebra (HCA). As we just derived and explained, *any* Hamiltonian Clifford algebra of dimension $N = p + q$ in which the SUM γ_0 is a generating element has dimension $Cl(N - 1, 1)$ and therefore produces a metric of Minkowski type.

Real 4×4 matrices may represent either $Cl(2, 2)$ or $Cl(3, 1)$, each having 4 basis elements, but only $Cl(3, 1)$ is Hamiltonian, i.e. has a Clifford basis γ_μ consisting exclusively Hamiltonian elements⁴⁰.

Dirac introduced 4×4 matrices in order to *reproduce* the already known RDR $\mathcal{E}^2 - \vec{p}^2 = \text{const} = m^2$ (using $c = 1$). But no other Dirac Clifford algebra but only $Cl(3, 1)$ is based exclusively on (real) Hamiltonian matrices. In the conventional, historically oriented lore, Lorentz covariance is a more or less surprising requirement for the invariance of Maxwell’s equations, which have been discovered experimentally and combined piece by piece by Faraday, Maxwell, Heaviside and others. Then it was Einstein’s principle of the constancy of the speed of light, that required to establish the Lorentz transformations as something fundamental[154]:

The real importance of Einstein’s work was that he introduced Lorentz transformations as something fundamental in physics.

However, there are always different perspectives possible, as described by Swann [155]:

In any presentation of a branch of modern physics, two courses are open. The first is the historical. This has the disadvantage that, usually, it does not represent a sequence of logical developments. The ways in which conclusions are reached are founded frequently upon considerations of special cases, and sometimes are based upon experiments whose representatives have completely evaporated in the more general fields in which the conclusions are subsequently used. The alternative method is to take the results which have

³⁹ A detailed demonstration of the inevitability exceeds the scope of this paper, but has been given in Ref. [12, 13].

⁴⁰ This is required in order to have a “dimension” that is able to act as a Hamiltonian generator. See below.

been stumbled upon in the historical development, review the path by which they have been reached, remove from it as much as possible of the obsolete debris with which it is encumbered, and try to construct some more satisfactory path by which the results might have been reached even to the extent of possible modifications in the starting points.

or by Peter Ball in his recently published brilliant book[156]:

Yet most popular descriptions of quantum theory have been too wedded to its historical evolution. There is no reason to believe that the most important aspects of the theory are those that were discovered first, and plenty of reason to think that they are not.

Levy-Leblond summarized it even shorter[116]:

The chronological building of order of a physical theory, however, rarely coincides with its logical structure.

Though this could be regarded as a platitude, the lesson apparently has not yet been understood. Why else should it be emphasized?

Nothing in the usual presentation of the matter suggests, that the RDR can, with the help of the Clifford algebras, be obtained from classical Hamiltonian notions. But it is a math fact, that the Clifford algebraic structure $Cl(3, 1)$ stems from the symmetry of classical phase space [12, 13] and this suffices to derive the mathematical form of the Lorentz transformations. But then the core concepts of the physics of the 20th century, namely Lorentz covariance and wave mechanics, are barely more than applied Hamiltonian mechanics⁴¹.

But the usual textbook presentations of special relativity discusses coordinate transformations as something physical without recurring to the physical quantities which are the generators of these transformations⁴². It is

⁴¹ We are aware of the fact that the mere form of equations does not automatically provide an interpretation. The method suggested by Loewdin's contentless deductive theory implies that possible interpretations (and hence applications) have to be found on the basis of the formal relations. See also the discussion in Ref. [157].

⁴² Brent Mundy made a related remark [158]:

There are several respects in which the standard formulation may be considered as inadequate or misleading, from a philosophical viewpoint. In the first place, it leaves some uncertainty as to what the theory is a theory *of*. Taking the standard presentation literally, it seems to be a theory of coordinate systems and their properties and relations. This is somewhat disturbing, since a coordinate system is, after all, an arbitrary and artificial human construct, part of our conceptual apparatus for the *description* of nature, rather than a proper part of the subject matter of physics itself.

part of Hamiltonian methods to regard physically (and not merely mathematically) possible transformations as generated by observable physical quantities. The saying that the Hamiltonian function itself is “the generator of translation in time”, expresses the content of Eq. 41. As we shall demonstrate in Sec. V and Sec. VI, the generators of both, rotations and boosts, can be identified with physical observables, namely the magnetic and electric fields, respectively⁴³. In the last example in Sec. VI it will be shown that even Maxwell's equations can be obtained from Hamiltonian considerations [12].

We introduced the notion of the Hamiltonian Clifford basis, from which all other elements of a Clifford algebra are generated. All basis elements combined give a four-parameter matrix \mathbf{F} with the form⁴⁴:

$$\mathbf{F} = \omega \gamma_0 + k_1 \gamma_1 + k_2 \gamma_2 + k_3 \gamma_3, \quad (60)$$

where $\gamma_0^2 = -1$ and $\gamma_k^2 = 1$ with $k = [1, 2, 3]$ are mutually anti-commuting Hamiltonian matrices. Using only these basic elements from which the Clifford algebra is generated, the equations of motion (Eq. 35) have the general form

$$\dot{\psi} = (\omega \gamma_0 + k_1 \gamma_1 + k_2 \gamma_2 + k_3 \gamma_3) \psi \quad (61)$$

so that we obtain a “2-dimensional” stable oscillator

$$\ddot{\psi} = (\omega \gamma_0 + k_1 \gamma_1 + k_2 \gamma_2 + k_3 \gamma_3)^2 \psi = -\omega_0^2 \psi \quad (62)$$

with the invariant eigenfrequency $\omega_0^2 = \omega^2 - k_1^2 - k_2^2 - k_3^2$ for $\omega_0^2 > 0$. This allows to obtain a purely Hamiltonian dispersion relation and as a matter of fact it is the correct relativistic dispersion relation (RDR). The only remaining step is to show that the time variable τ in the time derivative $\dot{\psi} = \frac{d}{d\tau} \psi$ is not the laboratory but indeed the proper time. Then, with the de Broglie relations from above⁴⁵, one obtains

$$\hbar \dot{\psi} = \mathbf{F} \psi = (\mathcal{E} \gamma_0 + p_1 \gamma_1 + p_2 \gamma_2 + p_3 \gamma_3) \psi \quad (63)$$

so that the mass $m = \sqrt{\mathcal{E}^2 - \vec{p}^2}$ is both, an invariant eigenvalue of \mathbf{F} , but also a constant of motion. However, it is a constant of the motion of ψ , which can not be observed directly in a classical sense (we come back to this in Sec. VIII). To describe the motion of the unobservable quantities ψ is of limited physical value. It is therefore required to change the dynamical variables and to introduce a new Hamiltonian that depends on the observables \mathcal{E} and \vec{p} , i.e. on the second moments \mathbf{S} . This step converts the status of the mass, the value of the first Hamiltonian, into a mere invariant parameter⁴⁶.

⁴³ However, electric and magnetic field are the generators of boosts and rotations not in coordinate (“physical”) space, but in *energy-momentum-space*, as immediately obvious from the mathematical form of the Lorentz force.

⁴⁴ The explicit form is given in Eq. B1.

⁴⁵ We shall come back to this in part two.

⁴⁶ This kind of flexibility to chose Hamiltonians is well established in the kind of classical mechanics developed for accelerators [16].

As we shall show below (Eq. 74), skew-symmetric Hamiltonian generators yield rotations and symmetric ones generate boosts. Since (as shown above) all skew-symmetric Hamiltonian generators⁴⁷ commute with γ_0 , they can't change the value of \mathcal{E} (see Eq. 76). Hence \mathcal{E} is the only rotationally invariant vector component known so far, and it is therefore nearby to use it as next Hamiltonian function. The canonical conjugate of the energy \mathcal{E} is a new time coordinate t . The relation between old (τ) and new time variable t follows from $\mathcal{H} = m c^2 = \sqrt{\mathcal{E}^2 - \vec{p}^2 c^2}$:

$$\begin{aligned} \frac{dt}{d\tau} &= \dot{t} = \frac{\partial \mathcal{H}}{\partial \mathcal{E}} \\ &= \frac{\mathcal{E}}{\mathcal{H}} = \frac{\mathcal{E}}{m c^2} \end{aligned} \quad (64)$$

We now use the ‘‘quantization rules’’ (Eq. 24) to replace the total derivative on the left side of Eq. 63 by the corresponding partial derivatives on the right to obtain the Dirac equation in the usual notation

$$\begin{aligned} -i m c^2 \psi &= i \hbar (\partial_t \gamma_0 - \partial_1 \gamma_1 - \partial_2 \gamma_2 - \partial_3 \gamma_3) \psi \\ m c^2 \psi &= i \hbar (\partial_t \Gamma_0 - \partial_1 \Gamma_1 - \partial_2 \Gamma_2 - \partial_3 \Gamma_3) \psi \end{aligned} \quad (65)$$

where $\Gamma_\mu = i \gamma_\mu$ are the conventional complex Dirac matrices corresponding to the conventional metric tensor $g_{\mu\nu} = \text{Diag}(1, -1, -1, -1)$. Hence the unit imaginary is, within our approach, an artifact of the preference for the metric $g_{\mu\nu} = \text{Diag}(1, -1, -1, -1)$ instead of the use of a metric $g_{\mu\nu} = \text{Diag}(-1, 1, 1, 1)$: The use of the unit imaginary in the Dirac equation is an exercise in redundancy.

It is mostly agreed that the sign of the metric has no physical significance⁴⁸. However, the conventional metric leads to a notation that suggests that the unit imaginary is a meaningful and necessary ingredient in Dirac's theory, something that generates ‘‘quantumness’’. But as we demonstrated, Dirac's theory allows for, but neither suggests nor requires the explicit use of the unit imaginary⁴⁹.

Coming back to the ‘‘particle picture’’ one obtains the new Hamiltonian dispersion relation $\mathcal{H}(\vec{p})$, in the new time coordinate t , which then reads

$$\mathcal{H} = \sqrt{m^2 c^4 + \vec{p}^2 c^2}, \quad (66)$$

which results in the Hamiltonian velocity of a free particle (Eq. 19):

$$\vec{v} = \vec{\nabla}_p \mathcal{H}(\vec{p}) = \frac{\vec{p} c^2}{\sqrt{m^2 c^4 + \vec{p}^2 c^2}} = \frac{\vec{p} c^2}{\mathcal{E}}. \quad (67)$$

where the velocity is, using the new Hamiltonian, the temporal derivative with respect to the coordinate time t (and not τ):

$$\vec{v} = \frac{d\vec{x}}{dt}. \quad (68)$$

If we scale to the constant c , then this reads as

$$\vec{\beta} = \frac{\vec{v}}{c} = \frac{\vec{p} c}{\mathcal{E}}. \quad (69)$$

Solving for \mathcal{E} and \vec{p} , one readily obtains

$$\begin{aligned} \mathcal{E} &= m c^2 \gamma \\ \vec{p} &= m c \gamma \vec{\beta} \end{aligned} \quad (70)$$

using the usual definition of $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. Combining Eq. 64 and Eq. 70, we obtain ‘‘time dilation’’ $dt = \gamma d\tau$ as a result of a canonical transformation. In a preceding paper we elaborated in detail that the Lorentz transformations are canonical symplectic similarity transformations and have their (conceptually) simplest representation in the 4×4 real Dirac algebra [134]. In the next section we will sketch the general setting.

As well-known, one arrives at the Newtonian expression in the usual approximation, taking only the first terms of the Taylor serie of $\mathcal{E}(\vec{p})$:

$$\mathcal{E} = m c^2 + \frac{\vec{p}^2}{2m} + \dots \quad (71)$$

which yields, due to $\vec{v} = \vec{\nabla}_p(E)$ Newton's $\vec{p} = m \vec{v}$. Furthermore the theory defines, what may and what may not be constant. If \mathbf{S} and \mathbf{F} commute, then both \mathcal{E} and \vec{p} and hence the velocity \vec{v} is constant. This, in some sense, (re-) establishes Newton's first axiom – from a Hamiltonian point of view.

V. FOURTH EXAMPLE: LORENTZ TRANSFORMATIONS

It is well known and understood from the theory of Lie algebras that Hamiltonian observables are generators of canonical transformations. Usually, when we employ a Hamiltonian description of a system of classical oscillators, our (macroscopic) description of the involved masses and spring constants determines the exact form of the matrix \mathbf{F} , i.e. which of the 10 possible parameters of \mathbf{F} vanish, which do not, and to what physical quantity they are related. But since we aim for the most general description, we have no reason to assume that certain elements of \mathbf{F} have some specific value. Since there are 10 free parameters in \mathbf{F} in total, six parameters are left to be discussed.

⁴⁷ It will be shown below that skew-symmetric matrices generate rotations while symmetric matrices generate boosts.

⁴⁸ Most textbooks use the metric $g_{\mu\nu} = \text{Diag}(1, -1, -1, -1)$. Weinberg's books on quantum field theory however uses $g_{\mu\nu} = \text{Diag}(-1, 1, 1, 1)$ [159].

⁴⁹ Since Schrödinger's original equation does not use spinors, the wave function must be complex in order to provide a canonical pair [13, 31, 132, 160].

These 6 parameters can be divided into two groups, firstly a set of three symmetric matrices

$$\begin{aligned}\gamma_4 &= \gamma_0 \gamma_1 \\ \gamma_5 &= \gamma_0 \gamma_2 \\ \gamma_6 &= \gamma_0 \gamma_3\end{aligned}\quad (72)$$

and secondly a set of three skew-symmetric matrices:

$$\begin{aligned}\gamma_7 &= \gamma_{14} \gamma_0 \gamma_1 = \gamma_2 \gamma_3 \\ \gamma_8 &= \gamma_{14} \gamma_0 \gamma_2 = \gamma_3 \gamma_1 \\ \gamma_9 &= \gamma_{14} \gamma_0 \gamma_3 = \gamma_1 \gamma_2.\end{aligned}\quad (73)$$

It is a math fact that bi-vectors, products of two Hamiltonian basis elements γ_ν , are also Hamiltonian, while 3-vectors and 4-vectors are skew-Hamiltonian [12, 13, 136, 137]. Therefore the 6 missing Hamiltonian parameters come in two sets of 3 bi-vector elements each. Note that this grouping into 3-vectors results from Hamiltonian symmetries.

If we consider the general properties of transformations using Eq. 41 with single Hamiltonian Clifford elements γ_a for which $\gamma_a^2 = \pm \mathbf{1}$:

$$\begin{aligned}\mathbf{M}_a(\tau) &= \mathbf{1} \cos(\tau) + \gamma_a \sin(\tau) & \text{if } \gamma_a^2 &= -\mathbf{1} \\ &= \mathbf{1} \cosh(\tau) + \gamma_a \sinh(\tau) & \text{if } \gamma_a^2 &= \mathbf{1}\end{aligned}\quad (74)$$

Note that $\mathbf{M}^{-1}(\tau) = \mathbf{M}(-\tau)$ holds for all transformations of Eq. 74. Whether such a transformation leaves some element constant or not, depends exclusively on the commutation properties of the algebra. Since the transformation matrices for pure transformations Eq. 74 contain only $\mathbf{1}$ and γ_a , they commute with some γ_b exactly, if γ_a and γ_b commute. Then the coefficient of γ_b remains unchanged by the similarity transformation

$$\begin{aligned}\tilde{\gamma}_b &= \mathbf{M}_a \gamma_b \mathbf{M}_a^{-1} \\ \tilde{\gamma}_b &= \gamma_b.\end{aligned}\quad (75)$$

If γ_a and γ_b anti-commute ($\gamma_a \gamma_b = -\gamma_b \gamma_a$), however, we obtain (rotations, $\gamma_a^2 = -\mathbf{1}$):

$$\begin{aligned}\tilde{\gamma}_b &= \mathbf{M}_a(\tau/2) \gamma_b \mathbf{M}_a^{-1}(\tau/2) \\ &= (\mathbf{1} \cos(\tau/2) + \gamma_a \sin(\tau/2)) \gamma_b \\ &\quad \times (\mathbf{1} \cos(\tau/2) - \gamma_a \sin(\tau/2)) \\ &= (\cos^2(\tau/2) - \sin^2(\tau/2)) \gamma_b \\ &\quad - 2 \sin(\tau/2) \cos(\tau/2) \gamma_b \gamma_a \\ &= \cos(\tau) \gamma_b - \sin(\tau) \gamma_b \gamma_a\end{aligned}\quad (76)$$

and boosts, correspondingly, for $\gamma_a^2 = \mathbf{1}$ [134]:

$$\tilde{\gamma}_b = \cosh(\tau) \gamma_b - \sinh(\tau) \gamma_b \gamma_a \quad (77)$$

Hence any symplectic similarity transformation with pure Clifford elements results in a rotation in phase space

for skew-symmetric matrices $\gamma_a^2 = -\mathbf{1}$ and in a boost for symmetric matrices $\gamma_a^2 = \mathbf{1}$. Other, polynomial solutions are also possible, but they do not represent non-singular systems and we do not address them here [161].

Many textbooks on QED do not elaborate the Lorentz transformation of Dirac spinors in detail⁵⁰. We therefore refer to a preceding paper in which we explicitly elaborated the Lorentz transformations on the basis of these Hamiltonian notions [134]. It is both, a result of these investigations, but also well-known in Dirac's theory that the components of the symmetric bi-vector are generators of boosts and transform like the electric field, i.e. like a so-called "radial" bi-vector $E_x \gamma_4 + E_y \gamma_5 + E_z \gamma_6$. The components of the skew-symmetric "axial" bi-vector are generators of rotations and transform like the components of the magnetic field vector $\vec{B} = B_x \gamma_7 + B_y \gamma_8 + B_z \gamma_9$.

Hence there is another matrix \mathbf{F} , which consists of electromagnetic bi-vector components⁵¹:

$$\mathbf{F} = E_x \gamma_4 + E_y \gamma_5 + E_z \gamma_6 + B_x \gamma_7 + B_y \gamma_8 + B_z \gamma_9. \quad (78)$$

The eigenfrequencies of this matrix are the known relativistic invariants

$$\omega = \pm \sqrt{\vec{B}^2 - \vec{E}^2 \pm 2 \sqrt{-(\vec{E} \cdot \vec{B})^2}}. \quad (79)$$

Of course, this equation makes only sense, if we can express fields in units of frequencies. But the required physical scaling constants are known today and effectively this means little more than to express the electromagnetic fields in units of Schwinger's limiting fields [160].

The representation of structure by numbers as implemented by the use of the Dirac algebra *automatically* delivers the most compact form of the Lorentz transformations [134], but also the invariants of electromagnetic fields, even before we derived or even considered Maxwell's equations at all.

Then it should not be surprising that also the Lorentz force and Maxwell's equations pop out [12]. In order to better distinguish vector components (Eq. 60) from the bi-vectors components (Eq. 78) and the total Hamiltonian matrix, we use a bold \mathbf{P} for the 4-momentum:

$$\mathbf{P} = \mathcal{E} \gamma_0 + p_x \gamma_1 + p_y \gamma_2 + p_z \gamma_3 \quad (80)$$

and $\frac{q}{m} \mathbf{F}$ for the bi-vectors (Eq. 78). The factor $\frac{q}{m}$ enters to obtain the equations in the usual system of units (see also Ref. [160]). Then Eq. 48 can be written as follows:

$$\dot{\mathbf{P}} = \frac{q}{2m} (\mathbf{F} \mathbf{P} - \mathbf{P} \mathbf{F}). \quad (81)$$

⁵⁰ The best presentation known to the author, albeit in German, can be found in Schmüser's book [162].

⁵¹ The explicit form is given in Eq. B2.

Written explicitly in vector components we have [12, 13, 136]:

$$\begin{aligned}\frac{d\mathcal{E}}{d\tau} &= \frac{q}{m} \vec{p} \cdot \vec{E} \\ \frac{d\vec{p}}{d\tau} &= \frac{q}{m} \left(\mathcal{E} \vec{E} + \vec{p} \times \vec{B} \right)\end{aligned}\quad (82)$$

Using the lab frame time $dt = \gamma d\tau$ these equations are identical to the usual Lorentz force equations (for $c = 1$). Hence also the Lorentz force can be obtained purely on Hamiltonian grounds, even without knowledge of Maxwell's equations.

VI. LAST EXAMPLE: MAXWELL'S EQUATIONS

Eq. 48 has another important implication: The change of a Hamiltonian of the left is side is connected to a product, namely the skew-symmetric product, of two Hamiltonian matrices on the right side. This is important, because it connects the time evolution of k -vectors with $k \pm j$ -vectors by a multiplication.

We call a Hamiltonian Clifford algebra irreducible, if the maximal number of variables in the matrix representation $2n \times 2n = 4n^2$ corresponds to the number of elements of the Clifford algebra, which is 2^N . Equating these numbers $4n^2 = 2^N$ provides evidence that all irreducible Hamiltonian Clifford algebras have an even dimension N . In $p + q = N$ is even, then it is impossible to obtain all elements from bi-vectors only. No multiplication of any number of even elements may produce odd elements (vectors, 3-vectors). If in Eq. 44, both \mathbf{F} and \mathbf{S} , are even and Hamiltonian, i.e. bi-vectors, then the left side is either a scalar or a pseudoscalar or another bi-vector. It can not be a vector.

Interpreting Eq. 44 physically, we can construct bi-vectors from the interaction of vector quantities but not vice-versa. Hence we may regard vectors as representations of particles and bi-vectors as representations of fields, generated by particles. Bi-vectors are the generators of rotations and boosts of vectors, but they can not directly be used to establish vectors by any kind of Lorentz covariant multiplication as in Eq. 44.

It is part of Hamiltonian theory to distinguish mechanical and canonical momentum. The (possible) difference appeared before in Eq. 21: The relation between velocity and momentum allows for additional components \vec{A} ; correspondingly the energy may contain an additional term ϕ . When established by Eq. 21, then we have to consider an additional vector type quantity ϕ, \vec{A} that depends on coordinates only. It follows that we must in general regard those quantities that do not depend on the momentum, i.e. the bi-vector coefficients, as dependent on the corresponding canonical coordinates:

$$\begin{aligned}\vec{E} &= \vec{E}(\vec{x}, t) \\ \vec{B} &= \vec{B}(\vec{x}, t)\end{aligned}\quad (83)$$

and we should expect that these components can be obtained from vector type quantities ϕ and \vec{A} .

Again, as in the first two examples, the Hamiltonian method allows to derive equations of motion for new variables, this time for the Maxwellian bi-vector fields. First we need a derivative operator that is compatible with the Hamiltonian-Clifford framework elaborated so far: It must allow for the described symplectic similarity transformation. The derivative operator is, of course, a vector type quantity:

$$\partial \equiv -\partial_t \gamma_0 + \partial_x \gamma_1 + \partial_y \gamma_2 + \partial_z \gamma_3. \quad (84)$$

As established by Eq. 44, Hamiltonian motion is connected to symmetric products (anti-commutators) and skew-symmetric products. Then matrix multiplication from the right combined with a derivative ∂ requires to indicate the direction in which the differentiation acts. We indicate the direction by arrows in what follows. The commutative derivative is

$$\partial \wedge \mathbf{A} \equiv \frac{1}{2} \left(\vec{\partial} \mathbf{A} - \mathbf{A} \overleftarrow{\partial} \right) \quad (85)$$

and the *anti-commutative*

$$\partial \cdot \mathbf{A} \equiv \frac{1}{2} \left(\vec{\partial} \mathbf{A} + \mathbf{A} \overleftarrow{\partial} \right) \quad (86)$$

Four different derivatives are possible with following results:

$$\begin{aligned}\partial \wedge \text{vector} &\Rightarrow \text{bi-vector} \\ \partial \wedge \text{bi-vector} &\Rightarrow \text{vector} \\ \partial \cdot \text{vector} &\Rightarrow \text{scalar} = 0 \\ \partial \cdot \text{bi-vector} &\Rightarrow \text{axial vector} = 0\end{aligned}\quad (87)$$

There is only one unique way to express bi-vector fields from such a derivative – it is the commutative derivative of a four vector, according to the first of Eq. 87. This demonstrates the rigidity of Hamiltonian notions. We may now write this equation, using the vector type “potential” $\mathbf{A} = \gamma_0 \phi + \vec{\gamma} \vec{A}$

$$\mathbf{F} = \partial \wedge \mathbf{A}, \quad (88)$$

or explicitly in components:

$$\begin{aligned}\vec{E} &= -\vec{\nabla} \phi - \partial_t \vec{A} \\ \vec{B} &= \vec{\nabla} \times \vec{A}.\end{aligned}\quad (89)$$

This is the only possible linear Hamiltonian definition of the electromagnetic field from vector type quantities and it explains the meaning of the “integration constants” appearing in Eq. 21.

The second of Eq. 87 suggests that the “source” of a bi-vector field is again a vector:

$$\partial \wedge \mathbf{F} = 4\pi \mathbf{J}, \quad (90)$$

which can be regarded as a *definition* of the *vector current*

$$\mathbf{J} = \rho \gamma_0 + j_x \gamma_1 + j_y \gamma_2 + j_z \gamma_3. \quad (91)$$

Written explicitly in components, Eq. 90 is given by⁵²:

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= 4\pi\rho \\ \vec{\nabla} \times \vec{B} - \partial_t \vec{E} &= 4\pi\vec{j}. \end{aligned} \quad (92)$$

The third of Eq. 87 then yields the continuity equation and likewise the Lorentz gauge. It is a trivial consequence of Eq. 90:

$$\partial \cdot \mathbf{J} = \frac{1}{16\pi} \left(\overset{\leftarrow}{\partial}^2 \mathbf{F} - \overset{\rightarrow}{\partial} \mathbf{F} \overset{\leftarrow}{\partial} + \overset{\rightarrow}{\partial} \mathbf{F} \overset{\leftarrow}{\partial} - \mathbf{F} \overset{\leftarrow}{\partial}^2 \right) = 0. \quad (93)$$

Note that $\overset{\leftarrow}{\partial}^2$ and $\overset{\rightarrow}{\partial}^2$ are scalars (d'Alembert's operator $\square = \vec{\nabla}^2 - \partial_t^2$) and hence $\overset{\rightarrow}{\partial} \mathbf{F} - \mathbf{F} \overset{\leftarrow}{\partial} = 0$. Written in components, Eq. 93 is equal to

$$\partial_t \rho + \vec{\nabla} \cdot \vec{j} = 0. \quad (94)$$

Finally, the last of Eq. 87 gives

$$\partial \cdot \mathbf{F} = 0 \quad (95)$$

which are the homogeneous Maxwell equations, when written in components:

$$\begin{aligned} \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} + \partial_t \vec{B} &= 0. \end{aligned} \quad (96)$$

From a rigorous Hamiltonian point of view, this is the proper way to establish Maxwell's equations, namely a way that *inherently implies* the nature of their "covariance".

Note that neither the autocorrelation matrix \mathbf{S} nor the Hamiltonian matrix \mathbf{F} may contain non-zero coefficients for the skew-Hamiltonian elements of the Dirac algebra, i.e. for the scalar $\gamma_{15} \equiv \mathbf{1}$, pseudo-scalar γ_{14} and the axial vector components $\gamma_{14} \gamma_\nu$. Hence we must demand that the corresponding derivatives vanish (in the linear approximation we discuss here) as indicated in Eq. 87. But as we have seen, this comes out automatically from the formalism as a consequence of the fact that the space-time-derivative must be a vector in the Hamiltonian Clifford Algebra $Cl(3, 1)$.

VII. AFTERMATH

The usual mind-set of modern physics as to be found in standard textbooks suggests that theorizing

⁵² We have shown in Ref. [160] that these equations are compatible with the Dirac current.

in fundamental physics starts with the presumption of some background space-time, some kind of mathematical space, often equipped with fancy mathematical features⁵³. Minkowski's space-time is such a background and it (re-) produces the mathematical feature of Lorentz covariance. Mathematically there is nothing wrong with this. But this mode of thinking is, from a logical point of view, disturbing: In textbooks on QM it is asserted that the "majority" of physicists has accepted the Copenhagen interpretation⁵⁴ of QM, which asserts that at the atomic level physical processes can not be described as happening objectively in space and time. But on the next page, the conventional lore insists that, on the fundamental level, there must be a space-time with unexplainable ("given") physical properties. This excludes the principle possibility that the dimensionality of space-time has itself a *dynamical* reason.

We also started with the assumption of a Newtonian space-time in the second example: we presumed some Euclidean space-time and a distributed amount of matter in it, described by a (normalizable) density distribution. In the third example however, we addressed the question whether we can derive some general kind of dispersion relation from nothing but Hamiltonian (i.e. dynamical) notions. Once the idea to consider the space of possible linear canonical transformations with two abstract classical canonical pairs of dynamical variables – as suggested by Dirac – is considered, the structure of spacetime as described by special relativity, follows with logical necessity. Therefore the Hamiltonian approach described here is based on a different kind of fundamental background, which has the form of a (symplectic) phase space, similar to the Γ -space of classical statistical mechanics. However, this phase space received, in the course of reasoning, a new kind of ontological significance. In other cases, like accelerator physics, the terms that can possibly be found in the matrix \mathbf{F} , can be (and actually are) derived from spelling out the consequences of classical 3-dimensional reasoning. However, in the presented approach the logic needs to be reversed: All *mathematically possible* terms of the abstract phase space are considered on equal footing. Nonetheless they allow to derive the dynamical symmetries and relations that hold in a space of higher dimension, namely in Minkowski space-time.

In the historical presentation of special relativity, the lack of a physical/logical legitimization of the Lorentz transformations left room for quite a number of alternative "space-time" transformations⁵⁵. Per-Olov Löwdin has shown, that few general and reasonable assumptions about space suffice to constrain the possibilities to two

⁵³ Despite the fact that the majority of physicists is said to have accepted the Copenhagen interpretation of QM, which asserts that at the atomic level physical processes can not be described as being objective in space and time.

⁵⁴ Despite the fact that the real content of this interpretation is still controversial.

⁵⁵ See Ref. [163] and references therein.

forms of space-time transformations, namely those of Galileo and Lorentz [164]. But the Hamiltonian framework that we described is even more restrictive and does not require the assumption of space-time at all. Furthermore it incorporates the Hamiltonian viewpoint that physically possible coordinate transformations of dynamical systems must be canonical and are generated by physical quantities.

Since the structure of Minkowski’s space-time can be derived from little more than the most general linear interaction of two canonical pairs, this suggests a presentation in which space-time (geometry) is derived from dynamics and not vice-versa. While the second example was based on the Euklidian/Newtonian meta-physics of absolute space and time, the prioritization has changed with the Dirac equation: now the nature of space-time stems from the structure of the underlying phase space as represented by the Dirac Clifford algebra. But we do not simply postulate to use some Clifford algebra, possibly for reasons of convenience. We have shown that the use of Clifford algebras can be motivated from Hamiltonian symmetries only⁵⁶, and they receive additional and important constraints from these Hamiltonian symmetries, namely the distinction between Hamiltonian and skew-Hamiltonian (matrix) elements.

Lorentz transformations, the Lorentz force and even Maxwell’s equations are obtained by this type of Hamiltonian deduction. This raises the question, if one could possibly formulate a similar approach, based on larger phase spaces, for a hypothetical world with more or less than 3+1 dimensions. We are not going to discuss this in detail, we just mention some restrictions resulting from Hamiltonian notions⁵⁷.

Neither Newtonian physics nor Einstein’s relativity provide any *intrinsic* argument for the dimensionality of space-time. In both theories space-time is postulated as if it was one of the ten commandments⁵⁸. Here we want to raise awareness for the fact, that other coherent narratives indeed do exist: Maybe it is wrong to think that physical theorizing must presume a specific space-time dimension to begin with.

Clifford algebras $Cl(p, q)$ with Hamiltonian basis exist only in dimension $Cl(N - 1, 1)$. But it is a math fact called “Bott-periodicity” that Clifford algebras with real matrix representations exist only for certain dimensions, namely with $q = 1$ and $p = N - 1$ we can only have

$$p - q = N - 2 = 0, 2 \pmod{8}. \quad (97)$$

This means that irreducible space-times in analogy to the Hamiltonian derivation of Minkowski space-time exist only for a subset of (hypothetical) space-times, namely

for 1+1, 3+1, 9+1, 11+1, . . . , 25+1, 27+1 etc. dimensions⁵⁹. We think that the mentioned points are remarkable results, which demonstrate how *restrictive* Hamiltonian notions actually are (see also Fig. 3).

VIII. WHY “HAMILTONIAN” NOTIONS, NOT “LAGRANGIAN”?

The power of Lagrangian mechanics has caused generations of students to wonder why it is necessary or even desirable, to recast mechanics in Hamiltonian form. The answer [...] is that the Hamiltonian formulation is a much better basis from which to build more advanced methods. The Hamiltonian equations have an elegant symmetry that the Lagrangian equations lack [167].

Most common textbooks on classical mechanics follow a “historical” approach which usually starts with Newtonian mechanics, continues with Lagrangian mechanics as a formal generalization of Newton’s, then Hamiltonian mechanics (as a variant of Lagrange’s) and eventually spend a few words on Hamiltonian-Jacobi theory as yet another formal method. But after having solved the standard textbook examples using Newtonian and Lagrangian methods, it seems to be an exercise in redundancy to re-iterate known solutions of standard problems using yet another method. It is hardly possible that students grasp the fundamental differences between these theories, which can not be found in the solutions of age-old textbook problems but in their respective logical construction. Taught this way, the students must disentangle on their own that Hamiltonian mechanics is not at all about mechanics, but has a much broader scope and applicability, especially when combined with statistical physics. The concepts of force and inertia, central to Newtonian physics, are mostly absent in Hamiltonian physics.

Newtonian mechanics has, strictly speaking, been falsified by special relativity and can therefore not any more be regarded as fundamental to physics. Though it is possible to force the notation of relativistic mechanics into a Newtonian form, this can not belie the fact that Newtonian physics requires fundamental revision in relativity. Not so with Hamiltonian notions. The Hamiltonian function $\mathcal{H}(q_i, p_i)$ is different in relativistic mechanics compared to non-relativistic mechanics, but Hamiltonian methods do not require a specific ontology to be applicable: Hamiltonians are by no means intrinsically restricted to the form $\mathcal{H}(q, p) = T(p) + V(q)$.

So what about Lagrangian mechanics? Some modern textbooks on dynamics drop the variational approach

⁵⁶ For more details see Refs. [12, 13].

⁵⁷ For more details see Refs. [12, 13, 160].

⁵⁸ See also Stenger [165].

⁵⁹ Apparently also string theorists found reasons to consider “space-times” of the some of these dimensions, namely of 10 and 26 [166].

completely, mostly due to mathematical drawbacks⁶⁰. Many bright physicists and mathematicians have understood that the Hamiltonian form is conceptually superior. Cornelius Lanczos for instance wrote that⁶¹

[Hamilton's] equations are entirely equivalent to the original Lagrangian equations and are merely a mathematically new form. Yet the new equations are vastly superior to the originals. For derivatives with respect to t appear only on the left-hand sides of the equations, since the Hamiltonian function does not contain any derivatives of q_i or p_i with respect to t .

He further continues⁶²

A further reason why the Hamiltonian equations are superior to the Lagrangian equations in their transformation properties is that the number of variables is doubled. If at first sight this increase seems more of a loss than a gain, the procedure of coordinate transformations turns the liability into an asset. It is of great advantage that we can widen the realm of possible transformations by having a larger number of variables at our disposal.

Finally, in Lagrangian mechanics we do not possess any systematic method for the simplification of the Lagrangian function. We may hit on ignorable variables by lucky guesses, but there is no systematic way of producing them. In Hamiltonian mechanics a definite method can be devised for the systematic production of ignorable variables and the simplification of the Hamiltonian function. This method, which reduces the entire integration problem to the finding of one fundamental function, finally, in the generating function of a certain transformation, plays a central role in the theory of canonical equations and opens wide perspectives [...].

But there is another problem, rarely spelled out explicitly, namely that the *principle of least action*, despite its admitted elegance and beauty, and its widespread use in theoretical physics, is inappropriate to serve as *the* foundational principle of physics. As Pulte reported [8], not only Jacobi but many classical physicists and mathematicians criticized this principle: “Lagrange’s formulation and (or) demonstration of the principle of virtual velocities posed a challenge for a number of mathematicians from Fourier (1798), de Prony (1798), Laplace (1799), L.

Carnot (1803), and Ampere (1806) to Cournot (1829), Gauss (1829), Poisson (1833), Ostrogradsky (1835, 1838), and Poinso (1806, 1838, 1846)”. According to Pulte, Lagrange’s attempt to base mechanics on the principle of virtual velocities “leads inevitably to a conflict with the traditional meaning of axiom as a self-evident first proposition, which is neither provable nor in need of a proof.” Hence if we recall the “original goal” of science, namely “clearing up mysteries” [170], then Hamiltonian notions should play a central role.

We have been trained to swallow the “axioms” of quantum theory, and it therefore appears somewhat old-fashioned to demand that axioms should be *self-evident first propositions, which are neither provable nor in need of a proof*. Many physicists seem to have lost trust in such apparently naive positions. Wasn’t this exactly Newton’s mistake – to believe that space, time and motion were self-evident? The fundamental theory of nature, quantum theory, we have been told, is counter-intuitive and requires even a new non-classical logic. Why then should we expect that the theory could be based on axioms in the classical sense? But as we could show, quantum theory can be developed mathematically straight from generalized classical Hamiltonian notions and classical arguments of (local) causality. The foundation of this formalism are Hamiltonian notions which are so vastly general in their essence, that (when correctly applied) they cannot fail to describe physical reality.

In its very core, Hamiltonian methods are based on one simple definition, namely that, for any stable closed physical system with the dynamical variables ψ , there exists a function $\mathcal{H}(\psi) = \text{const}$. This is neither a postulate nor a principle, it is mostly a definition of the term “closed physical system”.

The value of any axiomatic and deductive method depends on the self-evidence of the axioms, whether this is admitted or not⁶³. We are not the first to criticize the arbitrariness of the axioms of quantum theory. Fuchs, to pick out a prominent name, wrote about the “standard axioms” of quantum mechanics [171]:

The task is not to make sense of the quantum axioms by heaping more structure, more definitions, more science-fiction imagery on top of them, but to throw them away wholesale and start afresh.

⁶⁰ See for instance Ref. [168], §12.98, page 140.

⁶¹ See page 167 in Ref. [169].

⁶² See page 194 in Ref. [169].

⁶³ The attribution of developments to individual scientists is not always as clear as naming conventions suggest: “The symplectic formulation of Hamiltonian mechanics can be retraced (in embryonic form) to the work of Lagrange between 1808 and 1811; what we today call ‘Hamilton’s equations’ were in fact written down by Lagrange who used the letter H to denote the ‘Hamiltonian’ to honor Huygens – not Hamilton, who was still in his early childhood at that time! It is however undoubtedly Hamilton’s great merit to have recognized the importance of these equations, and to use them with great efficiency in the study of planetary motion, and of light propagation” [92].

As our examples illustrate, it seems possible to drop the idea that the finding of “deep” and “profound” principles will eventually provide the corner stones of this theory. The decisive starting point of physical science is not necessarily a list of counter-intuitive axioms that are “inscribed on stone tablet” [165]. The decisive starting point, in our opinion, are the primary distinctions that the theory is based upon.

In the conventional lore Newtonian physics is based on axioms that provide some substantial knowledge about the physical world, namely that force and acceleration are related by $F = m \cdot a$. This equation however represents not a fact about nature (as general relativity has shown), it is not even a counter-factual assertion about nature: in the first place it is a definition and a distinction. It is a definition of what the theory is about. Even if Newton’s definitions would be circular, as occasionally suspected [172], they nonetheless provide an account of the conceptional framework to be used. It is a distinction between quantities that are contingent (initial position and velocity) and those that require (and allow for) explanation, namely acceleration. As Poincare argued, in many ways Newton’s theory contains core elements of Hamiltonian mechanics: The initial state of a physical object is specified by pairs of values for each coordinate, namely by coordinate position and -velocity. This indirect insight is closely related to the notion of phase space in Hamiltonian mechanics⁶⁴. Classically can be defined without Newton’s axioms and then it leads, as we have shown, directly to the pinnacle of modern physics, namely to Dirac’s incredible spinor theory.

With the development of thermodynamics, the principle of energy conservation *practically* became a defining principle in physics: no patent office (and no properly educated physicist) will accept any suggested device or theory that is in conflict with energy conservation. A violation of the principle of energy conservation is, by definition, “un-physical”. But is energy conservation really a self-evident first proposition, which is neither provable nor in need of a proof? Why then did it take so long to discover this principle? We argue here that there is no need to specify the kind of conserved quantity. It fully suffices to accept that the whole universe (or whatever fraction of it) could, at least in principle, be regarded as a closed system (though only *in theory*). Then there must be a universal scale, i.e. a universal quantity that is conserved over all subsystems. This is what “energy” means after all, at least operationally. If one regards energy conservation from a more general point of view, it is essentially identical to what is otherwise called “object

permanence”: It says, from a bird’s eye view, that any “substantial” quantity must be preserved in a physical world.

Hence it suffices that Hamiltonian mechanics derives from *some* universal additive conserved quantity; there is no need to introduce the *physical* notion of energy a priori. The essential distinction that Hamiltonian mechanics rests upon, is between dynamical variables and constants of motion, or vulgo: between those things that change and those that do not. Neither can we think of any “deeper” physical distinction, nor of any that is more essential and more trivial. However, a constant can only be incorporated properly into pure Hamiltonian physics as a constant of motion as we shall argue in Sec. VIII A. Furthermore, this distinction has a corollary: A causal theory requires that change must be derivable smoothly from the actual state of affairs.

As we mentioned in the beginning, Hamiltonian mechanics is further profaned (metaphysically) by the “theorem due to Lie and Koenigs on the reduction of any system of ordinary differential equations to the Hamiltonian form.”⁶⁵. Since any classical dynamical theory has the form of a set of ordinary differential equations, any classical theory can be reduced to the Hamiltonian form, at least *locally*. Then Hamiltonian mechanics is little more than a “method” and boils down to the mere possibility to describe some physical system by a number ν of variables $\psi = (\psi_1, \psi_2, \dots, \psi_\nu)^T$ that obey some set of ordinary differential equations

$$\dot{\psi} = f(\psi). \quad (98)$$

Then, within the limits of applicability of this theorem, any dynamical law can be constructed from a conservation law. These facts suggest that Hamiltonian mechanics is in itself a “law without law”⁶⁶.

A. Units, Artifacts and Constants of Motion

Metrology truly is the Mother of Science! [174]

There is another logical reason to prefer – at the foundation of physics – a conservation law over Eq. 98 and ‘Hamiltonian’ over ‘Lagrangian’ methods. This reason is so basic and simple that it is rarely acknowledged at all. Einstein casually raised the issue in a contemplation on special relativity [175]:

It is striking that the theory (except for the four-dimensional space) introduces two kinds of things, i.e. (1) measuring rods and clocks, (2) all other things, e.g., the electromagnetic field, the material point, etc. This,

⁶⁴ It is not our intention to question the value of Newton’s (or Heisenberg’s) work or to violate the feelings of those who desire to celebrate the achievements of these giants. But beyond legitimate hero worship, students should not be misled to confuse a modern understanding of classical physics with point particles and Newtonian “common sense”.

⁶⁵ See page 275ff in Ref. [1], and Ref. [173].

⁶⁶ We shall come back to this in part two.

in a certain sense, is inconsistent; strictly speaking, measuring rods and clocks should emerge as solutions of the basic equations [...], not, as it were, as theoretically self-sufficient entities.

While good introductory textbooks on physics should contain a passage on weights and measures, most advanced textbooks (and theories) take their existence for granted. However, as Bunge remarked,

Being a theoretical concept, the notion of a unit should be elucidated within a theory proper. And being a generic notion, i. e. one occurring in every branch of quantitative science, its elucidation through a mathematical theory is a task for the foundations of science[176].

The short version of our argument is as follows: Weights and measures are, evidently, not only the practical but also the logical basis of physics as a quantitative “exact” experimental science. If we are supposed to provide a theoretical account of weights and measures, for instance of a measuring rod of a certain length, then this *implies* that we presume further *underlying* (maybe yet unknown) laws of physics, equations from a *more* fundamental level of reality, from which standards *can* emerge, at least in principle. The mass of a weight and the length of a rigid rod can only be constant, if they are functions of other conserved quantities that originate on some more fundamental level. We have no other idea of how they could physically *emerge*, within Hamiltonian theory, in any other way than as constants of motion of the underlying microphysical dynamical system: Either one finds a physical system in which a distance or radius (or some other physical quantity) is a conserved quantity or one derives mathematical relationships which allow to express a constant distance as a function of other conserved quantities. The precondition for the measurement of a physical quantity is the existence of some physical object or artifact for which a property of the same physical dimension is a (conserved) constant property. But if the precondition to measure some quantity is the availability of an artifact providing a constant reference standard of the same type and dimension, then there must necessarily be one more level of dynamical quantities – *below* the level of the most fundamental measurable quantities.

Eventually this implies that either there is no fundamental level at all, or if there is something like a most basic level, then it consists of dynamical variables that can not be directly measured, because by definition no level *below* this most basic level may exist that could provide a measurement standard. This most fundamental level must therefore be represented by variables for which no reference standard is available. Then they can not be measured “directly” and the corresponding variables must remain abstract [12]: These most fundamental variables are phase space coordinates, but in an *abstract*

phase space, in a space without predefined ontological interpretation.

Hence, also from the standpoint of metrology, we must inevitably presuppose the validity of conservation laws before we can even start to make our first measurement: But since *some* conservation law has to be presumed *anyhow*, what motivation remains to “derive” them from a “least action principle”?

IX. SUMMARY AND CONCLUSIONS

The prevalent historically oriented presentation of physical theories has drawbacks. Not only does the historical account presented in standard textbooks often lack historical precision, it is evidently not even intended to give a correct *historical* account, not to speak of an account that would be subscribed by professional historians of science [177–181].

Furthermore, as Ralston pointed out, *the order matters* [77]. To place something in an early stage suggests fundamental relevance. For instance, since the order of presentation in textbooks on classical mechanics usually begins with Newton’s axioms, continues with Lagrangian mechanics and finally ends up in Hamiltonian mechanics (sometimes followed by Hamilton-Jacobi theory), students are lead to conclude that force is fundamental and that Hamiltonian notions are but an abstract reformulation of Newtonian mechanics. This however is a misrepresentation of the logical hierarchy. Hamiltonian methods neither require Newton’s axioms nor his space-time metaphysics to achieve meaning and validity. As we have shown, it is rather the other way around: Newtonian mechanics can be reconstructed as an approximation of results that can be derived using pure Hamiltonian methods. The advantage of the latter approach is the possibility to avoid obscure metaphysical axioms.

Not only did the Hamiltonian methods survive the scientific revolutions of the 20th century, due to their abstractness and generality one must conclude that they can be applied to any imaginable physical reality. As Whittaker taught a century ago, they are but a general mathematical set of analytical methods that can be applied whenever we consider dynamical systems that depend on a timelike parameter. Therefore, we think, it is inappropriate to merely distinguish between classical physics and quantum physics. There is the old “phenomenological” classical mechanics (Newton’s) and the new abstract classical mechanics (Hamilton’s). They differ as much as the old “phenomenological” quantum theory of Bohr and Heisenberg differs from the new abstract quantum theory of Schrödinger and Dirac⁶⁷. In both cases there is an old theory that is contaminated with metaphysical presuppositions and a new theory in which this extra baggage can be dismissed as dead weight [182]:

⁶⁷ See also Chap. 1-3 in Ref. [77].

In 1924, Louis de Broglie proposed that particles like electrons also have wave properties. The new quantum theory by Erwin Schrödinger in 1926 is a wave theory. But the developers of the old quantum theory had no thought that matter might be wave-like. For them, matter followed deterministic classical orbits, restricted only by the quantum conditions imposed on the action variables of the action-angle theory. Given that major conceptual flaw, it is surprising that the old quantum theory was able to explain as much of the experimental physics of the day as it did.

Coming back to the question raised in the setup: Apparently one can find Hamiltonian descriptions with unexpected predictive power because different levels of physical description are “vertically” connected by their respective Hamiltonian constraints. In the proper Hamiltonian formulation of Dirac’s theory no constraints from a spatio-temporal level exist and therefore all mathematically possible terms, all *Hamiltonian* terms, can (and do) have physical significance. In this sense, the phase space of Dirac spinors is, from a logical point of view, more fundamental than Minkowski’s space-time: As we have shown, one can derive the Minkowski-metric from Hamiltonian notions via Dirac’s theory, but the reverse, the introduction of the Dirac matrices was always regarded as an ingenious ad-hoc move, as something that can not be derived but can only be postulated.

It is a historical contingency, namely the historical order in which quantum theory was developed, that lead quantum physicist to believe that imaginary numbers had some new and mysterious “non-classical” significance in quantum mechanics. But if complex numbers were anything but pairs of real numbers, then one could not simulate quantum theory with conventional computers, which are known to be based on integer and floating-point numbers only. Furthermore, if imaginary numbers had special significance, then there were 15 canonical (traceless) generators in Dirac’s theory while there are only ten. The conviction that complex numbers are, somehow, more fundamental than real numbers, puts forth strange blossoms and as of today there is barely any mention of real representations of anything in theoretical physics. The Hamiltonian account of Dirac’s theory however uncovers an intrinsic redundancy: Dirac’s theory does not require complex wave-functions (though the complex notation might be easier to handle): The six skew-Hamiltonian elements, namely the 4 components of the axial vector, the scalar, and the pseudo-scalar are no canonical generators in real linear Hamiltonian theory, while they would be allowed in a complex theory in which matrix transposition is connected with complex conjugation. Only the use of the real matrices and classical notation uncovers that the unit imaginary is an artifact from Schrödinger’s theory that does not intrinsically belong to the Dirac equation.

Freeman Dyson complained already in 1962 [183]:

Probably all these connections would have been clarified long ago, if quantum physicists had not been hampered by a prejudice in favor of complex and against real numbers.

And also Dirac had few tendency to be mystified by the appearance of complex numbers in quantum theory [184]:

Thus a complex Hilbert vector is not a more general kind of quantity than a real one. A real Hilbert space is the more elementary concept. A complex Hilbert space should be looked upon as a real one in which a certain structure is introduced, namely a pairing of the coordinates, each pair being then considered as a complex number. Changing the phase factors of these complex numbers then provides a special kind of rotation in the Hilbert space.

and he continues:

In a structureless real Hilbert space there are no special linear transformations. All are on the same footing. This is the most suitable basis for a general mathematical theory. The existence of special transformations would complicate the discussion of the fundamental ideas. We shall therefore deal with a real Hilbert space, where the vectors have real coordinates.

However, vice versa, if we are to derive Schrödinger’s equation from Dirac’s, it is convenient to invent the unit imaginary, simply to avoid the spinor notation. Hence the unit imaginary entered quantum mechanics as an artifact of the historical order.

It has been shown in Ref. [160], the Dirac current is the source of the field terms in \mathbf{S} : Maxwell’s theory, the Dirac “particle”, quantum theory and special relativity can be developed and presented in one single cast [12, 13]. Then it might also be possible to explain the statistical properties of the “quantum world”: Since the “particle” properties are defined by the second moments of the phase-space distribution, it is clear why $\vec{p} = 0$ does not imply $\langle \vec{p}^2 \rangle = 0$: In a phase space distribution we may as well have $\langle pq \rangle = 0$ and $\langle p^2 q^2 \rangle > 0$. To regard this as an intrinsic “uncertainty” or “indeterminacy” is possible but somewhat willful, because fourth order moments do not necessarily indicate “uncertainty” or “indeterminacy” of second order moments.

We claim that a proper understanding of the mathematical form of special relativity and QM requires little more than to understand the fundamental significance of conserved quantities and Hamiltonian notions. Then it becomes evident that the math does not describe a particle without volume but a set of four related quantities (energy and three momentum components) which

are connected in the Dirac theory: They are (linear combinations of) matrix elements. One can look at them *as if* they would represent a massive point particle. The spin indicates motion, but not *spatial* motion. The Lorentz transformations are, in the first place, not space-time coordinate transformations but canonical transformations in spinor space. They can also be looked upon *as if* they would describe the transformations between “inertial frames”. Many physicists seek for an understanding that goes beyond mere formal derivations, for a visualizable setup to begin with. But on the other hand, natural philosophy was always driven by the desire to understand the last principles *behind* the apparent phenomena. However, if these principles are supposed to be found *behind* the apparent phenomena, then it would be odd to expect that they can be obtained directly from the phenomena: It is no surprize that these principles are not suitable for a spatio-temporal visualization. But the principles are not “deep”: In order to serve as axioms, principles must be (self-) evident, which is the opposite of deep. “Deep” are those consequences that were, due to the simplicity of the axioms, initially unexpected.

So why do we call the Hamiltonian logic “vertical”? It is vertical insofar it concerns the logic between two (or more) different levels of description, namely that of the dynamical variables ψ of spinorial phase space and that of moments (observables) of this space, namely \mathbf{S} . Let us reconsider the emergence of Eq. 44: The time derivative of the matrix \mathbf{S} on the left side is proportional to elements of \mathbf{F} and to matrix elements of \mathbf{S} on the right. But the equations of motion that lead to Eq. 44 stem from one level below, namely from the equations of motion of ψ (and the *same* elements of \mathbf{F}). This is the vertical connection between the different levels. On the level of the Dirac spinor, i.e. the supposed fundamental level, the only restriction of the equations of motion is their Hamiltonian form. On the next level, the equations of motion

Due to the universality of the Hamiltonian method, it is practically always possible to find a Hamiltonian formulation, even if the underlying level is either unknown or regarded as metaphysically obscure. However, if one finds that a level in which *all possible* Hamiltonian terms have physical significance and none is missing, this is a strong indication that this respective level is irreducible and hence in this sense *fundamental*. The phase space of Dirac spinors appears to be such a level.

Now the ten Clifford parameters f_i (Eq. 59) of some Hamiltonian matrix of $sp(4, \mathbb{R})$, also of \mathbf{S} , can be written in vector form $\mathbf{x} = (f_0, f_1, \dots, f_9)^T$ and then Eq. 44 can be reformulated as [136]:

$$\dot{\mathbf{x}}_i = \mathcal{F}_{ij} \mathbf{x}_j. \quad (99)$$

Doing this one may recover the conventional relativistic “tensor” notation: The upper left 4×4 sub-matrix of the 10×10 -matrix \mathcal{F} is the electromagnetic field tensor [136]. But the footprint of the “spinorial phase space”, remains on this “higher” level, namely the specific form of \mathcal{F} and

the necessity to use two types of indices, i.e. to implement the signature of the Clifford algebra via the use of the so-called “metric tensor”. The matrix \mathbf{F} has the size 10×10 , hence many more *possible* than actual terms compared to the matrix \mathbf{S} . If an agent (“observer”) has direct access only to (even) moments of ψ , namely here \mathbf{S} (or \mathbf{x} , respectively), she is lead to think that those variables are fundamental and that the form of \mathcal{F} , reflects some fundamental law of nature, while “in reality” all that is just statistics on a lower dimensional space in which no such laws exist. No doubt that such an agent might face some unexpected correlations, namely between moments of the variables in \mathbf{x} .

The usual textbook presentation of physics does less than something to clarify the vertical logic of Hamiltonian methods that we tried to point out in this article; it does not even seem to acknowledge that any original Hamiltonian logic exists.

ACKNOWLEDGMENTS

We thank H. Röcken for comments and critical remarks. L^AT_EX has been used to write this article, XFig 3.2.5 for the figures. Mathematica[®] has originally been used for parts of the symbolic calculations.

Appendix A: Clifford Algebra in a Nutshell

A Clifford algebra $Cl(p, q)$ is generated from $N = p + q$ mutually anti-commuting “basis” elements γ_ν where $\nu \in [0, \dots, N - 1]$, such that $\gamma_\nu^2 = \mathbf{1}$ for $\nu \in [0, \dots, p - 1]$ and $\gamma_\nu^2 = -\mathbf{1}$ for $\nu \in [p, \dots, p + q]$. It follows from this definition that the anti-commutators of the basis elements can be summarized by the so-called “metric tensor” $g_{\mu\nu}$:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} = 2 \text{Diag}(1, 1, \dots, -1, -1, \dots). \quad (\text{A1})$$

where $g_{\mu\nu}$ is a diagonal matrix with p diagonal elements equal to $+1$ and q diagonal elements equal to -1 , corresponding to the signature of the basis elements γ_ν .

Only these basic elements γ_ν are required to generate all other elements as (multiple) products of basic elements. From combinatorics it is known that a system of N elements allows for $\binom{N}{k}$ products of k elements and hence generates a multiplicative group with a total number of

$$\sum_{k=0}^N \binom{N}{k} = 2^N \quad (\text{A2})$$

elements. The elements are called k -vectors, if they are proportional to products of k basis-elements γ_ν . The product of all basis elements, the N -vector, is called pseudo-scalar. This means that Clifford algebras are related to Pascal’s triangle. The unit matrix is called

scalar and the product of all basis elements is the so-called *pseudo-scalar* (see Fig. 3). As freely defined math-

n	N	Cl	10	45	120	210	252	210	120	45	10	1
1	1	\mathbb{R}										
1	2	\mathbb{C}										
1	2	$Cl(1,1)$										
3	3											
2	4	$Cl(3,1)$										
5	5											
4	6											
7	7											
8	8											
9	9											
16	10	$Cl(9,1)$										

FIG. 3. Pascal’s triangle for Clifford algebras. Left: Number of “spinor components” n , corresponding dimension N of Clifford algebra and special cases: $\mathbb{R} = Cl(0, 0)$, $\mathbb{C} = Cl(0, 1)$. The rows of the Hamiltonian Clifford algebras are indicated in gray. Note that a “Lorentz 4-vector” in tensor algebra is just a 1-vector with 4 components in a Clifford algebra, while the “4-vector” in the Dirac algebra has only a single component called “pseudoscalar”.

ematical entities, the unit elements γ_μ do not require any representation beyond a mere symbol and the definition given above. Framed just mathematically one may define and analyze CAs with an arbitrary number of dimension and any signature. This is certainly an interesting (and active) field of research in its own right, but it is not of specific interest here⁶⁸. As physicists we are most often interested in (matrix) representations of CAs. But matrix elements may, according to the prevalent reading, either be real or complex numbers and even quaternions. Regarded this way, also matrices may be matrix elements and since the complex numbers and quaternions⁶⁹ are in themselves representations of Clifford algebras, one may also use (why not?) Clifford algebraic elements within matrices. Yet again, representation theory is an interesting (and active) field of research in its own right, but here we are only interested in CAs insofar as they allow for the analysis of classical Hamiltonian symmetries⁷⁰. This suggests a restriction to real matrices, but this is not really a reduction of the possibilities: Any Clifford algebra can, in some way, be represented by real matrices, because, as we just mentioned, also the complex numbers and the quaternions are Clifford algebras in themselves and have real matrix representations.

The complex numbers, for instance, require a single unit element i with $i^2 = -1$. We could also say, it consists only of the SUM γ_0 and the unit matrix $\mathbf{1}$. This

is the Clifford algebra $Cl(0, 1)$. A representation by real matrices is possible, but “incomplete” insofar as the required matrices allow for a larger algebra than the complex numbers: Regarded this way, the complex numbers are a sub-algebra of the real Pauli algebra.

The next step would be an algebra with two basis elements, say the Pauli matrices η_0 and η_1 with $\eta_0^2 = -\mathbf{1}$ and $\eta_1^2 = \mathbf{1}$. The only other element (besides the neutral element, i.e. unit matrix), according to Pascal’s triangle, then is $\eta_0 \eta_1$, which then is both, the only existing bi-vector and the pseudo-scalar (see Fig. 3).

From a conceptional point of view, representations based on complex numbers and quaternions are “tricky” because they use structures inside structures. As we have shown in Sec. IV, the Hamiltonian way to regard CAs is based on the idea to charge numbers with structural meaning. But it is somewhat pointless to charge structures with structural meaning. Therefore, from a puristic Hamiltonian point of view, only those Clifford algebras which have an irreducible real matrix representation are of primary concern.

Now let’s consider the algebra $Cl(3, 0)$ which consists of 3 basis elements, \mathbf{e}_1 , \mathbf{e}_2 and \mathbf{e}_3 and regard the “vectors” $\mathbf{x} = x \mathbf{e}_1 + y \mathbf{e}_2 + z \mathbf{e}_3$ and $\mathbf{p} = p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3$ just as we would write vectors in classical vector algebra. Let us have a look at the respective (anti-) commutative products of two such vectors (compare to Eq. 73):

$$\begin{aligned} \mathbf{x} \mathbf{p} &= (x p_x + y p_y + z p_z) \mathbf{1} \\ &+ (y p_z - z p_y) \mathbf{e}_2 \mathbf{e}_3 \\ &+ (z p_x - x p_z) \mathbf{e}_3 \mathbf{e}_1 \\ &+ (x p_y - y p_x) \mathbf{e}_1 \mathbf{e}_2, \end{aligned} \quad (\text{A3})$$

The result contains, firstly, a scalar component equal to the scalar product of classical vector algebra, and secondly the vector (“cross”) product $\mathbf{x} \times \mathbf{p}$ appearing in the coefficients of the bi-vectors.

Hence we find the “meaning” of commutative (outer) and anti-commutative (inner) products

$$\begin{aligned} \mathbf{x} \cdot \mathbf{p} &= (\mathbf{x} \mathbf{p} + \mathbf{p} \mathbf{x})/2 \\ \mathbf{x} \wedge \mathbf{p} &= (\mathbf{x} \mathbf{p} - \mathbf{p} \mathbf{x})/2 \end{aligned} \quad (\text{A4})$$

This gives a first glimpse of why Clifford algebras are said to have *geometric* content. For a detailed discussion of the general Lorentz covariance as represented by $Cl(3, 1)$ see Ref. [134].

Appendix B: The General Hamiltonian Matrix

In the chosen matrix representation, the particle’s matrix is (compare Eq. 60, Eq. 63):

$$\mathbf{P} = \begin{pmatrix} -P_z & \mathcal{E} - P_x & 0 & P_y \\ -\mathcal{E} - P_x & P_z & P_y & 0 \\ 0 & P_y & -P_z & \mathcal{E} + P_x \\ P_y & 0 & -\mathcal{E} + P_x & P_z \end{pmatrix} \quad (\text{B1})$$

⁶⁸ Standard textbooks are, for instance, Ref. [152, 153].

⁶⁹ The complex numbers \mathbb{C} and the quaternions \mathbb{H} have no irreducible representation by real matrices, but they are Clifford algebras, $Cl(0, 1)$ and $Cl(0, 2)$, respectively. \mathbb{C} can be represented by a subalgebra of the real Pauli algebra and \mathbb{H} can be represented by two different subalgebras of $Cl(3, 1)$.

⁷⁰ See, for instance, Ref. [185].

and that of the electromagnetic fields is (Eq. 78):

$$\mathbf{F} = \begin{pmatrix} -E_x & E_z + B_y & E_y - B_z & B_x \\ E_z - B_y & E_x & -B_x & -E_y - B_z \\ E_y + B_z & B_x & E_x & E_z - B_y \\ -B_x & -E_y + B_z & E_z + B_y & -E_x \end{pmatrix} \quad (\text{B2})$$

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