

Bigraded Betti numbers and Generalized Persistence Diagrams

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July 26, 2022

Abstract

Commutative diagrams of vector spaces and linear maps over \mathbb{Z}^2 are objects of interest in topological data analysis (TDA) where this type of diagrams are called 2-parameter persistence modules. Given that quiver representation theory tells us that such diagrams are of wild type, studying informative invariants of a 2-parameter persistence module M is of central importance in TDA. One of such invariants is the generalized rank invariant, recently introduced by Kim and Mémoli. Via the Möbius inversion of the generalized rank invariant of M , we obtain a collection of connected subsets $I \subset \mathbb{Z}^2$ with signed multiplicities. This collection generalizes the well known notion of *persistence barcode* of a persistence module over \mathbb{R} from TDA. In this paper we show that the bigraded Betti numbers of M , a classical algebraic invariant of M , are obtained by counting the corner points of these subsets I s. Along the way, we verify that an invariant of 2-parameter persistence modules called the interval decomposable approximation (introduced by Asashiba et al.) also encodes the bigraded Betti numbers in a similar fashion.

1 Introduction

Multiparameter persistent homology. Theoretical foundations of *persistent homology*, one of the main protagonists in topological data analysis (TDA), have been rapidly developed in the last two decades, allowing a large number of applications. Persistent homology is obtained by applying the homology functor to an \mathbb{R} (or \mathbb{Z}) -indexed increasing family of topological spaces [12, 28]. This parametrized family of topological spaces, for example, often arises as either a *sublevel set filtration* of a real-valued map on a topological space, or the *Vietoris-Rips simplicial filtration* of a metric space.

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With more complex input data, we obtain \mathbb{R}^d -indexed increasing families ($d > 1$) of topological spaces, e.g. a sublevel set filtration of a topological space that is filtered by multiple real-valued functions, or a Vietoris-Rips-sublevel simplicial filtration of a metric space equipped with a map [12, 16]. By applying the homology functor (with coefficients in a fixed field k) to such a multiparameter filtration, we obtain a d -parameter persistence module \mathbb{R}^d (or \mathbb{Z}^d) $\rightarrow \mathbf{vec}$, a functor from the poset \mathbb{R}^d (or \mathbb{Z}^d) to the category \mathbf{vec} of finite dimensional vector spaces and linear maps over the field k . In contrast to the case of $d = 1$, there is no discrete and complete invariant for \mathbb{R}^d (or \mathbb{Z}^d) $\rightarrow \mathbf{vec}$ for $d > 1$ [16]. In quiver representation theory, functors \mathbb{R}^d (or \mathbb{Z}^d) $\rightarrow \mathbf{vec}$ ($d > 1$) are of *wild type*, implying that there is no simple invariant which completely encodes the isomorphism type of \mathbb{R}^d (or \mathbb{Z}^d) $\rightarrow \mathbf{vec}$ [23, 32]. Nevertheless, there have been many studies on the invariants of d -parameter persistence modules, e.g. [16, 18, 33, 39, 47, 53, 54].

Special attention has been placed on the case of $d = 2$ [1, 2, 8, 9, 20, 26, 31, 42] in part because 2-parameter filtrations arise in the study of interlevel set persistence [8, 14], in the study of point cloud data with non-uniform density [4, 11, 15, 16], or in applications in material science and chemical engineering [31, 35, 36]. The software RIVET [42] can efficiently compute and visualize the dimension function (a.k.a. the Hilbert function), the fibered barcode, and the bigraded Betti numbers of a 2-parameter persistence module.

Multigraded Betti numbers. Multigraded Betti numbers encode important information about the algebraic structure of a multigraded module over the polynomial ring in n variables [30, 48]. For multiparameter persistence modules that arise from data, multigraded Betti numbers provide insight about the coarse-scale topological features of the data (cf. [11]). For 2-parameter persistence modules, the multigraded Betti numbers are also called the *bigraded* Betti numbers. RIVET [42] represents the bigraded Betti numbers of a 2-parameter persistence module as a collection of colored dots in the plane. More interestingly, RIVET employs the bigraded Betti numbers to implement an interactive visualization of the fibered barcode. Recently, Lesnick-Wright [43] and Kerber-Rolle [37] developed efficient algorithms for computing minimal presentations and the bigraded Betti numbers of 2-parameter persistence modules.

Persistence diagram and its generalizations. In most applications of 1-parameter persistent homology, the notion of *persistence diagram* [29, 40] (or equivalently *barcode* [17]; cf. Definition 2.3) plays a central role. The persistence diagram of any $M : \mathbb{R} \rightarrow \mathbf{vec}$ is not only a visualizable topological summary of M , but also a *stable* and *complete* invariant of M [21]. In contrast, as mentioned before, there is no simple complete invariant for d -parameter persistence modules when $d > 1$.

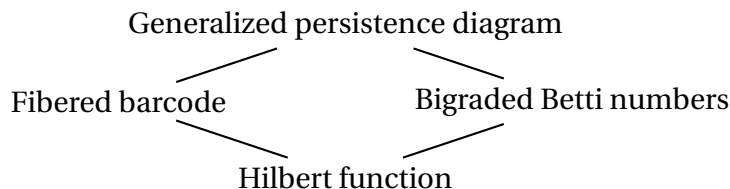
Patel introduced the notion of *generalized persistence diagram* for *constructible* functors $\mathbb{R} \rightarrow \mathcal{C}$, in which \mathcal{C} satisfies certain properties [51]. Construction of the generalized persistence diagram is based on the observation that the persistence diagram of $M : \mathbb{R} \rightarrow \mathbf{vec}$ [29] is an instance of the Möbius inversion of the *rank invariant* [16] of M . McCleary and Patel showed that the generalized persistence diagram is stable when \mathcal{C} is a skeletally small abelian category [45]. Kim and Mémoli further extended Patel’s generalized persistence diagram to the setting of functors $\mathbb{P} \rightarrow \mathcal{C}$ in which \mathbb{P} is a *essentially* finite poset such as a finite n -dimensional grid [38]. The generalized persistence diagram of $\mathbb{P} \rightarrow \mathcal{C}$ is defined as the Möbius inversion of the *generalized rank invariant* of $\mathbb{P} \rightarrow \mathcal{C}$. The generalized persistence diagram is not only a com-

plete invariant of interval decomposable persistence modules $\mathbb{P} \rightarrow \mathbf{vec}$ (Theorem 2.20), but is also well-defined regardless of the interval decomposability. The generalized rank invariant of \mathbb{R}^d (or \mathbb{Z}^d) $\rightarrow \mathbf{vec}$ is proven to be stable with respect to a certain generalization of the erosion distance [51] and the interleaving distance [41] (see the latest version of the arXiv preprint of [38]).

Our contributions. Assume that a given $M : \mathbb{Z}^2 \rightarrow \mathbf{vec}$ is finitely generated. We establish a combinatorial formula for extracting the bigraded Betti numbers of M from the generalized persistence diagram of M (Theorem 3.5). More interestingly, the formula we found is a generalization of a well-known formula for extracting the bigraded Betti numbers from interval decomposable persistence modules (Theorem 2.9).

Namely, for any finitely generated *interval decomposable* $M : \mathbb{Z}^2 \rightarrow \mathbf{vec}$, there is a visually intuitive way to find the bigraded Betti numbers of M from the indecomposable summands of M . An example of this process is shown in Fig. 1 (A)-(C). For *any* finitely generated $N : \mathbb{Z}^2 \rightarrow \mathbf{vec}$, which may not be interval decomposable, we utilize a similar process to find the bigraded Betti numbers of N from the (**Int**-)generalized persistence diagram of N . This process is shown in Fig. 1 (A')-(C'). In a sense, Theorem 3.5 thus reinforces the viewpoint that the (**Int**-)generalized persistence diagram is a proxy for the *barcode* (Definition 2.3) of persistence modules [2, 38].

One implication of Theorem 3.5 is that all invariants of 2-parameter persistence modules that are computed by the software RIVET [42] are encoded by the generalized persistence diagram. In other words, we obtain the following hierarchy of invariants for *any* finitely generated $M : \mathbb{Z}^2 \rightarrow \mathbf{vec}$, where invariant A is placed above invariant B if invariant B can be recovered from invariant A:



We remark that the generalized persistence diagram is equivalent to the generalized rank invariant (Definitions 2.17 and 2.18). Also, the fibred barcode is equivalent to the (standard) rank invariant [16]. Hence, in the diagram above, *generalized persistence diagram* and *fibred barcode* can be replaced by *generalized rank invariant* and *rank invariant*, respectively.

In the course of establishing Theorem 3.5, we verify that the *interval decomposable approximation* of 2-parameter persistence modules (introduced by Asashiba et al. [2]) also encodes the bigraded Betti numbers (Remark 2.19).

Remark 1.1. It should not be construed that Theorem 3.5 provides a practically efficient way to compute the bigraded Betti numbers. Rather, we hope that the aforementioned efficient algorithms to compute the bigraded Betti numbers could be useful for approximating the generalized persistence diagram.

Other related work. McCleary and Patel utilized the Möbius inversion formula for establishing a functorial pipeline to summarize simplicial filtrations over finite lattices into persistence

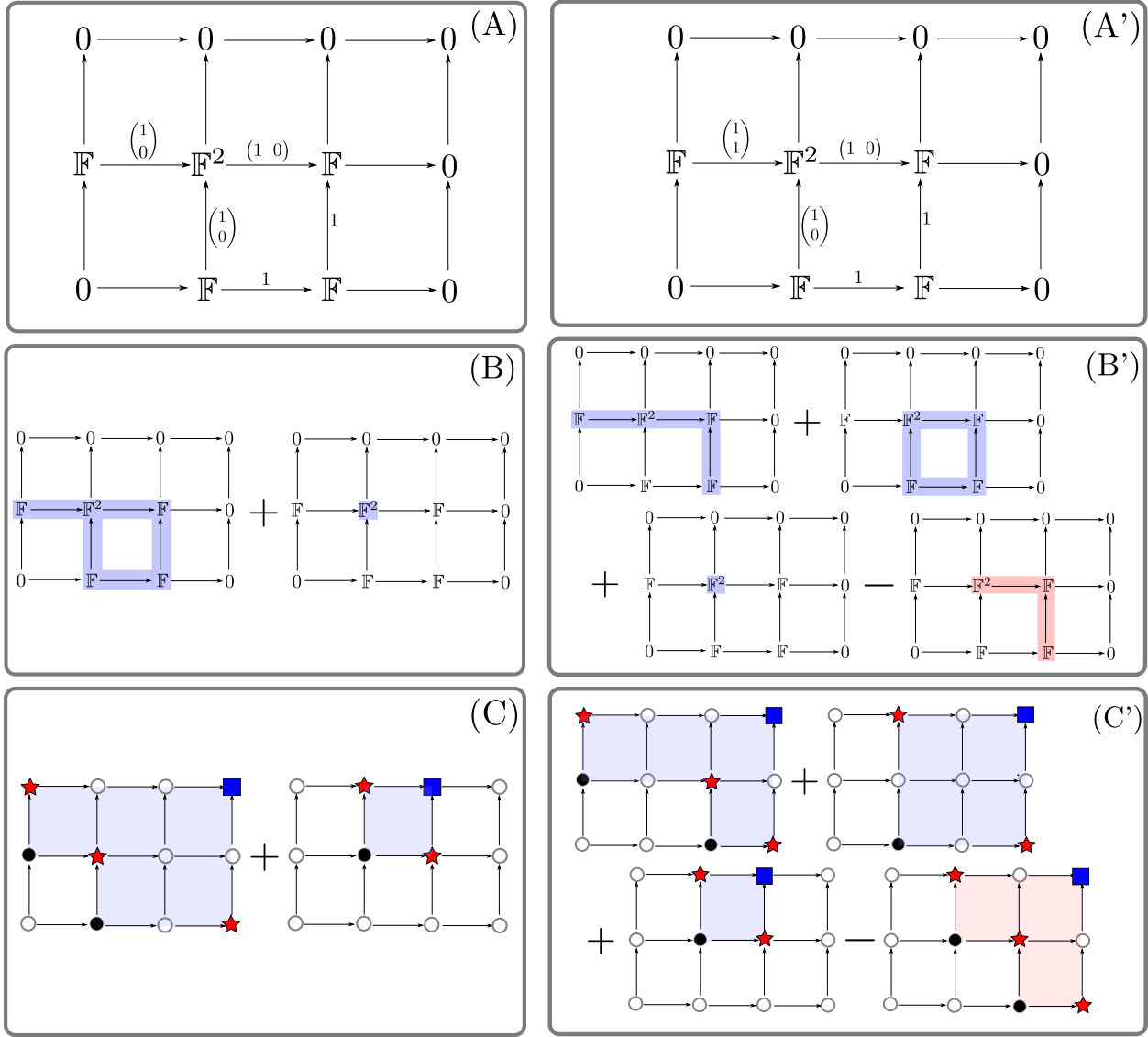


Figure 1: (A) A \mathbb{Z}^2 -indexed persistence module M whose support is contained in a 3×4 grid. (B) M is interval decomposable, and the barcode of M consists of the two blue intervals of \mathbb{Z}^2 (Definitions 2.2 and 2.3). (C) Expand each of the blue intervals from (B) to intervals in \mathbb{R}^2 as follows: Each point $p = (p_1, p_2)$ in the two intervals is expanded to the unit square $[p_1, p_1 + 1) \times [p_2, p_2 + 1) \subset \mathbb{R}^2$. Black dots, red stars and blue squares indicate three different *corner types* of the expanded intervals (see Fig. 2). The bigraded Betti numbers of M can be read from these corner types; for each $p \in \mathbb{Z}^2$, $\beta_j(M)(p)$ is equal to the number of black dots, red stars, and blue squares at p when $j = 0, 1, 2$, respectively. (A') Another \mathbb{Z}^2 -indexed persistence module N whose support is contained in a 3×4 grid. N is not interval decomposable. (B') The **Int**-generalized persistence diagram of N (Definition 2.18) is shown, where the multiplicity of the red interval is -1 and the multiplicity of each blue interval is 1 . (C') is similarly interpreted as in (C), where corner points of the red interval negatively contribute to the counting of the bigraded Betti numbers. More details are provided in Example 3.6.

diagrams [46]. Botnan et al. introduced notions of signed barcode and rank decomposition for encoding the rank invariant of multiparameter persistence modules as a linear combination of rank invariants of indicator modules [10]. In their paper, Möbius inversion was utilized for computing the rank decomposition, characterizing the generalized persistence diagram in terms of rank decompositions. Asashiba et al. provided a criterion for determining whether or not a given multiparameter persistence module is interval decomposable without having to explicitly compute indecomposable decompositions [1]. Dey and Xin proposed an efficient algorithm for decomposing multiparameter persistence modules and introduced a notion of *persistent* graded Betti numbers, a refined version of the graded Betti numbers [27]. Dey et al. reduced the problem of computing the generalized rank invariant of a given 2-parameter persistence module to computing the indecomposable decompositions of zigzag persistence modules [24]. Blanchette et al. developed a theoretical framework for building new invariants of a persistence module over a poset using homological algebra [6].

Organization. In Section 2, we review the notions of persistence modules, bigraded Betti numbers, and generalized persistence diagrams. In Section 3, we show that the bigraded Betti numbers can be recovered from the generalized persistence diagram. In Section 4, we discuss open questions.

Acknowledgments. Samantha Moore is supported by a National Science Foundation Graduate Research Fellowship under Grant No. 1650116. The authors would like to thank Dr. Ezra Miller, Dr. Richárd Rimányi, and Dr. Facundo Mémoli for their invaluable comments.

2 Preliminaries

In Section 2.1, we review the notions of persistence modules and interval decomposability. In Section 2.2, we recall the notion of bigraded Betti numbers (an invariant of 2-parameter persistence modules). In Section 2.3, we review the Möbius inversion formula in combinatorics. In Section 2.4, we review the notions of generalized rank invariant and generalized persistence diagram. In Section 2.5, we provide a formula of the (**Int**-)generalized persistence diagram in a certain setting, which will be useful in the next section.

2.1 Persistence modules and their interval decomposability

Let \mathbb{P} be a poset. We regard \mathbb{P} as the category that has points of \mathbb{P} as its objects and for $p, q \in \mathbb{P}$ there is a unique morphism $p \rightarrow q$ if and only if $p \leq q$ in \mathbb{P} . For $d \in \mathbb{N}$, let \mathbb{R}^d and subsets of \mathbb{R}^d (such as \mathbb{Z}^d) be given the partial order defined by $(a_1, a_2, \dots, a_d) \leq (b_1, b_2, \dots, b_d)$ if and only if $a_i \leq b_i$ for $i = 1, 2, \dots, d$.

Every vector space in this paper is over some fixed field k . Let \mathbf{vec} denote the category of *finite dimensional* vector spaces and linear maps over k .

A \mathbb{P} -**indexed persistence module**, or simply a \mathbb{P} -**module**, refers to a functor $M : \mathbb{P} \rightarrow \mathbf{vec}$. In other words, to each $p \in \mathbb{P}$, a vector space $M(p)$ is associated, and to each pair $p \leq q$ in \mathbb{P} , a linear map $\varphi_M(p, q) : M(p) \rightarrow M(q)$ is associated. Importantly, whenever $p \leq q \leq r$ in \mathbb{P} , it is

required that $\varphi_M(p, r) = \varphi_M(q, r) \circ \varphi_M(p, q)$. When $\mathbb{P} = \mathbb{R}^d$ or \mathbb{Z}^d , M is also called a **d -parameter persistence module**.

Consider a *zigzag poset* of n points,

$$\bullet_1 \leftrightarrow \bullet_2 \leftrightarrow \dots \leftrightarrow \bullet_{n-1} \leftrightarrow \bullet_n \quad (1)$$

where \leftrightarrow stands for either \leq or \geq . A functor from a zigzag poset (of n points) to \mathbf{vec} is called a **zigzag module** (of length n) [13].

A **morphism** between \mathbb{P} -modules M and N is a natural transformation $f : M \rightarrow N$ between M and N . That is, f is a collection $\{f_p : M(p) \rightarrow N(p)\}_{p \in \mathbb{P}}$ of linear maps such that for every pair $p \leq q$ in \mathbb{P} , the following diagram commutes:

$$\begin{array}{ccc} M(p) & \xrightarrow{\varphi_M(p,q)} & M(q) \\ \downarrow f_p & & \downarrow f_q \\ N(p) & \xrightarrow{\varphi_N(p,q)} & N(q). \end{array}$$

The **kernel** of f , denoted by $\ker(f) : \mathbb{P} \rightarrow \mathbf{vec}$, is defined as follows: For $p \in \mathbb{P}$, $\ker(f)(p) := \ker(f_p) \subseteq M(p)$. For $p \leq q$ in \mathbb{P} , $\varphi_{\ker(f)}(p, q)$ is the restriction of $\varphi_M(p, q)$ to $\ker(f_p)$. Two \mathbb{P} -modules M and N are (naturally) **isomorphic**, denoted by $M \cong N$, if there exists a natural transformation $\{f_p\}_{p \in \mathbb{P}}$ from M to N where each f_p is an isomorphism.

The **direct sum** $M \oplus N$ of $M, N : \mathbb{P} \rightarrow \mathbf{vec}$ is the \mathbb{P} -module where $(M \oplus N)(p) = M(p) \oplus N(p)$ for $p \in \mathbb{P}$ and $\varphi_{M \oplus N}(p, q) = \varphi_M(p, q) \oplus \varphi_N(p, q)$ for $p \leq q$ in \mathbb{P} . A nonzero \mathbb{P} -module M is **indecomposable** if whenever $M = M_1 \oplus M_2$ for some \mathbb{P} -modules M_1 and M_2 , either $M_1 = 0$ or $M_2 = 0$.

Theorem 2.1 (Krull-Remak-Schmidt-Azumaya [3]). *Any \mathbb{P} -module M has a direct sum decomposition $M \cong \bigoplus_i M_i$ where each M_i is indecomposable. Such a decomposition is unique up to isomorphism and reordering of the summands.*

In what follows, we review the notion of interval decomposability.

Definition 2.2. Let \mathbb{P} be a poset. An **interval**¹

I of \mathbb{P} is a subset $I \subseteq \mathbb{P}$ such that: (i) I is nonempty. (ii) If $p, q \in I$ and $p \leq r \leq q$, then $r \in I$. (iii) I is **connected**, i.e. for any $p, q \in I$, there is a sequence $p = p_0, p_1, \dots, p_\ell = q$ of elements of I with either $p_i \leq p_{i+1}$ or $p_{i+1} \leq p_i$ for each $i \in [0, \ell - 1]$. By $\mathbf{Int}(\mathbb{P})$, we denote the set of all intervals of \mathbb{P} .

For example, any interval of a zigzag poset in (1) is a set of consecutive points in $\{\bullet_1, \bullet_2, \dots, \bullet_n\}$. For an interval I of a poset \mathbb{P} , the **interval module** $V_I : \mathbb{P} \rightarrow \mathbf{vec}$ is defined as

$$V_I(p) = \begin{cases} k & \text{if } p \in I \\ 0 & \text{otherwise,} \end{cases} \quad \varphi_{V_I}(p, q) = \begin{cases} \text{id}_k & \text{if } p, q \in I, p \leq q \\ 0 & \text{otherwise.} \end{cases}$$

It is well-known that any interval module is indecomposable [8, Proposition 2.1].

¹This definition of interval is not the standard definition of interval used in order theory but it is often used in the literature concerned with persistence modules over posets; e.g. [7].

Definition 2.3. A \mathbb{P} -module M is said to be **interval decomposable** if there exists a multiset $\text{barc}(M)$ of intervals of \mathbb{P} such that $M \cong \bigoplus_{I \in \text{barc}(M)} V_I$. We call $\text{barc}(M)$ the **barcode** of M .

Theorem 2.4 ([3, 22, 32]). For $d = 1$, any $M: \mathbb{R}^d$ (or \mathbb{Z}^d) $\rightarrow \mathbf{vec}$ is interval decomposable and thus admits a (unique) barcode. However, for $d \geq 2$, M may not be interval decomposable. Lastly, any zigzag module is interval decomposable and thus admits a (unique) barcode.

The following notation is useful in the rest of the paper.

Notation 2.5. Assume that a \mathbb{P} -module M is isomorphic to the direct sum $\bigoplus_{i \in \mathcal{I}} M_i$ for some indexing set \mathcal{I} where each M_i is indecomposable. For $I \in \mathbf{Int}(\mathbb{P})$, we define $\text{mult}(I, M)$ as the cardinality of the set $\{i \in \mathcal{I} : M_i \cong V_I\}$. In words, $\text{mult}(I, M)$ is the number of those summands M_i which are isomorphic to the interval module V_I .

2.2 Bigraded Betti numbers

In this section we review the notion of bigraded Betti numbers [30].

Fix any $p \in \mathbb{Z}^2$. Then, the *upper* set $p^\uparrow := \{x \in \mathbb{Z}^2 : p \leq x\}$ determines an interval of \mathbb{Z}^2 . An \mathbb{Z}^2 -module F is **free** if there exists p_1, p_2, \dots, p_n in \mathbb{Z}^2 such that $F \cong \bigoplus_{i=1}^n V_{p_i^\uparrow}$.

Let M be an \mathbb{Z}^2 -module. An element $v \in M(p)$ for some $p \in \mathbb{Z}^2$ is called a **homogeneous** element of M . Assume that M is **finitely generated**, i.e. there exist $p_1, \dots, p_n \in \mathbb{Z}^2$ and $v_i \in M(p_i)$ for $i = 1, \dots, n$ such that for any $p \in \mathbb{Z}^2$ and for any nonzero $v \in M(p)$, there exist $c_i \in k$ for $i = 1, \dots, n$ with

$$v = \sum_{i=1}^n c_i \cdot \varphi_M(p_i, p)(v_i).$$

The collection $\{v_1, \dots, v_n\}$ is called a (homogeneous) **generating set** for M .

Let us assume that $\{v_1, \dots, v_n\}$ is a *minimal* homogeneous generating set for M , i.e. there is no homogeneous generating set for M that includes fewer than n elements. Let $F_0 := \bigoplus_{i=1}^n V_{p_i^\uparrow}$. For $i = 1, \dots, n$, let $1_{p_i} \in V_{p_i^\uparrow}(p_i)$. Then, the set $\{1_{p_1}, \dots, 1_{p_n}\}$ generates F_0 and the morphism $\eta_0: F_0 \rightarrow M$ defined by $\eta_0(1_{p_i}) = v_i$ for $i = 1, \dots, n$ is surjective. Let $K_0 := \ker(\eta_0) \subseteq F_0$ and let $\iota_0: K_0 \hookrightarrow F_0$ be the inclusion map. Iterate this process using K_0 in place of M .²

Namely, identify a minimal homogeneous generating set $\{v'_1, \dots, v'_m\}$ for K_0 where $v'_j \in (K_0)_{p'_j}$ for some $p'_1, \dots, p'_m \in \mathbb{Z}^2$ and consider the free module $F_1 := \bigoplus_{j=1}^m V_{p'_j}$ and the surjection $\eta_1: F_1 \rightarrow K_0$. Then we have the map $\iota_0 \circ \eta_1: F_1 \rightarrow F_0$. By repeating this process, we obtain a **minimal free resolution of M** :

$$\dots \rightarrow F_2 \xrightarrow{\iota_1 \circ \eta_2} F_1 \xrightarrow{\iota_0 \circ \eta_1} F_0 \xrightarrow{\eta_0} M \rightarrow 0.$$

This resolution is unique up to isomorphism [30, Theorem 1.6]. Hilbert's Syzygy Theorem guarantees that $F_j = 0$ for $j > 2$ [34].

²We remark that K_0 is finitely generated by the following two facts. (1) A submodule of any finitely generated module over a Noetherian ring is finitely generated. (2) A finitely generated \mathbb{Z}^2 -module can be viewed as a module over the polynomial ring $k[x_1, x_2]$ in two variables x_1 and x_2 , which is Noetherian; this viewpoint can be found in [16, 48] for example.

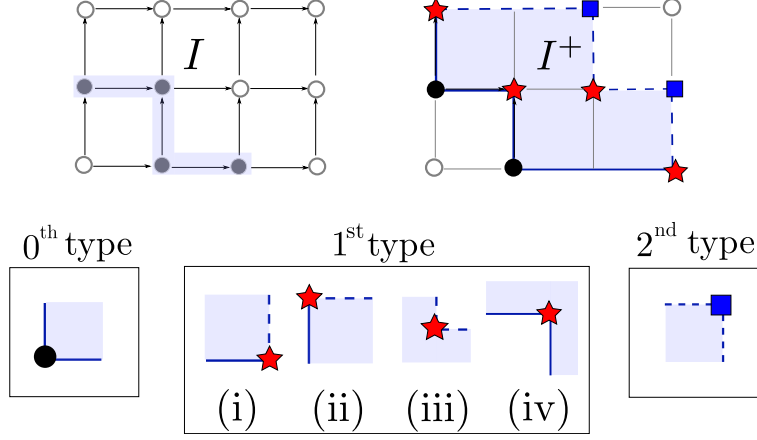


Figure 2: An interval $I \in \mathbf{Int}(\mathbb{Z}^2)$ and its corresponding region $I^+ \subset \mathbb{R}^2$ with its corner points. Points on the upper boundary (dashed lines) do not belong to I^+ , while points on the lower boundary (solid lines) belong to I^+ . Points which lie on both boundaries do not belong to I^+ .

Definition 2.6. For $j = 0, 1, 2$, the j^{th} **bigraded Betti number** $\beta_j(M) : \mathbb{Z}^2 \rightarrow \mathbf{vec}$ of M is defined by mapping each $p \in \mathbb{Z}^2$ to

$$\beta_j(M)(p) := \text{mult}(p^\perp, F_j) \quad (\text{Notation 2.5}).$$

We will see that for an interval decomposable \mathbb{Z}^2 -module M , its bigraded Betti numbers can be extracted from $\text{barc}(M)$. To this end, we will make use of a certain regions that arise by “blowing-up” intervals from $\text{barc}(M)$:

Definition 2.7. Given any $I \in \mathbf{Int}(\mathbb{Z}^2)$, the subset of \mathbb{R}^2

$$I^+ := \bigcup_{(p_1, p_2) \in I} [p_1, p_1 + 1) \times [p_2, p_2 + 1) \quad (2)$$

will be referred to as the region corresponding to I in \mathbb{R}^2 .

The following remarks are well-known; e.g. [11, Remarks 2.4 and 3.10].

- Remark 2.8.** (i) For any finitely generated $M, N : \mathbb{Z}^2 \rightarrow \mathbf{vec}$, we have $\beta_j(M \oplus N) = \beta_j(M) + \beta_j(N)$ for $j = 0, 1, 2$.
- (ii) Let $I \in \mathbf{Int}(\mathbb{Z}^2)$. For the interval module $V_I : \mathbb{Z}^2 \rightarrow \mathbf{vec}$, the j^{th} graded Betti number $\beta_j(V_I)(p)$ is equal to 1 if p is a j^{th} type corner point of I^+ and is equal to 0 otherwise; see Fig. 2.

Remark 2.8 directly implies:

Theorem 2.9. *Given any finitely generated interval decomposable module $M : \mathbb{Z}^2 \rightarrow \mathbf{vec}$, the bigraded Betti numbers of M can be extracted from $\text{barc}(M)$. More specifically, the bigraded Betti numbers of M can be extracted from the corner points of the elements in the multiset*

$$\{I^+ \subset \mathbb{R}^2 : I \in \text{barc}(M)\}.$$

In Theorem 3.5, we remove the assumption that M be interval decomposable and generalize Theorem 2.9 to the setting of *any* finitely generated \mathbb{Z}^2 -modules.

2.3 The Möbius inversion formula in combinatorics

In this section, we briefly review the Möbius inversion formula, a fundamental concept in combinatorics [5, 52].

A poset \mathbb{A} is said to be **locally finite** if for all $p, q \in \mathbb{A}$ with $p \leq q$, the set $[p, q] := \{r \in \mathbb{A} : p \leq r \leq q\}$ is finite. Let \mathbb{A} be a locally finite poset. The Möbius function $\mu_{\mathbb{A}} : \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{Z}$ of \mathbb{A} is defined³ recursively as

$$\mu_{\mathbb{A}}(p, q) = \begin{cases} 1, & p = q, \\ -\sum_{p \leq r < q} \mu_{\mathbb{A}}(p, r), & p < q, \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

For $q_0 \in \mathbb{A}$, consider the principal ideal $q_0^\downarrow := \{q \in \mathbb{A} : q \leq q_0\}$. Note that if we assume that q^\downarrow is finite for *all* $q \in \mathbb{A}$, then \mathbb{A} must be locally finite. To see this, note that, for any $p, q \in \mathbb{A}$ with $p \leq q$, the set $[p, q]$ is a subset of the finite set q^\downarrow .

Theorem 2.10 (Möbius Inversion formula). *Assume that q^\downarrow is finite for all $q \in \mathbb{A}$. Let k be a field. For any pair of functions $f, g : \mathbb{A} \rightarrow k$,*

$$g(q) = \sum_{r \leq q} f(r) \text{ for all } q \in \mathbb{A}$$

if and only if

$$f(q) = \sum_{r \leq q} g(r) \cdot \mu_{\mathbb{A}}(r, q) \text{ for all } q \in \mathbb{A}.$$

The function f is called the *Möbius inversion* of g .

One interpretation of the Möbius inversion formula is that of a discrete analogue of the derivative of a real-valued map in elementary calculus, as explained in the following example:

Example 2.11. Let $[m] = \{0, 1, \dots, m\}$ with the usual order. Then $\mu_{[m]}(a, b) = \begin{cases} 1, & a = b, \\ -1, & a = b - 1, \\ 0, & \text{otherwise.} \end{cases}$

Hence, for any function $g : [m] \rightarrow \mathbb{R}$, its Möbius inversion $f : [m] \rightarrow \mathbb{R}$ is given by $f(a) = g(a) - g(a - 1)$ for $a \neq 0$ and $f(0) = g(0)$. Hence, at each point $a \neq 0$, $f(a)$ captures the rate of change of g around that point.

2.4 Generalized rank invariant and generalized persistence diagrams

In this section we review the notions of generalized rank invariant and generalized persistence diagram [38, 51].

Throughout this subsection, let \mathbb{P} denote a *finite connected* poset (Definition 2.2 (iii)).

³More precisely, the codomain of $\mu_{\mathbb{A}}$ is the multiple of 1 in a specified base ring.

Consider any \mathbb{P} -module M . Then M admits a limit and a colimit of M : $\varprojlim M = (L, (\pi_p : L \rightarrow M(p))_{p \in \mathbb{P}})$ and $\varinjlim M = (C, (\iota_p : M(p) \rightarrow C)_{p \in \mathbb{P}})$; see the appendix for a review of the definitions of limits and colimits (Definitions A.4 and A.6). This implies that, for every $p \leq q$ in \mathbb{P} ,

$$M(p \leq q) \circ \pi_p = \pi_q \text{ and } \iota_q \circ M(p \leq q) = \iota_p.$$

Since \mathbb{P} is connected, these equalities imply that $\iota_p \circ \pi_p = \iota_q \circ \pi_q : L \rightarrow C$ for any $p, q \in \mathbb{P}$. In words, the composition $\iota_p \circ \pi_p$ is independent of p . The **canonical limit-to-colimit map** $\psi_M : \varprojlim M \rightarrow \varinjlim M$ is therefore defined to be *the* linear map $\iota_p \circ \pi_p$ where p is *any* point in \mathbb{P} .

Definition 2.12 ([38]). The **rank** of $M : \mathbb{P} \rightarrow \mathbf{vec}$ is defined as the rank of the canonical limit-to-colimit map $\psi_M : \varprojlim M \rightarrow \varinjlim M$.

The rank of $M : \mathbb{P} \rightarrow \mathbf{vec}$ counts the multiplicity of the fully supported interval module $V_{\mathbb{P}}$ in a direct sum decomposition of M into indecomposable modules:

Theorem 2.13 ([19, Lemma 3.1]). *For any $M : \mathbb{P} \rightarrow \mathbf{vec}$, the rank of M is equal to $\text{mult}(\mathbb{P}, M)$.*

Let $p, q \in \mathbb{P}$. We say that p *covers* q and write $q \triangleleft p$ if $q < p$ and there is no $r \in \mathbb{P}$ such that $q < r < p$.

A subposet $I \subseteq \mathbb{P}$ is said to be **path-connected** in \mathbb{P} if for any $p \neq q$ in I , there exists a sequence $p = p_0, p_1, \dots, p_n = q$ in I such that either $p_i \triangleleft p_{i+1}$ or $p_{i+1} \triangleleft p_i$ in \mathbb{P} for $i = 0, \dots, n-1$. For example, the set $\{0, 2\}$ is a *connected* (Definition 2.2 (iii)) subposet of $\{0, 1, 2\}$ equipped with the usual order, but is not path-connected in $\{0, 1, 2\}$.

By $\mathbf{Con}(\mathbb{P})$ we denote the poset of all path-connected subposets of \mathbb{P} that is ordered by inclusions. We remark that, since \mathbb{P} is finite, $\mathbf{Con}(\mathbb{P})$ is finite. For example, assume that \mathbb{P} is the zigzag poset $\{\bullet_1 < \bullet_2 > \bullet_3\}$. Then, $\mathbf{Con}(\mathbb{P})$ consists of the six elements: $\{\bullet_1\}$, $\{\bullet_2\}$, $\{\bullet_3\}$, $\{\bullet_1, \bullet_2\}$, $\{\bullet_2, \bullet_3\}$, and $\{\bullet_1, \bullet_2, \bullet_3\}$. All of these are also intervals of $\{\bullet_1 < \bullet_2 > \bullet_3\}$, i.e. $\mathbf{Int}(\mathbb{P}) = \mathbf{Con}(\mathbb{P})$ (Definition 2.2). In general, $\mathbf{Int}(\mathbb{P})$ is a subposet of $\mathbf{Con}(\mathbb{P})$.

Definition 2.14. The **generalized rank invariant** of $M : \mathbb{P} \rightarrow \mathbf{vec}$ is the function

$$\text{rk}(M) : \mathbf{Con}(\mathbb{P}) \rightarrow \mathbb{Z}_{\geq 0}$$

which maps $I \in \mathbf{Con}(\mathbb{P})$ to the rank of the restriction $M|_I$ of M .

In fact, in order to define the generalized rank invariant, \mathbb{P} does not need to be finite [38, Section 3]. However, for this work, it suffices to consider the case when \mathbb{P} is finite.

Remark 2.15. Let $I \in \mathbf{Con}(\mathbb{P})$. For any \mathbb{P} -module M , the following hold:

- (i) By Theorem 2.13, if there exists $p \in I$ such that $M(p) = 0$, then $\text{rk}(M)(I) = 0$.
- (ii) Let $I, J \in \mathbf{Con}(\mathbb{P})$ with $J \supseteq I$. Then $\text{rk}(M)(J) \leq \text{rk}(M)(I)$, i.e. $\text{rk}(M)$ is order-reversing. This is because the canonical limit-to-colimit map $\varprojlim M|_I \rightarrow \varinjlim M|_I$ is a factor of the canonical limit-to-colimit map $\varprojlim M|_J \rightarrow \varinjlim M|_J$ [38, Proposition 3.7]. This monotonicity implies that if $\text{rk}(M)(I) = 0$, then $\text{rk}(M)(J) = 0$.

The following is a corollary of Theorem 2.13.

Proposition 2.16 ([38, Proposition 3.17]). Let $M : \mathbb{P} \rightarrow \mathbf{vec}$ be interval decomposable. Then for any $I \in \mathbf{Con}(\mathbb{P})$,

$$\mathrm{rk}(M)(I) = \sum_{\substack{J \supseteq I \\ J \in \mathbf{Int}(\mathbb{P})}} \mathrm{mult}(J, M).$$

In words, $\mathrm{rk}(M)(I)$ equals the total multiplicity of intervals J in $\mathrm{barc}(M)$ that contain I .

For any poset \mathbb{A} , let \mathbb{A}^{op} denote the **opposite poset** of \mathbb{A} , i.e. $p \leq q$ in \mathbb{A} if and only if $q \leq p$ in \mathbb{A}^{op} . By virtue of Theorem 2.10 we have:

Definition 2.17. Let \mathbb{P} be a finite connected poset. The **generalized persistence diagram** of $M : \mathbb{P} \rightarrow \mathbf{vec}$ is the unique function $\mathrm{dgm}(M) : \mathbf{Con}(\mathbb{P}) \rightarrow \mathbb{Z}$ that satisfies, for any $I \in \mathbf{Con}(\mathbb{P})$,

$$\mathrm{rk}(M)(I) = \sum_{\substack{J \supseteq I \\ J \in \mathbf{Con}(\mathbb{P})}} \mathrm{dgm}(M)(J). \quad (4)$$

In other words, $\mathrm{dgm}(M)$ is the Möbius inversion of $\mathrm{rk}(M)$ over $\mathbf{Con}^{\mathrm{op}}(\mathbb{P})$. That is, for $I \in \mathbf{Con}(\mathbb{P})$,

$$\mathrm{dgm}(M)(I) := \sum_{\substack{J \supseteq I \\ J \in \mathbf{Con}(\mathbb{P})}} \mu_{\mathbf{Con}^{\mathrm{op}}(\mathbb{P})}(J, I) \cdot \mathrm{rk}(M)(J). \quad (5)$$

The function $\mu_{\mathbf{Con}^{\mathrm{op}}(\mathbb{P})}$ has been precisely computed in [38, Section 3].

Next, we restrict the domain of $\mathrm{rk}(M)$ and $\mathrm{dgm}(M)$ to the collection $\mathbf{Int}(\mathbb{P})$ of all intervals of \mathbb{P} . For $M : \mathbb{P} \rightarrow \mathbf{vec}$, let $\mathrm{rk}_{\mathbb{I}}(M)$ denote the restriction of $\mathrm{rk}(M) : \mathbf{Con}(\mathbb{P}) \rightarrow \mathbb{Z}_{\geq 0}$ to $\mathbf{Int}(\mathbb{P})$. We consider the Möbius inversion of $\mathrm{rk}_{\mathbb{I}}(M)$ over the poset $\mathbf{Int}^{\mathrm{op}}(\mathbb{P})$. Again by virtue of Theorem 2.10 we have:

Definition 2.18. Let \mathbb{P} be a finite connected poset. The **Int-generalized persistence diagram** of $M : \mathbb{P} \rightarrow \mathbf{vec}$ is the unique function $\mathrm{dgm}_{\mathbb{I}}(M) : \mathbf{Int}(\mathbb{P}) \rightarrow \mathbb{Z}$ that satisfies, for any $I \in \mathbf{Int}(\mathbb{P})$,

$$\mathrm{rk}_{\mathbb{I}}(M)(I) = \sum_{\substack{J \supseteq I \\ J \in \mathbf{Int}(\mathbb{P})}} \mathrm{dgm}_{\mathbb{I}}(M)(J).$$

In other words, by Theorem 2.10, $\mathrm{dgm}_{\mathbb{I}}(M)$ is the Möbius inversion of $\mathrm{rk}_{\mathbb{I}}(M)$ over $\mathbf{Int}^{\mathrm{op}}(\mathbb{P})$, i.e. for $I \in \mathbf{Int}(\mathbb{P})$,

$$\mathrm{dgm}_{\mathbb{I}}(M)(I) := \sum_{\substack{J \supseteq I \\ J \in \mathbf{Int}(\mathbb{P})}} \mu_{\mathbf{Int}^{\mathrm{op}}(\mathbb{P})}(J, I) \cdot \mathrm{rk}_{\mathbb{I}}(M)(J). \quad (6)$$

Recall from Example 2.11 that, for $m \in \mathbb{Z}_{\geq 0}$, $[m]$ is defined as the set $\{0 < 1 < \dots < m\}$.

Remark 2.19. In Definition 2.18, let \mathbb{P} be the finite product poset $[m] \times [n]$ for any $m, n \in \mathbb{N} \cup \{0\}$. Then, $\mathrm{dgm}_{\mathbb{I}}(M)$ is equivalent to the *interval decomposable approximation* $\delta^{\mathrm{tot}}(M)$ given in [2]; this is a direct corollary of Theorem 2.13. The Möbius function $\mu_{\mathbf{Int}^{\mathrm{op}}([m] \times [n])}$ has been precisely computed in [2], which leads to Theorem 2.22 below.

Although we do not require $M : \mathbb{P} \rightarrow \mathbf{vec}$ to be interval decomposable in order to define $\mathrm{dgm}(M)$ or $\mathrm{dgm}_{\mathbb{I}}(M)$, these two diagrams generalize the notion of barcode (Definition 2.3):

Theorem 2.20. *Let $M : \mathbb{P} \rightarrow \mathbf{vec}$ be interval decomposable. Then we have:*

$$\mathrm{dgm}(M)(I) = \begin{cases} \mathrm{mult}(I, M) & I \in \mathbf{Int}(\mathbb{P}) \\ 0 & I \in \mathbf{Con}(\mathbb{P}) \setminus \mathbf{Int}(\mathbb{P}), \text{ and} \end{cases} \quad (7)$$

$$\mathrm{dgm}_{\mathbb{1}}(M)(I) = \mathrm{mult}(I, M) \text{ for all } I \in \mathbf{Int}(\mathbb{P}). \quad (8)$$

The equality given in Equation (7) was first proved in [38, Theorem 3.14], but we include a proof here for completeness.

Proof. By Proposition 2.16, we have that $\mathrm{rk}(M)(I) = \sum_{\substack{J \supseteq I \\ J \in \mathbf{Int}(\mathbb{P})}} \mathrm{mult}(I, M)$. By the uniqueness of $\mathrm{dgm}(M)$ in Definition 2.18, $\mathrm{dgm}(M)(I) = \mathrm{mult}(I, M)$ for all $I \in \mathbf{Int}(\mathbb{P})$ and $\mathrm{dgm}(M)(I) = 0$ for $I \in \mathbf{Con}(\mathbb{P}) \setminus \mathbf{Int}(\mathbb{P})$. By a similar argument, we have that $\mathrm{dgm}_{\mathbb{1}}(M)(I) = \mathrm{mult}(I, M)$ for all $I \in \mathbf{Int}(\mathbb{P})$. \square

In the restricted case when $\mathbb{P} = [m] \times [n]$, the equality in Equation (8) has been also independently proved in [2, Theorem 5.10].

Theorem 2.20 implies that both $\mathrm{dgm}(M)$ and $\mathrm{dgm}_{\mathbb{1}}(M)$ are able to completely determine the isomorphism type of an interval decomposable persistence module M (which also implies that each of $\mathrm{rk}(M)$ and $\mathrm{rk}_{\mathbb{1}}(M)$ is strong enough to determine the isomorphism type of M). However, in general, the generalized persistence diagram $\mathrm{dgm}(M)$ is more discriminative than the \mathbf{Int} -generalized persistence diagram $\mathrm{dgm}_{\mathbb{1}}(M)$; see Example A.2 in the appendix.

In Section 3, the case when \mathbb{P} is a zigzag poset of length 3 will be useful.

Example 2.21 ([38, Section 3.2,2]). Assume that \mathbb{P} is any zigzag poset of length 3, i.e. $\bullet_1 \leftrightarrow \bullet_2 \leftrightarrow \bullet_3$ where \leftrightarrow stands for either \leq or \geq . Then, $\mathrm{dgm}(M)$ is computed as follows:

$$\begin{aligned} \mathrm{dgm}(M)(\{\bullet_1\}) &= \mathrm{rk}(M)(\{\bullet_1\}) - \mathrm{rk}(M)(\{\bullet_1, \bullet_2\}), \\ \mathrm{dgm}(M)(\{\bullet_2\}) &= \mathrm{rk}(M)(\{\bullet_2\}) - \mathrm{rk}(M)(\{\bullet_1, \bullet_2\}) - \mathrm{rk}(M)(\{\bullet_2, \bullet_3\}) + \mathrm{rk}(M)(\{\bullet_1, \bullet_2, \bullet_3\}), \\ \mathrm{dgm}(M)(\{\bullet_3\}) &= \mathrm{rk}(M)(\{\bullet_3\}) - \mathrm{rk}(M)(\{\bullet_2, \bullet_3\}), \\ \mathrm{dgm}(M)(\{\bullet_1, \bullet_2\}) &= \mathrm{rk}(M)(\{\bullet_1, \bullet_2\}) - \mathrm{rk}(M)(\{\bullet_1, \bullet_2, \bullet_3\}), \\ \mathrm{dgm}(M)(\{\bullet_2, \bullet_3\}) &= \mathrm{rk}(M)(\{\bullet_2, \bullet_3\}) - \mathrm{rk}(M)(\{\bullet_1, \bullet_2, \bullet_3\}), \\ \mathrm{dgm}(M)(\{\bullet_1, \bullet_2, \bullet_3\}) &= \mathrm{rk}(M)(\{\bullet_1, \bullet_2, \bullet_3\}). \end{aligned}$$

Since M is a zigzag module, it is interval decomposable (Theorem 2.4). Thus, we have $\mathrm{dgm}(M)(I) = \mathrm{mult}(I, M)$ for $I \in \mathbf{Con}(\mathbb{P})$, the multiplicity of I in $\mathrm{barc}(M)$. Since $\mathbf{Con}(\mathbb{P}) = \mathbf{Int}(\mathbb{P})$, each $\mathrm{dgm}(M)$ above can be replaced by $\mathrm{dgm}_{\mathbb{1}}(M)$.

2.5 Int-Generalized persistence diagram of an $([m] \times [n])$ -module.

In this section we review a formula of the \mathbf{Int} -generalized persistence diagram of an $([m] \times [n])$ -module for any fixed integers $m, n \geq 0$.

Let us consider the poset $\mathbf{Int}([m] \times [n])$. Then, given any two distinct $I, J \in \mathbf{Int}([m] \times [n])$, we say that J covers I if $J \supseteq I$ and there is no interval K such that $J \supseteq K \supseteq I$. For $I \in \mathbf{Int}([m] \times [n])$,

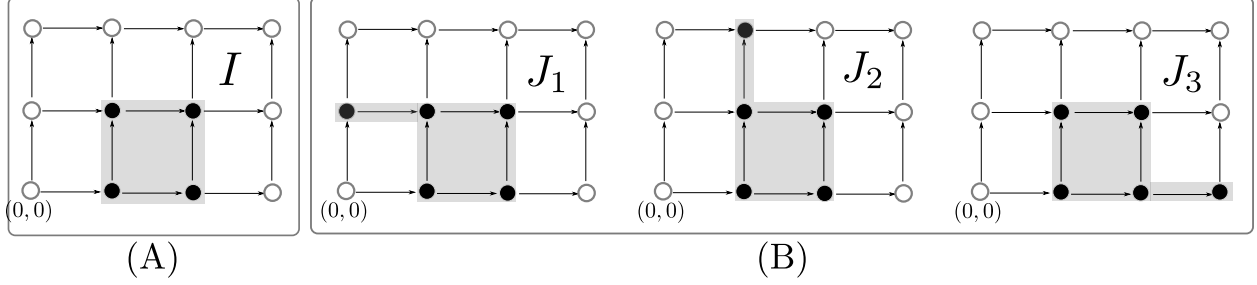


Figure 3: Illustrations for Example 2.23.

let us define $\text{cov}(I)$ as the collection of all $J \in \mathbf{Int}([m] \times [n])$ that cover I . Given any nonempty $S \subseteq \mathbf{Int}([m] \times [n])$, by $\bigvee S$, we denote the smallest interval J that contains all $I \in S$.

The following theorem is established by invoking Remark 2.19 and finding an explicit formula for the Möbius function $\mu_{\mathbf{Int}^{\text{op}}(\mathbb{P})}$ that appears in Equation (6) with $\mathbb{P} = [m] \times [n]$.

Theorem 2.22 ([2, Theorem 5.3]). *For any $([m] \times [n])$ -module M ,*

$$\text{dgm}_{\mathbb{Q}}(M)(I) = \text{rk}_{\mathbb{Q}}(M)(I) + \sum_{\substack{S \subseteq \text{cov}(I) \\ S \neq \emptyset}} (-1)^{|S|} \text{rk}_{\mathbb{Q}}(M)(\bigvee S). \quad (9)$$

Example 2.23. Let $I \in \mathbf{Int}([3] \times [2])$ depicted as in Fig. 3 (A). Note that $\text{cov}(I) = \{J_1, J_2, J_3\}$ where J_1, J_2 and J_3 are depicted as in Fig. 3 (B). For any $([3] \times [2])$ -module M , we have:

$$\begin{aligned} \text{dgm}_{\mathbb{Q}}(M)(I) &= \text{rk}_{\mathbb{Q}}(M)(I) - \sum_{i=1}^3 \text{rk}_{\mathbb{Q}}(M)(J_i) + \sum_{i \neq j} \text{rk}_{\mathbb{Q}}(M)(\bigvee \{J_i, J_j\}) \\ &\quad - \text{rk}_{\mathbb{Q}}(M)(\bigvee \{J_1, J_2, J_3\}), \end{aligned}$$

where $\bigvee \{J_1, J_2\} = J_1 \cup J_2 \cup \{(0, 2)\}$, $\bigvee \{J_1, J_3\} = J_1 \cup J_3$, $\bigvee \{J_2, J_3\} = J_2 \cup J_3$, and $\bigvee \{J_1, J_2, J_3\} = J_1 \cup J_2 \cup J_3 \cup \{(0, 2)\}$.

The following remark will be useful in the next section.

Remark 2.24. Let M be an $([m] \times [n])$ -module and let $I \in \mathbf{Int}([m] \times [n])$. By Remark 2.15 (ii) and Equation (9), if $\text{rk}_{\mathbb{Q}}(M)(I) = 0$, then $\text{dgm}_{\mathbb{Q}}(M)(I) = 0$.

3 Extracting the bigraded Betti numbers from the generalized persistence diagram

In this section we aim at establishing Theorem 3.5, as a generalization of Theorem 2.9.

Let M be a finitely generated \mathbb{Z}^2 -module.⁴ We may assume that $M(p) = 0$ for $p \neq (0, 0)$. Then, all algebraic information of M can be recovered from the restricted module $M' := M|_{[m] \times [n]}$ for some large enough positive integers m and n . We will show that the generalized persistence diagram of M' determines the bigraded Betti numbers of M .

⁴Main results in this section (which are Proposition 3.2 and Theorem 3.5) also hold for finitely presented \mathbb{R}^2 -modules upto rescaling parameters [16, 42].

Definition 3.1. A given \mathbb{Z}^2 -module M is said to be **encoded** by $M' : [m] \times [n] \rightarrow \mathbf{vec}$ if the following hold:

- If $p \in \mathbb{Z}^2$ is *not* greater than equal to $(0, 0)$, then $M(p) = 0$.
- For $(0, 0) \leq p$ in \mathbb{Z}^2 , we have that $M(p) = M'(q)$ where q is the maximal element of $[m] \times [n]$ such that $q \leq p$ (we write $q = \lfloor p \rfloor_{m,n}$ in this case).
- For $(0, 0) \leq p_1 \leq p_2$ in \mathbb{Z}^2 , the map $\varphi_M(p_1, p_2)$ is equal to $\varphi_{M'}(\lfloor p_1 \rfloor_{m,n} \leq \lfloor p_2 \rfloor_{m,n})$.

The \mathbb{Z}^2 -module M described above is clearly finitely generated and its restriction $M|_{[m] \times [n]}$ coincides with M' . The following proposition is the key to obtain Theorem 3.5. Let $e_1 := (1, 0)$ and $e_2 := (0, 1)$ in \mathbb{Z}^2 .

Proposition 3.2. Assume that a \mathbb{Z}^2 -module M is encoded by $M' : [m] \times [n] \rightarrow \mathbf{vec}$. Then, $\text{dgm}(M')$ determines the bigraded Betti numbers of M via the following formulas:

For $p \notin [m+1] \times [n+1]$, we have $\beta_j(M)(p) = 0$, $j = 0, 1, 2$. For $p \in [m+1] \times [n+1]$, we have:

$$\beta_j(M)(p) = \begin{cases} \sum_{\substack{J \ni p \\ J \not\ni p-e_1, p-e_2}} \text{dgm}(M')(J) & j = 0 \\ \sum_{\substack{J \ni p-e_1 \\ J \not\ni p-e_1-e_2, p-e_2, p}} \text{dgm}(M')(J) + \sum_{\substack{J \ni p-e_2 \\ J \not\ni p-e_1-e_2, p-e_1, p}} \text{dgm}(M')(J) + \sum_{\substack{J \ni p-e_1-e_2, p-e_1, p-e_2 \\ J \not\ni p}} \text{dgm}(M')(J) + 2 \sum_{\substack{J \ni p-e_1, p-e_2 \\ J \not\ni p-e_1-e_2, p}} \text{dgm}(M')(J) & j = 1 \\ \sum_{\substack{J \ni p-e_1, p-e_2, p \\ J \not\ni p-e_1-e_2}} \text{dgm}(M')(J) - \sum_{\substack{J \ni p-e_1-e_2, p-e_1, p \\ J \not\ni p-e_2}} \text{dgm}(M')(J) - \sum_{\substack{J \ni p-e_1-e_2, p-e_2, p \\ J \not\ni p-e_1}} \text{dgm}(M')(J) & j = 1 \\ \sum_{\substack{J \ni p-e_1-e_2 \\ J \not\ni p-e_1, p-e_2}} \text{dgm}(M')(J) & j = 2, \end{cases} \quad (10)$$

where each sum is taken over $J \in \mathbf{Con}([m] \times [n])$.⁵ Moreover, each $\text{dgm}(M')$ above can be replaced by $\text{dgm}_{\square}(M')$ where each sum is taken over $J \in \mathbf{Int}([m] \times [n])$.

One direct consequence of this proposition is that if $p \in [m+1] \times [n+1]$ is outside of $[m] \times [n]$, then $\beta_0(M)(p) = 0$: no $J \in \mathbf{Con}([m] \times [n])$ can include p and thus the sum $\sum_{\substack{J \ni p \\ J \not\ni p-e_1, p-e_2}} \text{dgm}(M')(J)$ is zero.

We defer the proof of Proposition 3.2 to the end of this section.

Remark 3.3. In Proposition 3.2, the equation for $\beta_1(M)$ with respect to $\text{dgm}_{\square}(M')$ can be further simplified by removing the fourth, sixth, and seventh sums, i.e.

$$\beta_1(M) = \sum_{\substack{J \ni p-e_1 \\ J \not\ni p-e_1-e_2, p-e_2, p}} \text{dgm}_{\square}(M')(J) + \sum_{\substack{J \ni p-e_2 \\ J \not\ni p-e_1-e_2, p-e_1, p}} \text{dgm}_{\square}(M')(J) + \sum_{\substack{J \ni p-e_1-e_2, p-e_1, p-e_2 \\ J \not\ni p}} \text{dgm}_{\square}(M')(J) + \sum_{\substack{J \ni p-e_1, p-e_2, p \\ J \not\ni p-e_1-e_2}} \text{dgm}_{\square}(M')(J).$$

⁵For example, when $j = 0$, the sum is taken over every $J \in \mathbf{Con}([m] \times [n])$ that contains p and does not contain $p - e_1$ and $p - e_2$.

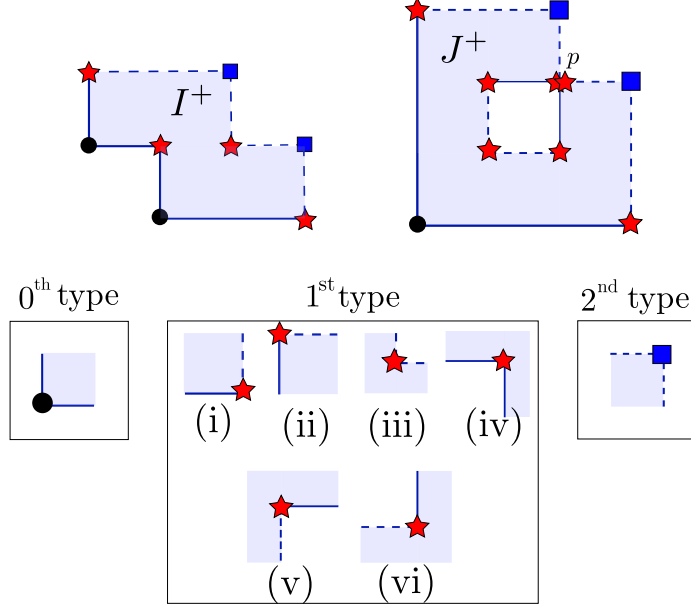


Figure 4: The three different types of corner points in $I^+ \subset \mathbb{R}^2$ and $J^+ \subset \mathbb{R}^2$. Note that two different 1^{st} type corner points of J are located at p . See Definition A.1 for a rigorous description of each of the three types of corner points.

This is because the connected sets J (Definition 2.2 (iii)) over which the fourth, sixth, and seventh sums are taken cannot be intervals of $[m] \times [n]$ (those connected sets J cannot satisfy Definition 2.2 (ii)). Similarly, if $\text{dgm}(M')(J) = 0$ for all non-intervals $J \in \mathbf{Con}([m] \times [n])$, then the fourth, sixth, and seventh sums can be eliminated in the equation for $\beta_1(M)$.⁶

By virtue of Remark 3.3, Proposition 3.2 admits a simple pictorial interpretation which generalizes Remark 2.8 (ii) and Theorem 2.9. To state this interpretation, we introduce the following notation.

Notation 3.4. Given any $I \in \mathbf{Con}(\mathbb{Z}^2)$, let $I^+ \subset \mathbb{R}^2$ be the corresponding region (cf. equation (2)). Then I^+ admits the 3 types of corner points depicted in Figure 4. For $j = 0, 1, 2$, we define functions $\tau_j(I^+) : \mathbb{Z}^2 \rightarrow \{0, 1, 2\}$ as follows: for $j = 0, 2$, let $\tau_j(I^+)(p) := 1$ if p is a j^{th} type corner point of I^+ , and 0 otherwise. For $j = 1$, let

$$\tau_1(I^+)(p) := \begin{cases} 2, & p \text{ is a } 1^{\text{st}}\text{-type corner point of } I^+ \text{ with multiplicity 2} \\ 1, & p \text{ is a } 1^{\text{st}}\text{-type corner point of } I^+ \text{ with multiplicity 1} \\ 0, & \text{otherwise.} \end{cases}$$

Our main theorem below says that the bigraded Betti numbers of a given \mathbb{Z}^2 -module M encoded by an $([m] \times [n])$ -module M' can be read off from the corner points of the elements in either of

$$\{I^+ \subset \mathbb{R}^2 : \text{dgm}(M')(I) \neq 0\} \text{ and } \{I^+ \subset \mathbb{R}^2 : \text{dgm}_1(M')(I) \neq 0\}.$$

⁶We remark that, in general, there can exist a non-interval $J \in \mathbf{Con}([m] \times [n])$ where $\text{dgm}(M')(J) \neq 0$; see Example A.2 in the appendix.

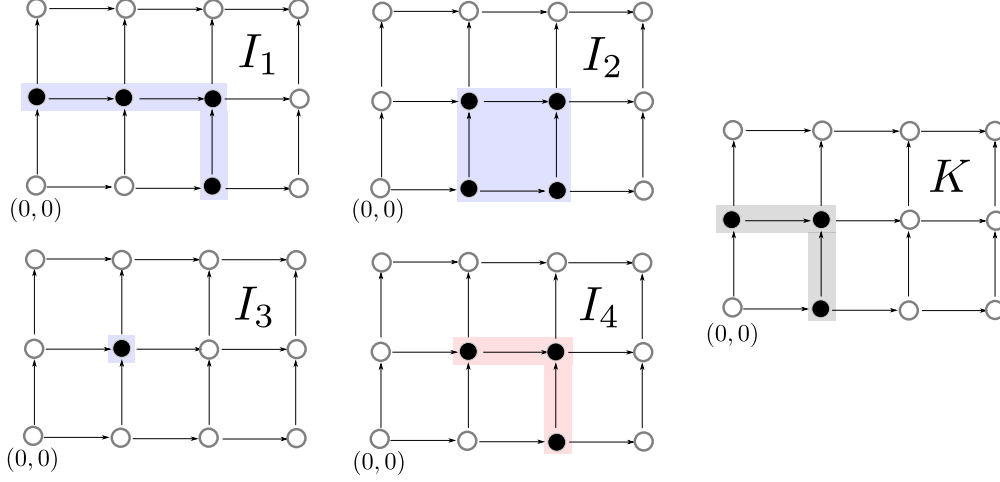


Figure 5: $I_1, I_2, I_3,$ and I_4 are the intervals corresponding to Fig. 1 (B').

Theorem 3.5. Assume that a \mathbb{Z}^2 -module M is encoded by $M' : [m] \times [n] \rightarrow \mathbf{vec}$. Then, for every $j = 0, 1, 2$ and for every $p \in \mathbb{Z}^2$, we have

$$\beta_j(M)(p) = \sum_{I \in \mathbf{Con}([m] \times [n])} \text{dgm}(M')(I) \times \tau_j(I^+)(p). \quad (11)$$

Also we have:

$$\beta_j(M)(p) = \sum_{I \in \mathbf{Int}([m] \times [n])} \text{dgm}_{\mathbb{q}}(M')(I) \times \tau_j(I^+)(p). \quad (12)$$

Notice that, by Theorem 2.20, the theorem above is a generalization of Theorem 2.9. We prove Theorem 3.5 at the end of this section.

Example 3.6. Recall that $[3] \times [2] = \{0, 1, 2, 3\} \times \{0, 1, 2\} \subset \mathbb{Z}^2$ and assume that a \mathbb{Z}^2 -module N is encoded by the module $N' : [3] \times [2] \rightarrow \mathbf{vec}$ depicted in Fig. 1 (A). Then, Fig. 1 (B') and (C') are explained as follows:

(B') For $I_1, I_2, I_3, I_4 \in \mathbf{Int}([3] \times [2])$ in Fig. 5, we have that $\text{dgm}_{\mathbb{q}}(N')(I_i) = 1$ for $i = 1, 2, 3$ and $\text{dgm}_{\mathbb{q}}(N')(I_4) = -1$ and $\text{dgm}_{\mathbb{q}}(N')(J) = 0$ for the other $J \in \mathbf{Int}([3] \times [2])$ (more details are provided after this example).

(C') For $i = 1, 2, 3, 4$, expand I_i to its corresponding region I_i^+ in \mathbb{R}^2 (cf. Definition 2.7). The corner points of each I_i^+ are marked according to their types as described in Fig. 4. By Theorem 3.5, for each $p \in \mathbb{Z}^2$ and $j = 0, 1, 2$, $\beta_j(N)(p)$ is equal to the number of black dots, red stars, and blue squares at p respectively, where the corner points of the red interval I_4^+ negatively contribute to the counting. The net sum is illustrated in Fig. 6.

Details about Example 3.6. For $I_1, I_2, I_3, I_4 \in \mathbf{Int}([3] \times [2])$ in Fig. 5, we show that $\text{dgm}_{\mathbb{q}}(N')(I_i) = 1$ for $i = 1, 2, 3$ and $\text{dgm}_{\mathbb{q}}(N')(I_4) = -1$ and $\text{dgm}_{\mathbb{q}}(N')(J) = 0$ for the other $J \in \mathbf{Int}([3] \times [2])$.

- (i) If $J \in \mathbf{Int}([2] \times [3])$ contains any point $p \in [3] \times [2]$ such that $N'(p) = 0$, then $\text{dgm}_{\mathbb{q}}(N')(J) = 0$ by Remarks 2.15 (i) and 2.24.

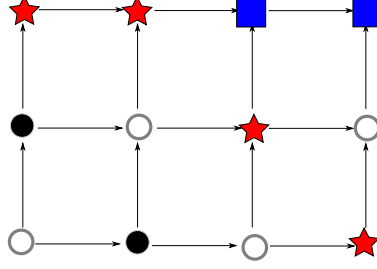


Figure 6: For $N : \mathbb{Z}^2 \rightarrow \mathbf{vec}$ in Example 3.6, a black dot at p indicates $\beta_0(N)(p) = 1$, a red star at p indicates $\beta_1(N)(p) = 1$, and a blue square at p indicates $\beta_2(N)(p) = 1$. For all other j and p , $\beta_j(N)(p) = 0$.

- (ii) Consider $K := \{(0, 1), (1, 1), (1, 0)\} \in \mathbf{Int}([3] \times [2])$ that is depicted in Fig. 5. We claim that for all $J \supseteq K$, $\mathrm{dgm}_{\mathbb{q}}(N')(J) = 0$: By Remarks 2.15 (ii) and 2.24, it suffices to show that $\mathrm{rk}_{\mathbb{q}}(N')(K) = 0$. This follows from Theorem 2.13 and the fact that the zigzag module $N'|_K$ does not admit a summand that is isomorphic to the interval module $V_K : K \rightarrow \mathbf{vec}$. An alternative way to prove $\mathrm{rk}_{\mathbb{q}}(N')(K) = 0$ is to show that $\varprojlim N'|_K$ is trivial: Note that $\varprojlim N'|_K \cong (L, \{\pi_p\}_{p \in K})$, where

$$L = \{(v_1, v_2, v_3) \in N'(0, 1) \oplus N'(1, 1) \oplus N'(1, 0) : \varphi_{N'}((0, 1), (1, 1))(v_1) = v_2 = \varphi_{N'}((1, 0), (1, 1))(v_3)\}$$

and $\pi_p : L \rightarrow N'(p)$ are the canonical projections for $p \in K$. Then, we have:

$$\begin{aligned} L &= \{(x_1, (x_2, x_3), x_4) \in k \oplus (k^2) \oplus k : x_1 = x_2, x_3 = 0, x_2 = x_3 = x_4\} \\ &= \{(0, (0, 0), 0)\}. \end{aligned}$$

- (iii) We claim that $\mathrm{dgm}_{\mathbb{q}}(N')(I_1) = \mathrm{dgm}_{\mathbb{q}}(N')(I_2) = 1$. Fix any $i \in \{1, 2\}$. By invoking Theorem 2.13, one can check that $\mathrm{rk}_{\mathbb{q}}(N')(I_i) = \mathrm{rk}(N'|_{I_i}) = 1$. Let us observe that any interval $J \supseteq I_i$ must contain either K or a point $p \in [3] \times [2]$ such that $N'(p) = 0$. Hence, by Remarks 2.15 (i) and (ii), we have that $\mathrm{rk}_{\mathbb{q}}(N')(J) = 0$. Therefore, by Theorem 2.22, we have:

$$\mathrm{dgm}_{\mathbb{q}}(N')(I_i) = \mathrm{rk}_{\mathbb{q}}(N')(I_i) + \sum_{\substack{S \subseteq \mathrm{cov}(I_i) \\ S \neq \emptyset}} (-1)^{|S|} \mathrm{rk}_{\mathbb{q}}(N')(\bigvee S) = 1 + \sum_{\substack{S \subseteq \mathrm{cov}(I_i) \\ S \neq \emptyset}} (-1)^{|S|} \cdot 0 = 1.$$

Similarly, one can compute $\mathrm{dgm}_{\mathbb{q}}(N')(I_3) = 1$, $\mathrm{dgm}_{\mathbb{q}}(N')(I_4) = -1$, and $\mathrm{dgm}_{\mathbb{q}}(N')(L) = 0$ for any $L \in \mathbf{Int}([3] \times [2])$ that has not been considered so far. \square

Proofs of Proposition 3.2 and Theorem 3.5. Lemma 3.7 below will be used in the proof of Proposition 3.2. Let M be any finitely generated \mathbb{Z}^2 -module. For any $p \in \mathbb{Z}^2$, consider the subposet $\{p - e_1 \leq p \leq p - e_2\}$ where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The restriction of M to $\{p - e_1 \leq p \leq p - e_2\}$ is a zigzag module and thus admits a barcode (Theorem 2.4). Let n_p be the multiplicity of $\{p\}$ in the barcode of $M|_{\{p - e_1 \leq p \leq p - e_2\}}$. Similarly, we also consider the subposet $\{p + e_1 \geq p \geq p + e_2\}$ and define m_p to be the multiplicity of $\{p\}$ in the barcode of $M|_{\{p + e_1 \geq p \geq p + e_2\}}$.

Lemma 3.7 ([49, 50]). Given any finitely generated \mathbb{Z}^2 -module M , for every $p \in \mathbb{Z}^2$, we have:

$$\beta_j(M)(p) = \begin{cases} n_p & j = 0 \\ n_p - \dim(M(p)) + \dim(M(p - e_1)) + \dim(M(p - e_2)) \\ \quad - \dim(M(p - e_1 - e_2)) + m_{p-e_1-e_2} & j = 1 \\ m_{p-e_1-e_2} & j = 2. \end{cases} \quad (13)$$

A combinatorial proof of this lemma can be found in [49, Corollary 2.3]. This lemma can be also proved by utilizing machinery from commutative algebra as follows (see [30, Section 2A.3] for details): A finitely generated 2-parameter persistence module M can equivalently be considered as an \mathbb{N}^2 graded module over $k[x_1, x_2]$. The bigraded Betti numbers of M can be defined using tensor products, after which Lemma 3.7 follows by tensoring M with the Koszul complex on x_1 and x_2 .

Proof of Proposition 3.2. We consider the case $j = 1$, as the other cases are similar. By Lemma 3.7,

$$\beta_1(M)(p) = n_p - \dim(M(p)) + \dim(M(p - e_1)) + \dim(M(p - e_2)) - \dim(M(p - e_1 - e_2)) + m_{p-e_1-e_2}. \quad (14)$$

Let $p \in \mathbb{Z}^2$ where $p \notin [m] \times [n]$. Then, we claim that $\beta_1(M)(p) = 0$. This fact can be shown by checking that $0 = n_p = m_{p-e_1-e_2}$ and

$$0 = -\dim(M(p)) + \dim(M(p - e_1)) + \dim(M(p - e_2)) - \dim(M(p - e_1 - e_2)).$$

Next, let $p \in [m] \times [n]$. We will now find a formula for each term in the right-hand side (RHS) of Equation (14) in terms of the generalized rank invariant of $M|_{[m] \times [n]} = M'$. Notice that for every $q \in [m] \times [n]$,

$$\dim(M(q)) = \text{rk}(M')(\{q\}). \quad (15)$$

Next, consider $M|_{\{p-e_1 \leq p \geq p-e_2\}}$, which is a zigzag module and thus it is interval decomposable (Theorem 2.4). Recall that n_p is the multiplicity of $\{p\}$ in the barcode of $M|_{\{p-e_1 \leq p \geq p-e_2\}}$. From Example 2.21, we know that:

$$n_p = \text{rk}(M')(\{p\}) - \text{rk}(M')(\{p - e_1 \leq p\}) - \text{rk}(M')(\{p \geq p - e_2\}) + \text{rk}(M')(\{p - e_1 \leq p \geq p - e_2\}). \quad (16)$$

Similarly, we have:

$$m_{p-e_1-e_2} = \text{rk}(M')(\{p - e_1 - e_2\}) - \text{rk}(M')(\{p - e_1 - e_2 \leq p - e_1\}) \\ - \text{rk}(M')(\{p - e_1 - e_2 \leq p - e_2\}) + \text{rk}(M')(\{p - e_1 \geq p - e_1 - e_2 \leq p - e_2\}). \quad (17)$$

Combining equations (14), (15), (16), (17) yields:

$$\beta_1(M)(p) = -\text{rk}(M')(\{p - e_1 \leq p\}) - \text{rk}(M')(\{p \geq p - e_2\}) + \text{rk}(M')(\{p - e_1 \leq p \geq p - e_2\}) \\ + \text{rk}(M')(\{p - e_1\}) + \text{rk}(M')(\{p - e_2\}) - \text{rk}(M')(\{p - e_1 - e_2 \leq p - e_1\}) \\ - \text{rk}(M')(\{p - e_1 - e_2 \leq p - e_2\}) + \text{rk}(M')(\{p - e_1 \geq p - e_1 - e_2 \leq p - e_2\}).$$

Since $p \in [m] \times [n]$, by invoking Equation (4), we obtain:

$$\begin{aligned}
\beta_1(M)(p) = & - \sum_{J \ni p-e_1, p} \text{dgm}(M')(J) - \sum_{J \ni p-e_2, p} \text{dgm}(M')(J) + \sum_{J \ni p-e_1, p-e_2, p} \text{dgm}(M')(J) \\
& + \sum_{J \ni p-e_1} \text{dgm}(M')(J) + \sum_{J \ni p-e_2} \text{dgm}(M')(J) - \sum_{J \ni p-e_1-e_2, p-e_1} \text{dgm}(M')(J) \\
& - \sum_{J \ni p-e_1-e_2, p-e_2} \text{dgm}(M')(J) + \sum_{J \ni p-e_1-e_2, p-e_1, p-e_2} \text{dgm}(M')(J)
\end{aligned} \tag{18}$$

Let $J \in \mathbf{Con}([m] \times [n])$. The multiplicity of $\text{dgm}(M')(J)$ in the RHS of Equation (18) is fully determined by the intersection of J and the four-point set $\{p-e_1-e_2, p-e_1, p-e_2, p\}$. For example, if $p-e_1, p-e_1-e_2 \in J$ and $p-e_2, p \notin J$, then $\text{dgm}(M')(J)$ occurs only in the fourth and sixth sums, and has an overall multiplicity of zero in the RHS of Equation (18). For another example, if $p-e_1 \in J$ and $p-e_1-e_2, p-e_2, p \notin J$, then $\text{dgm}(M')(J)$ occurs only in the fourth summand, which yields the first sum of the RHS in Equation (19) below. Considering all possible 2^4 combinations of the intersection of J and the four-point set $\{p-e_1-e_2, p-e_1, p-e_2, p\}$ yields

$$\begin{aligned}
\beta_1(M)(p) = & \sum_{\substack{J \ni p-e_1 \\ J \not\ni p-e_1-e_2, p-e_2, p}} \text{dgm}(M')(J) + \sum_{\substack{J \ni p-e_2 \\ J \not\ni p-e_1-e_2, p-e_1, p}} \text{dgm}(M')(J) + \sum_{\substack{J \ni p-e_1-e_2, p-e_1, p-e_2 \\ J \not\ni p}} \text{dgm}(M')(J) \\
& + 2 \sum_{\substack{J \ni p-e_1, p-e_2 \\ J \not\ni p-e_1-e_2, p}} \text{dgm}(M')(J) + \sum_{\substack{J \ni p-e_1, p-e_2, p \\ J \not\ni p-e_1-e_2}} \text{dgm}(M')(J) - \sum_{\substack{J \ni p-e_1-e_2, p-e_1, p \\ J \not\ni p-e_2}} \text{dgm}(M')(J) \\
& - \sum_{\substack{J \ni p-e_1-e_2, p-e_2, p \\ J \not\ni p-e_1}} \text{dgm}(M')(J),
\end{aligned} \tag{19}$$

as claimed. \square

Proof of Theorem 3.5. We only prove equation (12) with $j = 1$, as the other cases are similar. By Remark 3.3,

$$\beta_1(M)(p) = \sum_{\substack{J \ni p-e_1 \\ J \not\ni p-e_1-e_2, p-e_2, p}} \text{dgm}_{\square}(M')(J) + \sum_{\substack{J \ni p-e_2 \\ J \not\ni p-e_1-e_2, p-e_1, p}} \text{dgm}_{\square}(M')(J) + \sum_{\substack{J \ni p-e_1-e_2, p-e_1, p-e_2 \\ J \not\ni p}} \text{dgm}_{\square}(M')(J) + \sum_{\substack{J \ni p-e_1, p-e_2, p \\ J \not\ni p-e_1-e_2}} \text{dgm}_{\square}(M')(J), \tag{20}$$

where each sum is taken over $J \in \mathbf{Int}(\mathbb{Z}^2)$.

Furthermore, for $I \in \mathbf{Int}(\mathbb{Z}^2)$, observe that $\tau_1(I^+)(p) = 1$ if and only if one of the following is true about I : (i) $p-e_1 \in I$ and $p-e_1-e_2, p-e_2, p \notin I$, (ii) $p-e_2 \in I$ and $p-e_1-e_2, p-e_1, p \notin I$, (iii) $p-e_1-e_2, p-e_1, p-e_2 \in I$ and $p \notin I$, or (iv) $p-e_1, p-e_2, p \in I$ and $p-e_1-e_2 \notin I$. Otherwise, $\tau_1(I^+)(p) = 0$. These four cases (i), (ii), (iii), and (iv) correspond to the four sums on the RHS of Equation (20) in order, and also correspond to the 1st corner types (i), (ii), (iii), (iv) given in Figure 4. Therefore:

$$\beta_1(M)(p) = \sum_{I \in \mathbf{Int}([m] \times [n])} \text{dgm}_{\square}(M')(I) \times \tau_1(I^+)(p).$$

\square

4 Conclusions

The formula in Theorem 3.5 for computing the bigraded Betti numbers reinforces the fact that the (\mathbf{Int} -)generalized persistence diagram and the *interval decomposable approximation* by Asashiba

et al. (Remark 2.19) are a proxy for the “barcode” of M in a novel way. Some open questions follow.

- (i) Note that when M is a finitely generated \mathbb{Z}^2 -module, $\text{dgm}(M)$ can recover $\text{dgm}_1(M)$ by construction while $\text{dgm}_1(M)$ may not be able to recover $\text{dgm}(M)$. However, if M is interval decomposable, then both $\text{dgm}(M)$ and $\text{dgm}_1(M)$ are equivalent to the barcode of M by Theorem 2.20. Are there other settings in which $\text{dgm}_1(M)$ can recover $\text{dgm}(M)$?
- (ii) The d -graded Betti numbers for d -parameter persistence modules are defined in a similar way that the bigraded Betti numbers are defined for 2-parameter persistence modules. When $d > 2$, can the (**Int**-)generalized persistence diagram recover the d -graded Betti numbers, extending Theorem 3.5?

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A Appendix

Types of corner points. We define the 0th, 1st and 2nd type corner points that are depicted in Fig. 4. Given any $A \subset \mathbb{R}^2$, let $\mathbb{1}_A : \mathbb{R}^2 \rightarrow \{0, 1\}$ be the indicator function of A , i.e. $\mathbb{1}_A(p) = 1$ if $p \in A$ and zero otherwise.

Definition A.1. Let $I \in \mathbf{Con}(\mathbb{Z}^2)$ and let $I^+ := \bigcup_{(p_1, p_2) \in I} [p_1, p_1 + 1) \times [p_2, p_2 + 1) \subset \mathbb{R}^2$. Fix $p \in \mathbb{R}^2$. This p is a **0-th type corner point** of I^+ if

$$\mathbb{1}_{I^+}(p) = 1, \quad \lim_{\varepsilon \rightarrow 0^+} \mathbb{1}_{I^+}(p - (\varepsilon, 0)) = \lim_{\varepsilon \rightarrow 0^+} \mathbb{1}_{I^+}(p - (0, \varepsilon)) = \lim_{\varepsilon \rightarrow 0^+} \mathbb{1}_{I^+}(p - (\varepsilon, \varepsilon)) = 0.$$

The point p is a **1-st type corner point with multiplicity k** ($k = 1, 2$) of I^+ if one of the following two conditions holds:

- (i) ($k = 1$) Either the following is evaluated to be -1

$$\mathbb{1}_{I^+}(p) - \lim_{\varepsilon \rightarrow 0^+} \mathbb{1}_{I^+}(p - (\varepsilon, 0)) - \lim_{\varepsilon \rightarrow 0^+} \mathbb{1}_{I^+}(p - (0, \varepsilon)) + \lim_{\varepsilon \rightarrow 0^+} \mathbb{1}_{I^+}(p - (\varepsilon, \varepsilon)) \quad (21)$$

(cf. (i)-(iv) in the panel corresponding to the 1st type in Figure 4), or the following holds:

$$\mathbb{1}_{I^+}(p) = \lim_{\varepsilon \rightarrow 0^+} \mathbb{1}_{I^+}(p - (\varepsilon, \varepsilon)) = 1 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0^+} \mathbb{1}_{I^+}(p - (\varepsilon, 0)) \neq \lim_{\varepsilon \rightarrow 0^+} \mathbb{1}_{I^+}(p - (0, \varepsilon))$$

(cf. (v) and (vi) in the panel corresponding to the 1st type in Figure 4).

- (ii) ($k = 2$) The formula given in (21) is evaluated to be -2 (cf. the point p in Figure 4).

The point p is a **2-nd type corner point** of I^+ if

$$\mathbb{1}_{I^+}(p) = \lim_{\varepsilon \rightarrow 0^+} \mathbb{1}_{I^+}(p - (\varepsilon, 0)) = \lim_{\varepsilon \rightarrow 0^+} \mathbb{1}_{I^+}(p - (0, \varepsilon)) = 0, \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0^+} \mathbb{1}_{I^+}(p - (\varepsilon, \varepsilon)) = 1.$$

Definition A.1 is closely related to the *differential* of an interval introduced in [25].

The generalized persistence diagram is more discriminative than the Int-generalized persistence diagram. We provide a pair of persistence modules that are distinguished by their generalized rank invariants (and hence by their generalized persistence diagrams) but have the same **Int**-generalized rank invariant (and hence the same **Int**-generalized persistence diagram).

Example A.2. Let $M, N : [2]^2 \rightarrow \mathbf{vec}$ and $J \in \mathbf{Con}([2]^2)$ be defined as in Figure 7. Then, $\text{rk}(M)(J) = 1$ whereas $\text{rk}(N)(J) = 0$; this directly follows from Theorem 2.13. Note also that for all $I \supsetneq J$ in $\mathbf{Con}([2]^2)$, we have that $\text{rk}(M)(I) = \text{rk}(N)(I) = 0$. This implies, by equations (3), (5), (6), that $\text{dgm}(M)(J) = 1 \neq 0 = \text{dgm}(N)(J)$.

Invoking Theorem 2.13 again, one can check that $\text{rk}(M)(I) = \text{rk}(N)(I)$ for all $I \in \mathbf{Int}([2]^2)$, i.e. $\text{rk}_{\mathbb{1}}(M) = \text{rk}_{\mathbb{1}}(N)$ and thus $\text{dgm}_{\mathbb{1}}(M) = \text{dgm}_{\mathbb{1}}(N)$.

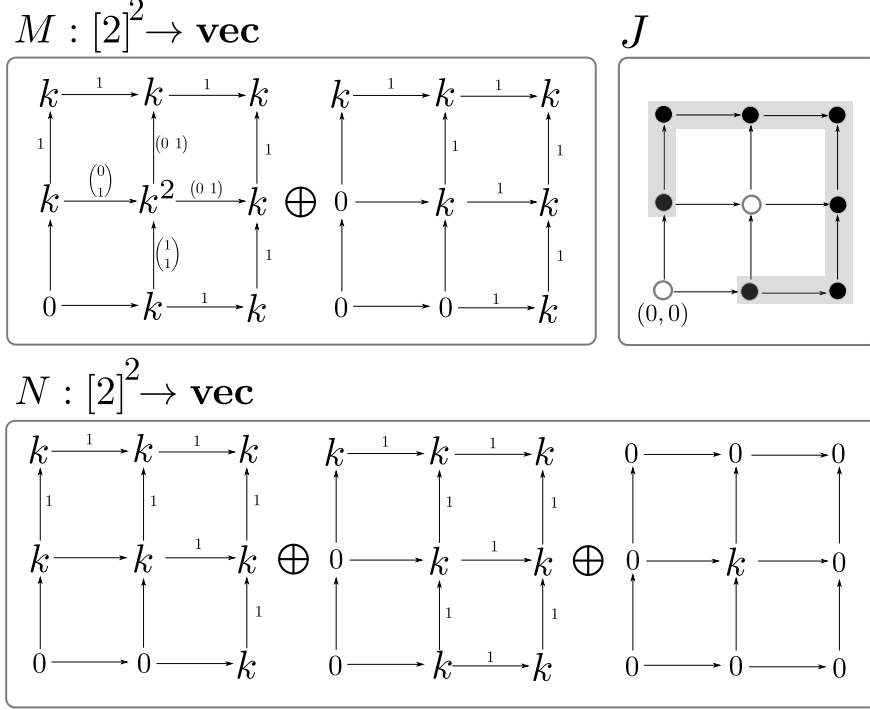


Figure 7: An illustration for Example A.2 illustrating that, in general, $\text{dgm}(M)$ is a stronger invariant than $\text{dgm}_{\mathfrak{q}}(M)$

Limits and colimits. We recall the notions of limit and colimit [44, Chapter V]. In what follows, I stands for a small category, i.e. I has a set of objects and a set of morphisms. Let \mathcal{C} be any category.

Definition A.3 (Cone). Let $F : I \rightarrow \mathcal{C}$ be a functor. A *cone* over F is a pair $(L, (\pi_x)_{x \in \text{ob}(I)})$ consisting of an object L in \mathcal{C} and a collection $(\pi_x)_{x \in \text{ob}(I)}$ of morphisms $\pi_x : L \rightarrow F(x)$ that commute with the arrows in the diagram of F , i.e. if $g : x \rightarrow y$ is a morphism in I , then $\pi_y = F(g) \circ \pi_x$ in \mathcal{C} . Equivalently, the diagram below commutes:

$$\begin{array}{ccc}
 F(x) & \xrightarrow{F(g)} & F(y) \\
 \swarrow \pi_x & & \searrow \pi_y \\
 & L &
 \end{array}$$

A limit of $F : I \rightarrow \mathcal{C}$ is a terminal object in the collection of all cones over F :

Definition A.4 (Limit). Let $F : I \rightarrow \mathcal{C}$ be a functor. A *limit* of F is a cone over F , denoted by $(\varprojlim F, (\pi_x)_{x \in \text{ob}(I)})$ or simply $\varprojlim F$, with the following *terminal* property: If there is another cone $(L', (\pi'_x)_{x \in \text{ob}(I)})$ of F , then there is a *unique* morphism $u : L' \rightarrow \varprojlim F$ such that $\pi'_x = \pi_x \circ u$ for all $x \in \text{ob}(I)$.

It is possible that a functor does not have a limit at all. However, if a functor does have a limit then the terminal property of the limit guarantees its uniqueness up to isomorphism. For

this reason, we sometimes refer to a limit as *the* limit of a functor. When I is a finite category and $\mathcal{C} = \mathbf{vec}$, any functor $F : I \rightarrow \mathbf{vec}$ admits a limit in \mathbf{vec} .

Cocones and colimits are defined in a dual manner:

Definition A.5 (Cocone). Let $F : I \rightarrow \mathcal{C}$ be a functor. A *cocone* over F is a pair $(C, (i_x)_{x \in \text{ob}(I)})$ consisting of an object C in \mathcal{C} and a collection $(i_x)_{x \in \text{ob}(I)}$ of morphisms $i_x : F(x) \rightarrow C$ that commute with the arrows in the diagram of F , i.e. if $g : x \rightarrow y$ is a morphism in I , then $i_x = i_y \circ F(g)$ in \mathcal{C} , i.e. the diagram below commutes.

$$\begin{array}{ccc}
 & C & \\
 i_x \nearrow & & \nwarrow i_y \\
 F(x) & \xrightarrow{F(g)} & F(y)
 \end{array}$$

A colimit of a functor $F : I \rightarrow \mathcal{C}$ is an initial object in the collection of cocones over F :

Definition A.6 (Colimit). Let $F : I \rightarrow \mathcal{C}$ be a functor. A *colimit* of F is a cocone, denoted by $(\varinjlim F, (i_x)_{x \in \text{ob}(I)})$ or simply $\varinjlim F$, with the following *initial* property: If there is another cocone $(C', (i'_x)_{x \in \text{ob}(I)})$ of F , then there is a *unique* morphism $u : \varinjlim F \rightarrow C'$ such that $i'_x = u \circ i_x$ for all $x \in \text{ob}(I)$.

It is possible that a functor does not have a colimit at all. However, if a functor does have a colimit then the initial property of the colimit guarantees its uniqueness up to isomorphism. For this reason, we sometimes refer to a colimit as *the* colimit of a functor. When I is a finite category and $\mathcal{C} = \mathbf{vec}$, any functor $F : I \rightarrow \mathbf{vec}$ admits a colimit in \mathbf{vec} .