Double Auctions with Two-sided Bandit Feedback

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Abstract

Double Auction enables decentralized transfer of goods between multiple buyers and sellers, thus underpinning functioning of many online marketplaces. Buyers and sellers compete in these markets through bidding, but do not often know their own valuation a-priori. As the allocation and pricing happens through bids, the profitability of participants, hence sustainability of such markets, depends crucially on learning respective valuations through repeated interactions. We initiate the study of Double Auction markets under bandit feedback on both buyers' and sellers' side. We show with confidence bound based bidding, and 'Average Pricing' there is an efficient price discovery among the participants. In particular, the buyers and sellers exchanging goods attain $O(\sqrt{T})$ regret in T rounds. The buyers and sellers who do not benefit from exchange in turn only experience $O(\log T/\Delta)$ regret in T rounds where Δ is the minimum price gap. We augment our upper bound by showing that even with a known fixed price of the good – a simpler learning problem than Double Auction – $\omega(\sqrt{T})$ regret is unattainable in certain markets.

1 Introduction

Online marketplaces, such as eBay, Craigslist, Task Rabbit, Doordash, Uber, enables allocation of resources between supply and demand side agents at a scale through market mechanisms, and dynamic pricing. In many of these markets, the valuation of the resources are often personalized across agents (both supply and demand side), and remain apriori unknown. The agents learn their own respective valuations through repeated interactions while competing in the marketplace. In turn, the learning influences the outcomes of the market mechanisms. In a recent line of research, this interplay between learning and competition in markets has been studied in multiple systems, such as two-sided matching markets [21, 22, 28, 4], centralized basic auctions [18, 14]. The previous works, mainly focus on one-sided uncertainty in the market.

In this paper, we study the decentralized Double Auction market where multiple sellers and buyers, each with their own valuation, trades an indistinguishable good. In each round, the sellers and the buyers present bids for the goods.² The auctioneer is then tasked with creating an allocation, and pricing for the goods. All sellers with bids smaller than the price set by the auctioneer sell at that price, whereas all the buyers with higher bids buy at that price. Double auction is used in e-commerce [32] – including stock exchanges, business-to-business and peer-to-peer markets, bandwidth allocation [16, 15], power allocation [23]. The desiderata for Double Auction is [26]:

- Individually Rational: No participant should loose from joining the auction.
- Balanced Budget: The auctioneer should not loose money.
- Incentive Compatible: for all agents, reporting the true valuation is a Nash equilibrium.
- Economic Efficiency: The item goes to the participants that value it the highest.

^{*}Work done outside AWS

²In some literature, the bids of the sellers is called 'asks', but we use bids for both sellers and buyers.

However, a classical result of Myerson et al. [26] states that in Double Auction it is impossible to attain all four. For example, VCG [29] mechanism is not budget balanced, and needs to subsidize the trade which is arguably unsustainable. We focus on the 'Average Mechanism', detailed in Section 3.2, which satisfies all but the incentive compatibility desiderata. In this mechanism, first an allocation is found, by maximizing K such that the K highest bidding buyers all bid higher than the K lowest bidding sellers. The price is set as the average of the lowest bid among the K buyers, and the highest bid among the K sellers.

We have two-sided uncertainty in the market. Each buyer and seller, is oblivious to all the prices including her own. If a buyer, or a seller participates in the market the value of the good is revealed. The uncertainty in bids manifests in two ways. Firstly, each buyer need to compete with others by bidding high enough to get allotted so that she can discover her own price. Similarly, the sellers compete by bidding lower for price discovery. The competition driven increase of buyers' bids, and decrease of the sellers' bids may decrease the utility that a buyer or seller generates. Secondly, as the valuation needs to be estimated, the price set in each round as a function of these estimated valuations (communicated to the auctioneer in the form of bids) remains noisy. This noise in price also decreases the utility. However, when price discovery is slow the noise in price increases. Therefore, the main challenge in Double auction with learning is to strike a balance between the competition driven increase/decrease of bids, and controlling the noise in price in a decentralized way.

1.1 Main Contributions

Our main contributions in this paper are as follows.

- 1. We study the Double Auction markets under Average Mechanism with both seller and buyer learning from bandit feedback. To the best of out knowledge, this initiates the study of bandits in markets with two-sided uncertainty with fixed mean utilities for agents. We establish that the algorithm where sellers bid the lower confidence bound (LCB), and buyers bid the upper confidence bound (UCB) on their respective valuation is a no-regret strategy. We note that this algorithm design does not follow *optimism in the face of uncertainty* [1], but depends on the central idea of *domination of information flow*, i.e. ensuring that the number of allocation in each round is larger or equal to the same under the true valuation. Indeed, by using UCB bids the buyers, and using the LCB bids the sellers decrease their reward, while increasing the chance of trade and price discovery in the system.
- 2. We show that all sellers and buyers who do not participate under the true valuations incur $O(\log(T)/\Delta)$ regret in T rounds with a minimum reward gap Δ . However, the true participating buyers and sellers incur a $O(\sqrt{T\log(T)})$ regret. Our upper bound holds for heterogeneous confidence widths, making it robust against the choices of the individual agents.
- 3. We show that the price discovery itself is $\Omega(\sqrt{T})$ hard in the minimax sense. Specifically, we consider the system where the price of the good is known to all. Also, there is infinite pool of resource, so any buyer willing to pay the price gets to buy, and any seller willing to sell at the price does so. Fix a buyer or seller. We show under this setup there exists a system where $\Omega(\sqrt{T})$ for that agent.

2 Related Work

Classical mechanism design in double auctions: There is a large body of work on mechanism design for double auctions, following Myerson et al. [26]. The average-mechanism, which is the subject of focus in this paper achieves all the above desiderata except for being incentive compatible. The VCG mechanism was developed in a series of works [29], [8] and [13] achieves all desiderata except being budget balanced. This mechanism requires the auctioneer to subsidize the trade. More sophisticated trade mechanisms known as the McAfee mechanism [24], trade reduction, and the probabilistic mechanism [2] all trade-off some of the desiderata for others. However, the key assumption in all of these lines of work was that all participants know their own valuations, and do not need to learn through repeated interactions.

Bandit learning in matching markets: In recent times, online learning for the two-sided matching markets have been extensively studied in [21], [22], [28], [4], [11]. This line of work studies two sided markets when one of the side does not know of their preferences apriori and learn it through interactions. However, unlike the price-discovery aspect of the present paper, the space of preferences that each participant has to learn is discrete and finite, while the valuations that agents need to learn form a continuum. This model was improved upon by [6] that added notions of price and transfer.

The paper of [17] studied a contextual version of the two-sided markets where the agents preferences depend on the reveled context through an unknown function that is learnt through interactions.

Learning in auctions: Online learning in simple auctions has a rich history - [3], [9], [25], [5], [10], [30], [14], [12] to name a few, each of which study a separate angle towards learning from repeated samples in auctions. However, unlike the our setting, one side of the market knows of their true valuations apriori. The work of [18] is the closest to ours, where the participants apriori do not know their true valuations. However, they consider the VCG mechanism in the centralized setting, i.e., all participants can observe the utilities of all participants in every round.

Learning in bi-lateral trade: One of the first studies on learning in bilateral trade is [7]. However, in this paper the focus is on setting a price for an incoming buyer and seller pair to facilitate trade while maximizing gain from trade, not on multi-agent auction. In particular, in each round a buyer and a seller draw her own valuation i.i.d. from their respective distributions. Then an arbiter sets the price, with trade happening if the price is between the seller's and buyer's price. They show with full-information follow-the-leader type algorithm achieves $O(\sqrt{T})$ regret, where as with realistic (bandit-like) feedback by learning the two distributions approximately $O(T^{2/3})$ regret can be achieved.

3 System Model

The market consists of $N \geq 1$ buyers and $M \geq 1$ sellers, in a market for a single type of item which are indistinguishable across sellers. This set of M+N market participants, repeatedly participate in the market for T rounds. Each buyer $i \in [N]$ has valuation $B_i \geq 0$, for the item and each seller $j \in [M]$ has valuation $S_i \geq 0$. No participant knows of their valuation apriori and learn it while repeatedly participating in the market over T rounds.

3.1 Interaction Protocol between the buyers and sellers

The interaction is parameterized by a bilateral trade mechanism that runs every time. Throughout this paper, we consider the market in which the bilateral trade mechanism is the average mechanism induced by the natural ordering, whose precise definition we give in the sequel.

At each round t > 1.

- (i) All buyers and sellers submit bids and asking prices respectively to the auctioneer. The buyers' bids are denoted by $(b_i(t))_{i=1}^N$ and the sellers' asking prices are denoted by $(s_j(t))_{j=1}^M$.
- (ii) The auctioneer implements the average price mechanism bilateral trade mechanism, described in the sequel in Section 3.2 and outputs
- A subset $\mathcal{P}_b(t) \subseteq [N]$ of participating buyer, and $\mathcal{P}_s(t) \subseteq [M]$ of participating seller, both with size $|\mathcal{P}_b(T)| = |\mathcal{P}_s(t)| = K(t) \le \min(M, N)$,
- The trading price $p(t) \in \mathbb{R}$ for the participating buyers and sellers in this round. The participating buyers (those in the set $\mathcal{P}_b(T)$) pay price p(t), and participating sellers (those in set $\mathcal{P}_s(t)$) get price p(t), and each one of these buyers (sellers) in $\mathcal{P}_b(t)(\mathcal{P}_s(t))$ transact one good. The platform/auctioneer herself is not part of the trade, and does not get paid by the trade.
- (iii) Every buyer $i \in [N]$ is either part of the trade at time t, i.e., $i \in \mathcal{P}_b(t)$ in which case they get utility $r_i^{(B)}(t) := Y_{b,i}(t) p(t)$, where $Y_{b,i}(t) = B_i + \nu_{b,i}(t)$ with $\nu_{b,i}(t)$ for all $i \in [N]^3$, and $t \in [T]$ are i.i.d., 0 mean, 1 sub-Gaussian random variables. Otherwise, they are not part of the trade, i.e., if $i \notin \mathcal{P}_b(t)$, they are given that signal that they are not part of the trade, and receive 0 utility.⁴
- (iv) Every seller $j \in [M]$ that is part of the trade at time t, i.e., $j \in \mathcal{P}_s(t)$ get reward of $r_j^{(S)}(t) = p(t) Y_{s,j}(t)$, where $Y_{s,j}(t) = S_j + \nu_{s,j}(t)$ with $\nu_{s,j}(t)$ being i.i.d. 0 mean, 1 sub-gaussian random variables for all $i \in [N]$, and $t \in [T]$. For sellers not part of the trade, i.e., for $j \notin \mathcal{P}_s(t)$, are given that signal that they are not part of the trade, and receive 0 reward.

In the rest of this article, we denote by all the buyers and sellers that are matched in any given round as the set of participating agents in round t. From a technical perspective, the randomn variables in

³For a non-negative integer $A, [A] := \{1, \dots, A\}$ denotes the set of natural numbers till A

⁴We study the system with 1-sub-Gaussian to avoid clutter. Extension to general σ -sub-Gaussian is trivial when σ or an upper bound to it is known.

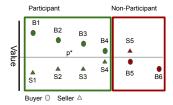


Figure 1: Average Mechanism with 6 Buyers and 5 Sellers

the system are all a measurable function of the collection of i.i.d. 0 mean, 1 sub-gaussian random variables $(\nu_{b,i}(t))_{i\in[N],t\in[T]}$ and $(\nu_{s,j}(t))_{j\in[M],t\in[T]}$.

3.2 Average price mechanism

In this section, we detail the mechanism implemented by the auctioneer in every round t (Step 2) above. Under this mechanism, at each round t, the auctioneer receives bids $(b_i(t))_{i=1}^N$ from the buyers and $(s_j(t))_{j=1}^M$ from the sellers. The auctioneer then orders these by the 'natural order', i.e., sorts the buyers bids in descending order and the seller's bids in ascending order. Denote by the sorted bids from the buyer and seller as $b_{i_1}(t) \geq \cdots b_{i_N}(t)$ and the sorted sellers bids by $s_{j_1}(t) \leq \cdots s_{j_M}(t)$. Denote by the index K(t) to be the largest index such that $b_{i_K(t)}(t) \geq s_{j_{K(t)}}(t)$. In words, K(t) is the 'break-even index' such that all buyers $i_1, \cdots, i_{K(t)}$ have placed bids offering to buy at a price strictly larger than the asking price submitted by sellers $j_1, \cdots, j_{K(t)}$. The auctioneer then selects the participating buyers $\mathcal{P}_b(t) = \{i_1, \cdots, i_{K(t)}\}$, and participating sellers $\mathcal{P}_s(t) = \{j_1, \cdots, j_{K(t)}\}$.

The price is set to $p(t) := \frac{b_{i_K(t)} + s_{j_K(t)}}{2}$, and thus the name of the mechanism is deemed as the average mechanism.

3.3 Regret definition

For the given bilateral trade mechanism, and true valuations $(B_i)_{i \in [N]}$ and $(S_j)_{j \in [M]}$, denote by $K^* \leq \min(M,N)$ be the number of matches and by p^* to be the price under the average mechanism when all the buyers and sellers bid their true valuations. Let \mathcal{P}_b^* to be set of the *true participating buyers*, and \mathcal{P}_s^* to be set of the *true participating sellers*. For any buyer $i \in [N]$, we denote by $(B_i - p^*)$ to be the true utility of the buyer. Similarly, for any seller $j \in [N]$, we denote by $(p^* - S_j)$ to be the true utility of seller j. From the description of the average mechanism, when they all bid their true valuations, all agents have non-negative true utilities.

Recall from the protocol description in Section 3.1 that at any time t, if buyer $i \in [N]$ participates, then she receives a mean utility of $(B_i - p(t))$. For a participating seller $j \in [M]$ her mean utility of $(p(t) - S_j)$ in round t. If in any round t, if a buyer $i \in [N]$ or a seller $j \in [M]$ does not participate, then she receives a deterministic utility 0. The expected regret of a buyer i, namely $R_{b,i}(T)$, and a seller j, namely $R_{s,j}(T)$ is defined as

$$R_{b,i}(T) = \mathbb{E}\Big[\sum_{t=1}^T (B_i - p(t))\Big], \text{ and } R_{s,j}(T) = \mathbb{E}\Big[\sum_{t=1}^T (p(t) - S_j)\Big].$$

Auctioneer has no regret as average mechanism is *budget balanced*, i.e. auctioneer does not gain or lose any utility during the process.

4 Decentralized Bidding with Confidence Bound Bids

We consider the decentralized system where each market participant bids based on their own observation, without any additional communication. We consider a simple algorithm. Each seller $j \in [M]$, with $n_{s,j}(t)$ participation upto round t, at time t+1 bids the lower confidence bound (scaled by $\alpha_{s,j}$), LCB($\alpha_{b,i}$) in short, of its own valuation of the item as

$$s_j(t+1) = \hat{s}_j(t) - \sqrt{\frac{\alpha_{s,j} \log(t)}{n_{s,j}(t)}},$$

where $n_{s,j}(t) := \sum_{\substack{t' \leq t \ 1}} \mathbf{1}(j \in \mathcal{P}_s(t'))$ is the number of times seller j participated upto and including time t and $\hat{s}_j(t) = \frac{1}{n_{s,j}(t)} \sum_{\substack{t' \leq t: j \in \mathcal{P}_s(t') \ s}} Y_{s,j}(t')$ is the observed empirical valuation of the item by seller j.

Each buyer $i \in [N]$, with $n_{s,j}(t)$ participation upto round t, at time t+1, bids the upper confidence bound (scaled by $\alpha_{b,i}$), UCB $(\alpha_{b,i})$ in short, of its own valuation of the item as

$$b_j(t+1) = \hat{b}_i(t) + \sqrt{\frac{\alpha_{b,i} \log(t)}{n_{b,i}(t)}},$$

where $n_{b,i}(t) := \sum_{t' \leq t} \mathbf{1}(i \in \mathcal{P}_b(t'))$ is the number of times buyer i participated upto and including time t $\hat{b}_i(t) = \frac{1}{n_{b,i}(t)} \sum_{t' \leq t: i \in \mathcal{P}_b(t')} Y_{b,i}(t')$ is the empirical mean utility observed by buyer i.

We now contrast our algorithm design from standard multi-armed-bandit (MAB) problems. In a typical MAB problems, including other multi-agent settings, the Algorithm is designed based on optimism in the face of uncertainty in learning (OFUL) principle [1]. The UCB-type indices under OFUL arises as optimistic estimate of the rewards of arms/actions. However, our algorithm does not rely on the OFUL principle directly. Instead, the central idea is *domination of information flow*, i.e. higher or equal allocation than under true valuation. Guided by this in our algorithm, a buyer uses *UCB index*, and a seller uses *LCB index* as their respective bids. Note that this is not optimistic as UCB for a buyer and LCB for a seller decreases their respective rewards. Instead, this choice ensures that the system increases the chance for buyer and seller to participate in each round, i.e. there is domination of information flow, and consequently they discover their own valuation in the market.

However, too much aggression can disrupt the price setting process. In particular, if even one non-participating buyer is bidding high enough (due to UCB bids) to exceed a non-participating seller's price, firstly she participates and accrues regret. More importantly, the participating sets deviates, resulting in a deviation of the price of the good from p^* . Thus resulting in regret for all participating agents as well. Similar problems arise if one or more non-participating seller predicts lower. On the other hand, too low aggression is also harmful as the price discovery may not happen resulting in deviation of participating set, deviation of price, and high regret. In the next section we show the aggression remains within desired range, i.e. the regret of the agents remain low, even with heterogeneous UCBs and LCBs.

5 Regret Upper Bound

In this section, we derive the regret upper bound for all the buyers and sellers in the system. Without loss of generality, let us assume that the buyers are sorted in decreasing order of valuation $B_1 \geq B_2 \geq ... \geq B_N$. The sellers are sorted in increasing order of valuation $S_1 \leq S_2 \leq ... \leq S_N$. The the buyers $i=1,\ldots,K^*$ are the true participating buyers, and, similarly, the sellers $j=1,\ldots,K^*$ are the true participating sellers. Let us define $\alpha_{\min} := \min\{\alpha_{b,i},\alpha_{s,j}\}$, and $\alpha_{\max} := \max\{\alpha_{b,i},\alpha_{s,j}\}$. We define the minimum distance of an agent's true valuation from the true price p^* as

$$\Delta = \min_{i, \in [N], j \in [M]} \{ |p^* - S_j|, |B_i - p^*| \}.$$

The following theorem provides the regret bounds for all sellers and buyers in T rounds.

Theorem 1. The regret of the Average mechanism with buyers bidding $UCB(\alpha_b)$, and sellers bidding $LCB(\alpha_s)$ of their estimated valuation, for $\alpha_{\min} \geq 4$, we have the expected regret is bounded as:

- for a participating buyer $i \in [K^*]$ as $R_{b,i}(T) \leq (2 + \sqrt{\alpha_{\max}})\sqrt{T\log(T)} + C_{b',i}\log(T)$,
- for a participating seller $j \in [K^*]$ as $R_{s,j}(T) \le (2 + \sqrt{\alpha_{\max}})\sqrt{T\log(T)} + C_{s',j}\log(T)$,
- for a non-participating buyer $i \ge (K^*+1)$ as $R_{b,i}(T) \le \frac{\sqrt{(M-K^*+1)}(2+\sqrt{\alpha_{\max}})^2}{(B_{K^*}-B_i)}\log(T)$,
- for a non-participating seller $j \ge (K^* + 1)$ as $R_{s,j}(T) \le \frac{\sqrt{(N-K^*+1)}(2+\sqrt{\alpha_{\max}})^2}{(S_j S_{K^*})} \log(T)$.

Here $C_{b',i}$ and $C_{s',j}$ are $O\left(\frac{(M-K^*+1)(N-K^*+1)}{\Delta}\right)$ constants whose exact expressions are given in Theorem 19 in the Appendix A.

Several comments on our main theorem are in order.

Participation is encouraged for $\Delta = \omega(\sqrt{\log(T)/T})$: We observe that the non-participating buyers and sellers, are only having regret $O(\log(T)/\Delta)$. And the participating buyers and sellers are having regret $O(\sqrt{T\log(T)})$. This is reassuring as this does not discourage buyers and sellers from participation for a large range of system a-priori. Indeed, for the last participating buyer and seller the utility is $(B_K^* - S_K^*)/2$, which is close to Δ . Hence, as long as $\Delta T = \omega(\max\{\log(T)/\Delta, \sqrt{T\log(T)}\})$ or $\Delta = \omega(\sqrt{\log(T)/T})$ a non-participating buyer or seller prefers entering the market then discovering her price and getting out, as compared to not participating in the beginning. Also, participating buyer or seller is guaranteed return through participation.

Scaling of Regret with N, M, Δ for Non-Participants: The regret scales as $O(\sqrt{M}\log(T)/\Delta)$ for a non-participant buyer, whereas it scales as $O(\sqrt{N}\log(T)/\Delta)$ for a non-participant seller. For any non-participant buyer once it has $O(\log(T)/\Delta^2)$ samples it no longer falls in top K^* buyer. However, it can keep participating until each non-participant seller collects enough samples, i.e. $O(\log(T)/\Delta^2)$ samples. Finally, regret is shown to be $O(\sqrt{\# \text{ participation } \log(T)})$ which leads to the $O(\sqrt{M}\log(T)/\Delta)$ regret.

Scaling of Regret with N, M, and Δ for Participants: The leading $O(\sqrt{T\log(T)})$ term in the regret for each true participating buyer and seller do not scale with the size of the system. This leading term depends mainly on the random fluctuation of the bid of the lowest bidding participating buyer and the highest bidding participating seller. The $O\left(\frac{MN}{\Delta}\log(T)\right)$ regret for the participating buyer and seller comes because each time a non-participating buyer or seller ends up participating the price deviates.

Scaling in a system with $K^* \approx N \approx M$: We see that when the system the number of participants is very high, i.e. $(N-K^*), (M-K^*) = O(1)$, then the $O(\log(T)/\Delta)$ component, hence the regret, per buyer/seller does not scale with the system size. This indicates that there is rapid learning, as all the participants are discovering her own price almost every round.

5.1 Proof Outline of Regret Upper Bound (Theorem 1)

We now present an outline to the proof of Theorem 1. The full proof is given in Appendix A. Due to the UCB and LCB property with high probability, the bids lie close to the true mean, with roughly $\sqrt{\alpha_{b,i}\log(t)/n_{b,i}(t)}$ deviation for buyer i, and $\sqrt{\alpha_{s,j}\log(t)/n_{s,j}(t)}$ deviation for seller j. Let us first note that the regret of a non-participant buyer i, can be bound by

$$\sum_{t:i\in\mathcal{P}_b(t)}(p(t)-B_i) \leq \sum_{t:i\in\mathcal{P}_b(t)}(b_i(t)-B_i) \lessapprox \sum_{t:i\in\mathcal{P}_b(t)}\sqrt{\frac{\alpha_{b,i}\log(t)}{n_{b,i}(t)}} \lessapprox \sqrt{\alpha_{b,i}n_{b,i}(T)\log(T)}.$$

We have p(t) lesser than $b_i(t)$ because *i*-th buyer participates in round t. A similar argument shows for a non-participating seller j the regret is roughly $\sqrt{\alpha_{s,j}n_{s,j}(T)\log(T)}$.

For a participating buyer i, we have the regret bounded as

$$\sum_{t:i \notin \mathcal{P}_b(t)} (B_i - p^*) + \sum_{t:i \in \mathcal{P}_b(t)} (p(t) - p^*) = (T - n_{b,i}(T))(B_i - p^*) + \sum_{t:i \in \mathcal{P}_b(t)} (p(t) - p^*).$$

For a participating seller j, the regret bound is $(T - n_{s,j}(T))(p^* - S_j) + \sum_{t:j \in \mathcal{P}_s(t)} (p^* - p(t))$.

The cornerstore of our proof is the *domination of the information flow* brought about by the UCB and LCB based bids from the buyers, and sellers, respectively. We leverage monotonicity of the average mechanism, which says if the bid of each buyer is equal or higher, and simultaneously bid of each seller is equal or lower the number of participation increases. For our bidding algorithm, this leads to $K(t) \geq K^*$ w.h.p.⁵

The next challenge in our proof comes from two-sided uncertainty. In Double auction, the outcomes of the buyers' and sellers' sides are inherently coupled. Hence error in buyers' side propagates to the sellers' side, and vice versa, under two-sided uncertainty. For example, if the buyers are bidding lower than their true valuation then the true participating sellers, even with perfect knowledge of their

 $^{^5}$ In fact, this relies only on the natural ordering property of the system which is used in most mechanisms including VCG, Trade reduction mechanism, and McAfee's mechanism mentioned in Section 2. Thus $K(t) \ge K^*$ holds w.h.p. for the confidence bound based algorithms therein.

own valuation, ends up participating. Our first key lemma, Lemma 14 in Appendix A, decouples the true participating buyers, and true participating sellers. It lower bounds the number of participation for true participating buyers and sellers. We state the result informally here.

Lemma 2 (Informal). For $\alpha_{\min} > 4$, with high probability for any $i, j \in [K^*]$, we have

$$(T - n_{b,i}(T)) \lessapprox \sum_{i' \ge K^* + 1} \frac{\alpha_{b,i'} \log(T)}{(B_i - B_{i'})^2} \text{ and } (T - n_{s,j}(T)) \lessapprox \sum_{j' \ge K^* + 1} \frac{\alpha_{s,j'} \log(T)}{(S_{j'} - S_{j})^2}.$$

It argues after a true non-participant i' gets $O(\frac{\log(T)}{(B_i - B_{i'})^2})$ samples it can not replace i from $\mathcal{P}_b(t)$. Similar reasoning holds for seller side.

Unlike true participating agents, the effect of uncertainties on the true non-participating buyers and sellers cannot be decoupled directly. With at least one non-participating buyer with large estimation error in her valuation present, the non-participating sellers can keep participating even with perfect knowledge. However, this does not ensure directly that the estimation error of this non-participating buyer decreases. Next, in Lemma 16 in Appendix A, we resolve this problem and upper bound the number of times a non-participant can participate. Informally we have

Lemma 3 (Informal). For $\alpha_{\min} > 4$, with high probability for any $i, j \geq (K^* + 1)$,

$$n_{b,i}(T) \lessapprox \frac{\alpha_{b,i}\log(T)}{(B_{K^*} - B_i)^2} + \sum_{j' \ge (K^* + 1)} \frac{\alpha_{s,j'}\log(T)}{(S_{j'} - B_i)^2}, n_{s,j}(T) \lessapprox \frac{\alpha_{s,j}\log(T)}{(S_{j} - S_{K^*})^2} + \sum_{i' \ge (K^* + 1)} \frac{\alpha_{b,i'}\log(T)}{(S_{j} - B_{i'})^2}$$

We use a pigeon-hole style argument instead. A non-participant buyer i after obtaining $O(\frac{\log(T)}{(B_{K^*}-B_i)^2})$ samples does not belong to top K^* w.h.p. Hence, she participates with non-negligible probability only if a seller $j' \geq (K^*+1)$ participates. However, this implies that with each new match of buyer i, additionally at least one non-participating seller is matched, decreasing both their uncertainties. For this buyer i, we argue such spurious participation happens a total of $\sum_{j' \geq (K^*+1)} O(\frac{\log(T)}{(S_{j'}-B_i)^2})$ times. After that all the non-participating sellers $j \geq (K^*+1)$ will have enough samples so that their LCB bids will separate from the i-th buyer's UCB bid. Reversing sellers' and buyers' roles does the rest.

The final part of the proof establishes bound on the cumulative variation of price from the true price p^* . This is done in Lemma 18 in Appendix A.

Lemma 4. (Informal) For $\alpha_{\min} > 4$, with high probability,

$$\sum_{t=1}^{T} (p(t) - p^*) \lesssim C_b \log(T) + \sqrt{\alpha_{max} T \log(T)}, \sum_{t=1}^{T} (p^* - p(t)) \lesssim C_s \log(T) + \sqrt{\alpha_{max} T \log(T)},$$

where the constants C_b and C_s are $O(\frac{(M-K^*)(N-K^*)}{\Delta})$.

Let us focus on the first upper bound, i.e. of the cumulative value of $(p(t) - p^*)$. The proof breaks down the difference into two terms,

$$2(p(t) - p^*) = \min_{i \in \mathcal{P}_b(t)} (b_i(t) - B_{K^*}) + (S_{K^*} - \max_{j \in \mathcal{P}_s(t)} s_j(t)).$$

For rounds when the buyer K^* is present (which happens all but $O(\frac{(N-K^*)\log(T)}{(B_i-p^*)^2})$ rounds) we can replace $\min_{i\in\mathcal{P}_b(t)}b_i(t)$ with $b_{K^*}(t)$. Finally noticing that $\sum_t (b_{K^*}(t)-B_{K^*})\lessapprox \sqrt{\alpha_{max}}\sqrt{T\log(T)}$ takes care of the first term. For the second term, we need to study the process $\max_{j\in\mathcal{P}_s(t)}s_j(t)$. First we bound the number of times sellers 1 to (K^*-1) crosses the seller K^* . Next we eliminate all the rounds where at least one seller $j\geq (K^*+1)$ are participating. Such an elimination comes at a cost of $O(\frac{(M-K^*)}{\Delta})$ For any seller $j\geq (K^*+1)$ this happens $O(\frac{(N-K^*)\log(T)}{\Delta^2})$ times for some appropriate Δ_j , and gives Δ_j regret in each round. This final step gives us the dominating $O(\frac{(M-K^*)(N-K^*)}{\Delta})$ term. The bound for the cumulative value of $(p^*-p(t))$ follows analogously. Combining all these bounds our main theorem follows.

5.2 Deviations and Incentives of Agents

We now discuss how agents may deviate under average mechanism under their own incentive. We limit ourselves to deviations from *myopic oracle* agents, each of whom optimizes her single round reward, and possess an oracle knowledge of her own valuation. Our incentives are shaped by symmetric equilibrium in double auction [31], where all non-strategic buyers employ the same common strategy, and similarly all the non-strategic sellers employ a common strategy. In particular, we assume in this discussion that all the non-strategic agents use confidence-based bidding described in Section 4. The strategic deviant agents do not deviate from a strategy if incremental deviations in bids do not improve their own reward.

Under the above setup, for average mechanism only the price-setting agents (i.e. the K^* -th buyer and K^* -th seller) have incentive to deviate from their true valuation to increase their single-round reward (c.f. Section 5 in [31]). For other agents, deviations from reporting their true value does not improve their instantaneous reward. Thus average mechanism is *truthful for non-price-setting agents*.

For the price-setting agents, the incentives, and impact on regret are as follows.

- 1. Only K^* -th buyer deviates: The K^* -th buyer has an incentive to set her bid $close^6$ to $\max(S_{K^*}, B_{K^*+1})$. The long term average price is now set close to $(S_{K^*} + \max(S_{K^*}, B_{K^*+1}))/2$. When compared to Average mechanism outcomes this leads to $(B_{K^*} \max(S_{K^*}, B_{K^*+1}))/2$ average surplus in each round for participating buyers, and the same average deficit in each round for participating sellers.
- 2. Only K^* -th seller deviates: The K^* -th seller has an incentive to set her bid close to $\min(B_{K^*}, S_{K^*+1})$. With a long term average price of $(B_{K^*} + \min(B_{K^*}, S_{K^*+1}))/2$, each seller has a per round $(S_{K^*} \min(B_{K^*}, S_{K^*+1}))/2$ surplus, and each buyer has the same average deficit in each round.
- 3. Both K^* -th seller and buyer deviate: The K^* -th seller has an incentive to set her bid close to $\min((B_{K^*}+S_{K^*})/2,S_{K^*+1})$. Whereas, for the K^* -th buyer the bid is close to $\max((B_{K^*}+S_{K^*})/2,B_{K^*+1})$. We can derive the long term average price, and surplus and deficit for the agents similarly. We leave out the exact expressions due to space limitations.

We acknowledge that the study of incentive compatibility under the notions of equilibrium in sequential games with incomplete information – where an agent can strategize thinking about her long term consequences (c.f. [27, 19]) in presence of learning is out of scope for this paper. This is similar to other contemporary works on learning in repeated games [21, 28, 22, 4].

6 Minimax Regret Lower Bound

The main result of this section is Lemma 8 that shows a minimax regret lower bound of $\Omega(\sqrt{T})$ by considering a simpler system that decouples learning and competition. This result illustrates that even in the absence of competition, $O(\sqrt{T})$ regret bound is un-avoidable in the minimax sense. Thus, our regret upper bound of $O(\sqrt{T})$ is order-wise optimal, despite the presence of competition.

6.1 Simple setup: A System with one buyer and a fixed seller

In order to set ideas, we consider a simple system with one buyer and seller, repeatedly interacting in multiple rounds. We consider the system in which the seller is assumed to (i) know her exact valuation, and (ii) always ask her true valuation as the selling price, i.e., is truthful in her asking price in all the rounds. Furthermore, the pricing mechanism at every round is fixed to be the average mechanism, i.e., a transaction occurs if and only if $B_t \geq S$ at price $p_t = \frac{B_t + S_t}{2}$. As before, the utility of the buyer at time t is defined as $U_t = (p_t - B)\mathbf{1}(B_t \geq S)$. The first observation we make is that an oracle buyer that knows her true valuation will play the following strategy.

Lemma 5. The utility maximizing optimal action of an oracle buyer that knows B and S is to bid $B_t = S\mathbf{1}(B \ge S) + (S - \varepsilon)\mathbf{1}(B < S)$ for any $\varepsilon > 0$, at all times. In words, the oracle buyer either bids S and matches with the seller at price S or "abstains" by bidding strictly less than S.

The proof is in the Appendix in Section B.1. In what follows, we denote the regret of the buyer to be the difference in cumulative utility obtained by the oracle buyer of Lemma 5 and the obtained utility.

⁶In this context, *close* means ϵ larger bids for buyers, and ϵ smaller bids for sellers for $\epsilon \gtrsim 0$.

Corollary 6. For the system described in Section 6.1, in order to minimize expected utility over a time horizon for the buyer, it suffices for the buyer in each round to decide to either bid S and participate in the market by paying price S, or abstain without participation and obtain no reward.

6.2 Reduction to a two armed bandit problem

Corollary 6 gives that at each time, it suffices for the buyer that does not know her true valuation to either bid S and participate at price S, or abstain from participating. In any round t that the buyer participates, she obtains a mean reward of B-S, while she receives 0 reward in rounds she abstains from participating. Thus, the actions of the buyer are equivalent to a two armed bandit, one with mean B-S and the other is deterministic 0 mean. The reduction is formalized in the following Corollary.

Corollary 7. Any bidding policy given in Definition 21 describes a two-armed bandit policy (Definition 22) with arm-means B - S and 0.

The proof is in the Appendix in Section B.3.

6.3 Minimax lower bound for the two armed bandit problem

Lemma 8. For every bidding policy, there exists a system such that $\mathbb{E}[R_T] \geq \frac{1}{36}\sqrt{T}$.

Corollary 7 states that it suffices to show that for every bandit policy, there exists a two armed system with expected regret exceeding $1/36\sqrt{T}$. The lower bound proof for the bandit problems follows the standard recipe (see for ex. [20]) and is reproduced in the Appendix B.2 for completeness.

6.4 Extension to the multi-agent setting

The simplified system in 6.1 specified a single buyer and seller system that had no competition as the seller's behaviour was fixed. In this section, we show that a \sqrt{T} minimax lower bound is inevitable even in an appropriately simplified multi-agent system that decouples learning from competition.

In this simplified setting, we assume (i) the selling price $p_t := p^*$ is set constant for every round t, and (ii) there is no shortage of goods, i.e., any buyer(seller) that wants to participate in a given round by paying(selling at) p^* , can buy(sell) so. Thus, under this setup, the entire market is a decoupled union of individual agents interacting against a fixed environment dictated by a price p^* . It can be observed, that for any buyer(seller) in this simplified setting, the lower bound from Lemma 8 applies verbatim following the same coupling arguments.

Corollary 9. For every bidding policy for the general system, for any seller $j \in [M]$ there exists a system such that $\mathbb{E}[R_{s,j}(T)] \geq \frac{1}{36}\sqrt{T}$. Similarly, for any buyer $i \in [N]$ there exists a system such that $\mathbb{E}[R_{b,i}(T)] \geq \frac{1}{36}\sqrt{T}$.

7 Simulation Study

We perform some synthetic studies to augment our theoretical guarantees. Our objective is to study behavior of different systems with heterogeneous α , varying gaps, and system sizes.

We create an instance as a function of N buyers, M sellers, K^* participants, and Δ gap. The rewards are Bernoulli, with means varying between [0,1]. We randomly generate an instance that satisfies the above property. We vary the confidence width of the buyers, α_b , and seller, α_s , in $[\alpha_1,\alpha_2]$. Next we simulate the performance of the UCB(α_b) and LCB(α_s) over 100 independent sample paths with T=50k. We report the mean, 25% and 75% value of the trajectories. We plot the cumulative regret of the buyers, $R_{b,i}(t)$, and the sellers, $R_{s,j}(t)$. We also plot the number of matches of the buyers ans sellers as a function of time. Finally, we plot the convergence of the number of matches in the system K(t), and the price difference $(p(t)-p^*)$.

In Figure 2 in Appendix C, we have a 8×8 system with 5 matches. We see that K(t) converges to 5, where as $(p(t)-p^*)$ converges to 0. The non-participant regret, for both buyers and sellers, converges and assumes the log(T). The participant regret, for both buyers and sellers, has more noise and the envelope grows as $O(\sqrt{T})$. Note that the regret that comes from price difference has opposite sign for buyers and sellers in each sample path. Hence, if regret plot of buyers is increasing with T then it will decrease for sellers, and vice versa.

8 Conclusion and Future Work

We study the Double auction with Average mechanism where the buyers and sellers need to know their own valuation from her own feedback. Using confidence based bounds – UCB for the buyers and LCB for the sellers, we show that it is possible to obtain $O(\sqrt{T\log(T)})$ regret for the true participant buyers and sellers in T rounds. Whereas, the true non-participant buyers and sellers obtain a $O(\log(T)/\Delta)$ regret where Δ is the smallest gap. We show that there are simpler systems where each buyer and seller must obtain a $\Omega(\sqrt{T})$ regret in the minimax sense. We conclude with a few open directions. Firstly, we do not obtain a minimax matching regret for the proposed strategy (i.e. regret for any $\Delta>0$). Can we obtain a $O(\sqrt{T})$ minimax bound? More importantly coming up with a framework and bidding strategy that provably provides 'good' regret for general Double auction mechanisms remain open.

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A Proofs from Section 5

We setup some notation for the analysis. The total number of samples collected by buyer i is given as $n_{b,i}(t)$, and similarly for seller j, it is $n_{s,j}(t)$. Let K(t) be the number of participants in round t. For any $i \in [N]$ and $j \in [M]$, let $\chi_{b,i}(t)$ be the indicator of buyer i participating in round t, and $\chi_{s,j}(t)$ be the indicator of seller j participating in round t. Let $\mathcal{P}_b(t)$ denote the set of participating buyers, and $\mathcal{P}_s(t)$ denote the set of participating sellers.

For any buyer $i \in [N]$ and any time t, we have

$$\mathcal{E}_{i,t}^{(\beta)} := \left\{ |\widehat{b}_i(t) - B_i| \le \sqrt{\frac{\beta \log(t)}{n_{b,i}(t)}} \right\}, \quad \mathbb{P}[\mathcal{E}_{i,t}] \ge 1 - 1/t^{\beta/2}.$$

Similarly, for any seller $j \in [M]$ we have

$$\mathcal{E}_{j,t}^{(\beta)} := \left\{ |\widehat{s}_j(t) - S_j| \le \sqrt{\frac{\beta \log(t)}{n_{s,j}(t)}} \right\}, \quad \mathbb{P}[\mathcal{E}_{j,t}] \ge 1 - 1/t^{\beta/2}.$$

Without loss of generality, let true valuation of the buyers be in the descending order, and for the seller in the ascending order. Then under the Average mechanism with the true bids (a.k.a. oracle Average mechanism) the buyers 1 to K^* , and the sellers 1 to K^* participate, while the others do not participate. Let $\alpha_{\min} = \min\{\alpha_{s,j}, \alpha_{b,i} : j \in [M], i \in [N]\}$, and $\alpha_{\max} = \min\{\alpha_{s,j}, \alpha_{b,i} : j \in [M], i \in [N]\}$.

Lemma 10. Under the event $\mathcal{E}_t^{(\beta)} := \bigcap_{i \in [N]} \bigcap_{j \in [M]} \mathcal{E}_{i,t}^{(\beta)} \cap \mathcal{E}_{j,t}^{(\beta)}$, in time t, the bid for the i-th buyer and the j-th seller admits the (random) bounds with $\alpha_{\min} \geq \beta$

$$b_{i}(t) \in \left[B_{i} + (\sqrt{\alpha_{b,i}} - \sqrt{\beta}) \sqrt{\frac{\log(t)}{n_{b,i}(t)}}, B_{i} + (\sqrt{\alpha_{b,i}} + \sqrt{\beta}) \sqrt{\frac{\log(t)}{n_{b,i}(t)}} \right],$$

$$s_{j}(t) \in \left[S_{j} - (\sqrt{\alpha_{s,j}} + \sqrt{\beta}) \sqrt{\frac{\log(t)}{n_{s,j}(t)}}, S_{j} - (\sqrt{\alpha_{s,j}} - \sqrt{\beta}) \sqrt{\frac{\log(t)}{n_{s,j}(t)}} \right].$$

In particular, for any buyer $i \in [N]$, any seller $j \in [M]$, and $\alpha_{max} \ge \beta$, we have $b_i(t) \ge B_i$ and $s_j(t) \le S_j$.

Proposition 11. For any round t, under the event $\mathcal{E}_t^{(\beta)}$, the following events are true, for $\min\{\alpha_{s,j},\alpha_{b,i}\} \geq \beta$, the number of participants $K(t) \geq K^*$.

Proof. Under the conditions, from Lemma 10 we know that $b_i(t) \geq B_i$ and $s_j(t) \leq S_j$ for all buyers i and sellers j. The number of participant in the system given any value profile is given as $K(t) = \max_p \min(|\{i \in [N] : b_i(t) \geq p\}|, |\{j \in [M] : s_j(t) \leq p\}|)$. Let $p^* \in (S_{K^*}, B_{K^*})$. Then we have $|\{i \in [N] : b_i(t) \geq p^*\}| \geq |\{i \in [N] : B_i \geq p^*\}| = K^*$. Similarly, $|\{j \in [M] : s_j(t) \leq p^*\}| \geq |\{j \in [M] : S_j \leq p^*\}| = K^*$. Therefore, we have $K(t) \geq K^*$.

Let us define the gaps for the seller j from the minimum seller as $\Delta_{s,j}=(S_j-S_{K^*})$, and the gap for the buyer i from the maximum buyer as $\Delta_{b,i}=(B_{K^*}-B_i)$. Also, we define for the seller j, $\Delta_{s,b,j}=(S_j-B_{K^*})$, and for the buyer i, $\Delta_{b,i}=(S_{K^*}-B_i)$. We have $S_{K^*}\leq B_{K^*}$. We now introduce a definition next to ease exposition of simultaneous participations of two buyers or two sellers.

Definition 12. A buyer i (a seller j) precedes a buyer i' (resp., a seller j') if and only if buyer i (resp., seller j) participates, and buyer i' (resp., seller j') does not participate.

The next lemma states that once a true non-participant buyer (seller) have enough samples, this buyer (resp., seller) never precede the true participant buyers (resp., sellers).

Lemma 13. For any round t, under the event $\mathcal{E}_t^{(\beta)}$, the following events are true, for $\alpha_{\min} > \beta$

• for any specific buyer $i' \in [K^*]$, if for a buyer $i \geq (K^* + 1)$, $n_{b,i}(t) \geq \frac{(\sqrt{\alpha_{b,i}} + \sqrt{\beta})^2}{(B_{i'} - B_i)^2} \log(t)$ then buyer i does not precede buyer i'.

• for any specific seller $j' \in [K^*]$, if for any $j \geq (K^* + 1)$, $n_{s,j}(t) \geq \frac{(\sqrt{\alpha_{s,j}} + \sqrt{\beta})^2}{(S_j - S_{j'})^2} \log(t)$ then seller j does not precede seller j'.

Proof. Let K(t) be the number of participants in round t. We know that under $\mathcal{E}_t^{(\beta)}$ and $\alpha_{\min} > \beta$, $s_j(t) < S_j$ and $b_i(t) > B_i$ for all $i \in [N]$ and $j \in [M]$. We note that if for any $i \geq (K^* + 1)$ and $i' \leq K^*$, if $n_{b,i}(t) \geq \frac{(\sqrt{\alpha_{b,i}} + \sqrt{\beta})^2}{(B_{i'} - B_i)^2} \log(t)$ then $b_i(t) \leq B_{i'}$. For buyers i' we have $b_{i'}(t) > B_{i'}$, hence buyer i can not precede any of the buyers i'. This is true as under the current mechanism for any i' with $b_{i'}(t) > b_i(t)$, it can not happen that buyer i' participates but buyer i does not. A similar argument proves the seller side statement.

Furthermore, the above Lemma 13 can be used in conjunction to Proposition 11, to show the true participant buyers (or sellers) fail to match only logarithmically many times.

Lemma 14. Under the event $\mathcal{E}^{(\beta)} = \bigcup_{t=1}^T \mathcal{E}_t^{(\beta)}$ and $\alpha_{\min} > \beta$, we have

•
$$n_{b,i}(T) \ge T - \sum_{i' \ge K^* + 1} \frac{(\sqrt{\alpha_{b,i'}} + \sqrt{\beta})^2}{(B_i - B_{i'})^2} \log(T)$$
 for any $i \in [K^*]$,

•
$$n_{s,j}(T) \ge T - \sum_{j' \ge K^* + 1} \frac{(\sqrt{\alpha_{s,j'}} + \sqrt{\beta})^2}{(S_{j'} - S_j)^2} \log(T)$$
 for any $j \in [K^*]$.

Proof. We know that under the condition of the lemma, $K(t) \geq K^*$ for all $1 \leq t \leq T$. Therefore, we have $\sum_{i \in [N]} n_{b,i}(T) \geq K^*T$. Furthermore, as $K(t) \geq K^*$ in each round, a buyer $i \in [K^*]$ does not participate, only if there exists at least one participant $i' \geq (K^*+1)$ that precedes buyer i. This is because if in some round no participant $i' \geq (K^*+1)$ precedes buyer i and buyer i does not match then that implies at most there can be (K^*-1) matches in that round. This leads to a contradiction of $K(t) \geq K^*$. However, under event $\mathcal{E}^{(\beta)}$ and $\alpha_{\min} > \beta$, from Lemma 13 we know that any $i' \geq (K^*+1)$ can precede buyer i only $\frac{(\sqrt{\alpha_{b,i'}}+\sqrt{\beta})^2}{(B_i-B_{i'})^2}\log(T)$ many times. This implies that $(T-n_{b,i}(T)) \leq \sum_{i' \geq K^*+1} \frac{(\sqrt{\alpha_{b,i'}}+\sqrt{\beta})^2}{(B_i-B_{i'})^2}\log(T)$.

A similar treatment of the sellers give us the remaining result.

We next show that the true non-participants only match logarithmically many times. This means K(t) converges to K^* fast. It is important to note that the previous argument does not conclude that where we mainly relied upon $K(t) \geq K^*$ for all t w.h.p. To that end we introduce the following definition.

Definition 15. A buyer i, and a seller j co-participates in a given round, if and only if both buyer i, and seller j participates in that round.

Lemma 16. Under the event $\mathcal{E}^{(\beta)} = \bigcup_{t=1}^T \mathcal{E}_t^{(\beta)}$ and $\alpha_{\min} > \beta$, we have

•
$$n_{b,i}(T) \leq \left(\frac{(\sqrt{\alpha_{b,i}} + \sqrt{\beta})^2}{(B_{K^*} - B_i)^2} + \sum_{j \geq (K^* + 1)} \frac{(\sqrt{\alpha_{s,j}} + \sqrt{\beta})^2}{(S_j - B_i)^2}\right) \log(T) \text{ for any } i \geq (K^* + 1) \text{ and }$$

$$\sum_{i \geq (K^* + 1)} n_{b,i}(T) \leq \left(\sum_{i \geq (K^* + 1)} \frac{(\sqrt{\alpha_{b,i}} + \sqrt{\beta})^2}{(B_{K^*} - B_i)^2} + \sum_{j \geq (K^* + 1)} \frac{(\sqrt{\alpha_{s,j}} + \sqrt{\beta})^2}{(S_j - B_{K^*})^2}\right) \log(T),$$

•
$$n_{s,j}(T) \leq \left(\frac{(\sqrt{\alpha_{s,j}} + \sqrt{\beta})^2}{(S_j - S_{K^*})^2} + \sum_{i \geq (K^* + 1)} \frac{(\sqrt{\alpha_{b,i}} + \sqrt{\beta})^2}{(S_j - B_i)^2}\right) \log(T)$$
 for any $j \geq (K^* + 1)$, and
$$\sum_{j \geq (K^* + 1)} n_{s,j}(T) \leq \left(\sum_{j \geq (K^* + 1)} \frac{(\sqrt{\alpha_{s,j}} + \sqrt{\beta})^2}{(S_j - S_{K^*})^2} + \sum_{i \geq (K^* + 1)} \frac{(\sqrt{\alpha_{b,i}} + \sqrt{\beta})^2}{(S_{K^*} - B_i)^2}\right) \log(T).$$

Proof. Let us consider a non-participant buyer i. From Lemma 13 we know that if $n_{b,i}(t) \geq Th_i(t) \equiv \frac{(\sqrt{\alpha_{b,i}} + \sqrt{\beta})^2}{(B_{K^*} - B_i)^2} \log(t)$ then buyer i does not precede any buyer $i' \in [K^*]$. Therefore, for this buyer to match there should exist at least $(K^* + 1)$ sellers with bids no more than the bid of this buyer i, and at least $(K^* + 1)$ seller participates. This further implies that for this buyer to have additional participation, after $n_{b,i}(t) = Th_i(T)$:

- 1. There exists at least 1 seller $j \geq (K^* + 1)$ with bid no more than the bid of this buyer i.
- 2. There exists at least 1 seller $j \ge (K^* + 1)$ such that buyer i and seller j co-participates.

Let $\mathcal{S}_{i,j}$ be the rounds after $n_{b,i}(t) = Th_i(T)$, and buyer i and seller j co-participates. Therefore, from the 2nd point above we conclude that $n_{b,i}(T) \leq Th_i(T) + |\cup_{j \geq (K^*+1)} \mathcal{S}_{i,j}|$.

However, if the seller j has $n_{s,j}(t) > Th_{i,j}(t) \equiv \frac{(\sqrt{\alpha_{s,j}} + \sqrt{\beta})^2}{(S_j - B_i)^2} \log(t)$ many participation then under $\mathcal{E}_t^{(\beta)}$ and $\alpha_{\min} > \beta$ we have $s_j(t) > b_i(t)$. But then from the 1st point above we know that $|\mathcal{S}_{i,j}| \leq Th_{i,j}(T)$, because $|\mathcal{S}_{i,j}| \leq n_{s,j}(T)$, and buyer i and seller j can co-participate only if $n_{s,j}(t) \leq Th_{i,j}(T)$. Therefore, we have $n_{b,i}(T) \leq Th_i(T) + \sum_{j \geq (K^* + 1)} Th_{i,j}(T)$.

In fact, we can improve the cumulative bound. We have

$$\sum_{i>(K^*+1)} n_{b,i}(T) \le \sum_{i>(K^*+1)} Th_i(T) + |\cup_{i\ge(K^*+1)} \cup_{j\ge(K^*+1)} S_{i,j}|.$$

However, we know that after $n_{s,j}(t) > \max_i Th_{i,j}(T)$ a seller $j \ge (K^* + 1)$ can not co-participate for any seller $i \ge (K^* + 1)$. Thus we can bound $|\bigcup_{i>(K^*+1)} S_{i,j}| \le \max_i Th_{i,j}(T)$.

$$\sum_{i \ge (K^*+1)} n_{b,i}(T) \le \sum_{i \ge (K^*+1)} Th_i(T) + \sum_{j \ge (K^*+1)} \max_{i \ge (K^*+1)} Th_{i,j}(T).$$

A similar treatment proves the lemma for a non-participant seller j.

Corollary 17. Under the event $\mathcal{E}^{(\beta)} = \bigcup_{t=1}^T \mathcal{E}_t^{(\beta)}$ and $\alpha_{\min} > \beta$, we have

•
$$n_{b,i}(T) \le \frac{(M-K^*+1)(\sqrt{\alpha_{\max}}+\sqrt{\beta})^2}{(B_{K^*}-B_i)^2}\log(T)$$
 for any $i \ge (K^*+1)$,

•
$$n_{s,j}(T) \le \frac{(N-K^*+1)(\sqrt{\alpha_{\max}}+\sqrt{\beta})^2}{(S_j-S_{K^*})^2}\log(T)$$
 for any $j \ge (K^*+1)$.

We have shown, up to this point, that the true participating buyers and sellers participate in all but O(log(T)) rounds. Moreover, true non-participating buyers and sellers participate in O(log(T)) rounds. This suffices to show the regret for non-participating buyers and sellers is O(log(T)) (we will state this precisely later). However, to compute the regret for participating buyers and sellers we next need to understand how the price is set in each round. We next argue that if in any round t, sellers $j \geq (K^* + 1)$ do not participate, and buyer K^* participates then the regret of a true participating buyer is small. Similarly, if in any round t, buyers $i \geq (K^* + 1)$ do not participate, and seller K^* participates then the regret of a true participating seller is small.

Recall that $p^* = (S_{K^*} + B_{K^*})/2$ be the price under true bids for the average mechanism, and $p(t) = (\min_{i' \in \mathcal{P}_b(t)} b_{i'}(t) + \max_{j \in \mathcal{P}_s(t)} s_j(t))/2$ denotes the price in round t. Further, $\chi_{b,i}(t)$, and $\chi_{s,j}(t)$ is the participation indicator for buyer i, and seller j respectively.

The regret for any true non-participating buyer or seller can be computed easily. We will present it later.

The regret for any true participating buyer $i \in [K^*]$ can be decomposed in rounds where the buyer i does not participate, and where the buyer i participates.

$$r_{b,i}(T) = \sum_{t:\chi_{b,i}(t)=0} (B_i - (B_{K^*} + S_{K^*})/2) + \sum_{t:\chi_{b,i}(t)=1} (p(t) - p^*).$$

Similarly, for any true participating seller $j \in [K^*]$ the regret is bounded as

$$r_{s,j}(T) = \sum_{t:\chi_{s,j}(t)=0} ((B_{K^*} + S_{K^*})/2 - S_j) + \sum_{t:\chi_{s,j}(t)=1} (p^* - p(t)).$$

We first focus on the regret of the buyers which implies upper bounding $\sum_t (p(t) - p^*)$ in the next lemma.

Lemma 18. Under the event $\mathcal{E}^{(\beta)} = \bigcup_{t=1}^T \mathcal{E}_t^{(\beta)}$ and $\alpha_{\min} > \beta$, we have for all $\tilde{i} \in [N]$ and $\tilde{j} \in [M]$

$$\sum_{t:\chi_{b,\tilde{i}}(t)=1} (p(t)-p^*) \le C_b \log(T) + (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{n_{b,\tilde{i}}(T) \log(T)},$$

⁷Note that for any $i, j \ge (K^* + 1)$ we have $S_j > B_i$.

$$\sum_{t:\chi_{s,\tilde{j}}(t)=1} (p^* - p(t)) \le C_s \log(T) + (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{n_{b,\tilde{j}}(T) \log(T)},$$

where

$$2C_b = \sum_{j < K^*} \frac{(\sqrt{\alpha_{max}} + \sqrt{\beta})^2}{(S_{K^*} - S_j)} + \sum_{i \ge (K^* + 1)} \frac{(\sqrt{\alpha_{max}} + \sqrt{\beta})^2 \sqrt{(M - K^* + 1)}}{(B_{K^*} - B_i)}$$

$$+ \sum_{j \ge (K^* + 1)} \left(\frac{(N - K^* + 1)(\sqrt{\alpha_{max}} + \sqrt{\beta})^2}{(S_j - S_{K^*})} + \frac{\sqrt{(N - K^* + 1)}(\sqrt{\alpha_{max}} + \sqrt{\beta})^2}{(S_j - S_{K^*})} \right),$$

$$2C_s = \sum_{i < K^*} \frac{(\sqrt{\alpha_{max}} + \sqrt{\beta})^2}{(B_i - B_{K^*})} + \sum_{j \ge (K^* + 1)} \frac{(\sqrt{\alpha_{max}} + \sqrt{\beta})^2 \sqrt{(N - K^* + 1)}}{(S_j - S_{K^*})}$$

$$+ \sum_{i > (K^* + 1)} \left(\frac{(M - K^* + 1)(\sqrt{\alpha_{max}} + \sqrt{\beta})^2}{(B_{K^*} - B_i)} + \frac{\sqrt{(M - K^* + 1)}(\sqrt{\alpha_{max}} + \sqrt{\beta})^2}{(B_{K^*} - B_i)} \right)$$

We are now in a position to prove out main theorem.

Theorem 19. The regret of the Average mechanism with buyers bidding $UCB(\alpha)$, and sellers bidding $LCB(\alpha)$ of their estimated valuation, for $\alpha_{\min} > \beta \geq 4$, we have the expected regret is bounded as:

- for a participating buyer $i \in [K^*]$ as $R_{b,i}(T) \leq (\sqrt{\alpha_{max}} + \sqrt{\beta})\sqrt{T\log(T)} + C_{b',i}\log(T)$,
- for a participating seller $j \in [K^*]$ as $R_{s,j}(T) \leq (\sqrt{\alpha_{max}} + \sqrt{\beta})\sqrt{T\log(T)} + C_{s',j}\log(T)$,
- for a non-participating buyer $i \ge (K^* + 1)$ as $R_{b,i}(T) \le \frac{\sqrt{(M K^* + 1)}(\sqrt{\alpha_{max}} + \sqrt{\beta})^2}{(B_{K^*} B_i)} \log(T),$
- for a non-participating seller $j \ge (K^* + 1)$ as $R_{s,j}(T) \le \frac{\sqrt{(N-K^*+1)}(\sqrt{\alpha_{max}} + \sqrt{\beta})^2}{(S_j S_{K^*})} \log(T).$

Here C_b and C_s is as defined in Lemma 18, and

$$C_{b',i} = \left((N - K^*) \frac{(\sqrt{\alpha_{max}} + \sqrt{\beta})^2}{(B_i - p^*)} + C_b \right),$$

$$C_{s',j} = \left((M - K^*) \frac{(\sqrt{\alpha_{max}} + \sqrt{\beta})^2}{(p^* - S_j)} + C_s \right).$$

Proof of Regret Upper Bound. We first bound the regret under the event $\mathcal{E}^{(\beta)} = \cup_{t=1}^T \mathcal{E}_t^{(\beta)}$ and $\alpha_{\min} > \beta$. Applying the bounds in Lemma 18, we bound the of a true participating buyer $i \in [K^*]$ as

$$\begin{split} r_{b,i}(T) &= \sum_{t:\chi_{b,i}(t)=0} (B_i - (B_{K^*} + S_{K^*})/2) + \sum_{t:\chi_{b,i}(t)=1} (p(t) - p^*) \\ &\leq \sum_{i' \geq K^*+1} \frac{(\sqrt{\alpha_{max}} + \sqrt{\beta})^2 (B_i - (B_{K^*} + S_{K^*})/2)}{(B_i - B_{i'})^2} \log(T) + C_b \log(T) + (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{T \log(T)} \\ &\leq \left((N - K^*) \frac{(\sqrt{\alpha_{max}} + \sqrt{\beta})^2}{(B_i - p^*)} + C_b \right) \log(T) + (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{T \log(T)} \end{split}$$

Similarly, for any true participating seller $j \in [K^*]$ the regret is bounded as

$$\begin{split} r_{s,j}(T) &= \sum_{t:\chi_{s,j}(t)=0} ((B_{K^*} + S_{K^*})/2 - S_j) + \sum_{t:\chi_{s,j}(t)=1} (p^* - p(t)) \\ &\leq \sum_{j' \geq K^*+1} \frac{(\sqrt{\alpha_{max}} + \sqrt{\beta})^2 ((B_{K^*} + S_{K^*})/2 - S_j)}{(S_{j'} - S_j)^2} \log(T) + C_s \log(T) + (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{T \log(T)} \\ &\leq \left((M - K^*) \frac{(\sqrt{\alpha_{max}} + \sqrt{\beta})^2}{(p^* - S_j)} + C_s \right) \log(T) + (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{T \log(T)} \end{split}$$

The regret for any true non-participating buyer $i \geq (K^* + 1)$ is non negative only when the buyer i participates. Under $\mathcal{E}^{(\beta)}$ and $\alpha_{\min} > \beta$ we have

$$\begin{split} r_{b,i}(T) &= \sum_{t:\chi_{b,i}(t)=1} (p(t) - B_i) \\ &\leq \sum_{t:\chi_{b,i}(t)=1} (b_i(t) - B_i) \\ &\leq \sum_{t:\chi_{b,i}(t)=1} (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{\frac{\log(t)}{n_{b,i}(t)}} \\ &\leq \sum_{n=1}^{n_{b,i}(T)} (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{\frac{\log(T)}{n}} \\ &\leq (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{n_{b,i}(T) \log(T)} \end{split}$$

Where the first inequality is due to the fact that if buyer i participates in round t then bid $b_i(t) \ge p(t)$. Also, the regret for any true non-participating buyer $j \ge (K^* + 1)$ is non negative only when the buyer i participates. Under $\mathcal{E}^{(\beta)}$ and $\alpha_{\min} > \beta$ we have similarly

$$r_{s,j}(T) = \sum_{t:\chi_{s,j}(t)=1} (S_j - s_j(t)) \le (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{n_{s,j}(T) \log(T)}$$

The terms $n_{b,i}(T)$ and $n_{s,j}(T)$ above can be bounded using Lemma 16 when \mathcal{E}^{β} holds for $\alpha_{max} \geq \beta$ Therefore, the expected regret can be bounded for $\beta \geq 4$ as

$$R_{b,i}(T) \leq \mathbb{E}[r_{b,i}(T)|\mathcal{E}^{(\beta)}] + b_{max}(1 - \mathbb{P}[\mathcal{E}^{(\beta)}])$$

$$\leq \mathbb{E}[r_{b,i}(T)|\mathcal{E}^{(\beta)}] + b_{max} \sum_{t=1}^{T} MN/t^{\beta/2}$$

$$\leq \mathbb{E}[r_{b,i}(T)|\mathcal{E}^{(\beta)}] + MNb_{max}\pi^{2}/6.$$

Similarly, for $\beta > 4$ we have

$$R_{s,j}(T) \leq \mathbb{E}[r_{s,j}(T)|\mathcal{E}^{(\beta)}] + MNs_{max}\pi^2/6.$$

This concludes the proof.

Let us recall the minimum gap is $\Delta = \min_{i, \in [N], j \in [M]} \{ |p^* - S_j|, |B_i - p^*| \}$. Then we can bound the constants associated with the logarithmic terms as

Corollary 20. The constants in Theorem 19 is upper bounded as

$$\begin{array}{l} \bullet \ \, C_{b'} \leq \frac{\left(N + (N - K^* + 1)\sqrt{M - K^* + 1} + \sqrt{N - K^* + 1}(M - K^* + 1) + (N - K^* + 1)(M - K^* + 1)\right)\left(\sqrt{\alpha_{max}} + \sqrt{\beta}\right)^2}{\Delta} \\ \bullet \ \, C_{s'} \leq \frac{\left(M + (N - K^* + 1)\sqrt{M - K^* + 1} + \sqrt{N - K^* + 1}(M - K^* + 1) + (N - K^* + 1)(M - K^* + 1)\right)\left(\sqrt{\alpha_{max}} + \sqrt{\beta}\right)^2}{\Delta} \end{array}$$

$$\bullet \ \ C_{s'} \leq \frac{\left(M + (N - K^* + 1)\sqrt{M - K^* + 1} + \sqrt{N - K^* + 1}(M - K^* + 1) + (N - K^* + 1)(M - K^* + 1)\right)\left(\sqrt{\alpha_{max}} + \sqrt{\beta}\right)^2}{\Delta}$$

A.1 Proof of Lemma 18

Proof. We first focus on the upper bound for some buyer $i \in [N]$.

$$\sum_{t:\chi_{b,\bar{i}}(t)=1} (p(t)-p^*) = \frac{1}{2} \sum_{t:\chi_{b,\bar{i}}(t)=1} \left(\min_{i' \in \mathcal{P}_b(t)} b_{i'}(t) - B_{K^*} \right) + \frac{1}{2} \sum_{t:\chi_{b,\bar{i}}(t)=1} (\max_{j \in \mathcal{P}_s(t)} s_j(t) - S_{K^*})$$

The rounds where buyer K^* participates, we have $\min_{i \in \mathcal{P}_b(t)} b_i(t) \leq b_{K^*}(t)$. Therefore, we can bound

$$\sum_{t} \left(\min_{i' \in \mathcal{P}_b(t)} b_{i'}(t) - B_{K^*} \right)$$

$$\leq \sum_{t:\chi_{b,K^*}(t)=0} \left(\min_{i' \in \mathcal{P}_b(t)} b_{i'}(t) - B_{K^*} \right) + \sum_{t:\chi_{b,K^*}(t)=1,\chi_{b,\tilde{i}}(t)=1} \left(b_{K^*}(t) - B_{K^*} \right)$$

Under $\mathcal{E}^{(\beta)} = \bigcup_{t=1}^T \mathcal{E}^{(\beta)}_t$ and $\alpha_{\min} > \beta$, we know that $\chi_{b,K^*}(t) = 0$ only if $\max_{i \geq (K^*+1)} \chi_{b,K^*}(t) = 1$. That is K^* does not participate, only if at least one of the non-participant buyers participate. Hence we further have,

$$\sum_{t:\chi_{b,K^*}(t)=0} \left(\min_{i' \in \mathcal{P}_b(t)} b_{i'}(t) - B_{K^*} \right)$$

$$\leq \sum_{i \geq (K^*+1)} \sum_{t:\chi_{b,i}(t)=1,\chi_{b,K^*}(t)=0} (b_i(t) - B_{K^*})$$

$$\leq \sum_{i \geq (K^*+1)} \sum_{t:\chi_{b,i}(t)=1} (b_i(t) - B_{K^*})$$

$$\leq \sum_{i \geq (K^*+1)} \sum_{t:\chi_{b,i}(t)=1} (b_i(t) - B_i)$$

Under the event $\mathcal{E}^{(\beta)}=\cup_{t=1}^T\mathcal{E}_t^{(\beta)}$ and $\alpha_{\min}>\beta$, we have

$$\begin{split} & \sum_{t:\chi_{b,K^*}(t)=1,\chi_{b,\tilde{i}}(t)=1} (b_{K^*}(t) - B_{K^*}) \\ & \leq \sum_{t:\chi_{b,K^*}(t)=1,\chi_{b,\tilde{i}}(t)=1} (\sqrt{\alpha_{b,K^*}} + \sqrt{\beta}) \sqrt{\frac{\log(t)}{n_{b,K^*}(t)}} \\ & \leq \sum_{n_{b,\tilde{i}}(T)} (\sqrt{\alpha_{b,K^*}} + \sqrt{\beta}) \sqrt{\frac{\log(T)}{n}} \leq (\sqrt{\alpha_{b,K^*}} + \sqrt{\beta}) \sqrt{n_{b,\tilde{i}}(T) \log(T)}, \end{split}$$

Above, we use the logic that the summation is minimized when $n_{b,K^*}(t)$ increases in unison with $n_{b,i}(t)$, otherwise we will have larger denominator.

From Corollary 17 we know that under the event $\mathcal{E}^{(\beta)}$ and $\alpha_{\min} > \beta$, the maximum number of time a buyer $i \geq (K^* + 1)$ can participate is

$$\tilde{T}h_i(T) = (M - K^* + 1)\frac{(\sqrt{\alpha_{\text{max}}} + \sqrt{\beta})^2}{(B_{K^*} - B_i)^2}\log(T).$$

Under the event $\mathcal{E}^{(\beta)} = \cup_{t=1}^T \mathcal{E}_t^{(\beta)}$ and $\alpha_{\min} > \beta$, we have

$$\sum_{t:\chi_{b,i}(t)=1} (b_i(t) - B_i)$$

$$\leq \sum_{t:\chi_{b,i}(t)=1} (\sqrt{\alpha_{\max}} + \sqrt{\beta}) \sqrt{\frac{\log(t)}{n_{b,i}(t)}}$$

$$\leq \sum_{n=1}^{\tilde{T}h_i(T)} (\sqrt{\alpha_{\max}} + \sqrt{\beta}) \sqrt{\frac{\log(T)}{n}}$$

$$\leq (\sqrt{\alpha_{\max}} + \sqrt{\beta}) \sqrt{\tilde{T}h_i(T) \log(T)}$$

$$\leq \frac{(\sqrt{\alpha_{\max}} + \sqrt{\beta})^2 \sqrt{(M - K^* + 1)}}{(B_{K^*} - B_i)} \log(T).$$

Let $j_{\max}(t) = \arg\max_{j \in [M]} \left((S_j - S_{K^*}) + (\sqrt{\alpha_{s,j}} + \sqrt{\beta}) \sqrt{\frac{\log(t)}{n_{b,j}(t)}} \right)$. We have under the event $\mathcal{E}^{(\beta)} = \cup_{t=1}^T \mathcal{E}_t^{(\beta)}$ and $\alpha_{\min} > \beta$,

$$\sum_{t} \left(\max_{j \in \mathcal{P}_s(t)} s_j(t) - S_{K^*} \right)$$

$$\leq \sum_{t} \max_{j \in \mathcal{P}_{s}(t)} \left((S_{j} - S_{K^{*}}) + (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{\frac{\log(t)}{n_{b,j}(t)}} \right) \\
\leq \sum_{j \in [M]} \sum_{t: \chi_{s,j}(t) = 1, j_{\max}(t) = j} \left((S_{j} - S_{K^{*}}) + (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{\frac{\log(t)}{n_{b,j}(t)}} \right)$$

Further, under the event $\mathcal{E}^{(\beta)}$ and $\alpha_{\min} > \beta$, we know that $K(t) \geq K^*$, which implies at least one seller $j \geq K^*$ is active. Hence, for any $j < K^*$, $j_{\max}(t) = j$ only if

$$n_{b,j}(t) \le \tilde{Th}_j(T) = \min_{j' \ge K^*} \frac{(\sqrt{\alpha_{max}} + \sqrt{\beta})^2}{(S_{j'} - S_j)^2} \log(T) = \frac{(\sqrt{\alpha_{max}} + \sqrt{\beta})^2}{(S_{K^*} - S_j)^2} \log(T).$$

Also, the maximum number of times any seller $j \ge (K^* + 1)$ participates under the event $\mathcal{E}^{(\beta)}$ and $\alpha_{\min} > \beta$ is $Th_j(T) \ge \frac{(N - K^* + 1)(\sqrt{\alpha_{max}} + \sqrt{\beta})^2}{(S_j - S_{K^*})^2} \log(T)$, according to 17.

Therefore, we can proceed as

$$\begin{split} & \sum_{t:\chi_{b,\hat{i}}(t)=1} (\max_{j \in \mathcal{P}_{s}(t)} s_{j}(t) - S_{K^{*}}) \\ & \leq \sum_{j \in [M]} \sum_{t:\chi_{s,j}(t)\chi_{s,\hat{i}}(t)=1, j_{\max}(t)=j} \left((S_{j} - S_{K^{*}}) + (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{\frac{\log(t)}{n_{b,j}(t)}} \right) \\ & \leq \sum_{j \geq (K^{*}+1)} \sum_{t:\chi_{s,j}(t)\chi_{b,\hat{i}}(t)=1} \left((S_{j} - S_{K^{*}}) + (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{\frac{\log(t)}{n_{b,j}(t)}} \right) \\ & + \sum_{j \leq K^{*}} \sum_{t:\chi_{s,j}(t)\chi_{b,\hat{i}}(t)=1, j_{\max}(t)=j} \left((S_{j} - S_{K^{*}}) + (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{\frac{\log(t)}{n_{b,j}(t)}} \right) \\ & \leq \sum_{j \geq (K^{*}+1)} \left(Th_{j}(T)(S_{j} - S_{K^{*}}) + \sum_{n=1}^{Th_{j}(T)} (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{\frac{\log(T)}{n}} \right) \\ & + \sum_{j < K^{*}} \sum_{n=1}^{\tilde{T}h_{j}(T)} (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{\frac{\log(T)}{n}} + \sum_{n=1}^{n_{b,\hat{i}}(T)} (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{\frac{\log(T)}{n}} \right) \\ & \leq \sum_{j \geq (K^{*}+1)} \left(Th_{j}(T)(S_{j} - S_{K^{*}}) + (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{Th_{j}(T)\log(T)} \right) \\ & + \sum_{j < K^{*}} \left((\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{\tilde{T}h_{j}(T)\log(T)} + (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{n_{b,\hat{i}}(T)\log(T)} \right) \\ & \leq \sum_{j \geq (K^{*}+1)} \left(\frac{(N - K^{*} + 1)(\sqrt{\alpha_{max}} + \sqrt{\beta})^{2}}{(S_{j} - S_{K^{*}})} + \frac{\sqrt{(N - K^{*} + 1)}(\sqrt{\alpha_{max}} + \sqrt{\beta})^{2}}{(S_{j} - S_{K^{*}})} \right) \log(T) \\ & + \sum_{j < K^{*}} \frac{(\sqrt{\alpha_{max}} + \sqrt{\beta})^{2}}{(S_{K^{*}} - S_{j})} \log(T) + (\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{n_{b,\hat{i}}(T)\log(T)} \end{split}$$

Combining the above bounds, we get

$$\begin{aligned} 2 \sum_{t: \chi_{b, \overline{i}}(t) = 1} (p(t) - p^*) \\ \leq \underbrace{2(\sqrt{\alpha_{max}} + \sqrt{\beta}) \sqrt{n_{b, \overline{i}}(T) \log(T)}}_{\text{price setting buyer and seller}} + \underbrace{\sum_{j < K^*} \frac{(\sqrt{\alpha_{max}} + \sqrt{\beta})^2}{(S_{K^*} - S_j)} \log(T)}_{\text{pron-price setting participant sellers}} \end{aligned}$$

$$+ \underbrace{\sum_{i \geq (K^*+1)} \frac{(\sqrt{\alpha_{max}} + \sqrt{\beta})^2 \sqrt{(M - K^* + 1)}}{(B_{K^*} - B_i)} \log(T)}_{\text{non-participating buyers}} \\ + \underbrace{\sum_{j \geq (K^*+1)} \left(\frac{(N - K^* + 1)(\sqrt{\alpha_{max}} + \sqrt{\beta})^2}{(S_j - S_{K^*})} + \frac{\sqrt{(N - K^* + 1)}(\sqrt{\alpha_{max}} + \sqrt{\beta})^2}{(S_j - S_{K^*})} \right) \log(T)}_{\text{Continuous}}.$$

Reversing the role of buyer and seller in the above derivation, and leveraging the symmetry in the system, we can get

$$\begin{split} 2\sum_{t:\chi_{s,\tilde{j}}(t)=1} &(p^*-p(t))\\ &\leq 2(\sqrt{\alpha_{max}}+\sqrt{\beta})\sqrt{n_{s,\tilde{j}}(T)\log(T)} + \sum_{i< K^*} \frac{(\sqrt{\alpha_{max}}+\sqrt{\beta})^2}{(B_i-B_{K^*})}\log(T)\\ &\text{price setting buyer and seller} \\ &+\sum_{j\geq (K^*+1)} \frac{(\sqrt{\alpha_{max}}+\sqrt{\beta})^2\sqrt{(N-K^*+1)}}{(S_j-S_{K^*})}\log(T)\\ &\text{non-participating sellers} \\ &+\sum_{i\geq (K^*+1)} \left(\frac{(M-K^*+1)(\sqrt{\alpha_{max}}+\sqrt{\beta})^2}{(B_{K^*}-B_i)} + \frac{\sqrt{(M-K^*+1)}(\sqrt{\alpha_{max}}+\sqrt{\beta})^2}{(B_{K^*}-B_i)}\right)\log(T). \end{split}$$

B Proofs From Section 6

B.1 Proof of Lemma 5

Proof. Assume the oracle buyer knows B, S, the average price mechanism and the fact that the seller is not strategizing.

Case I: B < S. If the buyer bids $B_t \ge S$, then the buyer will be matched with price $p_t \ge S$ and will receive an utility $U_t := B - p_t < 0$. If on the other hand, the buyer puts any bid $B_t < S$, then the buyer is not matched and receives an utility of 0. Thus, the optimal choice for the oracle buyer in this case is to place any bid $B_t < S$. Thus, bidding $B_t := S - \varepsilon$, for every $\varepsilon > 0$ is optimal.

Case II: $B \geq S$. If the buyer bids $B_t \geq S$, then the buyer will be matched with price $p_t = \frac{B_t + S}{2}$ and will receive an utility $U_t := B - p_t = \frac{2B - B_t - S}{2}$. Observe that the utility U_t is non-decreasing in the bid-price B_t . If $B_t < S$ however, no match occurs and the oracle buyer will receive 0 utility. If on the other-hand $S \leq B_t \leq B$, then $B - B_t \geq 0$ and $B - S \geq 0$. Thus, the utility $U_t \geq 0$. This along with the non-increasing nature of U_t gives that the optimal action is to play $B_t = S$.

B.2 Proof of Lemma 8

Proof. This follows the same recipe of bandit lower bounds [20]. Let the bidding policy be arbitrary as in the hypothesis of the lemma. Fix a system and denote by $\varepsilon := S - B$. We will choose an appropriate value of ε later in the proof. Denote by this system where $S - B = \varepsilon$ as ν . Denote by the system in which $B = S + \varepsilon$ as ν' . We denote by R_T and R_T' to be the regret obtained by the bidding policy in system ν and ν' respectively. The first observation we will make is that the divergence decomposition lemma gives

$$\mathbb{E}_{\nu}\left[\sum_{t=1}^{T} Z_{t}\right] D(-\varepsilon, \varepsilon) \ge \log\left(\frac{1}{2(\mathbb{P}_{\nu}(A) + \mathbb{P}_{\nu'}(A^{\complement}))}\right),\tag{1}$$

for any measurable event A. The proof of this claim follows from the well known Bretagnolle–Huber inequality (Theorem 14.2 [20]) and the divergence decomposition lemma (Lemma 15.1 [20]) to compute the divergence between the system ν and ν' . The formula from Lemma 15.1 of [20] when applied to our system simplifies to the LHS of Equation (1) by observing the fact that in both system ν and system ν' , not participating in the market gives a deterministic 0 reward. Thus, the KL divergence between the reward distributions for not participating is 0.

The rest of the proof is to verbatim follow the proof of Theorem 15.2 of [20] to re-arrange Equation (1) to yield the desired result. We denote by the event $A:=\left\{\sum_{t=1}^T Z_t \geq T/2\right\}$. Thus, trivially, $R_T \geq \frac{\varepsilon T}{2} \mathbb{P}_{\nu}[A]$ and $R_T^{'} \geq \frac{\varepsilon T}{2} \mathbb{P}_{\nu}[A^\complement]$. Furthermore, $\mathbb{E}\left[\sum_{t=1}^T Z_t\right] \leq T$. Now, using the fact that for unit variance gaussians, $D(-\varepsilon,\varepsilon)=2\varepsilon^2$ and plugging these estimates in Equation (1), we obtain that

$$R_T + R_T^{'} \ge \frac{\varepsilon T}{4} \exp\left(-2\varepsilon^2 T\right).$$

As $\varepsilon > 0$ was arbitrary, we can set it to be equal to $\frac{1}{\sqrt{T}}$, and use the well known fact that for any non-negative a, b, we have $a + b \le 2 \max(a, b)$, to get

$$\max(R_T, R_T^{'}) \ge \frac{1}{36}\sqrt{T}.$$

B.3 Proof of Corollary 6

Definition 21 (Bidding Policy). A sequence of binary random variables $(Z_t)_{t\geq 1}$, such that for all $t\geq 1, Z_t\in\{0,1\}$ denotes whether the buyer participates (by placing a bid of S) in the tth round or not such that Z_t is measurable with respect to the sigma-algebra generated by decisions and rewards $Z_1Y_{b,1},\cdots,Z_{t-1}Y_{b,t-1}$ observed till time t-1.

Definition 22 (Two-armed bandit policy:). A sequence of $\{0,1\}$ valued random variables $(\widehat{Z}_t)_{t\geq 1}$ such that for all time t, \widehat{Z}_t is measurable with respect to a sequence of \mathbb{R} valued random variables $\mathcal{F}_{t-1} := \sigma(\widehat{X}_1, \cdots, \widehat{X}_{t-1})$ such that for all time t, conditioned on \widehat{Z}_t , (i) \widehat{X}_t is independent of \mathcal{F}_{t-1} , and (ii) $\widehat{X}_t \sim \mathcal{P}_{\widehat{Z}_t}$, where \mathcal{P}_i , $i \in \{1,2\}$ are two fixed probability distributions on \mathbb{R} .

This definition formalizes the intuitive notion that a sequence of binary decisions made at each time, with the decision being measurable function of all past observations and the observations themselves being independent conditioned on the arm chosen.

Proof. A sequence of binary valued random variables $\{Z_{\cdot}, \cdots, Z_{T}\}$ and \mathbb{R} valued observation random variables $\{Y_{1}, \cdots, Y_{T}\}$ describes a policy for a two armed stochastic bandits if and only if (i) Z_{t} is measurable with respect to the sigma-algebra $\mathcal{F}_{t-1} := \sigma(Y_{1}, \cdots, Y_{t-1})$ generated by all observed rewards upto time t-1, and (ii) conditioned on the arm Z_{t} , the observed reward Y_{t} is conditionally independent of (Y_{1}, \cdots, Y_{t-1}) and (Z_{1}, \cdots, Z_{t-1}) the actions and rewards obtained in the past. It is easy to observe both of these for the bidding system described before.

C Additional Synthetic Experiments

In this section, we present some additional synthetic experiments to study the impact of different parameters on the performance of our algorithm. Our methodology is same as mentioned in Section 7. Recall, that the negative regret for participant buyers/sellers is expected, as the regret increases as $(p(t) - p^*)$ for sellers, and $(p^* - p(t))$ for buyers.

C.1 Convergence and Regret under Average Mechanism

We study an instance see this is Figure 3 where we have a 15×15 system with 10 matches. The rest of the behavior in Figure 3 is similar to Figure 2.

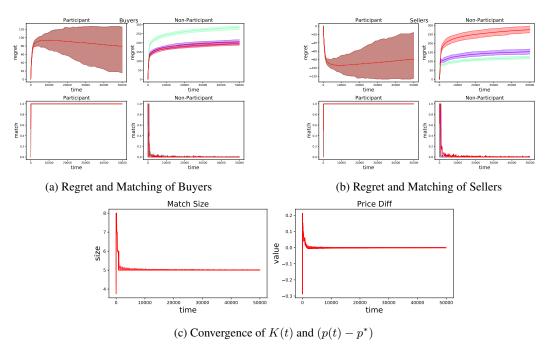


Figure 2: Double Auction $N=8, M=8, K^*=5, \Delta=0.2, \alpha_1=4,$ and $\alpha_2=8.$

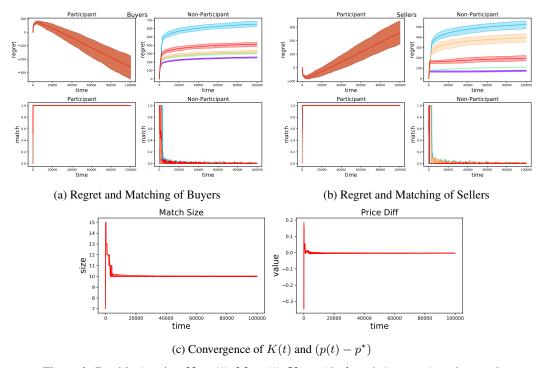


Figure 3: Double Auction $N=15,\,M=15,\,K^*=10,\,\Delta=0.4,\,\alpha_1=4,$ and $\alpha_2=8$

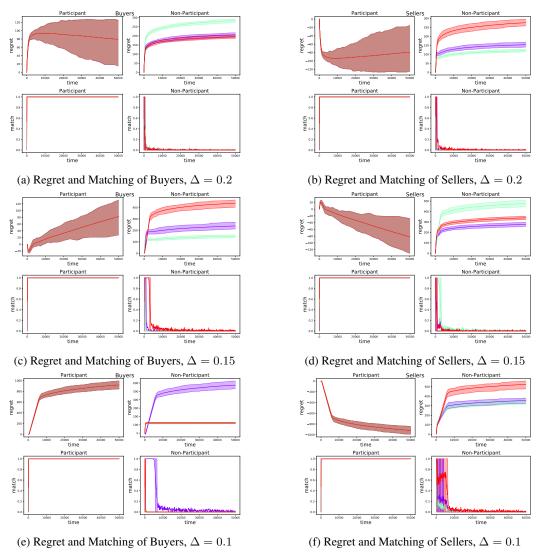


Figure 4: Double Auction with varying Δ ($N=M=8, K^*=5, \alpha_1=4, \text{ and } \alpha_2=8$).

C.2 Impact of the gap Δ

We first study the impact of changing the gap Δ on the performance, for a system of size 8x8 and $K^* = 5$. In Figure 4, we study three gaps, $\Delta \in \{0.1, 0.15, 0.2\}$. Here, we observe that the convergence in the number of times an agent participates is delayed as we decrease the gap. As a result, there is an increase in the regret of non-participants which is dominated by Δ , whereas the participant regret is not directly impacted by Δ (after the initial stage) as it is dominated by the $O(\sqrt{T\log(T)})$ term.

C.3 Impact of the sizes M, N, and K^*

We now study the impact of the size of the system, and number of true participants. First, with M=N, we vary M while keeping the $(M-K^*)$ fixed. We observe in Figure 5 that the regret of the agents do not vary a lot, which is as expected from the theory. Next, we keep M=N=8 fixed, while varying the participant size K^* . We see as $(M-K^*)$ increases, the regret increases in Figure 6 as suggested by the theory.

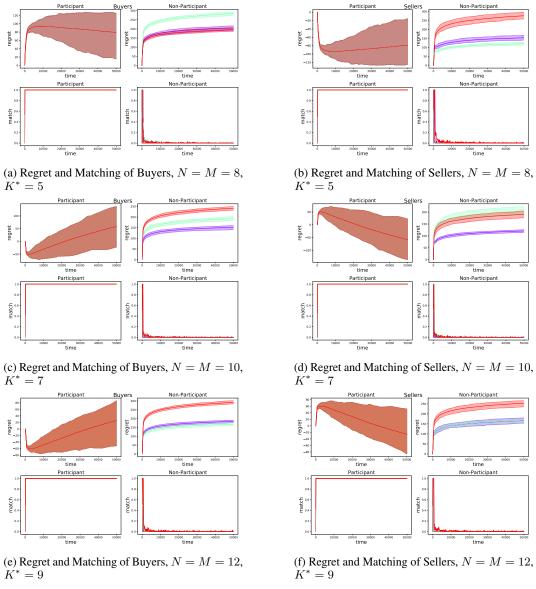


Figure 5: Double Auction with varying N=M, and fixed $(M-K^*)=3$ $(\Delta=0.2,\,\alpha_1=4,\,\alpha_2=8)$

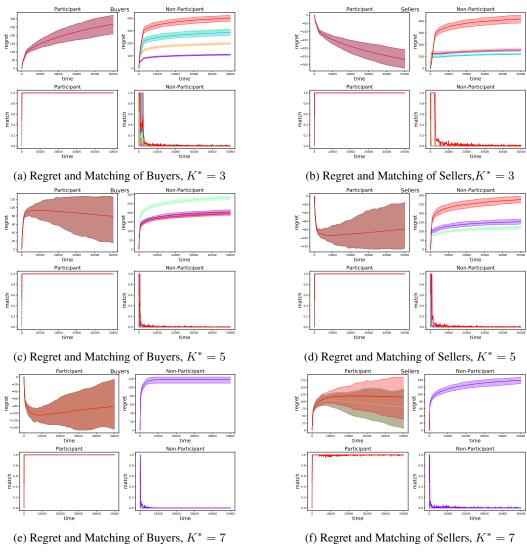


Figure 6: Double Auction with varying K^* ($N=M=8, \Delta=0.2, \alpha_1=4, \alpha_2=8$)

C.4 Impact of size difference in M and N.

In this part, we assess the size difference between number of sellers M, and number of buyers N. As the buyer and seller are symmetric, we keep N=8, and $K^*=5$ fixed. We next vary $M \in \{5,8,15\}$ to study the effect. In Figure 7, we observe the regret of non-participant buyers and sellers increases with increase in M. This is mainly because, the non-participant buyers presence of more non-participant sellers is allowing the non-participant buyers to match with these non-participant sellers.

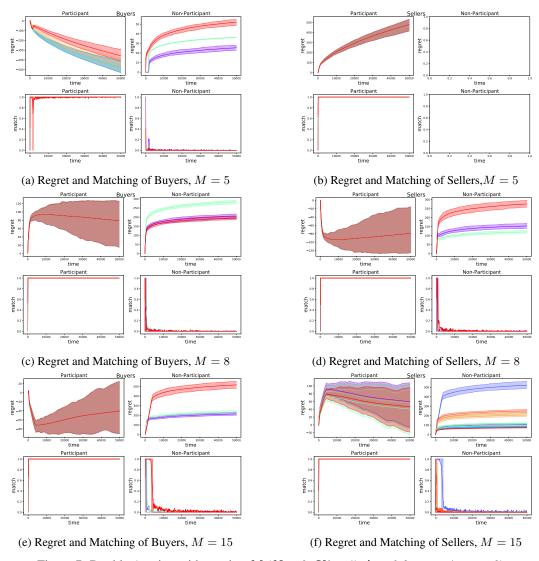


Figure 7: Double Auction with varying M ($N=8, K^*=5, \Delta=0.2, \alpha_1=4, \alpha_2=8$)