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Monshizadeh, Nima; De Persis, Claudio; van der Schaft, Abraham; Scherpen, Jacquelien M.A.

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A Networked Reduced Model for Electrical Networks with Constant Power Loads

Nima Monshizadeh   Claudio De Persis   Arjan J. van der Schaft   Jacquelien M.A. Scherpen

Abstract— **We consider structure preserving power networks with proper algebraic constraints resulting from constant power loads. Both for the linear and the nonlinear model of the network, we propose explicit reduced order models which are expressed in terms of ordinary differential equations. The relative frequencies among all the buses in the original power grid are readily tractable in the proposed reduced models. For deriving these reduced models, we introduce the "projected pseudo incidence" matrix which yields a novel decomposition of the reduced Laplacian matrix. With the help of this new matrix, we are able to eliminate the proper algebraic constraints while preserving the crucial frequency information of the loads.**

I. INTRODUCTION

The interdisciplinary field of power networks and microgrids has received lots of attentions from the control community in the last decade, see e.g. [7], [14], [13], [15], [4]. Principal components of a power grid are synchronous generators, inverters, and loads. The frequency behavior of the synchronous generators are often modelled by the so called "swing equation" [10]. It can be shown that the frequency of the droop-controlled inverters also admits a similar dynamics, see e.g. [13].

In the intuitive modelling of the power network, the generators and the loads are located at different subset of nodes. This corresponds to the so-called *structure preserving* model which is naturally expressed in terms of differential algebraic equations (DAE), see [1], [5]. The algebraic constraints in the structure preserving model are associated with the load dynamics.

Motivated by the fact that the presence of the algebraic constraints hinders the stability analysis of power networks, several aggregated models are reported in the literature where each bus of the grid is associated with certain load and generation; see e.g. [3], [12]. The advantage of these aggregated models is mainly due to the fact that they are described by ordinary differential equations (ODE) which facilitates the analysis of the network. However, the explicit relationship between the aggregated model and the original structure preserved model is often missing, which restricts the validity and applicability of the results.

Nima Monshizadeh, Claudio De Persis, and Jacquelien M.A. Scherpen are with the Engineering and Technology Institute, University of Groningen, 9747AG, The Netherlands, n.monshizadeh@rug.nl, c.de.persis@rug.nl, j.m.a.scherpen@rug.nl

Arjan J. van der Schaft is with the Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, 9700AK, The Netherlands, a.j.van.der.schaft@rug.nl

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Aiming at simplified ODE description of the model together with respecting the heterogenous structure of the power network has endorsed the use of Kron reduced models [6], [2]. In the Kron reduction method, the variables which are exclusive to the algebraic constraints are solved in terms of the rest of the variables. This results in a reduced graph, the (loopy) Laplaican matrix of which is the Schur complement of the (loopy) Laplacian matrix of the original graph. By construction, the Kron reduction technique restricts the class of the applicable load dynamics. The most notable subclass, for which Kron reduced models can be obtained, includes constant current and constant admittance loads in which each load is modelled as a constant current demand and a shunt admittance connected to the ground [6].

The algebraic constraints can also be solved in the case of frequency dependent loads where the active power drawn by each load consists of a constant term and a frequency-dependent term [1], [16], [11]. However, in the popular class of constant power loads, the algebraic constraints are "proper", meaning that they are not explicitly solvable. One way to avoid these proper algebraic constraints is to work with linear approximations. Otherwise, a remarkable method to cope with these constraints in the nonlinear model is to use the implicit function theorem, and study the approximated *implicit* ODE model around the point of interest [9]. To the best of our knowledge, for nonlinear power networks with proper algebraic constraints, an *explicit* reduced ODE model is absent in the literature.

In this paper, first we revisit the Kron reduction method for the linear case, where the Schur complement of the Laplacian matrix (which is again a Laplacian) naturally appears in the network dynamics. It turns out that the usual decomposition of the reduced Laplacian matrix leads to a state space realization which contains merely partial information of the original power network, and the frequency behavior of the loads is not visible. As a remedy for this problem, we introduce a new matrix, namely the projected pseudo incidence matrix, which yields a novel decomposition of the reduced Laplacian. Then, we derive reduced order models capturing the behavior of the original structure preserved model. Next, we turn our attention to the nonlinear case where the algebraic constraints are not readily solvable. Again by the use of the projected pseudo incidence matrix, we propose explicit reduced models expressed in terms of ordinary differential equations. We identify the loads embedded in the proposed reduced network by unveiling the conserved quantity of the system.

The structure of the paper is as follows. Section II describes the power network model we consider in this paper.

In Section III, we discuss the reduced models for the system obtained by linear approximations. Also in Subsection III-A, we introduce the projected pseudo incidence matrix and the new decomposition of the reduced Laplacian matrix. An explicit reduced order model for the nonlinear power network is established in Section IV. Finally, the paper closes with conclusions in Section V.

II. Power Network

The topology of the power network is represented by a connected and undirected graph $\mathcal{G}(\mathcal{V},\mathcal{E})$. There are two types of buses (nodes): synchronous generators $\mathcal{V}_G$ and loads $\mathcal{V}_L$, with $\mathcal{V} = \mathcal{V}_G \cup \mathcal{V}_L$. The edge set $\mathcal{E}$ is the set of unordered pairs $\{i,j\}$ accounting for the transmission lines which are assumed to be inductive. Let the matrix B denote the incidence matrix of $\mathcal{G}$. Recall that for an undirected graph $\mathcal{G}$, the incidence matrix is obtained by assigning an arbitrary orientation to the edges of $\mathcal{G}$ and defining

$$b_{ik} = \begin{cases} +1 & \text{if } i \text{ is the tail of arc } k \\ -1 & \text{if } i \text{ is the head of arc } k \\ 0 & \text{otherwise} \end{cases}$$

with b_{ik} being the $(i,k)^{th}$ element of B.

At each node $i \in \mathcal{V}$, the electrical active power is given by

$$p_i = \sum_{j\in\mathcal{N}_i} X_{ij}^{-1} V_i V_j \sin\theta_{ij}, \quad \theta_{ij} := \theta_i - \theta_j$$

where X_{ij} is the inductance of the transmission line $\{i,j\}$, V_i is the voltage magnitude at node i, and θ_i is the voltage angle with respect to the nominal reference $\theta^* = \omega^* t$. We assume that the transmission lines are lossless and the voltage magnitudes are constant. We consider synchronous generators admitting the so-called swing equation

$$M_i\ddot{\theta}_i = -A_i\dot{\theta}_i - p_i + u_i, \qquad i \in \mathcal{V}_G$$

where M_i is the angular momentum, A_i is the damping coefficient, and u_i is the net shaft power input to the generator. As for the loads, we consider the constant power loads admitting the *proper* algebraic constraint

$$0 = p_i - p_i^*, \qquad i \in \mathcal{V}_L$$

where p_i^* is constant. Now the network model can be written in compact as

$$M\ddot{\theta}_G = -A\dot{\theta}_G - B_G\Gamma\sin(B^T\theta) + u \tag{1a}$$
$$0 = -B_L\Gamma\sin(B^T\theta) + p \tag{1b}$$

where $\theta_G = \mathrm{col}(\theta_i)$ with $i \in \mathcal{V}_G$, and $\theta_L = \mathrm{col}(\theta_i)$ with $i \in \mathcal{V}_L$. The $\sin(\cdot)$ operator is defined element-wise. In addition, $\theta = \mathrm{col}(\theta_G,\theta_L)$, $B = \mathrm{col}(B_G, B_L)$, and $\Gamma = \mathrm{diag}(\gamma_k)$ with

$$\gamma_k = X_{ij}^{-1} V_i V_j$$

where k is the index of the edge $\{i,j\}$ in accordance with the incidence matrix B. We assume that the voltages are positive and constant, and thus the matrix Γ is positive definite. Note that the notation $\mathrm{col}(Y_1,Y_2)$ is used to denote in short the matrix $\begin{bmatrix}Y_1^T & Y_2^T\end{bmatrix}^T$ for given matrices Y_1 and Y_2.

Our goal here is to eliminate the load dynamics and embed it into the dynamics of the generators in order to obtain an explicit reduced order model described by ordinary differential equations.

III. Linear model

First, we consider the linear model where $\sin(\eta)$ is approximated by η, with $\eta = B^T\theta$. Then, the system (1a)-(1b) can be written as

$$\begin{bmatrix} M\ddot{\theta}_G + A\dot{\theta}_G \\ 0 \end{bmatrix} = -\begin{bmatrix} B_G\Gamma B_G^T & B_G\Gamma B_L^T \\ B_L\Gamma B_G^T & B_L\Gamma B_L^T \end{bmatrix}\begin{bmatrix}\theta_G \\ \theta_L\end{bmatrix} + \begin{bmatrix} u \\ p\end{bmatrix} \tag{2}$$

Note that the two by two block matrix on the right hand side of (2) can be associated with the Laplacian matrix of the graph $\mathcal{G}$, say L, where the weights on the edges are defined by the matrix Γ. In particular,

$$L = B\Gamma B^T = \begin{bmatrix} B_G\Gamma B_G^T & B_G\Gamma B_L^T \\ B_L\Gamma B_G^T & B_L\Gamma B_L^T \end{bmatrix}.$$

The vector θ_L can be computed as

$$\theta_L = -(B_L\Gamma B_L^T)^{-1}B_L\Gamma B_G^T\theta_G + (B_L\Gamma B_L^T)^{-1}p. \tag{3}$$

Note that $B_L\Gamma B_L^T$ is a principle submatrix of the Laplacian matrix and thus invertible. By replacing this back to (2), we obtain

$$M\ddot{\theta}_G = -A\dot{\theta}_G - L_S\theta_G + u - \hat{p} \tag{4}$$

where

$$L_S = B_G\Gamma B_G^T - B_G\Gamma B_L^T(B_L\Gamma B_L^T)^{-1}B_L\Gamma B_G^T$$

and

$$\hat{p} = B_G\Gamma B_L^T(B_L\Gamma B_L^T)^{-1}p.$$

Noting that L_S is equal to the Schur complement of the Laplacian matrix L, it is well-known that L_S is again a Laplacian matrix defined on a reduced graph $\hat{\mathcal{G}} = (\mathcal{V}_G, \hat{\mathcal{E}})$, and admits the decomposition

$$L_S = \hat{B}\hat{\Gamma}\hat{B}^T \tag{5}$$

where $\hat{B}$ is the incidence matrix of $\hat{\mathcal{G}}$.

A crucial issue in frequency regulation is to keep the frequency disagreement among the buses as small as possible, and steer the frequency back to the nominal frequency using a secondary control scheme. Notice that this frequency disagreement is not transparent in (4). Now, let $\omega_G = \dot{\theta}_G$, $\omega_L = \dot{\theta}_L$, and $\omega = \mathrm{col}(\omega_G, \omega_L)$. To capture the frequency disagreements in the original network (1a)-(1b), we define the vector δ as

$$\delta = B^T\omega. \tag{6}$$

Observe that δ_k indicates the difference between the (actual) frequencies of the nodes i and j, with $\{i,j\}$ being the k^{th} edge of $\mathcal{G}$. Then the network dynamics (1a)-(1b) admits the following model

$$\dot{\eta} = \delta = B^T\omega \tag{7a}$$
$$M\dot{\omega}_G = -A\omega_G - B_G\Gamma\eta + u \tag{7b}$$
$$0 = -B_L\Gamma\eta + p \tag{7c}$$

where $\eta = B^T\theta$. Similarly, for the aggregated model (4), the frequency disagreement vector is defined as

$$\hat{\delta} = \hat{B}^T \omega_G. \tag{8}$$

Then by (5) the system (4) has the following state-space representation

$$\dot{\hat{\eta}} = \hat{\delta} = \hat{B}^T \omega_G \tag{9a}$$

$$M\dot{\omega}_G = -A\omega_G - \hat{B}\hat{\Gamma}\hat{\eta} + u - \hat{p} \tag{9b}$$

where $\hat{\eta} = \hat{B}^T \theta_G$.

Although the Kron reduced model (9) provides an explicit aggregated model for the network (7), comparing the dynamics (9a) to (7a) reveals several disadvantages for this model:

i) Unlike the vector δ, the disagreement vector $\hat{\delta}$ captures only the mismatch among the frequencies of the generators, whereas, clearly one would like to monitor the mismatch of the frequencies in the entire network.

ii) Note that the vectors $\hat{\eta}$ and $\hat{\delta}$ typically have $\frac{N_g^2 - N_g}{2}$ elements where N_g is the total number of generators. Hence, the size of these vectors increases substantially by the increase in the size of the network, which makes the monitoring and simulations intractable.

Motivated by the above drawbacks, next we propose an alternative decomposition of the reduced Laplacian matrix L_S, instead of the customary one given by (5).

A. A novel decomposition of the reduced Laplacian

We make the result of this subsection self contained, and independent of the power network interpretation. To this end, let again $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote an undirected graph with n vertices and m edges, and assume that $\mathcal{G}$ is connected. As before, for each $k = 1, 2, \ldots, m$, let $\gamma_k > 0$ denote the weight associated to the k^{th} edge of $\mathcal{G}$. The Laplacian matrix of $\mathcal{G}$ is defined as $L = B\Gamma B^T$ where $\Gamma = \mathrm{diag}(\gamma_k)$. Suppose that the vertex set V is partitioned as $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ with $\mathcal{V}_1 \cap \mathcal{V}_2 = \varnothing$. Then the Laplacian matrix L can be partitioned as

$$L = \begin{bmatrix} L_{11} & L_{21} \\ L_{21}^T & L_{22} \end{bmatrix}$$

where $L_{11} \in \mathbb{R}^{|\mathcal{V}_G| \times |\mathcal{V}_G|}$. Note that the Schur complement of L with respect to L_{22} is given by

$$L_S = L_{11} - L_{21} L_{22}^{-1} L_{21}^T.$$

This can be rewritten as

$$L_S = B_1 \Gamma B_1^T - B_1 \Gamma B_2^T (B_2 \Gamma B_2^T)^{-1} B_2 \Gamma B_1^T \tag{10}$$

where $B = \mathrm{col}(B_1, B_2)$ is the incidence matrix of $\mathcal{G}$. For a connected graph $\mathcal{G}$, it is well known that L_S is well defined, and is the Laplacian matrix of an undirected graph $\hat{\mathcal{G}}$ with $|\mathcal{V}_1|$ vertices.

Now, given a partitioning $B = \mathrm{col}(B_1, B_2)$ and the positive weights Γ, we define the *Projected pseudo incidence matrix* of $\mathcal{G}$ as

$$B_S = B_1 (I - B_2^+ B_2) \tag{11}$$

where B_2^+ is the pseudo inverse of the matrix B_2 with respect to the inner product defined by Γ, i.e.

$$B_2^+ = \Gamma B_2^T (B_2 \Gamma B_2^T)^{-1}. \tag{12}$$

Observe that $B_S = B_1 \Pi$ where

$$\Pi = I - B_2^+ B_2$$

is the orthogonal projection to the kernel of B_2, with respect to the inner product defined by Γ. Similar to the incidence matrix, the matrix B_S has zero column sums. Some useful properties of the matrix B_S are captured in the following proposition.

Proposition 1 *Let B_S denote the projected pseudo incidence matrix of $\mathcal{G}$ with respect to the partitioning $B = \mathrm{col}(B_1, B_2)$ and the weights Γ as given by (11). Then the following statements hold:*

i) $\mathrm{im}\, \mathbb{1} = \ker B_S^T$

ii) $0 = B_S \Gamma B_2^T$

iii) $L_S = B_S \Gamma B_1^T$

iv) $L_S = B_S \Gamma B_S^T$

Proof. Clearly,

$$\begin{aligned} B_S \Gamma B_2^T &= B_1 (I - B_2^+ B_2) \Gamma B_2^T \\ &= B_1 \Gamma B_2^T - B_1 B_2^+ B_2 \Gamma B_2^T = 0, \end{aligned} \tag{13}$$

which proves the second statement. From (10), we have

$$\begin{aligned} L_S &= B_1 (I - \Gamma B_2^T (B_2 \Gamma B_2^T)^{-1} B_2) \Gamma B_1^T \\ &= B_1 (I - B_2^+ B_2) \Gamma B_1^T = B_S \Gamma B_1^T, \end{aligned}$$

which verifies the third statement.

The matrix $B_S \Gamma B_S^T$ is computed as

$$B_S \Gamma B_S^T = B_S \Gamma B_1^T - B_S \Gamma B_2^T (B_2^+)^T B_1^T. \tag{14}$$

By the third statement of the proposition, $B_S \Gamma B_1^T = L_S$. In addition, the second term on the right hand side of (14) is equal to zero by (13). Therefore, we obtain that $B_S \Gamma B_S^T = L_S$.

As the matrix L_S is the Laplacian matrix of a reduced graph $\hat{\mathcal{G}}$, we have $L_S \mathbb{1} = 0$. Then, by the forth statement of the proposition and positive definiteness of Γ, we have $B_S^T \mathbb{1} = 0$. Recall that L_S is the Schur complement of the Laplacian matrix L. As $\mathcal{G}$ is connected, the spectral interlacing property [8, Thm. 3.1] implies that $\hat{\mathcal{G}}$ is connected as well, and thus $\ker L_S = \ker B_S^T = \mathrm{im}\, \mathbb{1}$. ■

B. The new representation of the reduced order network

Now, consider again the model (4). Let B_S be the projected pseudo incidence matrix with respect to the partitioning $B = \mathrm{col}(B_G, B_L)$ and the weights Γ, as given by (11). Let the vector η_S be defined as

$$\eta_S = B_S^T \theta_G. \tag{15}$$

Observe that by (3) we have

$$\omega_L = -(B_G B_L^+)^T \omega_G \tag{16}$$

where again B_L^+ denote the pseudo inverse of B_L with respect to the inner product Γ, i.e. $B_L^+ = \Gamma B_L^T (B_L \Gamma B_L^T)^{-1}$. Hence, we have

$$\begin{aligned}\dot{\eta}_S &= B_S^T \omega_G = (I - B_L^+ B_L)^T B_G^T \omega_G \\ &= B_G^T \omega_G - B_L^T (B_G B_L^+)^T \omega_G = B_G^T \omega_G + B_L^T \omega_L \\ &= B^T \omega = \delta\end{aligned}$$

where we have used (16) and (6). Also note that, by Proposition 1, we have

$$L_S \theta_G = B_S \Gamma B_S^T \theta_G = B_S \Gamma \eta_S.$$

Therefore the system (4) admits the following state space model

$$\dot{\eta}_S = \delta = B_S^T \omega_G \tag{17a}$$
$$M\dot{\omega}_G = -A\omega_G - B_S \Gamma \eta_S + u - \hat{p} \tag{17b}$$

where $\hat{p}$ has the same expression as before. The main advantage of the reduced model (17) over (9) is that the model (17) readily reflects the properties of the full network (7). Most importantly, notice that both the frequency disagreement vector δ and the weight matrix Γ are preserved in the reduced model. Therefore, one can easily deduce the behavior of the full network by looking into the model (17), and the aforementioned drawbacks for the model (9) do not apply to this case.

IV. Nonlinear model

In this section, we consider the nonlinear model (1a)-(1b), and investigate possible elimination of purely algebraic constraints resulting form the constant power load dynamics (1b). Notice that unlike the linear case, the state components θ_L cannot be explicitly solved here in terms of θ_G and p.

Before proceeding with the establishment of a reduced order model, it is necessary to assume that (1a)-(1b) admits a solution. To make this assumption, more explicit we write the differential algebraic system (1a)-(1b) as

$$\dot{\theta}_G = \omega_G \tag{18a}$$
$$M\dot{\omega}_G = -A\omega_G - B_G \Gamma \sin(B^T\theta) + u \tag{18b}$$
$$0 = -B_L \Gamma \sin(B^T \theta) + p \tag{18c}$$

Suppose that $\overline{\theta} = \mathrm{col}(\overline{\theta}_G . \overline{\theta}_L)$ and $\overline{u}$ are constant vectors satisfying

$$0 = -B_G \Gamma \sin(B^T \overline{\theta}) + \overline{u} \tag{19a}$$
$$0 = -B_L \Gamma \sin(B^T \overline{\theta}) + p \tag{19b}$$

Then, the point $\theta = \overline{\theta}$, $\omega_G = 0$, and $u = \overline{u}$ identify an equilibrium of (18). Let the right hand side of (18c) be denoted by $g(\theta)$. To investigate the regularity of (18c) and the existence of the (local) solutions to the DAE (18), we compute the Jacobian of g with respect to θ_L as

$$\frac{\partial g}{\partial \theta_L} = -B_L \Gamma \mathbf{cos}(\eta) B_L^T \tag{20}$$

where $\eta = B^T\theta = B_G^T \theta_G + B_L^T \theta_L$, and $\mathbf{cos}(\eta) = \mathrm{diag}(\cos(\eta_k))$. Observe that the matrix $B_L \Gamma \mathbf{cos}(\eta) B_L^T$ is a principle submatrix of the Laplacian matrix

$$L' = B\Gamma' B^T$$

where $\Gamma'(\eta) = \Gamma \mathbf{cos}(\eta)$. Hence, $\frac{\partial g}{\partial \theta_L}$ is nonsingular if Γ' is positive definite. Therefore, by denoting

$$\Omega := (-\frac{\pi}{2}, \frac{\pi}{2})^m,$$

the existence of the equilibrium and the regularity of (18c) impose the following assumption.

Assumption 1 There exists a constant vector $\overline{\theta}$ with $B^T\overline{\theta} \in \Omega$ such that (19) is satisfied.

Under the assumption above, the DAE (18) admits a unique (local) solution, see [9] for more details. Also note that the assumption $B^T\overline{\theta} \in \Omega$ is ubiquitous in the power grid literature and is sometimes referred to as a security constraint [7].

Next, we establish a reduced model for the system (18). Clearly, the differential algebraic system (18) admits the dynamics

$$\dot{\eta} = B^T\omega = B_G^T \omega_G + B_L^T \omega_L \tag{21a}$$
$$M\dot{\omega}_G = -A\omega_G - B_G \Gamma \sin(\eta) + u \tag{21b}$$
$$0 = -B_L \Gamma \sin(\eta) + p. \tag{21c}$$

where $\eta = B^T\theta$ and $\omega = \mathrm{col}(\omega_G, \omega_L)$ as before. Taking the time derivative of (21c) yields

$$\begin{aligned}0 &= -B_L \Gamma \mathbf{cos}(\eta) B^T \omega \\ &= -B_L \Gamma \mathbf{cos}(\eta) B_G^T \omega_G - B_L \Gamma \mathbf{cos}(\eta) B_L^T \omega_L\end{aligned} \tag{22}$$

where $\mathbf{cos}(\eta) = \mathrm{diag}(\cos(\eta_k))$. Assuming that $\eta \in \Omega$, the matrix $\mathbf{cos}(\eta)$ is nonsingular, and thus ω_L is obtained as

$$\omega_L = -(B_L \Gamma' B_L^T)^{-1} B_L \Gamma' B_G^T \omega_G \tag{23}$$

where

$$\Gamma'(\eta) = \Gamma \mathbf{cos}(\eta).$$

Note that by Assumption 1 and equality (20) there exists a neighborhood around $\overline{\eta} = B^T\overline{\theta}$ such that $B_L \Gamma \mathbf{cos}(\eta) B_L^T$ is nonsingular, and there exists a solution to (18), and thus to (21), for a nonzero interval of time. This means that (22) and (23) are well defined.

By substituting (23) in (21a), we have

$$\dot{\eta} = B_G^T \omega_G - B_L^T (B_L \Gamma' B_L^T)^{-1} B_L \Gamma' B_G^T \omega_G. \tag{24}$$

Now, let B_S denote the projected pseudo incidence matrix with respect to the partitioning $B = \mathrm{col}(B_G, B_L)$ and the weights Γ'. Then, it is easy to see that the right hand side of (24) is equal to $B_S^T(\eta)\omega_G$, and hence we obtain the following reduced model

$$\dot{\eta} = B_S^T(\eta)\omega_G \tag{25a}$$

$$M\dot{\omega}_G = -A\omega_G - B_G\Gamma\sin(\eta) + u. \tag{25b}$$

This defines a valid state space model for (η, ω_G) with ordinary differential equations, and in particular we have the following theorem.

Theorem 2 *Suppose that $(\eta, \omega_G, \omega_L, u)$ is a solution to the differential algebraic equations* (21)*, defined on the interval $\mathcal{I} = [0, T)$. Assume that $\eta(t) \in \Omega$, $\forall t \in \mathcal{I}$. Then (η, ω_G, u) is a solution to the ordinary differential equations* (25)*, defined on the interval $\mathcal{I}$.*

Proof. The proof follows from the construction of the reduced model (25). ■

Note that at the first glance it seems that the constant power loads are missing in the reduced model (25). However, these loads are actually embedded in the reduced dynamics. To see this, we make the following important observation.

Proposition 3 *Let $\eta(0) \in \Omega$. Then the vector $B_L\Gamma\sin(\eta)$ is a conserved quantity of the dynamical system* (25) *over the domain of the existence of the solution.*

Proof. By taking the time derivative of $B_L\Gamma\sin(\eta)$ along the solutions of (25), we obtain that

$$\frac{d}{dt}B_L\Gamma\sin(\eta) = B_L\Gamma\mathbf{cos}(\eta)\dot{\eta} = B_L\Gamma'(\eta)B_S^T(\eta)\omega_G$$

Note that the matrix Γ' is positive definite, and the matrix B_S is well defined over the domain of the existence of the solution. Then the second statement of Proposition 1 yields $B_L\Gamma' B_S^T = 0$ which completes the proof. ■

Remark 4 Notice that solutions to (25) may not always exist for all time, and in particular the system trajectories may have finite escape time at the closure of Ω. To rule out this finite escape time and guarantee the existence of the solution for all time, one need to assume that there exists a subset $\mathcal{A}$ of the state space $\Omega \times \mathbb{R}^{|V_G|}$ that is forward invariant along the solutions to (25). This forward invariance condition can be fulfilled by establishing the attractivity of the equilibrium $(\overline{\eta}, \overline{\omega}_G)$.[1]

Proposition 3 suggests that the constant vector $B_L\Gamma\sin(\eta)$ can indeed be interpreted as the constant power loads of the reduced network.

[1]This will be postponed to a future work.

Assume that $u = \overline{u}$ is constant. Then for an equilibrium $(\overline{\eta}, \overline{\omega}_G)$ of (25) with $\overline{\eta} \in \Omega$, we have

$$0 = B_S^T(\overline{\eta})\overline{\omega}_G \tag{26a}$$

$$0 = -A\overline{\omega}_G - B_G\Gamma\sin(\overline{\eta}) + \overline{u}. \tag{26b}$$

Hence, by Proposition 1, we have $\overline{\omega}_G = \mathbb{1}\omega^0$ for some constant ω^0. By multiplying both sides of (26b) from the left by $\mathbb{1}^T$, we obtain that

$$\omega^0 = \frac{-\mathbb{1}^T B_G\Gamma\sin(\overline{\eta}) + \mathbb{1}^T\overline{u}}{\mathbb{1}^T A\mathbb{1}},$$

which boils down to

$$\omega^0 = \frac{\mathbb{1}^T B_L\Gamma\sin(\eta) + \mathbb{1}^T\overline{u}}{\mathbb{1}^T A\mathbb{1}},$$

where we have used the fact that $\mathbb{1}^T B_G = -\mathbb{1}^T B_L$,and $B_L\Gamma\sin(\eta)$ is constant. Hence, $\mathbb{1}^T B_L\Gamma\sin(\eta) + \mathbb{1}^T\overline{u}$ has to be identically zero to avoid frequency deviation. This corresponds to the well-known demand and supply matching condition which again elucidates the fact that the vector $B_L\Gamma\sin(\eta)$ plays the role of the loads in the reduced network (25).

By the discussion above, and the results of Theorem 2 and Proposition 3, we conclude that the original network (21) is *embedded* in the reduced network (25). This enables us to deduce the properties of the original model by looking at the explicit reduced ODE model (25). Moreover, one can design controllers based on the reduced ODE model (25) rather than the DAE model (21).

Remark 5 Despite the fact that the reduced network (25) is expressed in terms of ordinary differential equations, the analysis/control schemes available in the power network literature, in particular for aggregated modes, are not readily applicable to this case. This is mainly due to the presence of the state dependent map B_S in (25a) instead of the ordinary time-independent incidence matrix. However, one can show that this does not hinder the analysis thanks to the nice properties of the projected pseudo incidence matrix captured in Proposition 1 as well as the invariance observation made in Proposition 3. The analysis and control of the reduced order model (25) will be discussed in a future work.

V. Conclusions

We have considered structure preserving power networks expressed as differential algebraic equations, where the proper algebraic constraints are the result of the presence of constant power loads. We have introduced the notion of the projected pseudo incidence matrix, which provides a novel decomposition of the reduced Laplacian matrix. For the linear network model, by exploiting this new matrix, we have proposed a reduced model in which the frequency disagreements among all the buses of the network are readily tractable. We have also addressed the elimination of the purely algebraic constraints in the nonlinear network model. Again, by using the projected pseudo incidence matrix, we have established a reduced model under a suitable regularity assumption. The proposed explicit reduced order model is expressed in terms of ordinary differential equations, and thus facilitates the understanding, analysis, and the controller design of the power network. Frequency regulation and active power sharing of the proposed reduced models will be investigated in a future work.

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