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Adaptive Neural Network Finite-Time Control of Uncertain Fractional-Order Systems with Unknown Dead-Zone Fault via Command Filter

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Abstract: In this paper, the adaptive finite-time control problem for fractional-order systems with uncertainties and unknown dead-zone fault was studied by combining a fractional-order command filter, radial basis function neural network, and Nussbaum gain function technique. First, the fractional-order command filter-based backstepping control method is applied to avoid the computational complexity problem existing in the conventional recursive procedure, where the fractional-order command filter is introduced to obtain the filter signals and their fractional-order derivatives. Second, the radial basis function neural network is used to handle the uncertain nonlinear functions in the recursive design step. Third, the Nussbaum gain function technique is considered to handle the unknown control gain caused by the unknown dead-zone fault. Moreover, by introducing the compensating signal into the control law design, the virtual control law, adaptive laws, and the adaptive neural network finite-time control law are constructed to ensure that all signals associated with the closed-loop system are bounded in finite time and that the tracking error can converge to a small neighborhood of origin in finite time. Finally, the validity of the proposed control law is confirmed by providing simulation cases.

Keywords: uncertain fractional-order systems; finite-time control; unknown dead-zone fault; neural network; command filter

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1. Introduction

Over the past several decades, control problems of uncertain nonlinear systems [1], nonsmooth nonlinear systems [2], strict-/nonstrict-feedback systems [3,4], and pure-feedback systems [5] have been widely studied, and to achieve the specified control objectives, various control laws have been constructed by scholars. It should be pointed out that the order of the above-mentioned systems is integer order, namely, the so-called integer-order systems. In fact, some systems, such as hyper-chaotic economic systems and heat conduction and viscoelastic structures [6,7], cannot be modeled by integer-order systems. Therefore, as the extension of integer-order systems, the control problems of fractional-order systems have been developed by many scholars. Currently, whether it is the solution problem of fractional calculus or the control problem, the research results of fractional calculus can be found in many literatures [8–12].

Because fractional-order systems break through the limitation of integer-order systems, they can better describe the historical information of control objects [13,14], which have attracted more and more attention in recent years [15–17]. An adaptive control law based on neural network was presented in [15], which guarantees that the tracking error of the switched fractional-order nonlinear systems can converge to a small

neighborhood of the origin under arbitrary switching. In [16], an $\mathcal{L}_1$ adaptive control law for the control problem of fractional-order systems with matched uncertainties and external disturbances was solved. The authors of [17] addressed an adaptive sliding mode observer for a class of Takagi–Sugeno fuzzy descriptor fractional-order systems, in which the assumption that the local input matrices are identical was eliminated by applying a fuzzy sliding surface. In addition, some excellent control strategies, such as the adaptive backstepping control law [18,19], the adaptive event-triggered control law [20], the observer-based adaptive fuzzy control law [21], the active disturbance rejection control-based backstepping control law [22], and their references have also been studied and applied.

It should be emphasized that the occurrence of actuator faults can sometimes not be predicted in advance. How to solve the control problems of fractional-order systems with unknown actuator faults is a problem worth studying. Moreover, for the actual control needs, it is usually hoped that the given systems can achieve the desired control in finite time. However, these problems have not been deeply studied in the above-mentioned literature.

Actuator faults are inevitable in most engineering problems. If these faults are not handled in time, these cases may lead to the weakening of the system's performance or even the complete failure of the control system. Therefore, it is important and necessary to study the control problems of fault systems (see [3,5,23–25], for example). The same is true for fractional-order systems. Recently, many interesting results have been gained for control schemes for fractional-order systems. In [26,27], adaptive fault-tolerant control laws with fuzzy logic systems were designed to solve the control problems of fractional-order systems, where the actuator faults involve partial failures, the loss of control effectiveness, and stuck faults. Considering the existence of saturation fault, the adaptive neural network constraint control law for fractional-order nonstrict-feedback systems was addressed in [28]. In [29], a stabilization criterion with linear matrix inequalities was proposed. This guarantees the robust stability of a class of variable-order fractional interval systems. Based on the designed neural network decentralized state observer and decentralized control law, the authors of [30] investigated the output–feedback control problem for fractional-order nonstrict-feedback large-scale systems with unknown dead-zone faults. Also, it should be pointed out that the problem of unknown control direction may be triggered when the system appears as an unknown failure. Since the sign of the control direction is unknown, this will bring great difficulties to the design of control laws of the systems. To solve the control problem of unknown control direction, the Nussbaum gain function control technique was proposed [31], and many related results have been proposed by scholars to solve the control problems of the systems with unknown control directions [32–35].

It should be noted that the solution to the above control problems is achieved in infinite time. However, some practical engineering applications, such as the chemical reaction process and spacecraft attitude control, need to achieve stability within finite time. Compared with the infinite time control strategy, the finite-time control strategy has a faster convergence rate and better robustness against uncertainty [36–38]. Correspondingly, some interesting results on the finite-time control of fractional-order systems were developed in [39–41]. Based on the backstepping control technique, a fractional finite-time adaptive fuzzy sliding control scheme for uncertain fractional order systems with uncertainties and external disturbances was designed in [39]. This ensures that the closed-loop system reaches the desired sliding mode surface in finite time. Different from [39], an adaptive finite-time control law with a fractional-order command filter was presented in [40], which can eliminate the computational complexity problem in the traditional backstepping design and the tracking error can be guaranteed to converge in a finite time. In [41], the finite-time event-triggered control problem for fractional-order systems was studied, and a finite-time control law combined with the event-triggered mechanism and the neural network was proposed.

Moreover, the control problems of fractional-order systems with unknown actuator fault can be found in some papers without further discussion of the finite time control problems [26,27,30]. Although a few studies have investigated the finite-time control problems of fractional-order systems [39–41], they did not consider the existence of unknown dead-zone fault. Inspired by the above discussion, the objective of this paper is to address the finite-time control for fractional-order systems with unknown dead-zone fault and uncertain dynamics. Based on the application of the fractional-order command filter, the radial basis function neural network, and the Nussbaum gain function technique, an adaptive neural network finite-time control law was developed. The main contributions of this paper are as follows:

(1) A class of uncertain fractional-order systems with unknown dead-zone fault is investigated. Compared with [30,39,40], the model considered in this paper is more general.

(2) A fractional-order command filter is introduced to obtain the filter signals and their fractional-order derivatives, which avoids the computational complexity problem existing in the conventional backstepping recursive procedure.

(3) To deal with uncertain nonlinear functions in the step of recursive design and unknown control gain caused by the unknown dead-zone fault, the radial basis function (RBF) neural network and Nussbaum gain function technique are applied in this paper. Then, the virtual control laws, adaptive laws and finial adaptive neural network finite-time control law are designed.

(4) By using the designed adaptive neural network finite-time control law, it can be guaranteed that all signals associated with the closed-loop system are bounded in finite time, and the tracking error converges to a small neighborhood of origin in finite time.

The rest of this paper consists of the following sections. The problem formulation and preliminaries are given in Section 2. In Section 3, the main design processes of the control law are provided, and the stability analysis is also shown in this section. In what follows, we give the simulation results and brief conclusions in Sections 4 and 5, respectively.

Notations: Throughout this paper, R, $\mathbb{C}$, and $\mathbb{N}$ represent, respectively, the sets of real numbers, complex numbers, and integers; R^n represents the set of n-dimensional real vectors; $|\cdot|$ stands for the absolute value of a constant; $\|\cdot\|$ is the induction norm of a matrix or the Euclidean norm of a vector; C^T stands for the transpose of matrix C or vector C; and $\min(X)$ or $\max(X)$ represent the minimum value or maximum value of X.

2. Problem Formulation and Preliminaries

This section will introduce the problem formulation for uncertain fractional-order systems, and some preliminaries, such as the fractional calculation, Nussbaum gain function technique, and some lemmas are given for the subsequent analysis.

2.1. Problem Formulation

Consider the uncertain fractional-order systems with unknown dead-zone fault, which is described as

$$\begin{aligned}
&{}^{C}\mathcal{D}_t^{\alpha} x_1 = g_1(\bar{\boldsymbol{x}})x_2 + f_1(\bar{\boldsymbol{x}}) + \boldsymbol{\gamma}_1^T \boldsymbol{\varphi}_1(\bar{\boldsymbol{x}}) \\
&{}^{C}\mathcal{D}_t^{\alpha} x_2 = g_2(\bar{\boldsymbol{x}})x_3 + f_2(\bar{\boldsymbol{x}}) + \boldsymbol{\gamma}_2^T \boldsymbol{\varphi}_2(\bar{\boldsymbol{x}}) \\
&\quad\vdots \\
&{}^{C}\mathcal{D}_t^{\alpha} x_{n\text{-}1} = g_{n-1}(\bar{\boldsymbol{x}})x_n + f_{n-1}(\bar{\boldsymbol{x}}) + \boldsymbol{\gamma}_{n-1}^T \boldsymbol{\varphi}_{n-1}(\bar{\boldsymbol{x}}) \\
&{}^{C}\mathcal{D}_t^{\alpha} x_n = g_n(\bar{\boldsymbol{x}})u^F(t) + f_n(\bar{\boldsymbol{x}}) + \boldsymbol{\gamma}_n^T \boldsymbol{\varphi}_n(\bar{\boldsymbol{x}}) \\
&y = x_1
\end{aligned} \tag{1}$$

where α is the fractional order; $\bar{\boldsymbol{x}} = [x_1, \cdots, x_n]^T \in R^n$, $u^F(t) \in R$, and $y \in R$ are the state vector, the control input, and the output of system, respectively; $g_i(\bar{\boldsymbol{x}})$ and $f_i(\bar{\boldsymbol{x}})$, $i = 1, \cdots, n$, represent the known nonzero smooth functions and uncertain nonlinear functions, respectively; $\boldsymbol{\gamma}_i$ and $\boldsymbol{\varphi}_i(\bar{\boldsymbol{x}})$, for $i = 1, \cdots, n$, stand for the unknown constant vectors and known nonlinear function vectors, respectively. For convenience, the functions $g_i(\bar{\boldsymbol{x}})$, $f_i(\bar{\boldsymbol{x}})$ and $\boldsymbol{\varphi}_i(\bar{\boldsymbol{x}})$ are denoted by g_i, f_i, and $\boldsymbol{\varphi}_i$, respectively.

In this paper, the control input $u^F(t)$ is subjected to the dead-zone fault, where "F" is the first letter of "Fault". Based on [42], $u^F(t)$ is given as

$$u^F(t) = \begin{cases} k_d\left(u(t) - b_r\right), & u(t) \geq b_r \\ 0, & -b_l < u(t) < b_r \\ k_d\left(u(t) + b_l\right), & u(t) \leq -b_l \end{cases} \tag{2}$$

where $k_d > 0$ represents an unknown bounded constant and is defined as the slope of the dead zone; $b_l > 0$ is the left breakpoint of dead-zone, and $b_r > 0$ is the right breakpoint of the dead zone.

By applying the mean value theorem, the control input (2) can be rewritten as

$$u^F(t) = k_d u(t) + \phi(t) \tag{3}$$

and there exists $\left|u^F(t)\right| \leq \left|u(t)\right| \leq \bar{U}$, where $\bar{U}$ represents the maximum value allowed by the system; $\phi(t)$ is a bounded function that satisfies $\left|\phi(t)\right| \leq \bar{\phi}$, and $\phi(t)$ is shown as

$$\phi(t) = \begin{cases} -k_d b_r, & u(t) \geq b_r \\ -k_d u(t), & -b_l < u(t) < b_r \\ k_d b_l, & u(t) \leq -b_l \end{cases} \tag{4}$$

For the system (1), the control goal of this paper is to construct an adaptive neural network finite-time control law $u(t)$ such that all signals of the closed-loop system are bounded in finite time, and the system output $y = x_1$ can track the reference signal y_d in finite time.

To achieve the desired control objective, some assumptions are provided as follows.

Assumption 1. *The reference signal y_d and its fractional-order derivative ${}^C\mathcal{D}_t^{\alpha} y_d$ are smooth and bounded.*

Assumption 2. *The smooth functions g_i, $i = 1, \cdots, n$ are bounded and the signs are identical; namely, there exist positive constants $g_{i,\min}$ and $g_{i,\max}$ such that $g_{i,\min} \leq \left|g_i\right| \leq g_{i,\max}$.*

Remark 1. *Assumptions 1 and 2 are common in the control law design of fractional-order systems and can be found in most existing results* [18,20,28]. *Assumption 2 implies that the time-varying control gains g_i are either strictly positive or strictly negative with the same sign. Moreover, the purpose of introducing positive constants $g_{i,\min}$ and $g_{i,\max}$ is to analyze the boundlessness of all signals and the stability of the system.*

2.2. Fractional Calculation

Definition 1 ([43]). *The αth Caputo derivative of a smooth function $f(t)$ is described as*

$${}^C\mathcal{D}^{\alpha} f(t) = \frac{1}{\Gamma(q-\alpha)} \int_0^t (t-s)^{q-\alpha-1} f^{(q)}(s) ds \tag{5}$$

where ${}^{C}\mathcal{D}^{\alpha}$ *denotes the Caputo fractional operator with* $q-1<\alpha<q$ *for* $q\in\mathbb{N}$; $\Gamma(\cdot)$ *is the Gamma function, which is given as* $\Gamma(q)=\int_0^{+\infty}s^{q-1}e^{-s}ds$.

For the Caputo fractional operator, the following properties hold.

$$ {}^{C}\mathcal{D}_t^{\alpha}\lambda_1=0 \tag{6}$$

$$ {}^{C}\mathcal{D}_t^{\alpha}\left(\lambda_1x_1(t)+\lambda_2x_2(t)\right)=\lambda_1\,{}^{C}\mathcal{D}_t^{\alpha}x_1(t)+\lambda_2\,{}^{C}\mathcal{D}_t^{\alpha}x_2(t) \tag{7}$$

where λ_1 and λ_2 are constants; $x_1(t)$ and $x_2(t)$ are smooth nonlinear functions.

Remark 2. *In the following analysis, only the case of* $0<\alpha<1$ *is considered. In addition, the notation* ${}^{C}\mathcal{D}^{\alpha}$ *for the Caputo operator is replaced by* $\mathcal{D}^{\alpha}$.

Definition 2 ([43]). *The two-parameter Mittag–Leffler function is*

$$E_{a_1,a_2}(\chi)=\sum_{k=0}^{\infty}\frac{\chi^k}{\Gamma(a_1k+a_2)} \tag{8}$$

where $a_1>0$, $a_2>0$, *and* $\chi\in\mathbb{C}$. *In particular,* $E_{1,1}(\chi)=e^{\chi}$. *The Laplace transform of (8) is*

$$\mathcal{L}\left(t^{a_2-1}E_{a_1,a_2}(-bt^{a_1})\right)=\frac{s^{a_1-a_2}}{s^{a_1}+b},\ b\in R \tag{9}$$

Lemma 1 ([43]). *There exist* $a_1\in(0,2)$ *and* $a_2\in R$ *such that if* $\pi a_1/2<a_3\le\min\{\pi,\pi a_1\}$ *is satisfied, then*

$$\left|E_{a_1,a_2}(\chi)\right|\le\frac{d}{1+|\chi|} \tag{10}$$

where $d>0$, $a_3\le|\arg(\chi)|\le\pi$, *and* $|\chi|\ge0$.

Lemma 2 ([44]). *Let* $h(t)$ *be a smooth function, then*

$$\frac{1}{2}\mathcal{D}_t^{\alpha}\left(h^T(t)h(t)\right)\le h^T(t)\mathcal{D}_t^{\alpha}h(t) \tag{11}$$

Lemma 3 ([45]). *If the* αth *fractional derivative of a smooth function* $V(t):[0,\infty)\to R$ *satisfies*

$$\mathcal{D}_t^{\alpha}V(t)\le-a_4V(t)+a_0 \tag{12}$$

where $0<\alpha<1$, $a_0>0$ *and* $a_4>0$, *then one can obtain*

$$V(t)\le\frac{a_0\varsigma}{a_4} \tag{13}$$

where $\varsigma=\max\{1,d\}$, *and* d *is defined as shown in Lemma 1.*

Lemma 4 ([46]). *Consider the fractional-order system* $\mathcal{D}_t^{\alpha}\boldsymbol{x}(t)=f\left(\boldsymbol{x}(t)\right)$, $0<\alpha<1$ *and* $\boldsymbol{x}(t)\in R^n$. *If there exist continuous and positive-definite function* $V\left(\boldsymbol{x}(t)\right)$, *K-functions* c_1 *and* c_2, *and constants* $b_1>0$, $b_2>0$, $0<\beta<1$ *with* β *being a constant to be designed, satisfying*

$$\begin{aligned}&c_1\left(\|\boldsymbol{x}(t)\|\right)\le V\left(\boldsymbol{x}(t)\right)\le c_2\left(\|\boldsymbol{x}(t)\|\right)\\&\mathcal{D}_t^{\alpha}V\left(\boldsymbol{x}(t)\right)\le-b_1V\left(\boldsymbol{x}(t)\right)^{\beta}+b_2\end{aligned} \tag{14}$$

and $\|\boldsymbol{x}(t)\|\le X^*$ *holds with* X^* *being a sufficient small positive constant, and there exists*

$$V(\boldsymbol{x}(t)) \le \left[\frac{b_2}{b_1(1-\mu)}\right]^{\frac{1}{\beta}}, t \ge T_f \tag{15}$$

where $\mu \in (0,1)$, T_f *is the finite setting time, which satisfies*

$$T_f \le \left[V_0^{1-\beta} - \left(\frac{b_2}{b_1(1-\mu)}\right)^{\frac{1-\beta}{\beta}}\right]^{\frac{1}{\alpha}} \left[\frac{\Gamma(2-\beta)\Gamma\left(1+\frac{1}{1-\beta}\right)\Gamma(1+\alpha)}{\Gamma\left(1+\frac{1}{1-\beta}-\alpha\right)b_1\mu}\right]^{\frac{1}{\alpha}} \tag{16}$$

where $V_0 = V(\boldsymbol{x}(0))$.

2.3. Nussbaum-Type Gain Function

Definition 3 ([47]). *A function* $\mathcal{N}(s)$ *is defined as a Nussbaum-type gain function if the following properties satisfy*

$$\begin{aligned} \limsup_{s\to\infty} \frac{1}{s}\int_0^s \mathcal{N}(s)ds = +\infty \\ \liminf_{s\to\infty} \frac{1}{s}\int_0^s \mathcal{N}(s)ds = -\infty \end{aligned} \tag{17}$$

Lemma 5 ([9,48]). *Let* $V(t)$ *and* $\kappa_i(t)$, *for* $i = 1,\cdots,n$, *be smooth functions defined on* $[0,t_0)$ *with* $V(t) \ge 0$ *for* $\forall t \in [0,t_0)$. $\mathcal{N}(\kappa_i)$ *is a special Nussbaum-type gain function, if the following inequality holds:*

$$\mathcal{D}_t^{\alpha} V(t) \le -\varpi V + \sum_{i=1}^{n}\left(\xi_i(t)\mathcal{N}(\kappa_i)+1\right)\dot{\kappa}_i + C_1 \tag{18}$$

where $\varpi > 0$ *and* $C_1 > 0$ *are constants, and* $\xi_i(t)$ *stands for a bounded smooth function that has* $\xi_{i,\min} \le |\xi_i(t)| \le \xi_{i,\max}$ *with* $\xi_{i,\min} > 0$ *and* $\xi_{i,\max} > 0$. *Then* $\kappa_i(t)$, $V(t)$, *and* $\sum_{i=1}^{n}\left(\xi_i(t)\mathcal{N}(\kappa_i)+1\right)\dot{\kappa}_i$ *will be bounded on* $[0,t_0)$ *for* $i = 1,\cdots,n$. *Particularly, for* $i = 1$, *the boundedness of* $\left(\xi(t)\mathcal{N}(\kappa)+1\right)\dot{\kappa}(t)$ *can be maintained.*

To facilitate the analysis of finite time problems, the following lemmas are provided.

Lemma 6 ([30]). *For any continuous function* $F(\boldsymbol{x})$ *over a compact set* $\Omega \in R^n$, *there exists an RBF neural network* $(\boldsymbol{W}^*)^T\boldsymbol{\Phi}(\boldsymbol{x})$ *such that*

$$F(\boldsymbol{x}) = (\boldsymbol{W}^*)^T\boldsymbol{\Phi}(\boldsymbol{x}) + \varepsilon(\boldsymbol{x}), \forall \boldsymbol{x} \in \Omega \tag{19}$$

where $\boldsymbol{W}^* \in R^l$ *is the optimal weight vector,* $l > 1$ *is the neural network node number,* $\varepsilon(\boldsymbol{x})$ *is the approximation error and there exists* $|\varepsilon(\boldsymbol{x})| \le \varepsilon^*$, *and* $\boldsymbol{\Phi}(\boldsymbol{x}) = [\varphi_1(\boldsymbol{x}),\cdots,\varphi_l(\boldsymbol{x})]^T \in R^l$ *represents a Gaussian-like basis function vector with*

$$\varphi_i(\boldsymbol{x}) = \exp\left(-\frac{(\boldsymbol{x}-\boldsymbol{\iota}_i)^T(\boldsymbol{x}-\boldsymbol{\iota}_i)}{\hbar^2}\right) \tag{20}$$

where $\boldsymbol{\iota}_i = [\iota_{i1},\cdots,\iota_{in}]^T$ *and* $\hbar$ *are the center of the basis function and the width of the Gaussian function, respectively.*

Lemma 7 ([48]). *A fractional-order second-order command filter with* $\psi_{i,1}(0) = \upsilon_i(0)$ *and* $\psi_{i,2}(0) = 0$ *as its initial conditions is given as*

$$\begin{aligned} \mathcal{D}_t^{\alpha}\psi_{i,1} &= \omega\psi_{i,2} \\ \mathcal{D}_t^{\alpha}\psi_{i,2} &= -2\tau\omega\psi_{i,2} - \omega(\psi_{i,1} - \upsilon_{i-1}) \end{aligned} \tag{21}$$

where $i = 2,\cdots,n$, υ_{i-1}, *and* $\psi_{i,1}$ *are the input and output of the fractional-order command filter, respectively. Then for any* $\lambda_i > 0$, *there exist* $\omega > 0$ *and* $0 < \tau \le 1$ *such that* $|\psi_{i,1} - \upsilon_{i-1}| \le \lambda_i$ *in finite time.*

Lemma 8 ([40]). *For any real variables* x *and* y, *and any positive constants* o_1, o_2, *and* o_3, *the following inequality holds:*

$$|x|^{o_1}|y|^{o_2} \le \frac{o_1}{o_1+o_2}o_3|x|^{o_1+o_2} + \frac{o_2}{o_1+o_2}o_3^{-\frac{o_1}{o_2}}|y|^{o_1+o_2} \tag{22}$$

Lemma 9 ([38]). *For* $\alpha_k \in R$, $k = 1,\cdots,n$ *and* $0 < p < 1$, *the following relationship holds:*

$$\left(\sum_{k=1}^{n}|\alpha_k|\right)^p \le \sum_{k=1}^{n}|\alpha_k|^p \le n^{1-p}\left(\sum_{k=1}^{n}|\alpha_k|\right)^p \tag{23}$$

Lemma 10 ([19]). *Let* $b \in R$ *and* $\vartheta > 0$; *for the hyperbolic tangent function* tanh, *there exists* $0 < |b| - b\tanh(b/\vartheta) \le 0.2785\vartheta$.

3. Control Law Design Process and Stability Analysis

For this section, the adaptive neural network finite-time control law for uncertain fractional-order systems with unknown dead-zone fault (1) is proposed. This can not only ensure that all signals of the closed-loop system are bounded in finite time, but it also makes the output of the system track the reference signal in finite time.

3.1. Adaptive Neural Network Finite-Time Control Law Design

We define the following coordinate transformation:

$$e_i = x_i - y_{i,d}, i = 1,\cdots,n \tag{24}$$

Where $y_{1,d} = y_d$, $y_{i,d}$ for $i = 2,\cdots,n$, and $y_{i,d} = \psi_{i,1}$ is the output of the fractional-order second-order command filter (see Lemma 7) with the virtual control law υ_{i-1} as the input.

The compensated tracking error z_i is defined as

$$z_i = e_i - s_i, i = 1,\cdots,n \tag{25}$$

where s_i is the compensating signal to be designed.

Step 1 ($i = 1$)**:** Considering (1), (24), and (25), the αth fractional-order derivative of z_1 is

$$\begin{aligned} \mathcal{D}_t^{\alpha}z_1 &= \mathcal{D}_t^{\alpha}e_1 - \mathcal{D}_t^{\alpha}s_1 \\ &= \mathcal{D}_t^{\alpha}x_1 - \mathcal{D}_t^{\alpha}y_d - \mathcal{D}_t^{\alpha}s_1 \\ &= g_1e_2 + g_1\left(y_{2,d} - \upsilon_1\right) + g_1\upsilon_1 + f_1 + \boldsymbol{\gamma}_1^T\boldsymbol{\varphi}_1 - \mathcal{D}_t^{\alpha}y_d - \mathcal{D}_t^{\alpha}s_1 \end{aligned} \tag{26}$$

According to Lemma 6, an RBF neural network is introduced to approximate the unknown nonlinear function f_1. Then we have

$$f_1 = (\boldsymbol{W}_1^*)^T\boldsymbol{\Phi}_1 + \varepsilon_1, |\varepsilon_1| \le \varepsilon_1^* \tag{27}$$

Design the compensating signal s_1 as

$$\mathcal{D}_t^{\alpha}s_1 = -\lambda_1 s_1 + g_1 s_2 + g_1\left(y_{2,d} - \upsilon_1\right) - \ell_1 sign(s_1) \tag{28}$$

where $\lambda_1 > 0$ and $\ell_1 > 0$ are design constants.

Substituting (27) and (28) into (26) yields

$$\mathcal{D}_t^{\alpha} z_1 = \lambda_1 s_1 + g_1 z_2 + g_1 \upsilon_1 + (\boldsymbol{W}_1^*)^T \boldsymbol{\Phi}_1 + \varepsilon_1 + \boldsymbol{\gamma}_1^T \boldsymbol{\varphi}_1 - \mathcal{D}_t^{\alpha} y_d + \ell_1 sign(s_1) \tag{29}$$

Design the Lyapunov function candidate as

$$V_1 = \frac{1}{2} z_1^2 + \frac{1}{2\mathrm{A}_1} \tilde{\boldsymbol{W}}_1^T \tilde{\boldsymbol{W}}_1 + \frac{1}{2\mathrm{B}_1} \tilde{\boldsymbol{\gamma}}_1^T \tilde{\boldsymbol{\gamma}}_1 \tag{30}$$

where A_1 and B_1 are the designed positive constants; $\tilde{\boldsymbol{W}}_1 = \boldsymbol{W}_1^* - \hat{\boldsymbol{W}}_1$ and $\tilde{\boldsymbol{\gamma}}_1 = \boldsymbol{\gamma}_1 - \hat{\boldsymbol{\gamma}}_1$, where $\hat{\boldsymbol{W}}_1$ and $\hat{\boldsymbol{\gamma}}_1$ are the estimations of $\boldsymbol{W}_1^*$ and $\boldsymbol{\gamma}_1$, respectively. Considering (6), (7), and Lemma 2, the fractional derivative of V_1 is given as

$$\begin{aligned} \mathcal{D}_t^{\alpha} V_1 &\le z_1 \left(\mathcal{D}_t^{\alpha} z_1 \right) + \frac{1}{\mathrm{A}_1} \tilde{\boldsymbol{W}}_1^T \left(\mathcal{D}_t^{\alpha} \tilde{\boldsymbol{W}}_1 \right) + \frac{1}{\mathrm{B}_1} \tilde{\boldsymbol{\gamma}}_1^T \left(\mathcal{D}_t^{\alpha} \tilde{\boldsymbol{\gamma}}_1 \right) \\ &= \lambda_1 z_1 s_1 + g_1 z_1 z_2 + z_1 \left(g_1 \upsilon_1 + (\boldsymbol{W}_1^*)^T \boldsymbol{\Phi}_1 + \varepsilon_1 + \boldsymbol{\gamma}_1^T \boldsymbol{\varphi}_1 - \mathcal{D}_t^{\alpha} y_d \right) + \ell_1 z_1 sign(s_1) \\ &\quad - \frac{1}{\mathrm{A}_1} \tilde{\boldsymbol{W}}_1^T \left(\mathcal{D}_t^{\alpha} \hat{\boldsymbol{W}}_1 \right) - \frac{1}{\mathrm{B}_1} \tilde{\boldsymbol{\gamma}}_1^T \left(\mathcal{D}_t^{\alpha} \hat{\boldsymbol{\gamma}}_1 \right) \end{aligned} \tag{31}$$

Design the virtual control law υ_1 as

$$\upsilon_1 = \frac{1}{g_1} \left(-\lambda_1 e_1 - c_1 z_1^{2\beta - 1} - \hat{\boldsymbol{W}}_1^T \boldsymbol{\Phi}_1 - \hat{\boldsymbol{\gamma}}_1^T \boldsymbol{\varphi}_1 - \varepsilon_1^* \tanh\left(\frac{\varepsilon_1^* z_1}{\vartheta}\right) + \mathcal{D}_t^{\alpha} y_d \right) \tag{32}$$

where $c_1 > 0$ and $\beta \in (0,1)$.

Substituting (32) into (31) has

$$\begin{aligned} \mathcal{D}_t^{\alpha} V_1 &\le -\lambda_1 z_1^2 - c_1 z_1^{2\beta} + g_1 z_1 z_2 + \ell_1 z_1 sign(s_1) + \frac{1}{\mathrm{A}_1} \tilde{\boldsymbol{W}}_1^T \left(\mathrm{A}_1 z_1 \boldsymbol{\Phi}_1 - \mathcal{D}_t^{\alpha} \hat{\boldsymbol{W}}_1 \right) \\ &\quad + \frac{1}{\mathrm{B}_1} \tilde{\boldsymbol{\gamma}}_1^T \left(\mathrm{B}_1 z_1 \boldsymbol{\varphi}_1 - \mathcal{D}_t^{\alpha} \hat{\boldsymbol{\gamma}}_1 \right) + \varepsilon_1 z_1 - \varepsilon_1^* z_1 \tanh\left(\frac{\varepsilon_1^* z_1}{\vartheta}\right) \end{aligned} \tag{33}$$

Design the adaptive laws $\hat{\boldsymbol{W}}_1$ and $\hat{\boldsymbol{\gamma}}_1$ as

$$\mathcal{D}_t^{\alpha} \hat{\boldsymbol{W}}_1 = \mathrm{A}_1 z_1 \boldsymbol{\Phi}_1 - \eta_1 \hat{\boldsymbol{W}}_1 \tag{34}$$

$$\mathcal{D}_t^{\alpha} \hat{\boldsymbol{\gamma}}_1 = \mathrm{B}_1 z_1 \boldsymbol{\varphi}_1 - \delta_1 \hat{\boldsymbol{\gamma}}_1 \tag{35}$$

where $\eta_1 > 0$ and $\delta_1 > 0$ are design constants.

Substituting (34) and (35) into (33), and considering Lemma 10**,** gives

$$\mathcal{D}_t^{\alpha} V_1 \le -\lambda_1 z_1^2 - c_1 z_1^{2\beta} + g_1 z_1 z_2 + \frac{\eta_1}{\mathrm{A}_1} \tilde{\boldsymbol{W}}_1^T \hat{\boldsymbol{W}}_1 + \frac{\delta_1}{\mathrm{B}_1} \tilde{\boldsymbol{\gamma}}_1^T \hat{\boldsymbol{\gamma}}_1 + \ell_1 z_1 sign(s_1) + 0.2785\vartheta \tag{36}$$

Step i ($i = 2, \cdots, n-1$)**:** Considering (1), (24), and (25), the fractional derivative of z_i is

$$\mathcal{D}_t^{\alpha} z_i = g_i e_{i+1} + g_i \left(y_{i+1,d} - \upsilon_i \right) + g_i \upsilon_i + f_i + \boldsymbol{\gamma}_i^T \boldsymbol{\varphi}_i - \mathcal{D}_t^{\alpha} y_{i,d} - \mathcal{D}_t^{\alpha} s_i \tag{37}$$

Similarly, an RBF neural network is introduced to approximate the unknown nonlinear function f_i. Then we obtain

$$f_i = (\boldsymbol{W}_i^*)^T \boldsymbol{\Phi}_i + \varepsilon_i, \left| \varepsilon_i \right| \le \varepsilon_i^* \tag{38}$$

Design the compensating signal s_i as

$$\mathcal{D}_t^{\alpha} s_i = -\lambda_i s_i + g_i s_{i+1} - g_{i-1} s_{i-1} + g_i \left(y_{i+1,d} - \upsilon_i \right) - \ell_i sign(s_i) \tag{39}$$

where $\lambda_i > 0$ and $\ell_i > 0$ are design constants.

Substituting (38) and (39) into (37) yields

$$\mathcal{D}_t^{\alpha} z_i = \lambda_i s_i + g_i z_{i+1} + g_{i-1} s_{i-1} + g_i \upsilon_i + (\boldsymbol{W}_i^*)^T \boldsymbol{\Phi}_i + \varepsilon_i + \boldsymbol{\gamma}_i^T \boldsymbol{\varphi}_i - \mathcal{D}_t^{\alpha} y_{i,d} + \ell_i sign(s_i) \tag{40}$$

Design the Lyapunov function candidate as

$$V_i = \frac{1}{2} z_i^2 + \frac{1}{2\mathrm{A}_i} \tilde{\boldsymbol{W}}_i^T \tilde{\boldsymbol{W}}_i + \frac{1}{2\mathrm{B}_i} \tilde{\boldsymbol{\gamma}}_i^T \tilde{\boldsymbol{\gamma}}_i \tag{41}$$

where A_i and B_i are the designed positive constants; $\tilde{\boldsymbol{W}}_i = \boldsymbol{W}_i^* - \hat{\boldsymbol{W}}_i$ and $\tilde{\boldsymbol{\gamma}}_i = \boldsymbol{\gamma}_i - \hat{\boldsymbol{\gamma}}_i$, where $\hat{\boldsymbol{W}}_i$ and $\hat{\boldsymbol{\gamma}}_i$ are the estimations of $\boldsymbol{W}_i^*$ and $\boldsymbol{\gamma}_i$, respectively. Then, the fractional derivative of V_i is

$$\begin{aligned} \mathcal{D}_t^{\alpha} V_i &\le z_i \left(\mathcal{D}_t^{\alpha} z_i\right) + \frac{1}{\mathrm{A}_i} \tilde{\boldsymbol{W}}_i^T \left(\mathcal{D}_t^{\alpha} \tilde{\boldsymbol{W}}_i\right) + \frac{1}{\mathrm{B}_i} \tilde{\boldsymbol{\gamma}}_i^T \left(\mathcal{D}_t^{\alpha} \tilde{\boldsymbol{\gamma}}_i\right) \\ &= \lambda_i z_i s_i + g_i z_i z_{i+1} + g_{i-1} s_{i-1} z_i + z_i \left(g_i \upsilon_i + (\boldsymbol{W}_i^*)^T \boldsymbol{\Phi}_i + \varepsilon_i + \boldsymbol{\gamma}_i^T \boldsymbol{\varphi}_i - \mathcal{D}_t^{\alpha} y_{i,d}\right) \\ &\quad - \frac{1}{\mathrm{A}_i} \tilde{\boldsymbol{W}}_i^T \left(\mathcal{D}_t^{\alpha} \hat{\boldsymbol{W}}_i\right) - \frac{1}{\mathrm{B}_i} \tilde{\boldsymbol{\gamma}}_i^T \left(\mathcal{D}_t^{\alpha} \hat{\boldsymbol{\gamma}}_i\right) + \ell_i z_i sign(s_i) \end{aligned} \tag{42}$$

Design the virtual control law υ_i as

$$\upsilon_i = \frac{1}{g_i} \left(-\lambda_i e_i - g_{i-1} e_{i-1} - c_i z_i^{2\beta-1} - \hat{\boldsymbol{W}}_i^T \boldsymbol{\Phi}_i - \hat{\boldsymbol{\gamma}}_i^T \boldsymbol{\varphi}_i - \varepsilon_i^* \tanh\left(\frac{\varepsilon_i^* z_i}{\vartheta}\right) + \mathcal{D}_t^{\alpha} y_{i,d} \right) \tag{43}$$

where $c_i > 0$ and $\beta \in (0,1)$.

Substituting (43) into (42) has

$$\begin{aligned} \mathcal{D}_t^{\alpha} V_i &\le -\lambda_i z_i^2 - c_i z_i^{2\beta} + g_i z_i z_{i+1} - g_{i-1} z_{i-1} z_i + \varepsilon_i z_i - \varepsilon_i^* z_i \tanh\left(\frac{\varepsilon_i^* z_i}{\vartheta}\right) + \ell_i z_i sign(s_i) \\ &\quad + \frac{1}{\mathrm{A}_i} \tilde{\boldsymbol{W}}_i^T \left(\mathrm{A}_i z_i \boldsymbol{\Phi}_i - \mathcal{D}_t^{\alpha} \hat{\boldsymbol{W}}_i\right) + \frac{1}{\mathrm{B}_i} \tilde{\boldsymbol{\gamma}}_i^T \left(\mathrm{B}_i z_i \boldsymbol{\varphi}_i - \mathcal{D}_t^{\alpha} \hat{\boldsymbol{\gamma}}_i\right) \end{aligned} \tag{44}$$

Design the adaptive laws $\hat{\boldsymbol{W}}_i$ and $\hat{\boldsymbol{\gamma}}_i$ as

$$\mathcal{D}_t^{\alpha} \hat{\boldsymbol{W}}_i = \mathrm{A}_i z_i \boldsymbol{\Phi}_i - \eta_i \hat{\boldsymbol{W}}_i \tag{45}$$

$$\mathcal{D}_t^{\alpha} \hat{\boldsymbol{\gamma}}_i = \mathrm{B}_i z_i \boldsymbol{\varphi}_i - \delta_i \hat{\boldsymbol{\gamma}}_i \tag{46}$$

where $\eta_i > 0$ and $\delta_i > 0$ are design constants.

Substituting (45) and (46) into (44), and considering Lemma 10, one has

$$\mathcal{D}_t^{\alpha} V_i \le -\lambda_i z_i^2 - c_i z_i^{2\beta} + g_i z_i z_{i+1} - g_{i-1} z_{i-1} z_i + \frac{\eta_i}{\mathrm{A}_i} \tilde{\boldsymbol{W}}_i^T \hat{\boldsymbol{W}}_i + \frac{\delta_i}{\mathrm{B}_i} \tilde{\boldsymbol{\gamma}}_i^T \hat{\boldsymbol{\gamma}}_i + \ell_i z_i sign(s_i) + 0.2785\vartheta \tag{47}$$

Step n $(i = n)$**:** In this step, the adaptive neural network finite-time control law is derived. Considering (1), (3), (24), and (25), the fractional derivative of z_n is given as

$$\mathcal{D}_t^{\alpha} z_n = k_d g_n u(t) + \phi(t) g_n + f_n + \boldsymbol{\gamma}_n^T \boldsymbol{\varphi}_n - \mathcal{D}_t^{\alpha} y_{n,d} - \mathcal{D}_t^{\alpha} s_n \tag{48}$$

The unknown nonlinear function f_n in (48) is approximated by using the RBF neural network, that is

$$f_n = (\boldsymbol{W}_n^*)^T \boldsymbol{\Phi}_n + \varepsilon_n, \left|\varepsilon_n\right| \le \varepsilon_n^* \tag{49}$$

Design the compensating signal s_n as

$$\mathcal{D}_t^{\alpha} s_n = -\lambda_n s_n - g_{n-1} s_{n-1} - \ell_n sign(s_n) \tag{50}$$

where $\lambda_n > 0$ and $\ell_n > 0$ are the design constants.

Substituting (49) and (50) into (48) yields

$$\mathcal{D}_t^{\alpha} z_n = G_n u(t) + (\boldsymbol{W}_n^*)^T \boldsymbol{\Phi}_n + \bar{\varepsilon}_n + \boldsymbol{\gamma}_n^T \boldsymbol{\varphi}_n - \mathcal{D}_t^{\alpha} y_{n,d} + \lambda_n s_n + g_{n-1} s_{n-1} + \ell_n sign(s_n) \tag{51}$$

where $G_n = k_d g_n$ and $\bar{\varepsilon}_n = g_n \phi(t) + \varepsilon_n^*$. Considering the boundlessness of g_n, k_d, and $\phi(t)$, there exist $|G_n| \le G^*$ and $|\bar{\varepsilon}_n| \le \bar{\varepsilon}_n^*$ with unknown constants $G^* > 0$ and $\bar{\varepsilon}_n^* > 0$.

Design the Lyapunov function candidate as

$$V_n = \frac{1}{2} z_n^2 + \frac{1}{2\mathrm{A}_n} \tilde{\boldsymbol{W}}_n^T \tilde{\boldsymbol{W}}_n + \frac{1}{2\mathrm{B}_n} \tilde{\boldsymbol{\gamma}}_n^T \tilde{\boldsymbol{\gamma}}_n \tag{52}$$

where A_n and B_n are the designed positive constants; $\tilde{\boldsymbol{W}}_n = \boldsymbol{W}_n^* - \hat{\boldsymbol{W}}_n$ and $\tilde{\boldsymbol{\gamma}}_n = \boldsymbol{\gamma}_n - \hat{\boldsymbol{\gamma}}_n$, where $\hat{\boldsymbol{W}}_n$, and $\hat{\boldsymbol{\gamma}}_n$ are the estimations of $\boldsymbol{W}_n^*$ and $\boldsymbol{\gamma}_n$, respectively. The fractional derivative of V_n is given as

$$\begin{aligned} \mathcal{D}_t^\alpha V_n \le{}& \lambda_n z_n s_n + g_{n-1} s_{n-1} z_n + G_n z_n u(t) + z_n \left((\boldsymbol{W}_n^*)^T \boldsymbol{\Phi}_n + \bar{\varepsilon}_n + \boldsymbol{\gamma}_n^T \boldsymbol{\varphi}_n - \mathcal{D}_t^\alpha y_{n,d} \right) \\ & - \frac{1}{\mathrm{A}_n} \tilde{\boldsymbol{W}}_n^T \left(\mathcal{D}_t^\alpha \hat{\boldsymbol{W}}_n \right) - \frac{1}{\mathrm{B}_n} \tilde{\boldsymbol{\gamma}}_i^T \left(\mathcal{D}_t^\alpha \hat{\boldsymbol{\gamma}}_n \right) + \ell_n z_n sign(s_n) \end{aligned} \tag{53}$$

Design the adaptive neural network finite-time control law $u(t)$ as

$$u(t) = \mathcal{N}(\kappa)\theta(t) \tag{54}$$

$$\theta(t) = \lambda_n e_n + g_{n-1} e_{n-1} + c_n z_n^{2\beta-1} + \hat{\boldsymbol{W}}_n^T \boldsymbol{\Phi}_n + \hat{\boldsymbol{\gamma}}_n^T \boldsymbol{\varphi}_n + \bar{\varepsilon}_n^* \tanh\left(\frac{\bar{\varepsilon}_n^* z_n}{\vartheta}\right) - \mathcal{D}_t^\alpha y_{n,d} \tag{55}$$

$$\dot{\kappa}(t) = z_n \theta(t) \tag{56}$$

Substituting (54)–(56) into (53), one gets

$$\begin{aligned} \mathcal{D}_t^\alpha V_n \le{}& -\lambda_n z_n^2 - g_{n-1} z_{n-1} z_n - c_n z_n^{2\beta} + \left(G_n \mathcal{N}(\kappa) + 1 \right) \dot{\kappa}(t) + \bar{\varepsilon}_n z_n - \bar{\varepsilon}_n^* z_n \tanh\left(\frac{\bar{\varepsilon}_n^* z_n}{\vartheta}\right) \\ & + \frac{1}{\mathrm{A}_n} \tilde{\boldsymbol{W}}_n^T \left(\mathrm{A}_n z_n \boldsymbol{\Phi}_n - \mathcal{D}_t^\alpha \hat{\boldsymbol{W}}_n \right) + \frac{1}{\mathrm{B}_n} \tilde{\boldsymbol{\gamma}}_i^T \left(\mathrm{B}_n z_n \boldsymbol{\varphi}_n - \mathcal{D}_t^\alpha \hat{\boldsymbol{\gamma}}_n \right) + \ell_n z_n sign(s_n) \end{aligned} \tag{57}$$

Design the adaptive laws $\hat{\boldsymbol{W}}_n$ and $\hat{\boldsymbol{\gamma}}_n$ as

$$\mathcal{D}_t^\alpha \hat{\boldsymbol{W}}_n = \mathrm{A}_n z_n \boldsymbol{\Phi}_n - \eta_n \hat{\boldsymbol{W}}_n \tag{58}$$

$$\mathcal{D}_t^\alpha \hat{\boldsymbol{\gamma}}_n = \mathrm{B}_n z_n \boldsymbol{\varphi}_n - \delta_n \hat{\boldsymbol{\gamma}}_n \tag{59}$$

where $\eta_n > 0$ and $\delta_n > 0$ are the design constants.

Substituting (58) and (59) into (57), and considering Lemma 10, one obtains

$$\begin{aligned} \mathcal{D}_t^\alpha V_n \le{}& -\lambda_n z_n^2 - c_n z_n^{2\beta} - g_{n-1} z_{n-1} z_n + \left(G_n \mathcal{N}(\kappa) + 1 \right) \dot{\kappa}(t) + \frac{\eta_n}{\mathrm{A}_n} \tilde{\boldsymbol{W}}_n^T \hat{\boldsymbol{W}} + \frac{\delta_n}{\mathrm{B}_n} \tilde{\boldsymbol{\gamma}}_n^T \hat{\boldsymbol{\gamma}}_n \\ & + \ell_n z_n sign(s_n) + 0.2785\vartheta \end{aligned} \tag{60}$$

3.2. Stability Analysis

Based on the virtual control laws, adaptive laws, and adaptive neural network finite-time control law designed above, the main results can be summarized as follows.

Theorem 1. *Consider an uncertain fractional-order system (1) that is subject to unknown dead-zone fault (2). Under Assumptions 1 and 2, if the compensating signals are selected as (28), (39), and (50), the virtual control laws are designed as shown in (32) with adaptive laws (34) and (35), and (43) with adaptive laws (45) and (46), and the adaptive neural network finite-time control law is designed as shown in (54) with adaptive laws (58) and (59). Then, all signals of the closed-loop system are bounded in finite time and the tracking error e_1 can converge to a small neighborhood of origin in finite time.*

Proof. Design the following Lyapunov function as

$$V = \sum_{i=1}^{n} V_i \tag{61}$$

Invoking (36), (47), and (60), the αth fractional-order derivative of V is

$$\begin{aligned} \mathcal{D}_t^{\alpha} V &= \sum_{i=1}^{n} \mathcal{D}_t^{\alpha} V_i \\ &\le -\sum_{i=1}^{n} \lambda_i z_i^2 - \sum_{i=1}^{n} c_i z_i^{2\beta} + \sum_{i=1}^{n} \frac{\eta_i}{\mathrm{A}_i} \tilde{\boldsymbol{W}}_i^T \hat{\boldsymbol{W}}_i + \sum_{i=1}^{n} \frac{\delta_i}{\mathrm{B}_i} \tilde{\boldsymbol{\gamma}}_i^T \hat{\boldsymbol{\gamma}}_i + \sum_{i=1}^{n} \ell_i z_i sign(s_i) \\ &\quad + \left(G_n \mathcal{N}(\kappa) + 1\right) \dot{\kappa}(t) + 0.2785 \vartheta n \end{aligned} \tag{62}$$

By applying Lemma 8, the following results can be obtained:

$$\tilde{\boldsymbol{W}}_i^T \hat{\boldsymbol{W}}_i = \tilde{\boldsymbol{W}}_i^T \left(\boldsymbol{W}_i^* - \tilde{\boldsymbol{W}}_i\right) \le \frac{1}{2} (\boldsymbol{W}_i^*)^T \boldsymbol{W}_i^* - \frac{1}{2} \tilde{\boldsymbol{W}}_i^T \tilde{\boldsymbol{W}}_i \tag{63}$$

$$\tilde{\boldsymbol{\gamma}}_i^T \hat{\boldsymbol{\gamma}}_i = \tilde{\boldsymbol{\gamma}}_i^T \left(\boldsymbol{\gamma}_i - \tilde{\boldsymbol{\gamma}}_i\right) \le \frac{1}{2} \boldsymbol{\gamma}_i^T \boldsymbol{\gamma}_i^T - \frac{1}{2} \tilde{\boldsymbol{\gamma}}_i^T \tilde{\boldsymbol{\gamma}}_i \tag{64}$$

$$\ell_i z_i sign(s_i) \le \ell_i z_i^2 + \frac{1}{4} \ell_i \tag{65}$$

Substituting (63)–(65) into (62) yields

$$\begin{aligned} \mathcal{D}_t^{\alpha} V \le & -\sum_{i=1}^{n} \left(\lambda_i - \ell_i\right) z_i^2 - \sum_{i=1}^{n} c_i z_i^{2\beta} - \sum_{i=1}^{n} \frac{\eta_i}{2\mathrm{A}_i} \tilde{\boldsymbol{W}}_i^T \tilde{\boldsymbol{W}}_i - \sum_{i=1}^{n} \frac{\delta_i}{2\mathrm{B}_i} \tilde{\boldsymbol{\gamma}}_i^T \tilde{\boldsymbol{\gamma}}_i + \left(G_n \mathcal{N}(\kappa) + 1\right) \dot{\kappa}(t) \\ & + \nu_1 \left(\sum_{i=1}^{n} \frac{1}{2\mathrm{A}_i} \tilde{\boldsymbol{W}}_i^T \tilde{\boldsymbol{W}}_i \right)^{\beta} - \nu_1 \left(\sum_{i=1}^{n} \frac{1}{2\mathrm{A}_i} \tilde{\boldsymbol{W}}_i^T \tilde{\boldsymbol{W}}_i \right)^{\beta} + \nu_2 \left(\sum_{i=1}^{n} \frac{1}{2\mathrm{B}_i} \tilde{\boldsymbol{\gamma}}_i^T \tilde{\boldsymbol{\gamma}}_i \right)^{\beta} \\ & - \nu_2 \left(\sum_{i=1}^{n} \frac{1}{2\mathrm{B}_i} \tilde{\boldsymbol{\gamma}}_i^T \tilde{\boldsymbol{\gamma}}_i \right)^{\beta} + \sum_{i=1}^{n} \frac{\eta_i}{2\mathrm{A}_i} (\boldsymbol{W}_i^*)^T \boldsymbol{W}_i^* + \sum_{i=1}^{n} \frac{\delta_i}{2\mathrm{B}_i} \boldsymbol{\gamma}_i^T \boldsymbol{\gamma}_i^T \\ & + \sum_{i=1}^{n} \frac{\ell_i}{4} + 0.2785 \vartheta n \end{aligned} \tag{66}$$

where ν_1 and ν_2 are positive constants. □

Considering Lemma 8 again, let $x = 1$, $y = \sum_{i=1}^{n} \left(\tilde{\boldsymbol{W}}_i^T \tilde{\boldsymbol{W}}_i\right) / 2\mathrm{A}_i$ or $y = \sum_{i=1}^{n} \left(\tilde{\boldsymbol{\gamma}}_i^T \tilde{\boldsymbol{\gamma}}_i\right) / 2\mathrm{B}_i$, $o_1 = 1 - \beta$, $o_2 = \beta$ and $o_3 = \beta^{\beta/(1-\beta)}$, respectively. Thus, the following inequalities hold

$$\nu_1 \left(\sum_{i=1}^{n} \frac{1}{2\mathrm{A}_i} \tilde{\boldsymbol{W}}_i^T \tilde{\boldsymbol{W}}_i \right)^{\beta} \le \nu_1 \sum_{i=1}^{n} \frac{1}{2\mathrm{A}_i} \tilde{\boldsymbol{W}}_i^T \tilde{\boldsymbol{W}}_i + \nu_1 (1 - \beta) \beta^{\frac{\beta}{1-\beta}} \tag{67}$$

$$\nu_2 \left(\sum_{i=1}^{n} \frac{1}{2\mathrm{B}_i} \tilde{\boldsymbol{\gamma}}_i^T \tilde{\boldsymbol{\gamma}}_i \right)^{\beta} \le \nu_2 \sum_{i=1}^{n} \frac{1}{2\mathrm{B}_i} \tilde{\boldsymbol{\gamma}}_i^T \tilde{\boldsymbol{\gamma}}_i + \nu_2 (1 - \beta) \beta^{\frac{\beta}{1-\beta}} \tag{68}$$

By substituting (67) and (68) into (66), and applying Lemma 9, the following result is satisfied by choosing appropriate parameters satisfying $\lambda_i > \ell_i$, $\eta_i > \nu_1$, and $\delta_i > \nu_2$, that is

$$\begin{aligned}\mathcal{D}_t^{\alpha}V \le & -\sum_{i=1}^{n}(\lambda_i-\ell_i)z_i^2-\sum_{i=1}^{n}\frac{\eta_i-\nu_1}{2\mathrm{A}_i}\tilde{\boldsymbol{W}}_i^{T}\tilde{\boldsymbol{W}}_i-\sum_{i=1}^{n}\frac{\delta_i-\nu_2}{2\mathrm{B}_i}\tilde{\boldsymbol{\gamma}}_i^{T}\tilde{\boldsymbol{\gamma}}_i-\sum_{i=1}^{n}c_i z_i^{2\beta}-\nu_1\left(\sum_{i=1}^{n}\frac{1}{2\mathrm{A}_i}\tilde{\boldsymbol{W}}_i^{T}\tilde{\boldsymbol{W}}_i\right)^{\beta}\\ & -\nu_2\left(\sum_{i=1}^{n}\frac{1}{2\mathrm{B}_i}\tilde{\boldsymbol{\gamma}}_i^{T}\tilde{\boldsymbol{\gamma}}_i\right)^{\beta}+\sum_{i=1}^{n}\frac{\eta_i}{2\mathrm{A}_i}(\boldsymbol{W}_i^{*})^{T}\boldsymbol{W}_i^{*}+\sum_{i=1}^{n}\frac{\delta_i}{2\mathrm{B}_i}\boldsymbol{\gamma}_i^{T}\boldsymbol{\gamma}_i^{T}+\nu_1(1-\beta)\beta^{\frac{\beta}{1-\beta}}\\ & +\nu_2(1-\beta)\beta^{\frac{\beta}{1-\beta}}+\left(G_n\mathcal{N}(\kappa)+1\right)\dot{\kappa}(t)+\sum_{i=1}^{n}\frac{\ell_i}{4}+0.2785\vartheta n\\ = & -\bar{a}V-\bar{b}V^{\beta}+\left(G_n\mathcal{N}(\kappa)+1\right)\dot{\kappa}(t)+D_1\end{aligned} \tag{69}$$

where $\bar{a}$, $\bar{b}$ and D_1 are respectively given as

$$\bar{a}=\min\left\{2(\lambda_i-\ell_i),(\eta_i-\nu_1),(\delta_i-\nu_2)\right\}$$

$$\bar{b}=\min\left\{2^{\beta}c_i,\nu_1,\nu_2\right\}$$

$$D_1=\sum_{i=1}^{n}\frac{\eta_i}{2\mathrm{A}_i}(\boldsymbol{W}_i^{*})^{T}\boldsymbol{W}_i^{*}+\sum_{i=1}^{n}\frac{\delta_i}{2\mathrm{B}_i}\boldsymbol{\gamma}_i^{T}\boldsymbol{\gamma}_i^{T}+(\nu_1+\nu_2)(1-\beta)\beta^{\frac{\beta}{1-\beta}}+\sum_{i=1}^{n}\frac{\ell_i}{4}+0.2785\vartheta n$$

Next, we verify our results in three steps.

Step 1. Considering (69) and the definition of V, it can be easily obtained that $\bar{b}V^{\beta}\ge 0$. Then, we have

$$\mathcal{D}_t^{\alpha}V\le -\bar{a}V+\left(G_n\mathcal{N}(\kappa)+1\right)\dot{\kappa}(t)+D_1 \tag{70}$$

By applying Lemma 5, there exist a positive constant G^{*} such that $\max\left(G_n\mathcal{N}(\kappa)+1\right)\dot{\kappa}(t)=G^{*}$ for $t\in[0,t_0)$. Therefore, (69) can be written as

$$\mathcal{D}_t^{\alpha}V\le -\bar{a}V-\bar{b}V^{\beta}+D_1^{*} \tag{71}$$

where $D_1^{*}=G^{*}+D_1$.

Step 2. Based on the results of Step 1, from (71), we have

$$\mathcal{D}_t^{\alpha}V\le -\bar{a}V+D_1^{*} \tag{72}$$

Applying Lemma 1 and Lemma 3, then there is a positive constant ς such that

$$V\le\frac{D_1^{*}\varsigma}{\bar{a}} \tag{73}$$

which means that V is bounded, and it further implies that the signals z_i, $\tilde{\boldsymbol{W}}_i$, and $\tilde{\boldsymbol{\gamma}}_i$ are also bounded. Noting $\tilde{\boldsymbol{W}}_i=\boldsymbol{W}_i^{*}-\hat{\boldsymbol{W}}_i$ and $\tilde{\boldsymbol{\gamma}}_i=\boldsymbol{\gamma}_i-\hat{\boldsymbol{\gamma}}_i$, then the boundlessness of $\hat{\boldsymbol{W}}_i$ and $\hat{\boldsymbol{\gamma}}_i$ can be also obtained.

Step 3. From the definition of $z_1=e_1-s_1$, if z_1 and s_1 are finite-time stable, then the tracking error e_1 is also finite-time stable. Considering (71) and the fact that $\bar{a}V\ge 0$, then we have

$$\mathcal{D}_t^{\alpha}V\le -\bar{b}V^{\beta}+D_1^{*} \tag{74}$$

By applying Lemma 4, it can be held that

$$V(t)\le\left[\frac{D_1^{*}}{\bar{b}(1-\mu_1)}\right]^{\frac{1}{\beta}} \tag{75}$$

and the setting time T_{f1} is

$$T_{f1} \le \left[V_0^{1-\beta} - \left(\frac{D_1^*}{\bar{b}(1-\mu_1)} \right)^{\frac{1-\beta}{\beta}} \right]^{\frac{1}{\alpha}} \left[\frac{\Gamma(2-\beta)\Gamma\left(1+\frac{1}{1-\beta}\right)\Gamma(1+\alpha)}{\Gamma\left(1+\frac{1}{1-\beta}-\alpha\right)\bar{b}\mu_1} \right]^{\frac{1}{\alpha}} \tag{76}$$

where $\mu_1 \in (0,1)$ and $V_0 = V(0)$.

According to the definition of V, one gives

$$|z_1| \le \sqrt{2} \left[\frac{D_1^*}{\bar{b}(1-\mu_1)} \right]^{\frac{1}{2\beta}} \tag{77}$$

Now, we show that the compensated signal s_1 is finite-time stable.

Choose the following Lyapunov function candidate:

$$Y = \sum_{n=1}^{n} \frac{1}{2} s_i^2 \tag{78}$$

Invoking (28), (39), and (50), the αth fractional-order derivative of Y is

$$\begin{aligned} \mathcal{D}_t^{\alpha} Y &= \sum_{n=1}^{n} s_i \left(\mathcal{D}_t^{\alpha} s_i \right) \\ &\le -\lambda_1 s_1^2 + g_1 s_1 s_2 + s_1 g_1 \left(y_{2,d} - \upsilon_1 \right) - \ell_1 s_1 sign(s_1) - \lambda_2 s_2^2 + g_2 s_2 s_3 - g_1 s_1 s_2 \\ &\quad + s_2 g_2 \left(y_{3,d} - \upsilon_2 \right) - \ell_2 s_2 sign(s_2) + \cdots - \lambda_n s_n^2 - g_{n-1} s_{n-1} s_n - \ell_n s_n sign(s_n) \\ &= -\sum_{n=1}^{n} \lambda_i s_i^2 + \sum_{n=1}^{n-1} s_i g_i \left(y_{i+1,d} - \upsilon_i \right) - \sum_{n=1}^{n} \ell_i |s_i| \end{aligned} \tag{79}$$

Considering Lemma 7, it can be obtained that $|y_{i+1,d} - \upsilon_i| \le \hat{\lambda}_i$ in finite time T_{f2}. In view of Assumption 2 and Lemma 8, from (79), we have

$$\begin{aligned} \mathcal{D}_t^{\alpha} Y &\le -\sum_{n=1}^{n} \lambda_i s_i^2 + \sum_{n=1}^{n-1} \left(g_{i,\max} \hat{\lambda}_i \right) |s_i| - \sum_{n=1}^{n} \ell_i |s_i| + \left(g_{n,\max} \hat{\lambda}_n \right) |s_n| \\ &= -\sum_{n=1}^{n} \lambda_i s_i^2 + \sum_{n=1}^{n} \left(g_{i,\max} \hat{\lambda}_i - \ell_i \right) |s_i| \\ &\le -\sum_{n=1}^{n} \left(\lambda_i - \Lambda_i \right) s_i^2 + \sum_{n=1}^{n} \frac{1}{4\Lambda_i} \left(g_{i,\max} \hat{\lambda}_i - \ell_i \right)^2 \end{aligned} \tag{80}$$

where $\left(g_{i,\max} \hat{\lambda}_i - \ell_i \right) |s_i| \le \left(g_{i,\max} \hat{\lambda}_i - \ell_i \right)^2 / 4\Lambda_i + \Lambda_i s_i^2$ is applied.

Let $\bar{c} = \min\{2(\lambda_i - \Lambda_i)\} > 0$, and considering Lemma 8 again, we get

$$\left(\sum_{i=1}^{n} \frac{1}{2} s_i^2 \right)^{\beta} \le \sum_{i=1}^{n} \frac{1}{2} s_i^2 + (1-\beta) \beta^{\frac{\beta}{1-\beta}} \tag{81}$$

Substituting (81) into (80) and applying Lemma 9, one has

$$\begin{aligned} \mathcal{D}_t^{\alpha} Y &\le -\bar{c} \sum_{n=1}^{n} \left(\frac{1}{2} s_i^2 \right)^{\beta} + \sum_{n=1}^{n} \frac{1}{4\Lambda_i} \left(g_{i,\max} \hat{\lambda}_i - \ell_i \right)^2 + \bar{c}(1-\beta) \beta^{\frac{\beta}{1-\beta}} \\ &= -\bar{c} Y^{\beta} + D_2^* \end{aligned} \tag{82}$$

where $D_2^* = \sum_{i=1}^{n} \left(g_{i,\max} \hat{\lambda}_i - \ell_i \right)^2 / 4\Lambda_i + \bar{c}(1-\beta) \beta^{\beta/(1-\beta)}$.

Noting Lemma 4, and similar to the proof of z_1, it can be obtained that the compensating signal s_1 is finite time stable and satisfies

$$|s_1| \le \sqrt{2} \left[\frac{D_2^*}{\bar{c}(1-\mu_2)} \right]^{\frac{1}{2\beta}} \tag{83}$$

and the setting time T_{f3} is

$$T_{f3} \le \left[Y_0^{1-\beta} - \left(\frac{D_2^*}{\bar{c}(1-\mu_2)} \right)^{\frac{1-\beta}{\beta}} \right]^{\frac{1}{\alpha}} \left[\frac{\Gamma(2-\beta)\Gamma\left(1+\frac{1}{1-\beta}\right)\Gamma(1+\alpha)}{\Gamma\left(1+\frac{1}{1-\beta}-\alpha\right)\bar{c}\mu_2} \right]^{\frac{1}{\alpha}} \tag{84}$$

where $\mu_2 \in (0,1)$ and $Y_0 = Y(0)$.

Considering (25), (77), and (83), the tracking error e_1 satisfies

$$|e_1| \le |z_1| + |s_1| \le \sqrt{2}\left[\frac{D_1^*}{\bar{b}(1-\mu_1)} \right]^{\frac{1}{2\beta}} + \sqrt{2}\left[\frac{D_2^*}{\bar{c}(1-\mu_2)} \right]^{\frac{1}{2\beta}} \tag{85}$$

Observing (85), it can be seen that the tracking error e_1 is the sum of the compensating signal s_1 and the compensated tracking error z_1. Accordingly, the convergence time T also satisfies this relationship. Moreover, it can be found that the tracking error e_1 depends on parameters λ_i, ℓ_i, c_i, η_i, δ_i, A_i and B_i, $i = 1, \cdots, n$. It also implies that the tracking error e_1 converge to the specified small neighborhood of origin in finite time within the setting time $T = T_{f1} + T_{f2} + T_{f3}$ by selecting the appropriate parameters. This completes the proof.

Remark 3. *Noting (85), the tracking error e_1 can be made arbitrarily small by adjusting parameters λ_i, ℓ_i, c_i, η_i, δ_i, A_i and B_i, $i = 1, \cdots, n$. We can decrease D_1^* by decreasing the values of parameters η_i and δ_i or increasing A_i and B_i, and we can decrease D_2^* by increasing ℓ_i. We can also increase $\bar{b}$ by increasing the value of parameter c_i, and we can increase $\bar{c}$ by increasing λ_i. Based on the adjustment of D_1^*, D_2^*, $\bar{b}$, and $\bar{c}$, it can be guaranteed that the tracking error e_1 can converge to the specified small neighborhood of origin in finite time within the setting time. However, it should be emphasized that the change of ℓ_i simultaneously affects D_1^* and D_2^*, and the change of λ_i simultaneously affects D_2^* and $\bar{c}$. Moreover, the adjustment of these parameters may be bringing about an increase in the amplitude of the control signal. Therefore, when selecting suitable parameters, a trade-off should be made between the control performance of the tracking and the amplitude of the control signal.*

4. Simulation Analysis

In this section, the simulation cases are given to verify the validity of the control law designed in this paper.

Case 1: Consider a class of uncertain fractional-order systems as follows:

$$\begin{aligned} \mathcal{D}_t^{\alpha} x_1 &= g_1 x_2 + f_1 + + \boldsymbol{\gamma}_1^T \boldsymbol{\varphi}_1 \\ \mathcal{D}_t^{\alpha} x_2 &= g_2 x_3 + f_2 + \boldsymbol{\gamma}_2^T \boldsymbol{\varphi}_2 \\ \mathcal{D}_t^{\alpha} x_3 &= g_3 u^F(t) + f_3 + \boldsymbol{\gamma}_3^T \boldsymbol{\varphi}_3 \\ y &= x_1 \end{aligned} \tag{86}$$

where $g_1 = 0.9$, $g_2 = 0.5$, $g_3 = 1 + 0.7\sin t$, $f_1 = -x_2 \sin(x_1)$, $f_2 = e^{-x_2^2/15}$, $f_3 = 2x_2 - 2x_3 - \sin(x_1 x_3)$, $\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_2 = \boldsymbol{\gamma}_3 = [0.5, 1]^T$, $\boldsymbol{\varphi}_1 = [\cos(x_1), x_2]^T$, $\boldsymbol{\varphi}_2 = [-\sin(x_1 x_2), -x_2 \cos(x_3)]^T$, and $\boldsymbol{\varphi}_3 = \left[x_2, x_1^2\right]^T$. The dead-zone fault model is shown in (2), and $k_d = 1.5$, $b_l = 0.15$ and $b_r = 0.3$. The reference signal is $y_d = 1.5(\sin t + \sin 2t)$; the initial states are $x_1(0) = 1.2$, $x_2(0) = 0.5$ and $x_3(0) = 0.25$; and the simulation time is $t = 20\mathrm{s}$.

The RBFNN is used to approximate the unknown nonlinear functions f_1, f_2 and f_3. The node number for each RBF neural network is considered to be 9 with the width of basis function being $\hbar = 4$. The centers of the basis function $\boldsymbol{\iota}_i$ ($i = 1, \cdots, 9$) for the function f_1 are evenly spaced in $[-8,8]\times[-8,8]$; for the function f_2, they are evenly spaced in $[-8,8]$; and for the function f_3, they are evenly spaced in $[-8,8]\times[-8,8]\times[-8,8]$.

The other design parameters are $\vartheta = 0.01$, $\beta = 0.95$, $\lambda_1 = 1.5$, $\lambda_2 = 3.0$, $\lambda_3 = 4.5$, $\ell_1 = 1.5$, $\ell_2 = 2.0$, $\ell_3 = 0.5$, $c_1 = 20$, $c_2 = 12$, $c_3 = 2.0$, $\mathrm{A}_1 = 1.5$, $\mathrm{A}_2 = 2.5$, $\mathrm{A}_3 = 0.5$, $\eta_1 = 0.4$, $\eta_2 = 1.5$, $\eta_3 = 1.6$, $\mathrm{B}_1 = 4.5$, $\mathrm{B}_2 = 2.9$, $\mathrm{B}_3 = 1.5$, $\delta_1 = 0.5$, $\delta_2 = 3.0$, $\delta_3 = 5.5$, $\varepsilon_1^* = \varepsilon_2^* = 0.5$, $\bar{\varepsilon}_3^* = 1.0$. The parameters for the second-order command filter are set as $\omega = 3.0$ and $\tau = 0.7$. The initial conditions for adaptive laws are set as $s_1(0) = s_2(0) = s_3(0) = 0.01$, $\hat{\boldsymbol{\gamma}}_1(0) = \hat{\boldsymbol{\gamma}}_2(0) = \hat{\boldsymbol{\gamma}}_3(0) = [0.01]_{2\times 1}$, $\hat{\boldsymbol{W}}_1(0) = \hat{\boldsymbol{W}}_2(0) = \hat{\boldsymbol{W}}_3(0) = [0.01]_{9\times 1}$ and $\kappa(0) = 0$.

The simulation results for this case are shown in Figures 1–6. Figures 1 and 2 give the curves of the tracking performance and the tracking error e_1. It can be seen from Figure 1 that the system (86) can obtain a good tracking performance in finite time, although the system suffers from the unknown dead-zone fault. From Figure 2, we can see that the tracking error can converge to a small neighborhood of zero in finite time under the proposed control law. The results of these two figures also further verify the validity of the designed control law. Furthermore, the trajectories of the state variables x_1, x_2 and x_3 are displayed in Figure 3, the curves of the control law $u(t)$ and adaptive laws $\|\hat{\boldsymbol{W}}_i\|$ and $\|\hat{\boldsymbol{\gamma}}_i\|$ ($i = 1, 2, 3$) are shown in Figures 4–6. Noting Figures 2–6, the signals of the closed-loop system are bounded in finite time, which shows the validity of the theoretical analysis.

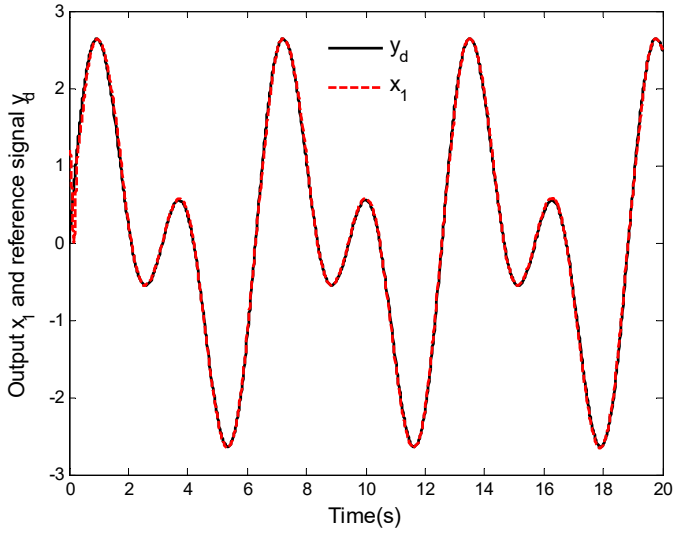

Figure 1. Curves of tracking performance.

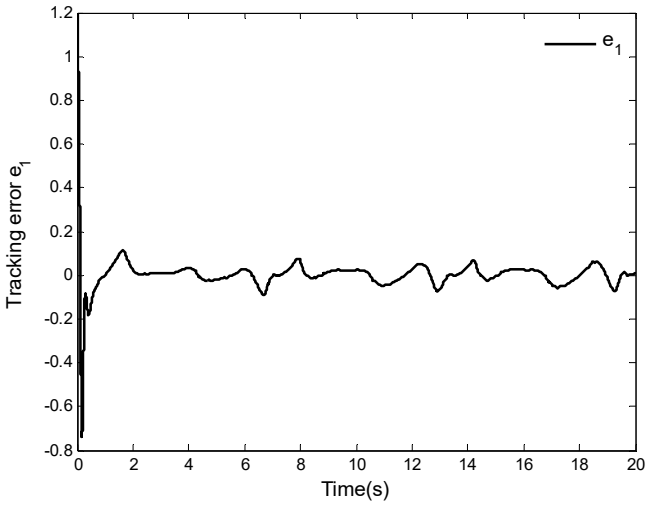

Figure 2. Tracking error e_1.

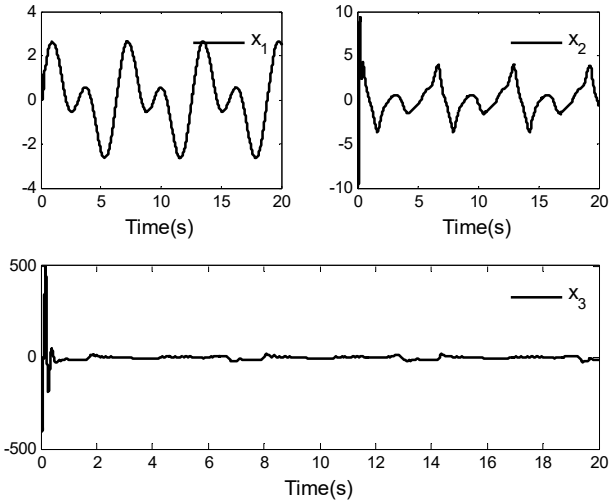

Figure 3. System states x_1, x_2 and x_3.

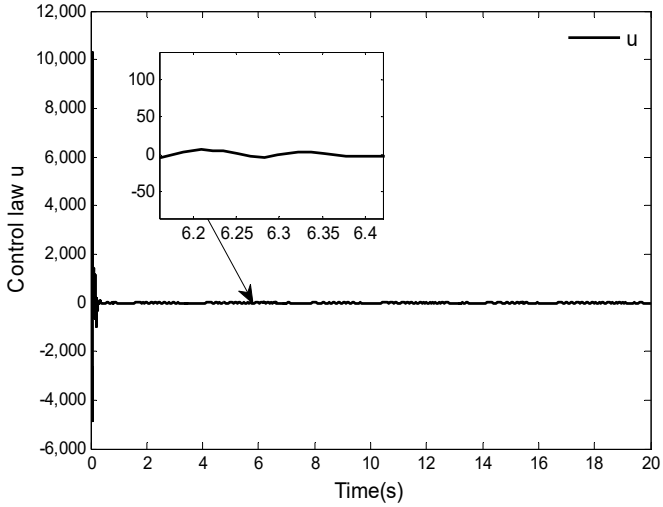

Figure 4. Control law $u(t)$.

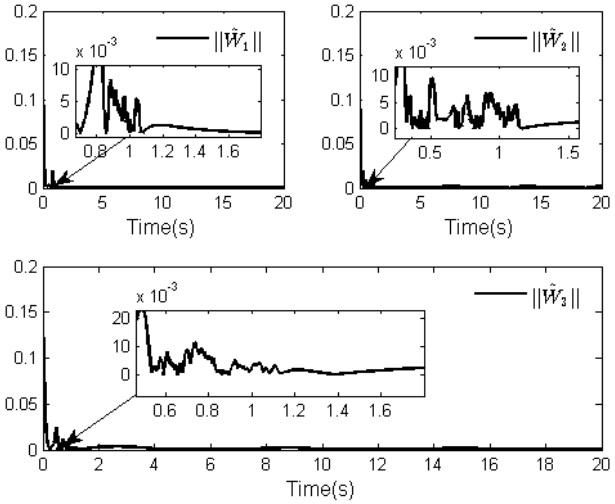

Figure 5. Norms of adaptive laws $\left\|\hat{\boldsymbol{W}}_i\right\|$.

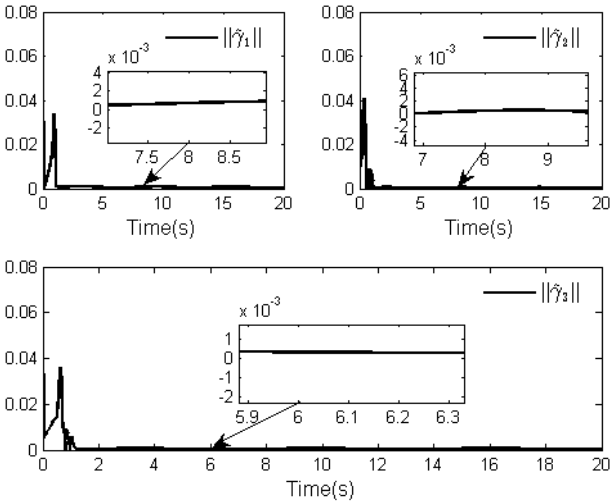

Figure 6. Norms of adaptive laws $\left\|\hat{\boldsymbol{\gamma}}_i\right\|$.

Case 2: Consider the uncertain fractional-order Arneodo system as [49]

$$\begin{aligned}
&\mathcal{D}_t^{\alpha} x_1 = g_1 x_2 + f_1 + + \boldsymbol{\gamma}_1^T \boldsymbol{\varphi}_1 \\
&\mathcal{D}_t^{\alpha} x_2 = g_2 x_3 + f_2 + \boldsymbol{\gamma}_2^T \boldsymbol{\varphi}_2 \\
&\mathcal{D}_t^{\alpha} x_3 = -q_1 x_1 - q_2 x_2 - q_3 x_3 - q_4 x_1^3 + g_3 u^F(t) + f_3 + \boldsymbol{\gamma}_3^T \boldsymbol{\varphi}_3 \\
&y = x_1
\end{aligned} \tag{87}$$

If $f_1 = f_2 = f_3 = 0$, $\boldsymbol{\gamma}_1^T \boldsymbol{\varphi}_1 = \boldsymbol{\gamma}_2^T \boldsymbol{\varphi}_2 = \boldsymbol{\gamma}_3^T \boldsymbol{\varphi}_3 = 0$, $u^F(t) = 0$ and $\alpha = 0.98$, $g_1 = g_2 = 1$, $q_1 = -5.5$, $q_2 = 3.5$, $q_3 = 0.8$, $q_4 = -1.0$, the initial conditions are considered to be $x_1(0) = -0.2$, $x_2(0) = 0.5$, and $x_3(0) = 0.2$. The system (87) will appear to have a chaotic phenomenon, as shown in Figure 7.

In system (87), let $g_3 = 1.2\sin x_1 + 2$, $f_1 = -x_1 e^{-10x_2}$, $f_2 = -2.5x_2 \cos x_3$, $f_3 = -x_1 \sin(x_3)$, $\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_2 = \boldsymbol{\gamma}_3 = [0.5, 1]^T$, $\boldsymbol{\varphi}_1 = [-\sin x_1, 0]^T$, $\boldsymbol{\varphi}_2 = [0, -\sin(x_1 x_2)]^T$, and $\boldsymbol{\varphi}_3 = [3x_2, -2x_3]^T$; the initial states are $x_1(0) = 0.5$, $x_2(0) = 0.25$ and $x_3(0) = 0.1$. The parameters of the dead-zone fault model, the reference signal, and the simulation time are consistent with Case 1.

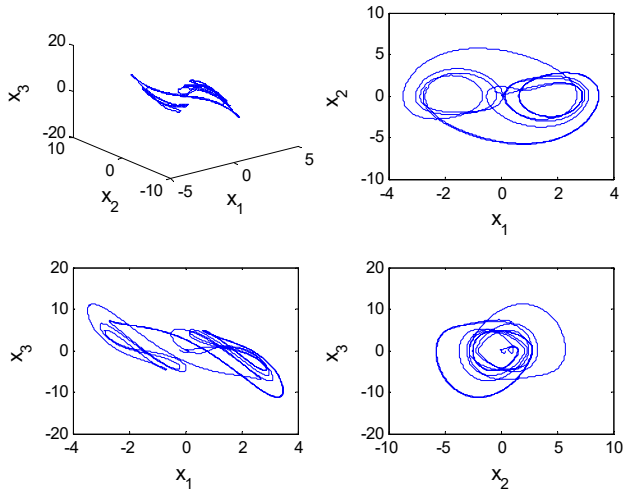

Figure 7. Phrase plots of x_1, x_2, and x_3.

The RBF neural network is applied to approximate the unknown nonlinear functions f_1, f_2 and f_3. Since there are only two variables in functions f_1, f_2, and f_3, the node number for each RBF neural network is chosen to be 9 with the centers of the basis function ι_i ($i=1,\cdots,9$) evenly spaced in $[-8,8]\times[-8,8]$ and the width being $\hbar=4$.

The other design parameters are $\vartheta=0.01$, $\beta=0.95$, $\lambda_1=10$, $\lambda_2=5.5$, $\lambda_3=7.5$, $\ell_1=2.5$, $\ell_2=1.5$, $\ell_3=2.0$, $c_1=25$, $c_2=18$, $c_3=1.5$, $\mathrm{A}_1=0.9$, $\mathrm{A}_2=1.2$, $\mathrm{A}_3=0.2$, $\eta_1=0.5$, $\eta_2=2.5$, $\eta_3=3.0$, $\mathrm{B}_1=2.2$, $\mathrm{B}_2=1.6$, $\mathrm{B}_3=0.5$, $\delta_1=0.7$, $\delta_2=1.5$, $\delta_3=2.1$, $\varepsilon_1^*=0.5$, $\varepsilon_2^*=0.7$, and $\bar{\varepsilon}_3^*=1.0$. The parameters selection of the second-order command filter and the initial conditions of the adaptive laws are the same as those described for Case 1.

The simulation results of this case are displayed in Figures 8–13. Figure 8 shows the curves of the system output x_1 and the reference signal y_d. It is not difficult to see from Figure 8 that the system (87) can obtain a good tracking performance by applying the proposed control law. The tracking error curve is given in Figure 9. One observes that the tracking error e_1 can converge to a small neighborhood of zero in finite time. From Figures 8 and 9, although the system (87) is affected by unknown dead zone fault, the tracking performance of the system can be guaranteed under the designed control law. This also proves the effectiveness of the proposed control law from another perspective. Furthermore, the curves of state variables x_1, x_2 and x_3 are given in Figure 10, and the curves of the control law $u(t)$ and adaptive laws $\|\hat{W}_i\|$ and $\|\hat{\gamma}_i\|$ ($i=1,2,3$) are depicted in Figures 11–13. It can be found that the signals of the closed-loop system shown in these figures are bounded, which verifies the validity of the theoretical analysis. However, it is not difficult to observe in Figures 9–11 that there are oscillations in these simulation results. In fact, considering the existence of unknown dead-zone faults and uncertain dynamics in the system, this makes it necessary to make a reasonable trade-off between system tracking performance and control output.

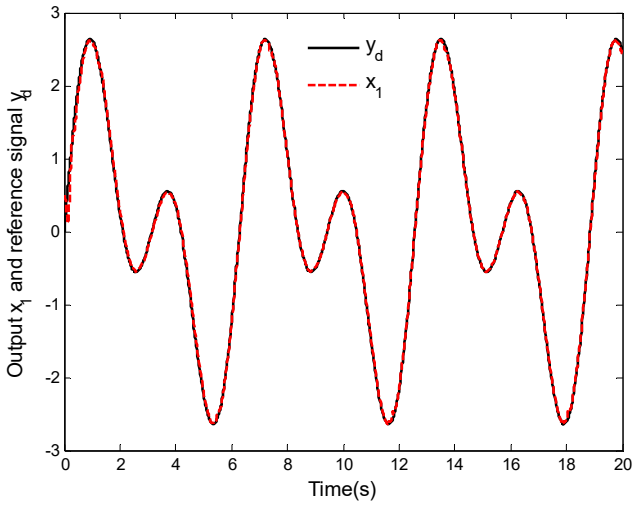

Figure 8. Curves of the tracking performance.

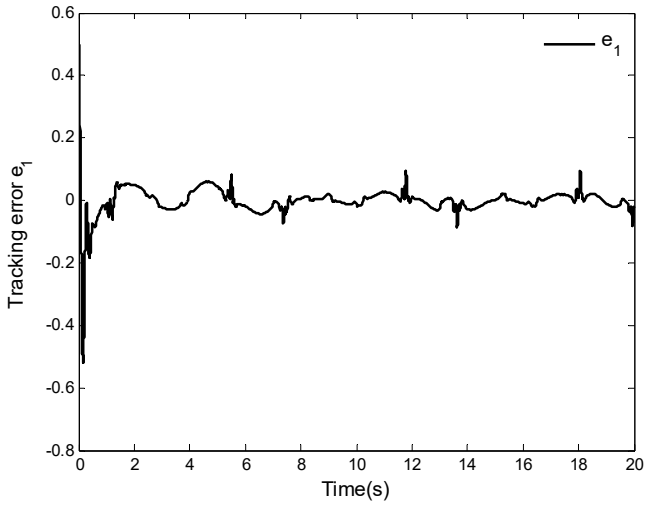

Figure 9. Tracking error e_1.

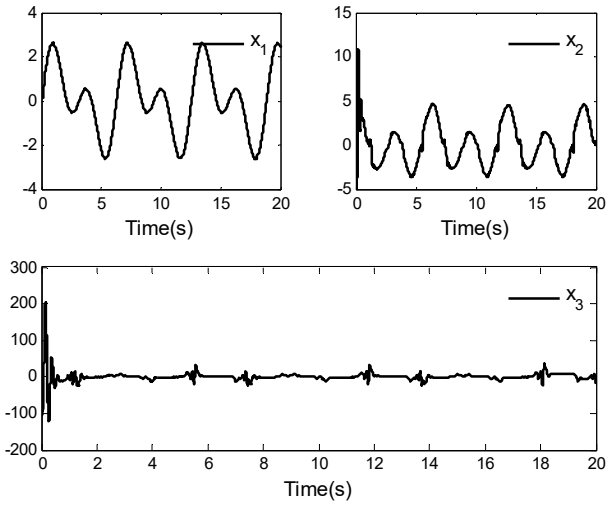

Figure 10. System states x_1, x_2, and x_3.

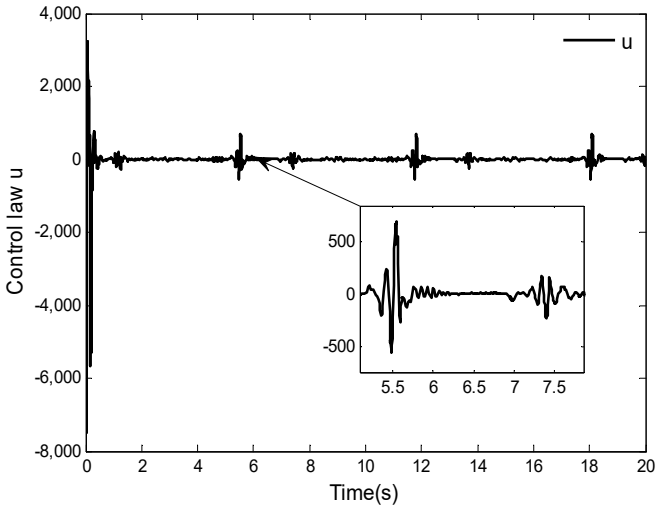

Figure 11. Control law $u(t)$.

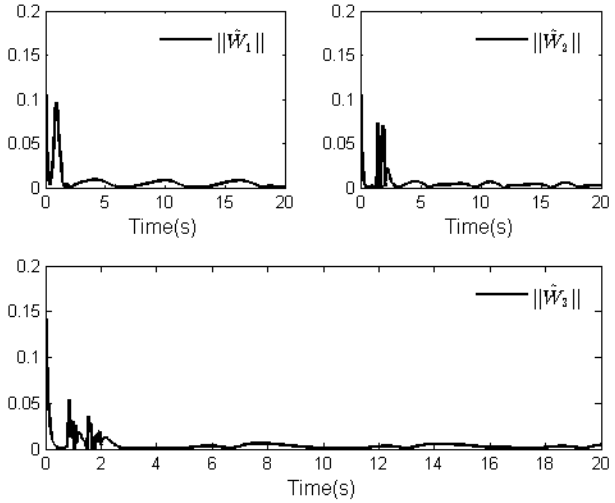

Figure 12. Norms of adaptive laws $\left\|\hat{\boldsymbol{W}}_i\right\|$.

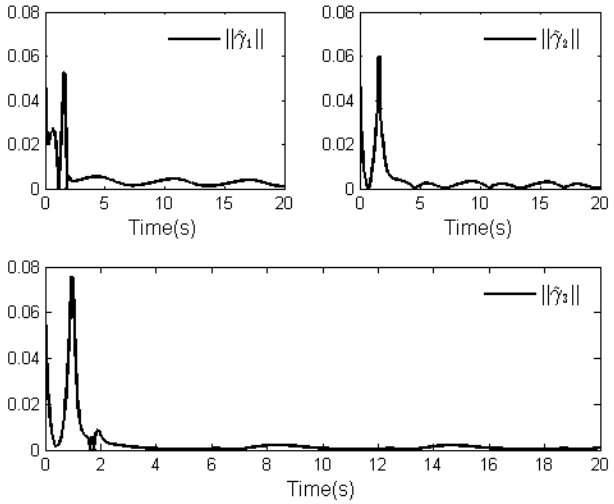

Figure 13. Norms of adaptive laws $\|\hat{\gamma}_i\|$.

5. Conclusions

The adaptive finite-time tracking control for uncertain fractional-order systems with unknown dead-zone fault was considered in this paper. The fractional-order command filter was applied to avoid the computational complexity problem existing in conventional recursive procedures, and the neural network approximator was used to approximate the unknown uncertain nonlinear functions. Through the application of the Nussbaum gain function technique, the adaptive neural network finite-time control law was developed to solve the finite-time control problem of the given fractional-order systems. It has been proven that the desinged control law can not only ensure that all signals of the closed-loop system are bounded in finite time but can also ensure that the tracking error converges to a small neighborhood of the origin in finite time. However, it should be pointed out that the control law presented in this paper is only suitable for the systems with known state gains and measurable states. When the nonlinear system under consideration has unknown state gains and unmeasurable states, the proposed control law will not work effectively. Therefore, one of our future research directions is to design feasible control laws to realize the adaptive finite-time control of uncertain fractional-order systems with unknown control gain and partially unmeasurable states.

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