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Research Article

On The Generating Functions of the Powers of the K-Fibonacci Numbers

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Abstract: In this paper we will introduce the k–Fibonacci factorial function $F_{k,n}!$ and the k–Fibonomial numbers. Then we present the generating function of the powers ofthe k–Fibonacci numbers. Finally, we study the numerators and the denominators of these functions. As consequence of this study, we find out several integer sequences some of the mare listed in the Online Encyclopedia of Integer Sequences (OEIS) and we introduce some more.
Keywords: k–Fibonacci numbers, k–Fibonomial numbers, k–Lucas numbers ,Generating function.
MSC2000: 15A36; 11C20; 11B39

INTRODUCTION

k-Fibonacci numbers [2,3] are defined as the sequence $\{F_{k,n}\}_{n\in\mathbb{N}}$ such that

$F_{k,0}=0,\ F_{k,1}=1$ and $F_{k,n+1}=k\,F_{k,n}+F_{k,n-1}$ for $n\geq 1$.

The recurrence relation of this formula is $r^2-k\,r-1=0$ whose solutions are $\sigma_1=\dfrac{k+\sqrt{k^2+4}}{2}$ and $\sigma_2=\dfrac{k-\sqrt{k^2+4}}{2}$.

The Binet formula for these numbers is $F_{k,n}=\dfrac{\sigma_1^n-\sigma_2^n}{\sigma_1-\sigma_2}$ and the convolution formula is

$$F_{k,n+m}=F_{k,n+1}F_{k,m}+F_{k,n}F_{k,m-1} \quad (1)$$

From the definition of the k-Fibonacci numbers, the firsts of them are presented in Table 1.

Table 1: First k-Fibonacci numbers

$$F_{k,0}=0$$
$$F_{k,1}=1$$
$$F_{k,2}=k$$
$$F_{k,3}=k^2+1$$
$$F_{k,4}=k^3+2k$$
$$F_{k,5}=k^4+3k^2+1$$
$$F_{k,6}=k^5+4k^3+3k$$
....

So, the first k-Fibonacci sequences are, $F=F_1=\{0,1,1,2,3,5,8,...\}$ that is the classical Fibonacci sequence, A000045 in [5], OEIS from now.

$P = F_2 = \{0,1,2,5,12,29,70,...\}$, the Pell sequence, A000129 in OEIS.

$F_3 = \{0,1,3,10,33,109,360,...\}$, A006190 in OEIS, etc.

In the same way, the k–Lucas numbers [1] are defined by mean of the formula $L_{k,n+1} = k\,L_{k,n} + L_{k,n-1}$ with the initial conditions $L_{k,0} = 2$ and $L_{k,1} = k$.

Then, the first k–Lucas numbers are presented in the following table:

Table 2: First k-Fibonacci numbers

$$L_{k,0} = 2$$
$$L_{k,1} = k$$
$$L_{k,2} = k^2 + 2$$
$$L_{k,3} = k^3 + 3k$$
$$L_{k,4} = k^4 + 4k^2 + 2$$
$$L_{k,5} = k^5 + 5k^3 + 5k$$

For k = 1 we obtain the classical Lucas numbers $L = L_1 = \{2,1,3,4,7,11,18,...\}$, A000204, and for k = 2 the sequence $PL = L_2 = \{2,2,6,14,34,82,...\}$, A002203, is the
Pell-Lucas sequence.

k–Fibonacci numbers and k–Lucas numbers are related by mean of the formula

$$L_{k,n} = F_{k,n-1} + F_{k,n+1}$$

K-FIBONOMIAL NUMBERS

In the development of this article, we have followed the ideas developed in [7].

We introduce the k–Fibonacci factorial $F_{k,n}!$ as the product of the k–Fibonacci numbers from $F_{k,n}$, $F_{k,n-1}$, down to $F_{k,1} = 1$.

That is $F_{k,n}! = F_{k,n} \cdot F_{k,n-1} \cdots F_{k,1}$

For example $F_{3,5}! = F_{3,5} \cdot F_{3,4} \cdot F_{3,3} \cdot F_{3,2} \cdot F_{3,1} = 109 \cdot 33 \cdot 10 \cdot 3 \cdot 1 = 107910$

Assume $F_{k,0}! = 1$, being the empty product, evaluates to 1 [6].

We define a new form of *binomial numbers* but using this definition of factorial instead.

We define the k–Fibonomial numbers as $k - Fibonomial(n, j) = \begin{bmatrix} n \\ j \end{bmatrix}_{F_k} = \dfrac{F_{k,n} \cdot F_{k,n-1} \cdots F_{k,n-j+1}}{F_{k,j} \cdot F_{k,j-1} \cdots F_{k,2} \cdot F_{k,1}} = \dfrac{F_{k,n}!}{F_{k,j}!\,F_{k,n-j}!}$

where n and j are non-negative integers, $0 \le j \le n$.

We will suppose $\begin{bmatrix} n \\ j \end{bmatrix}_{F_k} = 0, \ \forall j > n$.

Taking into account $F_{k,r} \mid F_{k,nr}$ [4], we deduce all k–Fibonomial number is integer.

From the definition and taking into account $F_{k,0}! = 1$, it is easy to prove $\begin{bmatrix} n \\ 0 \end{bmatrix}_{F_k} = \begin{bmatrix} n \\ n \end{bmatrix}_{F_k} = 1$ and

$$\begin{bmatrix} n \\ 1 \end{bmatrix}_{F_k} = \begin{bmatrix} n \\ n-1 \end{bmatrix}_{F_k} = F_{k,n}$$

We can put the k–Fibonomial numbers in triangular form as follows:

Table 3: The k–Fibonomial triangle

n = 0					1			
n = 1				1		1		
n = 2			1		k		1	
n = 3		1		k^2+2		k^2+2		1
n = 4	1	k^3+2k			k^4+3k^2+2		k^3+2k	1

In particular, for k = 1, we obtain the following table.

Table 4: 1–Fibonomial triangle or Fibonomial triangle

n = 0				1			
n = 1				1	1		
n = 2			1	1	1		
n = 3			1	2	2	1	
n = 4		1	3	6	3	1	
n = 5		1	5	15	15	5	1
n = 6	1	8	40	60	40	8	1

Diagonal sequences are listed in OEIS. The sequence of row sums of Fibonomial triangle is $\{1,2,3,6,14,42,...\}$, A056569.
For k = 2, we have

Table 5: 2–Fibonomial triangle or Pellonomial triangle

n = 0				1			
n = 1				1	1		
n = 2			1	2	1		
n = 3			1	5	5	1	
n = 4		1	12	30	12	1	
n = 5		1	29	174	174	29	1
n = 6	1	70	1015	2436	1015	70	1

Also, these diagonal sequences and the sequence of row sums of Pellonomial triangle $\left(\{1,2,4,12,56,408,4608,...\},\ A099928\right)$, are listed in OEIS.

Some properties of the k-Fibonomial numbers
In the sequel we present some properties of the k-Fibonomial numbers

Symmetry property: $\begin{bmatrix} n \\ i \end{bmatrix}_{F_k} = \begin{bmatrix} n \\ n-i \end{bmatrix}_{F_k}$

It is obvious.

$$F_{k,n-i}\begin{bmatrix} n \\ i \end{bmatrix}_{F_k} = F_{k,n}\begin{bmatrix} n-1 \\ i \end{bmatrix}_{F_k}$$

Proof.

$$LHS = \frac{F_{k,n}F_{k,n-1}\cdots F_{k,n-i+1}}{F_{k,i}\,!}F_{k,n-i} = F_{k,n}\frac{F_{k,n-1}F_{k,n-2}\cdots F_{k,n-i+1}F_{k,n-i}}{F_{k,i}\,!} = RHS$$

This property can also be written in the form $\begin{bmatrix} n \\ i \end{bmatrix}_{F_k} = \dfrac{F_{k,n}}{F_{k,n-i}}\begin{bmatrix} n-i \\ i \end{bmatrix}_{F_k}$

In similar form we can prove $F_{k,i}\begin{bmatrix} n \\ i \end{bmatrix}_{F_k} = F_{k,n}\begin{bmatrix} n-i \\ i-1 \end{bmatrix}_{F_k}$

Addition formula: $\begin{bmatrix} n \\ i \end{bmatrix}_{F_k} = F_{k,i-1}\begin{bmatrix} n-1 \\ i \end{bmatrix}_{F_k} + F_{k,n-i+1}\begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_{F_k}$

$$(a) = F_{k,i-1}\begin{bmatrix} n-1 \\ i \end{bmatrix}_{F_k} = F_{k,i-1}\frac{F_{k,n-1}F_{k,n-2}\cdots F_{k,n-i+1}F_{k,n-i}}{F_{k,i}F_{k,i-1}\cdots F_{k,1}}$$

Proof. $(b) = F_{k,n-i+1}\begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_{F_k} = F_{k,n-i+1}\frac{F_{k,n-1}F_{k,n-2}\cdots F_{k,n-i+1}}{F_{k,i-1}F_{k,i-2}\cdots F_{k,1}}$

$$(a)+(b) = \frac{F_{k,n-1}F_{k,n-2}\cdots F_{k,n-i+1}\left(F_{k,n-i}F_{k,i-1}+F_{k,n-i+1}F_{k,i}\right)}{F_{k,i}F_{k,i-1}\cdots F_{k,1}} = \frac{F_{k,n-1}F_{k,n-2}\cdots F_{k,n-i+1}F_{k,n}}{F_{k,i}!} = \begin{bmatrix} n \\ i \end{bmatrix}_{F_k}$$

since $F_{k,n-i}F_{k,i-1}+F_{k,i}F_{k,n-i+1} = F_{k,n-i+i} = F_{k,n}$ from Formula (1). For instance:

$$\begin{bmatrix} 7 \\ 5 \end{bmatrix}_{F_2} = F_{2,4}\begin{bmatrix} 6 \\ 5 \end{bmatrix}_{F_2} + F_{2,3}\begin{bmatrix} 6 \\ 4 \end{bmatrix}_{F_2} = 12\cdot 70 + 5\cdot 1015 = 5915$$

A GENERATING FUNCTION FOR THE POWERS OF THE K–FIBONACCI NUMBERS

Following the process used in [2] to find the generating function of the k–Fibonacci numbers, we can also find the generating function of the natural powers of these numbers. As a result we will see that there is a relationship between these generating functions and the k–Fibonomials numbers.

Here is a table summarising these results, where $gf\left[F_{k,n}^r\right]$ is the generating function of the r–th power of the k–Fibonacci numbers (without $F_{k,0}=0$):

$$gf\left[F_{k,n}\right] = \frac{1}{1-\begin{bmatrix} 2 \\ 1 \end{bmatrix}_{F_k} x - x^2}$$

$$gf\left[F_{k,n}^2\right] = \frac{1-x}{1-\begin{bmatrix} 3 \\ 1 \end{bmatrix}_{F_k} x - \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{F_k} x^2 + x^3}$$

$$gf\left[F_{k,n}^3\right] = \frac{1-2kx-x^2}{1-\begin{bmatrix} 4 \\ 1 \end{bmatrix}_{F_k} x - \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{F_k} x^2 + \begin{bmatrix} 4 \\ 3 \end{bmatrix}_{F_k} x^3 + x^4}$$

$$gf\left[F_{k,n}^4\right] = \frac{1-(3k^2+1)x-(3k^2+1)x^2+x^3}{1-\begin{bmatrix} 5 \\ 1 \end{bmatrix}_{F_k} x - \begin{bmatrix} 5 \\ 2 \end{bmatrix}_{F_k} x^2 + \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{F_k} x^3 + \begin{bmatrix} 5 \\ 4 \end{bmatrix}_{F_k} x^4 - x^5}$$

$$gf\left[F_{k,n}^5\right] = \frac{1-(4k^3+3k)x-(3k^2+1)(2k^2+2)x^2+(4k^3+3k)x^3+x^4}{1-\begin{bmatrix} 6 \\ 1 \end{bmatrix}_{F_k} x - \begin{bmatrix} 6 \\ 2 \end{bmatrix}_{F_k} x^2 + \begin{bmatrix} 6 \\ 3 \end{bmatrix}_{F_k} x^3 + \begin{bmatrix} 6 \\ 4 \end{bmatrix}_{F_k} x^4 - \begin{bmatrix} 6 \\ 5 \end{bmatrix}_{F_k} x^5 - x^6}$$

with

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_{F_k} = k$$

$$\begin{bmatrix}3\\1\end{bmatrix}_{F_k}=\begin{bmatrix}3\\2\end{bmatrix}_{F_k}=k^2+1$$

$$\begin{bmatrix}4\\1\end{bmatrix}_{F_k}=\begin{bmatrix}4\\3\end{bmatrix}_{F_k}=k^3+2k \qquad \begin{bmatrix}4\\2\end{bmatrix}_{F_k}=k^4+3k^2+2$$

$$\begin{bmatrix}5\\1\end{bmatrix}_{F_k}=\begin{bmatrix}5\\4\end{bmatrix}_{F_k}=k^4+3k^2+1 \qquad \begin{bmatrix}5\\2\end{bmatrix}_{F_k}=\begin{bmatrix}5\\3\end{bmatrix}_{F_k}=k^6+5k^4+7k^2+2$$

$$\begin{bmatrix}6\\1\end{bmatrix}_{F_k}=\begin{bmatrix}6\\5\end{bmatrix}_{F_k}=k^5+4k^3+3k \qquad \begin{bmatrix}6\\2\end{bmatrix}_{F_k}=\begin{bmatrix}6\\4\end{bmatrix}_{F_k}=k^8+7k^6+16k^4+13k^2+3$$

$$\begin{bmatrix}6\\3\end{bmatrix}_{F_k}=k^9+8k^7+22k^5+23k^3+6k$$

For instance, for k = 1, the generating functions of the first powers of the classical Fibonacci numbers are:

$$gf[F_n]=\frac{1}{1-x-x^2}$$

$$gf[F_n^2]=\frac{1-x}{1-2x-2x^2+x^3}$$

$$gf[F_n^3]=\frac{1-2x-x^2}{1-3x-6x^2+3x^3+x^4}$$

$$gf[F_n^4]=\frac{1-4x-4x^2+x^3}{1-5x-15x^2+15x^3+5x^4-x^5}$$

And the generating functions of the first powers of the Pell numbers are

$$gf[P_n]=\frac{1}{1-2x-x^2}$$

$$gf[P_n^2]=\frac{1-x}{1-5x-5x^2+x^3}$$

$$gf[P_n^3]=\frac{1-4x-x^2}{1-12x-30x^2+12x^3+x^4}$$

$$gf[P_n^4]=\frac{1-13x-13x^2+x^3}{1-29x-174x^2+174x^3+29x^4-x^5}$$

Study of the denominators of the generating functions

With help of MATHEMATICA, below is the polynomial factorization of the denominators of the above generating functions, where D_j are the consecutive denominators:

$$D_1=-x^2-kx+1$$

$$D_2=\left(x^2-(k^2+2)x+1\right)(x+1)$$

$$D_3=\left(x^2+(k^3+3k)x-1\right)\left(x^2-k\,x-1\right)$$

$$D_4=\left(x^2-\left(k^4+4k^2+2\right)x+1\right)\left(x^2+\left(k^2+2\right)x+1\right)\left(x-1\right)$$

$$D_5=\left(x^2+\left(k^5+5k^3+5k\right)x-1\right)\left(x^2-\left(k^3+3k\right)x-1\right)\left(-x^2-kx+1\right)$$

We can look at the coefficients of these polynomials, without the signs and according to Table 5, are the k–Lucas numbers. Consequently, we can write, according the subscripts are odd or even

$$D_1=\left(x^2+L_{k,1}x-1\right)(-1)$$

$$D_3 = \left(x^2 + L_{k,3}x - 1\right)\left(x^2 - L_{k,1}x - 1\right)$$
$$D_5 = \left(x^2 + L_{k,5}x - 1\right)\left(x^2 - L_{k,3}x - 1\right)\left(x^2 + L_{k,1}x - 1\right)(-1)$$
$$D_7 = \left(x^2 + L_{k,7}x - 1\right)\left(x^2 - L_{k,5}x - 1\right)\left(x^2 + L_{k,3}x - 1\right)\left(x^2 - L_{k,1}x - 1\right)$$

and

$$D_2 = \left(x^2 - L_{k,2}x + 1\right)(x+1)$$
$$D_4 = \left(x^2 - L_{k,4}x + 1\right)\left(x^2 + L_{k,2}x + 1\right)(x-1)$$
$$D_6 = \left(x^2 - L_{k,6}x + 1\right)\left(x^2 + L_{k,4}x + 1\right)\left(x^2 - L_{k,2}x + 1\right)(x+1)$$
$$D_8 = \left(x^2 - L_{k,8}x + 1\right)\left(x^2 + L_{k,6}x + 1\right)\left(x^2 - L_{k,4}x + 1\right)\left(x^2 + L_{k,2}x + 1\right)(x-1)$$

And finally, we can summarizing these relations in the formulas

$$D_{2r+1} = (-1)^{r+1}\prod_{j=0}^{r}\left(x^2 + (-1)^j L_{k,2(r-j)+1}x - 1\right)$$

$$D_{2r} = (x - (-1)^r)\prod_{j=0}^{r-1}\left(x^2 - (-1)^j L_{k,2(r-j)}x + 1\right)$$

Study of the nominators of the generating functions

Below is a table with the coefficients of the numerators of the successive generating functions (without the signs):

Table 6: Coefficients of the numerators of the generating functions $gf\left[F_{k,n}^r\right]$

0	1				
1	1	1			
2	1	2k	1		
3	1	$3k^2+1$	$3k^2+1$	1	
4	1	$4k^3+3k$	$6k^4+8k^2+2$	$4k^3+3k$	1

We will indicate as $T_{r,j}$ the elements of this table, where r is the row and jis the number of order of this element in this row for r = 0, 1, 2, 3… andj = 0, 1, 2…r.

With the conditions $T_{r,0} = T_{r,r} = 1$and $T_{r,j} = 0$ for j > r, these elements verify the relation

$$T_{r,j} = F_{k,r-j+1}T_{r-1,j-1} + F_{k,j+1}T_{r-1,j} \quad (2)$$

For instance, the elements of the fifth row of Table 6 verify the relation $T_{5,j} = F_{k,6-j}T_{4,j-1} + F_{k,j+1}T_{4,j}$ and so its elements are

$$T_{5,0} = T_{5,5} = 1 \; T_{5,1} = T_{5,4} = F_{k,5}T_{4,0} + F_{k,2}T_{4,1} = \left(k^4 + 3k^2 + 1\right)1 + k\left(4k^3 + 3k\right) = 5k^4 + 6k^2 + 1$$

$$T_{5,2} = T_{5,3} = F_{k,4}T_{4,1} + F_{k,3}T_{4,2} = \left(k^3 + 2k\right)\left(4k^3 + 3k\right) + \left(k^2 + 1\right)\left(6k^4 + 8k^2 + 2\right)$$
$$= 10k^6 + 25k^4 + 16k^2 + 2$$

Formula (2) is similar that formula used in the construction of the classical Pascal triangle, $T_{r,j} = T_{r-1,j-1} + T_{r-1,j}$ deduced from the relation $\binom{r-1}{j-1} + \binom{r-1}{j} = \binom{r}{j}$

Symmetry property of the coefficients Tr,j

The elements $T_{r,j}$ verify the symmetry property $T_{r,r-j} = T_{r,j}$ that we will prove by induction.

For r = 1, 2, 3, 4 this relation is proved in Table 6.

Let us suppose this relation is true until r − 1, that is $T_{r-1,r-1-j} = T_{r-1,j}$

Then $T_{r-1,r-j} = T_{r-1,(r-1)-(j-1)} = T_{r-1,j-1}$ and

$$T_{r,r-j} = F_{k,r-(r-j)+1}T_{r-1,r-j-1} + F_{k,r-j+1}T_{r-1,r-j} = F_{k,j+1}T_{r-1,j} + F_{k,r-j+1}T_{r-1,j-1} = T_{r,j}$$

So, the successive numerators of the generating functions $gf\left[F_{k,n}^{r}\right]$ are of the form $N_r = \sum_{j=0}^{r}(-1)^{\frac{j(j+1)}{2}}T_{r,j}x^j$

In particular, for k = 1 and k = 2 we obtain the respective triangle

Table 7: Coefficients of the numerators of the Generating Function of F_n^r

r = 0						1					
r = 1					1		1				
r = 2				1		2		1			
r = 3			1		4		4		1		
r = 4		1		7		16		7		1	
r = 5	1		12		53		53		12		1

Only the first five diagonal sequences and the row sums {1, 2, 4, 10, 32…} are listed in OEIS. The sixth diagonal sequence is

{1, 20, 492, 9288, 163504, 2616064, 39369649, 561744656, 7690052788, 101717711304…}

For the Pell numbers, we obtain the following table:

Table 8: Coefficients of the numerators of the Generating Function of P_r^n

r = 0						1					
r = 1					1		1				
r = 2				1		4		1			
r = 3			1		13		13		1		
r = 4		1		38		130		38		1	
r = 5	1		105		1106		1106		105		1

Only the first diagonal sequence {1, 4, 13, 38, 105…} is listed in OEIS.

The following four diagonal sequences begin as follows:

{1, 13, 130, 1106, 8575, 62475, 435576, 2939208, 19342285, 124800361…}

{1, 38, 1106, 26544, 567203, 11179686, 207768576, 3692419776, 63361188037…}

{1, 105, 8575, 567203, 32897774, 1736613466, 85474679858, 3985272984490…}

{1, 280, 62475, 11179686, 1736613466, 243125885240, 31464032862802…}

CONCLUSIONS

We have defined the k–Fibonomial numbers in a similar form to the Binomial numbers and obtain formulas for generating these numbers from the Generating function.

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