Classical verification of quantum depth

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Abstract

We present two protocols for classical verification of quantum depth. Our protocols allow a purely classical verifier to distinguish devices with different quantum circuit depths even in the presence of classical computation. We show that a device with quantum circuit depth at most d will be rejected by the verifier even if the prover applies additional polynomial-time classical computation to cheat. On the other hand, the verifier accepts a device which has quantum circuit depth d' for some d' > d. In our first protocol, we introduce an additional untrusted quantum machine which shares entanglements with the target machine. Applying a robust self-test, our first protocol certifies the depth of the target machine with information theoretic security and nearly optimal separation. The protocol relies on the oracle separation problem for quantum depth by Chia, Chung and Lai [STOC 2020] and a transformation from an oracle separation problem to a two-player non-local game. Our second protocol certifies the quantum depth of a single device based on quantum hardness of learning with errors. The protocol relies on the noisy trapdoor claw-free function family and the idea of pointer chasing to force the prover to keep quantum coherence until all preceding message exchanges are completed. To our knowledge, we give the first constructions for distinguishing hybrid quantum-classical computers with different circuit depths in unrelativized models.

1 Introduction

Quantum circuit depth is an essential consideration when evaluating the power of near-term quantum devices. Quantum computers with many qubits have been recently implemented [1, 17, 10, 2]; however, these computers have limited quantum circuit depth due to the noisy gates and short coherence time. Hence, how to leverage the power of these small-depth quantum devices becomes a practical challenge as well as a fascinating question in quantum complexity theory.

Indeed, Aaronson and Chen showed that small-depth quantum computers can demonstrate so-called "Quantum Supremacy" [3] on the random circuit sampling problem, which means that quantum computers can efficiently solve the problem that is intractable for classical machines. Arute et al. [5] reported the results of experiments on demonstrating quantum supremacy by using superconducting quantum computers of Google. In the near term, the coherence time seriously limits the usable lifespan of quantum states. Thus information processing with a small and noisy quantum device has become a central topic in field of quantum computing.

¹New classical algorithms are found for solving the problem in few days (by estimation) [20], which implies that random circuit sampling of the size in the experiment in [5] might not be classically intractable. However, even these new classical algorithms are slower than the quantum one (that solves the problem in 200 seconds); therefore, the experiments showed quantum advantages on the problem.

Among the computational models that use small quantum devices, hybrid quantum-classical computing that interleaves classical computers with quantum devices is a natural approach to use the power of small-depth quantum circuits. This hybrid approach has been gaining much attention recently and might be able to surpass the capabilities of classical machines on some real-world problems, such as molecular simulation [19], optimization problems [14], etc. Notably, Cleve and Watrous [12] proved that the quantum Fourier transform can be implemented in logarithmic quantum depth in this model. This implies that quantum algorithms for Abelian hidden subgroup problems, such as Shor's factoring algorithm, can be implemented in logarithmic quantum circuit depth.

The results above indicate that quantum computers with circuit depth beyond certain thresholds are able to demonstrate quantum advantages. Seeing the possible applications of small-depth quantum devices, one might start wondering:

Can we certify if a computer has sufficient quantum depth for quantum advantages?

An answer to the question is to find some problem, give an efficient algorithm that only requires small-depth quantum circuits, and prove that no algorithm using strictly smaller quantum depth achieves the same time complexity. For instance, the aforementioned results [3, 12] showed separations between small-depth quantum circuits and classical computers under plausible computational assumptions. That is, based on the assumption that a problem is classically hard, a device which can solve some problem in a reasonable time frame must exhibit quantum power.

Another possible approach is designing cryptographic protocols that demonstrate the quantumness of a quantum device [7, 8, 16, 18]. In these protocols, the classical verifier sends the description of a cryptographic hash function f and random coins to challenge the prover to answer information about f. It is guaranteed that only a prover which performs quantum computation will successfully answer these challenges with high probability. While these protocols seem to be satisfying proposals for demonstrating quantumness, there is a caveat: for a quantum prover to succeed, it is required to evaluate f coherently, and thus the implementation of f with quantum gates sets a lower bound on the resource requirement. To address the issue, in subsquent works, Hirahara and Le Gall [16] and Liu and Gheorghiu [18] independently showed that these protocols only requires a hybrid computation that only uses a constant-depth quantum circuit using different approaches.

It is worth noting that these two approaches, in state of the art, mainly focus on distinguishing quantum computers from classical ones. They do not directly translate into ones that separate quantum computers with different quantum resources. It is unclear if we can show that these protocols or problems cannot be tackled using smaller quantum depth.

In this work, we give "fine-grained" solutions to the question in the following scenario: Suppose Bob claims that he has a computer with quantum circuit depth larger than d. Can Alice, who only has a classical machine, catch Bob is cheating if Bob only has quantum circuit depth at most d? Of course, Alice shall assume that Bob might use a powerful classical machine to cheat. Here, we are actually asking for protocols that allow a classical verifier to verify if a prover has a quantum circuit with depth at least d in the presence of the prover's powerful classical machine. We call such protocols Classical Verification of Quantum Depth (CVQD).

The problem for separating quantum depth. Chia, Chung, and Lai [11] introduced the d-Shuffling Simon's Problem Problem (d-SSP) that separates d- from (2d+1)-depth quantum circuits in the presence of polynomial-time classical computation. One straightforward approach is to use this problem to certify quantum depth by asking the computer to solve d-SSP. However, this approach does not lead to a solution to classical verification of quantum depth since d-SSP is an oracle

problem that requires quantum access to the oracle for efficient quantum algorithms. Therefore, we need new ideas for our purpose.

1.1 Main results

In this work, we give an affirmative answer to the question by showing two CVQD protocols that can distinguish quantum circuits with different depth in the presence of polynomial-time classical computation. We first give definitions of the two CVQD protocols that we consider in this work.

We consider the setting where a single quantum machine is being tested by a classical verifier. The verifier should reject if the quantum depth no more than d, and accept if the quantum depth is at least d' > d. The machines are promised to be in one of the cases.

Definition 1.1 (CVQD(d, d'), informal). Let $d, d' \in \mathbb{N}$ and d' > d. Let P_A be a bounded-depth quantum circuit with classical polynomial-time computation. Let V be a classical verifier. A CVQD(d, d') protocol that separates quantum circuit depth d from d' satisfies the following properties:

- Completeness: If P_A has quantum circuit depth at least d', then $\langle V, P_A \rangle$ accepts with probability at least 2/3.
- Soundness: If P_A has quantum circuit depth at most d, then for any polynomial-time P_A , $\langle V, P_A \rangle$ accepts with probability at most 1/3.

We also consider protocols that consist of two provers which are not allowed to communicate with each other, but may share entanglements. In this setting, one prover P_A is the target machine being tested. We add another prover P_O to help certify the quantum depth, but neither of the provers is trusted by the classical verifier.

Definition 1.2 (CVQD₂(d, d'), informal). Let $d, d' \in \mathbb{N}$ and d' > d. Let P_A be a bounded-depth quantum circuit with classical polynomial-time computation. Let P_O be an unbounded quantum prover and V be a classical verifier. A CVQD₂(d, d') protocol that separates quantum circuit depth d from d' satisfies the following properties:

- Non-locality: P_O and P_A share arbitrarily many EPR pairs and are not allowed to communicate with each other once the protocol starts.
- Completeness: If P_A has quantum circuit depth at least d', then there exists P_O and P_A such that $\langle V, P_O, P_A \rangle$ accepts with probability at least 2/3.
- Soundness: If P_A has quantum circuit depth at most d, then for any P_O and polynomial-time P_A , $\langle V, P_O, P_A \rangle$ accepts with probability at most 1/3.

In both definitions, the verifier accepts if P_A has a quantum circuit with depth at least d', and rejects any dishonest prover which might interleave its small-depth quantum circuit (depth at most d) with a polynomial-time classical algorithm. As defined by Chia, Chung and Lai [11], there are two schemes, where the quantum process interleaves a quantum machine with a classical one, called d-CQ and d-QC schemes. Briefly, the d-CQ scheme allows a classical algorithm to query a d-depth quantum circuit polynomially many times, and the d-QC scheme allows a d-depth quantum circuit to access polynomial-time classical algorithms after each layer of 1-depth circuit. We aim to design protocols to rule out cheating provers using both schemes.

We show there exist constructions of CVQD and $CVQD_2$. In particular, we show the following result.

Theorem 1.3 (Informal). Let $d \in \mathbb{N}$.

- 1. There exists a two-prover $\mathsf{CVQD}_2(d, d+3)$ protocol $\langle V, P_A, P_O \rangle$ that is unconditionally secure with inefficient honest P_O and V. Moreover, honest P_O and V can be efficient assuming the existence of quantum-secure pseudorandom permutation (qPRP).
- 2. For polynomially bounded function d and constant d_f , there exists a $CVQD(d, d + d_f)$ protocol under the QLWE assumption.

Here, the QLWE assumption assumes that the Learning With Error (LWE) problems in hard for any quantum polynomial-time algorithms.² The constant d_f is the quantum circuit depth for implementing a particular function.

Comparing the two results in Theorem 1.3. The two results in Theorem 1.3 are incomparable. The second result achieves single-prover $CVQD(d, d + d_f)$ under QLWE. It is worth noting that we only know that P_A that implements $(d + d_f)$ -QC schemes can be accepted in this protocol; in contrast, d'-CQ schemes might require d' to be larger than $d + d_f$ to be accepted. On the other hand, although the first result (a construction of $CVQD_2(d, d + 3)$) requires an additional (and untrusted) quantum prover, it has the following advantages: (1) its separation is nearly optimal (d versus d + 3), (2) it achieves information theoretic security, (3) P_O and the verification can be made efficient by only assuming the existence of qPRPs in a query model, and (4) P_A that implements either (d + 3)-CQ or (d + 3)-QC schemes can be accepted.

To prove the first result in Theorem 1.3, we provide a framework that transforms a quantum oracle separation into a two-prover protocol.

Theorem 1.4 (Informal). Let C and C' be two complexity classes. Let L^O be an oracle problem such that $L^O \in C^O$ and $L^O \notin C'^O$. Then, there exists a two-prover protocol $\langle V, P_A, P_O \rangle$ two real numbers $c, s \in [0, 1]$ satisfying $c - s = 1/\operatorname{poly}(n)$ for size n of input such that the following conditions hold.

- Completeness: If P_A can solve problems in C, then there exists P_O such that $\langle V, P_A, P_O \rangle$ accepts with probability at least c.
- Soundness: If P_A can only solve problems in C', then for any P_O , $\langle V, P_A, P_O \rangle$ accepts with probability at most s.
- Classical verification: V is classical, and the runtimes of V and the honest P_O depend on the number of queries for solving L^O and the complexity for implementing O.

We can transform d-SSP (an quantum oracle problem by Chia, Chung and Lai [11] for separating quantum depth) into a two-prover CVQD protocol under the framework in Theorem 1.4. However, the separation is not as good as in Theorem 1.3. We further show that by modifying the original d-SSP, the separation can be improved to d versus d+3. We call the new problem "in-place d-SSP." Then, we transform in-place d-SSP into a two-prover CVQD(d, d+3) protocol following Theorem 1.4.

1.2 Technical overview

In this section, we give a brief overview for proving the main theorems.

²In fact, it is sufficient to assume that QLWE is hard for a *d*-depth hybrid machines.

1.2.1 A two-prover protocol with optimal depth separation

The problem d-SSP for separating quantum depth. Our protocol can be seen as a two-player instantiation of the algorithms for solving d-SSP in [11], an oracle problem that distinguishes d- from (2d + 1)-depth quantum circuits.

The problem is a shuffled version of the Simon's problem. Recall that for the "plain" Simon's problem, a constant-depth algorithm is sufficient to output the hidden shift. To turn the problem into one that certifies large quantum query depth, Chia, Chung and Lai proposed the d-Shuffling Simon's problem (d-SSP) [11]. The algorithm is given oracle access to d+1 functions f_0, \ldots, f_d , where f_1, \ldots, f_{d-1} are random permutations on an exponentially larger set, and the last function f_d is a 2-to-1 function such that $f_d \circ \cdots \circ f_2 \circ f_1(x) = f(x)$ for a Simon's function f. We call the functions f_0, \ldots, f_d to be a d-shuffling of a Simon's function f. The task is to find the hidden shift.

It is obvious that d-SSP remains easy for a (2d+1)-depth quantum algorithm which simulates a query to f using two queries to each function in $\{f_0, \ldots, f_{d-1}\}$ and one query to f_d : first query f_0, \ldots, f_d in sequence to get

$$|x,y\rangle \mapsto |x, f_0(x), f_1 \circ f_0(x), \dots, f_{d-1} \circ \dots \circ f_0(x), f(x) \oplus y\rangle,$$
 (1)

and then query the first d functions in the reverse order to reset the intermediate registers to back zero states. On the other hand, any polynomial-time algorithm with quantum depth at most d cannot solve d-SSP. This follows from the intuitions that one needs to make (d+1)-sequential quantum queries to f_0, \ldots, f_d in order for evaluating f on a uniform superposition, and only f_d in an exponentially small random subset of the domain has information about f. Thus, any polynomial-time algorithm without sufficient quantum depth cannot even evaluate f in superposition.

To turn the problem into a protocol that certifies quantum depth, an idea is to have the verifier V play the role of the oracle, and checks if prover P_A outputs the hidden shift. The resulting protocol is quite straightforward: the prover P_A is allowed to perform arbitrary quantum computation (subject to its quantum resources) between message exchanges with the verifier. In the intermediate rounds, V computes the quantum circuits of the oracles on the state given by P_A , and sends the resulting quantum state back. At the end, the verifier accepts if P_A outputs the hidden shift. The analysis of the protocol is also straightforward. As long as the verifier implements the quantum-accessible oracles f_0, \ldots, f_d reliably between the computation performed by the prover, the completeness and soundness directly follows from the result of Chia, Chung and Lai [11].

However, this approach has two drawbacks: First, the verifier needs to reliably implement a large QRAM to support quantum access to the oracle. This requires a reliable large-scale quantum computer that can solve problems in quantum polynomial time. Moreover, it requires reliable quantum communication between the prover and the verifier. None of the requirements seems to be within the reach in the near future.

In this paper, we give constructions that allows a purely classical verifier to certify quantum depth. Our first protocol is to rely on the technique of self-testing to certify the untrusted quantum servers sharing entanglements. In particular, we apply a sequence of transformations from the aforementioned straightforward approach into one that has a weak requirement on the verifier, i.e., it runs in probabilistic polynomial time. We briefly introduce the techniques as follows.

Delegating the oracle to another quantum prover. To achieve purely classical verification, we introduce another untrusted prover, denoted P_O , which may share entanglements with P_A but they are not allowed to communicate with each other once the protocol starts. The verifier delegates the oracle computations to P_O , and checks if P_A outputs the hidden shift in the end. To make "queries," P_A forwards a quantum state by quantum teleportation.

To ensure that P_O behaves honestly, we modify the EPR protocol by Broadbent [9] to verify the computation of P_O . To understand how this works, let us recall some idea of the protocol. The original Broadbent protocol allows a weakly quantum verifier to delegate a quantum computation to the prover. To show that the prover has to be honest, the computation is made indistinguishable from two tests (X-test and Z-test). These tests are used to check if the prover's attack is trivial on the single bit the verifier aims to learn from the prover.

However, to apply the protocol to our problem, there are two caveats that remain to solve. First, the Broadbent protocol is designed for verifying a BQP-complete language. An instance in the language is a classical description of a unitary U with the promise that sampling the first qubit of $U|0\rangle$ by performing a standard basis measurement yields a 0 with probability at least 2/3, or at most 1/3. In our setting, we do not have such a promise. Secondly, the protocol only guarantees that the output b is an encryption of the random variable close to sampling $U|0\rangle$ by performing a standard basis measurement on the first qubit, provided the prover passes the tests with high probability. For our purposes, we would need to show if P_O 's output state ρ_i on each query $|\psi_i\rangle$ is close to $O|\psi_i\rangle$ for each query $i \in \{0, 1, \ldots, d\}$ in a reasonable metric.

We show that with a modification, our variant of the Broadbent protocol is rigid in the sense that every prover that is accepted in our variant with probability $1 - \epsilon$ must output a state ρ_i which is $O(\epsilon)$ -close the ideal state $O|\psi_i\rangle$ in trace distance. The modification requires a quantum channel which allows a transmission of poly(n) qubits between P_O and V, but the requirement is not necessary when we turn the protocol into a purely classical verification.

Dequantizing the verification. We further dequantize the quantum verification and communication by applying the Verifier-on-a-Leash protocol (also called the Leash protocol) by Coladangelo, Grilo, Jeffery and Vidick [13]. In a high level, the idea is to add another prover to perform the measurements by the quantum verifier in the Broadbent protocol, and check if the added prover behaves honestly.

To transform an oracle separation into purely classical verification, one possible approach is to add a third prover P_V to help certify that P_O behaves as intended. More concretely, the classical verifier asks P_A to perform the computation between queries, and P_O to apply the quantum circuit of the oracle. To check P_O behaves as intended, a third player P_V is added to perform the measurements in the bases determined from the rules of the Broadbent protocol. In our settings, none of the provers are assumed to be trusted. Thus it is necessary to verify P_V performs the measurements in the correct bases. Thus the verifier challenges P_V and P_O to run either the protocol for verifying P_O or a rigidity test to certify P_V , and the two choices are made indistinguishable to P_V 's viewpoint.

However, this approach does not work directly. More specifically, for the security to hold, it is crucial that P_O does not distinguish the computational round and the test rounds. In this approach, P_O interacts with P_A via quantum teleportation to implement the original query algorithms in the computation round, whereas to certify P_O 's behavior, the classical verifier must ask P_O to interact with P_V in the test round. Hence, P_O can determine the round type and cheat. Moreover, another drawback with this approach is that it requires three provers.

We can fix the issues about the aforementioned three-prover protocol by asking P_A to play the role of P_V simultaneously. To explain how this works, we consider the following protocol for a single-query algorithm: Initially, the verifier chooses to run the computation, X-test, Z-test or rigidity test. The prover P_A prepares an (arbitrary) initial n-bit quantum state $|\psi_0\rangle$ and teleports three states $|\psi_0\rangle$, $|0^n\rangle$, $|+^n\rangle$ to disjoint random subsets of P_O 's halves of EPR pairs (the other halves are held by P_A). Note that the subsets are chosen by the verifier, but it does not reveal the underlying states. If any of the first three tests is chosen, the prover P_O performs the computation O on one of the

subsets specified by the verifier. Note that since these three states are encrypted with quantum one-time pad, P_O cannot distinguish them and thus the round type. To perform the computation O, a set S of EPR pairs shared by P_O and P_A is used to implement gadgets for computing O. To be more specific, the verifier asks P_A to perform measurements on S in random bases, and chooses a subset (of S) on which P_A is specified to perform measurements in desirable bases determined by the rules of the Broadbent protocol according to the round type. The verifier then tells P_O to use the subset in S to compute O. The random-basis measurement on S is to certify the behavior of P_A in the rigidity test. Roughly speaking, when the rigidity test is chosen, the verifier can check (with help of P_O) if P_A performs the measurements on the EPR pairs in the random bases chosen by the verifier. Since P_A 's behavior for all four tests are measurements in random bases, whether a rigidity test is executed is unknown to P_A . Note that although P_O can learn if a rigidity test is executed, it does not affect the security since P_O has no chance to reveal this to P_A .

However, there are a couple of issues that remain to address when considering multiple rounds of interaction between P_O and P_A . First, some tests running on more than one query can potentially reveal the type of the test. More specifically, if the verifier chooses to run the rigidity test, then P_O would certainly learn an application of the oracle unitary O is not necessary for this round. The prover P_A can possibly detect the choice of the test by observing the input state and the resulting state using, say, a swap test. Furthermore, to reflect the actual performance of the query algorithm, it is crucial that with sufficiently large probability, no test has been applied throughout the protocol. This is because when the computation is not performed in this round, computation applied in the following rounds will not yield a useful result (e.g., outputting the hidden shift for d-SSP), even when the provers opt to follow the protocol honestly. If the tests are nevertheless executed with very low probability, the provers may deviate from the protocol seriously.

We show that it suffices that the verifier randomly selects to certify one random query and trusts all the other queries, and with probability $\Theta(1/q)$, no test is executed for a q-query protocol. For the selected query, the verifier either asks P_O to certify P_A 's measurements, or P_A to certify P_O performs the oracle unitary O by running the test phases in our variant of the Broadbent protocol. If the provers pass the test, the verifier accepts and terminates the protocol. Since the knowledge of the round type for certifying a query can only lead to an attack on the following queries, verification on a random query can prevent these issues from breaking the soundness of the protocol.

Putting things together. We then combine our aforementioned tests to turn an oracle separation problem into a two-player protocol. In particular, we show that with a suitable choice of the weights of entering each test, the completeness-soundness gap shrinks by at most an inverse polynomial multiplicative factor in the number of queries. More formally, we prove the implication by reduction. Suppose that in the protocol, there are provers P_A, P_O such that P_A is subject to its quantum resources and they break the soundness. Then we construct a query algorithm which succeeds with sufficiently large probability to break the soundness guarantee in the associated relativized world. Since the oracle separation problem distinguishes the quantum complexity classes, the resulting protocol yields a completeness-soundness gap $1/\operatorname{poly}(q)$ for a q-query algorithm.

Given that d-SSP is a oracle separation problem between a hybrid d-depth and a hybrid (2d+1)-depth computation, we conclude that our transformation yields a construction of $\mathsf{CVQD}_2(d, 2d+1)$ with gap $1/\mathsf{poly}(d)$. We apply a sequential repetition to amplify the gap to constant. The repetition itself does not require an increase of the quantum depth of P_A since the same hybrid computation can be reused.

Efficient instantiation. We have shown that an oracle separation problem implies a two-player protocol that distinguishes hybrid quantum computation with different quantum depth. However,

to succeed in the protocol honestly, V must sample an oracle from a distribution \mathcal{D} which is not known to be efficiently samplable, and P_O must perform a quantum circuit that implements O. In the problem d-SSP, the oracles consist of random permutations. By a counting argument, most of the permutations does not have an efficient implementation.

To address the issue, we leverage oracle indistinguishability in the associated relativized world. More concretely, suppose that for the distribution \mathcal{D} of random d-shuffling of a random Simon's function, there is an efficiently samplable distribution \mathcal{D}' which is indistinguishable from \mathcal{D} . Then in the two-player protocol, when the efficient verifier samples the oracle according to \mathcal{D}' , the soundness error is increased negligibly. The idea for showing this directly follows from our proof for showing an oracle separation implies a two-player protocol. For every query algorithm \mathcal{A} that has small quantum depth, it succeeds with probability at most p when the oracle is sampled from \mathcal{D} . Then replacing \mathcal{D} with \mathcal{D}' , by the oracle indistinguishability, \mathcal{A} succeeds with probability at most negligibly close to p. Applying the transformation with \mathcal{D}' yields a sound two-player protocol with efficient P_O and V.

We show how to give a distribution \mathcal{D}' using quantum-secure pseudorandom permutations (qPRP) against adversaries making queries to the permutation and its inverse. In particular, to sample a pseudorandom d-shuffling of a Simon's function, first sample d independent keys k_0, \ldots, k_{d-1} from the key space of the pseudorandom permutation P and let $f_i = P(k_i, \cdot)$ for $i \in \{0, 1, \ldots, d-1\}$. For the last function, again by a counting argument, not every Simon's function has an efficient implementation. We observe that every Simon's function can be computed by composing a permutation and an efficiently computable function that is constant on every one dimensional affine subspace of the form $\{x, x \oplus s\}$. This implies that sampling a random Simon's function can be done by sampling a random shift and a random permutation. Replacing the random permutation with a pseudorandom permutation yields a pseudorandom Simon's function.

A nearly optimal separation. As mentioned previously, the problem d-SSP provides an oracle separation between d- and (2d+1)-depth quantum circuits in the presence of polynomial-time classical computation. Next we further improve the separation to distinguish d- from (d+3)-depth hybrid quantum computation. In particular, we modify the problem to allow a (d+3)-depth algorithm to succeed with high probability, while at the same time, it remains hard for a d-depth prover to learn the hidden shift.

First we recall that a (2d+1)-depth quantum algorithm is needed because to simulate a query to f, the algorithm queries f_0, f_1, \ldots, f_d followed by queries to f_{d-1}, \ldots, f_0 to uncompute the intermediate values. To avoid the need of extra depth for uncomputation, our idea is to replace the standard access to f_0, \ldots, f_d with "in-place oracles." In this model, the algorithm is given access to $|x\rangle \xrightarrow{f_i} |f_i(x)\rangle$, and thus the intermediate queries have been erased automatically. While in-place oracle access to an arbitrary is not a unitary in general, in our case, perhaps fortunately, the functions f_0, \ldots, f_d are either permutations or 2-to-1 functions. It is clear that the in-place oracle access for permutations is a unitary. Furthermore, we modify the last function f_d such that f_d is bijective, but a depth (d+1)-depth algorithm can simulate a query to the underlying Simon's function f with constant probability. We call the same problem with in-place oracle access the in-place d-Shuffling Simons Problem (in-place d-SSP, see Definition 5.10). Finally, we show that that the in-place d-SSP cannot be solved by any hybrid quantum-classical computers with quantum circuit depth at most d.

1.2.2 Single-prover protocol from LWE

The second protocol relies on the assumption that the Learning-with-Errors (LWE) problem is hard for quantum computers (also called the QLWE assumption). In a breakthrough [7], Brakerski, Christiano, Mahadev, Vazirani and Vidick showed that the QLWE assumption implies the existence of a noisy trapdoor claw-free function (NTCF). A function f is trapdoor claw-free if it is 2-to-1, and given a pair (x, y) such that f(x) = y, it is computationally intractable to find the other preimage of y. Furthermore, the function f is also equipped with a strong property called the adaptive hardcore bit property. In a nutshell, the property states that no quantum adversary given access to a description of f can output (y, x, e) such that x is a preimage of y and $e \cdot (x_0 + x_1) = 0$ with probability non-negligibly better than 1/2, where x_0, x_1 are the preimages of y. In contrast, there exist quantum processes which allows an efficient quantum device to output either (y, x) or (y, e).

This observation leads to a proof-of-quantumness protocol: the verifier on receiving y requests the prover to present a preimage x or an equation e. An efficient quantum prover can succeeds with nearly perfect probability. For proving classical hardness, the idea is that one can rewind a classical prover which succeeds with probability non-negligibly more than 1/2 to extract both x and e with non-negligibly probability: For every prover \mathcal{A} , let the state before receiving the challenge be a random variable σ_y . The adversary challenges \mathcal{A} to use the same state σ_y to output both a preimage and an equation. Any prover \mathcal{A} that wins the test with non-negligible advantage would imply that the adversary breaks the property.

In subsequent works, Hirahara and Le Gall [16] and Liu and Gheorghiu [18] showed that the same protocol only requires a quantum prover of constant depth. The ideas behind these constructions basically follow from presenting NTCFs that can be evaluated with constant quantum depth.

A proof-of-quantumness protocol can be viewed as a protocol which separates a prover of non-zero quantum depth from one of zero quantum depth (i.e., a classical device). It seems natural to rely on the same hardness assumption to separate a high-depth quantum device from a low-depth one with the following protocol:

- 1. The verifier samples the functions f_1, \ldots, f_d and sends these functions to the prover.
- 2. The prover outputs y_1, \ldots, y_d .
- 3. For $i = 1 \dots d$, the verifier sequentially samples a random bit c_i which indicates the request to send a preimage x_i or a equation e_i for y_i . The verifier rejects if in any of the rounds the prover fails.

In this protocol, the prover must increase its quantum depth by 1 in each round of Step 3, since the operation the prover performs depend on the challenge bit c_i , which depends on the previous message from the prover. It is straightforward to see a $(d_f + d)$ -depth prover succeeds with nearly perfect probability, where d_f is the depth required for the evaluation of f. However, to show the hardness for any small-depth device, since the device is no longer purely classical, the same rewinding argument does not directly apply.

We formalize an observation that a (d-1)-depth prover cannot stay coherent throughout the protocol, and has to "reset" (i.e., to destroy all its coherence and to continue with a purely classical state) in an intermediate round j. Thus from round $i = j \dots d$, the prover begins with an intermediate classical state σ , and responds with its quantum power. To break the adaptive hardcore bit property, the reduction simulates the protocol to compute the state σ , and rewinds on σ to compute both a preimage and an equation for f_d .

1.3 Discussion and open problems

We give protocols that allows a classical verifier to distinguish quantum machines with different circuit depths and polynomial-time classical computation. Our first two-prover protocol can achieve nearly optimal separation and information theoretic security by adding an additional untrusted quantum helper, and the verification can be made efficient if quantum-secure pseudorandom functions exist. The second protocol achieves a single-prover CVQD based on QLWE, and works for a slightly larger constant promise gap on the depth. We note that the two protocols we present in this paper are incomparable. The first protocol makes no additional assumption to certify that the target machine runs in small quantum depth. In contrast, the second requires no additional prover to achieve the same task based on a widely-held assumption.

We include a few open questions. First, the two-prover protocol has separation d versus d+3 and the single-prover protocol has separation d versus $d+d_f+1$. It is interesting to know if these separations can be improved. Secondly, it would be interesting to know if we can directly instantiate d-SSP from (standard) computational assumptions. This is similar to the case for instantiating abelian hidden subgroup problems by factoring. If we can find such instantiation, then we can obtain a single-prover CVQD protocol that is different from the one in this work.

In the single-prover CVQD protocol, we only know that an honest prover that implements $(d+d_f)$ -QC schemes can succeed. It is open if a prover can convince the verifier by implementing a $(d+d_f)$ -CQ scheme. Moreover, our single-prover verification scheme requires the use of randomized encoding of an NTCF family to achieve depth-efficient function evaluation, but since the number of qubits scales with the depth threshold, the honest prover would need a large space to succeed in the protocol. Can we give a more space efficient protocol such that a demonstration of quantum depth can be implemented in a near-term quantum device?

Finally, the round complexity of our protocol scales with the depth. In particular, to determine that a device has quantum depth no more than d, the round complexity is O(d). Can we give CVQD protocols for which the round complexity does not scale with the depth?

Related work. In an independent work [15], Atsuya Hasegawa and François Le Gall defined the d-Bijective Shuffling Simon's Problem that improves the quantum depth separation in [11] to d versus d+1 using the similar idea as in-place d-SSP (Definition 5.10). For in-place d-SSP, the gap is d versus d+1 if we consider the same models as in Definition 3.8 and Definition 3.10 in [11]. However, the models in [11] count the depth of quantum queries to the oracle. In this work, we also count the two layers of Hadamard transforms at the beginning and the end of Simon's algorithm. This results in the gap d versus d+3 in our first result in Theorem 1.3 (see Theorem 5.11 and Corollary 5.12 for formal statements).

1.4 Organization

The rest of the paper is organized as follows. Section 2 includes the required technical background knowledge for this paper, and our modifications of the previous protocols which will be useful for our contributions. Section 3 defines a transformation from a quantum oracle separation to a two-prover protocol that preserves completeness and soundness. Section 4 presents a framework that transforms a quantum oracle separation to a two-prover protocol with a classical verifier. Section 5 shows a protocol for classical verification of quantum depth under the framework developed in Section 3 and Section 4. Finally, in Section 6, we present a new single-prover protocol from QLWE.

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2 Preliminaries

For finite set \mathcal{X} , we denote $x \leftarrow_R \mathcal{X}$ the process of sampling a random variable x uniformly from \mathcal{X} . For distribution \mathcal{D} over a finite set \mathcal{X} , we denote $x \leftarrow_R \mathcal{D}$ the process of sampling a random variable $x \in \mathcal{X}$ according to \mathcal{D} . For a classical or quantum process \mathcal{A} , we denote $y \leftarrow \mathcal{A}(x)$ to specify that \mathcal{A} on input x outputs y. A function $f: \mathbb{N} \to \mathbb{R}$ is negligible, denoted f(n) = negl(n), if there exists an integer n_0 such that for $n \geq n_0$, $f(n) \leq n^{-c}$ for every constant c. In other words, f if negligible if $f(n) = n^{-\omega(1)}$. We use the notation 1_P to denote 1 if P is true and 0 if P is false.

2.1 Oracle separation for quantum depth

We first introduce the two models for interleaving d-depth quantum circuits and classical polynomial-time computation.

Definition 2.1 (d-CQ scheme [11]). Let k = poly(n). Let $\mathcal{A}_c^1, \ldots, \mathcal{A}_c^k$ be a sequence of classical polynomial-time algorithms and $\mathcal{A}_q^1, \ldots, \mathcal{A}_q^k$ be a sequence of d-depth quantum circuits. A d-CQ scheme can be represented as following:

$$\mathcal{A}_c^k \circ (\Pi_{0/1} \circ \mathcal{A}_a^k) \circ \cdots \circ \mathcal{A}_c^2 \circ (\Pi_{0/1} \circ \mathcal{A}_a^2) \circ \mathcal{A}_c^1 \circ (\Pi_{0/1} \circ \mathcal{A}_a^1),$$

where, $\Pi_{0/1}$ is a measurement on all qubits in the computational basis.

Definition 2.2 (d-QC scheme [11]). Let k = poly(n). Let $\mathcal{A}_c^0, \mathcal{A}_c^1, \ldots, \mathcal{A}_c^d$ be a sequence of classical polynomial-time algorithms and $\mathcal{A}_q^1, \ldots, \mathcal{A}_q^d$ be a sequence of 1-depth quantum circuits. A d-CQ scheme can be represented as following:

$$\mathcal{A}_c^d \circ (\Pi_{0/1} \otimes I) \circ \mathcal{A}_q^d \circ \cdots \circ \mathcal{A}_c^2 \circ (\Pi_{0/1} \otimes I) \circ \mathcal{A}_q^2 \circ \mathcal{A}_c^1 \circ (\Pi_{0/1} \otimes I) \circ \mathcal{A}_q^1 \circ \mathcal{A}_c^0,$$

where, $\Pi_{0/1} \otimes I$ is a computational basis measurement that only operates on part of the qubits. The input of \mathcal{A}_q^i includes the output quantum state of \mathcal{A}_q^{i-1} for $i=2,\ldots,d$ and the classical information from \mathcal{A}_c^j and the measurement outcome of \mathcal{A}_q^j for j < i. The input of \mathcal{A}_c^i includes the measurement outcome of \mathcal{A}_q^i and other classical information from \mathcal{A}_c^j and \mathcal{A}_q^j for j < i for all $i \in [d]$.

Remark 2.3. In this work, we generally choose the universal gateset to be all one- and two-qubit gates. In particular, the impossibility results in *Theorem 1.3* showing all d-CQ and d-QC schemes fail the CVQD protocols hold for any universal gateset with bounded fan-in gates.

Roughly speaking, d-CQ schemes allow a classical algorithm to access a d-depth quantum circuit polynomially many times; however, all the qubits of the quantum circuit need to be measured after each access (and thus no quantum state can be passed to following d-depth quantum circuits). On the other hand, d-QC schemes let a quantum circuit to access classical algorithms after each depth and pass quantum states to the rest of the circuits for at most d depths.

In [11], the class of problems that can be solved by d-CQ schemes is defined as $\mathsf{BPP}^{\mathsf{BQNC}_d}$, and the class of problems that can be solved by d-QC schemes is defined as $\mathsf{BQNC}_d^{\mathsf{BPP}}$.

Chia, Chung and Lai [11] presented an oracle problem that can separate schemes in Definition 2.1 and Definition 2.2 with different quantum circuit depths. We briefly introduce the oracle separation in the following.

Definition 2.4 (d-shuffling [11, Definition 4.1]). Let $f: \{0,1\}^n \to \{0,1\}^n$ be any function. A d-shuffling of f is defined by $\mathcal{F} := (f_0, \ldots, f_d)$, where f_0, \ldots, f_{d-1} are random permutations over $\{0,1\}^{(d+2)n}$. The last function f_d is a fixed function satisfying the following properties: let $S_d := \{f_{d-1} \circ \cdots \circ f_0(x') : x' \in \{0,1\}^n\}$.

- For $x \in S_d$, let $f_{d-1} \circ \cdots \circ f_0(x') = x$, and choose the function $f_d : S_d \to [0, 2^n 1]$ such that $f_d \circ f_{d-1} \circ \cdots \circ f_0(x') = f(x')$.
- For $x \notin S_d$, $f_d(x) = \perp$.

Then, we recall the definition of Simon's function.

Definition 2.5 (Simon's function). For a finite set S and $s \in \{0,1\}^n$ (also called the hidden shift), the Simon's function $f: \{0,1\}^n \to S$ satisfies that f(x) = f(x') if and only if $x' = \{x, x \oplus s\}$.

The Simon's problem is to compute the hidden shift s given oracle access to a Simon's function f. The quantum algorithm for Simon's problem uses one quantum query to sample a random vector y satisfying $y \cdot s = 0$. Making O(n) queries suffices to find a generating set of the subspace $H = \{y : y \cdot s = 0\}$ with overwhelming probability, and thus the hidden shift is uniquely determined from the generators. It is worth noting that any classical algorithm that finds s with high probability requires $\Omega(\sqrt{2^n})$ queries even if the Simon's function is given uniformly randomly. A random Simon's function is defined as a function drawn uniformly from the set of Simon's functions from $\{0,1\}^n$ to S and we choose $S = \{0,1\}^n$.

We now define the d-Shuffling Simon's problem (d-SSP) that separates $\mathsf{BPP}^{\mathsf{BQNC}_{2d+3}} \cap \mathsf{BQNC}_{2d+3}^{\mathsf{BPP}}$ from $\mathsf{BPP}^{\mathsf{BQNC}_d} \cup \mathsf{BQNC}_d^{\mathsf{BPP}}$ relative to an oracle.

Problem 1 (d-shuffling Simon's problem (d-SSP) [11, Definition 4.9]). Let $n \in \mathbb{N}$ and $f : \{0,1\}^n \to \{0,1\}^n$ be a random Simon's function. Given oracle access to the d-shuffling $\mathcal{F} := \{f_0, f_1, \ldots, f_d\}$ of f, the problem is to find the hidden shift s of f.

Chia, Chung and Lai showed the following theorem [11].

Theorem 2.6 ([11]). Let d = poly(n). The d-SSP problem can be solved by (2d + 3)-CQ and (2d + 3)-QC schemes with oracle access to the d-shuffling oracle of f. Furthermore, for any d'-CQ and d'-QC schemes A with with oracle access to the d-shuffling oracle of f and $d' \leq d$, the probability that A solves the problem is negligible.

Remark 2.7. In [11], it said that d-SSP can be solved by (2d+1)-CQ and -QC schemes because the models defined in Definition 3.8 and Definition 3.10 in [11] mainly considered the depth for querying the oracle. Here, for our purpose, we count the two Hadamard transforms at the beginning and the end of Simon's algorithm, which gives additional two depths.

This means that when there is a quantum algorithm of (2d+3) quantum circuit depth (including access to the oracle) succeeding with probability at least 2/3 (in fact the success probability is 1 - negl(n)). The second part of Theorem 2.6 shows that every quantum algorithm of quantum circuit depth at most d outputs the hidden shift with negligible probability, even if it makes an arbitrary polynomial number of queries.

2.2 Quantum-secure pseudorandom permutations

For our task, we also want the oracle can be implemented efficiently. However, by a counting argument, a random permutation cannot be computed efficiently with overwhelming probability. We use pseudorandom permutations to address this issue. In a query model, we can replace a random permutation with a pseudorandom one without decreasing the performance of a query algorithm by non-negligible difference.

Definition 2.8 (Quantum-secure pseudorandom permutations (qPRP) [21]). For security parameter λ and a polynomial $m = m(\lambda)$, a pseudorandom permutation P over $\{0,1\}^m$ is a keyed function $\mathcal{K} \times \{0,1\}^m \to \{0,1\}^m$ such that there exists a negligible function ϵ such that for every quantum adversary \mathcal{A} , it holds that

$$\left| \Pr_{F \leftarrow_B \mathcal{P}} [\mathcal{A}^{O_F, O_{F^{-1}}} = 1] - \Pr_{k \leftarrow_B \mathcal{K}} [\mathcal{A}^{O_{P(k,\cdot)}, O_{P^{-1}(k,\cdot)}} = 1] \right| \le \epsilon(\lambda), \tag{2}$$

where \mathcal{P} is the set of permutations over $\{0,1\}^m$ and $O_Q: |x,y\rangle \mapsto |x,y\oplus Q(x)\rangle$ for permutation $Q: \{0,1\}^m \to \{0,1\}^m$ and $x,y\in \{0,1\}^m$.

2.3 The Broadbent protocol for verifying quantum computation

In this section, we briefly introduce the Broadbent protocol for verifying quantum computation [9]. The protocol consists of two parties, the prover P which is untrusted but can perform arbitrary quantum computation, and the verifier V which is almost classical. In particular, V can perform measurements in certain bases. The prover and the verifier interact, and at the end of the protocol, the verifier outputs a bit which is either "accept" or "reject." The protocol can be used to verify a complete language in BQP (more precisely, PromiseBQP):

- Completeness: if the computation U satisfies $\|\Pi_0 U|_0 > \|^2 \ge 2/3$, then there exists a quantum prover which makes V accept with probability at least c.
- Soundness: if the computation U satisfies $\|\Pi_0 U|0\rangle\|^2 \le 1/3$, then for every prover, the verifier accepts with probability no more than s.

Here the projector $\Pi_0 = |0\rangle\langle 0| \otimes \mathbb{1}$ refers to the event that measuring the first qubit of the state $U|0\rangle$ in the standard basis yields an outcome 0. The parameters c, s are called the completeness and soundness respectively.

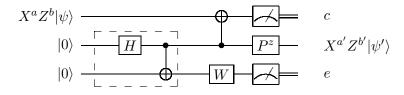


Figure 1: The quantum circuit of the T gadget. The dashed box prepares an EPR pair. The first two qubits are held by the prover and the third qubit is held by the verifier. The bit $z \leftarrow_R \{0,1\}$ is sampled independently for each T gadget by the verifier. For the choice of W, see Table 1.

In the Broadbent protocol, the prover and the verifier share (n+t) EPR pairs, where n is the number of qubits the computation U acts on, and t is the number of T gadgets in U. The quantum circuit for the T gadget is shown in Figure 1. The computation U is a quantum circuit consisting

Round type	Unitary W
Computation	$HP^{a'+c+z}T$
Even parity	1
Odd parity	HP^z

Table 1: The choice of unitary W in a T gadget. The table is modified from [13, Table 3].

Gate/Gadget	Key update rule
T gadget, computation	(a+c,b+e+a+c+(a+c)z)
T gadget, even parity	(e,0)
T gadget, odd parity	(0, b + e + z)
H gate	(b,a)
CNOT gate	(a, b + b', a + a', b')

Table 2: Rules for updating the one-time-pad keys after applying each gate in the protocol. The bit z is chosen uniformly by the verifier, and the measurement outcomes e, c are obtained by the prover (see Figure 1). For the T gadget and the Hadamard gate, before updating the key is (a, b). For a CNOT gate, the keys are (a, b) on the control qubit and (a', b') on the target.

of gates in $\{CNOT, H, T\}$ (which is a universal gate set), performed on a quantum state encrypted by quantum one-time pad. The prover is designated to perform the following operations for each gate.

- For each CNOT gate, the prover is designated to perform the gate on the associated qubits.
- ullet For each T gate, the prover performs a T gadget using a half of an EPR pair as an ancilla qubit.
- Each Hadamard gate is compiled into the sequence HTTHTTHTH of single-qubit gates. For each T gate in the sequence, they run a T gadget. For each Hadamard gate in the sequence, the prover applies a Hadamard gate. We will also refer to the implementation as a Hadamard gadget (or an H gadget for short).

The verifier V chooses to run one of the following rounds, and performs the operations on the other halves of the (n+t) EPR pairs depending on the choice of rounds. More specifically, V performs measurements on the EPR pairs to create the initial state and the states required for each T gadget, and the actions for each round type is specified as follows.

- Computation round (with constant probability p): the computation is delegated to the prover P. If P behaves honestly, the result can be recovered by V. In the beginning of the protocol, V measures the first n EPR pairs in the standard basis. The half held by the prover collapses to a quantum one-time pad of the zero state. Then the (honest) prover performs the quantum computation on this state. For each T-gadget (used to apply a T gate or a Hadamard gate), the prover uses the a half of the EPR pair associated with the T-gadget as an ancilla qubit, and the verifier performs the associated measurement so the T-gadget implements a T gate, up to a key update (Table 2).
- Test round (with probability 1-p): V tests if P behaves honestly, and rejects if any error is detected. A test round has two types, each of which is executed with probability 1/2, outlined as follows.

- An X-test round is used to detect if there is a bit flip error. In an X-test round, the verifier measures the first n EPR half in the standard basis to create a quantum one-time pad of the zero state $|0\rangle^{\otimes n}$. For each T-gadget, the verifier performs the measurement in the basis such that the T-gadget acts trivially (as the identity operation) up to a key update (Table 2). In the end, the verifier applies the key update rule to compute the key, and decrypts the first qubit. The verifier accepts if the bit is 0 and rejects otherwise.
- A Z-test round is used to detect if there is a phase flip error. The operations are the same as the X-test round except that they are performed in the Hadamard basis. The verifier measures the first n EPR pairs in the Hadamard basis to create a quantum one-time pad of the plus state $|+\rangle^{\otimes n}$. For each T gadget, the verifier performs the measurement in some basis such that the T gadget acts trivially up to a key update (Table 2). In the end, the verifier applies the key update rule to compute the key, and decrypts the first qubit. The verifier disregards the result and rejects only if any error was detected throughout the computation.

The protocol performs quantum computation on encrypted data by quantum one-time pad. Depending on the round type, different key update rules are adopted. For a Hadamard gadget, six T-gadgets are performed. Recall that in an X-test round, a T-gadget acts trivially on the zero state. Passing an odd number of Hadamard gates yields an encrypted plus state, and thus the T-gadget used in this case will be the same as a T gate application in a Z-test round. Thus for convenience, we may define the parity of a T-gadget as follows. A T-gadget is of even parity if it is not part of an Hadamard gadget, or an even (resp. odd) number of Hadamard gates has been applied before in an Hadamard gadget in an X-test (resp. a Z-test) round; otherwise it is of odd parity. The key update rules are formally defined in Table 2.

The original Broadbent protocol is used to verify BQP languages (more precisely PromiseBQP). Now we consider the following modification to show a rigidity result: the last message from the prover to the verifier is an n-qubit quantum state ρ . The verifier computes the key according to the key update rule (Table 2), and applies one-time pad decryption on ρ , with the following modification after the last message from the prover is sent.

- In an X-test round, the verifier accepts if measuring the state in the standard basis yields 0^n .
- In a Z-test round, the verifier accepts if measuring the state in the Hadamard basis yields 0^n .

The differences are instead of checking only the first qubit, the verifier determines, in a test round, if an attack has been applied on any of the qubits. The new verification procedure requires a larger quantum channel for the prover to send the entire state to the verifier. As we will see, the extra cost is not necessary when the protocol is made purely classical with two provers.

Our rigidity statement is formally stated as follows: if the prover is accepted with probability $1 - \epsilon$ in test rounds, the prover in computation run implements a quantum channel \mathcal{E}_C that is $O(\epsilon)$ -close to the honest computation. First recall the following fact about Pauli twirls.

Lemma 2.9 (Pauli twirls). Let $P_a := X^{a_1}Z^{a_2}$ denote a Pauli operator for $a = (a_1, a_2)$ where $a_1, a_2 \in \{0, 1\}^n$. For any quantum state ρ and quantum channel Φ , it holds that

$$\sum_{a \in \{0,1\}^{2n}} P_a^{\dagger} \Phi(P_a \rho P_a^{\dagger}) P_a = \sum_a r_a P_a \rho P_a^{\dagger}, \tag{3}$$

for some distribution r over Pauli matrices.

Broadbent shows that without loss of generality, any attack performed by the prover can be written as a honest execution C followed by a quantum channel Φ . The prover first performs C and yields a quantum state $\rho = C|\psi\rangle\langle\psi|C^{\dagger}$, one-time padded with key Q. The prover then applies any quantum channel Φ , followed by decryption performed by the verifier. By Lemma 2.9, the prover's attack can be written as a probabilistic mixture of Pauli unitaries. We prove the following theorem.

Theorem 2.10. For $\epsilon \in [0, 1/2]$, any prover who succeeds with probability $1 - \epsilon$ in the test rounds if and only if it implements a quantum channel \mathcal{E}_C satisfying $\|\mathcal{E}_C - \mathcal{C}\|_{\diamond} \leq 4\epsilon$ where $\mathcal{C}(\rho) := C\rho C^{\dagger}$ in the computational round.

Proof. Without loss of generality, we may express the action of any prover given access to circuit description C and the verifier's decryption as a quantum channel

$$\mathcal{E}_{C,i} = \mathbb{E}_{k}(\mathcal{O}_{k} \circ \Phi \circ \mathcal{O}_{k}) \circ \mathcal{C}_{i} = \mathcal{P} \circ \mathcal{C}_{i}, \tag{4}$$

for some Pauli channel \mathcal{P} with Kraus form $\{r_a^{1/2}P_a: a \in \{0,1\}^{2n}\}$ by Lemma 2.9 and round type i. Also recall that here we denote $P_a = X^{a_1}Z^{a_2}$ for $a = (a_1, a_2)$ and $a_1, a_2 \in \{0,1\}^n$. Since these rounds look completely identical to the prover, the quantum channel Φ must be identical among the choices of round type.

Let the success probability of the prover in an X- and a Z-test round be $1 - \epsilon_X$ and $1 - \epsilon_Z$ respectively. If the prover succeeds with probability $1 - \epsilon$ conditioned on the event that a test round is chosen, it must hold that $\epsilon_X, \epsilon_Z \leq 2\epsilon$ since otherwise $\frac{1}{2}(1 - \epsilon_X) + \frac{1}{2}(1 - \epsilon_Z) = 1 - \frac{\epsilon_X + \epsilon_Z}{2} < 1 - \epsilon$.

In an X-test round, $C_X = \mathcal{I}$, the identity channel, and

$$1 - \epsilon_X = \langle 0^n | \mathcal{P}(|0^n \rangle \langle 0^n |) | 0^n \rangle$$

$$= \sum_a r_a |\langle 0^n | P_a | 0^n \rangle|^2$$

$$= \sum_{a:a_1=0} r_a \ge 1 - 2\epsilon.$$
(5)

Similarly, in a Z-test round, $\sum_{a:a_2=0} r_a \geq 1-2\epsilon$. Combining the inequalities, we conclude that $r_0 \geq 1-4\epsilon$. Therefore, for any prover who succeeds with probability $1-\epsilon$, it holds that in the computation round

$$\|\mathcal{P} \circ \mathcal{C} - \mathcal{C}\|_{\diamond} \leq \|\mathcal{P} - \mathcal{I}\|_{\diamond}$$

$$= \max_{\rho: \text{tr}\rho = 1} \text{tr} \left| \sum_{a} r_{a} (P_{a} \rho P_{a}^{\dagger} - \rho) \right|$$

$$\leq \sum_{a \neq 0} r_{a} \|\mathcal{P}_{a} - \mathcal{I}\|_{\diamond}$$

$$\leq 4\epsilon, \tag{6}$$

where $\mathcal{P}_a(\rho) = P_a \rho P_a^{\dagger}$. The above reasoning shows the "only if" direction.

If the prover implements $\mathcal{E}_C = \mathcal{P} \circ \mathcal{C}$ which is δ -close to \mathcal{C} in the computation round. Then in the X-test round, we have $\|\mathcal{P}(|0^n\rangle\langle 0^n|) - |0^n\rangle\langle 0^n|\|_{\mathrm{tr}} \leq \|\mathcal{E}_C - \mathcal{C}\|_{\diamond} \leq \delta$. Similarly, in the Z-test round, we have $\|\mathcal{P}(|+^n\rangle\langle +^n|) - |+^n\rangle\langle +^n|\|_{\mathrm{tr}} \leq \|\mathcal{E}_C - \mathcal{C}\|_{\diamond} \leq \delta$. Then we conclude that the success probability is at $1 - \delta$ when a X-test or a Z-test round is chosen. This implies that the success probability in a test round is at least $1 - \delta$.

2.4 The Verifier-on-a-Leash protocol

The Broadbent protocol can be used to verify arbitrary quantum computation, but it requires the verifier to have the capability to perform measurements in the bases listed in Table 1. To achieve purely classical verification, Coladangelo, Grilo, Jeffery and Vidick presented a new protocol called the Verifier-on-a-Leash protocol (or the Leash protocol). In the Leash protocol, two provers called PP and PV share entanglement, but are not allowed to communicate with each other after the protocol starts. PP plays the role of the prover in the Broadbent protocol, and PV performs the measurements by the verifier in the Broadbent protocol. To certify PV plays honestly, a new rigidity test is introduced to verify Clifford measurements, in particular, the observables in $\Sigma = \{X,Y,Z,F,G\}$ where X,Y,Z are Pauli matrices, $F = \frac{1}{\sqrt{2}}(-X+Y)$ and $G = \frac{1}{\sqrt{2}}(X+Y)$. The idea is to use a non-local game for certifying the standard basis measurement (i.e., the observable X) and the Hadamard basis measurement (the observable Z). We will modify a test called RIGID' (Σ, m) which certifies Clifford measurements by Coladangelo, Grilo, Jeffery and Vidick [13] to only act on a subset of qubits. In particular, the following theorem holds with the test.

Theorem 2.11 ([13, Theorem 4]). There exists a test $\mathsf{RIGID}'(\Sigma, m)$ such that the following holds: suppose a strategy for the players succeeds in test $\mathsf{RIGID}'(\Sigma, m)$ with probability at least $1 - \epsilon$. Then for $D \in \{A, B\}$, there exists an isometry V_D such that

$$\|(V_A \otimes V_B)|\psi\rangle_{AB} - |EPR\rangle_{A'B'}^{\otimes m} |AUX\rangle_{\hat{A}\hat{B}}\|^2 \le O(\sqrt{\epsilon}), \tag{7}$$

and

$$\mathbb{E}_{W \in \Sigma^m} \sum_{u \in \{\pm\}^m} \left\| V_A \operatorname{tr}_B((\mathbb{1}_A \otimes W_B^u) | \psi \rangle \langle \psi |_{AB}(\mathbb{1}_A \otimes W_B^u)) V_A^{\dagger} - \sum_{\lambda \in \{\pm\}} \left(\bigotimes_{i=1}^m \frac{\sigma_{W_i, \lambda}^{u_i}}{2} \otimes \tau_{\lambda} \right) \right\|_1 = O(\operatorname{poly}(\epsilon)).$$
(8)

Moreover, players employing the honest strategy succeed with probability $1 - e^{-\Omega(m)}$ in the test.

Theorem 2.11 shows that a strategy that succeeds in $\mathsf{RIGID}'(\Sigma,m)$ with probability at least $1-\epsilon$ must satisfy the following conditions: The players' joint state is $O(\sqrt{\epsilon})$ -close to a tensor product of m EPR pairs together with an arbitrary ancilla register. Moreover, on average over uniformly chosen basis $W \leftarrow_R \Sigma^m$, the provers' measurement is $\mathsf{poly}(\epsilon)$ -close to a probabilistic mixture of ideal measurements and their conjugates. More specifically, the probabilistic mixture can be realized as follows: performing a measurement on the ancilla register yields a post-measurement state τ_λ and a single bit outcome $\lambda \in \{\pm\}$ that specifies whether the ideal measurement associated with the observable $\sigma_{W_{i,+}}$ or that with its conjugate $\sigma_{W_{i,-}}$ is performed.³ Then applying $\sigma_{W_{i,\lambda}}$ yields a post-measurement state $\sigma_{W_{i,\lambda}}^{u_i}$ and an outcome $u_i \in \{\pm\}$ for each index $i \in [m]$. We can only hope to certify that the strategy is close a probabilisite mixture of the ideal strategy and its conjugate because a protocol with classical communication does not distinguish a strategy from its conjugate (more concretely, replacing $i = \sqrt{-1}$ with -i for any strategy by the prover does not change the score).

In Section 4, we will modify $RIGID'(\Sigma, m)$ to only apply on a random subset of indices in [m], chosen by the verifier. It is clear that for the subset, Theorem 2.11 holds.

³The conjugate of a matrix A is obtained by taking the complex conjugate of each element of A.

2.5 Proof of quantumness

Our protocol in the plain model will be based on a construction of noisy trapdoor claw-free function (NTCF) function family [7], based on QLWE.

Definition 2.12 (NTCF family [7, Definition 3.1]). Let λ be a security parameter and \mathcal{X}, \mathcal{Y} be finite sets. Let $\mathcal{K}_{\mathcal{F}}$ be a finite set of keys. A family of functions

$$\mathcal{F} = \{ f_{k,b} : \mathcal{X} \to \mathcal{D}_{\mathcal{Y}} \}_{k \in \mathcal{K}_{\mathcal{F}}, b \in \{0,1\}}$$

$$\tag{9}$$

is called a noisy trapdoor claw free (NTCF) family if the following conditions hold:

- 1. Efficient function generation. There exists an efficient probabilistic algorithm $Gen_{\mathcal{F}}$ which generates a key $k \in \mathcal{K}_{\mathcal{F}}$ together with a trapdoor $t: (k,t) \leftarrow Gen_{\mathcal{F}}(1^{\lambda})$.
- 2. Trapdoor injective pair. For all keys $k \in \mathcal{K}_{\mathcal{F}}$, the following conditions hold.
 - (a) Trapdoor: There exists an efficient deterministic algorithm INV_F such that for all $b \in \{0,1\}$, $x \in \mathcal{X}$ and $y \in \text{Supp}(f_{k,b}(x))$, INV_F(t,b,y) = x. Note that this implies that for all $b \in \{0,1\}$ and $x \neq x'$, Supp $(f_{k,b}(x)) \cap \text{Supp}(f_{k,b}(x')) = \emptyset$.
 - (b) Injective pair: there exists a perfect matching $\mathcal{R}_k \subseteq \mathcal{X} \times \mathcal{X}$ such that $f_{k,0}(x_0) = f_{k,1}(x_1)$ if and only if $(x_0, x_1) \in \mathcal{R}_k$.
- 3. Efficient range superposition. For all key $k \in \mathcal{K}_{\mathcal{F}}$ and $b \in \{0,1\}$, there exists a function $f'_{k,b}: \mathcal{X} \to \mathcal{D}_{\mathcal{Y}}$ such that the following hold.
 - (a) For all $(x_0, x_1) \in \mathcal{R}_k$ and $y \in \text{SUPP}(f'_{k,b}(x_b))$, $\text{INV}_{\mathcal{F}}(t, 0, y) = x_0$ and $\text{INV}_{\mathcal{F}}(t, 1, y) = x_1$.
 - (b) There exists an efficient deterministic procedure $Chk_{\mathcal{F}}$ that, on input $k, b \in \{0, 1\}$, $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, returns 1 if $y \in Supp(f'_{k,b}(x))$ and 0 otherwise. Note that $Chk_{\mathcal{F}}$ is not provided the trapdoor t.
 - (c) For every k and $b \in \{0, 1\}$,

$$\mathbb{E}_{x \leftarrow_{R} \mathcal{X}}[H^2(f_{k,b}(x), f'_{k,b}(x))] \le \mu(\lambda), \tag{10}$$

for some negligible function μ . Here H^2 is the Hellinger distance. Moreover, there exists an efficient procedure SAMP_F that on input k and $b \in \{0,1\}$, prepares the state

$$|\mathcal{X}|^{-1/2} \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \sqrt{f'_{k,b}(x)(y)} |x\rangle |y\rangle. \tag{11}$$

- 4. Adaptive hardcore bit. For all keys $k \in \mathcal{K}_{\mathcal{F}}$, the following conditions hold for some integer w that is a polynomially bounded function of λ .
 - (a) For all $b \in \{0,1\}$ and $x \in \mathcal{X}$, there exists a set $G_{k,b,x} \subseteq \{0,1\}^w$ such that $\Pr_{d \leftarrow_R \{0,1\}^w} [d \notin G_{k,b,x}]$ is negligible, and moreover there exists an efficient algorithm that checks for membership in $G_{k,b,x}$ given k,b,x and the trapdoor t.
 - (b) There is an efficiently computable injection $J: \mathcal{X} \to \{0,1\}^w$ such that J can be inverted efficiently on its range, and such that the following holds. If

$$H_k = \{(b, x_b, d, d \cdot (J(x_0) \oplus J(x_1))) | b \in \{0, 1\}, (x_0, x_1) \in \mathcal{R}_k, d \in G_{k, 0, x_0} \cap G_{k, 1, x_1}\}, \bar{H}_k = \{(b, x_b, d, c) | (d, x, d, c \oplus 1) \in H_k\},$$

then for any quantum polynomial-time procedure A, there exists a negligible function μ such that

$$\left| \Pr_{(k,t) \leftarrow \operatorname{Gen}_{\mathcal{F}}(1^{\lambda})} [\mathcal{A}(k) \in H_k] - \Pr_{(k,t) \leftarrow \operatorname{Gen}_{\mathcal{F}}(1^{\lambda})} [\mathcal{A}(k) \in \bar{H}_k] \right| \le \mu(\lambda). \tag{12}$$

In a breakthrough, Brakerski, Christiano, Mahadev, Vazirani and Vidick give a proof-of-quantumness protocol (the BCMVV protocol) based on NTCFs [7]. The protocol proceeds in the following steps.

- 1. The verifier samples $(k,t) \leftarrow \text{Gen}(1^{\lambda})$ and sends k to the prover.
- 2. The prover performs SAMP_F on the input state $|+\rangle$ and measures the image register to yield an outcome y.
- 3. The verifier samples a random coin $c \leftarrow_R \{0,1\}$ and sends c to the prover.
- 4. If c = 0, the prover performs a standard basis measurement; otherwise the prover performs a Hadamard basis measurement. The outcome w is then sent to the verifier.
- 5. The verifier outputs V(t,c,w), which is defined as

$$V(t, y, c, w) := \begin{cases} 1 & \text{if } c = 0 \text{ and } \text{Chk}_{\mathcal{F}}(k, y, b, x) = 1, \ w = (b, x) \\ 1 & \text{if } c = 1, \ d \in G_{k, y} \text{ and } d \cdot (J(x_0) \oplus J(x_1)) = u, \ w = (u, d) \\ 0 & \text{otherwise,} \end{cases}$$
 (13)

where $G_{k,y} := G_{k,0,x_0} \cap G_{k,1,x_1}$. Here we note that for c = 0, CHK_F does not require the trapdoor t, but the bit can be determined using the trapdoor.

The adaptive hardcore bit property (see (12)) implies that every adversary \mathcal{A} given access to k, outputs y, w_0, w_1 such that $V(y, 0, w_0) = V(y, 1, w_1) = 1$ with probability at most $1/2 + \text{negl}(\lambda)$. On the other hand, there exist efficient quantum processes to output valid (y, w_0) or (y, w_1) with probability $1 - \text{negl}(\lambda)$.

Hirahara and Le Gall [16] and Liu and Gheorghiu [18] independently proposed two different methods of transforming the BCMVV protocol [7] into one that can be computed using only constant quantum depth (interleaving with classical computation). In our second protocol for certifying quantum depth based on LWE, we will be using the theorem which states that there exists a construction of NTCF family for which the function evaluation takes constant depth, based on randomized encoding, defined as follows.

Definition 2.13 (Randomized encoding [4, 18]). Let $f: \{0,1\}^n \to \{0,1\}^\ell$ be a function and $r \leftarrow_R \{0,1\}^m$. A function $\hat{f}: \{0,1\}^n \times \{0,1\}^m \to \{0,1\}^s$ is a δ -correct, ϵ -private randomized encoding of f if it satisfies the following properties:

- Efficient generation: there exists a deterministic polynomial-time algorithm that, given a description of the circuit implementing \hat{f} , outputs a description of a circuit implementing \hat{f} .
- δ -correctness: there exists a deterministic polynomial-time algorithm DEC, called the decoder, such that for every input $x \in \{0,1\}^n$, $\Pr_r[\text{DEC}(\hat{f}(x,r)) \neq f(x)] \leq \delta$.
- ϵ -privacy: there exists a PPT algorithm S, called the simulator, such that for every $x \in \{0,1\}^n$, the total variation distance between S(f(x)) and $\hat{f}(x,r)$ is at most ϵ .

Furthermore, a perfect randomized encoding is one for which $\epsilon = \delta = 0$.

Theorem 2.14 ([18, Section 3.1]). There exists an efficient quantum process which uses gates of bounded fan-out in constant quantum depth and prepares the state

$$\sum_{b,x} |b\rangle|x\rangle|\hat{f}_k(b,x)\rangle,\tag{14}$$

where \hat{f} is a perfect randomized encoding of an NTCF f.

In particular, as shown in the same paper [18, Section 3.3], the proof-of-quantumness protocol takes total quantum depth 14 and three quantum-classical interleavings.

Theorem 2.15 ([18, Theorem 3.1]). There exists a perfect randomized encoding of an NTCF, which satisfies the randomness reconstruction property and is an NTCF.

In particular, Theorem 2.15 implies that with the new construction, the adaptive hardcore bit property holds.

3 Certifying a query algorithm

A q-query algorithm \mathcal{A} given access to O, denoted \mathcal{A}^O , without loss of generality, consists of a sequence of unitary maps $U_qO\ldots U_1OU_0$ acting on the zero state, followed by a standard basis measurement to extract the information about the oracle. In the real world, information is transmitted in a non-black-box way, and therefore one must instantiate the oracle with an efficient quantum process. However, any instantiation in the plain model would in general leak the information about the oracle, even based on computational assumptions [6]. Thus, the analysis or the conclusions obtained in a query model does not apply when the oracle is replaced with an instantiation.

On the other hand, the impossibility result does not apply when the oracle is implemented by an external device to which the algorithm has limited access. In this setting, another device, denoted P_O , is added to implement the oracle O, and the secret encoded in the oracle may be transparent to P_O . An equivalent two-player protocol that mimics the query computation can be described as the following protocol:

- 1. A player P_A performs U_0 on the zero state, and teleports the state to the other player P_O .
- 2. P_O performs O on the quantum state and teleport the output state back.
- 3. \mathcal{A} performs U_1 on the input state and teleports the state.
- 4. Repeat the last two steps (q-1) times with unitaries U_2, \ldots, U_q and outputs w.
- 5. The verifier accepts if the oracle O and w satisfies some relation R and rejects otherwise.

When the players behave honestly, i.e., P_O performs O in each iteration, and P_A performs U_0, \ldots, U_q in order, the verifier accepts with probability exactly the same as the query algorithm A solving R. When neither of the players is trusted, we ask the question: given a query algorithm A, can we construct an equivalent protocol against untrusted players? To answer the question, we specify the classes of problems which we consider in this work, and the equivalence between a query algorithm and a two-player protocol.

Oracle separation problems. For quantum complexity classes C_{yes} , C_{no} , an oracle separation problem R is a relation between the oracle O (drawn from a public distribution) and a binary string w such that

- there is a quantum query algorithm \mathcal{A} that is given access to O and runs in \mathcal{C}_{yes} , outputs w such that R(O, w) = 1 with probability at least p, and
- no quantum algorithm that is given access to O and runs in C_{no} , outputs w such that R(O, w) = 1 with probability at most p'.

Here, the probabilities p and p' are defined by C_{yes} and C_{no} , e.g., if $C_{yes} = PP$ and $C_{no} = BPP$, p = 1/2 and p' = 1/3.

Two-player protocol for separation. A two-player protocol distinguishes C_{yes}^{4} from C_{no} if there exists a classical verifier V interacting with two provers P_{O} and P_{A} such that the following conditions hold.

- Non-locality. P_O and P_A share an arbitrary quantum state and there is no quantum and classical channel between once the protocol starts.
- Completeness. There exists P_O and P_A that runs in C_{yes} such that $\Pr[\langle V, P_O, P_A \rangle = accept] \geq c$.
- Soundness. For every P_A that runs in C_{no} and any P_O , $\Pr[\langle V, P_O, P_A \rangle = accept] \leq s$. Note that the protocol is sound for unbounded P_O .

Transforming oracle separation into two-player protocols. Our goal is to transform an oracle separation problem R into a two-player protocol that distinguishes C_{yes} from C_{no} . In this work, we focus on the cases where the oracle O is quantum accessible, and the predicate R can be computed deterministically. For real numbers $s \geq 0$ and $c \geq s + n^{-O(1)}$, we say a two-player protocol is (c, s)-equivalent to a problem R if there exists a classical verifier V such that the following conditions hold.

- Completeness. For every \mathcal{A}^O that runs in \mathcal{C}_{yes}^O and solves R with probability 2/3, there exist P_A that runs in \mathcal{C}_{yes} and a quantum prover P_O , such that V accepts with probability at least c.
- Soundness. For every P_A that runs in C_{no} , if V accepts with probability more than s, there exists an oracle algorithm \mathcal{A}^O that runs in C_{no}^O and solves R with probability more than 1/3.

The completness statement together with the contrapositive of the soundness statement implies that if there is an oracle separation problem between C_{yes} and C_{no} , then there is a two-player protocol that separates C_{yes} and C_{no} . We note the thresholds 2/3 and 1/3 are unimportant for our analysis; they can be replaced for giving a contradiction.

⁴In this work, we only focus on the machines that can at least teleport quantum states.

4 A two-player non-local game for oracle separation

In this section, we show how to classically verify a query algorithm. More specifically, we show how to turn a query algorithm into a two-player protocol that preserves the completeness and soundness, up to an inverse polynomial multiplicative factor. Our protocol consists of two quantum provers P_A and P_O sharing entanglements and interacting with a purely classical verifier V. The prover P_O is designated to perform quantum computation O, and the prover P_A performs arbitrary quantum computation to learn the information about O.

The protocol consists of two phases: in a query phase, P_O receives a teleported quantum state from P_A . An honest P_O performs the unitary map of the oracle O and teleports the resulting state to P_A . Next, in the computation phase, P_A may perform any quantum computation, subject to its resource constraints.

To classically verify whether P_A , P_O instantiates a query algorithm in the two-player model, it is crucial that the verifier V can classically verify the P_O behaves as intended. To this end, we rely on the Leash protocol by Coladangelo, Grilo, Jeffery and Vidick [13], as introduced in Section 2.4. In this protocol, the classical verifier interacts with two entangled provers PP and PV. PP is designated to perform a quantum computation U on encrypted classical input $|x\rangle$ (using quantum one-time pad). The Leash protocol relies on the Broadbent protocol to verify that PP performs U, conditioned on PV performing the correct single-qubit measurement for each T-gadget. To classically verify PV's measurements, the verifier chooses either to run either a rigidity test to verify PV, or to execute the Broadbent protocol. It is crucial that a rigidity test looks indistinguishable to PV from the real execution of the Broadbent protocol, while at the same time PP needs to play differently for these two tests. On the other hand, when running the Broadbent protocol, PP and PV are required to perform one of three indistinguishable tests introduced in the previous sections. This allows the verifier to make sure PP behaves as intended.

To verify a query algorithm, a naive approach is to run the Leash protocol to check in every query, P_O honestly run the unitary O, and P_A performs the correct measurements (for the T-gadgets). However, note that in the Leash protocol, the tests hide the round type from the provers, and only one "query" to PP is made. Between the queries, P_A performs any computation on the plaintext (which can be done by applying the Pauli correction on the teleported state from P_O). When integrating the Leash protocol for multiple queries, the round type could possibly be leaked, at least to P_O : imagine that P_A teleports its quantum state $|\psi\rangle$ to P_O . When the verifier chooses to run a rigidity test, P_O actually performs corresponding measurements on fresh EPR pairs, and leaves the state $|\psi\rangle$ unchanged. Thus P_O on receiving the teleported quantum state, can run a swap test to check if the returned state has been changed.

To overcome the issue, our approach is to run the tests in the Leash protocol only once: in the beginning of the protocol, the verifier tosses a biased coin γ such that $\Pr[\gamma = 0] = O(1/q)$. If $\gamma = 1$, the verifier chooses a uniformly random index $\ell \in \{1, \ldots, q\}$, and performs a random test. In particular, when $\gamma = 1$, the verifier chooses a random iteration $i \in [q]$, and runs the Leash protocol for the i^{th} query, i.e., running a X-test, a Z-test, or a rigidity test with probability determined later. If the provers pass the test, then the verifier simply accepts and terminates (i.e., ignores the rest of the queries). If $\gamma = 0$, the verifier only performs the same check as in the original query algorithm, i.e., it checks whether P_A 's final output w satisfies R(O, w) = 1.

We describe the protocol in the following steps. Note that though the provers do not necessarily follow the steps, we include the honest behavior for concreteness and clarity.

Protocol 1. [QUERY (q, \mathcal{O}, V)] Let p = 1/3 and $\alpha = \Theta(1/q)$.

- 1. The verifier V samples a oracle $O \leftarrow_R \mathcal{O}$ for oracle ensemble \mathcal{O} and a bit γ with $\Pr[\gamma = 0] = \alpha$. If $\gamma = 1$, then V samples a random index $\ell \leftarrow_R [q]$.
- 2. If $\gamma = 1$, for $i \in \{1, \dots, \ell 1\}$, run $\mathsf{COMP}(\Sigma, O, m)$. For $i = \ell$,
 - (a) V chooses a random protocol from $\{X-\mathsf{TEST}(\Sigma,O,m),\mathsf{Z-TEST}(\Sigma,O,m),\mathsf{RIGID}(\Sigma,m)\}$ with probability $p_X=p, p_Z=p, p_R=1-2p$ respectively.
 - (b) If the test succeeds then the verifier accepts and terminates; otherwise it rejects.

Otherwise, $\gamma = 0$, for $i \in \{1, ..., q\}$, run $\mathsf{COMP}(\Sigma, O, m)$. Finally P_A sends an answer w, and the verifier outputs 1 if V(O, w) = 1 and 0 otherwise.

The protocol QUERY (Protocol 1) describes V's behavior to initiate the protocol. More specifically, V randomly decides to do computation or tests first. If V decides to do tests, it randomly picks one iteration ℓ from [q] and randomly selects a test (X, Z, or rigidity) with probability p_X, p_Z, p_R respectively. Then V applies corresponding protocols X-TEST (Protocol 3), Z-TEST (Protocol 4), and RIGID (Protocol 5) at iteration ℓ and always applies COMP (Protocol 2) to other rounds.

Steps 1 to 4 in these four protocols are the same. In particular, the verifier V divides the EPR pairs shared between P_O and P_A into subsets N_C , N_X , and N_Z for running COMP (Protocol 2), X-TEST (Protocol 3), and Z-TEST (Protocol 4) with corresponding measurements (chosen from the set Σ for each EPR pair). The registers B_1, \ldots, B_d are used for applying the T gadgets. Then, the verifier asks P_A to measure its halves of the EPR pairs according to W, to teleport his queries to P_O , and to send the corresponding Pauli correction to the V. The verifier V sets N according to the round type and reveals N only to P_O , but the underlying state remains unknown to P_O since it is encrypted by a quantum one-time pad. On the other hand, P_A cannot distinguish COMP (Protocol 2), X-TEST (Protocol 3) or Z-TEST (Protocol 4) since it does not know N. Moreover, from P_A 's view, RIGID (Protocol 5) looks the same as the other tests. This implies that P_A cannot know if V is chooses to run RIGID (Protocol 5) to check its behavior.

Protocol 2. $[COMP(\Sigma, O, m)]$

1. Setup:

- (a) V samples a random subset $N_C \subseteq [m]$ of size n, where n is the number of qubits O acts on.
- (b) For each $i \in \bar{N}_C$, V samples $W_i \leftarrow_R \Sigma$.
- (c) Then V samples two random subsets $N_X \subseteq \{i : W_i = X\}$ and $N_Z \subseteq \{i : W_i = Z\}$, both of size n.
- (d) V sets $N = N_C$.
- (e) V partitions $[m]\setminus (N_C \cup N_X \cup N_Z)$ into subsets of equal size B_1, \ldots, B_d , where d is the number of T layers in O.
- 2. $(V \to P_A)$ V sends $N_C, W_{\bar{N}_C}$ to P_A sequentially.

- 3. $(P_A \to V)$ P_A teleports its quantum state $|\psi\rangle$ using EPR pairs with indices in N, and sends the Pauli correction (a_{N_C}, b_{N_C}) . For each $i \in \bar{N}_C$, perform measurements on the i^{th} EPR pair in basis W_i . Finally P_A sends the outcomes $e_{\bar{N}_C}$.
- 4. $(V \to P_O)$: V sends N to P_O .
- 5. For each $\ell = 1, \ldots, d$,
 - (a) $(V \to P_O)$ V chooses a random subset $T_{\ell} \subseteq \{i \in B_{\ell} : W_i \in \{G, F\}\}$, and sends T_{ℓ} to P_O . For each Clifford gate in the ℓ -th layer, perform the appropriate key update.
 - (b) $(P_O \to V)$: P performs the Clifford operations in the ℓ -th layer. For each T-gadget in the ℓ -th layer, P_O runs the T-gadget on $(i,j) \in T_\ell \times N$, and sends the measurement outcome c_{T_ℓ} to V.
 - (c) $(V \to P_O)$ For each $i \in T_\ell$, set $z_i = a_j + 1_{W_i = F} + c_i$. V sends z_{T_ℓ} to P_O .
- 6. $(P_O \to V)$ Let $|\phi\rangle$ be the resulting state after the evaluation of O is done. P_O teleports $|\phi\rangle$ to P_A and sends the Pauli correction a', b' to V.

In the following steps in Protocol 2, V basically guides P_O to apply the oracle on $|\psi\rangle$ using corresponding gadgets and updates the keys according to the measurement outcomes of P_O and P_A .

Protocol 3. $[X-TEST(\Sigma, O, m)]$

- 1. Setup:
 - (a) V samples a random subset $N_C \subseteq [m]$ of size n.
 - (b) For each $i \in \bar{N}_C$, V samples $W_i \leftarrow_R \Sigma$.
 - (c) Then V samples two random subsets $N_X \subseteq \{i : W_i = X\}$ and $N_Z \subseteq \{i : W_i = Z\}$, both of size n.
 - (d) V sets $N = N_X$.
 - (e) V partitions $[m]\setminus (N_C \cup N_X \cup N_Z)$ into subsets of equal size B_1, \ldots, B_d , where d is the number of T layers in O.
- 2. $(V \to P_A)$ V sends $N_C, W_{\bar{N}_C}$ to P_A sequentially.
- 3. $(P_A \to V)$ P_A teleports its quantum state $|\psi\rangle$ using EPR pairs with indices in N, and sends the Pauli correction (a_{N_C}, b_{N_C}) . For each $i \in \bar{N}_C$, perform measurements on the i^{th} EPR pair in basis W_i . Finally P_A sends the outcomes $e_{\bar{N}_C}$.
- 4. $(V \to P_O)$: V sends N to P_O .
- 5. For each $\ell = 1, \ldots, d$,
 - (a) $(V \to P_O)$ V chooses a random subset $T_\ell = T_\ell^0 \cup T_\ell^1$ such that T_ℓ^0 is a random subset of $\{i: W_i = Z\}$, and T_ℓ^1 is a random subset of $\{i: W_i \in \{X,Y\}\}$. V sends T_ℓ to P_O . For each Clifford gate in the ℓ -th layer, perform the appropriate key update.
 - (b) $(P_O \to V)$: P_O performs the Clifford operations in the ℓ -th layer. For each T-gadget in the ℓ -th layer, P_O runs the T-gadget on $(i,j) \in T_\ell \times N$, and sends the measurement outcome c_{T_ℓ} to V.

- (c) $(V \to P_O)$ For each $i \in T_\ell^0$, set $z_i \leftarrow_R \{0,1\}$; if $i \in T_\ell^1$, $z_i = 1_{W_i = Y}$. V sends z_{T_ℓ} to P_O . P_O teleports $|\phi\rangle$ to P_A and sends the Pauli correction a', b' to V.
- 6. $(P_O \to V)$ Let $|\phi\rangle$ be the resulting state after the evaluation of O is done. P_O teleports $|\phi\rangle$ to P_A and sends the Pauli correction a', b' to V.
- 7. $(V \to P_A)$ V requests P_A to perform a standard basis measurement on every qubit of the teleported state.
- 8. $(P_A \to V)$ P_A performs a standard basis measurement on every qubit of the teleported state and sends the outcome d. V accepts if and only if $d \oplus a' \oplus a'' = 0$, where a'' is calculated by the key update rule.

Protocol 4. $[Z-TEST(\Sigma, O, m)]$

- 1. Setup:
 - (a) V samples a random subset $N_C \subseteq [m]$ of size n.
 - (b) For each $i \in \bar{N}_C$, V samples $W_i \leftarrow_R \Sigma$.
 - (c) Then V samples two random subsets $N_X \subseteq \{i : W_i = X\}$ and $N_Z \subseteq \{i : W_i = Z\}$, both of size n.
 - (d) V sets $N = N_Z$.
 - (e) V partitions $[m]\setminus (N_C \cup N_X \cup N_Z)$ into subsets of equal size B_1, \ldots, B_d , where d is the number of T layers in O.
- 2. $(V \to P_A)$ V sends $N_C, W_{\bar{N}_C}$ to P_A sequentially.
- 3. $(P_A \to V)$ P_A teleports its quantum state $|\psi\rangle$ using EPR pairs with indices in N, and sends the Pauli correction (a_{N_C}, b_{N_C}) . For each $i \in \bar{N}_C$, perform measurements on the i^{th} EPR pair in basis W_i . Finally P_A sends the outcomes $e_{\bar{N}_C}$.
- 4. $(V \to P_O)$: V sends N to P_O .
- 5. For each $\ell = 1, \ldots, d$,
 - (a) $(V \to P_O)$ V chooses a random subset $T_\ell = T_\ell^0 \cup T_\ell^1$ such that T_ℓ^0 is a random subset of $\{i: W_i = \{X,Y\}\}$, and T_ℓ^1 is a random subset of $\{i: W_i \in Z\}$. V sends T_ℓ to P_O . For each Clifford gate in the ℓ -th layer, perform the appropriate key update.
 - (b) $(P_O \to V)$: P_O performs the Clifford operations in the ℓ -th layer. For each T-gadget in the ℓ -th layer, P_O runs the T-gadget on $(i,j) \in T_\ell \times N$, and sends the measurement outcome c_{T_ℓ} to V.
 - (c) $(V \to P_O)$ For each $i \in T_\ell^1$, set $z_i \leftarrow_R \{0,1\}$; if $i \in T_\ell^0$, $z_i = 1_{W_i = Y}$. V sends z_{T_ℓ} to P_O . P_O teleports $|\phi\rangle$ to P_A and sends the Pauli correction a', b' to V.
- 6. $(P_O \to V)$ Let $|\phi\rangle$ be the resulting state after the evaluation of O is done. P_O teleports $|\phi\rangle$ to P_A and sends the Pauli correction a', b' to V.
- 7. $(V \to P_A)$ V requests P_A to perform a Hadamard basis measurement on every qubit of the teleported state.

8. $(P_A \to V)$ P_A performs a Hadamard basis measurement on every qubit of the teleported state and sends the outcome d. V accepts if and only if $d \oplus b' \oplus b'' = 0$, where b'' is calculated by the key update rule.

Protocol 3 and Protocol 4 originate from the Broadbent protocol to check if P_O is consistent with O. The crucial idea, as introduced in Section 2.3, is that P_O acts as applying identity on an all-zero or an all-plus state up to key update. Therefore, the verifier can detect if there is an attack applied to the computation: First V asks P_A to perform a standard basis or a Hadamard basis measurement on the state received from P_O . Then it applies the key update rules to compute the decryption key, and check if the measurement outcomes from P_A decrypts to zero.

Since the above steps relies on reliable measurements performed by P_A , it is essential to enforce P_A to perform the measurements correctly. We include our modification $\mathsf{RIGID}(\Sigma,m)$ of the rigidity test in Section 2.4. The test is the same as $\mathsf{RIGID}'(\Sigma,|\bar{N}|)$ on a random subset \bar{N} of [m]. The purpose of RIGID (Protocol 5) is to check if P_A measures its EPR pairs in bases W'. From the collection of measurement outcomes for questions W to P_A and W' to P_O , V checks if the outcomes follows the relation specified in Protocol 5. By Theorem 2.11, passing with probability $1-\epsilon$ ensures P_A 's output is $\mathsf{poly}(\epsilon)$ -close in total variation distance to a measurement performed on EPR pairs in the correct bases W. Note that P_O knows that V chooses to execute RIGID (Protocol 5) after receiving W from V. However, P_A does not know this since it only receives random partitions and W, indistinguishable from the messages he obtained in other protocols.

Protocol 5. $[RIGID(\Sigma, m)]$

- 1. Setup:
 - (a) V samples a random subset $N_C \subseteq [m]$ of size n.
 - (b) For each $i \in \bar{N}_C$, V samples $W_i \leftarrow_R \Sigma$.
 - (c) Then V samples two random subsets $N_X \subset \{i : W_i = X\}$ and $N_Z \subseteq \{i : W_i = Z\}$, both of size n.
 - (d) V sets $N = N_C$.
 - (e) V partitions $[m]\setminus (N_C \cup N_X \cup N_Z)$ into subsets of equal size B_1, \ldots, B_d , where d is the number of T layers in O.
- 2. Execute RIGID' $(\Sigma, |\bar{N}|)$ on the subset \bar{N} and output the outcome.

4.1 Completeness

The completeness immediately follows: for algorithm \mathcal{A} , the provers P_O performs O and P_A performs U_i in each iteration i. Then the provers with probability 1 if X-TEST or Z-TEST is chosen. By Theorem 2.11, when RIGID is chosen, they succeeds with probability $1 - \exp(-\Omega(n+t))$. We give a proof as follows.

Theorem 4.1 (Completeness). For every q-query algorithm \mathcal{A}^O which outputs w satisfying R(O, w) = 1 with probability at least 2/3, there exist provers P_O, P_A which succeed with probability at least $1 - \frac{\alpha}{3} - \exp(-\Omega(n+t))$.

Proof. For every algorithm \mathcal{A} that succeeds with probability at least $c' \geq 2/3$, in each iteration i, P_O runs O and P_A runs U_i . The provers passes X-TEST and Z-TEST, when they are chosen,

with probability 1. By Theorem 2.11, when the verifier chooses to execute RIGID (with probability $(1-\alpha)/3$), honest provers succeed with probability $1 - \exp(-\Omega(n+t))$ since CHSH games are run in sequential repetition. Thus the success probability of the provers is

$$\alpha \cdot c' + \frac{2(1-\alpha)}{3} + \frac{1-\alpha}{3} (1 - \exp(-\Omega(n+t))) = 1 - \alpha(1-c') - \frac{1-\alpha}{3} \exp(-\Omega(n+t))$$

$$\geq 1 - \frac{\alpha}{3} - \exp(-\Omega(n+t)). \tag{15}$$

4.2 Soundness

To show the soundness, recall that it suffices to show that if the provers succeed with probability more than s, then there exists a query algorithm which is accepted with probability more than 1/3. First we show a simpler case in which P_A behaves honestly.

Lemma 4.2 (Soundness with honest P_A). There exists $\alpha = \Theta(1/q)$ such that, for every provers P_O that succeeds with success probability $s > 1 - \frac{2\alpha}{3}$, there exists a query algorithm that is given access to O and outputs w satisfying R(O, w) = 1 with probability more than 1/3.

Proof. Let the success probability be $1 - \epsilon_i$ conditioned on $\gamma = 1$ and the chosen index is i. The the probability of failing X-TEST or Z-TEST is at most $p^{-1}\epsilon_i$. Then the quantum channel \mathcal{E}_i that P_A implements at iteration i satisfies $\|\mathcal{E}_i - \mathcal{O}\|_{\diamond} \leq 2p^{-1}\epsilon_i$.

Then let \mathcal{A} be $\mathcal{U}_q \circ \mathcal{O} \circ \mathcal{U}_{q-1} \circ \cdots \circ \mathcal{U}_1 \circ \mathcal{O} \circ \mathcal{U}_0$. By union bound,

$$\|\mathcal{A} - \mathcal{U}_q \circ \mathcal{E}_q \circ \mathcal{U}_{q-1} \circ \dots \circ \mathcal{U}_1 \circ \mathcal{E}_1 \circ \mathcal{U}_0\|_{\diamond} \le \frac{2}{p} \sum_{i=1}^t \epsilon_i = 2qp^{-1}\epsilon, \tag{16}$$

where $\epsilon := \frac{1}{q} \sum_{i} \epsilon_{i} \in [0, 1]$. Then the success probability of P_{O} is

$$s = (1 - \alpha)(1 - \epsilon) + \alpha \delta \le (1 - \alpha)(1 - \epsilon) + \alpha(p_{\mathcal{A}} + 2q\epsilon/p), \tag{17}$$

where δ is the probability that the provers output w such that such that R(O, w) = 1 conditioned on $\gamma = 0$ (i.e., the second quantum channel in (16)), and $p_{\mathcal{A}}$ is the success probability of \mathcal{A} . This implies that

$$p_{\mathcal{A}} \ge \frac{s}{\alpha} - \left(\frac{1}{\alpha} - 1\right)(1 - \epsilon) - 2q\epsilon/p = 1 - \frac{1 - s}{\alpha} + \epsilon\left(\frac{1}{\alpha} - 1 - \frac{2q}{p}\right). \tag{18}$$

Setting $\alpha = \frac{1}{1+2q\cdot c/p}$ for any constant c>0, we conclude that $p_{\mathcal{A}} \geq 1-\frac{1-s}{\alpha}$. If $s>1-\frac{2\alpha}{3}$, $p_{\mathcal{A}}>1/3$.

Now we consider the effect of a cheating P_A . The crucial idea is if P_A chooses to deviate non-trivially from the protocol by ϵ in total variation distance, then the probability it is accepted when RIGID (Protocol 5) is chosen is then at most $1 - \epsilon$. As argued previously, since P_A does not learn whether RIGID is selected, the same strategy must have been applied for other tests. This implies that in the delegation game (where X-TEST, Z-TEST or COMP is chosen), the score can be at most increased by at most poly(ϵ). This is because as shown in Theorem 2.11, the rigidity test guarantees that for every pair of provers succeeds with probability $1 - \epsilon$, the output transcript must

be $\operatorname{poly}(\epsilon)$ -close to the that from an honest strategy in total variance distance. More formally, for every pair of provers P_O and P_A such that they succeed in the rigidity test with probability $1-\epsilon$ and in the delegation game with probability q, there exist P'_O and P'_A such that P'_A plays honestly (i.e., performs a correct measurement on a half of every EPR pairs) and they succeed in the delegation game with probability $q - \operatorname{poly}(\epsilon)$. We use the result to prove the following theorem.

Theorem 4.3 (Soundness). For constant p, there exists $\alpha = 1/\operatorname{poly}(q)$, such that for every pair of provers that succeeds with probability $s > 1 - \frac{2\alpha}{3}$, there exists a query algorithm that is given access to O and outputs w satisfying R(O, w) = 1 with probability more than 1/3.

Proof. Let the success probability be $1-\epsilon_i$ conditioned on $\gamma=1$ and the chosen index being $i\in[q]$. Thus the failure probabilities are at most $\frac{\epsilon_i}{p},\frac{\epsilon_i}{p}$ and $\frac{\epsilon_i}{1-2p}$ respectively conditioned on the events that an X-TEST, an Z-TEST and an rigidity test RIGID is chosen. Also note that when an X-TEST or a Z-TEST is chosen, the provers do not distinguish the test from COMP until V asks a measurement from P_A . When RIGID is chosen, P_A does not distinguish it from COMP, X-TEST, Z-TEST until V accepts or rejects.

By Theorem 2.11, there exist P_A', P_O' such that P_A' plays honestly, and P_O' successfully passes X-TEST and Z-TEST with probability at least $1 - \delta_i = 1 - \frac{\epsilon_i}{p} - \text{poly}\left(\frac{\epsilon_i}{1-2p}\right)$. Thus by Theorem 2.10, P_O' implements a quantum channel \mathcal{E}_i such that $\|\mathcal{E}_i - \mathcal{O}\|_{\diamond} \leq 1 - 2\delta_i$.

Conditioned on $\gamma = 0$, let the process of P'_A on receiving a teleported state $\rho_{in}^{(i)}$, produces the output state $\rho_{out}^{(i)}$ be $\mathcal{U}_i : \rho_{in}^{(i)} \mapsto \rho_{out}^{(i)}$. Now let the algorithm $\mathcal{A} := \mathcal{U}_q \circ \mathcal{O}_q \circ \cdots \circ \mathcal{U}_1 \circ \mathcal{O}_1 \circ \mathcal{U}_0$. By union bound,

$$\|\mathcal{A} - \mathcal{U}_q \circ \mathcal{E}_q \circ \mathcal{U}_{q-1} \circ \dots \circ \mathcal{U}_1 \circ \mathcal{E}_1 \circ \mathcal{U}_0\|_{\diamond} \le 2 \sum_{i=1}^q \delta_i, = 2q\delta, \tag{19}$$

where $\delta = \frac{1}{q} \sum_{i=1}^{q} \delta_i = \epsilon/p + \operatorname{poly}(\frac{\epsilon}{1-2p}) \leq g(\epsilon) = \operatorname{poly}(\epsilon)$ for some monotonically increasing g in $[0,\infty)$ (e.g., $c \cdot \epsilon^a$ for constants $a \leq 1$ and c). Since g is monotonically increasing for $\epsilon \geq 0$, we note that the inverse g^{-1} exists. The success probability of the provers is upper bounded by

$$s \le (1 - \alpha)(1 - \epsilon) + \alpha \cdot \max\{(p_A + 2q \cdot g(\epsilon)), 1\} \tag{20}$$

where $\epsilon = \frac{1}{q} \sum_i \epsilon_i$ and $p_{\mathcal{A}}$ is the probability that measuring the associated qubits on \mathcal{A} 's output state yields an outcome w satisfying R(O, w) = 1. Since $g(\epsilon)$ is monotonically increasing, there exists $\epsilon^* \geq 0$ such that $2q\delta(\epsilon^*) = 1 - p_{\mathcal{A}} \leq 2q \cdot g(\epsilon^*)$. This implies that

$$s \le (1 - \alpha)(1 - \epsilon^*) + \alpha = 1 - (1 - \alpha)\epsilon^* \le 1 - (1 - \alpha) \cdot g^{-1} \left(\frac{1 - p_A}{2q}\right),\tag{21}$$

and

$$p_{\mathcal{A}} \ge 1 - 2q \cdot g\left(\frac{1-s}{1-\alpha}\right). \tag{22}$$

By (22), if
$$s > 1 - (1 - \alpha) \cdot g^{-1}(\frac{1}{3q}), p_{\mathcal{A}} > 1/3$$
. For $\alpha > \frac{g^{-1}(\frac{1}{3q})}{\frac{2}{3} + g^{-1}(\frac{1}{3q})}$, we have $1 - (1 - \alpha) \cdot g^{-1}(\frac{1}{3q}) > 1 - \frac{2\alpha}{3}$. It suffices to choose $\alpha = \frac{2 \cdot g^{-1}(\frac{1}{3q})}{\frac{2}{3} + g^{-1}(\frac{1}{3q})} = 1/\operatorname{poly}(q)$.

Setting p = 1/3, we conclude with the following corollary, a direct consequence of Theorem 4.1 and Theorem 4.3.

Corollary 4.4. Let C_{yes} , C_{no} be two complexity classes. If there exists an oracle \mathcal{O} and a relation R such that R is solvable in $C_{yes}^{\mathcal{O}}$ using q queries but not in $C_{no}^{\mathcal{O}}$. Then, there exists $\alpha = 1/\operatorname{poly}(q)$ and a protocol $\langle V, P_O, P_A \rangle$ such that the following statements hold.

- There exist P_O that runs in $O(q \cdot CC(\mathcal{O})) + \operatorname{poly}(n)$ and P_A runs in C_{yes} such that the verifier accepts with probability at least $1 \frac{\alpha}{3} e^{-\Omega(n)}$.
- For any P_A that runs in C_{no} and any unbounded P_O , the verifier accepts with probability at most $1 \frac{2\alpha}{3}$.

Here, $CC(\mathcal{O})$ is the quantum circuit complexity for implementing \mathcal{O} .

5 Application: classical verification of quantum depth

In this section, we will prove the existence of $CVQD_2(d, d')$ for integers d' > d. First we give the formal definition as follows.

Definition 5.1 (CVQD₂(d, d')). Let d, d' $\in \mathbb{N}$ and d' > d. A two-prover protocol CVQD₂(d, d') that separates quantum circuit depth d from d' consists of a classical verifier V and two provers P_O , P_A such that the following properties hold:

- Non-locality: P_O and P_A share an arbitrary quantum state, and there is no quantum and classical channel between them once the protocol starts.
- Completeness: There exist an integer $\hat{d} \geq d'$, a quantum prover P_O and a \hat{d} -QC or \hat{d} -CQ scheme P_A such that $\Pr[\langle V, P_O, P_A \rangle = accept] \geq 2/3$.
- Soundness: For integer $\hat{d} \leq d$ and any \hat{d} -QC or \hat{d} -CQ scheme P_A and any P_O , $\Pr[\langle V, P_O, P_A \rangle = accept] \leq 1/3$.

We prove the following theorem by Corollary 4.4 and the oracle separation in [11]. Note that Definition 5.1 does not specify the power of V and honest P_O except that V is a classical algorithm and P_O is a quantum machine that can store EPR pairs. we first show the existence of an information-theoretically secure $\mathsf{CVQD}_2(d, 2d+3)$ for any $d = \mathsf{poly}(n)$ with inefficient verification, i.e., V and honest P_O need to run in exponential time. Then, we show a $\mathsf{CVQD}_2(d, 2d+3)$ for any $d = \mathsf{poly}(n)$ with efficient verification under the assumption that qPRP (Definition 2.8) exists.

Theorem 5.2. Let d = poly(n).

- 1. There exist $\alpha = 1/\operatorname{poly}(d)$ and an unconditionally secure $\mathsf{CVQD}_2(d, 2d + 3)$ (Definition 5.1), in which the verifier V runs in probabilistic $O(2^n)$ time such that the following holds.
 - Completeness: There exist P_A which has quantum depth at least 2d+3 and P_O which runs in quantum $O(2^n)$ time such that $\Pr[\langle V, P_O, P_A \rangle = accept] \ge 1 \frac{\alpha}{3}$.
 - Soundness: For any unbounded P_O and P_A that are d-CQ or d-QC schemes, $\Pr[\langle V, P_O, P_A \rangle = accept] \leq 1 \frac{2\alpha}{3}$.
- 2. Assume that there exist quantum-secure pseudorandom permutations (qPRP) as defined in Definition 2.8. There exist $\alpha = 1/\operatorname{poly}(d)$ and $\mathsf{CVQD}_2(d, 2d+3)$ (Definition 5.1), in which V runs in probabilistic polynomial time such that the following holds.

- Completeness: There exist P_A that has quantum depth at least 2d+3 and P_O that runs in quantum polynomial time such that $\Pr[\langle V, P_O, P_A \rangle = accept] \geq 1 \frac{\alpha}{3}$.
- Soundness: For any unbounded P_O and P_A that are d-CQ or d-QC schemes, $\Pr[\langle V, P_O, P_A \rangle = accept] \leq 1 \frac{2\alpha}{3}$.

For efficient instantiations, it is required that the functions f_0, \ldots, f_d are efficiently samplable and computable. Any construction of qPRP satisfying Definition 2.8 (e.g., [21]) can be used to construct a pseudorandom d-shuffling of a pseudorandom Simon's function. We now give constructions.

In the problem d-SSP, the functions f_0, \ldots, f_{d-1} are random permutations. These functions can be replaced with pseudorandom permutations. For the last function f_d , we note that f_d can be written as $f \circ f_0^{-1} \circ \cdots \circ f_{d-1}^{-1}$, where f is a random Simon's function, when the domain is restricted to a hidden subset. It then suffices to show a construction of a pseudorandom Simon's function.

Definition 5.3 (Pseudorandom Simon's function). For finite set \mathcal{Y} , let \mathcal{S} be the set of Simon's function from $\{0,1\}^n$ to \mathcal{Y} , i.e., $f \in \mathcal{S}$ if there exists $s \in \{0,1\}^n$ such that f(x) = f(x') if and only if $x = x' \oplus s$. For key space \mathcal{K} , a pseudorandom Simon's function is a keyed function $F: \mathcal{K} \times \{0,1\}^n \to \{0,1\}^m$ such that for every quantum adversary \mathcal{A} , it holds that

$$\left| \Pr_{F \leftarrow_R \mathcal{S}} [\mathcal{A}^F() = 1] - \Pr_{k \leftarrow_R \mathcal{K}} [\mathcal{A}^{F_k}() = 1] \right| \le \text{negl}(n).$$
 (23)

We note that by the definition of Simon's function, it must be the case that $m \ge n - 1$. Next we prove that there exists a pseudorandom Simon's function from qPRP.

Claim 5.4. Assume that qPRP exists as defined in Definition 2.8. Then there exists a pseudorandom Simon's function as defined in Definition 5.3.

Proof. We first show that a random Simon's function can be constructed from a random permutation, and thus replacing a random permutation with a qPRP, we obtain a pseudorandom Simon's function.

Let $H := \{x \in \{0,1\}^n : x < x \oplus s\}$ for total ordering < over $\{0,1\}^n$ defined as follows: For $x, y \in \{0,1\}^n$, x < y if the smallest index $i \in [n]$ where $x_i \neq y_i$ satisfies $x_i = 0$ and $y_i = 1$.

The subset H forms a subgroup of $\{0,1\}^n$ for group operation \oplus : It is clear that $0 \in H$ since 0 is smaller than any string in $\{0,1\}^n$. Let $i \in [n]$ be the smallest index such that $s_i = 1$. For $x, y \in H$, $x_i = y_i = 0$, and thus $(x \oplus y)_i = 0$. This implies that $x \oplus y \in H$. Since H is a subgroup, the cosets $\{H, s \oplus H\}$ forms a partition of $\{0,1\}^n$.

Now we show that for codomain $\mathcal{Y} = \{0,1\}^m$ where $m \geq n-1$, every Simon's function $f: \{0,1\}^n \to \mathcal{Y}$ can be constructed from a permutation and a hidden shift s. We then define the following function $T_s: \{0,1\}^n \to H$, $T_s(x) = x$ for $x \in H$, and $T_s(x) = x \oplus s$ for $x \in s \oplus H$. Let the mapping $W_s: H \to \{0,1\}^m$,

$$W_s(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, 0, \dots, 0),$$
(24)

where i is the smallest index such that $s_i = 1$. The padding has m - n + 1 zeros. For every Simon's function f with shift s, we know that $f = f \circ T_s$. When the domain is restricted to H, f is injective, and thus there exist $(2^m - |H|)!$ permutations $P : \{0,1\}^m \to \{0,1\}^m$ such that $f = P \circ W_s \circ T_s$ (we use the convention that 0! = 1). On the other hand, by definition, for every P, $P \circ W_s \circ T_s$ is a Simon's function. These facts imply that $(P,s) \mapsto P \circ W_s \circ T_s$ is a well-defined mapping from a pair of permutation and shift to a Simon's function and is $(2^m - |H|)!$ -to-1.

Thus a random Simon's function can be sampled (inefficiently) as follows: First sample a uniform shift $s \leftarrow_R \{0,1\}^n$ and a random permutation $P: \{0,1\}^m \to \{0,1\}^m$ and output $P \circ W_s \circ T_s$. A pseudorandom Simon's function can be sampled efficiently using a qPRP $F: \mathcal{K} \times \{0,1\}^m \to \{0,1\}^m$: Sample a random shift $s \leftarrow_R \{0,1\}^n$ and $k \leftarrow_R \mathcal{K}$ and output $g_{k,s} := F_k \circ W_s \circ T_s$.

To show that $g_{k,s}$ indeed yields a pseudorandom Simon's function, suppose toward contradiction there exists an adversary \mathcal{A} which distinguishes $g_{k,s}$ for uniform k,s from a random Simon's function with non-negligible probability ϵ . Then the reduction \mathcal{B} given oracle access to a permutation Q, samples $s \leftarrow_R \{0,1\}^n$ and outputs $b \leftarrow \mathcal{A}^{Q \circ W_s \circ T_s}$. If Q is random, $Q \circ W_s \circ T_s$ is a random Simon's function; otherwise let the key be k, and the Simon's function is $g_{k,s}$. By the assumption, \mathcal{B} distinguishes a random permutation from a qPRP with non-negligible advantage.

We then define the pseudorandom d-shuffling of a function f, which can be implemented in quantum polynomial time.

Definition 5.5 (Pseudorandom d-shuffling (cf. Definition 2.4)). For $f : \{0,1\}^n \to \{0,1\}^n$, pseudorandom d-shuffling of f is a tuple of functions (f_0,\ldots,f_d) , where f_0,\ldots,f_{d-1} are pseudorandom permutations over $\{0,1\}^{(d+2)n}$, and the last function f_d is a fixed function satisfying the following properties: Let $S_d := \{f_{d-1} \circ \cdots \circ f_0(x') : x' \in \{0,1\}^n\}$.

- For $x \in S_d$, let $f_{d-1} \circ \cdots \circ f_0(x') = x$, and choose the function $f_d : S_d \to [0, 2^n 1]$ such that $f_d \circ f_{d-1} \circ \cdots \circ f_0(x') = f(x')$.
- For $x \notin S_d$, $f_d(x) = \bot$.

Now we are ready to define a pseudorandom d-shuffling Simon's problem.

Problem 2 (Pseudorandom d-SSP (cf. Problem 1)). Given oracle access to a pseudorandom d-shuffling (Definition 5.5) of a pseudorandom Simon's function (Definition 5.3), the problem is to find the hidden shift.

By a simple hybrid argument, pseudorandom d-SSP separates a depth-(2d + 3) quantum computation from a depth-d one.

Corollary 5.6. Let d = poly(n). Pseudorandom d-SSP (Problem 2) can be solved by a QNC_{2d+3} circuit with classical post-processing. Furthermore, for any \hat{d} -CQ and $\hat{d}d$ -QC schemes \mathcal{A} with $\hat{d} \leq d$, the probability that \mathcal{A} solves the problem is negligible.

Now, we can prove Theorem 5.2.

Proof of Theorem 5.2. By Theorem 2.6, d-SSP separates the complexity classes $\mathsf{BPP}^{\mathsf{BQNC}_d} \cup \mathsf{BQNC}_d^{\mathsf{BPP}}$ and $\mathsf{BPP}^{\mathsf{BQNC}_{2d+3}} \cap \mathsf{BQNC}_{2d+3}^{\mathsf{BPP}}$ relative to the d-shuffling oracle of f. Furthermore, teleporting quantum states only takes one circuit depth by choosing the gateset properly or considering the gateset including all one- and two-qubit gates as in Remark 2.3. Therefore, by setting q = 2d + 1, \mathcal{O} to be the shuffling oracle, and R to be d-SSP, Protocol 1 separates depth-(2d + 3) quantum computation from depth-d and the relation R(O, w) = 1 if and only if O is the shuffling oracle and w is the hidden shift according (see Corollary 4.4). Here, V and P_O are inefficient since describing and implementing the shuffling oracle are inefficient.

Following the same proof, we can also show that Protocol 1 separates depth-(2d + 3) quantum computation from depth-d by replacing d-SSP by pseudorandom d-SSP. This follows from the fact that pseudorandom d-SSP also separates depth-(2d + 3) quantum computation from depth-d by

Corollary 5.6. Then, we follow the same proof for d-SSP by using Corollary 4.4 except that we set \mathcal{O} to be a pseudorandom-shuffling oracle that can be described and implemented efficiently, and thus both V and P_O are efficient.

Note that the algorithm is allowed to make parallel queries which do not increase the query depth. It is straightforward to adapt Protocol 1 to allow parallel queries: let t denote the largest number of parallel queries. For any query algorithm \mathcal{A} of depth q, there is a query algorithm \mathcal{A}' that is given access to $O^{\otimes t}$ and achieves the same performance as \mathcal{A} . The equivalent two-player protocol to \mathcal{A}' is QUERY(2d+3, O', R), where sampling O' can be performed by sampling $O \leftarrow_R \mathcal{O}$ and outputting $O' = O^{\otimes t}$.

Furthermore, we emphasize that while the protocol only has a small completeness-soundness gap $\alpha/3 = 1/\operatorname{poly}(d)$, by sequential repetition for $O(\alpha^{-2} \cdot \log^2 \lambda)$ times it suffices to amplify the gap to $1 - \operatorname{negl}(\lambda)$.

5.1 A nearly optimal separation

In this section, we modify the original d-SSP to give an oracle separation that reduces the gap from d versus 2d + 3 to d versus d + 3.

First, instead of considering standard quantum query model, we define d-shuffling in the "inplace" quantum oracle model.

Definition 5.7 (In-place d-shuffling). Let $f: \{0,1\}^n \to \{0,1\}^n$ be a Simon's function with shift s. Let $\mathcal{F} := \{f_0,\ldots,f_d\}$ be a d-shuffling of f. We define the in-place (d,f)-shuffling $U:=\{U_{f_0},\ldots,U_{f_d}\}$ as follows:

- 1. For i = 0, let U_{f_0} be a unitary such that for all $x \in \{0,1\}^{(d+2)n}$, $U_{f_0}|x\rangle |0\rangle = |x\rangle |f_0(x)\rangle$.
- 2. For i = 1, ..., d-1, let U_{f_i} be a unitary in $\mathbb{C}^{2^{(d+2)n} \times 2^{(d+2)n}}$ such that for all $x \in \{0, 1\}^{(d+2)n}$, $U_{f_i}|x\rangle = |f_i(x)\rangle$.
- 3. Let U_{f_d} be a unitary in $\mathbb{C}^{2^{(d+2)n+1}\times 2^{(d+2)n+1}}$ such that for all $x\in\{0,1\}^{(d+2)n}$ and $b\in\{0,1\}$, $U_{f_d}|x,b\rangle=|f_d(x)\rangle|b\oplus b'\rangle$, where b'=1 if $x\in H$ (see the definition of H in the proof of Claim 5.4).

We note that an in-place pseudorandom permutation exists if there exists a qPRP defined in Definition 2.8. First we give the definition of in-place qPRPs.

Definition 5.8 (In-place qPRP (cf. Definition 2.8)). For security parameter λ and polynomial $m = m(\lambda)$, a pseudorandom permutation P over $\{0,1\}^m$ is a keyed function $\mathcal{K} \times \{0,1\}^m \to \{0,1\}^m$ such that there exists a negligible function such that for every quantum adversary \mathcal{A} , it holds that

$$\left| \Pr_{F \leftarrow_{R} \mathcal{P}} [\mathcal{A}^{I_{F}, I_{F}-1} = 1] - \Pr_{k \leftarrow_{R} \mathcal{K}} [\mathcal{A}^{I_{P(k,\cdot)}, I_{P}-1}(k,\cdot)} = 1] \right| \le \operatorname{negl}(\lambda), \tag{25}$$

where $I_Q: |x\rangle \mapsto |Q(x)\rangle$ for $x \in \{0,1\}^m$ and permutation $Q: \{0,1\}^m \to \{0,1\}^m$.

Theorem 5.9. If there exists a qPRP as defined in Definition 2.8, then an in-place qPRP defined in Definition 5.8 exists.

Proof. It suffices to show that an in-place oracle of a permutation P can be implemented using two queries to the standard oracles $O_P: |x,y\rangle \mapsto |x,y\oplus P(x)\rangle$ and $O_{P^{-1}}$. A query to the in-place oracle of P can be computed in the following steps:

$$|x,0\rangle \xrightarrow{O_P} |x,P(x)\rangle \xrightarrow{O_{P^{-1}}} |x \oplus P^{-1}(P(x)),P(x)\rangle = |0,P(x)\rangle.$$
 (26)

Swapping the registers and removing the ancilla yields a mapping $|x\rangle \mapsto |P(x)\rangle$.

Now we show that if $F: \mathcal{K} \times \{0,1\}^n \to \{0,1\}^n$ is a qPRP as defined in Definition 2.8, then F is an in-place qPRP. Suppose toward contradiction that a quantum adversary \mathcal{A} which distinguishes $|x\rangle \mapsto |F_k(x)\rangle$ for $k \leftarrow_R \mathcal{K}$ and $|x\rangle \mapsto |P(x)\rangle$ for uniform permutation P with advantage ϵ . Then we show there exists an adversary \mathcal{B} which distinguishes $O_{F_k}, O_{F_k^{-1}}$ from $O_P, O_{P^{-1}}$. The adversary \mathcal{B} simulates the mapping in (26) using two queries to either $O_{F_k}, O_{F_k^{-1}}$ or $O_P, O_{P^{-1}}$ and runs \mathcal{A} . By the assumption, \mathcal{B} distinguishes the oracles with advantage ϵ .

We define in-place d-SSP as follows:

Definition 5.10 (In-place d-SSP). Let $n \in \mathbb{N}$ and $f : \{0,1\}^n \to \{0,1\}^n$ be a random Simon's function. Given access to the in-place d-shuffling oracle of f, the problem is to find the hidden shift of f.

Theorem 5.11. Let d = poly(n). In-place d-SSP can be solved using (d+3)-CQ and (d+3)-QC schemes with access to the in-place d-shuffling oracle of f. Furthermore, for any d'-CQ or d'-QC schemes A with access to the in-place d-shuffling oracle of f and $d' \leq d$, the probability that A solves the problem is negligible.

Note that if we consider the models defined in Definition 3.8 and Definition 3.10 in [11], the quantum depth separation will be d versus d+1. See Remark 2.7 for the detailed discussion.

We present the proof of Theorem 5.11 in Appendix A. By Corollary 4.4 and Theorem 5.11, we have a construction of $CVQD_2(d, d + 3)$.

Corollary 5.12. For d = poly(n), there exists an unconditionally secure $CVQD_2(d, d + 3)$ which is sound as in Theorem 5.2, 1 and complete with P_O and V running in $O(2^n)$ time. Moreover, if qPRP (Definition 2.8) exists, then there exists $CVQD_2(d, d + 3)$ which is sound as in Theorem 5.2, 2 and complete with P_O and V running in polynomial time.

Proof. Following the same idea, we set q = d + 1, and we choose \mathcal{O} to be the in-place shuffling oracle and R to be in-place d-SSP. Then, Protocol 1 separates d + 3-depth quantum circuit from d-depth quantum circuit in the presence of polynomial-time classical computation by Corollary 4.4 and Theorem 5.11.

Again, V and P_O are not efficient since implementing random permutations is expensive. Following the same idea for proving the second result of Theorem 5.2, we can make both V and P_O efficient using qPRP in the in-place oracle model.

6 Certifying quantum depth from learning with errors

In this section, we describe a protocol for certifying quantum depth using NTCFs. First we define a single-prover protocol for certifying quantum depth.

Definition 6.1. Let $d, d' \in \mathbb{N}$ and d' > d. A single-prover protocol $\mathsf{CVQD}(d, d')$ that separates quantum circuit depth d from d' consists of a classical verifier V and a prover P such that the following properties hold:

- Completeness: There exist an integer $\hat{d} \geq d$ and a \hat{d} -QC or \hat{d} -CQ scheme P such that $\Pr[\langle V, P \rangle] \geq 2/3$.
- Soundness: For integer $\hat{d} \leq d$ and any P that are \hat{d} -CQ or \hat{d} -QC schemes, $\Pr[\langle V, P \rangle = accept] \leq 1/3$.

We give a prototocol based on a randomized encoding of NTCFs, and include the honest behavior of the prover in the description of the protocol (but the prover does not necessarily follow the instructions).

Protocol 6. $[CVQD(d, d + d_0)]$

- 1. The verifier samples $\{(k_i, t_i) \leftarrow \text{Gen}(1^{\lambda}) : i \in [d+1]\}$ and sends (k_1, \dots, k_{d+1}) .
- 2. The prover performs the quantum process in Theorem 2.14 and prepares the state

$$\frac{1}{(2|\mathcal{X}|)^{d/2}} \bigotimes_{i=1}^{d+1} \left(\sum_{b_i \in \{0,1\}, x_i \in \mathcal{X}} |b_i\rangle_{B_i} |x_i\rangle_{X_i} |\hat{f}_{k_i}(b_i, x_i)\rangle_{Y_i} \right),\tag{27}$$

and performs a standard basis measurement on the registers Y_1, \ldots, Y_{d+1} to yield y_1, \ldots, y_{d+1} , which is sent to the verifier.

- 3. For $i = 1 \dots d + 1$, the verifier and the prover proceed as follows.
 - (a) The verifier samples a random bit $c_i \in \{0,1\}$ and sends c_i to the verifier.
 - (b) If $c_i = 0$, the prover performs a standard basis measurement on B_iX_i ; otherwise the prover performs a Hadamard basis measurement on B_iX_i . The prover then sends the outcome w_i to the verifier.
 - (c) The verifier computes $V(t_i, y_i, c_i, w_i) = a$, where V is defined in (13). If a = 0, then the verifier rejects and aborts.

If the verifier does not reject for each $i \in [d]$, then it accepts.

Theorem 6.2 (Completeness). There exists a constant d_0 such that for security parameter λ and polynomially bounded function d, a negligible function ϵ , there is a prover which is a $(d_0 + d(\lambda))$ -QC scheme and succeeds with probability $1 - \epsilon(\lambda)$.

Proof. By Theorem 2.14, the preparation of the state in (27) can be done in constant depth d'. Let $d_0 = d' + 1$. The prover performs standard basis measurement on the registers Y_1, \ldots, Y_{d+1} to sample y_1, \ldots, y_{d+1} . For every $i \in [d+1]$, the prover measures the state in the i^{th} coordinate in the standard basis if $c_i = 0$ and in the Hadamard basis otherwise. There exists a negligible function μ such that if $c_i = 0$, with probability at least $1 - \text{negl}(\lambda)$, performing a standard basis on $B_i X_i$ yields a preimage; if $c_i = 1$, with probability at least $1 - \mu(\lambda)$, performing a Hadamard basis measurement on $B_i X_i$ yields an outcome that passes the equation test $V(t_i, y_i, 1, \cdot)$. By the union bound, the

prover succeeds with probability

$$\Pr[\text{success}] \ge 1 - \sum_{i} \Pr[\text{Prover fails the } i^{th} \text{ round}]$$

$$\ge 1 - d(\lambda) \cdot \mu(\lambda) = 1 - \text{negl}(\lambda)$$
(28)

for polynomially bounded function d.

As an example, we present the quantum circuit for d=2 in Figure 2.

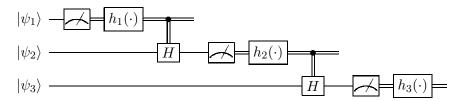


Figure 2: The quantum circuit for d=2. In each layer $i \in [3]$, the verifier's action can be viewed as a classical computation h_i , which takes the measurement outcome as input. If the measurement outcome is accepted, then it outputs a random string $c_i \leftarrow_R \{0,1\}^{\ell(\lambda)}$; otherwise it outputs \bot indicating rejection. Each $|\psi_i\rangle$ is the post-measurement state of (27) after a standard basis measurement on Y_1, Y_2, Y_3 is performed. By Theorem 2.14, the state $|\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle$ can be prepared in constant depth.

Next we show a lower bound on the quantum depth. In each round i, let the prover's action on receiving challenge $c_i = c$ be an isometry $U_{i,c}$ acting on the quantum state $|\psi_{i,T}\rangle$, which depends on the previous transcript T, followed by a standard basis measurement. We show that if there exists i such that the quantum depth does not increase by 1 for round i, then there is a quantum adversary which breaks the adaptive hardcore bit property.

Theorem 6.3 (Soundness). There exists a negligible function μ such that for sufficiently large λ , for any prover that is either a d-CQ or d-QC scheme succeeds with probability at most $\frac{3}{4} + \mu(\lambda)$.

Proof. Suppose toward contradiction that there exists a prover P which succeeds with probability $3/4 + \epsilon$ for non-negligible ϵ . First we show that the operation in each round must have non-zero quantum depth. If this is not the case for some round $i \in [d+1]$, then without loss of generality, P's operation consists of

- 1. a standard basis measurement on some intermediate quantum state ρ_i to yield an outcome v_i , followed by
- 2. a classical algorithm \mathcal{A}_i which on input v_i and the challenge $c_i = c$, outputs the response $w_i \leftarrow \mathcal{A}_i(c_i, v_i, T)$ for previous transcript T.

Then there is a reduction \mathcal{A} which uses P to break the adaptive hardcore bit property. Since the probability of passing d rounds is at least $3/4 + \epsilon(\lambda)$, the probability of winning the first i^{th} round is at least $3/4 + \epsilon$. Thus \mathcal{A} , on input challenge key k, samples $k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_{d+1}$ and sets $k_i = k$ and computes v_i . Using v_i , \mathcal{A} runs \mathcal{A}_i on $w_b \leftarrow \mathcal{A}_i(b, v_i, T)$ for each $b \in \{0, 1\}$. Let the probability that w_b is a valid response be p_b . By the assumption, $p_0 + p_1 \geq 3/2 + 2\epsilon$. This implies that with probability $1 - (1 - p_0) - (1 - p_1) = p_0 + p_1 - 1 \geq 1/2 + 2\epsilon$ both w_0 and w_1 are valid.

Thus with probability at least $1/2 + 2\epsilon$, w_0 is a valid preimage and w_i is a valid equation, and thus the adaptive hardcore bit is broken.

Since P must have non-zero quantum depth in each round and it has total quantum depth d, there must exist a round $j \in [d+1]$ such that P must destroy all its coherence after receiving c_j , and continue answering the remaining rounds with an intermediate classical information σ_j . Now the reduction \mathcal{A}' samples $c_1, \ldots, c_{d+1} \leftarrow_R \{0, 1\}$ and simulates the verifier in the protocol, and runs the following steps to break the adaptive hardcore bit property:

- 1. \mathcal{A}' , on receiving k, samples k_1, \ldots, k_d and sets $k_{d+1} = k$, and runs P to get y_1, \ldots, y_{d+1} and some quantum information ρ .
- 2. \mathcal{A}' continues running P on input ρ and c_1, \ldots, c_i to compute σ_i .
- 3. \mathcal{A}' continues running P on input $\sigma_j, c_j, \ldots, c_d, c_{d+1} = b$, to yield a response w'_b in round d for each $b \in \{0, 1\}$.
- 4. \mathcal{A}' outputs (y_{d+1}, w'_0, w'_1) as the response.

To analyze the performance of \mathcal{A}' , we apply the same idea as that for calculating the performance of \mathcal{A} . Let the probability that w_b' be a valid response for round d be p_d' . By the assumption, $p_0' + p_1' \geq 3/2 + 2\epsilon$. Thus both w_0' and w_1' are valid with probability at least $1 - (1 - p_0') - (1 - p_1') \geq 1/2 + 2\epsilon$. This implies that the adaptive hardcore bit property is broken.

Theorem 6.2 and Theorem 6.3 immediately imply the following theorem.

Theorem 6.4. Assuming that LWE is hard for any d-CQ and d-QC schemes, Protocol 6 satisfies the following conditions.

- Completeness: There exists a prover which is a $(d_0 + d)$ -QC scheme and succeeds with probability at least $1 \text{negl}(\lambda)$.
- Soundness: Every prover that are d-CQ and d-QC schemes succeed with probability at most $\frac{3}{4} + \text{negl}(\lambda)$.

By sequential repetition, the completeness-soundness gap can be amplified to $1 - \text{negl}(\lambda)$. We note that we do not know if the function evaluation can be done by using $(d_0 + d)$ -CQ schemes and leave it as an open question.

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A Proof of Theorem 5.11

Lemma A.1. In-place d-SSP can be solved by a (d+3)-depth quantum circuit with classical post-processing.

Proof. The algorithm is as follows:

$$\begin{split} \sum_{x \in \mathbb{Z}_2^n} |x\rangle |0\rangle |0\rangle & \xrightarrow{U_{f_0}} \sum_{x \in \mathbb{Z}_2^n} |x\rangle |f_0(x)\rangle |0\rangle \\ & \xrightarrow{U_{f_1}} \sum_{x \in \mathbb{Z}_2^n} |x\rangle |f_1(x)\rangle |0\rangle \\ & \xrightarrow{U_{f_d}} \sum_{x \in \mathbb{Z}_2^n} |x\rangle |f(x)\rangle |b(x)\rangle \\ & \xrightarrow{\text{measure the second register}} & \frac{1}{\sqrt{2}} (|x\rangle |b(x)\rangle + |x \oplus s\rangle |b(x \oplus s)\rangle) \\ & \xrightarrow{\text{Apply H on the last qubit and measure}} & \frac{1}{\sqrt{2}} (|x\rangle + |x \oplus s\rangle) \text{ with probability } 1/2. \\ & \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{j \in \mathbb{Z}_2^n} ((-1)^{x \cdot j} + (-1)^{(x+s)j}) |j\rangle \end{split}$$

Then, the rest of the algorithm follows from Simon's algorithm. The additional two depths come from the Hadamard gates an the beginning and at the end of the above algorithm. \Box

We sketch a proof for the lower bound in the following.

Lemma A.2. For any d-CQ or d-QC schemes A, the probability that A solves in-place d-SSP is negligible.

To prove that in-place d-SSP is hard for d-depth quantum circuit, we need to prove the oneway-to-hiding lemma (Lemma 5.7 in [11]) for in-place shuffling oracle.

Claim A.3 (in-place oracle version of Lemma 5.7 in [11]). Let \mathcal{F} be a d-shuffling of a random Simon's function f and $\mathcal{U} := \{U_{f_0}, \ldots, U_{f_d}\}$ be the corresponding in-place d-shuffling. Let $\mathbf{S} = \{\bar{S}^{(0)}, \ldots, \bar{S}^{(d)}\}$ be a sequence of hidden sets as defined in Definition 5.2 in [11]. Then, for all $k = 0, \ldots, d$, there exists a shadow \mathcal{G} of \mathcal{F} in $\bar{S}^{(k)}$ such that for any single-depth quantum circuit U_c , initial state ρ , and any binary string t,

$$\begin{aligned} |\Pr[\Pi_{0/1} \circ \mathcal{U}_{\mathcal{F}} U_c(\rho) = t] - \Pr[\Pi_{0/1} \circ \mathcal{U}_{\mathcal{G}} U_c(\rho) = t]| &\leq B(\mathcal{U}_{\mathcal{F}} U_c(\rho), \mathcal{U}_{\mathcal{G}} U_c(\rho)) \\ &\leq \sqrt{2 \Pr[find \ \bar{S}^{(k)} : U^{\mathcal{F}} \setminus \bar{S}^{(k)}, \rho]}. \end{aligned}$$

Here, $B(\cdot,\cdot)$ is the Bures distance between quantum states, and $U^{\mathcal{F}\setminus\bar{S}^{(k)}}$ is defined in Definition 5.6 in [11]. $\mathcal{U}_{\mathcal{G}}$ is the in-place oracle for \mathcal{G} .

Proof. We first define the shadow \mathcal{G} of \mathcal{F} in $\bar{S}^{(k)}$. The definition follows the same spirit of the shadow in [11]. In the original definition, the shadow \mathcal{G} maps $x \in \bar{S}^{(k)}$ to a special symbol \bot and is consistent with \mathcal{F} for $x \notin \bar{S}^{(k)}$. This definition of shadow does not work in the in-place oracle setting since the corresponding oracle is not a unitary.

So, here, we define \mathcal{G} as a random function satisfying the following: If $x \notin \bar{S}^{(k)}$, we let $\mathcal{G}(x) = \mathcal{F}(x)$; else if $x \in \bar{S}^{(k)}$, we let $\mathcal{G}(x)$ to be independent of $\mathcal{F}(x)$, and the in-place oracle of \mathcal{G} , $U_{\mathcal{G}}$, is still a unitary. In particular, for shadows corresponding to f_1, \ldots, f_{d-1} in $\bar{S}^{(k)}$, we pick another random permutation that is independent of f_1, \ldots, f_{d-1} in $\bar{S}^{(k)}$ and is consistent with f_1, \ldots, f_{d-1} for $x \notin \bar{S}^{(k)}$; for f_d , we can pick another 2-to-1 function that results in no hidden shift or a different hidden shift.

Then, the rest of the proof directly follows from the proof for Lemma 5.7 in [11].

The rest of the analysis to show that in-place d-SSP is hard for d-depth quantum circuit in the presence of classical computation follows the proof in [11] by using the new shadow we construct in the proof for Claim A.3.