

# FINITE 3-CONNECTED-SET-HOMOGENEOUS LOCALLY $2\mathbf{K}_n$ GRAPHS AND $s$ -ARC-TRANSITIVE GRAPHS

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**ABSTRACT.** In this paper, all graphs are assumed to be finite. For  $s \geq 1$  and a graph  $\Gamma$ , if for every pair of isomorphic connected induced subgraphs on at most  $s$  vertices there exists an automorphism of  $\Gamma$  mapping the first to the second, then we say that  $\Gamma$  is  $s$ -connected-set-homogeneous, and if every isomorphism between two isomorphic connected induced subgraphs on at most  $s$  vertices can be extended to an automorphism of  $\Gamma$ , then we say that  $\Gamma$  is  $s$ -connected-homogeneous. For  $n \geq 1$ , a graph  $\Gamma$  is said to be locally  $2\mathbf{K}_n$  if the subgraph  $[\Gamma(u)]$  induced on the set of vertices of  $\Gamma$  adjacent to a given vertex  $u$  is isomorphic to  $2\mathbf{K}_n$ .

Note that 2-connected-set-homogeneous but not 2-connected-homogeneous graphs are just the half-arc-transitive graphs which are a quite active topic in algebraic graph theory. Motivated by this, we posed the problem of characterizing or classifying 3-connected-set-homogeneous graphs of girth 3 which are not 3-connected-homogeneous in (Eur. J. Combin. 93 (2021) 103275). Until now, there have been only two known families of 3-connected-set-homogeneous graphs of girth 3 which are not 3-connected-homogeneous, and these graphs are locally  $2\mathbf{K}_n$  with  $n = 2$  or  $4$ . In this paper, we complete the classification of finite 3-connected-set-homogeneous graphs which are locally  $2\mathbf{K}_n$  with  $n \geq 2$ , and all such graphs are line graphs of some specific 2-arc-transitive graphs. Furthermore, we give a good description of finite 3-connected-set-homogeneous but not 3-connected-homogeneous graphs which are locally  $2\mathbf{K}_n$  and have solvable automorphism groups. This is then used to construct some new 3-connected-set-homogeneous but not 3-connected-homogeneous graphs as well as some new 2-arc-transitive graphs.

**Keywords** 3-connected-set-homogeneous, 3-connected-homogeneous, 2-geodesic-transitive, Cayley graph, 2-arc-transitive

## 1. INTRODUCTION

The main purpose of this paper is to give a partial answer to a problem raised by the author in [57] about the 3-connected-set-homogeneous graphs. As a by-product of this investigation, we disprove a conjecture posed by Feng and Kwak in their 2006 paper on trivalent symmetric graphs of order twice a prime power [21], and answer a question on 3-arc-transitive graphs posed by Li, Seress and Song in [36], and we also correct an error in [41] about tetravalent 3-arc-regular Cayley graphs. Before proceeding, we give some background to this topic, and set some notation.

A graph is said to be *regular* if each of its vertices is adjacent to  $k$  vertices for some constant positive integer  $k$ . Let  $\Gamma$  be a graph. We use  $V(\Gamma)$ ,  $E(\Gamma)$  and  $\text{Aut}(\Gamma)$  to denote its vertex set, edge set and full automorphism group, respectively. For  $B \subseteq V(\Gamma)$ ,  $[B]$  denotes the subgraph induced by  $B$ . For a vertex  $v$  of  $\Gamma$ , let  $\Gamma(v)$  be the set of vertices adjacent to  $v$ . Let  $G \leq \text{Aut}(\Gamma)$ . Denote by  $G_v$  the subgroup of  $G$  fixing  $v$ , by  $G_v^{\Gamma(v)}$  the

permutation subgroup on  $\Gamma(v)$  induced by  $G_v$ , and by  $G_v^{[1]}$  the subgroup of  $G_v$  fixing every vertex in  $\Gamma(v)$ . For a positive integer  $n$ , we say that  $\Gamma$  is *locally*  $2\mathbf{K}_n$  if  $[\Gamma(v)] \cong 2\mathbf{K}_n$  for each  $v \in V(\Gamma)$ , where  $2\mathbf{K}_n$  means the disjoint union of two copies of  $\mathbf{K}_n$ .

For a positive integer  $n$ , denote by  $C_n$  the cyclic group of order  $n$ , and by  $A_n$  and  $S_n$  the alternating group and symmetric group of degree  $n$ , respectively. For two groups  $M$  and  $N$ ,  $N \rtimes M$  denotes a semidirect product of  $N$  by  $M$ , and  $N \wr M$  the wreath product of  $N$  by  $M$ . See Section 2 for other unexplained terms.

For a positive integer  $s$  and a graph  $\Gamma$ , if for any pair of isomorphic connected induced subgraphs of  $\Gamma$  on at most  $s$  vertices there is an automorphism of  $\Gamma$  mapping the first to the second, then we say that  $\Gamma$  is *s-connected-set-homogeneous*, or *s-CSH*, and if every isomorphism between two isomorphic connected induced subgraphs on at most  $s$  vertices can be extended to an automorphism of  $\Gamma$ , then we say that  $\Gamma$  is *s-connected-homogeneous*, or *s-CH*. A graph is said to be *connected-set-homogeneous* or *connected-homogeneous* if it is *s-CSH* or *s-CH*, respectively, for all positive integers  $s$ .

*s-CSH* or *s-CH* graphs have received a lot of attention in the literature. For example, in 1978, Gardiner [23] gave a classification of finite connected-set-homogeneous graphs, in 2009, Gray [25] classified infinite 3-CSH or 3-CH graphs with more than one end, and Devillers et al. [13, 40] investigated the finite  $k$ -CH graphs with  $k \geq 3$ . For more results related to *s-CSH* or *s-CH* graphs, we refer the reader to [17, 19, 26, 30].

Clearly, a graph is 1-CSH or 1-CH if and only if it is vertex-transitive. Furthermore, 2-CH graphs are precisely regular arc-transitive graphs, and every 2-CSH graph is vertex- and edge-transitive. A graph is said to be *half-arc-transitive* if it is 2-CSH but not 2-CH. In 1966, Tutte [51] initiated the study of half-arc-transitive graphs, and he proved that the valency of a half-arc-transitive graph must be even, and a few years later, Brouwer [5] constructed the first family of half-arc-transitive graphs. Following this pioneering work, half-arc-transitive graphs have been extensively studied over the last half a century, and numerous papers have been published on this class of graphs (see, for example, the survey papers [11, 46] and recent papers [48, 49, 54, 56]).

In this paper, we are interested in 3-CSH but not 3-CH graphs which are a natural generalization of half-arc-transitive graphs. Note that a graph of girth at least 4 is 3-CSH but not 3-CH if and only if it is 2-path-transitive but not 2-arc-transitive. In 1996, Conder and Praeger [12] initiated the study of 2-path-transitive graphs, and more than ten years later, Li and Zhang [38, 39] systematically investigated 2-path-transitive graphs which are not 2-arc-transitive. Motivated by this, the author [57] began the study of 3-CSH but not 3-CH graphs of girth 3, and we proved the existence of such graphs and proposed the following problem.

**Problem 1.1.** [57, Problem B] *Characterize or classify 3-connected-set-homogeneous graphs of girth 3 which are not 3-connected-homogeneous.*

In this paper, we shall partially solve this problem by classifying 3-CSH graphs which are locally  $2\mathbf{K}_n$  with  $n \geq 2$ . This was partially motivated by our previous work in [57], where we proved the existence of 3-CSH but not 3-CH graphs of girth 3 by constructing some 3-CSH graphs which are locally  $2\mathbf{K}_n$  with  $n = 2$  or 4. Our main results show that there is a close relationship between 3-CSH graphs which are locally  $2\mathbf{K}_n$  and *s*-arc-transitive graphs with  $s \geq 2$ . Constructing or classifying *s*-arc-transitive graphs with  $s \geq 2$  has been a perennially active topic in the area of algebraic graph theory; see, for

example, [31, 32, 36, 45]. This is another motivation for us to study 3-CSH graphs which are locally  $2\mathbf{K}_n$ . Before stating our main results, we introduce some terminology.

For  $s \geq 0$ , an  $s$ -arc in  $\Gamma$  is an ordered  $(s+1)$ -tuple  $(v_0, v_1, \dots, v_{s-1}, v_s)$  of vertices of  $\Gamma$  such that  $v_{i-1}$  is adjacent to  $v_i$  for  $1 \leq i \leq s$  and  $v_{i-1} \neq v_{i+1}$  for  $1 \leq i \leq s-1$ . For some group  $G$  of automorphisms of  $\Gamma$ , we say that the graph  $\Gamma$  is  $(G, s)$ -arc-transitive if  $\Gamma$  is regular and  $G$  is transitive on the set of  $s$ -arcs in  $\Gamma$ ;  $\Gamma$  is  $(G, s)$ -arc-regular if  $G$  is regular on the set of  $s$ -arcs of  $\Gamma$ . When  $G = \text{Aut}(\Gamma)$ , a  $(G, s)$ -arc-transitive or  $(G, s)$ -arc-regular graph  $\Gamma$  is simply called  $s$ -arc-transitive or  $s$ -arc-regular, respectively.

The line graph  $L(\Gamma)$  of a graph  $\Gamma$  is the graph whose vertices are the edges of  $\Gamma$ , with two edges adjacent in  $L(\Gamma)$  if they have a vertex in common. A graph  $\Gamma$  is said to be locally 3-transitive if the vertex-stabilizer  $\text{Aut}(\Gamma)_v$  of  $v \in V(\Gamma)$  acts 3-transitively on  $\Gamma(v)$ .

Now we state our first main theorem.

**Theorem 1.2.** *Let  $\Gamma$  be a locally  $2\mathbf{K}_n$  graph with  $n \geq 2$ . Then  $\Gamma$  is 3-connected-homogeneous if and only if  $\Gamma$  is isomorphic to the line graph  $L(\Sigma)$  of a 3-arc-transitive and locally 3-transitive graph  $\Sigma$ .*

By Theorem 1.2, to construct 3-CH locally  $2\mathbf{K}_n$  graphs with  $n \geq 2$ , it is equivalent to construct 3-arc-transitive and locally 3-transitive graphs. The following theorem characterizes vertex stabilizers of 3-arc-transitive graphs.

**Theorem 1.3.** [36, Theorem 4.2] *For a  $(G, 3)$ -arc-transitive graph  $\Gamma$  of valency  $k$  and a 2-arc  $(w, u, v)$ , at least one of the following holds:*

- (i)  $G_u^{[1]}$  is transitive on  $\Gamma(w) - \{u\}$ , or
- (ii)  $G_u = A_7$  or  $S_7$ , and  $k = 7$ , or
- (iii)  $C_p^f \trianglelefteq G_u^{\Gamma(u)} \leq \text{AGL}_1(p^f)$ , the number of  $G_u^{[1]}$ -orbits on  $\Gamma(w) - \{u\}$  divides  $\gcd(p^f - 1, f)^2$ , and  $(G_u^{\Gamma(u)})_{wv} \leq C_f$ .

There are infinitely many 3-arc-transitive and locally 3-transitive graphs satisfying the condition in Theorem 1.3 (i). For example, the complete bipartite graph  $\mathbf{K}_{n,n}$  is a 3-arc-transitive and locally 3-transitive graph for each  $n \geq 3$ . For more examples of 3-arc-transitive and locally 3-transitive graphs satisfying the condition in Theorem 1.3 (i), we refer the reader to [31, 37]. It is easy to see that every 3-arc-transitive graph satisfying the condition in Theorem 1.3 (ii) is locally 3-transitive. Recently, Giudici and King [28] gave a classification of edge-primitive 3-arc-transitive graphs satisfying the condition in Theorem 1.3 (ii), and there are two sporadic and eight infinite families of such graphs.

In part (c) of [36, Remarks on Theorem 4.2], Li et al. wrote ‘‘It is not known whether there are 3-arc-transitive graphs satisfying the condition in part (iii) of Theorem 4.2’’. It is easy to see that there does not exist a 3-arc-transitive and locally 3-transitive graph satisfying the condition in Theorem 1.3 (iii). However, our next result shows that there do exist  $(G, 3)$ -arc-transitive graphs satisfying the condition in Theorem 1.3 (iii).

**Proposition 1.4.** *Let  $p$  be a prime and  $f$  be a positive integer. If  $p^f - 1$  is not coprime to  $f$ , then there exists  $G \leq \text{Aut}(\mathbf{K}_{p^f, p^f})$  such that  $\mathbf{K}_{p^f, p^f}$  is a  $(G, 3)$ -arc-transitive graphs satisfying the condition in Theorem 1.3 (iii).*

Following [47], we say that a pentavalent symmetric graph  $\Gamma$  is of type  $\mathcal{Q}_2^5$  if  $\text{Aut}(\Gamma)_u \cong \text{Frob}(20) \times C_2$  and  $\text{Aut}(\Gamma)_{\{u,v\}} \cong M_{16}$ , where  $\{u, v\}$  is an edge of  $\Gamma$ ,

$$\text{Frob}(20) = \langle a, b \mid a^5 = b^4 = 1, b^{-1}ab = a^2 \rangle \text{ and } M_{16} = \langle a, b \mid a^8 = b^2 = 1, bab = a^5 \rangle.$$

A trivalent symmetric graph  $\Gamma$  is said to be of *type*  $2^2$  if  $\text{Aut}(\Gamma)_u \cong S_3$  and  $\text{Aut}(\Gamma)_{\{u,v\}} \cong C_4$ , where  $\{u, v\}$  is an edge of  $\Gamma$  (see [10]). Now we state our next main theorem.

**Theorem 1.5.** *Let  $\Gamma$  be a locally  $2\mathbf{K}_n$  graph with  $n \geq 2$ . Then  $\Gamma$  is 3-connected-set-homogeneous but not 3-connected-homogeneous if and only if  $\Gamma$  is isomorphic to the line graph  $L(\Sigma)$  of a graph  $\Sigma$  such that one of the following holds:*

- (1)  $\Sigma$  is a tetravalent 3-arc-regular graph;
- (2)  $\Sigma$  is a pentavalent 3-arc-regular graph;
- (3)  $\Sigma$  is a 3-arc-transitive graph of valency 8 and  $\text{Aut}(\Sigma)_u^{\Sigma(u)} \cong C_2^3 \rtimes (C_7 \rtimes C_t)$  with  $t = 1$  or 3 and  $u \in V(\Sigma)$ ;
- (4)  $\Sigma$  is a 3-arc-transitive graph of valency 32 and  $\text{Aut}(\Sigma)_u^{\Sigma(u)} \cong C_2^5 \rtimes (C_{31} \rtimes C_5)$  with  $u \in V(\Sigma)$ ;
- (5)  $\Sigma$  is a 3-arc-transitive graph of valency  $q + 1$  and  $\text{Aut}(\Sigma)_u^{\Sigma(u)} \cong \text{PSL}(2, q). \langle \eta \rangle$ , where  $u \in V(\Sigma)$ ,  $q$  is an odd prime power such that  $q \equiv -1 \pmod{4}$  and  $\eta$  is a field automorphism of  $\text{GF}(q)$ ;
- (6)  $\Sigma$  is a pentavalent symmetric graph of type  $\mathcal{Q}_2^6$ ;
- (7)  $\Sigma$  is a trivalent symmetric graph of type  $2^2$ .

**Remark on Theorem 1.5 (5).** Let  $\Sigma$  be a 3-arc-transitive graph of valency  $q + 1$ , where  $q$  is a prime power. Let  $u \in V(\Sigma)$ . If  $\text{PGL}(2, q) \leq \text{Aut}(\Sigma)_u^{\Sigma(u)}$ , then  $\Sigma$  is locally 3-transitive, and then by Theorem 1.2, the line graph  $\Gamma$  of  $\Sigma$  is 3-connected homogeneous.

By Theorem 1.5, to construct 3-CSH but not 3-CH graphs which are locally  $2\mathbf{K}_n$  with  $n \geq 2$ , it is equivalent to construct graphs satisfying the conditions in each of (1)–(7) of Theorem 1.5. In [57, Remark 4.2], we gave a pentavalent 3-arc-regular graph of order  $5^3$  (see also Example 6.3). In 2010, C.H. Li et al. in [35] constructed a tetravalent 3-arc-regular graph with automorphism group  $\text{P}\Gamma\text{L}(2, 27)$ , and in a recent paper [41], J.J. Li et al. gave another six tetravalent 3-arc-regular graphs. For trivalent symmetric graphs of type  $2^2$ , by [9] there are only eight such graphs on up to 10000 vertices, and in 2020, Feng et al. [22] constructed an infinite family of trivalent symmetric graphs of type  $2^2$ . We are not aware of any other 2-arc-transitive graphs satisfying the conditions in (1)–(7) of Theorem 1.5.

Our third main theorem provides a useful method to construct 3-CSH but not 3-CH graphs which are locally  $2\mathbf{K}_n$ , and using it, we can give some new constructions of 2-arc-transitive graphs satisfying the conditions in (1)–(7) of Theorem 1.5. To state the result, we introduce the concept of Cayley graphs.

Given a finite group  $G$  and an inverse closed subset  $S \subseteq G \setminus \{1\}$ , the *Cayley graph*  $\text{Cay}(G, S)$  on  $G$  with respect to  $S$  is a graph with vertex set  $G$  and edge set  $\{\{g, sg\} \mid g \in G, s \in S\}$ . For any  $g \in G$ ,  $R(g)$  is the permutation of  $G$  defined by  $R(g) : x \mapsto xg$  for  $x \in G$ . Set  $R(G) = \{R(g) \mid g \in G\}$ . It is well known that  $R(G)$  is a regular subgroup of  $\text{Aut}(\text{Cay}(G, S))$ . In general, a vertex-transitive graph  $\Gamma$  is isomorphic to a Cayley graph on a group  $G$  if and only if its automorphism group has a subgroup isomorphic to  $G$ , acting regularly on the vertex set of  $\Gamma$  (see [3, Lemma 16.3]). Set  $A = \text{Aut}(\text{Cay}(G, S))$  and  $\text{Aut}(G, S) = \{\alpha \in \text{Aut}(G) \mid S^\alpha = S\}$ . Then  $N_A(R(G)) = R(G) \rtimes \text{Aut}(G, S)$ , and  $\Gamma$  is said to be a *normal Cayley graph* of  $G$  whenever  $N_A(R(G)) = A$  (see [24, 55]).

Now we state our last theorem which gives a good description for 3-CSH but not 3-CH graphs which are locally  $2\mathbf{K}_n$  and have solvable automorphism groups. We say that a graph  $\Gamma$  is *solvable* if  $\text{Aut}(\Gamma)$  is solvable.

**Theorem 1.6.** *Let  $n \geq 2$  and let  $\Gamma$  be a solvable locally  $2\mathbf{K}_n$  graph. Then  $\Gamma$  is 3-connected-set-homogeneous but not 3-connected-homogeneous if and only if  $\Gamma \cong \text{Cay}(H, S)$  such that the following hold:*

- (a)  $H$  is a group having two subgroups  $A, B$  such that  $G = \langle A, B \rangle$ ,  $A \cong B \cong C_p^f$ ,  $A \cap B = 1$  and  $S = (A \cup B) - \{1\}$ ; and
- (b) one of the following holds:
  - (1)  $(p, f) = (2, 2)$ ,  $\text{Aut}(H, S) \cong C_3 \wr C_2$ ;
  - (2)  $(p, f) = (5, 1)$ ,  $\text{Aut}(H, S) \cong C_4 \wr C_2$ ;
  - (3)  $(p, f) = (2, 3)$ ,  $C_7 \wr C_2 \leq \text{Aut}(H, S) \leq (C_7 \rtimes C_3) \wr C_2$ ;
  - (4)  $(p, f) = (2, 5)$ ,  $(C_{31} \times C_{31}) \rtimes C_{10} \leq \text{Aut}(H, S) \leq (C_{31} \rtimes C_5) \wr C_2$ ;
  - (5)  $(p, f) = (5, 1)$ ,  $\text{Aut}(H, S) \cong M_{16}$ ;
  - (6)  $(p, f) = (3, 1)$ ,  $\text{Aut}(H, S) \cong C_4$ .

**Remark on Theorem 1.6** (1) Applying Theorem 1.6, in Section 6 we shall show that there are infinitely many graphs  $\text{Cay}(H, S)$  satisfying the conditions in each of (1)–(6) of Theorem 1.6 (b).

(2) By [41, Theorem 1.1 & Corollary 1.2], every tetravalent 3-arc-regular Cayley graph is a normal cover of a Cayley graph on one of the following groups:  $C_3^{11} \times (C_2^{12} \cdot M_{11})$ ,  $S_{35}$  and  $A_{35}$ . This, however, is not true. Actually, we shall prove in Proposition 6.2 there are infinitely many graphs  $\Gamma = \text{Cay}(H, S)$  such that  $\Gamma$  satisfies the condition in (1) of Theorem 1.6 (b), and  $\Gamma = L(\Sigma)$  with  $\Sigma$  a tetravalent 3-arc-regular Cayley graph on a solvable group.

(3) By [47], there are nine types of pentavalent 2-arc-transitive graphs, characterized by the stabilizers of a vertex and an edge. We construct a pentavalent symmetric graphs of type  $\mathcal{Q}_2^6$ . To the best of our knowledge, this is the first known such graph.

(4) In 2006, Feng and Kwak [21, p.161] conjectured that every trivalent symmetric graph of order  $2 \cdot 3^n$  is a Cayley graph for each  $n \geq 1$ . In Lemma 6.8, we shall prove that there exists a Cayley graph  $\Gamma = \text{Cay}(H, S)$  on a group  $H$  of order  $3^{4n+1}$  for each  $n \geq 2$  satisfying the condition in (6) of Theorem 1.6 (b), and so  $\Gamma = L(\Sigma)$ , where  $\Sigma$  is a trivalent symmetric graph of order  $2 \cdot 3^{4n}$  of type  $2^2$ . Then every automorphism of  $\Sigma$  swapping any two adjacent vertices of  $\Sigma$  is not an involution, and so  $\Sigma$  is non-Cayley. This implies that Feng-Kwak's conjecture is not true.

## 2. PRELIMINARIES

Let  $G$  be a permutation group on a set  $\Omega$ . For a point  $\alpha \in \Omega$ , denote by  $G_\alpha$  the stabilizer of  $\alpha$  in  $G$ , and denote by  $\alpha^G$  the orbit of  $G$  on  $\Omega$  containing  $\alpha$ . Furthermore, for a subset  $\Delta \subseteq \Omega$ , denote by  $G_\Delta$  the subgroup of  $G$  fixing  $\Delta$  setwise. If  $G$  fixes  $\Delta$  setwise, then denote by  $G^\Delta$  the permutation group on  $\Delta$  induced by  $G$ .

We say that  $G$  is *semiregular* on  $\Omega$  if  $G_\alpha = 1$  for every  $\alpha \in \Omega$  and *regular* if  $G$  is transitive and semiregular. And,  $G$  is said to be *primitive* if  $G$  is transitive on  $\Omega$  and the only partitions of  $\Omega$  preserved by  $G$  are either the singleton subsets or the whole of  $\Omega$ . Let  $G$  be a transitive permutation group on a set  $\Omega$  and let  $u \in \Omega$ . The orbits of  $G_u$  on  $\Omega$  are called *suborbits* of  $G$ , and their sizes are called the *subdegrees* of  $G$ . The number  $r$  of the orbits of  $G_u$  on  $\Omega$  is called the permutation *rank* of  $G$  on  $\Omega$ .

A finite transitive permutation group  $G$  on a set  $\Omega$  is said to be  $\frac{3}{2}$ -*transitive* if all orbits of the stabilizer  $G_\alpha$  of any point  $\alpha \in \Omega$  on  $\Omega \setminus \{\alpha\}$  have the same size greater than 1. A

$\frac{3}{2}$ -transitive permutation group  $G$  on a set  $\Omega$  is said to be a *Frobenius group* if  $G_{\alpha\beta} = 1$  for each different point  $\alpha, \beta \in \Omega$ . By [2, Theorems 1.1–1.2], we obtain the following lemma.

**Lemma 2.1.** *Let  $G$  be a primitive  $\frac{3}{2}$ -transitive permutation group. Then  $G$  is either affine or almost simple. If  $G$  is almost simple, then one of the following holds:*

- (1)  $G$  is 2-transitive, or
- (2)  $n = 21$ ,  $G = A_7$  or  $S_7$  acting on the set of pairs in  $\{1, \dots, 7\}$ , or
- (3)  $n = \frac{1}{2}q(q-1)$  where  $q = 2^f \geq 8$ , and either  $G = \text{PSL}_2(q)$ , or  $G = \text{P}\Gamma\text{L}_2(q)$  with  $f$  prime; the size of the nontrivial subdegrees is  $q+1$  or  $f(q+1)$ , respectively.

A 2-arc  $(u, v, w)$  of a graph  $\Gamma$  is called a *2-geodesic* if  $u$  and  $w$  are at distance 2. A graph  $\Gamma$  is said to be *2-geodesic-transitive* if  $\Gamma$  has at least one 2-geodesic and  $\text{Aut}(\Gamma)$  is transitive on the set of  $t$ -geodesics of  $\Gamma$  for  $t \leq 2$ , where a 1-geodesic of  $\Gamma$  is an arc of  $\Gamma$ .

For a 2-arc  $(v_0, v_1, v_2)$  of a graph  $\Gamma$ ,  $(v_2, v_1, v_0)$  is also a 2-arc. If we identify these two arcs, then we obtain a *2-path*, denoted by  $[v_0, v_1, v_2]$ , and if  $v_0$  and  $v_2$  are adjacent then we get a triangle, denoted by  $\{v_0, v_1, v_2\}$ . The 2-path  $[v_0, v_1, v_2]$  is called a *2-geodesic-path* provided that the triple  $(v_0, v_1, v_2)$  is a 2-geodesic. We say that  $\Gamma$  is *2-path transitive* (*2-geodesic-path transitive*, respectively) if  $\text{Aut}(\Gamma)$  is transitive on the set of 2-paths (*2-geodesic-paths*, respectively) of  $\Gamma$ .

For a graph  $\Gamma$ , we use  $\Gamma^c$  to denote the complementary graph of  $\Gamma$ . From [57, Theorems 1.1–1.2, Corollary 1.4], we obtain the following lemma.

**Lemma 2.2.** *Let  $\Gamma$  be a connected 3-CSH non-complete graph of girth 3. Let  $G = \text{Aut}(\Gamma)$ . Then  $\Gamma$  is arc-transitive, and for any  $\{u, v\} \in E(\Gamma)$ , we have the following:*

- (1) *If  $[\Gamma(u)]$  is connected, then  $[\Gamma(u)]$  is of diameter 2, and if  $[\Gamma(u)]$  is disconnected, then  $[\Gamma(u)] \cong m\mathbf{K}_\ell$  for some positive integers  $m, \ell$ .*
- (2)  $G_u$  is edge-transitive on  $[\Gamma(u)]^c$ .
- (3)  $G_{uv}$  has  $s$  orbits on  $\Gamma(u) \cap \Gamma(v)$  with equal size, where  $s = 1, 2, 3$  or  $6$ .
- (4)  $G_{uv}$  has  $t$  orbits on  $\Gamma(u) - ((\Gamma(u) \cap \Gamma(v)) \cup \{v\})$  with equal size, where  $t = 1$  or  $2$ .
- (5) *If  $t = 1$ , then  $\Gamma$  is 2-geodesic-transitive.*

The following lemma gives a characterization of 2-geodesic-path transitive graphs.

**Lemma 2.3.** *Let  $\Gamma$  be a connected vertex-transitive graph of valency at least 2. Take  $u \in V(\Gamma)$ . Then  $\text{Aut}(\Gamma)$  is transitive on the set of 2-geodesic-paths if and only if  $\text{Aut}(\Gamma)_u$  is transitive on the edges of  $[\Gamma(u)]^c$ .*

**Proof.** Suppose first that  $\text{Aut}(\Gamma)$  is transitive on the set of 2-geodesic-paths. Take two edges, say  $\{x, y\}, \{x', y'\}$ , of  $[\Gamma(u)]^c$ . Then  $[x, u, y]$  and  $[x', u, y']$  are two 2-geodesic-paths of  $\Gamma$ . Then there exists  $g \in \text{Aut}(\Gamma)$  sending  $[x, u, y]$  to  $[x', u, y']$ . It follows that  $g$  fixes  $u$  and sends  $\{x, y\}$  to  $\{x', y'\}$ . Therefore,  $\text{Aut}(\Gamma)_u$  is transitive on the edges of  $[\Gamma(u)]^c$ .

Conversely, assume that  $\text{Aut}(\Gamma)_u$  is transitive on the edges of  $[\Gamma(u)]^c$ . Take an edge, say  $\{x, y\}$ , of  $[\Gamma(u)]^c$ . Then  $[x, u, y]$  is a 2-geodesic-path of  $\Gamma$ . For any 2-geodesic path, say  $[x', u', y']$  of  $\Gamma$ , by the vertex-transitivity, there exists a  $g \in \text{Aut}(\Gamma)$  sending  $u'$  to  $u$ , and so  $\{x', y'\}^g$  is an edge of  $[\Gamma(u)]^c$ . Since  $\text{Aut}(\Gamma)_u$  is transitive on the edges of  $[\Gamma(u)]^c$ , there exists  $h \in \text{Aut}(\Gamma)_u$  such that  $\{x', y'\}^{gh} = \{x, y\}$ , and so  $[x', u', y']^{gh} = [x, u, y]$ . This implies that  $\text{Aut}(\Gamma)$  is transitive on the set of 2-geodesic-paths.  $\square$

A *clique* of a graph  $\Gamma$  is a complete subgraph and a *maximal clique* is a clique which is not contained in a larger clique. The *clique graph*  $C(\Gamma)$  of  $\Gamma$  is a graph with vertices

the maximal cliques of  $\Gamma$  and with two different maximal cliques adjacent if they share at least one common vertex.

**Lemma 2.4.** *Let  $n \geq 2$  be an integer, and let  $\Gamma$  be a locally  $2\mathbf{K}_n$  graph. Then  $\Gamma$  is isomorphic to the line graph of  $C(\Gamma)$ , and  $\text{Aut}(\Gamma) \cong \text{Aut}(C(\Gamma))$ .*

**Proof.** By [14, Corollary 1.6], we know that  $\Gamma$  is isomorphic to the line graph of  $C(\Gamma)$ , and by [1, p.1455], we have  $\text{Aut}(\Gamma) \cong \text{Aut}(C(\Gamma))$ .  $\square$

From [15, Theorem 1.1] we deduce the following result.

**Proposition 2.5.** *Let  $\Gamma$  be a connected regular, non-complete graph of valency at least 3. Let  $s = 2$  or 3. Then  $\Gamma$  is  $s$ -arc-transitive if and only if the line graph of  $\Gamma$  is  $(s - 1)$ -geodesic-transitive.*

### 3. PROOFS OF THEOREM 1.2 AND PROPOSITION 1.4

In this section, we shall prove Theorem 1.2 and Proposition 1.4.

**Proof of Theorem 1.2** Let  $\Sigma = C(\Gamma)$ . By Lemma 2.4, for convenience, we shall identify  $\Gamma$  with the line graph of  $\Sigma$ .

Suppose first that  $\Sigma$  is 3-arc-transitive and locally 3-transitive. By Proposition 2.5,  $\Gamma$  is 2-geodesic-transitive and arc-transitive. To show that  $\Gamma$  is 3-CH, it suffices to prove that  $\text{Aut}(\Gamma)$  is transitive on the set of 3-tuples  $(e, f, g)$  such that  $\{e, f, g\}$  is a triangle. Let  $(e_1, e_2, e_3)$  and  $(f_1, f_2, f_3)$  be two 3-tuples of vertices in  $\Gamma$  such that both  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2, f_3\}$  induce two triangles. As we assume that  $\Gamma$  is the line graph of  $\Sigma$ , we may let  $e_1 = \{u, v_1\}$ ,  $e_2 = \{u, v_2\}$  and  $e_3 = \{u, v_3\}$ , and  $f_1 = \{x, y_1\}$ ,  $f_2 = \{x, y_2\}$  and  $f_3 = \{x, y_3\}$ , where  $u, v_i, x, y_i \in V(\Sigma)$  ( $i = 1, 2, 3$ ). Then  $(v_1, u, v_2)$  and  $(y_1, x, y_2)$  are 2-arcs of  $\Sigma$ . Since  $\Sigma$  is 3-arc-transitive, there exists  $\alpha \in \text{Aut}(\Sigma)$  such that  $(v_1, u, v_2)^\alpha = (y_1, x, y_2)$ , and so  $(e_1, e_2, e_3)^\alpha = (f_1, f_2, e_3^\alpha)$ , where  $e_3^\alpha = \{x, v_3^\alpha\}$ . Clearly,  $v_3^\alpha \in \Sigma(x)$ . Since  $\Sigma$  is locally 3-transitive, there exists  $\beta$  such that  $\beta$  fixes  $x, y_1, y_2$  and maps  $v_3^\alpha$  to  $y_3$ . So  $(e_1, e_2, e_3)^{\alpha\beta} = (f_1, f_2, f_3)$ .

Suppose now that  $\Gamma$  is 3-CH. Then  $\Gamma$  is 2-geodesic-transitive, and by Proposition 2.5,  $\Sigma$  is 3-arc-transitive. Let  $u \in V(\Sigma)$ . Then  $\text{Aut}(\Sigma)_u$  acts 2-transitively on  $\Sigma(u)$ . Take  $v, w \in \Sigma(u)$ . For any  $x_1, x_2 \in \Sigma(u) - \{v, w\}$ , both  $\{\{u, v\}, \{u, w\}, \{u, x_1\}\}$  and  $\{\{u, v\}, \{u, w\}, \{u, x_2\}\}$  induce two triangles of  $\Gamma$ . Since  $\Gamma$  is 3-CH, there exists  $\alpha \in \text{Aut}(\Sigma) = \text{Aut}(\Gamma)$  such that  $(\{u, v\}, \{u, w\}, \{u, x_1\})^\alpha = (\{u, v\}, \{u, w\}, \{u, x_2\})$ . It follows that  $\alpha \in \text{Aut}(\Sigma)_{uvw}$  and  $x_1^\alpha = x_2$ . Thus,  $\text{Aut}(\Sigma)_u$  acts 3-transitively on  $\Sigma(u)$ , and hence  $\Sigma$  is locally 3-transitive.  $\square$

**Proof of Proposition 1.4** Let  $p$  be a prime and  $f$  be a positive integer. Suppose that  $r$  is a common prime divisor of  $p^f - 1$  and  $f$ . Let  $\ell$  be a positive integer such that  $r^\ell \mid p^f - 1$  but  $r^{\ell+1} \nmid p^f - 1$ . Let  $\Gamma = \mathbf{K}_{p^f, p^f}$  with biparts  $U$  and  $W$ . Then  $\text{Aut}(\Gamma)$  has a subgroup  $A = \text{AGL}(1, p^f)$  which fixes  $U$  point-wise and is 2-transitive on  $W$ . Let  $g$  be any involution of  $\text{Aut}(\Gamma)$  swapping  $U$  and  $W$ , and let  $M = \langle A, g \rangle$ . Then  $M = (A \times A^g) \rtimes \langle g \rangle$  is 3-arc-transitive on  $\Gamma$ , and  $M_u^{[1]}$  is transitive on  $\Gamma(w) - \{u\} = U - \{u\}$ , where  $u \in U, w \in W$ .

For convenience, we assume that

$$A = \text{AGL}(1, p^f) = N \rtimes (\langle a \rangle \times \langle b \rangle) \cong C_p^f \rtimes (C_{p^f-1} \times C_f).$$

Let  $B = \langle N, a^{r^\ell}, a^{\frac{p^f-1}{r^\ell}}(b^{\frac{f}{r}})^g, g \rangle$ . We may let  $A_w = \langle a \rangle \rtimes \langle b \rangle$  and  $A_{wv} = \langle b \rangle$  with  $w, v \in W$ . Let  $u = w^g$  and  $x = v^g$ . Then  $u, x \in U$  and  $(w, u, v)$  is a 2-arc of  $\Gamma$ . Further,  $(A^g)_u = \langle a^g \rangle \rtimes \langle b^g \rangle$  and  $(A^g)_{ux} = \langle b^g \rangle$ . Now  $B_{wu} = \langle a^{r^\ell}, (a^{r^\ell})^g, a^{\frac{p^f-1}{r^\ell}}(b^{\frac{f}{r}})^g, (a^{\frac{p^f-1}{r^\ell}})^g b^{\frac{f}{r}} \rangle$ , and  $B_w = N^g \rtimes B_{wu}$  and  $B_u = N \rtimes B_{wu}$ . So  $B_w$  acts 2-transitively on  $U$ . Since  $g \in B$ ,  $\Gamma$  is  $(B, 2)$ -arc-transitive. Note that  $B_{wuv} = \langle (a^{r^\ell})^g, (a^{\frac{p^f-1}{r^\ell}})^g b^{\frac{f}{r}} \rangle$  and  $B_u^{[1]} = \langle (a^{r^\ell})^g, (a^{\frac{p^f-1}{r^\ell}})^g \rangle$ . So  $(B_u^{\Gamma(u)})_{wv} \cong B_{wuv}/B_u^{[1]} \leq C_f$ . Since  $\langle (a^{r^\ell})^g, (a^{\frac{p^f-1}{r^\ell}})^g \rangle$  is regular on  $U - \{u\}$ ,  $B_{wuv}$  is also transitive on  $\Gamma(v) - \{u\} = U - \{u\}$ . It follows that  $\Gamma$  is  $(B, 3)$ -arc-transitive. Note that  $B_u^{[1]} = \langle (a^{r^\ell})^g, (a^{\frac{p^f-1}{r^\ell}})^g \rangle$  has order  $\frac{p^f-1}{r}$ . Since  $\langle a^g \rangle$  is regular on  $\Gamma(w) - \{u\} = U - \{u\}$ , the number of orbits of  $B_u^{[1]}$  on  $\Gamma(w) - \{u\} = U - \{u\}$  is  $r$ , which is a divisor of  $\gcd(p^f - 1, f)$ . Now we conclude that  $\Gamma$  is a  $(G, 3)$ -arc-transitive graph satisfying the condition in part (3) of Theorem 1.3 with  $G = B$ .  $\square$

#### 4. PROOF OF THEOREM 1.5

We begin by proving two lemmas regarding  $\frac{3}{2}$ -transitive permutation groups.

**Lemma 4.1.** *Let  $G$  be a  $\frac{3}{2}$ -transitive permutation group on a set  $\Omega$ . Then the following hold:*

- (1) *if  $G$  has rank 3 or 4, then  $G$  is primitive;*
- (2) *if  $G$  has rank 7, then either  $G$  is primitive or  $|\Omega| = 25$  and  $|G| = 100$ .*

**Proof.** Suppose that  $G$  is imprimitive. By [53, Theorem 10.4],  $G$  is a Frobenius group. By [50, Proposition 4.4], there exist two different  $x, y \in \Omega$  such that the digraph  $\Gamma$  with vertex set  $\Omega$  and arc set  $\{(x^g, y^g) \mid g \in G\}$  is disconnected. Let  $\Gamma_1$  be a component of  $\Gamma$  with  $\Omega_1 = V(\Gamma_1)$ . Then  $|\Omega_1| \leq \frac{|\Omega|}{2}$ , and since  $G$  is transitive on  $\Omega$ , one has  $|\Omega| = t|\Omega_1|$  for some integer  $t > 0$ .

Let  $r$  be the rank of  $G$  and take  $\alpha \in \Omega_1$ . Since  $G$  is a Frobenius group, one has  $|G_\alpha| = \frac{|\Omega|-1}{r-1} \geq 2$ . It implies that  $|\Omega| \geq 2r - 1$ . Clearly,  $G_\alpha$  fixes  $\Omega_1$  setwise. Again, since  $G$  is a Frobenius group, we know that  $G_\alpha$  acts semiregularly on  $\Omega_1 - \{\alpha\}$ , and so  $|\Omega_1| = 1 + k|G_\alpha|$  for some positive integer  $k < r - 1$ . It follows that  $|\Omega_1| = 1 + \frac{k(|\Omega|-1)}{r-1}$ . Since  $|\Omega| = t|\Omega_1|$ , one has  $|\Omega| = t \cdot (1 + \frac{k(|\Omega|-1)}{r-1})$ , implying that

$$(r - 1 - tk)|\Omega| = t(r - 1 - k) > 0. \quad (1)$$

It follows that  $r - 1 - tk > 0$ . If  $t = 1$ , then Eq. (1) implies that  $|\Omega| = 1$ , contrary to  $|\Omega| \geq 2r - 1$ . Thus,  $t > 1$  and hence  $r > 2k + 1$ .

This implies that  $r > 3$ . If  $r = 4$ , then  $k = 1$ , and then by Eq. (1), we obtain that  $(3 - t)|\Omega| = 2t$ . It implies that  $t < 3$  and  $|\Omega| \leq 4$ , contrary to  $|\Omega| \geq 2r - 1 = 7$ . This proves part (1).

If  $r = 7$ , then  $2k + 1 < 7$ , and then  $k \leq 2$ . In case  $k = 2$ , by Eq. (1), we have  $(6 - 2t)|\Omega| = 4t$ . It follows that  $t < 3$  and  $|\Omega| \leq 4$ , contrary to  $|\Omega| \geq 2r - 1 = 13$ . Thus, we have  $k = 1$ . Again, by Eq. (1), we obtain that  $(6 - t)|\Omega| = 5t$ , implying  $t \leq 5$ . Since  $6 = r - 1$  is a divisor of  $|\Omega| - 1$ , it follows that  $t = 5$ , and hence  $|\Omega| = 25$ . Then  $|G| = 100$ . Part (3) holds.  $\square$



TABLE 1. 2-Transitive groups

Line	$G$	$ \Omega $	Remarks
1	$\mathrm{PSL}(2, q) \trianglelefteq G \leq \mathrm{P}\Gamma\mathrm{L}(2, q)$	$q + 1$	$q > 3$ a prime power
2	$\mathrm{Sz}(q)$	$q^2 + 1$	$q = 2^{2a+1} > 2$
3	$\mathrm{AGL}(1, 2^f)$	$2^f$	$f$ a prime

**Lemma 4.2.** *Let  $G$  be a 2-transitive permutation group on a set  $\Omega$  and take  $x \in \Omega$ . Suppose that  $G_x$  is a  $\frac{3}{2}$ -transitive permutation group on  $\Omega - \{x\}$  of rank 3, 4 or 7. Then  $G_x$  is primitive on  $\Omega - \{x\}$  and one of the following holds.*

- (1)  $G = \mathrm{PSL}(2, q) \cdot \langle \eta \rangle$  and  $|\Omega| = q + 1$ , where  $q > 3$  is an odd prime power and  $\eta$  is a field automorphism of  $\mathrm{GF}(q)$ . Moreover,  $G_x \cong C_{\frac{q-1}{2}} \cdot \mathcal{O}$  has rank 3 on  $\Omega - \{x\}$ , where  $\mathcal{O} \cong \langle \eta \rangle$ .
- (2)  $G \cong C_2^f \rtimes (C_{2^{f-1}} \rtimes C_f)$  and  $|\Omega| = 2^f$ , where  $f = 3$  or 5. Moreover,  $G_x$  has rank 3 or 7 on  $\Omega - \{x\}$ .

**Proof.** Suppose first that  $G_x$  is imprimitive on  $\Omega - \{x\}$ . By Lemma 4.1,  $G_x$  is a permutation group on  $\Omega - \{x\}$  with rank 7, and  $|\Omega - \{x\}| = 25$  and  $|G_x| = 100$ . However, by checking [16, Appendix B], there exist no 2-transitive permutation groups of degree 26 with point-stabilizer of order 100, a contradiction. Thus,  $G_x$  is primitive on  $\Omega - \{x\}$ .

Since  $G_x$  is  $\frac{3}{2}$ -transitive on  $\Omega - \{x\}$  with rank 3, 4 or 7, it follows that  $G$  is  $\frac{5}{2}$ -transitive but neither 3-transitive nor sharply 2-transitive on  $\Omega$ . By [42, Proposition 4], we conclude that  $G$  is one of the groups in Table 1.

If  $\mathrm{PSL}(2, q) \trianglelefteq G \leq \mathrm{P}\Gamma\mathrm{L}(2, q)$ , then  $|\Omega| = q + 1$  and  $[q] : C_{\frac{q-1}{s}} \trianglelefteq G_x$  with  $s = (2, q - 1)$ . Since  $G$  is not 3-transitive on  $\Omega$ , one has  $\mathrm{PGL}(2, q) \not\leq G$  and  $q > 3$  is odd. It follows that  $G = \mathrm{PSL}(2, q) \cdot \langle \eta \rangle$ , where  $\eta$  is a field automorphism of  $\mathrm{GF}(q)$ . Moreover,  $G_x \cong C_{\frac{q-1}{2}} \cdot \mathcal{O}$  with  $\mathcal{O} \cong \langle \eta \rangle$ , and  $G_x$  has rank 3 on  $\Omega - \{x\}$ . So part (1) holds.

If  $G = \mathrm{Sz}(q)$ , then  $G_x \cong [q^2] : C_{q-1}$ . By [16, Section 7.7], we deduce that the normal subgroup of  $G_x$  of order  $q^2$  is not an elementary abelian 2-group. This contradicts that  $G_x$  is primitive on  $\Omega - \{x\}$ .

Now let  $G = \mathrm{AGL}(1, 2^f)$  with  $f$  a prime. Then  $|\Omega| = 2^f$  and  $G_x \cong C_{2^{f-1}} \rtimes C_f$ . Since  $G_x$  is primitive on  $\Omega - \{x\}$ , one has  $|\Omega - \{x\}| = 2^f - 1$  and  $\mathrm{soc}(G_x) \cong C_{2^{f-1}}$ . It follows that  $2^f - 1$  is a prime. Furthermore, for an arbitrary  $y \in \Omega - \{x\}$ , we have  $G_{xy} \cong C_f$  and  $G_{xy}$  is semiregular on  $\Omega - \{x, y\}$ . Since  $G_x$  is  $\frac{3}{2}$ -transitive on  $\Omega - \{x\}$  with rank 3, 4 or 7, it follows that  $G_{xy}$  has  $s$  orbits of size  $f$  on  $\Omega - \{x, y\}$ , where  $s = 2, 3$  or 6. This implies that  $sf = |\Omega - \{x, y\}| = 2^f - 1$ .

Recall that  $f$  is a prime. If  $f > 5$ , then  $s = 2$  or 6, and hence  $2^{f-1} - 1 = f$  or  $3f$ . However, this is impossible because  $f > 5$ . If  $f = 2$ , then  $G_x \cong C_3 \rtimes C_2$  which is 2-transitive on  $\Omega - \{x\}$ . This is impossible because  $G_x$  is  $\frac{3}{2}$ -transitive group on  $\Omega - \{x\}$ . Thus,  $f = 3$  or 5, and  $G_x \cong C_{2^{f-1}} \rtimes C_f$ . Part (2) happens.  $\square$

In the next four lemmas, we shall prove the sufficiency of Theorem 1.5.

**Lemma 4.3.** *Let  $\Gamma$  be a connected 3-CSH locally  $2\mathbf{K}_n$  graph with  $n \geq 2$ . Then  $C(\Gamma)$  is 2-arc-transitive.*

**Proof.** By Lemma 2.4,  $\Gamma$  is isomorphic to the line graph of  $C(\Gamma)$ , and by Lemma 2.2,  $\Gamma$  is arc-transitive and so 1-geodesic-transitive. It then follows from Proposition 2.5 that  $C(\Gamma)$  is 2-arc-transitive.  $\square$

**Lemma 4.4.** *Let  $\Gamma$  be a connected locally  $2\mathbf{K}_q$  graph, where  $q > 3$  is an odd prime power. Let  $\Sigma = C(\Gamma)$  and let  $A = \text{Aut}(\Sigma)$ . Take  $u \in V(\Sigma)$  and  $v \in \Sigma(u)$ . Suppose that  $\Sigma$  is 3-arc-transitive and that  $A_u^{\Sigma(u)} \cong \text{PSL}(2, q). \langle \eta \rangle$ , where  $\eta$  is a field automorphism of  $\text{GF}(q)$ . Then  $\Gamma$  is 3-CSH if and only if  $q \equiv -1 \pmod{4}$ .*

**Proof.** By Lemma 2.4, we may assume that  $\Gamma$  is the line graph of  $\Sigma$ . Take an edge  $e = \{u_e, v_e\}$  of  $\Sigma$ . The number of triangles of  $\Gamma$  passing through  $e$  is equal to  $q(q-1)$ , which is just the number of edges of  $[\Gamma(e)]$ . Since  $\Sigma$  is 3-arc-transitive,  $\Gamma$  is vertex-transitive, and as every triangle contains three vertices, the total number of triangles of  $\Gamma$  is  $q(q-1)|V(\Gamma)|/3$ .

Let  $\Gamma(e) = \{e_i, e'_i \mid 1 \leq i \leq q\}$ , where  $e_1, e_2, \dots, e_q$  are edges of  $\Sigma$  incident with  $u_e$  and  $e'_1, e'_2, \dots, e'_q$  are edges of  $\Sigma$  incident with  $v_e$ . Then  $[\Gamma(e)] \cong 2\mathbf{K}_q$ , and  $\{e_i \mid i = 1, 2, \dots, q\}$  and  $\{e'_i \mid i = 1, 2, \dots, q\}$  are the bi-parts of  $[\Gamma(e)]^c = \mathbf{K}_{q,q}$  (see Figure (1)).

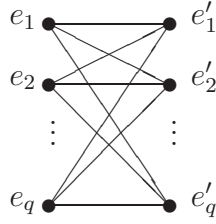


FIGURE 1. The complement  $[\Gamma(e)]^c$  of  $[\Gamma(e)]$

Let  $N_1 = \{e, e_i \mid 1 \leq i \leq q\}$ . Then  $N_1$  is just the set of edges of  $\Sigma$  incident with  $u_e$  and  $|N_1| = q+1$ . So we may view  $A_{u_e}^{\Sigma(u_e)}$  as a permutation group on  $N_1 = \{e, e_i \mid 1 \leq i \leq q\}$ .

Note that  $|\text{PSL}(2, q)| = \frac{1}{2}q(q^2 - 1)$ . This implies that  $3 \mid |\text{PSL}(2, q)|$ . Take an element, say  $x$ , of order 3 in  $\text{soc}(A_{u_e}^{\Sigma(u_e)})$ . We may assume that  $e_1^x \neq e_1$ . Then  $\{e_1, e_1^x, e_1^{x^2}\}$  induces a triangle of  $\Gamma$ , and so  $C_3 \leq A_{\{e_1, e_1^x, e_1^{x^2}\}}/A_{e_1 e_1^x e_1^{x^2}} \leq S_3$ . Since  $\Sigma$  is 3-arc-transitive, we obtain that  $\Gamma$  is arc-transitive. It follows that  $|A : A_{e_1}| = |V(\Gamma)|$  and  $|A_{e_1} : A_{e_1 e_1^x}| = |\Gamma(e_1)| = 2q$ . Noticing that  $A_{e_1 e_1^x}$  fixes  $e_1 \cap e_1^x = \{u_e\}$ , we have  $A_{e_1 e_1^x} \leq A_{u_e}$ . It implies that  $A_{e_1 e_1^x}$  fixes  $N_1$  setwise as  $N_1$  is the set of edges of  $\Sigma$  incident with  $u_e$ . So  $|A_{e_1 e_1^x} : A_{e_1 e_1^x e_1^{x^2}}| = |(A_{u_e}^{\Sigma(u_e)})_{e_1 e_1^x} : (A_{u_e}^{\Sigma(u_e)})_{e_1 e_1^x e_1^{x^2}}| = \frac{q-1}{2}$ . It follows that

$$|A : A_{e_1 e_1^x e_1^{x^2}}| = |A : A_{e_1}| |A_{e_1} : A_{e_1 e_1^x}| |A_{e_1 e_1^x} : A_{e_1 e_1^x e_1^{x^2}}| = q(q-1)|V(\Gamma)|.$$

If  $q \equiv 1 \pmod{4}$ , then there exists an involution  $y \in (\text{soc}(A_{u_e}^{\Sigma(u_e)}))_{e_1 e_1^x}$  and so  $y$  must interchange another two edges, say  $e_i$  and  $e_j$  in  $N_1$ . This implies that  $C_2 \leq A_{\{e_1, e_i, e_j\}}/A_{e_1 e_i e_j}$ . If  $\Gamma$  is 3-CSH, then  $A$  is transitive on the triangles of  $\Gamma$ , and then we would have  $A_{\{e_1, e_1^x, e_1^{x^2}\}}/A_{e_1 e_1^x e_1^{x^2}} \cong S_3$ . It then follows that  $|A_{\{e_1, e_1^x, e_1^{x^2}\}}| = 6|A_{e_1 e_1^x e_1^{x^2}}|$ . Consequently, the size of the orbit  $\{e_1, e_1^x, e_1^{x^2}\}^A$  of  $A$  acting on the set of triangles of  $\Gamma$  is

$$|A : A_{\{e_1, e_1^x, e_1^{x^2}\}}| = \frac{q(q-1)}{6}|V(\Gamma)|.$$

This, however, is impossible because the total number of triangles of  $\Gamma$  is  $q(q-1)|V(\Gamma)|/3$ .

If  $q \equiv -1 \pmod{4}$ , then  $(A_{u_e}^{\Sigma(u_e)})_{e_1 e_1^x}$  has odd order, and so every involution in  $A_{u_e}^{\Sigma(u_e)}$  does not fix any edge in  $N_1$ . Since  $A_{\{e_1, e_1^x, e_1^{x^2}\}}$  fixes  $e_1 \cap e_1^x \cap e_1^{x^2} = \{u_e\}$ , we have  $A_{e_1 e_1^x} \leq A_{u_e}$ . It follows that  $A_{\{e_1, e_1^x, e_1^{x^2}\}}$  fixes  $N_1$  setwise since  $N_1$  is the set of edges of  $\Sigma$  incident with  $u_e$ . So  $A_{\{e_1, e_1^x, e_1^{x^2}\}}^{\Gamma(e)} \leq A_{u_e}^{\Sigma(u_e)}$ . This implies that  $A_{\{e_1, e_1^x, e_1^{x^2}\}}/A_{e_1 e_1^x e_1^{x^2}} \cong C_3$ . Consequently, the size of the orbit  $\{e_1, e_1^x, e_1^{x^2}\}^A$  of  $A$  acting on the set of triangles of  $\Gamma$  is

$$|A : A_{\{e_1, e_1^x, e_1^{x^2}\}}| = \frac{q(q-1)}{3} |V(\Gamma)|.$$

Thus,  $\Gamma$  is 3-CSH. □

**Lemma 4.5.** *Let  $\Gamma$  be a connected locally  $2\mathbf{K}_q$  graph, where either  $q > 3$  is an odd prime power or  $q = 2^f - 1$  with  $f = 2, 3$  or  $5$ . Let  $\Sigma = C(\Gamma)$  and let  $A = \text{Aut}(\Sigma)$ . Take  $u \in V(\Sigma)$  and  $v \in \Sigma(u)$ . Suppose that  $\Sigma$  is 3-arc-transitive and satisfies one of the following:*

- (1)  $q$  is an odd prime power such that  $q \equiv -1 \pmod{4}$ , and  $A_u^{\Sigma(u)} \cong \text{PSL}(2, q). \langle \eta \rangle$ , where  $\eta$  is a field automorphism of  $\text{GF}(q)$ , or,
- (2)  $q = 2^2 - 1$  and  $A_u^{\Sigma(u)} \cong A_4$ , or,
- (3)  $q = 2^3 - 1$  and  $A_u^{\Sigma(u)} \cong C_2^3 \rtimes (C_7 \rtimes C_s)$  with  $s = 1$  or  $3$ , or,
- (4)  $q = 2^5 - 1$  and  $A_u^{\Sigma(u)} \cong C_2^5 \rtimes (C_{31} \rtimes C_5)$ .

Then  $\Gamma$  is 2-geodesic-transitive and 3-CSH but not 3-CH.

**Proof.** In view of Lemma 2.4, for convenience, we shall assume that  $\Gamma$  is the line graph of  $\Sigma$ . Since  $\Sigma$  is 3-arc-transitive, by Proposition 2.5,  $\Gamma$  is 2-geodesic-transitive. Observe that in each of (1)–(4),  $A_u^{\Sigma(u)}$  is not 3-transitive. It implies that  $\Sigma$  is not locally 3-transitive. So  $\Gamma$  is not 3-CH by Theorem 1.2.

If  $\Sigma$  satisfies the condition in part (1), then by Lemma 4.4,  $\Gamma$  is 3-CSH.

Next we consider the case when  $\Sigma$  satisfies the condition in one of parts (2), (3) and (4). Let  $q = 2^f - 1$  with  $f = 2, 3$  or  $5$ . Let  $e = \{u, v\}$  and let  $\Gamma(e) = \{e_i, e'_i \mid i = 1, 2, \dots, q\}$ , where  $e_1, e_2, \dots, e_q$  are edges of  $\Sigma$  incident with  $u$  and  $e'_1, e'_2, \dots, e'_q$  are edges of  $\Sigma$  incident with  $v$ . Then  $[\Gamma(e)] \cong 2\mathbf{K}_q$ , and  $\{e_i \mid i = 1, 2, \dots, q\}$  and  $\{e'_i \mid i = 1, 2, \dots, q\}$  are the bi-parts of  $[\Gamma(e)]^c = \mathbf{K}_{q,q}$  (see Figure (2)).

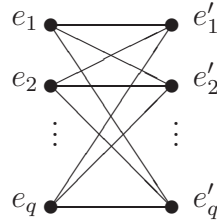


FIGURE 2. The complement  $[\Gamma(e)]^c$  of  $[\Gamma(e)]$

Assume first that  $\Sigma$  satisfies the condition in part (2). Then  $q = 3$  and  $A_u^{\Sigma(u)} \cong A_4$ . By [44, Theorem 4], we have that  $A_u \cong A_4 \times C_3$ . Then the arc stabilizer  $A_{(u,v)} \cong C_3 \times C_3$ , and  $A_{(u,v)}^{\Gamma(e)} = \langle (e_1, e_2, e_3) \rangle \times \langle (e'_1, e'_2, e'_3) \rangle$ . Since  $\Sigma$  is arc-transitive, there exists an involution  $g \in A_e$  such that  $g$  swaps the two bi-parts of  $[\Gamma(e)]^c = \mathbf{K}_{3,3}$ . This implies that  $A_e \cong (C_3 \times C_3) \rtimes C_2$  and that  $A_e$  is edge-transitive on both  $[\Gamma(e)]$  and  $[\Gamma(e)]^c$ . By [57, Lemma 3.1],  $\Gamma$  is 3-CSH.

Now assume that  $\Sigma$  satisfies the condition in one of parts (3) and (4). Then  $q = 2^f - 1$  with  $f = 3$  or  $5$ , and  $A_u^{\Sigma(u)} \cong C_2^f \times (C_q \times C_s)$ , where  $s = 1$  or  $3$  if  $f = 3$ , and  $s = 5$  if  $f = 5$ . To complete the proof, it suffices to show that  $A$  is transitive on the triangles of  $\Gamma$ . Note that the number of triangles of  $\Gamma$  passing through  $e$  is equal to  $q(q-1)$ , which is just the number of edges of  $[\Gamma(e)]$ . Since  $\Gamma$  is vertex-transitive and every triangle contains three vertices, the total number of triangles of  $\Gamma$  is  $q(q-1)|V(\Gamma)|/3$ .

Let  $N_1 = \{e, e_i \mid 1 \leq i \leq q\}$ . Then  $N_1$  is just the set of edges of  $\Sigma$  incident with  $u$  and  $|N_1| = q+1$ . So we may view  $A_u^{\Sigma(u)}$  as a permutation group on  $N_1$ . Then  $A_u^{[1]}$  is the kernel of  $A_u$  acting on  $N_1$  and  $A_u^{\Sigma(u)} \cong A_u/A_u^{[1]}$ . For any distinct  $e_i, e_j \in N_1$ ,  $\{e, e_i, e_j\}$  induces a triangle of  $\Gamma$ . Since  $\Sigma$  is 3-arc-transitive, it follows that  $A_u^{\Sigma(u)} \cong C_2^f \times (C_q \times C_s)$  is 2-transitive on  $\Sigma(u)$ . So  $A_u^{\Sigma(u)}$  is also 2-transitive on  $N_1$ . It follows that  $(A_u^{\Sigma(u)})_\alpha \cong C_q \times C_s$  for all  $\alpha \in \{e, e_i, e_j\}$ ,  $(A_u^{\Sigma(u)})_{ee_i} \cong C_s$  and  $(A_u^{\Sigma(u)})_{ee_ie_j} = 1$ .

Noting that  $e \cap e_i \cap e_j = \{u\}$ , we have  $A_{\{e, e_i, e_j\}} \leq A_u$ . This implies that  $A_{\{e, e_i, e_j\}}$  fixes  $N_1$  setwise. As  $(A_u^{\Sigma(u)})_{ee_ie_j} = 1$ , one has  $A_{ee_ie_j} = A_u^{[1]} \cap A_{\{e, e_i, e_j\}}$ . It follows that  $A_{\{e, e_i, e_j\}}/A_{ee_ie_j} \cong A_{\{e, e_i, e_j\}}A_u^{[1]}/A_u^{[1]} \leq A_u^{\Sigma(u)}$ . As  $(A_u^{\Sigma(u)})_\alpha \cong C_q \times C_s$  for all  $\alpha \in \{e, e_i, e_j\}$ , it follows that  $2 \nmid |A_{\{e, e_i, e_j\}}/A_{ee_ie_j}|$  since  $qs$  is odd. Thus,  $A_{\{e, e_i, e_j\}}/A_{ee_ie_j} \leq C_3$ .

If  $(s, f) \neq (3, 3)$ , then  $3 \nmid |A_u^{\Sigma(u)}|$  and so  $3 \nmid |A_{\{e, e_i, e_j\}}/A_{ee_ie_j}|$ . Then  $|A_{\{e, e_i, e_j\}}/A_{ee_ie_j}| = 1$ .

If  $(s, f) = (3, 3)$ , then we may take an element  $x$  of  $A_u^{\Sigma(u)}$  of order 3 such that  $e^x \neq e$ . By the arbitrariness of  $e_i$  and  $e_j$ , we may assume that  $\{e, e^x, e^{x^2}\} = \{e, e_i, e_j\}$ . Then we have  $A_{\{e, e_i, e_j\}}/A_{ee_ie_j} \cong C_3$ .

Now  $|A : A_{\{e, e_i, e_j\}}| = \frac{1}{3}|A : A_{ee_ie_j}|$  when  $s = f = 3$ , and otherwise,  $|A : A_{\{e, e_i, e_j\}}| = |A : A_{ee_ie_j}|$ . As  $e \cap e_i = \{u\}$ , we have  $A_{ee_i} \leq A_u$ , and so  $|A_{ee_i} : A_{ee_ie_j}| = |(A_u^{\Sigma(u)})_{ee_i} : (A_u^{\Sigma(u)})_{ee_ie_j}| = s$ . Since  $\Gamma$  is arc-transitive, one has  $|V(\Gamma)| = |A : A_e|$  and  $2q = |\Gamma(e)| = |A_e : A_{ee_i}|$ . It follows that

$$|A : A_{ee_ie_j}| = |A : A_e| |A_e : A_{ee_i}| |A_{ee_i} : A_{ee_ie_j}| = 2qs|V(\Gamma)|.$$

As a result, the size of the orbit  $\{e, e_i, e_j\}^A$  of  $A$  acting on the set of triangles of  $\Gamma$  is

$$|A : A_{\{e, e_i, e_j\}}| = \frac{q(q-1)}{3}|V(\Gamma)|.$$

It follows that  $A$  is transitive on the set of triangles of  $\Gamma$ . □

**Lemma 4.6.** *Let  $\Gamma$  be a connected locally  $2\mathbf{K}_n$  graph with  $n \geq 2$ . If  $C(\Gamma)$  is a 3-arc-regular graph of valency 5, then  $\Gamma$  is 2-geodesic-transitive and 3-CSH but not 3-CH.*

**Proof.** Let  $\Sigma = C(\Gamma)$ . In view of Lemma 2.4, for convenience, we shall assume that  $\Gamma$  is the line graph of  $\Sigma$ . Let  $A = \text{Aut}(\Sigma)$ . By Proposition 2.5,  $\Gamma$  is 2-geodesic-transitive. Take  $u \in V(\Sigma)$  and  $v \in \Sigma(u)$ . Since  $\Sigma$  is a 3-arc-regular graph of valency 5, by [59, Theorem 4.1], we have  $A_u \cong (C_5 \times C_4) \times C_4$  and  $A_{\{u, v\}} \cong (C_4 \times C_4) \times C_2$ . By [57, Theorem 1.5(2)],  $\Gamma$  is 3-CSH not 3-CH. □

**Lemma 4.7.** *Let  $\Gamma$  be a connected locally  $2\mathbf{K}_n$  graph with  $n \geq 2$ . If  $C(\Gamma)$  is either a trivalent symmetric graph of type  $2^2$  or a pentavalent symmetric graph of type  $\mathcal{Q}_2^6$ , then  $\Gamma$  is 3-CSH but not 2-geodesic-transitive.*

**Proof.** Let  $\Sigma = C(\Gamma)$ . In view of Lemma 2.4, for convenience, we shall assume that  $\Gamma$  is the line graph of  $\Sigma$ .

Assume first that  $\Sigma$  is a trivalent symmetric graph of type  $2^2$ . Then  $\Sigma$  is 2-arc-regular with edge-stabilizer isomorphic to  $C_4$ . By [57, Theorem 1.5(2)],  $\Gamma$  is 3-CSH, and by Proposition 2.5,  $\Gamma$  is not 2-geodesic-transitive.

Assume now that  $\Sigma$  is a pentavalent symmetric graph of type  $\mathcal{Q}_2^6$ . Let  $A = \text{Aut}(\Sigma)$ . Take an edge  $e = \{u_e, v_e\}$  of  $\Sigma$ . Then  $\Sigma$  is 2-arc-transitive with vertex stabilizer  $A_{u_e} \cong \text{Frob}(20) \times C_2$  and edge stabilizer  $A_e \cong M_{16}$  (see [47, Table 1]), where

$$M_{16} = \langle a, b \mid a^8 = b^2 = 1, bab = a^5 \rangle.$$

Again, by Proposition 2.5,  $\Gamma$  is arc-transitive but not 2-geodesic-transitive. To complete the proof, it suffices to prove that  $\Gamma$  is both 2-geodesic-path transitive and triangle transitive.

Let  $\Gamma(e) = \{e_i, e'_i \mid 1 \leq i \leq 4\}$ , where  $e_1, e_2, e_3, e_4$  are the four edges of  $\Sigma$  incident with  $u_e$  and  $e'_1, e'_2, e'_3, e'_4$  are the other four edges of  $\Sigma$  incident with  $v_e$ . Then  $[\Gamma(e)] \cong 2\mathbf{K}_4$ , see Figure 3. Since  $A_{u_e} \cong \text{Frob}(20) \times C_2$  and  $\Sigma$  is 2-arc-transitive of valency 5, we have

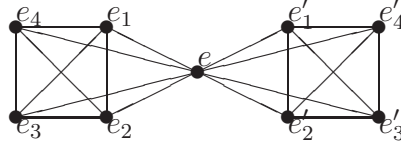


FIGURE 3. The subgraph of  $\Gamma$  induced by  $\{e\} \cup \Gamma(e)$

$A_{u_e}^{[1]} \cong C_2$  and  $A_{u_e}^{[1]} \cap A_{v_e}^{[1]} = 1$ . It follows that  $A_e$  acts faithfully on  $\Gamma(e)$ . Since  $A_e \cong M_{16}$ , without loss of generality, we may assume that  $A_e = M_{16}$ . Then  $\langle a \rangle$  acts regularly on  $\Gamma(e)$ . Let  $B_0 = \{e_i \mid 1 \leq i \leq 4\}$  and  $B_1 = \{e'_i \mid 1 \leq i \leq 4\}$ . Then  $B_0$  and  $B_1$  are the two bi-parts of  $[G(e)]^c \cong \mathbf{K}_{4,4}$ , where  $[\Gamma(e)]^c$  is the complement of the induced subgraph  $[\Gamma(e)]$  of  $\Gamma$ . The subgroup of  $\langle a \rangle$  fixing  $B_0$  setwise is  $\langle a^2 \rangle$ . So we may view  $[G(e)]^c$  as the Cayley graph  $\Delta = \text{Cay}(\langle a \rangle, \{a, a^3, a^5, a^7\})$  on  $\langle a \rangle$ . Furthermore, we may identify  $B_0$  with  $\langle a^2 \rangle$  and identify  $B_1$  with  $a\langle a^2 \rangle$ .

Then  $\langle a \rangle$  acts on  $V(\Delta) = \langle a \rangle$  by right multiplication and  $\langle b \rangle$  acts on  $V(\Delta) = \langle a \rangle$  by conjugation. Then

$$E(\Delta) = \{1, a\}^{(a)} \cup \{1, a^3\}^{(a)} \cup \{1, a^5\}^{(a)} \cup \{1, a^7\}^{(a)}.$$

Note that  $\{1, a\}^{a^7} = \{1, a^7\}$ ,  $\{1, a^3\}^{a^5} = \{1, a^5\}$  and  $\{1, a\}^b = \{1, a^5\}$ . It follows that  $E(\Delta) = \{1, a\}^{A_e}$ . This implies that  $A_e = M_{16}$  is transitive on the edges of  $[\Gamma(e)]^c$ . By Lemma 2.3,  $\Gamma$  is 2-geodesic-path transitive.

Note that the subgroup of  $A_e$  fixing  $B_0$  setwise is  $\langle a^2 \rangle \times \langle b \rangle$ , which induces a regular action on  $B_0$ . It follows that for each  $1 \leq i \leq 4$ ,  $A_{ee_i}$  fixes all  $e_1, e_2, e_3, e_4$ . So  $A_{ee_1} = A_{ee_1e_2}$ . Consider the triangle of  $\Gamma$  induced by  $\{e, e_1, e_2\}$ . If there exists  $g \in A$  such that  $g$  cyclically permutes  $e, e_1$  and  $e_2$ , then  $g$  must fix the maximal clique of  $\Gamma$  induced by  $e_1, e_2, e_3, e_4, e$ . As  $u_e$  is the intersection of these five edges, we have  $g \in A_{u_e}$ , forcing that  $A_{u_e}^{\Sigma(u_e)}$  would contain an element of order 3. This, however, is impossible because  $A_{u_e} \cong \text{Frob}(20) \times C_2$ . Thus,  $A_{\{e, e_1, e_2\}}/A_{ee_1e_2} \cong C_k$  with  $k = 1$  or  $2$ . Since  $A_{ee_1} = A_{ee_1e_2}$ , one has  $|A_{e_1e_2} : A_{e_1e_2e_3}| = 1$ . Since  $\Gamma$  is arc-transitive, one has  $|A : A_e| = |V(\Gamma)|$  and  $|A_e : A_{ee_1}| = |\Gamma(e)| = 8$ . It follows that

$$|A : A_{\{e_1, e_2, e_3\}}| = \frac{1}{k} |A : A_{e_1}| |A_{e_1} : A_{e_1e_2}| |A_{e_1e_2} : A_{e_1e_2e_3}| = \frac{8}{k} |V(\Gamma)| \geq 4|V(\Gamma)|.$$

The number of triangles of  $\Gamma$  is  $24|V(\Gamma)|/6 = 4|V(\Gamma)|$ . As  $|\{e_1, e_2, e_3\}^A| = |A : A_{\{e_1, e_2, e_3\}}| \leq 4|V(\Gamma)|$ , it follows that  $|\{e_1, e_2, e_3\}^A| = 4|V(\Gamma)|$  and so  $\Gamma$  is triangle transitive. This completes the proof.  $\square$

Now we are ready to prove Theorem 1.5. Due to Lemma 2.4, it is enough to prove the following theorem.

**Theorem 4.8.** *Let  $n \geq 2$ . Let  $\Gamma$  be a connected locally  $2\mathbf{K}_n$  graph. Let  $\Sigma = C(\Gamma)$  and  $\alpha \in V(\Sigma)$ . Then  $\Gamma$  is 3-CSH but not 3-CH if and only if one of the following holds:*

- (1)  $\Sigma$  is a tetravalent 3-arc-regular graph and  $\text{Aut}(\Sigma)_\alpha \cong A_4 \times C_3$ ;
- (2)  $\Sigma$  is a pentavalent 3-arc-regular graph and  $\text{Aut}(\Sigma)_\alpha \cong \text{Frob}(20) \times C_4$ ;
- (3)  $\Sigma$  is a 3-arc-transitive graph of valency 8 and  $\text{Aut}(\Sigma)_\alpha^{\Sigma(\alpha)} \cong C_2^3 \times (C_7 \times C_t)$  with  $t = 1$  or 3;
- (4)  $\Sigma$  is a 3-arc-transitive graph of valency 32 and  $\text{Aut}(\Sigma)_\alpha^{\Sigma(\alpha)} \cong C_2^5 \times (C_{31} \times C_5)$ ;
- (5)  $\Sigma$  is a 3-arc-transitive graph of valency  $q + 1$  and  $\text{Aut}(\Sigma)_\alpha^{\Sigma(\alpha)} \cong \text{PSL}(2, q). \langle \eta \rangle$ , where  $q$  is an odd prime power such that  $q \equiv -1 \pmod{4}$  and  $\eta$  is a field automorphism of  $\text{GF}(q)$ ;
- (6)  $\Sigma$  is a pentavalent symmetric graph of type  $\mathcal{Q}_2^6$  and  $\text{Aut}(\Sigma)_\alpha \cong \text{Frob}(20) \times C_2$ ;
- (7)  $\Sigma$  is a trivalent symmetric graph of type  $2^2$  and  $\text{Aut}(\Sigma)_\alpha \cong S_3$ .

**Proof.** From Lemmas 4.5–4.7 we immediately obtain the sufficiency. So we only need to prove the necessity.

Suppose that  $\Gamma$  is a locally  $2\mathbf{K}_n$  graph which is 3-CSH. By Lemma 4.3,  $\Sigma$  is 2-arc-transitive. Take  $u \in V(\Gamma)$ . In what follows, we always assume that  $\alpha$  and  $\beta$  are the two maximal cliques of  $\Gamma$  containing  $u$  (see Figure (4)). Let  $A = \text{Aut}(\Gamma)$ . Then every element of  $A_u$  either fixes or interchanges  $\alpha - \{u\}$  and  $\beta - \{u\}$  as  $\Gamma(u) = (\alpha - \{u\}) \cup (\beta - \{u\})$ . Clearly,  $\{\alpha, \beta\}$  is an edge of  $\Sigma$  and  $\alpha \cap \beta = \{u\}$ . Furthermore, for any  $v \in \alpha - \{u\}$ , we have  $\alpha - \{u, v\} = \Gamma(u) \cap \Gamma(v)$ . By Lemma 2.2,  $A_u$  is both vertex-transitive and edge-transitive on  $[\Gamma(u)]^c$ .

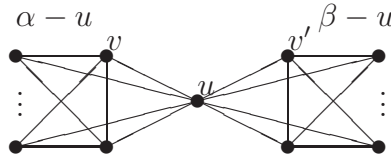


FIGURE 4. The subgraph of  $\Gamma$  induced by  $\{u\} \cup \Gamma(u)$

We shall divide the proof into the following two cases:

**Case 1.**  $A_u$  is arc-transitive on  $[\Gamma(u)]^c$ .

By Lemma 2.2 (5),  $\Gamma$  is 2-geodesic transitive, and then by Proposition 2.5,  $\Sigma$  is 3-arc-transitive. If  $A_u$  is arc-transitive on  $[\Gamma(u)]$ , then by [40, Proposition 2.1],  $\Gamma$  is 3-CH, a contradiction.

Thus,  $A_u$  is not arc-transitive on  $[\Gamma(u)]$ . By Lemma 2.2 (3),  $A_{vu}$  has  $s$  orbits of equal size on  $\alpha - \{u, v\}$  with  $s = 1, 2, 3$  or 6, where  $v \in \alpha - \{u\}$ . Note that  $\alpha - \{u, v\} = \Gamma(u) \cap \Gamma(v)$ . If  $s = 1$ , then  $A_{vu}$  is transitive on  $\Gamma(u) \cap \Gamma(v)$ , and then  $A_u$  would be arc-transitive on  $[\Gamma(u)]$  as  $A_u$  is transitive on  $\Gamma(u)$ , a contradiction. Thus,  $s \neq 1$ . Since  $A_u$  is transitive on  $\Gamma(u) = (\alpha - \{u\}) \cup (\beta - \{u\})$ , it follows that  $A_{\alpha - \{u\}}$  is transitive on  $\alpha - \{u\}$ . This implies that the permutation group  $(A_\alpha^\alpha)_u$  is either regular or  $\frac{3}{2}$ -transitive on  $\alpha - \{u\}$ .

Assume first that  $(A_\alpha^\alpha)_u$  is regular on  $\alpha - \{u\}$ . Then  $A_{uv}$  fixes every vertex in  $\alpha - \{u\}$ . Since  $A_{vu}$  has  $s$  orbits of equal size on  $\alpha - \{u, v\}$  with  $s = 2, 3$  or  $6$ , it follows that  $|\alpha - \{u, v\}| = 2, 3$  or  $6$ , and hence  $|\alpha - \{u\}| = 3, 4$  or  $7$ . If  $|\alpha - \{u\}| = 3$ , then we have  $A_\alpha^\alpha \cong A_4$  and  $|\alpha| = 4$ . So  $\Sigma$  has valency 4. Since  $\Sigma$  is 3-arc-transitive, by [44, Theorem 4], we have that  $A_\alpha \cong A_4 \times C_3$  and so  $\Sigma$  is a tetravalent 3-arc-regular graph. This implies part (1). If  $|\alpha - \{u\}| = 4$ , then we have  $|\alpha| = 5$  and  $A_\alpha^\alpha \cong \text{Frob}(20)$ . So  $\Sigma$  has valency 5. Since  $\Sigma$  is 3-arc-transitive, by [59, Theorem 4.1], we have  $A_\alpha \cong \text{Frob}(20) \times C_4$ , and hence  $\Sigma$  is 3-arc-regular. Then part (2) happens. If  $|\alpha - \{u\}| = 7$ , then we have  $|\alpha| = 8$  and  $A_\alpha^\alpha \cong C_2^3 \rtimes C_7$ . So  $\Sigma$  has valency 8 and part (3) happens.

Now assume that  $(A_\alpha^\alpha)_u$  is  $\frac{3}{2}$ -transitive on  $\alpha - \{u\}$ . Then  $(A_\alpha^\alpha)_u$  has rank 3, 4 or 7 on  $\alpha - \{u\}$  as  $A_{uv}$  has  $s$  orbits of equal size on  $\alpha - \{u, v\}$  with  $s = 2, 3$  or  $6$ . Clearly,  $A_\alpha^\alpha$  is a 2-transitive permutation group on  $\alpha$ . By Lemma 4.2,  $(A_\alpha^\alpha)_u$  is primitive on  $\alpha - \{u\}$  and Lemma 4.2 (1) or (2) happens. If Lemma 4.2 (2) happens, then either part (3) or part (4) holds. If Lemma 4.2 (1) happens, by Lemmas 4.4–4.5, we obtain part (5).

**Case 2.**  $A_u$  is not arc-transitive on  $[\Gamma(u)]^c$ .

In this case,  $\Gamma$  is not 2-geodesic transitive, and then by Proposition 2.5,  $\Sigma$  is 2-arc-transitive but not 3-arc-transitive. Recall that  $A_u$  is both vertex-transitive and edge-transitive on  $[\Gamma(u)]^c$ . It follows that  $A_u$  is half-arc-transitive on  $[\Gamma(u)]^c$ . Since  $[\Gamma(u)] \cong 2\mathbf{K}_n$ , one has  $[\Gamma(u)]^c \cong \mathbf{K}_{n,n}$ , and so  $n$  is even. Let  $B = \alpha - \{u\}$  and  $C = \beta - \{u\}$ . Then  $|B| = |C| = n$ . Since  $A_u$  is transitive on  $\Gamma(u)$ ,  $(A_u)_B$  is transitive on  $B$ . Since  $n - 1 = |B| - 1$  is odd, by Lemma 2.2 (3),  $A_{uv}$  has  $s$  orbits of equal size on  $B - \{v\} = \Gamma(u) \cap \Gamma(v)$  with  $s = 1$  or  $3$ , where  $v \in B$ . In particular, the permutation group  $(A_\alpha^\alpha)_u$  is either regular or  $\frac{3}{2}$ -transitive on  $\alpha - \{u\}$ .

Suppose first that  $A_{uv}$  has 3 orbits of equal size on  $B - \{v\}$ . Clearly,  $A_\alpha^\alpha$  is a 2-transitive permutation group on  $\alpha$ . By Lemma 4.2,  $(A_\alpha^\alpha)_u$  can not be a  $\frac{3}{2}$ -transitive permutation group on  $B = \alpha - \{u\}$  of rank 4. So  $(A_\alpha^\alpha)_u$  is regular on  $B$ , and hence all orbits of  $A_{uv}$  on  $B - \{v\}$  have size 1. Then  $|(A_\alpha^\alpha)_u| = |B| = 4$  and  $|\alpha| = 5$ . It then follows that  $A_\alpha^\alpha \cong \text{Frob}(20)$  and  $\Sigma$  is a pentavalent 2-arc-transitive graph. Since  $A_u$  is half-arc-transitive on  $[\Gamma(u)]^c \cong \mathbf{K}_{4,4}$ , one has  $A_u^{\Gamma(u)}$  is a 2-group. Note that  $A_u$  is just the stabilizer of the edge  $\{\alpha, \beta\}$  of  $\Sigma$  in  $A$ . By [47, Theorem 1.2], we see that  $A_\alpha \cong \text{Frob}(20) \times C_2$  and  $A_{\{\alpha, \beta\}} \cong M_{16}$ . We obtain part (6).

Now suppose that  $A_{uv}$  has only one orbit on  $B - \{v\}$ . Then  $(A_\alpha^\alpha)_u$  is 2-transitive on  $B = \alpha - \{u\}$ . Since  $A_u$  is transitive on  $\Gamma(u)$ , it follows that  $(A_u^{\Gamma(u)})_B$  is 2-transitive on  $B$  and  $(A_u^{\Gamma(u)})_C$  is also 2-transitive on  $C$ . If  $(A_u^{\Gamma(u)})_B$  is not faithful on  $B$ , then the kernel of  $(A_u^{\Gamma(u)})_B$  on  $B$  would be transitive on  $C$ . However, this is impossible since  $A_u$  is half-arc-transitive on  $[\Gamma(u)]^c$ . Thus,  $(A_u^{\Gamma(u)})_B$  is faithful on  $B$ . Similarly,  $(A_u^{\Gamma(u)})_C$  is faithful on  $C$ . Note that  $(A_u^{\Gamma(u)})_B = (A_u^{\Gamma(u)})_C$ . By [8],  $(A_u^{\Gamma(u)})_B$  is either affine or almost simple.

We now claim that the actions of  $(A_u^{\Gamma(u)})_B$  on  $B$  and  $C$  are equivalent. Take  $v \in B$ . Since  $A_u$  is half-arc-transitive on  $[\Gamma(u)]^c \cong \mathbf{K}_{n,n}$ ,  $A_{uv}$  has two orbits on  $C$  of equal size. In particular,  $|B| = |C| = n$  is even.

Assume first that  $(A_u^{\Gamma(u)})_B$  is affine. Then  $\text{soc}((A_u^{\Gamma(u)})_B)$  is elementary abelian of order  $|B|$ . Since  $|B|$  is even, we have  $\text{soc}((A_u^{\Gamma(u)})_B) \cong C_2^r$  for some integer  $r > 0$ . If the actions of  $(A_u^{\Gamma(u)})_B$  on  $B$  and  $C$  are not equivalent, then by inspecting the affine 2-transitive permutation groups (see [7, Table 7.3]), we conclude that  $r > 2$  and one of the following may happens:

- (i)  $A_{uv}^{\Gamma(u)} \leq \Gamma L(1, 2^r) \cong C_{2^{r-1}} \rtimes C_r$ ;
- (ii)  $\mathrm{SL}(d, q) \trianglelefteq A_{uv}^{\Gamma(u)} \leq \Gamma L(d, q) (2^r = q^d, d \geq 2)$ ;
- (iii)  $\mathrm{Sp}(d, q) \trianglelefteq A_{uv}^{\Gamma(u)} \leq (\mathbb{Z}_{q-1} \circ \mathrm{Sp}(2d, q)) \cdot (\mathbb{Z}_{(2, q-1)} \times \mathbb{Z}_{r/2d}) (q^{2d} = 2^r, d \geq 2)$ ;
- (iv)  $\mathrm{G}_2(q) \trianglelefteq A_{uv}^{\Gamma(u)} \leq (\mathbb{Z}_{q-1} \circ \mathrm{G}_2(q)) \cdot \mathbb{Z}_{r/6} (q^6 = 2^r)$ ;
- (v)  $A_{uv}^{\Gamma(u)} = \mathrm{A}_6$  and  $r = 4$ ;
- (vi)  $A_{uv}^{\Gamma(u)} = \mathrm{PSU}(3, 3)$  and  $r = 6$ .

In case (i), as  $A_{uv}$  has two orbits on  $C$  of length  $2^{r-1}$ , it follows that  $2^{r-1} \mid |A_{uv}^{\Gamma(u)}|$ , and so  $2^{r-1} \mid r(2^r - 1)$ . It implies that  $2^{r-1} \mid r$ . This, however, is impossible because  $r > 2$ . In cases (ii), (iii) or (iv), the center  $Z$  of  $A_{uv}^{\Gamma(u)}$  has order dividing  $q-1$ . Take  $w \in C$ . As  $A_{uv}$  has two orbits on  $C$  of length  $2^{r-1}$ , it follows that  $|w^Z|$  is divisor of  $\gcd(q-1, 2^{r-1}) = 1$ , and hence  $|w^Z| = 1$ . This implies that  $Z$  fixes every vertex in  $C$ . Since  $(A_u^{\Gamma(u)})_B$  is faithful on  $C$ , one has  $Z = 1$ . Since  $r > 2$ ,  $A_{uv}^{\Gamma(u)}$  is almost simple. Now we conclude that  $\mathrm{soc}(A_{uv}^{\Gamma(u)})$  is one of the following groups:

$$\mathrm{PSL}(d, q) (2^r = q^d), \mathrm{soc}(\mathrm{PSp}(d, q)) (q^{2d} = 2^r), \mathrm{G}_2(q) (q^6 = 2^r), \mathrm{A}_6, \mathrm{PSU}(3, 3). \quad (2)$$

On the other hand, since  $(A_u^{\Gamma(u)})_B$  is faithful on  $C$  and  $A_{uv}$  has two orbits on  $C$  of length  $2^{r-1}$ , it follows that  $\mathrm{soc}(A_{uv}^{\Gamma(u)})$  has an orbit on  $C$  of length dividing  $2^{r-1}$ . This implies that  $\mathrm{soc}(A_{uv}^{\Gamma(u)})$  has a maximal subgroup of index  $2^t$  with  $1 < t \leq r-1$ . By [27] (or [33, Theorem 2.2]), we have  $\mathrm{soc}(A_{uv}^{\Gamma(u)}) \cong \mathrm{A}_{2^t}$  or  $\mathrm{PSL}(2, \ell)$  with  $\ell$  a prime and  $\ell+1 = 2^t$ . Since  $\mathrm{soc}(A_{uv}^{\Gamma(u)})$  is also one of the groups in (2), by the Classification Theorem for Finite Simple Groups (see for example [52, p.3]) we conclude that either  $r = 4$  and  $\mathrm{SL}(4, 2) = \mathrm{soc}(A_{uv}^{\Gamma(u)}) \cong \mathrm{A}_8$ , or  $r = 3$  and  $\mathrm{SL}(3, 2) = \mathrm{soc}(A_{uv}^{\Gamma(u)}) \cong \mathrm{PSL}(2, 7)$ . For the former, we have  $r = 4$  and  $A_u^{\Gamma(u)} = \mathrm{AGL}(4, 2)$ . By **Magma** [4], all subgroups of  $\mathrm{AGL}(4, 2)$  isomorphic to  $\mathrm{A}_8$  are conjugate, and so the actions of  $(A_u^{\Gamma(u)})_B$  on  $B$  and  $C$  are equivalent, a contradiction. For the latter, we have  $r = 3$  and  $\mathrm{soc}(A_{uv}^{\Gamma(u)}) \cong \mathrm{PSL}(2, 7)$ , but  $\mathrm{PSL}(2, 7)$  does not have a subgroup of index no more than  $2^2$ , a contradiction.

Assume now that  $(A_u^{\Gamma(u)})_B$  is almost simple. If the actions of  $(A_u^{\Gamma(u)})_B$  on  $B$  and  $C$  are not equivalent, then by checking [8, Theorem 5.3], one of the following holds:

- (a)  $\mathrm{soc}((A_u^{\Gamma(u)})_B) = \mathrm{A}_6$  and  $|B| = 6$ ;
- (b)  $\mathrm{soc}((A_u^{\Gamma(u)})_B) = \mathrm{PSL}(d, q) (d > 2)$  and  $|B| = \frac{q^d-1}{q-1}$ ;
- (c)  $(A_u^{\Gamma(u)})_B = \mathrm{M}_{12}$  (Mathieu) and  $|B| = 12$ ;
- (d)  $(A_u^{\Gamma(u)})_B = \mathrm{HS}$  (Higman-Sims) and  $|B| = 176$ .

In cases (a) and (c), by **Magma** [4], we can obtain that  $(A_u^{\Gamma(u)})_v$  is transitive on  $C$ , a contradiction. In case (d), by **Magma** [4],  $(A_u^{\Gamma(u)})_v$  has exactly two orbits on  $C$  with size 50 and 126, respectively. This is contrary to the fact that the two orbits of  $(A_u^{\Gamma(u)})_v$  on  $C$  has equal size.

In case (b), we have  $\mathrm{soc}((A_u^{\Gamma(u)})_B) = \mathrm{PSL}(d, q) (d > 2)$  and  $|B| = \frac{q^d-1}{q-1}$ . Here we may assume that  $B$  and  $C$  are the set of points and the set of hyperplanes of the projective space  $PG(d-1, q)$ , respectively. Then the hyperplanes containing  $v$  form an orbit  $C_1$  of  $\mathrm{soc}((A_u^{\Gamma(u)})_B)_v$  on  $C$ , while the hyperplanes not containing  $v$  form another orbit  $C_2$  of  $\mathrm{soc}((A_u^{\Gamma(u)})_B)_v$  on  $C$ . Then  $|C_1| = \frac{q^{d-1}-1}{q-1}$ , and  $|C_2| = \frac{q^d-q^{d-1}}{q-1}$ . Since the two orbits of



$\text{soc}((A_u^{\Gamma(u)})_B)_v$  on  $C$  have equal size, we have  $|C_1| = |C_2|$ , and hence  $q^{d-1} - 1 = q^d - q^{d-1}$ , namely,  $q^d + 1 = 2q^{d-1}$ . This, however, is impossible.

By now, we have shown that the actions of  $(A_u^{\Gamma(u)})_B$  on  $B$  and  $C$  are equivalent. So  $(A_u^{\Gamma(u)})_v$  also fixes at least one vertex in  $C$ . Again, since  $(A_u^{\Gamma(u)})_v$  has exactly two orbits of equal size on  $C$ , we have  $|C| = 2$ . Then  $\Gamma$  has valency 4 and  $\Sigma$  has valency 3. Since  $\Sigma$  is 2- but not 3-arc-transitive, one has  $|A_u| = 4$ . Again, since  $A_u$  acts half-arc-transitively on  $[\Gamma(u)]^c \cong \mathbf{K}_{2,2}$ , we must have  $A_u \cong C_4$ , and so  $\Sigma$  is a trivalent symmetric graph of type  $2^2$ . Part (7) happens.  $\square$

## 5. PROOF OF THEOREM 1.6

The goal of this section is to characterize solvable 3-CSH but not 3-CH graphs and prove Theorem 1.6. We first give several lemmas about arc-transitive graphs. Let  $\Gamma$  be a  $(G, s)$ -arc-transitive graph with  $G \leq \text{Aut}(\Gamma)$  and  $s \geq 2$ , and let  $N$  be a normal subgroup of  $G$ . The *quotient graph*  $\Gamma_N$  of  $\Gamma$  relative to  $N$  is defined as the graph with vertices the orbits of  $N$  on  $V(\Gamma)$  and with two different orbits adjacent if there exists an edge in  $\Gamma$  between the vertices lying in those two orbits. If  $\Gamma_N$  and  $\Gamma$  have the same valency, then we say that  $\Gamma$  is a *normal cover* of  $\Gamma_N$ . In view of [45, Theorem 4.1] or [34, Lemma 2.5], we have the following.

**Lemma 5.1.** *Let  $\Gamma$  be a connected  $(G, 2)$ -arc-transitive graph with  $G \leq \text{Aut}(\Gamma)$ . Suppose that  $N \trianglelefteq G$  has at least three orbits on  $V(\Gamma)$ . Then*

- (1)  $N$  acts semiregularly on  $V(\Gamma)$  and  $\Gamma$  is a normal cover of  $\Gamma_N$ .
- (2)  $N$  is the kernel of  $G$  acting on  $V(\Gamma_N)$ ,  $G/N \leq \text{Aut}(\Gamma_N)$  and  $\Gamma_N$  is  $(G/N, 2)$ -arc-transitive.

**Lemma 5.2.** *Let  $\Gamma$  be a connected graph of valency  $k > 2$ . Suppose that  $G \leq \text{Aut}(\Gamma)$  is abelian and acts regularly on the edge set of  $\Gamma$ . Then  $\Gamma \cong \mathbf{K}_{k,k}$ .*

**Proof.** Take an edge  $\{u, v\} \in E(\Gamma)$ . Since  $G$  is regular on the edge set of  $\Gamma$ , one has  $E(\Gamma) = \{\{u^g, v^g\} \mid g \in G\}$  and  $|G| = |E(\Gamma)|$ . Then  $V(\Gamma) = u^G \cup v^G$ . Since  $k > 2$ , one has  $|V(\Gamma)| < |E(\Gamma)|$ , and so  $u^G, v^G$  are two distinct orbits of  $G$  on  $V(\Gamma)$ . Furthermore,  $\Gamma$  is a bipartite graph with two bi-parts  $u^G$  and  $v^G$ . By the edge-transitivity of  $G$  on  $\Gamma$ , we have  $G_u$  is transitive on  $\Gamma(u)$ . Clearly,  $v \in \Gamma(u)$ . For any  $w \in \Gamma(v) - \{u\}$ , since  $G$  is abelian, we have  $G_u = G_w$ , and so  $w$  is adjacent to all vertices in  $\Gamma(u)$  by the transitivity of  $G_u$  on  $\Gamma(u)$ . By the arbitrariness of  $w$ , we see that the subgraph induced by  $\Gamma(u) \cup \Gamma(v)$  is isomorphic to  $\mathbf{K}_{k,k}$ . Since  $\Gamma$  is connected, one has  $\Gamma \cong \mathbf{K}_{k,k}$ .  $\square$

**Lemma 5.3.** *Let  $\Gamma$  be a connected graph of valency  $k$ . Suppose that  $G \leq \text{Aut}(\Gamma)$  is regular on  $E(\Gamma)$  and intransitive on  $V(\Gamma)$ . Let  $\{u, v\}$  be an edge of  $\Gamma$ . Then the following hold.*

- (1)  $G = \langle G_u, G_v \rangle$ .
- (2)  $G_u \cap G_v = 1$ .
- (3)  $|G_u| = |G_v| = k$ , and  $G_u$  and  $G_v$  act regularly on  $\Gamma(u)$  and  $\Gamma(v)$ , respectively.
- (4) The line graph of  $\Gamma$  is isomorphic to the Cayley graph  $\text{Cay}(G, S)$  with  $S = (G_u \cup G_v) - \{1\}$ .

**Proof.** Since  $G$  is regular on  $E(\Gamma)$  but intransitive on  $V(\Gamma)$ , the connectedness of  $\Gamma$  implies that  $G = \langle G_u, G_v \rangle$ . This proves part (1).

Again since  $G$  is regular on  $E(\Gamma)$ , one has  $G_{\{u,v\}} = 1$  and  $|G| = |E(\Gamma)|$ . Moreover,  $V(G) = u^G \cup v^G$ . Since  $G$  is intransitive on  $V(\Gamma)$ , one has  $u^G \cap v^G = \emptyset$ . It follows that  $G_u \cap G_v = G_{\{u,v\}} = 1$ , proving part (2).

Since  $u^G \cap v^G = \emptyset$ , neither  $u^G$  nor  $v^G$  contains an edge of  $\Gamma$ . It follows that  $\Gamma(u) \subseteq v^G$  and  $\Gamma(v) \subseteq u^G$ . As  $G$  is transitive on  $E(G)$ ,  $G_u$  and  $G_v$  are transitive on  $\Gamma(u)$  and  $\Gamma(v)$ , respectively. Moreover,  $|E(G)| = |u^G||\Gamma(u)| = |v^G||\Gamma(v)|$ . Note that  $|G| = |u^G||G_u| = |v^G||G_v|$ . Since  $|G| = |E(\Gamma)|$ , it follows that  $|G_u| = |G_v| = |\Gamma(u)| = |\Gamma(v)| = k$ . Then  $G_u$  and  $G_v$  act regularly on  $\Gamma(u)$  and  $\Gamma(v)$ , respectively. We obtain part (3).

As  $G$  is regular on  $E(\Gamma)$ , the line graph, say  $\Sigma$ , of  $\Gamma$  is a Cayley graph on  $G$ . Note that  $V(\Sigma) = E(\Gamma)$  and the set of vertices of  $\Sigma$  adjacent to  $\{u, v\}$  is

$$F = \{\{u, x\}, \{v, y\} \mid v \neq x \in \Gamma(u), u \neq y \in \Gamma(v)\}.$$

Clearly,  $|F| = |\Gamma(u) - \{v\}| + |\Gamma(v) - \{u\}| = 2k - 2$ . Let  $S = \{g \in G \mid \{u, v\}^g \in F\}$ . Then  $\Sigma \cong \text{Cay}(G, S)$  and  $|S| = |F|$ . For any  $g \in G_u - \{1\}$ , we have  $\{u, v\}^g = \{u, v^g\}$  with  $v \neq v^g \in \Gamma(u)$ , and hence  $\{u, v\}^g \in F$ . It follows that  $g \in S$  and so  $G_u - \{1\} \subseteq S$ . Similarly, we have  $G_v - \{1\} \subseteq S$ . Since  $|(G_u \cup G_v) - 1| = |F|$ , one has  $S = (G_u \cup G_v) - 1$ . This proves part (4).  $\square$

**Lemma 5.4.** *Let  $\Gamma$  be a connected  $(G, s)$ -arc-transitive graph, where  $s \geq 2$  and  $G \leq \text{Aut}(\Gamma)$  is solvable. Then either*

- (1)  $G$  has a normal subgroup which is semiregular on  $V(\Gamma)$  with at most 2 orbits and for each vertex  $v$  of  $\Gamma$ ,  $G_v$  acts faithfully on  $\Gamma(v)$ ; or
- (2)  $\Gamma$  is a normal cover of  $\mathbf{K}_{p^n, p^n}$  with  $p$  a prime and  $n$  a positive integer, and  $G$  has a normal subgroup, say  $M$  such that the following hold:
  - (i)  $M$  is regular on  $E(\Gamma)$  and intransitive on  $V(\Gamma)$ ,  $M = \langle M_u, M_w \rangle$ ,  $M_u \cap M_w = 1$  and  $M_u \cong M_w \cong C_p^n$ , where  $\{u, w\} \in E(\Gamma)$ ; and
  - (ii) the line graph of  $\Gamma$  is isomorphic to the Cayley graph  $\text{Cay}(M, S)$  with  $S = (M_u \cup M_w) - \{1\}$ .

**Proof.** Let  $N \trianglelefteq G$  be maximal subject to the condition that  $N$  has at least three orbits on  $V(\Gamma)$ . Let  $\Gamma_N$  be the quotient graph of  $\Gamma$  relative to  $N$ . Since  $G$  is 2-arc-transitive on  $\Gamma$ , by Lemma 5.1,  $\Gamma$  is a normal cover of  $\Gamma_N$ , and  $N$  is semiregular on  $V(\Gamma)$  and  $N$  is the kernel of  $G$  acting on  $V(\Gamma_N)$ . Furthermore,  $G/N$  is a group of automorphisms of  $\Gamma_N$  acting transitively on the 2-arcs of  $\Gamma_N$ . Let  $M/N$  be a minimal normal subgroup of  $G/N$ . The solvability of  $G$  implies that  $M/N \cong C_p^r$  with  $p$  a prime and  $r$  a positive integer. By the maximality of  $N$ , either  $M/N$  is transitive on  $V(\Gamma_N)$  or  $M/N$  has two orbits on  $V(\Gamma_N)$ . If the former happens, then  $M/N$  is regular on  $V(\Gamma_N)$ , and then by the semiregularity of  $N$  on  $V(\Gamma)$ ,  $M$  is a normal subgroup of  $G$  acting regularly on  $V(\Gamma)$ . So  $\Gamma$  is a Cayley graph on  $M$ . Take an arbitrary  $u \in V(\Gamma)$ , and let  $S = \{g \in M \mid \{u, u^g\} \in E(\Gamma)\}$ . Then  $\Gamma \cong \text{Cay}(M, S)$ . Without loss of generality, we may let  $\Gamma = \text{Cay}(M, S)$ . Then  $\Gamma(1) = S$ . Since  $M \trianglelefteq G$ , one has  $G_1 \leq \text{Aut}(M, S) = \{\alpha \in \text{Aut}(M) \mid S^\alpha = S\}$  (see [24]). Since  $\Gamma$  is connected, one has  $M = \langle S \rangle$ . This implies that the vertex stabilizer  $G_1$  acts faithfully on  $\Gamma(1) = S$ . Since  $\Gamma$  is vertex-transitive, for each vertex  $v$  of  $\Gamma$ ,  $G_v$  acts faithfully on  $\Gamma(v)$ , as claimed in part (1).

Now let  $M/N$  have two orbits on  $V(\Gamma_N)$ . Then  $M$  has two orbits, say  $U$  and  $W$ , on  $V(\Gamma)$ . So  $U, W$  are blocks of imprimitivity of  $G$  on  $V(\Gamma)$ . Since  $G$  is 2-arc-transitive on  $\Gamma$ ,  $U$  and  $W$  contain no edges of  $\Gamma$ , and so  $\Gamma$  is bipartite with  $U$  and  $W$  as its two bi-parts. If  $M/N$  is semiregular on  $V(\Gamma_N)$ , then since  $N$  is semiregular on  $V(\Gamma)$ ,  $M$  is also semiregular on  $V(\Gamma)$ . By [32, Lemma 2.4], we see that for each vertex  $v$  of  $\Gamma$ ,  $G_v$  acts faithfully on  $\Gamma(v)$ , as claimed in part (1).

Now suppose that  $M/N$  is not semiregular on  $V(\Gamma_N)$ . Let  $\{u, w\}$  be an edge of  $\Gamma$  such that  $u \in U$  and  $w \in W$ . Let  $B = u^N$  and  $C = w^N$ . Then  $M_B = M_u N$  and  $M_C = M_w N$ . Recall that  $N$  is semiregular on  $V(\Gamma)$ . It follows that  $M_u \cong M_B/N$  and  $M_w \cong M_C/N$ . As  $M/N$  is not semiregular on  $V(\Gamma_N)$ ,  $M_B/N$  is a non-trivial normal subgroup of  $G_B/N$ . Since  $G/N$  is 2-arc-transitive on  $\Gamma_N$ ,  $M_B/N$  is transitive on the neighbors of  $B$  in  $\Gamma_N$ . It follows that  $M/N$  is transitive on the edges of  $\Gamma_N$ . Then  $M/N$  is regular on the edges of  $\Gamma_N$  since  $M/N \cong C_p^r$ . By Lemma 5.2, we have  $\Gamma_N \cong \mathbf{K}_{p^n, p^n}$ , and  $M/N = M_B/N \times M_C/N \cong C_p^n \times C_p^n$  with  $r = 2n$ . Furthermore,  $M_u \cong M_B/N \cong C_p^n$  and  $M_w \cong M_C/N \cong C_p^n$ . Since  $\Gamma$  is a normal cover of  $\Gamma_N$  and since  $M/N$  is regular on  $E(\Gamma_N)$ , it implies that  $M$  is regular on  $E(\Gamma)$ . Since  $M/N$  has two orbits on  $V(\Gamma_N)$ , it follows that  $M$  is intransitive on  $V(\Gamma)$ . Applying Lemma 5.3 to  $\Gamma$  and  $M$  we can obtain (i) and (ii) of part (2).  $\square$

Now we are ready to prove Theorem 1.6. We first prove the necessity of Theorem 1.6 in the following lemma.

**Lemma 5.5.** *Let  $n \geq 2$  and let  $\Gamma$  be a solvable locally  $2\mathbf{K}_n$  graph. If  $\Gamma$  is 3-CSH, then  $\Gamma$  is isomorphic to an arc-transitive normal Cayley graph  $\text{Cay}(H, S)$  on a group  $H$  such that the following hold:*

- (a)  $H$  has two subgroups  $A, B$  such that  $H = \langle A, B \rangle$ ,  $A \cong B \cong C_p^f$ ,  $A \cap B = 1$  and  $S = (A \cup B) - \{1\}$ ; and
- (b) if  $\Gamma$  is not 3-CH, then one of (1) – (6) of Theorem 1.6 (b) holds.

**Proof.** Suppose that  $\Gamma$  is 3-CSH. If  $\Gamma$  is 3-CH, then by Theorem 1.2,  $C(\Gamma)$  is 3-arc-transitive and locally 3-transitive, and if  $\Gamma$  is not 3-CH, then by Theorem 4.8,  $C(\Gamma)$  is a 2-arc-transitive graph satisfying the conditions in one of parts (1)–(4), (6) and (7) of Theorem 4.8. Let  $\Sigma = C(\Gamma)$ . By Lemma 2.4,  $\Gamma$  is isomorphic to the line graph of  $\Sigma$ . For convenience, in the following, we shall identify  $\Gamma$  with the line graph of  $\Sigma$ . Due to Lemma 2.4, we may also view  $\text{Aut}(\Gamma)$  as the full automorphism group of  $\Sigma$ . Take a vertex  $u$  of  $\Sigma$  and take  $v \in \Sigma(u)$ .

We begin by proving that  $\text{Aut}(\Gamma)_u$  is not faithful on  $\Sigma(u)$  if  $\Sigma$  is not the case in part (7) of Theorem 4.8. First, if  $\Sigma$  is 3-arc-transitive and locally 3-transitive, then since  $\text{Aut}(\Gamma)$  is solvable, it follows that  $\text{Aut}(\Gamma)_u^{\Sigma(u)}$  is a solvable 3-transitive permutation group on  $\Sigma(u)$ . By checking the list of finite affine 2-transitive permutation groups obtained by Hering (see for example [7, Table 7.3]), we see that either  $\Sigma$  has valency 3 and  $\text{Aut}(\Gamma)_u^{\Sigma(u)} \cong S_3$ , or  $\Sigma$  has valency 4 and  $\text{Aut}(\Gamma)_u^{\Sigma(u)} \cong S_4$ . Since  $\Sigma$  is 3-arc-transitive, by Theorem 1.3,  $\text{Aut}(\Gamma)_u$  is not faithful on  $\Sigma(u)$ , as claimed.

If  $\Sigma$  is a graph in part (2) of Theorem 4.8, then  $\Sigma$  is a pentavalent 3-arc-regular graph, and then by [47, Table 2], we see that  $\text{Aut}(\Gamma)_u \cong \text{Frob}(20) \times C_4$ ,  $\text{Aut}(\Gamma)_{\{u, v\}} \cong C_4 \wr C_2$  and  $\text{Aut}(\Gamma)_u^{[1]} \cong C_4$ . Similarly, if  $\Sigma$  is a graph in part (1) of Theorem 4.8, then  $\Sigma$  is a tetravalent 3-arc-regular graph, and by [44, Theorem 4], we see that  $\text{Aut}(\Gamma)_u \cong A_4 \times C_3$ ,

$\text{Aut}(\Gamma)_{\{u,v\}} \cong C_4 \wr C_2$  and  $\text{Aut}(\Gamma)_u^{[1]} \cong C_3$ . Moreover, if  $\Sigma$  is a graph in part (6) of Theorem 4.8, then it is a pentavalent 3-arc-transitive graph of type  $\mathcal{Q}_2^6$ . By [47, Table 1], we obtain that  $\text{Aut}(\Gamma)_u \cong \text{Frob}(20) \times C_2$ ,  $\text{Aut}(\Gamma)_{\{u,v\}} \cong M_{16}$  and  $\text{Aut}(\Gamma)_u^{[1]} \cong C_2$ . For the graph  $\Sigma$  in part (3) of Theorem 4.8, since  $\Sigma$  is 3-arc-transitive, one has  $8 \cdot 7^2 \mid |\text{Aut}(\Gamma)_u|$ , and in view of [36, Theorem 2.1], we see that

$$\begin{aligned} (C_2^3 \rtimes C_7) \times C_7 &\leq \text{Aut}(\Gamma)_u \leq (C_2^3 \rtimes (C_7 \times C_3)) \times (C_7 \times C_3), \\ (C_7 \times C_7) \rtimes C_2 &\leq \text{Aut}(\Gamma)_{\{u,v\}} \leq ((C_7 \times C_3) \times (C_7 \times C_3)) \rtimes C_2. \end{aligned}$$

Similarly, for the graph  $\Sigma$  in part (4) of Theorem 4.8, we have  $2^5 \cdot 31^2 \mid |\text{Aut}(\Gamma)_u|$ , and by [36, Theorem 2.1], we have

$$\begin{aligned} (C_2^5 \rtimes (C_{31} \times C_5)) \times C_{31} &\leq \text{Aut}(\Gamma)_u \leq (C_2^5 \rtimes (C_{31} \times C_5)) \times (C_{31} \times C_5), \\ (C_{31} \times C_{31}) \rtimes C_{10} &\leq \text{Aut}(\Gamma)_{\{u,v\}} \leq ((C_{31} \times C_5) \times (C_{31} \times C_5)) \rtimes C_2. \end{aligned}$$

So far we have shown that if  $\Sigma$  is not the case in part (7) of Theorem 4.8, then  $\Sigma$  is a 2-arc-transitive graph and  $\text{Aut}(\Gamma)_u$  is not faithful on  $\Sigma(u)$ . Since  $\text{Aut}(\Gamma)$  is solvable, applying Lemma 5.4 to  $\text{Aut}(\Gamma)$  we see that Lemma 5.4 (2) happens. It follows that  $\Gamma$  is a normal Cayley graph  $\text{Cay}(H, S)$  on a group  $H$ , where  $S = (A \cup B) - \{1\}$  with  $A = H_u$  and  $B = H_v$ . Furthermore,  $H = \langle A, B \rangle$ ,  $A \cap B = 1$ , and  $A \cong B \cong C_p^f$ , where  $p$  is a prime and  $p^f$  is just the valency of  $\Sigma$ . So  $H$  and  $S$  satisfies the condition in part (a).

If part (7) of Theorem 4.8 happens, then  $\Sigma$  is a trivalent symmetric graph of type  $2^2$ . Then  $\text{Aut}(\Gamma)_u \cong S_3$ , and for any  $v \in \Sigma(u)$ , the edge stabilizer  $\text{Aut}(\Gamma)_{\{u,v\}}$  is isomorphic to  $C_4$ . By [20, Corollary 1.2],  $\text{Aut}(\Gamma)$  has a normal subgroup, say  $N$ , such that the quotient graph  $\Sigma_N$  of  $\Sigma$  relative to  $N$  is isomorphic to  $\mathbf{K}_{3,3}$ . So  $\text{Aut}(\Gamma)/N$  is 2-arc-transitive on  $\Sigma_N \cong \mathbf{K}_{3,3}$ . Let  $P/N$  be the Sylow 3-subgroup of  $\text{Aut}(\Gamma)/N$ . Then  $P/N \cong C_3^2$ , which is normal in  $\text{Aut}(\Gamma)/N$  and is regular on the edges of  $\Sigma_N \cong \mathbf{K}_{3,3}$ . Then  $P$  is regular on  $E(\Sigma)$  but intransitive on  $V(\Sigma)$ . Furthermore,  $P_u \cong P_v \cong C_3$  and  $P = \langle P_u, P_v \rangle$ . Now applying Lemma 5.3 to  $P$  and observing that  $P \trianglelefteq \text{Aut}(\Gamma)$ , we see that  $\Gamma$  is a normal Cayley graph  $\text{Cay}(H, S)$  on  $H$ , where  $H = P$ ,  $A = P_u$ ,  $B = P_v$ ,  $H = \langle A, B \rangle$  and  $S = (A \cup B) - \{1\}$ . So  $H$  and  $S$  satisfy the condition in part (a).

As a conclusion,  $\Gamma$  is a normal Cayley graph  $\text{Cay}(H, S)$  on a group  $H$ , where  $S = (A \cup B) - \{1\}$ ,  $A$  and  $B$  are subgroups of  $H$  such that  $H = \langle A, B \rangle$ ,  $A \cap B = 1$  and  $A \cong B \cong \mathbb{Z}_p^f$ . Note that  $\text{Aut}(H, S)$  is just the stabilizer of an edge of  $\Sigma$  in  $\text{Aut}(\Gamma)$ . From the argument in the above paragraphs, one may see that if  $\Gamma$  is not 3-CH, then one of (1) – (6) of Theorem 1.6 (b) holds.  $\square$

Finally, we prove the sufficiency of Theorem 1.6. The main thing that we need to prove is the Cayley graph  $\text{Cay}(H, S)$  satisfying the conditions in part (a) and part (b) of Theorem 1.6 is 3-CSH but not 3-CH. Actually, we can give more information about the symmetry of  $\text{Cay}(H, S)$ . This is done in the following lemma.

**Lemma 5.6.** *Let  $p$  be a prime and  $f$  be a positive integer. Let  $H$  be a group having two subgroups  $A, B$  such that  $H = \langle A, B \rangle$ ,  $A \cong B \cong C_p^f$ , and  $A \cap B = 1$ . Let  $\Gamma = \text{Cay}(H, S)$  with  $S = (A \cup B) - \{1\}$ , and let  $\Sigma = C(\Gamma)$ . Then  $\Sigma$  has valency  $p^f$ , and  $R(H)$  is regular on  $E(\Sigma)$  and intransitive on  $V(\Sigma)$ . Moreover, if  $\text{Aut}(\Gamma)$  is solvable and  $\Gamma$  satisfies the conditions in one of (1)–(6) of Theorem 1.6 (b), then the following hold.*

- (1) *If  $(p, f) = (2, 2)$  and  $\text{Aut}(H, S) \cong C_3 \wr C_2$ , then  $\Sigma$  is a tetravalent 3-arc-regular graph, and  $\Gamma$  is a 6-valent 3-CSH and 2-geodesic-transitive but not 3-CH graph;*

- (2) If  $(p, f) = (5, 1)$  and  $\text{Aut}(H, S) \cong C_4 \wr C_2$ , then  $\Sigma$  is a pentavalent 3-arc-regular graph, and  $\Gamma$  is an 8-valent 3-CSH and 2-geodesic-transitive but not 3-CH graph;
- (3) If  $(p, f) = (2, 3)$  and  $C_7 \wr C_2 \leq \text{Aut}(H, S) \leq (C_7 \rtimes C_3) \wr C_2$ , then  $\Sigma$  is a 3-arc-transitive graph of valency 8, and  $\Gamma$  is a 14-valent 3-CSH and 2-geodesic-transitive but not 3-CH graph;
- (4) If  $(p, f) = (2, 5)$  and  $(C_{31} \times C_{31}) \rtimes C_{10} \leq \text{Aut}(H, S) \leq (C_{31} \rtimes C_5) \wr C_2$ , then  $\Sigma$  is a 32-valent 3-arc-transitive graph, and  $\Gamma$  is a 62-valent 3-CSH and 2-geodesic-transitive but not 3-CH graph;
- (5) If  $(p, f) = (5, 1)$  and  $\text{Aut}(H, S) \cong M_{16}$ , then  $\Sigma$  is a pentavalent symmetric graph of type  $\mathcal{Q}_2^6$ , and  $\Gamma$  is an 8-valent 3-CSH but not 2-geodesic-transitive graph;
- (6) If  $(p, f) = (3, 1)$  and  $\text{Aut}(H, S) \cong C_4$ , then  $\Sigma$  is a trivalent symmetric graph of type  $2^2$ , and  $\Gamma$  is a tetravalent 3-CSH but not 2-geodesic-transitive graph.

**Proof.** Since  $\Gamma = \text{Cay}(H, S)$  with  $S = (A \cup B) - \{1\}$ , it follows that both  $A$  and  $B$  induce two subgraphs of  $\Gamma$  isomorphic to  $\mathbf{K}_{p^f}$  due to  $A \cong B \cong C_p^f$ . As  $S = (A \cup B) - \{1\}$  and  $A \cap B = 1$ ,  $[A]$  and  $[B]$  are two maximal cliques of  $\Gamma$  containing 1. Since  $R(H)$  acts transitively on  $V(\Gamma) = H$  by right multiplication,  $\{[Ah], [Bh] \mid h \in H\}$  is the set of all maximal cliques of  $\Gamma$ . For arbitrary  $h \in H$ , we have  $Ah \cap Bh = \{h\}$  and  $\Gamma(h) = (Ah \cup Bh) - \{h\}$  as  $A \cap B = \{1\}$  and  $S = (A \cup B) - \{1\}$ . This implies that every vertex  $h$  of  $\Gamma$  is contained in exactly two maximal cliques, namely,  $[Ah]$  and  $[Bh]$ . Let  $\Sigma = C(\Gamma)$  be the clique graph of  $\Gamma$ . Then  $V(\Sigma) = \{Ah, Bh \mid h \in H\}$  and  $E(\Sigma) = \{\{Ah, Bh\} \mid h \in H\}$ . It follows that  $\Sigma$  is a bipartite graph with two bi-parts  $B_0 = \{Ah \mid h \in H\}$  and  $B_1 = \{Bh \mid h \in H\}$ . Clearly,  $R(H)$  induces an action on  $V(\Sigma)$  with two orbits  $B_0$  and  $B_1$ . Since  $A \cap B = 1$ , one has  $R(H)_A \cap R(H)_B = \{R(g) \mid g \in A \cap B\} = 1$ . This implies that  $R(H)$  acts faithfully on  $V(\Sigma)$ . Since  $\Sigma(A) = \{Ba \mid a \in A\}$ , it follows that  $\Sigma$  has valency  $|A| = p^f$  and  $R(H)_A$  acts regularly on  $\Sigma(A)$  as  $R(H)_A \cap R(H)_B = 1$ . Thus,  $R(H)$  is regular on  $E(\Sigma)$  but not transitive on  $V(\Sigma)$ . By Lemma 2.4,  $\Gamma$  is isomorphic to the line graph  $L(\Sigma)$  of  $\Sigma$ . This proves the first half of our lemma.

In what follows, we shall prove the second half of this lemma. For convenience, we identify  $\Gamma$  with the line graph of  $\Sigma$ . Then  $\text{Aut}(H, S)$  fixes an edge, say  $\{u, v\}$ , of  $\Sigma$ . Let  $N = R(H) \rtimes \text{Aut}(H, S)$ . Since  $H = \langle A, B \rangle$  and  $S = (A \cup B) - \{1\}$ ,  $\text{Aut}(H, S)$  acts faithfully on  $S$ . Note that both  $[A]$  and  $[B]$  are two maximal cliques of  $\Gamma$ . Since  $A \cap B = 1$  and  $\text{Aut}(H, S)$  fixes 1, it implies that every element of  $\text{Aut}(H, S)$  either setwise fixes or interchanges the subsets  $A - \{1\}$  and  $B - \{1\}$  of  $S$ . Then  $\text{Aut}(H, S)$  acts on  $\{A, B\}$ , and let  $K$  be the kernel of this action. Then  $\text{Aut}(H, S) \cong \text{Aut}(H, S)^S \leq (K^A \times K^B) \rtimes C_2$ . Since  $\text{Aut}(H, S)$  acts faithfully on  $(A \cup B) - \{1\}$ , it implies that  $K^A$  is faithful on  $A$ . Since  $K \leq \text{Aut}(H, S) \leq \text{Aut}(H)$  and  $A \leq H$ , one has  $K^A \leq \text{Aut}(A)$ . Similarly,  $K^B \leq \text{Aut}(B)$ . It follows that  $\text{Aut}(H, S) \cong \text{Aut}(H, S)^S \leq (\text{Aut}(A) \times \text{Aut}(B)) \rtimes C_2$ .

We first deal with the case when  $(p, f) = (2, f)$  with  $f = 2, 3$  or  $5$ . Note that if  $f = 2$ , then  $\text{Aut}(H, S) \cong C_3 \wr C_2$ , and if  $f = 3$ , then  $C_7 \wr C_2 \leq \text{Aut}(H, S)$ , and if  $f = 5$ , then  $(C_{31} \times C_{31}) \rtimes C_{10} \leq \text{Aut}(H, S) \leq (C_{31} \rtimes C_5) \wr C_2$ . It follows that in these three cases,  $N_u$  is not faithful on  $\Sigma(u)$  and that  $N$  is 3-arc-transitive on  $\Sigma$ . Furthermore, if  $f = 2$ , then  $N_u^{\Sigma(u)} \cong A_4$ , if  $f = 3$ , then  $N_u^{\Sigma(u)} \cong C_2^3 \rtimes (C_7 \rtimes C_s)$  with  $s = 1$  or  $3$ , and if  $f = 5$ , then  $N_u^{\Sigma(u)} \cong C_2^5 \rtimes (C_{31} \rtimes C_5)$ . To prove that part (1), (3) and (4) hold, by Lemma 4.5, it suffices to prove that  $N = \text{Aut}(\Sigma)$ , namely,  $R(H) \trianglelefteq \text{Aut}(\Sigma)$ . Since  $\text{Aut}(\Gamma)$  is solvable, by Lemma 5.4, the following statements are true:

- (a)  $\text{Aut}(\Sigma)$  has a normal subgroup, say  $T$ , which is regular on  $E(\Sigma)$  and intransitive on  $V(\Sigma)$ ,
- (b) the edge-stabilizer  $\text{Aut}(\Sigma)_{\{u,v\}} \leq (C_{2f-1} \times C_f) \wr C_2$ , and,
- (c)  $\Sigma$  is a bipartite graph with two bi-parts of size  $\frac{|T|}{2^f} = \frac{|R(H)|}{2^f}$ .

To complete the proof, it suffices to prove that  $R(H) = T$ . Suppose by way of contradiction that  $R(H) \neq T$ . Noticing that both  $R(H)$  and  $T$  are edge- but not vertex-transitive on  $\Sigma$ ,  $R(H)T$  is edge- but not vertex-transitive on  $\Sigma$ . Let  $D$  be the stabilizer of the edge  $\{u, v\}$  of  $\Sigma$  in  $R(H)T$ . Then  $R(H)T = T \rtimes D$ , and  $D$  also fixes the arc  $(u, v)$ . Note that arc-stabilizer  $\text{Aut}(\Sigma)_{(u,v)} \leq (C_{2f-1} \times C_f) \times (C_{2f-1} \times C_f)$ . So  $1 \neq D \lesssim K \times K$ , where  $K = S_3$  if  $f = 2$ , or  $K = C_7 \times C_3$  if  $f = 3$  or  $K = C_{31} \times C_5$  if  $f = 5$ . Let  $M = R(H) \cap T$ . Then  $M \trianglelefteq N$ . Suppose that  $M$  is semiregular on  $V(\Sigma)$ . Then the number of orbits of  $M$  on  $V(\Sigma)$  is equal to  $2 \cdot \frac{|R(H)|}{2^f \cdot |M|} = \frac{1}{2^{f-1}} \cdot \frac{|R(H)T|}{|T|} = \frac{|D|}{2^{f-1}}$ . This means  $4 \mid |D|$ , forcing  $f = 2$  and  $D \leq S_3 \times S_3$ . It follows that the number of orbits of  $M$  on  $V(\Sigma)$  is equal to 2, 6 or 18. If  $M$  has two orbits on  $V(\Sigma)$ , then by [32, Lemma 2],  $N_u$  is faithful on  $\Sigma(u)$ , which is impossible since  $N$  is 3-arc-transitive on  $\Sigma$  and  $\Sigma$  has valency 4. If  $M$  has 6 or 18 orbits on  $V(\Sigma)$ , then by Lemma 5.1, the quotient graph of  $\Sigma$  relative to  $M$  is a tetravalent 2-arc-transitive graph of order 6 or 18. However, by [44, Table 3], no such graph exists. Thus,  $M$  is not semiregular on  $V(\Sigma)$ , and so  $M_u \neq 1$ . Since  $N$  is 3-arc-transitive on  $\Sigma$ , we see that  $N_u$  is 2-transitive on  $\Sigma(u)$ , and since  $1 \neq M_u \trianglelefteq N_u$ , it implies that  $M_u$  is transitive on  $\Sigma(u)$ . So  $M$  is edge-transitive on  $\Sigma$ . It follows that  $M = R(H) = T$  since  $R(H)$  and  $T$  are regular on  $E(\Sigma)$ . This completes the proof of part (1), (3) and (4).

Suppose that  $(p, f) = (5, 1)$ . Then  $\Sigma$  has valency 5, and  $|A - \{1\}| = |B - \{1\}| = 4$ . Recall that  $\text{Aut}(H, S) \cong \text{Aut}(H, S)^S \leq (\text{Aut}(A) \times \text{Aut}(B)) \times C_2$ . It follows that if  $\text{Aut}(H, S) \cong C_4 \wr C_2$  or  $M_{16}$ , then  $\text{Aut}(H, S)$  acts transitively on  $S$ , and then  $\Gamma$  is arc-transitive. By Lemma 2.5,  $\Sigma$  is 2-arc-transitive. Assume  $\text{Aut}(H, S) \cong C_4 \wr C_2$ . By [47, Theorem 1.2],  $N_u \cong \text{Frob}(20) \times C_4$  and  $N_u^{[1]} \cong C_4$ . Since  $\text{Aut}(\Sigma)$  is solvable, by [59, Theorem 4.1] or [47, Theorem 1.2], we have  $\text{Aut}(\Sigma) = N$ , and so  $\Sigma$  is 3-arc-regular. Again, by Lemma 4.6,  $\Gamma$  is 2-geodesic-transitive but not 3-CH. This implies part (2). Assume now  $\text{Aut}(H, S) \cong M_{16}$ . Then by [47, Theorem 1.2], we have  $N_u \cong \text{Frob}(20) \times C_2$  and  $N_u^{[1]} \cong C_2$ . Since  $\text{Aut}(\Gamma)$  is solvable, by [59, Theorem 4.1] or [47, Theorem 1.2], one has  $\text{Aut}(\Sigma)_u \leq \text{Frob}(20) \times C_4$ , and so  $\text{Aut}(\Sigma)_u$  has a unique Sylow 5-subgroup. Moreover, by Theorem 5.4,  $\text{Aut}(\Sigma)$  contains a normal subgroup, say  $T$ , such that  $T$  is regular on  $E(\Sigma)$ . It follows that  $T_u$  is a Sylow 5-subgroup of  $\text{Aut}(\Sigma)_u$ . Clearly,  $R(H)_u$  is also a Sylow 5-subgroup of  $\text{Aut}(\Sigma)_u$ , so  $R(H)_u = T_u$ . Similarly,  $R(H)_v = T_v$ . It follows that  $R(H) = \langle R(H)_u, R(H)_v \rangle = \langle T_u, T_v \rangle = T$ , and hence  $R(H) \trianglelefteq \text{Aut}(\Sigma)$ . Thus,  $N = R(H) \times \text{Aut}(H, S) = \text{Aut}(\Gamma)$ , and hence  $\Sigma$  is a pentavalent symmetric graph of type  $\mathcal{Q}_2^6$ . By Lemma 4.7,  $\Gamma$  is 3-CSH but not 2-geodesic-transitive. This proves part (5).

Finally, suppose that  $(p, f) = (3, 1)$  and  $\text{Aut}(H, S) \cong C_4$ . Then  $\Sigma$  is a trivalent graph. Since  $|S| = 4$ ,  $\text{Aut}(H, S)$  acts regularly on  $S$ , and so  $N$  is arc-transitive on  $\Gamma$ . It follows that  $N$  is 2-arc-transitive on  $\Sigma$ . Since  $\text{Aut}(H, S)$  is the stabilizer of the edge  $\{u, v\}$  in  $N$ , one has  $N_u \cong S_3$ . By [10, Theorem 5.1],  $\Sigma$  is 2- or 3-arc-regular. So,  $\text{Aut}(\Sigma)_u \leq S_3 \times C_2$ , and so  $\text{Aut}(\Sigma)_u$  has a unique Sylow 3-subgroup. By Theorem 5.4, the solvability of  $\text{Aut}(\Gamma)$  implies that  $\text{Aut}(\Sigma)$  has a normal subgroup, say  $T$ , acting regularly on  $E(\Sigma)$ . It follows that  $T_u$  is a Sylow 3-subgroup of  $\text{Aut}(\Sigma)_u$ . Clearly,  $R(H)_u$  is also a Sylow 3-subgroup of  $\text{Aut}(\Sigma)_u$ , so  $R(H)_u = T_u$ . Similarly,  $R(H)_v = T_v$ . It follows that  $R(H) = \langle R(H)_u, R(H)_v \rangle = \langle T_u, T_v \rangle = T$ , and hence  $R(H) \trianglelefteq \text{Aut}(\Sigma)$ . Thus,

$N = R(H) \rtimes \text{Aut}(H, S) = \text{Aut}(\Gamma)$ , and so  $\Sigma$  is trivalent symmetric graph of type  $2^2$ . By Lemma 4.7,  $\Gamma$  is 3-CSH but not 2-geodesic-transitive. This proves part (6).  $\square$

## 6. EXAMPLES OF 3-CSH BUT NOT 3-CH GRAPHS

We begin by constructing a graph satisfying the condition (1) of Theorem 1.6 (b).

**Construction I** Let  $\mathcal{H} = \langle a, b, c, d, e, f, g, h \rangle$  be a group with the following relations:

$$\begin{aligned} a^2 = b^2 = c^4 = d^4 = e^2 = f^2 = g^4 = h^4 = 1, [a, b] = [e, f] = 1, \\ c = [a, e], d = [a, f], g = [b, e], h = [b, f], [c, d]^2 = [c, g]^2 = [c, h]^2 = [d, g]^2 = [d, h]^2 = [g, h]^2 = 1, \\ [a, [c, d]] = [b, [c, d]] = [e, [c, d]] = [f, [c, d]] = [a, [c, g]] = [b, [c, g]] = [e, [c, g]] = [f, [c, g]] = 1, \\ [a, [c, h]] = [b, [c, h]] = [e, [c, h]] = [f, [c, h]] = [a, [g, d]] = [b, [g, d]] = [e, [g, d]] = [f, [g, d]] = 1, \\ [a, [h, d]] = [b, [h, d]] = [e, [h, d]] = [f, [h, d]] = [a, [g, h]] = [b, [g, h]] = [e, [g, h]] = [f, [g, h]] = 1, \\ [[a, e], [b, f]][f, [a, [b, e]]] = [[b, e], [ab, f]][f, [b, [ab, e]]] = [[ab, e], [a, f]][f, [ab, [a, e]]] = 1, \\ [[a, f], [b, ef]][ef, [a, [b, f]]] = [[b, f], [ab, ef]][ef, [b, [ab, f]]] = [[ab, f], [a, ef]][ef, [ab, [a, f]]] = 1, \\ [[a, ef], [b, e]][e, [a, [b, ef]]] = [[b, ef], [ab, e]][e, [b, [ab, ef]]] = [[ab, ef], [a, e]][e, [ab, [a, ef]]] = 1, \\ [[e, a], [f, b]][b, [e, [f, a]]] = [[f, a], [ef, b]][b, [f, [ef, a]]] = [[ef, a], [e, b]][b, [ef, [e, a]]] = 1, \\ [[e, b], [f, ab]][ab, [e, [f, b]]] = [[f, b], [ef, ab]][ab, [f, [ef, b]]] = [[ef, b], [e, ab]][ab, [ef, [e, b]]] = 1, \\ [[e, ab], [f, a]][a, [e, [f, ab]]] = [[f, ab], [ef, a]][a, [f, [ef, ab]]] = [[ef, ab], [e, a]][a, [ef, [e, ab]]] = 1. \end{aligned}$$

Let  $\Delta = \text{Cay}(\mathcal{H}, S)$  with  $S = \{a, b, ab, e, f, ef\}$ .

**Lemma 6.1.** *The group  $\mathcal{H}$  has order  $2^{17}$ , and the graph  $\Delta$  is a normal Cayley graph on  $\mathcal{H}$  with  $\text{Aut}(\mathcal{H}, S) \cong C_3^2 \times C_2$ . In particular,  $\Delta$  is 2-geodesic-transitive and 3-CSH but not 3-CH, and  $C(\Delta)$  is a tetravalent 3-arc-regular Cayley graph of order  $2^{16}$ .*

**Proof.** We shall make use of **Magma** [4] in the proof and see Appendix 1 for the programs. Let  $A = \langle a, b \rangle$  and  $B = \langle e, f \rangle$ . By using the **pQuotient** command in **Magma** [4], we obtain that  $|\mathcal{H}| = 2^{17}$ ,  $A \cong B \cong C_2^2$ ,  $H = \langle A, B \rangle$  and  $A \cap B = 1$ . Then  $S = (A \cup B) - 1$  and the subgraphs of  $\Delta$  induced by  $A$  and  $B$ , respectively, are two maximal cliques  $\mathbf{K}_4$ . This implies that  $\text{Aut}(\mathcal{H}, S)$  acts on  $\{A, B\}$ . Moreover,  $\text{Aut}(\mathcal{H}, S)$  acts faithfully on  $A \cup B$  since  $\mathcal{H} = \langle A, B \rangle$ . It follows that  $\text{Aut}(\mathcal{H}, S) \cong \text{Aut}(\mathcal{H}, S)^S \leq (\text{Aut}(A) \times \text{Aut}(B)) \rtimes C_2 \cong S_3 \wr C_2$ .

By using the **hom** command in **Magma** [4], we see that both the map  $a \mapsto b, b \mapsto a, e \mapsto e, f \mapsto f$  and the map  $a \mapsto b, b \mapsto a, e \mapsto f, f \mapsto e$  do not induce automorphisms of  $\mathcal{H}$ , but each of the following maps induces an automorphism of  $\mathcal{H}$ :

$$\begin{aligned} \alpha : a \mapsto b, b \mapsto ab, e \mapsto e, f \mapsto f, \\ \beta : a \mapsto a, b \mapsto b, e \mapsto f, f \mapsto ef, \\ \gamma : a \mapsto e, b \mapsto f, e \mapsto a, f \mapsto b. \end{aligned}$$

It follows that  $\text{Aut}(\mathcal{H}, S) = (\langle \alpha \rangle \times \langle \beta \rangle) \rtimes \langle \gamma \rangle \cong C_3^2 \times C_2$ .

Let  $\Sigma = C(\Delta)$ . Recall that  $R(H)$  acts on  $V(\Delta) = H$  by right multiplication. Since  $A \cap B = 1$ , for all  $h \in H$ , we have  $Ah \cap Bh = h$  and  $\Delta(h) = (Ah \cup Bh) - h$  as  $\Delta(1) = S = (A \cup B) - 1$ . This implies that  $\Sigma$  has vertex set  $V(\Sigma) = \{Ah, Bh \mid h \in \mathcal{H}\}$ , and edge set  $\{\{Ah, Bh\} \mid h \in H\}$ . Let  $V_0 = \{Ah \mid h \in H\}$  and  $V_1 = \{Bh \mid h \in H\}$ . Then  $|V_0| = |V_1| = 2^{15}$ , and so  $|V(\Sigma)| = 2^{16}$ . Moreover,  $V_0$  and  $V_1$  are two orbits of  $R(H)$  on  $V(\Sigma)$ . Note that the neighborhood of  $A$  in  $\Sigma$  is  $\Sigma(A) = \{Ba \mid a \in A\}$  while the neighborhood of  $B$  in  $\Sigma$  is  $\Sigma(B) = \{Ab \mid b \in B\}$ . As  $A \cong B \cong C_2^2$ , it follows that  $\Sigma$  has valency 4. Let  $K = \langle ae, bf, \mathcal{H}' \rangle$ . By **Magma** [4], we have  $A \cap K = B \cap K = 1$ . It

follows that  $|H : K| = 2^4$  and hence  $|H| = |V_0| = |V_1|$ . Since  $R(H)_A = \{R(a) \mid a \in A\}$  and  $R(H)_B = \{R(b) \mid b \in B\}$ , it implies that  $R(K) = \{R(g) \mid g \in K\}$  acts semiregularly on  $V(\Sigma)$  with two orbits  $V_0$  and  $V_1$ . Notice that  $\gamma$  swaps  $a$  and  $e$ , and swaps  $b$  and  $f$ . It follows that  $\gamma$  centralizes  $R(ae)$  and  $R(bf)$ . As  $\mathcal{H}' \trianglelefteq \mathcal{H}$ , we have  $\gamma$  normalizes  $R(K)$ , implying that  $|\langle R(K), \gamma \rangle| = |V(\Sigma)|$ . Note that  $\gamma$  also swaps  $A$  and  $B$ . This implies that  $\langle R(K), \gamma \rangle$  is transitive and so regular on  $V(\Sigma)$ . It follows that  $\Sigma$  is a Cayley graph.

Since  $\Sigma$  is a tetravalent graph, the stabilizer of any vertex of  $\Sigma$  in  $\text{Aut}(\Sigma)$  is a  $\{2, 3\}$ -group, and hence  $\text{Aut}(\Sigma)$  is also a  $\{2, 3\}$ -group since  $\Sigma$  has order  $2^{16}$ . It follows that  $\text{Aut}(\Sigma)$  is solvable. By Lemma 5.6 (1),  $\Delta$  is 2-geodesic-transitive and 3-CSH, but not 3-CH, and  $\Sigma$  is a tetravalent 3-arc-regular Cayley graph.  $\square$

The following proposition proves that there exist infinitely many solvable tetravalent 3-arc-regular graphs.

**Proposition 6.2.** *There exist infinitely many solvable tetravalent 3-arc-regular graphs.*

**Proof.** By [18, Theorem 2.11 (1)], for all primes  $p > 2^2 \cdot 3^4$  there exists a connected  $(X, 3)$ -arc-regular graph  $\Pi$  with  $X \leq \text{Aut}(\Pi)$  satisfying the following conditions:

- (1)  $X$  has a normal subgroup  $N \cong C_p^{\beta(\Gamma)}$ , where  $\beta(\Gamma) = |E(C(\Delta))| - |V(C(\Delta))| + 1$  is the Betti number of  $C(\Delta)$  and  $\Delta$  is the graph in Construction I;
- (2) the normal quotient  $\Pi_N \cong C(\Delta)$  and  $\Pi$  is a normal cover of  $\Pi_N$ ;
- (3)  $X/N \cong \text{Aut}(C(\Delta))$ .

We claim that  $\Pi$  is 3-arc-regular. Suppose on the contrary that  $\Pi$  is not 3-arc-regular. Then  $\text{Aut}(\Pi) > X$ . Since  $\Pi$  is a tetravalent 2-arc-transitive graph, by [44, Theorem 4] we have  $|\text{Aut}(\Pi)| \mid 2^4 \cdot 3^6 \cdot |V(C(\Delta))|$ . So  $N$  is a Sylow  $p$ -subgroup of  $\text{Aut}(\Pi)$ . Since  $X$  is 3-arc-regular on  $\Pi$ , one has  $|X| = 2^2 \cdot 3^2 \cdot |V(C(\Delta))|$ . Consequently,  $|\text{Aut}(\Pi) : X| \mid 2^2 \cdot 3^4$ . Since  $N \trianglelefteq X$ , one has  $|\text{Aut}(\Pi) : N_{\text{Aut}(\Pi)}(N)| \mid 2^2 \cdot 3^4$ , and since  $p > 2^2 \cdot 3^4$ , by Sylow's theorem, we have  $\text{Aut}(\Pi) = N_{\text{Aut}(\Pi)}(N)$ , and so  $N \trianglelefteq \text{Aut}(\Pi)$ . By Lemma 5.1, we would have  $\text{Aut}(\Pi)/N \leq \text{Aut}(C(\Delta))$ , which is impossible because  $C(\Delta)$  is 3-arc-regular.  $\square$

**Remark on Proposition 6.2.** From [41, Theorem 1.1 & Corollary 1.2] one may deduce that every tetravalent 3-arc-regular Cayley graph is a normal cover of a Cayley graph on one of the following groups:  $C_3^{11} \rtimes (C_2^{12} \cdot M_{11})$ ,  $S_{35}$  and  $A_{35}$ . This, however, is not true by Proposition 6.2.

Next we give two graphs satisfying the condition (2) of Theorem 1.6 (b), of which the first one appeared in [57, Remark 4.2].

**Example 6.3.** Let  $\mathcal{M} = \langle a, b, c \mid a^5 = b^5 = c^5 = 1, c = [a, b], [a, c] = [b, c] = 1 \rangle$ . Let  $\Theta = \text{Cay}(\mathcal{M}, S)$  with  $S = \{a, a^2, a^3, a^4, b, b^2, b^3, b^4\}$ . By Magma [4],  $\mathcal{M}$  has order  $5^3$ , and the graph  $\Theta$  is a normal Cayley graph on  $\mathcal{M}$  with  $\text{Aut}(\mathcal{M}, S) \cong C_4^2 \rtimes C_2$ . By Lemma 5.6 (2),  $\Theta$  is 2-geodesic-transitive but not 3-CH, and  $C(\Theta)$  is a pentavalent 3-arc-regular graph of order  $2 \cdot 5^2$ . (See Appendix 3 for the Magma programs used in this example.)

**Example 6.4.** Let  $\mathcal{T} = \langle a, b, c, d, e \mid a^5 = b^5 = c^5 = d^5 = e^5 = 1, c = [a, b], d = [a, c], e = [b, c], [a, d] = [b, d] = [a, e] = [b, e] = 1 \rangle$ . Let  $\Phi = \text{Cay}(\mathcal{T}, S)$  with  $S = \{a, a^2, a^3, a^4, b, b^2, b^3, b^4\}$ . By Magma [4],  $\mathcal{T}$  has order  $5^5$ , and the graph  $\Phi$  is a normal Cayley graph on  $\mathcal{T}$  with  $\text{Aut}(\mathcal{T}, S) \cong C_4^2 \rtimes C_2$ . By Lemma 5.6 (2),  $\Phi$  is 2-geodesic-transitive but not 3-CH, and  $C(\Phi)$  is a pentavalent 3-arc-regular graph of order  $2 \cdot 5^4$ . (See Appendix 4 for the Magma programs used in this example.)



The following example gives a graph satisfying the condition (3) of Theorem 1.6 (b).

**Example 6.5.** Let  $\mathcal{G} = \langle a, b, c, e, f, g, x, y, z \rangle$  be a group with the following relations:

$$\begin{aligned} a^2 = b^2 = c^2 = e^2 = f^2 = g^2 = x^2 = y^2 = z^2 = 1, [g, a] = x, [g, b] = y, [g, c] = z, \\ [a, b] = [a, c] = [b, c] = [e, f] = [e, g] = [f, g] = 1, \\ [e, a] = xyz, [e, b] = xz, [e, c] = x, [f, a] = xz, [f, b] = x, [f, c] = y. \end{aligned}$$

Let  $\Theta = \text{Cay}(\mathcal{G}, S)$ , where  $S = A \cup B - \{1\}$ ,  $A = \langle a, b, c \rangle$  and  $B = \langle e, f, g \rangle$ . By Magma [4],  $\mathcal{G}$  has order  $2^9$ , and the graph  $\Theta$  is a normal Cayley graph on  $\mathcal{G}$  with  $\text{Aut}(\mathcal{G}, S) \cong (C_7 \times C_7) \rtimes C_6$ . By Lemma 5.6 (3),  $\Theta$  is 2-geodesic-transitive and 3-CSH but not 3-CH, and  $C(\Theta)$  is a 3-arc-transitive graph. (See Appendix 5 for the Magma programs used in this example.)

The following example gives a graph satisfying the condition (4) of Theorem 1.6 (b).

**Example 6.6.** Let  $\mathcal{L} = \langle a, b, c, d, e, u, v, x, y, z, f, g, h, i, j \rangle$  be a group with the following relations:

$$\begin{aligned} a^2 = b^2 = c^2 = d^2 = e^2 = f^2 = g^2 = h^2 = i^2 = j^2 = u^2 = v^2 = x^2 = y^2 = z^2 = 1, \\ [a, b] = [a, c] = [a, d] = [a, e] = [b, c] = [b, d] = [b, e] = [c, d] = [c, e] = [d, e] = 1, \\ [u, v] = [u, x] = [u, y] = [u, z] = [v, x] = [v, y] = [v, z] = [x, y] = [x, z] = [y, z] = 1, \\ [a, z] = f, [b, z] = g, [c, z] = h, [d, z] = i, [e, z] = j, \\ [a, u] = fgh, [b, u] = ghi, [c, u] = hij, [d, u] = fghj, [e, u] = f, \\ [a, v] = ghi, [b, v] = hij, [c, v] = fghj, [d, v] = f, [e, v] = g \\ [a, x] = hij, [b, x] = fghj, [c, x] = f, [d, x] = g, [e, x] = h, \\ [a, y] = fghj, [b, y] = f, [c, y] = g, [d, y] = h, [e, y] = i. \end{aligned}$$

Let  $\Pi = \text{Cay}(\mathcal{L}, S)$ , where  $S = A \cup B - \{1\}$ ,  $A = \langle a, b, c, d, e \rangle$  and  $B = \langle u, v, x, y, z \rangle$ . By Magma [4],  $\mathcal{L}$  has order  $2^{15}$ , and the graph  $\Pi$  is a normal Cayley graph on  $\mathcal{L}$  with  $\text{Aut}(\mathcal{L}, S) \cong (C_{31} \times C_{31}) \rtimes C_{10}$ . By Lemma 5.6 (4),  $\Pi$  is 2-geodesic-transitive and 3-CSH but not 3-CH, and  $C(\Pi)$  is a 3-arc-transitive graph. (See Appendix 6 for the Magma programs used in this example.)

Now we give a graph satisfying the condition (5) of Theorem 1.6 (b).

**Example 6.7.** Let  $\mathcal{N} = \langle a, b, c, d, e, f, g, h, k \rangle$  be a group with the following relations:

$$\begin{aligned} a^5 = b^5 = c^5 = d^5 = e^5 = f^5 = g^5 = h^5 = k^5 = 1, \\ c = [a, b], d = [a, c], e = [b, c], [d, e] = 1, [a, d] = f, [b, d] = g, [a, e] = h, [b, e] = k, \\ [a, f] = [a, g] = [a, h] = [a, k] = [b, f] = [b, g] = [b, h] = [b, k] = 1, f = k^{-2}, g = h^{-2}. \end{aligned}$$

Let  $\Lambda = \text{Cay}(\mathcal{N}, S)$  with  $S = \{a, a^2, a^3, a^4, b, b^2, b^3, b^4\}$ . By Magma [4],  $\mathcal{N}$  has order  $5^6$ , and the graph  $\Lambda$  is a normal Cayley graph on  $\mathcal{N}$  with  $\text{Aut}(\mathcal{N}, S) \cong M_{16}$ . By Lemma 5.6 (5),  $\Lambda$  is 3-CSH but not 2-geodesic-transitive, and  $C(\Lambda)$  is a pentavalent symmetric graph of type  $\mathcal{Q}_2^6$ . (See Appendix 7 for the Magma programs used in this example.)

**Remark on Examples 6.3–6.7.** With a similar argument as in the proof of Proposition 6.2, by using the graphs given in Examples 6.3–6.7, one can see that there exist infinitely many graphs satisfying the conditions (2)–(5) of Theorem 1.6 (b).

Finally, we construct a family of graphs satisfying the condition (6) of Theorem 1.6 (b).

**Construction II** Let  $n \geq 2$  be an integer, and let  $\mathcal{R} = \langle a, b \rangle$  be a finite 3-group with the following relations:

$$\begin{aligned} a^3 &= b^3 = c^{3^n} = d^{3^n} = e^{3^n} = f^{3^n} = g^3 = h^3 = 1, c = [a, b], d = [b, a^2], e = [a^2, b^2], f = [b^2, a], \\ [c, d] &= c^{-3^{n-1}} d^{3^{n-1}}, [c, f] = c^{-3^{n-1}} f^{3^{n-1}}, [d, e] = d^{-3^{n-1}} e^{3^{n-1}}, [e, f] = e^{-3^{n-1}} f^{3^{n-1}}, \\ [d^{3^{n-1}}, c] &= [d^{3^{n-1}}, e] = [d^{3^{n-1}}, f] = 1, [e^{3^{n-1}}, c] = [e^{3^{n-1}}, d] = [e^{3^{n-1}}, f] = 1, \\ f^{3^{n-1}} &= c^{3^{n-1}} d^{-3^{n-1}} e^{3^{n-1}}, [c^{3^{n-1}}, d] = [c^{3^{n-1}}, e] = [c^{3^{n-1}}, f] = 1, \\ g &= [c, e], h = [d, f], [g, a] = [g, b] = [h, a] = [h, b] = 1, \\ h &= c^{3^{n-1}} d^{3^{n-1}} e^{3^{n-1}}, g^{-1} = d^{3^{n-1}} e^{3^{n-1}} f^{3^{n-1}}. \end{aligned}$$

Let  $\Upsilon = \text{Cay}(\mathcal{R}, S)$  with  $S = \{a, a^2, b, b^2\}$ .

**Lemma 6.8.** *The group  $\mathcal{R}$  has order  $3^{4n+1}$ , and the graph  $\Upsilon$  is a normal Cayley graph on  $\mathcal{R}$  with  $\text{Aut}(\mathcal{R}, S) \cong C_4$ . In particular,  $\Upsilon$  is 3-CSH but not 2-geodesic-transitive, and  $C(\Upsilon)$  is a trivalent symmetric graph of type  $2^2$ .*

**Proof.** Let  $D = \langle c, d, e, f \rangle$ . By a direct calculation, we obtain the following relations:

$$c^a = d^{-1}e^{-1}, d^a = c, e^a = f, f^a = f^{-1}e^{-1}, c^b = c^{-1}f^{-1}, d^b = e^{-1}d^{-1}, e^b = d, f^b = c.$$

Since  $\mathcal{R}$  is generated by  $a, b$ , one has  $D \trianglelefteq \mathcal{R}$  and  $\mathcal{R}/D = \langle aD, bD \rangle$ . Since  $c = [a, b] \in D$ , one has  $[aD, bD] = [a, b]D = D$  and hence  $\mathcal{R}/D$  is abelian. As both  $a$  and  $b$  have order 3,  $\mathcal{R}/D = \langle aD \rangle \times \langle bD \rangle \cong C_3 \times C_3$ . It follows that  $D = \Phi(\mathcal{R}) = \mathcal{R}'$ .

We now show the following two claims.

**Claim 1** Let  $a_1 = b$  and  $b_1 = a^2$ . Then  $a_1$  and  $b_1$  have the same relations as do  $a$  and  $b$ .

Let  $c_1 = [a_1, b_1]$ ,  $d_1 = [b_1, a_1^2]$ ,  $e_1 = [a_1^2, b_1^2]$  and  $f_1 = [b_1^2, a_1]$ . Then  $c_1 = d$ ,  $d_1 = e$ ,  $e_1 = f$  and  $f_1 = c$ . So  $c_1^{3^n} = d_1^{3^n} = e_1^{3^n} = f_1^{3^n} = 1$ . From the following relations

$$\begin{aligned} f^{3^{n-1}} &= c^{3^{n-1}} d^{-3^{n-1}} e^{3^{n-1}}, \\ [c^{3^{n-1}}, d] &= [c^{3^{n-1}}, e] = [c^{3^{n-1}}, f] = 1, \\ [d^{3^{n-1}}, c] &= [d^{3^{n-1}}, e] = [d^{3^{n-1}}, f] = 1, \\ [e^{3^{n-1}}, c] &= [e^{3^{n-1}}, d] = [e^{3^{n-1}}, f] = 1, \end{aligned}$$

we know that  $c^{3^{n-1}}, d^{3^{n-1}}, e^{3^{n-1}}, f^{3^{n-1}}$  are in the center of  $D$ . So

$$f_1^{3^{n-1}} = c^{3^{n-1}} = f^{3^{n-1}} d^{3^{n-1}} e^{-3^{n-1}} = c_1^{3^{n-1}} d_1^{-3^{n-1}} e_1^{3^{n-1}},$$

and  $c_1^{3^{n-1}}, d_1^{3^{n-1}}, e_1^{3^{n-1}}$  are in the center of  $D$ .

Let  $g_1 = [c_1, e_1]$  and  $h_1 = [d_1, f_1]$ . Then  $g_1 = [d, f] = h$  and  $h_1 = [e, c] = g^{-1}$ . Clearly,  $g_1, h_1$  are in the center of  $\mathcal{R}$ . Also, it is easy to check that  $g_1^{-1} = h^{-1} = d_1^{3^{n-1}} e_1^{3^{n-1}} f_1^{3^{n-1}}$ , and  $h_1 = g^{-1} = c_1^{3^{n-1}} d_1^{3^{n-1}} e_1^{3^{n-1}}$ .

Finally,  $[c_1, d_1] = [d, e] = d^{-3^{n-1}} e^{3^{n-1}} = c_1^{-3^{n-1}} d_1^{3^{n-1}}$ ,  $[c_1, f_1] = [d, c] = d^{-3^{n-1}} c^{3^{n-1}} = c_1^{-3^{n-1}} f_1^{3^{n-1}}$ ,  $[d_1, e_1] = [e, f] = e^{-3^{n-1}} f^{3^{n-1}} = d_1^{-3^{n-1}} e_1^{3^{n-1}}$ , and  $[e_1, f_1] = [f, c] = f^{-3^{n-1}} c^{3^{n-1}} = e_1^{-3^{n-1}} f_1^{3^{n-1}}$ . This proves Claim 1.

**Claim 2**  $\mathcal{R}$  has no automorphisms swapping  $a$  and  $b$ .

Suppose on the contrary that  $\alpha$  is an automorphism of  $\mathcal{R}$  such that  $a^\alpha = b$  and  $b^\alpha = a$ . Then  $c^\alpha = [b, a] = c^{-1}$ ,  $d^\alpha = [a, b^2] = f^{-1}$ ,  $e^\alpha = [b^2, a^2] = e^{-1}$  and  $f^\alpha = [a^2, b] = d^{-1}$ . Furthermore,  $g^\alpha = [c^\alpha, e^\alpha] = [c^{-1}, e^{-1}] = g^{e^{-1}c^{-1}} = g$  and  $h^\alpha = [d^\alpha, f^\alpha] = [f^{-1}, d^{-1}] = h^{-1}$ .

As  $g^{-1} = d^{3^{n-1}} e^{3^{n-1}} f^{3^{n-1}}$ , one has  $d^{3^{n-1}} e^{3^{n-1}} f^{3^{n-1}} = g^{-1} = (g^{-1})^\alpha = f^{-3^{n-1}} e^{-3^{n-1}} d^{-3^{n-1}}$ . This forces that  $g^{-1} = d^{3^{n-1}} e^{3^{n-1}} f^{3^{n-1}} = 1$ , a contradiction.

Now we are ready to complete the proof. Note that  $c^{3^{n-1}}, d^{3^{n-1}}, e^{3^{n-1}}$  are in the center of  $D$ , and  $f^{3^{n-1}} = c^{3^{n-1}} d^{-3^{n-1}} e^{3^{n-1}}$ . Set  $M = \langle c^{3^{n-1}}, d^{3^{n-1}}, e^{3^{n-1}} \rangle$ . Then  $M \cong C_3^3$ . By the following relations,

$$\begin{aligned} h &= c^{3^{n-1}} d^{3^{n-1}} e^{3^{n-1}}, g^{-1} = d^{3^{n-1}} e^{3^{n-1}} f^{3^{n-1}}, \\ [c, d] &= c^{-3^{n-1}} d^{3^{n-1}}, [c, f] = c^{-3^{n-1}} f^{3^{n-1}}, [d, e] = d^{-3^{n-1}} e^{3^{n-1}}, [e, f] = e^{-3^{n-1}} f^{3^{n-1}}. \end{aligned}$$

we conclude that  $M \leq D'$ . It follows that  $D/M = \langle cM \rangle \times \langle dM \rangle \times \langle eM \rangle \times \langle fM \rangle \cong C_{3^{n-1}}^4$ . Since  $\mathcal{R}/D \cong C_3^2$  and  $M \cong C_3^3$ , it follows that  $|\mathcal{R}| = 3^{4n+1}$ .

By Claim 1, the map  $a \mapsto b, b \mapsto a^2$  induces an automorphism, say  $\beta$ , of  $\mathcal{R}$ , and  $\beta$  cyclically permutes the elements in  $S$ . So,  $\beta \in \text{Aut}(\mathcal{R}, S)$ . Since  $S = \{a, a^2, b, b^2\}$ , one has  $\text{Aut}(\mathcal{R}, S) \leq D_8$ . By Claim 2,  $\mathcal{R}$  has no automorphisms swapping  $a$  and  $b$ . Consequently, we have  $\text{Aut}(\mathcal{R}, S) \cong C_4$ . Since  $\Upsilon$  is a tetravalent graph of order  $3^{4n+1}$ ,  $\text{Aut}(\Upsilon)$  is a  $\{2, 3\}$ -group, and so it is a solvable group. By Lemma 5.6,  $\Upsilon$  is 3-CSH but not 2-geodesic-transitive, and  $C(\Upsilon)$  is a trivalent symmetric graph of type  $2^2$ .  $\square$

**Remark on Lemma 6.8.** (1) We also verify Lemma 6.8 in case  $n = 2$  by using Magma [4], and the reader may see Appendix 8 for the Magma programs.

(2) In 2006, Feng and Kwak [21] posed the following conjecture.

**Conjecture** *Every connected trivalent symmetric graph of order  $2 \cdot 3^m$  is a Cayley graph for each  $m \geq 1$ .*

By Lemma 6.8,  $C(\Upsilon)$  is a trivalent symmetric graph of type  $2^2$  and of order  $2 \cdot 3^{4n}$  with  $n \geq 1$ . If  $C(\Upsilon)$  is a Cayley graph, then it would have an automorphism of order 2 which swaps the two vertices of an edge. However, this is impossible since every edge-stabilizer for  $C(\Upsilon)$  is isomorphic to  $C_4$ . Consequently,  $C(\Upsilon)$  is a non-Cayley graph. This implies that the above conjecture is not true.

## 7. APPENDIX: MAGMA PROGRAMS IN SECTION 6

**Appendix 1** (Programs for the graph  $\Delta$  in Construction I): First, we input a group  $G\langle a, b, c, d, e, f, g, h \rangle := \text{Group}\langle a, b, c, d, e, f, g, h \mid R \rangle$ , where  $R$  is a set of relations as given in Construction I.

Construction of the group  $\mathcal{H}$ :  $\text{H, q} := \text{pQuotient}(G, 2, 100);$

The order of group  $\mathcal{H}$ :  $\text{FactoredOrder}(\text{H});$

The derived subgroup of  $\mathcal{H}$ :  $\text{D} := \text{DerivedSubgroup}(\text{H});$

The derived subgroup of  $\mathcal{H}'$ :  $\text{DD} := \text{DerivedSubgroup}(\text{D});$

$\mathcal{H}/\mathcal{H}' \cong C_2^4$ ,  $\mathcal{H}'/\mathcal{H}'' \cong C_2^4 \times C_4^2$  and  $\mathcal{H}'' \cong C_2$ :

$\text{GroupName}(\text{H}/\text{D}); \text{GroupName}(\text{D}/\text{DD}); \text{GroupName}(\text{DD});$

Construction of subgroups  $A$  and  $B$ :

$\text{a} := \text{a@q}; \text{b} := \text{b@q}; \text{e} := \text{e@q}; \text{f} := \text{f@q}; \text{A} := \text{sub}\langle \text{H} \mid \text{a, b} \rangle; \text{B} := \text{sub}\langle \text{H} \mid \text{e, f} \rangle;$

Test  $A \cong B \cong C_2^2$  and  $A \cap B = 1$ :

$\#A; \text{IsElementaryAbelian}(\text{A});$

$\#B; \text{IsElementaryAbelian}(\text{B});$

$\#(\text{A meet B}); \text{H eq sub}\langle \text{H} \mid \text{a, b, e, f} \rangle;$

The following maps are not automorphisms of  $\mathcal{H}$ :

$\text{hom}\langle H \rightarrow H \mid a \rightarrow b, b \rightarrow a, e \rightarrow f, f \rightarrow e \rangle$ ;

$\text{hom}\langle H \rightarrow H \mid a \rightarrow b, b \rightarrow a, e \rightarrow e, f \rightarrow f \rangle$ ;

$\text{hom}\langle H \rightarrow H \mid a \rightarrow a, b \rightarrow b, e \rightarrow f, f \rightarrow e \rangle$ ;

The following maps are automorphisms of  $\mathcal{H}$ :

$\text{alpha} := \text{hom}\langle H \rightarrow H \mid a \rightarrow b, b \rightarrow a*b, e \rightarrow e, f \rightarrow f \rangle$ ;  $\text{alpha}$ ;  $\text{Kernel}(\text{alpha})$ ;  $\text{Image}(\text{alpha})$   
eq H;

$\text{beta} := \text{hom}\langle H \rightarrow H \mid a \rightarrow a, b \rightarrow b, e \rightarrow f, f \rightarrow e*f \rangle$ ;  $\text{beta}$ ;  $\text{Kernel}(\text{beta})$ ;  $\text{Image}(\text{beta})$  eq  
H;

$\text{gamma} := \text{hom}\langle H \rightarrow H \mid a \rightarrow e, b \rightarrow f, e \rightarrow a, f \rightarrow b \rangle$ ;  $\text{gamma}$ ;  $\text{Kernel}(\text{gamma})$ ;  $\text{Image}(\text{gamma})$   
eq H;

Construction of subgroup  $K$  and testing  $K \cap A = K \cap B = 1$ :

$K := \text{sub}\langle H \mid a*e, b*f, \text{DerivedSubgroup}(H) \rangle$ ;

$\#(K \text{ meet } A)$ ;  $\#(K \text{ meet } B)$ ;

**Appendix 2** (Programs for the construction of a Cayley graph):

$\text{Cay} := \text{function}(G, S)$ ;

$V := g : g \text{ in } G$ ;

$E := g, s * g : g \text{ in } G, s \text{ in } S$ ;

return  $\text{Graph}\langle V \mid E \rangle$ ;

end function;

**Appendix 3** (Programs for the graph  $\Theta$  in Example 6.3): First, we input a group

$G \langle a, b, c \rangle := \text{Group}\langle a, b, c \mid a^5, b^5, c^5, c = (a, b), (a, c) = (b, c) = 1 \rangle$ ;

Construction of the group  $\mathcal{M}$ :  $M, q := \text{pQuotient}(G, 5, 100)$ ;

The order of group  $\mathcal{M}$ :  $\text{FactoredOrder}(M)$ ;

The derived subgroup of  $\mathcal{M}$ :  $D := \text{DerivedSubgroup}(M)$ ;

The derived subgroup of  $\mathcal{M}'$ :  $DD := \text{DerivedSubgroup}(D)$ ;

$\mathcal{M}/\mathcal{M}' \cong C_5^2$ ,  $\mathcal{M}'/\mathcal{M}'' \cong C_5$  and  $\mathcal{M}'' = 1$ :

$\text{GroupName}(M/D)$ ;  $\text{GroupName}(D/DD)$ ;  $\text{GroupName}(DD)$ ;

Construction of  $\Theta = \text{Cay}(\mathcal{M}, S)$ :

$a := a@q$ ;  $b := b@q$ ;

$S := \{a, a^2, a^3, a^4, b, b^2, b^3, b^4\}$ ;

$\text{Theta} := \text{Cay}(M, S)$ ;

Automorphism group of  $\Theta$ :

$A := \text{AutomorphismGroup}(\text{Theta})$ ;

$\Theta$  is a normal Cayley graph on  $\mathcal{M}$  (We find that every Sylow 5-subgroup of  $\text{Aut}(\Theta)$  is normal and regular on  $V(\Theta)$ . This implies that  $\Theta$  is normal):

$P := \text{SylowSubgroup}(A, 5)$ ;

$\text{IsNormal}(A, P)$ ;

$\text{IsRegular}(P)$ ;

$\text{Aut}(\mathcal{M}, S) \cong C_4 \wr C_2$ :

$A1 := \text{Stabilizer}(A, 1)$ ;

$\text{GroupName}(A1)$ ;

**Appendix 4** (Programs for the graph  $\Phi$  in Example 6.4): First, we input a group

$G \langle a, b, c, d, e \rangle := \text{Group}\langle a, b, c, d, e \mid a^5, b^5, c^5, d^5, e^5, c = (a, b), d = (a, c), e = (b, c) \rangle$ ,

```

(a,d)=(b,d)=(a,e)=(b,e)=1>;
Construction of the group group  $\mathcal{T}$ : T,q:=pQuotient(G,5,100);
The order of group  $\mathcal{T}$ : FactoredOrder(T);
The derived subgroup of  $\mathcal{T}$ : D:=DerivedSubgroup(T);
The derived subgroup of  $\mathcal{T}'$ : DD:=DerivedSubgroup(T);
 $\mathcal{T}/\mathcal{T}' \cong C_5^2$ ,  $\mathcal{T}'/\mathcal{T}'' \cong C_5^3$  and  $\mathcal{T}'' = 1$ :
GroupName(T/D); GroupName(D/DD); GroupName(DD);
Construction of  $\Phi = \text{Cay}(\mathcal{T}, S)$ :
a:=a@q; b:=b@q;
S:={a,a^2,a^3,a^4,b,b^2,b^3,b^4};
Phi:=Cay(T,S);
Automorphism group of  $\Phi$ :
A:=AutomorphismGroup(Phi);
 $\Phi$  is a normal Cayley graph on  $\mathcal{T}$  (We find that every Sylow 5-subgroup of  $\text{Aut}(\Phi)$  is
normal and regular on  $V(\Phi)$ . This implies that  $\Phi$  is normal):
P:=SylowSubgroup(A,5);
IsNormal(A,P);
IsRegular(P);
 $\text{Aut}(\mathcal{T}, S) \cong C_4 \wr C_2$ :
A1:=Stabilizer(A,1);
GroupName(A1);

```

**Appendix 5** (Programs for the graph  $\Theta$  in Example 6.5): First, we input a group

```

G<a,b,c,d,e,f,g,x,y,z>:=Group<a,b,c,d,e,f,g,x,y,z | a^2,b^2,c^2,e^2,f^2,g^2,x^2,y^2,z^2,
(a,b)=(a,c)=(b,c)=1, (e,f)=(e,g)=(f,g)=1, (e,a)=x*y*z, (e,b)=x*z, (e,c)=x,
(f,a)=x*z, (f,b)=x, (f,c)=y, (g,a)=x, (g,b)=y, (g,c)=z >;

```

Construction of the group group  $\mathcal{G}$ :  $G,q:=pQuotient(G,2,100)$ ;

```

The order of group  $\mathcal{G}$ : FactoredOrder(G);
The derived subgroup of  $\mathcal{G}$ : D:=DerivedSubgroup(G);
The derived subgroup of  $\mathcal{G}'$ : DD:=DerivedSubgroup(G);
 $\mathcal{G}/\mathcal{G}' \cong C_2^6$ ,  $\mathcal{G}'/\mathcal{G}'' \cong C_2^3$  and  $\mathcal{G}'' = 1$ :
GroupName(G/D); GroupName(D/DD); GroupName(DD);

```

Construction of subgroups  $A$  and  $B$ , and testing  $A \cong B \cong C_2^3$ ,  $A \cap B = 1$  and  $\mathcal{G} = \langle A, B \rangle$ :

```

a:=a@q; b:=b@q; c:=c@q; e:=e@q; f:=f@q; g:=g@q;
A:=sub<G|a,b,c>; #A; IsElementaryAbelian(A);
B:=sub<G|e,f,g>; #B; IsElementaryAbelian(B);
#(A meet B);
H eq sub<G|A,B>;
Construction of  $\Theta = \text{Cay}(\mathcal{G}, S)$ :
S:={x:x in A|x ne G!1} join {y:y in B|y ne G!1};
Theta:=Cay(G,S);
Automorphism group of  $\Theta$ :
au:=AutomorphismGroup(Theta);

```

$\Theta$  is a normal Cayley graph on  $\mathcal{G}$  (Note that  $R(\mathcal{G})$  is a regular subgroup of  $\text{Aut}(\Theta)$ . We first list all regular subgroups of  $\text{Aut}(\Theta)$ , and then we find that among these subgroups, there is only one which is isomorphic to  $R(\mathcal{G})$  and this subgroup is normal in  $\text{Aut}(\Theta)$ ):

```

R:=RegularSubgroups(au);//Find all regular subgroups of Aut(Θ)
T:={};
for i in {1..#R} do
if #(DerivedSubgroup(G)) eq #(DerivedSubgroup(R[i]'subgroup)) then
Include(~T,i);
end if;
end for;
#T;//|T| = 1
i:=Random(T);
IsNormal(au,R[i]'subgroup);
Aut(G,S) ≅ C7 × C7 ⋊ C6:
au1:=Stabilizer(au,1);
GroupName(au1);

```

**Appendix 6** (Programs for the graph  $\Pi$  in Example 6.6): First, we input a group  $G\langle a, b, c, d, e, u, v, x, y, z, f, g, h, i, j \rangle := \text{Group}\langle a, b, c, d, e, u, v, x, y, z, f, g, h, i, j \mid R \rangle$ , where  $R$  is a set of relations as given in Example 6.6.

```

Construction of the group  $\mathcal{L}$ : L,q:=pQuotient(G,2,100);
The order of group  $\mathcal{L}$ : FactoredOrder(L);
The derived subgroup of  $\mathcal{L}$ : D:=DerivedSubgroup(L);
The derived subgroup of  $\mathcal{L}'$ : DD:=DerivedSubgroup(L);
 $\mathcal{L}/\mathcal{L}' \cong C_2^{10}$ ,  $\mathcal{L}'/\mathcal{L}'' \cong C_2^5$  and  $\mathcal{L}'' = 1$ :
GroupName(L/D); GroupName(D/DD); GroupName(DD);
Construction of subgroups  $A$  and  $B$ , and testing  $A \cong B \cong C_2^5$ ,  $A \cap B = 1$  and
 $\mathcal{L} = \langle A, B \rangle$ :
a:=a@q; b:=b@q; c:=c@q; d:=d@q; e:=e@q;
u:=u@q; v:=v@q; x:=x@q; y:=y@q; z:=z@q;
A:=sub<L|a,b,c,d,e>; #A; IsElementaryAbelian(A);
B:=sub<L|u,v,x,y,z>; #B; IsElementaryAbelian(B);
#(A meet B); H eq sub<L|A,B>;
Construction of the clique graph of  $\Pi = \text{Cay}(\mathcal{L}, S)$ :
V1:={};
for h in L do
Vh:={};
for w in A do
Include(~Vh,w*h);
end for;
Include(~V1,Vh);
end for;
V2:={};
for h in L do
Vh:={};
for w in B do

```

```

Include(~Vh,w*h);
end for;
Include(~V2,Vh);
end for;
V:=V1 join V2;
E:={};
for w1 in V1 do
for w2 in V2 do
if #(w1 meet w2) eq 1 then
Include(~E,{w1,w2});
end if;
end for;
end for;
CPi:=Graph<V|E>;//This is the clique graph of  $\Pi$ 
Automorphism group of  $C(\Pi)$ :
au:=AutomorphismGroup(CPi);
 $\Pi$  is a normal Cayley graph on  $\mathcal{L}$  (Note that  $R(\mathcal{L})$  is a subgroup of  $\text{Aut}(C(\Pi))$ ). We first
list all subgroups of  $\text{Aut}(C(\Pi))$  of order  $2^{15}$ , and we find that among these subgroups,
there is only one which is isomorphic to  $R(\mathcal{L})$  and this subgroup is normal in  $\text{Aut}(C(\Pi))$ ):
R:=Subgroups(au:OrderEqual:=215);
T:={};
for i in {1..#R} do
if #(DerivedSubgroup(L)) eq #(DerivedSubgroup(R[i]'subgroup)) then
Include(~T,i);
end if;
end for;
#T;//|T| = 1
i:=Random(T);
IsNormal(au,R[i]'subgroup);
 $\text{Aut}(\mathcal{R}, S) \cong (C_{31} \times C_{31}) \times C_{10}$  (Note that  $\text{Aut}(\mathcal{R}, S) \cong \text{Aut}(C(\Pi))/R(\mathcal{L})$ ):
GroupName(au/R[i]'subgroup);

```

**Appendix 7** (Programs for the graph  $\Lambda$  in Example 6.7): First, we input a group  $G\langle a, b, c, d, e, f, g, h, k \rangle := \text{Group}\langle a, b, c, d, e, f, g, h, k \mid a^5, b^5, c^5, d^5, e^5, f^5, g^5, h^5, k^5, c=(a,b), d=(a,c), e=(b,c), (d,e)=1, (a,d)=f, (b,d)=g, (a,e)=h, (b,e)=k, (a,f)=(a,g)=(a,h)=(a,k)=(b,f)=(b,g)=(b,h)=(b,k)=1, f=k^{-2}, g=h^{-2} \rangle$ ;

Construction of the group  $\mathcal{N}$ :  $N, q := \text{pQuotient}(G, 5, 100)$ ;

The order of group  $\mathcal{N}$ :  $\text{FactoredOrder}(N)$ ;

The derived subgroup of  $\mathcal{N}$ :  $D := \text{DerivedSubgroup}(N)$ ;

The derived subgroup of  $\mathcal{N}'$ :  $DD := \text{DerivedSubgroup}(N)$ ;

$\mathcal{N}/\mathcal{N}' \cong C_5^2, \mathcal{N}'/\mathcal{N}'' \cong C_5^4$  and  $\mathcal{N}''' = 1$ ;

$\text{GroupName}(N/D)$ ;  $\text{GroupName}(D/DD)$ ;  $\text{GroupName}(DD)$ ;

Construction of  $\Lambda = \text{Cay}(\mathcal{N}, S)$ :

$a := a@q$ ;  $b := b@q$ ;

$S := \{a, a^2, a^3, a^4, b, b^2, b^3, b^4\}$ ;

$\text{Lam} := \text{Cay}(N, S)$ ;

Automorphism group of  $\Lambda$ :

$A := \text{AutomorphismGroup}(\Lambda);$

$\Lambda$  is a normal Cayley graph on  $\mathcal{N}$  (We find that every Sylow 5-subgroup of  $\text{Aut}(\Lambda)$  is normal and regular on  $V(\Lambda)$ . This implies that  $\Lambda$  is normal):

$P := \text{SylowSubgroup}(A, 5);$

$\text{IsNormal}(A, P);$

$\text{IsRegular}(P);$

$\text{Aut}(\mathcal{N}, S) \cong M_{16}$  (Note that  $M_{16}$  is just the semidihedral group of order 16):

$A1 := \text{Stabilizer}(A, 1);$

$\text{GroupName}(A1);$

**Appendix 8** (Programs for the graph  $\Upsilon$  in Construction II (in case  $n = 2$ )): First, we input a group

$G \langle a, b, c, d, e, f, g, h \rangle := \text{Group} \langle a, b, c, d, e, f, g, h \mid a^3, b^3, c^9, d^9, e^9, f^9, g^3, h^3, \\ c = (a, b), d = (b, a^2), e = (a^2, b^2), f = (b^2, a), (c, d) = c^{-3} * d^3, (c, f) = c^{-3} * f^3, (d, e) = d^{-3} * e^3, \\ (e, f) = e^{-3} * f^3, f^3 = c^3 * d^{-3} * e^3, g = (c, e), h = (d, f), h = c^3 * d^3 * e^3, g^{-1} = d^3 * e^3 * f^3, \\ (c^3, d) = (c^3, e) = (c^3, f) = 1, (d^3, c) = (d^3, e) = (d^3, f) = 1, \\ (e^3, c) = (e^3, d) = (e^3, f) = 1 \rangle;$

Construction of the group  $\mathcal{R}$ :  $R, q := \text{pQuotient}(G, 3, 100);$

The order of group  $\mathcal{R}$ :  $\text{FactoredOrder}(R);$

The derived subgroup of  $\mathcal{R}$ :  $D := \text{DerivedSubgroup}(R);$

The derived subgroup of  $\mathcal{R}'$ :  $DD := \text{DerivedSubgroup}(R);$

$\mathcal{R}/\mathcal{R}' \cong C_3^5, \mathcal{R}'/\mathcal{R}'' \cong C_3^3 \times C_9$  and  $\mathcal{R}'' \cong C_3^2$ :

$\text{GroupName}(R/D); \text{GroupName}(D/DD); \text{GroupName}(DD);$

Construction of  $\Upsilon = \text{Cay}(\mathcal{R}, S)$ :

$a := a@q; b := b@q;$

$S := \{a, a^2, b, b^2\};$

$\text{Ups} := \text{Cay}(R, S);$

Automorphism group of  $\Upsilon$ :

$A := \text{AutomorphismGroup}(\text{Ups});$

$\Upsilon$  is a normal Cayley graph on  $\mathcal{R}$  (We find that every Sylow 3-subgroup of  $\text{Aut}(\Upsilon)$  is normal and regular on  $V(\Upsilon)$ . This implies that  $\Upsilon$  is normal):

$P := \text{SylowSubgroup}(A, 3);$

$\text{IsNormal}(A, P);$

$\text{IsRegular}(P);$

$\text{Aut}(\mathcal{R}, S) \cong C_4$ :

$A1 := \text{Stabilizer}(A, 1);$

$\text{GroupName}(A1);$

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