Compositional Controller Synthesis for Interconnected Stochastic Systems with Markovian Switching*

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ABSTRACT. In this work, we propose a compositional scheme for the safety controller synthesis of interconnected discrete-time stochastic systems with Markovian switching signals. Our proposed approach is based on a notion of so-called *control storage certificates* computed for individual subsystems, by leveraging which, one can synthesize state-feedback controllers for interconnected systems to enforce safety specifications over finite time horizons. To do so, we employ a sum-of-squares (SOS) optimization approach to search for *multiple* storage certificates of each switching subsystem while synthesizing its corresponding safety controller. We then utilize dissipativity theory to compositionally construct barrier certificates for interconnected systems based on storage certificates of individual subsystems. The proposed dissipativity-type compositional conditions can leverage the structure of the interconnection topology and be fulfilled independently of the number or gains of subsystems. We eventually employ the constructed barrier certificate and quantify upper bounds on the probability that the interconnected system reaches certain unsafe regions in a finite time horizon. We apply our results to a room temperature network of 200 rooms with Markovian switching signals while accepting multiple storage certificates. We compositionally synthesize safety controllers to maintain the temperature of each room in a comfort zone for a bounded time horizon.

1. INTRODUCTION

Large-scale stochastic switching systems have been considering as an important modeling framework to describe a broad range of safety-critical applications including (air) traffic networks, autonomous vehicles, delivery drones, vehicle platooning, robotic networks, etc., to name a few. Synthesizing automated controllers for this type of complex systems to enforce some high-level logic properties, *e.g.,* linear temporal logic (LTL) formulae [1], is inherently challenging, mainly due to (i) the stochasticity inside dynamics, (ii) high dimensionality, and (iii) the random behavior of switching signals.

To deal with the encountered complexity, (in)finite abstractions have been introduced as a promising tool in the controller synthesis procedure (*e.g.,* [2, 3, 4, 5, 6]). However, the proposed abstractions-based techniques rely on the discretization of state and input sets, and consequently, they suffer severely from the *curse of*

dimensionality especially when dealing with large-scale dynamical systems. Then compositional approaches for the construction of (in)finite abstractions for complex systems based on abstractions of smaller subsystems have been proposed in the relevant literature (*e.g.,* [7, 8, 9, 10, 11, 12, 13, 14]).

The proposed compositional scheme in the setting of (in)finite abstractions can mitigate the effects of the state-explosion problem; however, the curse of dimensionality may still exist in the level of subsystems given the range of state and input sets and their quantization parameters. Then *control barrier certificates* as a *discretization-free* approach have been proposed for formal verification and analysis of complex dynamical systems. More precisely, control barrier certificates are Lyapunov-like functions defined over the state space of the system to enforce some conditions on both the function itself and the one-step transition of the system. Starting from a given set of initial conditions, an appropriate level set of a barrier certificate can separate an unsafe region from all system trajectories. Consequently, the existence of such a function provides a formal probabilistic certificate for the safety of the system.

There have been some results, proposed in the past decade, on the verification and controller synthesis of stochastic systems via control barrier certificates. Discretization-free techniques based on barrier certificates for (stochastic) hybrid systems are initially proposed in [15, 16, 17]. Stochastic safety verification using barrier certificates for switched diffusion processes and stochastic hybrid systems is, respectively, proposed in [18] and [19]. Temporal logic controller synthesis of stochastic systems via control barrier certificates is proposed in [20]. A controller synthesis framework for stochastic systems based on control barrier functions is presented in [21]. Verification and control for finite-time safety of stochastic systems via barrier functions are discussed in [22]. Verification of uncertain partially-observable Markov decision processes (POMDPs) with uncertain transition and/or observation probabilities using barrier certificates is discussed in [23]. Compositional construction of safety controllers for networks of continuous-space POMDPs using control barrier certificates is recently proposed in [24].

An introduction and overview of relevant work on control barrier functions and their application to verify and enforce safety properties in the context of safety-critical controllers are presented in [25]. Compositional construction of control barrier certificates for stochastic discrete- and continuous-time systems is, respectively, presented in [26] and [27]. In comparison with the current work, the proposed results in [26, 27] only handle stochastic *control* systems, while we enlarge here the class of systems to stochastic *switching* system with Markovian switching signals. Compositional construction of control barrier certificates for discrete-time stochastic switched systems is studied in [28]. Although the proposed results in [28] are about stochastic switched systems, their switching signals are deterministic that are served as control inputs. In comparison, switching signals in our work are not control inputs and are randomly changing. As a result, the controller synthesis problem here is more challenging compared to [28] since it deals with two different sources of adversary inputs:

(i) disturbances as the effects of other subsystems, and (ii) switching signals which are randomly changing in a finite set of modes.

Our main contribution in this work is to propose, for the first time, a compositional framework based on dissipativity theory for the safety controller synthesis of interconnected stochastic switching systems with *Markovian switching* signals admitting *multiple* storage certificates. To this end, we first introduce notions of stochastic storage and barrier certificates for, respectively, stochastic switching subsystems and interconnected systems. We employ sum-of-squares (SOS) optimization problems to search for control storage certificates of each individual subsystem while synthesizing its corresponding safety controller. We then compositionally construct barrier certificates for interconnected systems based on storage certificates of individual subsystems by leveraging some dissipativity-type compositional conditions. We show that the proposed compositionality conditions can utilize the structure of the interconnection topology and be satisfied independently of the number or gains of subsystems (cf. Remark 4.2 and the case study). Given the constructed barrier certificate, we quantify upper bounds on the probability that the interconnected system reaches certain unsafe regions in finite time horizons. Proofs of all statements are omitted due to space limitations.

2. Discrete-Time Stochastic Switching Systems

2.1. **Preliminaries.** The probability space, in this work, is considered as $(\Omega, \mathcal{F}_\Omega, \mathbb{P}_\Omega)$, where Ω is the sample space, $\mathcal{F}_\Omega$ is a sigma-algebra on Ω consisting subsets of Ω as events, and $\mathbb{P}_\Omega$ is the probability measure that assigns probability to those events. Random variables are assumed to be measurable functions of the form $X : (\Omega, \mathcal{F}_\Omega) \to (S_X, \mathcal{F}_X)$. Any random variable X induces a probability measure on $(S_X, \mathcal{F}_X)$ as $Prob\{A\} = \mathbb{P}_\Omega\{X^{-1}(A)\}$ for any $A \in \mathcal{F}_X$. The topological space S is a Borel space if it is homeomorphic to a Borel subset of a Polish space, *i.e.*, a separable and completely metrizable space. The Borel sigma-algebra generated from Borel space S is denoted by $\mathbb{B}(S)$ and the map $f : S \to Y$ is measurable whenever it is Borel measurable.

2.2. **Notation.** We employ $\mathbb{R}, \mathbb{R}_{>0}$, and $\mathbb{R}_{\geq 0}$ to denote the set of real, positive and non-negative real numbers, respectively, while $\mathbb{R}^n$ represents a real space of the dimension n. The set of non-negative and positive integers are denoted by $\mathbb{N} := \{0, 1, \dots\}$ and $\mathbb{N}_{\geq 1} = \{1, 2, \dots\}$, respectively. Given N vectors $x_i \in \mathbb{R}^{n_i}$, we use $x = [x_1; \dots; x_N]$ to denote the corresponding column vector of the dimension $\sum_i n_i$. The identity matrix in $\mathbb{R}^{n \times n}$ is denoted by $\mathbb{I}_n$. Given functions $f_i : X_i \to Y_i$, for any $i \in \{1, \dots, N\}$, their Cartesian product $\prod_{i=1}^N f_i : \prod_{i=1}^N X_i \to \prod_{i=1}^N Y_i$ is defined as $(\prod_{i=1}^N f_i)(x_1, \dots, x_N) = [f_1(x_1); \dots; f_N(x_N)]$.

2.3. **Discrete-Time Stochastic Switching Systems.** In this work, we consider discrete-time stochastic switching systems as formalized in the following definition.

Definition 2.1. *A discrete-time stochastic switching system (dt-SS) is a tuple*

$$\Sigma = (X, U, W, P, \mathcal{P}, \varsigma, F, Y, h), \tag{2.1}$$

where:

- $X \subseteq \mathbb{R}^n$ *is a Borel space as a state space of the system;*
- $U \subseteq \mathbb{R}^{\bar{m}}$ *is a Borel space as an input space of the system;*
- $W \subseteq \mathbb{R}^{\bar{p}}$ *is a Borel space as a disturbance space of the system;*
- $P = \{1, \ldots, m\}$ *is the finite set of modes;*
- $\mathcal{P}$ *is a subset of $\mathcal{S}(\mathbb{N}, P)$ which denotes the set of functions from $\mathbb{N}$ to P;*
- ς *is a sequence of independent and identically distributed (i.i.d.) random variables from the sample space Ω to the measurable space $(\mathcal{V}_\varsigma, \mathcal{F}_\varsigma)$, i.e.,*

$$\varsigma := \{\varsigma(k) : (\Omega, \mathcal{F}_\Omega) \to (\mathcal{V}_\varsigma, \mathcal{F}_\varsigma), \ k \in \mathbb{N}\};$$

- $F = \{f_1, \ldots, f_m\}$ *is a collection of vector fields indexed by p. For all $p \in P$, the map $f_p : X \times U \times W \times \mathcal{V}_\varsigma \to X$ is a measurable function characterizing the state evolution of the system;*
- $Y \subseteq \mathbb{R}^q$ *is a Borel space as an output space of the system;*
- $h : X \to Y$ *is a measurable function that maps a state $x \in X$ to its disturbance output $y = h(x)$.*

We associate sets $\mathcal{U}$ and $\mathcal{W}$ to, respectively, sets U and W as collections of input and disturbance sequences $\{\nu(k) : \Omega \to U, \ k \in \mathbb{N}\}$ and $\{w(k) : \Omega \to W, \ k \in \mathbb{N}\}$. Both $\nu(k)$ and $w(k)$ are independent from the random variable $\varsigma(z)$ for all $k, z \in \mathbb{N}$ and $z \geq k$. The state evolution of dt-SS Σ for a given initial state $x(0) \in X$, an input sequence $\{\nu(k) : \Omega \to U, \ k \in \mathbb{N}\}$, a disturbance sequence $\{w(k) : \Omega \to W, \ k \in \mathbb{N}\}$, and a switching signal $\mathbf{p}(k) : \mathbb{N} \to P$, is characterized by:

$$\Sigma : \begin{cases} x(k+1) = f_{\mathbf{p}(k)}(x(k), \nu(k), w(k), \varsigma(k)), \\ y(k) = h(x(k)), \end{cases} \tag{2.2}$$

for any $k \in \mathbb{N}$. For any $p \in P$, we use Σ_p to refer to system (2.2) with a constant switching signal $\mathbf{p}(k) = p$ for all $k \in \mathbb{N}$. For a given initial state $a \in X$, $\nu(\cdot) \in \mathcal{U}$ and $w(\cdot) \in \mathcal{W}$, and $\mathbf{p}(k) : \mathbb{N} \to P$, a random sequence $x_{a\nu w}^p : \Omega \times \mathbb{N} \to X$ denotes the *solution process* of Σ under the influence of the input ν, the disturbance w, and the switching signal p, started from the initial state a.

Given the dt-SS in (2.2) with $p, p' \in P$, the transition probability between modes is described using the following Markovian switching:

$$\mathbb{P}\Big\{\mathbf{p}(k+1) = p' \,\big|\, \mathbf{p}(k) = p\Big\} = \pi_{pp'}, \tag{2.3}$$

where $\pi_{pp'} \geq 0$ for any $p, p' \in P$, and $\sum_{p'=1}^{m} \pi_{pp'} = 1$. Accordingly, the transition probability matrix is defined by

$$\pi = \begin{bmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1m} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2m} \\ \vdots & & \ddots & \\ \pi_{m1} & \pi_{m2} & \cdots & \pi_{mm} \end{bmatrix}.$$

The Markovian switching in (2.3) implies that the switching between different modes is governed by a discrete-time Markov chain with the transition probability matrix π [29]. We assume that the controller has an access to switching modes which is a standard assumption in the relevant literature [30]. In particular, it is supposed that there is a mode detection device which is capable of identifying the system mode in real time so that the controller can switch to the matched mode.

In this work, we are ultimately interested in studying interconnected dt-SS without disturbances that results from the interconnection of dt-SS having disturbance signals. Hence, we consider dt-SS in (2.2) as individual subsystems and provide a formal definition of interconnected dt-SS as the following.

Definition 2.2. *Suppose we are given $N \in \mathbb{N}_{\geq 1}$ switching subsystems $\Sigma_i = (X_i, U_i, W_i, P_i, \mathcal{P}_i, \varsigma_i, F_i, Y_i, h_i), i \in \{1, \ldots, N\}$, where $X_i \in \mathbb{R}^{n_i}$, $U_i \in \mathbb{R}^{\bar{m}_i}$, $W_i \in \mathbb{R}^{\bar{p}_i}$, $Y_i \in \mathbb{R}^{q_i}$ along with a matrix $\mathcal{M}$ that describes the coupling between the subsystems, with a well-posed interconnection constraint $\mathcal{M} \prod_{i=1}^{N} Y_i \subseteq \mathcal{M} \prod_{i=1}^{N} W_i$. Then the interconnection of subsystems $\Sigma_i, i \in \{1, \ldots, N\}$, denoted by $\mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$, is dt-SS $\Sigma = (X, U, P, \mathcal{P}, \varsigma, F)$ such that $X := \prod_{i=1}^{N} X_i$, $U := \prod_{i=1}^{N} U_i$, $P := \prod_{i=1}^{N} P_i$, $\mathcal{P} := \prod_{i=1}^{N} \mathcal{P}_i$, $\varsigma = [\varsigma_1; \ldots; \varsigma_N]$, and $F := \prod_{i=1}^{N} F_i$, with the following interconnection constraint:*

$$[w_1; \ldots; w_N] = \mathcal{M}[h_1; \ldots; h_N]. \tag{2.4}$$

Such an interconnected dt-SS can be represented by

$$\Sigma \colon x(k+1) = f_{\mathbf{p}(k)}(x(k), \nu(k), \varsigma(k)), \tag{2.5}$$

with $f_p : X \times U \times \mathcal{V}_\varsigma \to X$.

Remark 2.3. *Note that the role of h in (2.2) is mainly for the sake of interconnecting subsystems as appeared in (2.4). Accordingly, the full-state information is assumed to be available for the interconnected system (i.e., its output map is identity) for the sake of controller synthesis.*

In the next section, we present notions of control barrier and storage certificates for, respectively, interconnected dt-SS and individual subsystems.

3. Control Barrier and Storage Certificates

Here, we first present the notion of control barrier certificates for interconnected dt-SS without disturbance signals.

Definition 3.1. *Consider an interconnected system $\Sigma = (X, U, P, \mathcal{P}, \varsigma, F)$, and $X_0, X_u \subseteq X$ as, respectively, initial and unsafe sets of the interconnected system. A function $\mathcal{B} \colon X \times P \to \mathbb{R}_{\geq 0}$ is called a stochastic control barrier certificate (CBC) for Σ if, for all $p \in P$,*

$$\mathcal{B}(x, p) \leq \gamma, \qquad \forall x \in X_0, \tag{3.1}$$

$$\mathcal{B}(x, p) \geq \lambda, \qquad \forall x \in X_u, \tag{3.2}$$

and $\forall x := x(k) \in X$, $\exists \nu := \nu(k) \in U$ such that

$$\sum_{p'=1}^{M} \pi_{pp'} \mathbb{E}\Big[\mathcal{B}(x(k+1), p') \,\big|\, x, \nu, p\Big] \leq \kappa \mathcal{B}(x, p) + \psi, \tag{3.3}$$

for some $0 < \kappa < 1$, $\gamma, \lambda, \psi \in \mathbb{R}_{\geq 0}$, with $\lambda > \gamma$, and $M = \Pi_{i=1}^{N} m_i$, where m_i is the number of modes for each subsystem Σ_i in (2.2).

The next theorem, adapted from [31], shows the usefulness of CBC to quantify an upper bound on the probability that the interconnected system reaches certain unsafe regions in finite time horizons.

Theorem 3.2. *Let $\Sigma = (X, U, P, \mathcal{P}, \varsigma, F)$ be an interconnected dt-SS without disturbance signals. Suppose $\mathcal{B}(x, p)$ is a CBC for Σ as in Definition 3.1. Then the probability that the solution process of Σ starting from any initial state $a \in X_0$ and any initial mode p_0 reaches X_u under $\nu(\cdot)$ within the time step $k \in [0, \mathcal{T}]$ is quantified as*

$$\mathbb{P}_\nu^a\Big\{x^p(k) \in X_u \text{ for some } 0 \leq k \leq \mathcal{T} \,\big|\, x^p(0) = a, p_0\Big\} \leq \delta, \tag{3.4}$$

with

$$\delta = \begin{cases} 1 - (1 - \frac{\gamma}{\lambda})(1 - \frac{\psi}{\lambda})^{\mathcal{T}}, & \text{if } \lambda \geq \frac{\psi}{1-\kappa}, \\ (\frac{\gamma}{\lambda})\kappa^{\mathcal{T}} + (\frac{\psi}{(1-\kappa)\lambda})(1 - \kappa^{\mathcal{T}}), & \text{if } \lambda < \frac{\psi}{1-\kappa}. \end{cases}$$

In general, finding barrier certificates for large-scale dt-SS as in Definition 3.1 is computationally very expensive mainly due to the high dimension of the system. Accordingly, we present in the following definition a notion of stochastic control *storage* certificates (CSC) for individual subsystems with disturbance signals as in (2.2). We then propose in Section 4 our compositional approach based on dissipativity theory to construct a CBC of the interconnected dt-SS based on CSC of individual subsystems.

Definition 3.3. *Consider a dt-SS Σ_p, and sets $X_0, X_u \subseteq X$ as initial and unsafe sets of the subsystem, respectively. A function $\mathcal{B}_p : X \to \mathbb{R}_{\geq 0}$ is called a stochastic control storage certificate (CSC) for Σ_p if there exist $0 < \kappa_p < 1$, $\gamma_p, \lambda_p, \psi_p \in \mathbb{R}_{\geq 0}$, and a symmetric matrix $\mathcal{X}$ with conformal block partitions $\mathcal{X}^{l\bar{l}}$, $l, \bar{l} \in \{1, 2\}$, such that for all $p \in P$,*

$$\mathcal{B}_p(x) \leq \gamma_p, \qquad \forall x \in X_0, \tag{3.5}$$

$$\mathcal{B}_p(x) \geq \lambda_p, \qquad \forall x \in X_u, \tag{3.6}$$

and $\forall x := x(k) \in X$, $\exists \nu := \nu(k) \in U$ such that $\forall w := w(k) \in W$,

$$\sum_{p'=1}^{m} \pi_{pp'} \mathbb{E}\Big[\mathcal{B}_{p'}(x(k+1)) \,\big|\, x, \nu, w, p\Big] \leq \kappa_p \mathcal{B}_p(x) + \psi_p + \begin{bmatrix} w \\ h(x) \end{bmatrix}^{\top} \overbrace{\begin{bmatrix} \mathcal{X}^{11} & \mathcal{X}^{12} \\ \mathcal{X}^{21} & \mathcal{X}^{22} \end{bmatrix}}^{\mathcal{X}:=} \begin{bmatrix} w \\ h(x) \end{bmatrix}. \tag{3.7}$$

Remark 3.4. *The stochastic storage certificate satisfying conditions (3.5)-(3.7) is useless on its own to ensure the safety of the interconnected system. More precisely, stochastic storage certificates are some appropriate tools to construct overall control barrier certificates if some compositionality conditions are satisfied (cf. (4.1),(4.2)). The safety of the system can then be verified via Theorem 3.2 only via the constructed* control barrier certificate.

In the next section, we analyze networks of stochastic switching subsystems and show under which conditions one can construct a CBC of an interconnected system using CSC of subsystems.

4. COMPOSITIONAL CONSTRUCTION OF CBC

Here, we provide a compositional framework to obtain a CBC of an interconnected dt-SS Σ based on CSC of subsystems Σ_i. Let us assume that there exists a CSC $\mathcal{B}_{ip_i}$ as in Definition 3.3 for each subsystem $\Sigma_i, i \in \{1, \ldots, N\}$, with $0 < \kappa_{ip_i} < 1$, $\gamma_{ip_i}, \lambda_{ip_i}, \psi_{ip_i} \in \mathbb{R}_{\geq 0}$, and a symmetric matrix $\mathcal{X}_{ip_i}$ with conformal block partitions $\mathcal{X}_{ip_i}^{l\bar{l}}$, $l, \bar{l} \in \{1, 2\}$. We now propose the following theorem, as the main compositionality result of the work, to provide sufficient conditions for the construction of a CBC of the interconnected dt-SS Σ based on CSC of subsystems $\Sigma_i, i \in \{1, \ldots, N\}$.

Theorem 4.1. *Consider an interconnected dt-SS $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ composed of N switching subsystems $\Sigma_i, i \in \{1, \ldots, N\}$, with an interconnection matrix $\mathcal{M}$. Suppose that each switching mode Σ_{ip_i} admits a CSC*

$\mathcal{B}_{ip_i}$ as defined in Definition 3.3 with initial and unsafe sets X_{0_i} and X_{u_i}, respectively. If

$$\begin{bmatrix} \mathcal{M} \\ \mathbb{I}_{\tilde{q}} \end{bmatrix}^{\top} \mathcal{X}_{cmp} \begin{bmatrix} \mathcal{M} \\ \mathbb{I}_{\tilde{q}} \end{bmatrix} \preceq 0, \tag{4.1}$$

$$\sum_{i=1}^{N} \mu_i \min_{p_i \in P_i} \{\lambda_{ip_i}\} > \sum_{i=1}^{N} \mu_i \max_{p_i \in P_i} \{\gamma_{ip_i}\}, \tag{4.2}$$

where $\mu_i > 0, i \in \{1, \ldots, N\}$, then

$$\mathcal{B}(x, p) := \sum_{i=1}^{N} \mu_i \mathcal{B}_{ip_i}(x_i) \tag{4.3}$$

with $p = [p_1; \ldots; p_N], p_i \in \{1, \ldots, m_i\}$, is a CBC for the interconnected system $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$, where

$$\mathcal{X}_{cmp} := \begin{bmatrix} \mathcal{X}_1^{11} & & & \mathcal{X}_1^{12} & & \\ & \ddots & & & \ddots & \\ & & \mathcal{X}_N^{11} & & & \mathcal{X}_N^{12} \\ \mathcal{X}_1^{21} & & & \mathcal{X}_1^{22} & & \\ & \ddots & & & \ddots & \\ & & \mathcal{X}_N^{21} & & & \mathcal{X}_N^{22} \end{bmatrix}, \tag{4.4}$$

and $\tilde{q} = \sum_{i=1}^{N} q_i$, with q_i being the dimension of the output of Σ_i. In addition,

$$\gamma := \sum_{i=1}^{N} \mu_i \max_{p_i \in P_i} \{\gamma_{ip_i}\}, \ \ \lambda := \sum_{i=1}^{N} \mu_i \min_{p_i \in P_i} \{\lambda_{ip_i}\},$$

$$\kappa s := \max \Big\{ \sum_{i=1}^{N} \mu_i \max_{p_i \in P_i} \{\kappa_{ip_i}\} \mathcal{B}_{ip_i}(s_i) \big| s_i \geq 0, \sum_{i=1}^{N} \mu_i \mathcal{B}_{ip_i} = s \Big\},$$

$$\psi := \sum_{i=1}^{N} \mu_i \max_{p_i \in P_i} \{\psi_{ip_i}\}.$$

Remark 4.2. *Note that condition (4.1) is a well-established LMI, discussed in [32], as a compositional stability condition based on the dissipativity theory. As shown in [32], this condition holds independently of the number of subsystems in many physical applications with particular interconnection topologies (cf. the case study). Condition (4.2) is not also restrictive since constants μ_i in (4.3) play a significant role in rescaling CSC for subsystems while normalizing the effect of gains of other subsystems.*

5. Computation of CSC and Safety Controller

In this section, we provide an approach based on sum-of-squares (SOS) optimization to compute a CSC and synthesize its corresponding controller for Σ_p. In order to utilize an SOS optimization, we raise the following assumption.

Assumption 1. *Suppose that Σ_p has a continuous state set $X \subseteq \mathbb{R}^n$ and continuous input and disturbance sets $U \subseteq \mathbb{R}^{\bar{m}}$ and $W \subseteq \mathbb{R}^{\bar{p}}$. Moreover, $f_p : X \times U \times W \times \mathcal{V}_\varsigma \to X$ is a polynomial function of the state x and input and disturbance ν, w.*

Under Assumption 1, the following lemma reformulates conditions (3.5)-(3.7) as an SOS optimization problem.

Lemma 5.1. *Suppose Assumption 1 holds and sets X_0, X_u, X, U, W can be defined by vectors of polynomial inequalities as $X_0 = \{x \in \mathbb{R}^n \mid g_0(x) \geq 0\}$, $X_u = \{x \in \mathbb{R}^n \mid g_u(x) \geq 0\}$, $X = \{x \in \mathbb{R}^n \mid g(x) \geq 0\}$, $U = \{\nu \in \mathbb{R}^{\bar{m}} \mid g_\nu(\nu) \geq 0\}$, and $W = \{w \in \mathbb{R}^{\bar{p}} \mid g_w(w) \geq 0\}$, where the inequalities are defined element-wise. Suppose there exists a sum-of-square polynomial $\mathcal{B}_p(x)$, constants $0 < \kappa_p < 1$, $\gamma_p, \lambda_p, \psi_p \in \mathbb{R}_{\geq 0}$, a symmetric matrix $\mathcal{X}_p$ with conformal block partitions $\mathcal{X}_p^{l\bar{l}}$, $l, \bar{l} \in \{1, 2\}$, polynomials $l_{\nu_j}(x)$ corresponding to the jth input in $\nu = (\nu_1, \nu_2, \ldots, \nu_{\bar{m}}) \in U \subseteq \mathbb{R}^{\bar{m}}$, and vectors of sum-of-squares polynomials $l_{0_p}(x)$, $l_{u_p}(x)$, $l_p(x, \nu, w)$, $l_{\nu_p}(x, \nu, w)$, and $l_{w_p}(x, \nu, w)$ of appropriate dimensions such that the following expressions are sum-of-squares polynomials, for all $p \in P$:*

$$-\mathcal{B}_p(x) - l_{0_p}^\top(x) g_0(x) + \gamma_p \tag{5.1}$$

$$\mathcal{B}_p(x) - l_{u_p}^\top(x) g_u(x) - \lambda_p \tag{5.2}$$

$$-\sum_{p'=1}^{m} \pi_{pp'} \mathbb{E}\Big[\mathcal{B}_p'(x(k+1)) \,\big|\, x, \nu, w, p\Big] + \kappa_p \mathcal{B}_p(x) + \psi_p + \begin{bmatrix} w \\ h(x) \end{bmatrix}^\top \begin{bmatrix} \mathcal{X}^{11} & \mathcal{X}^{12} \\ \mathcal{X}^{21} & \mathcal{X}^{22} \end{bmatrix} \begin{bmatrix} w \\ h(x) \end{bmatrix}$$

$$-\sum_{j=1}^{\bar{m}} (\nu_j - l_{\nu_{j_p}}(x)) - l_p^\top(x, \nu, w) g(x) - l_{w_p}^\top(x, \nu, w) g_w(w) - l_{\nu_p}^\top(x, \nu, w) g_\nu(\nu). \tag{5.3}$$

Then, $\mathcal{B}_p(x)$ satisfies conditions (3.5)-(3.7) in Definition 3.3 and $\nu = [l_{\nu_{1_p}}(x); \ldots; l_{\nu_{\bar{m}_p}}(x)]$ is the corresponding controller employed at the mode $p \in P$.

6. Case Study: Room Temperature Network

To demonstrate the effectiveness of our proposed results, we apply them to a room temperature network in a circular building containing $N = 200$ rooms. The model of this case study is borrowed from [33] by including the stochasticity in the model as an additive noise. The evolution of the temperature in the interconnected system is governed by the following stochastic switching system:

$$\Sigma : x(k+1) = Ax(k) + \beta T_h \nu(k) + \alpha T_{\mathbf{p}(k)} + R_{\mathbf{p}(k)} \varsigma(k),$$

where $A \in \mathbb{R}^{n \times n}$ is a matrix with diagonal elements of $\bar{a}_{ii} = (1 - 2\theta - \alpha - \beta\nu_i(k))$, off-diagonal elements $\bar{a}_{i,i+1} = \bar{a}_{i+1,i} = \bar{a}_{1,n} = \bar{a}_{n,1} = \theta$, $i \in \{1, \ldots, N-1\}$, and all other elements being identically zero. Parameters $\theta = 0.005$, $\alpha = 0.06$, and $\beta = 0.145$ are conduction factors, respectively, between rooms $i \pm 1$ and i, the external environment and the room i, and the heater and the room i. Outside temperatures are the same

for all rooms: $T_{\mathbf{p}(k)} = \bar{T}_{p_i}\mathbb{I}_n$ with $\bar{T}_{p_i} = \begin{cases} -15\,°\text{C}, & \text{if } p_i = 1, \\ -20\,°\text{C}, & \text{if } p_i = 2. \end{cases}$ The heater temperature is $T_h = 45\,°\text{C}$, and

$R_{\mathbf{p}(k)} = \bar{R}_{p_i}\mathbb{I}_n$ with $\bar{R}_{p_i} = \begin{cases} 0.3, & \text{if } p_i = 1, \\ 0.5, & \text{if } p_i = 2. \end{cases}$ Moreover, $x(k) = [x_1(k); \dots; x_N(k)]$, $\nu(k) = [\nu_1(k); \dots; \nu_N(k)]$,

and $\varsigma = [\varsigma_1(k); \dots; \varsigma_N(k)]$.

Now by considering individual rooms as Σ_i represented by

$$\Sigma_i : \begin{cases} x_i(k+1) = \bar{a}_{ii}x_i(k) + \beta_i T_h \nu_i(k) + \theta_i w_i(k) + \alpha_i \bar{T}_{ip_i} + \bar{R}_{ip_i}\varsigma_i(k), \\ y_i(k) = x_i(k), \end{cases}$$

one can readily verify that $\Sigma = \mathcal{I}(\Sigma_1, \dots, \Sigma_N)$, where the coupling matrix $\mathcal{M}$ is defined as $\hat{m}_{i,i+1} = \hat{m}_{i+1,i} = \hat{m}_{1,N} = \hat{m}_{N,1} = 1$, $i \in \{1, \dots, N-1\}$, and all other elements are identically zero.

Transition probability matrix for switching between two modes is given as $\pi = \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}$. In addition, the regions of interest are given as $X_i \in [1, 50], X_{0_i} \in [19.5, 20], X_{u_i} = [1, 17] \cup [23, 50], \forall i \in \{1, \dots, 200\}$. The main goal is to find a CBC for the interconnected system and its corresponding safety controller such that the temperature of rooms remains in the comfort zone $[17, 23]^{200}$. To do so, we first search for CSC and accordingly design local controllers for subsystems Σ_i. Consequently, the controller for the interconnected system Σ would be a vector such that each of its components is a controller for subsystems Σ_i.

We employ the software tool SOSTOOLS [34] and the SDP solver SeDuMi [35] to compute CSC as described in Section 5. Based on Lemma 5.1, we compute CSC of an order 4 as $\mathcal{B}_{ip_i}(x_i) = 0.00242x_i^4 - 0.091x_i^3 + 0.7696x_i^2 + 1.4935x_i + 3.1329$ and the corresponding controller $\nu_{ip_i} = -0.0121x_i + 0.8$ for $p_i = 1$, and $\mathcal{B}_{ip_i}(x_i) = 0.00191x_i^4 - 0.0718x_i^3 + 0.5998x_i^2 + 1.2424x_i + 3.2433$ together with $\nu_{ip_i} = -0.02527x_i + 1.15$ for $p_i = 2, \forall i \in \{1, \dots, 200\}$. Moreover, the corresponding constants in Definition 3.3 satisfying conditions (3.5)-(3.7) are quantified as $\gamma_{ip_i} = 0.13, \lambda_{ip_i} = 4.4, \kappa_{ip_i} = 0.91, \psi_{ip_i} = 0.001$ for $p_i = 1$, and $\gamma_{ip_i} = 0.14, \lambda_{ip_i} = 4.3, \kappa_{ip_i} = 0.92, \psi_{ip_i} = 0.0015$ for $p_i = 2, \forall i \in \{1, \dots, 200\}$, and

$$\mathcal{X}_i = \begin{bmatrix} 0.005 & 0.003 \\ 0.003 & -0.035 \end{bmatrix}. \tag{6.1}$$

We now proceed with Theorem 4.1 to construct a CBC for the interconnected system using CSC of subsystems. By selecting $\mu_i = 1, \forall i \in \{1, \dots, 200\}$, and utilizing $\mathcal{X}_i$ in (6.1), the matrix $\mathcal{X}_{cmp}$ in (4.4) is reduced to

$$\mathcal{X}_{cmp} = \begin{bmatrix} 0.005\,\mathbb{I}_{200} & 0.003\,\mathbb{I}_{200} \\ 0.003\,\mathbb{I}_{200} & -0.035\,\mathbb{I}_{200} \end{bmatrix},$$
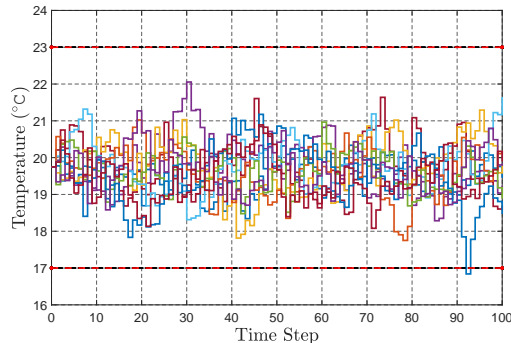
FIGURE 1. Closed-loop state trajectories of a representative room with 10 noise realizations in a network of 200 rooms for $\mathcal{T} = 100$.

and condition (4.1) is reduced to

$$\begin{bmatrix} \mathcal{M} \\ \mathbb{I}_{200} \end{bmatrix}^\top \mathcal{X}_{cmp} \begin{bmatrix} \mathcal{M} \\ \mathbb{I}_{200} \end{bmatrix} = 0.005\mathcal{M}^\top\mathcal{M} - 0.003(\mathcal{M} + \mathcal{M}^\top) - 0.035\,\mathbb{I}_{200} \preceq 0,$$

which is always satisfied without requiring any restrictions on the number or gains of subsystems. To show this, we employed the property of the interconnection topology as $\mathcal{M} = \mathcal{M}^\top$ and Gershgorin circle theorem [36]. Moreover, the compositionality condition (4.2) is also met since $\sum_{i=1}^{200} \mu_i \min_{p_i \in P_i}\{\lambda_{ip_i}\} > \sum_{i=1}^{200} \mu_i \max_{p_i \in P_i}\{\gamma_{ip_i}\}$. Then by employing the results of Theorem 4.1, one can conclude that $\mathcal{B}(x, p) := \sum_{i=1}^{200} \mathcal{B}_{ip_i}(x_i)$ is a CBC for the interconnected system with $\gamma = \sum_{i=1}^{200} \mu_i \max_{p_i \in P_i}\{\gamma_{ip_i}\} = 28, \lambda = \sum_{i=1}^{200} \mu_i \min_{p_i \in P_i}\{\lambda_{ip_i}\} = 860, \kappa = 0.92$, and $\psi = \sum_{i=1}^{200} \max_{p_i \in P_i}\{\psi_{ip_i}\} = 0.3$.

By employing Theorem 3.2, we guarantee that the temperature of the interconnected system Σ starting from initial conditions inside $X_0 = [19.5\ 20]^{200}$ remains in the safe set $[17\ 23]^{200}$ during the time horizon $\mathcal{T} = 100$ with the probability of at least 94%, *i.e.*,

$$\mathbb{P}_\nu^a \Big\{ x^p(k) \notin X_u \,\big|\, x^p(0) = a, p_0, \forall k \in [0, 100] \Big\} \geq 0.94. \tag{6.2}$$

Closed-loop state trajectories of a representative room with 10 different noise realizations are illustrated in Fig. 1. The computation of CSC and its corresponding controller for each individual room took almost 25 seconds with a memory usage of 1.4 MB on a Windows operating system (Intel i7@3.6GHz CPU and 32 GB of RAM). It is worth mentioning that in order to empirically verify the proposed probabilistic bound in (6.2), one can run Monte Carlo simulations with a high number of noise realizations.

7. Conclusion

In this work, we proposed a compositional framework based on dissipativity theory for the safety controller synthesis of large-scale discrete-time stochastic systems with *Markovian switching signals*. We first introduced the notion of control storage certificates, using which one can construct control barrier certificates of interconnected systems by leveraging dissipativity-type compositionality conditions. We then provided upper bounds on the probability that interconnected systems reach unsafe regions in finite time horizons. We formulated our proposed conditions to a sum-of-squares optimization problem to systematically search for storage certificates and corresponding local controllers enforcing safety properties. We finally verified our results on a room temperature network of 200 individual rooms while admitting Markovian switching signals.

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