A Basic Algorithm for Generating Individualized Numerical Scale (BAGINS)

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Abstract

Linguistic labels are effective means of expressing qualitative assessments because they account for the uncertain nature of human preferences. However, to perform computations with linguistic labels, they must first be converted to numbers using a scale function. Within the context of the Analytic Hierarchy Process (AHP), the most popular scale used to represent linguistic labels numerically is the linear 1-9 scale, which was proposed by Saaty. However, this scale has been criticized by several researchers, and various alternatives are proposed in the literature. There is a growing interest in scale *individualization* rather than relying on a generic fixed scale since the perceptions of the decision maker regarding these linguistic labels are highly subjective. The methods proposed in the literature for scale individualization focus on minimizing the transitivity errors, i.e., consistency. In this research, we proposed a novel, easy-to-learn, easy-to-implement, and computationally less demanding scale individualization approach based on compatibility. We also developed an experimental setup and introduced two new metrics that can be used by researchers that contribute to the theory of AHP. To assess the value of scale individualization in general, and the performance of the proposed novel approach in particular, numerical and two empirical studies are conducted. The results of the analyses demonstrate that the scale individualization outperforms the conventional fixed scale approach and validates the benefit of the proposed novel heuristic.

Keywords: Decision Processes, Linguistic Labels, Scale Individualization, Pairwise Comparisons, Multi Criteria Decision Making (MCDM)

1. Introduction

Humans prefer using linguistic labels as opposed to numbers in order to express their opinions since uncertainty associated with their preferences is better communicated using vague verbal terms that are more intuitive and natural (Mandel et al., 2021). Due to the uncertain nature of the entire decision problem, precise numerical values are generally avoided because they may imply a sense of precision that a decision-maker does not want. (Huizingh & Vrolijk, 1997). For example, people mostly think and talk about uncertainty in terms of verbal phrases (e.g., *likely, almost certainly not*, etc.,) and are more skilled in using the rules of language as compared to employing the rules of probability (Budescu & Wallsten, 1985).

Linguistic labels are used across different domains such as business, academia, intelligence, medicine and politics in order to express preferences and/or judgments (e.g., significance, probabilities, etc.). Some examples of linguistic labels are tabulated in Table 1. The perceptions of the decision-makers regarding these linguistic labels are highly subjective, i.e., they have different meanings for different individuals. For example, the pioneering work of Sherman Kent (Kent, 1964) regarding the perception of 11 probability phrases demonstrates that when the intelligence officers are asked to quantify these phrases, there is a significant degree of deviation associated with each one of the phrases (e.g., *Probable* corresponds to $75\% \pm 12\%$, *Almost Certain* corresponds to $93\% \pm 6\%$, etc.). A similar study which is conducted with thirty financial strategy experts also concludes

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that the perceptions of probability phrases are subjective and vary across different individuals (Tavana et al., 1997).

Table 1: Linguistic labels across different domains						
Analytic Hierarchy Process (AHP) (Saaty, 1977)	Numerical Equivalent					
Equal Importance	1					
Moderate Importance	3					
Strong Importance	5					
Very Strong Importance	7					
Extreme Importance	9					
Words of Estimate Probability in Intelligence (Kent, 1964)	Percentage Equivalent					
Almost Certain	$93\% (\pm 6\%)$					
Probable	$75\% (\pm 12\%)$					
Chance About Even	$50\% \ (\pm 10\%)$					
Probably Not	$30\%~(\pm 10\%)$					
Almost Certainly Not	$7\% \ (\pm 5\%)$					
Words of Estimate Probability in Medicine (WEP, 2020)	Quantitative Equivalent					
Likely	Expected to happen to more than 50% of subjects					
Frequent	Will probably happen to $10-50\%$ of subjects					
Occasional	Will happen to $1-10\%$ of subjects					
Rare	Will happen to less than 1% of subjects					

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In this research, we focus on the set of linguistic labels used in the realm of Multi-Criteria Decision Making (MCDM), in particular used as part of the Analytic Hierarchy Process (AHP) proposed by Thomas L. Saaty (1977). Being one of the most popular methods in MCDM, this technique has found diverse applications such as the selection of cloud computing service provider (Tanoumand et al., 2017), formulating cryptocurrency mining strategies (Hacioglu et al., 2021), decision support system for real-time ambulance relocation (Hajiali et al., 2022), selection of agricultural irrigation systems (Veisi et al., 2022), task-oriented crowdsourcing recommendations (Li et al., 2021), etc. Detailed and structured reviews of the various developments of AHP are available in Ishizaka & Labib (2011) and Emrouznejad & Marra (2017).

In AHP, expert preferences are elicited in the form of pairwise comparisons. Pairwise comparison is the preferred way of eliciting human preferences as the process deals with binary evaluations and is an easier cognitive task when compared to simultaneously evaluating all objects (Choo et al., 2016). In pairwise comparisons, two objects are evaluated at a time and preference intensities are provided in the form of linguistic labels. One of the main challenges lies in computing with these linguistic labels. Generally, the linguistic labels are transformed into numbers and the computations are carried out with these numbers. Both in practice and academic research the fixed generic scale proposed by Thomas L. Saaty (1977) is the most popular approach that is used for this purpose. Although the linear scale (i.e., Saaty scale of 1-9) is the most popular scale, it has been criticized by a number of researchers and various alternative generic fixed scales are proposed. Some of the popular scales are tabulated in Table 2.

Table 2: Numerical scales to quantify lexicons

Scale	Mathematical Formulation	Parameters
Linear (Saaty, 1977)	s = x	x = 1, 2,, 9
Power (Harker & Vargas, 1987)	$s = x^2$	x = 1, 2,, 9
Root Square (Harker & Vargas, 1987)	$s = \sqrt{x}$	x = 1, 2,, 9
Geometric (Lootsma, 1993)	$s = (\sqrt{2})^{x-1}$	x = 1, 2,, 9
Asymptotic (Dodd & Donegan, 1995)	$s = tanh^{-1}\left(\frac{\sqrt{3}(x-1)}{14}\right)$	x = 1, 2,, 9
Balanced (Salo & Hämäläinen, 1997)	$s = \frac{x}{1-x}$	x = 0.5, 0.55, 0.6,, 9
Logarithmic (Ishizaka et al., 2011)	$s = \log_2(x+1)$	x = 1, 2,, 9

Representing linguistic labels with any of the *fixed* generic scales tabulated in Table 2 is also disputable. It can be claimed that the numerical interpretation of linguistic labels is not same for all individuals due to the fact that words possess different meanings for different individuals. It is important for practical and theoretical reasons to evaluate this claim in the context of AHP, analogous to the case of probabilistic phrases (e.g.,

(Kent, 1964); (Tavana et al., 1997); (Budescu & Wallsten, 1985). Unfortunately, not many empirical studies are available in the literature. Occasionally, researchers refer to Huizingh & Vrolijk (1997) to substantiate this claim. However, in that study the authors *do not* provide empirical evidence of variation in numerical interpretations of linguistic labels, but rather they explore the *consequence* of different interpretations on the quality of the AHP analysis.

The actual empirical evidence which supports the claim that the linguistic labels used during the pairwise comparison process have different meanings for different individuals is provided by Pöyhönen et al. (1997). They explored the relationship between the linguistic labels and the numbers through an experiment in which 61 participants adjusted the heights of the two bars to represent a certain linguistic label. For example, when shown the label *moderately larger*, participants adjust the height of one bar to the other such that one bar is *moderately larger* than the other bar. The results of the experiment demonstrate that the numerical counterpart for the linguistic label *moderately larger* is 1.30 ± 0.24 , *strongly larger* is 2.02 ± 0.52 and *very strongly larger* is 3.65 ± 1.89 . Thus concluding that representative numerical counterparts for the verbal expressions vary across individuals (even to a degree where they overlap with each other e.g., someone's *very strongly larger* is smaller than someone else's *strongly larger*).

Once the scale is chosen, the linguistic pairwise comparison matrices are transformed into numerical matrices via the chosen scale in order to carry out the rest of the analysis. Therefore, the chosen scale directly influences the performance of AHP, and selecting the appropriate scale needs to be carried out diligently. Finan & Hurley (1999) motivated from a compelling experiment asserts that strict reliance on the Saaty scale can induce artificial inconsistency and harm the performance of the analysis. As a result, instead of the Saaty scale, they propose the use of a geometric scale, which can be calibrated with a parameter in order to induce as little inconsistency as possible by preferences solicited with the verbal scale. This parameter is determined for each pairwise comparison matrix separately, with a process that is conducted aside from the pairwise comparisons. In this process, which is referred to as *transitive calibrations*, the decision maker also provides the interpretation of moderately important by means of a percentage (denoted as s), and the rest of the scale is computed via the geometric progression. The parametric nature of the proposed scale, which can be calibrated for each pairwise comparison matrix, entitles Finan & Hurley (1999) as the first step towards the *individualization* of the scales in AHP to the best of our knowledge. Liang et al. (2008) also developed a parametric scale that spans a continuum of various fixed scales. The parameter of the developed scale is determined for each decision maker based on the user-specified degree of tolerance to inconsistency (ε). According to their approach, given the tolerance parameter and pairwise comparisons, one can obtain transitivity inequalities, and the corresponding parameter for the individualized scale is obtained as a solution to these inequalities.

In the literature, the pioneering work that directly addresses the individualization of the scale for AHP is Dong et al. (2013). Based on the transitive calibration concept of Finan & Hurley (1999) and 2-tuple linguistic modeling of Herrera & Martínez (2000), they propose a non-linear programming model that quantifies the linguistic labels at an individual level. For each linguistic label (e.g., for Saaty Scale, there are 17 such linguistic labels, i.e., those that have the values of 1 to 9 and their reciprocals) they form a theoretical transitive calibration matrix (17×17) . Each individual entry of the theoretical matrix is a linguistic label (or *null*) which has a value equal to the transitive calibration, i.e., the multiplication of the values of the linguistic labels that are associated with the corresponding row and the column. Whenever the resulting value of multiplication is beyond the spectrum of the scale, the entry is set to be null. In the example provided in the paper, they use the Saaty Scale as the gold standard in the determination of the theoretical transitive calibration matrix. Another 17×17 matrix is constructed from the *elicited* comparison matrices which represent the *individual characteristics* of the decision maker. In this matrix, the entries are the elicited (i.e., observed) linguistic labels corresponding to the transitive calibrations for each pair of linguistic labels associated with the corresponding row and the column. The overall objective is to determine the individualized scale that minimizes the total deviation between the theoretical and the observed transitive calibration matrices in order to achieve a *consistent* matrix. In the paper, a numerical and/or empirical study is not presented that demonstrates the performance of the proposed approach, but two illustrative examples are provided instead.

Zhou et al. (2018) also links individualization of the numerical scale to the *consistency* of the numerical

preferences and proposes a two-stage consistency-driven optimization model to individualize the numerical scale. In the first stage, the pairwise comparison matrices that are formed with linguistic labels are considered and their consistencies are checked based on *transitive properties*. In the second stage, a mathematical programming model that minimizes the inconsistency index of the numerical pairwise comparison matrices is used in order to determine the individualized scale. The mathematical model's objective is to minimize the sum of the normalized log deviations from the transitive calibrations for each observed pairwise comparison. The decision variables, i.e., the individualized scales, are constrained within a particular interval around the corresponding Saaty scale. The reciprocal property and monotonicity among the linguistic labels are also observed as constraints of the model. The model was later transformed into a linear programming model in order to be solved easier. A numerical study is conducted in order to compare the performance of the proposed approach with the Saaty scale.

With the advance of digital transformation in all aspects of our daily lives, digital assistants such as Apple's Siri, Google's Assistant, Amazon's Alexa, etc. are becoming part of our families. These digital assistants act as predictive chat-bots that use machine learning, natural language processing & understanding. They learn from the user's preferences, judgments, decisions, and from that learning, they personalize their interactions and provide recommendations to the users. Apparently, in the not-so-distant future, Human-Machine teams would be a more common reality in our business lives as well. Our machine partners should understand what we actually mean when we use a particular linguistic label. There is a need for individualization since linguistic labels have different meanings for different individuals. The individualization of the scales is receiving more attention from the researchers (e.g., Dong et al. (2013), Zhou et al. (2018)). However, there is still room for alternative approaches. First of all, the available alternatives are based on mathematical programming and not only the practitioners but also many researchers usually lack the theoretical knowledge to understand and implement them. They are also computationally demanding approaches. On the other hand, they focus on the transitivity errors, i.e., consistency, however other approaches might also be applicable to the individualization process. There is also a need for numerical and empirical studies that assess the value of individualization of the numerical scales in the context of AHP. Note that, unfortunately, there is particularly a lack of empirical studies in the literature. This scarcity has been pointed out by several researchers and such empirical studies are considered as valuable contributions (e.g., Brunelli (2018), Cavallo et al. (2019), Sato & Tan (2022),).

In this study, we focus on the individualization of the numerical scales in AHP and address the gaps that are mentioned above. The major contributions of this research are as follows:

- 1. A novel heuristic approach, which is easy-to-learn, easy-to-implement, and computationally less demanding that can be used for individualization of the numerical scales in AHP is introduced.
- 2. Numerical and empirical studies are conducted to assess the value of individualized scales in the context of AHP and test the performance of the proposed heuristic.
- 3. An experimental analysis framework is developed that can be adopted particularly by the researchers who are contributing to the theory of AHP (e.g., new scales, new priority vector derivation techniques, etc.), and two new performance metrics are proposed.

The remainder of this article is arranged as follows. Section 2 discuss the basic definitions and concepts. Section 3 introduces the proposed heuristic, presents an illustrative example regarding its implementation, and provides the motivation behind it. In section 4, we present the research methodology that is adopted to compare the performance of the proposed heuristic with the alternatives. Both the details of the numerical study and the two empirical studies are provided in this section. Results and discussions are presented in section 5. Section 6 concludes this research and highlights the future research areas.

2. Preliminaries

Let's introduce the basic definitions and knowledge required to understand the rest of the discussions.

Definition 1: Linguistic Pairwise Comparison Matrix (LPCM) Let $S = (S_k | k = 1, 2, ..., m)$ be a tuple with odd cardinality such that S_k is a linguistic label and $S_i > S_j$ if and only if i > j (Herrera & Martínez, 2000). The linguistic Pairwise Comparison Matrix LPCM, $L = (l_{ij})_{n \times n}$ represents pairwise comparisons provided by the decision maker, where $l_{ij} \in S$ for i, j = 1, 2, ..., n.

Definition 2: Numerical Pairwise Comparison Matrix (NPCM) Let $A = (a_{ij})_{n \times n}$ be a NPCM such that $a_{ij} > 0$ and $a_{ij} \times a_{ji} = 1$ (Saaty, 1977) and elements of the matrix $A = (a_{ij})_{n \times n}$ represent preference intensity of the *i*th criteria when compared with the *j*th criteria. Note that, A is a positive reciprocal matrix.

Definition 3: Scale Function Let $f_{(scale)}$ be the function that maps the linguistic labels $\in S$ to $a_{ij} \in \mathbb{R}^+$. Given a numerical scale, function $f_{(scale)}$ transforms linguistic labels into numbers so that $A = f_{(scale)}(L)$. For example, $f_{(saaty)}$ will transform all linguistic labels into numbers using a scale of 1-9. On the other hand, the inverse scale function $f_{(scale)}^{-1}$ converts all numeric numbers into the corresponding linguistic labels so that $L = f_{(scale)}^{-1}(A)$. Furthermore, $f_{(scale)}^k$ denotes the real positive number corresponding to the k^{th} linguistic label (i.e., S_k) for a particular scale.

Vlaev et al. (2011) provides an extensive list of models proposed in the domain of the theory of choice and maps them to three broad categories, i.e., *Value First View, Comparison Based Theories* and *Comparison Based Theories without Internal Scale.* According to this classification, well-known decision theories such as Utility Theory (Morgenstern & Von Neumann, 1953) and Prospect Theory (Kahneman & Tversky, 1979) fall under the category of *Value First View.* According to *Value First View*, the brain computes the value of alternatives and favors the ones with higher values. AHP assumes an underlying additive multi-attribute utility function in order to rank the alternatives in a multi-criteria setting. That is to say, according to AHP, a *true priority vector* exists that corresponds to the preferences of the decision maker in a particular context.

Definition 4: True Priority Vector $V = [v_1, v_2, ..., v_n]$ is the true priority vector that corresponds to the preference of the decision maker in a particular context.

Definition 5: Calculated Priority Vector AHP utilizes $L = (l_{ij})_{n \times n}$ and determines the priority vector $w = [w_1, w_2, ..., w_n]$ which corresponds to the weight of the criteria or score of the alternatives in terms of each criterion. Saaty (1977) proposes that w is the eigenvector associated with the maximum eigenvalue (λ_{max}) of the $A = f_{(Saaty)}(L)$ matrix. After the pioneering work of Saaty, some other approaches for priority vector derivation is also proposed in the literature (e.g., Logarithmic Least Squares Method (LLSM) (Crawford & Williams, 1985), Mean of Normalized Values Heuristics, etc.).

AHP aims to determine the *true* priority vector via the pairwise comparisons elicited from the decisionmakers. This process is illustrated in Figure 1.



Figure 1: Analytic Hierarchy Process

Definition 6: Consistency Index of NPCM Consistency Index denoted by CI is defined by Saaty as

follows:

$$CI = \frac{\lambda_{max} - n}{n - 1} \tag{1}$$

Definition 7: Consistency Ratio of NPCM Let RI be the Random Index which is the average of the CIs of the randomly generated NPCMs, then Consistency Ratio is defined by Saaty as $CR = \frac{CI}{RI}$. If A is sufficiently consistent then $CR \leq 0.1$ and if A is fully consistent, then CR = 0 (Saaty, 1977).

In order to address the consistency of NPCM, various other indices are proposed in the literature in addition to the one proposed by Saaty. Comprehensive reviews of various inconsistency indices are available in Brunelli (2018) and Sato & Tan (2022).

Definition 8: Theoretical Numerical Pairwise Comparison Matrix (TNPCM) Let priority vector $w = [w_1, w_2, ..., w_n]$ be calculated from $A = (a_{ij})_{n \times n}$. Then $W = (w_{ij})_{n \times n}$ is referred to as the theoretical numerical pairwise comparison matrix of A.

$$W = \begin{pmatrix} w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \cdots & w_2/w_n \\ \vdots & \vdots & \ddots & \vdots \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{pmatrix}$$
(2)

Definition 9: True Theoretical Numerical Pairwise Comparison Matrix (TTNPCM) Let priority vector $v = [v_1, v_2, ..., v_n]$ be the true priority vector. Then $V = (\frac{v_i}{v_j})_{n \times n}$ is referred to as the true theoretical pairwise comparison matrix.

Definition 10: Compatibility Index Value The Hadamard product (i.e., element-wise product) of matrix A and W^T is represented as $e^T A \circ W^T e$, where e is the vector of ones of size n. Then, the Compatibility Index Value (CIV) is defined as follows (Saaty, 1994);

$$CIV = n^{-2} \cdot e^T A \circ W^T e \tag{3}$$

Note that $e^T X e$ for any matrix X would be equal to the sum of all of the elements of the matrix X. Thus, we can explicitly write Equation 3 as follows:

$$CIV = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \cdot \frac{w_j}{w_i}$$
(4)

A fully consistent matrix A and the theoretical pairwise comparison matrix W constructed from the priority vector w of A is exactly same. Therefore, the CIV of A and W would be equal to 1 (see Equation 4). For matrix A which is inconsistent, CIV will be greater than 1 (Saaty, 1994).

In the framework of Saaty, where the weights are determined to be the principal right eigenvectors (i.e., $A \times w = \lambda_{max} \times w$), one can easily show that $CIV = \frac{\lambda_{max}}{n}$ (Due to the definition of eigenvector: $\sum_{j=1}^{n} a_{ij} \cdot w_j = \lambda_{max} \cdot w_i$). Therefore,

$$CIV = n^{-2} \cdot e^T A \circ W^T e = \frac{\lambda_{max}}{n}$$
(5)

Recall that CI deals with a single matrix (e.g., A) and measures how much the transitivity of the preferences is violated. On the other hand, CIV deals with the deviation between two matrices (e.g., A and W) that represent two priority vectors. However, As Equation 1 and Equation 5 hints, *CI* and CIV are interrelated measures. The relation between consistency and compatibility is nicely put forward by Saaty (1994) as: "... Consistency is concerned with the compatibility of a matrix of the ratios constructed from a principal right eigenvector with the matrix of judgments from which it is derived.". This interrelation also can be mathematically observed as follows:

$$CI = \frac{\lambda_{max} - n}{n - 1} \Longrightarrow \lambda_{max} = (n - 1).CI + n \Longrightarrow CIV = \frac{(n - 1)}{n}.CI + 1$$
(6)

In practice, the true priority vectors are unknown. Therefore, all we can use is the A matrix and/or the associated W matrix. As the above discussion indicates, analysis based on CI or CIV would actually be identical. However, in order to compare the performance of new proposals to the theoretical framework of AHP we can design experimental studies (numerical and/or empirical) in which we have the true priority vectors as well. Therefore, we are going to introduce a *generic* compatibility index which we can use as a performance metric in such experimental analysis.

Definition 11: Generic Compatibility Index Value (GCIV) Let X and Y be two $n \times n$ matrices. The Hadamard product of matrix X and Y^T is represented as $e^T X \circ Y^T e$. Then, the GCIV is formulated as follows;

$$GCIV = n^{-2} \cdot e^T X \circ Y^T e \tag{7}$$

In the context of compatibility in AHP, there are three possible GCIVs that can be considered. One of them corresponds to the CIV of Saaty introduced in definition 9 and is represented as GCIV-AW in Figure 2. The second one is GCIV-VW which addresses the similarity between matrix V derived from *true* priority vector v and the matrix W derived from *calculated* priority vector w. And the third one is GCIV-AV which addresses the similarity between matrix V derived from *true* priority vector v. In a perfectly consistent setting, all these GCIVs would be equal to one. The following sub-section provides an illustrative example for all the definitions presented above.



Figure 2: AHP & Compatibility

2.1. Illustrative Example

Let's assume that five criteria are to be evaluated and the true priority vector for these five criteria is given as v = [0.40, 0.30, 0.20, 0.05, 0.05]. Note that in most cases, this true priority vector will be unknown and the objective of AHP is to assess this vector. Given this true priority vector, let's assume that the linguistic pairwise comparison matrix elicited from the decision maker is as follows:

$$L = \begin{pmatrix} S9 & S10 & S11 & S15 & S16\\ S8 & S9 & S10 & S13 & S14\\ S7 & S8 & S9 & S14 & S13\\ S3 & S5 & S4 & S9 & S12\\ S2 & S4 & S5 & S6 & S9 \end{pmatrix}$$
(8)

Matrix L can be converted into an NPCM using a scale function $f_{(scale)}$ among those that are represented in Table 2. Given a Saaty scale (1 - 9), Matrix L would be converted to NPCM such that $A = f_{(Saaty)}(L)$ as follows:

$$A = \begin{pmatrix} 1 & 2 & 3 & 7 & 8\\ 1/2 & 1 & 2 & 5 & 6\\ 1/3 & 1/2 & 1 & 6 & 5\\ 1/7 & 1/5 & 1/6 & 1 & 4\\ 1/8 & 1/6 & 1/5 & 1/4 & 1 \end{pmatrix}$$
(9)

As proposed by Thomas L. Saaty (1977), calculated priority vector is the eigenvector corresponding to the maximum eigenvalue ($\lambda_{max} = 5.3436$) of A. This vector is calculated as w = [0.4329, 0.2671, 0.1975, 0.0669, 0.0356]. Also, the CI of the matrix A presented in definition 5 is calculated as $\frac{(5.3436-5)}{(5-1)} = 0.0859$. Furthermore, CR presented in definition 6 can be calculated as $\frac{0.0859}{1.12} = 0.0767$, whereas the value of Random Index (R.I) is 1.12 Saaty (1977).

From the *calculated* priority vector w = [0.4329, 0.2671, 0.1975, 0.0669, 0.0356], TNPCM W presented in definition 7 is given as follows;

$$W = \begin{pmatrix} 1.00 & 1.62 & 2.19 & 6.47 & 12.14 \\ 0.62 & 1.00 & 1.35 & 3.99 & 7.49 \\ 0.46 & 0.74 & 1.00 & 2.95 & 5.54 \\ 0.15 & 0.25 & 0.34 & 1.00 & 1.88 \\ 0.08 & 0.13 & 0.18 & 0.53 & 1.00 \end{pmatrix}$$
(10)

Furthermore, from the *true* priority vector v = [0.40, 0.30, 0.20, 0.05, 0.05], TTNPCM V presented in definition 8 is given as follows;

$$V = \begin{pmatrix} 1.00 & 1.33 & 2.00 & 8.00 & 8.00 \\ 0.75 & 1.00 & 1.50 & 6.00 & 6.00 \\ 0.50 & 0.67 & 1.00 & 4.00 & 4.00 \\ 0.13 & 0.17 & 0.25 & 1.00 & 1.00 \\ 0.13 & 0.17 & 0.25 & 1.00 & 1.00 \end{pmatrix}$$
(11)

From definitions 9 & 10, the similarity between any pair of matrices can be measured through a GCIV. In Figure 2, various measures of GCIV are presented. These values are calculated as GCIV-AV = 1.1174, GCIV-AW = 1.0687 and GCIV-VW = 1.0443.

3. Heuristic for Individualization of Numerical Scale

As discussed earlier, a fixed generic scale to convert linguistic labels into numbers diminishes the true meaning of the preferences of the decision maker. Scale individualization is one approach through which this issue can be addressed. Most of the previous studies (Dong et al. (2013); Zhou et al. (2018)) on scale individualization focus on reducing the inconsistency of the numerical pairwise comparison matrix. However, efforts to reduce inconsistency in a pairwise comparison matrix can distort the true meaning of the preferences in such a way that it no longer represents decision-maker preferences. A distinction should be made between the consistency of the preferences and the validity of the underlying decision process. Improving the consistency of an NPCM does not necessarily improve the validity of the results and thus consistency improving methods could be misleading (Saaty & Tran, 2007).

We utilize the special structure of pairwise comparison matrices to further extend the process illustrated in Figure 1 and propose a Basic Algorithm for Generating an Individualized Numerical Scale (BAGINS) to generate an individualized numerical scale. While comparing two objects, participants can accurately provide the rank or the order relationship, but capturing the degree of certitude is difficult. The novel heuristic proposed in this study measure this degree of certitude by constructing an individualized numerical scale during the process of estimating priority vectors from pairwise comparison matrices. This heuristic is presented in Algorithm 1 and graphically illustrated in Figure 3.

Algorithm 1 Basic Algorithm for Generating Individualized Numerical Scale (BAGINS)

1: Let $S = (S_k \mid k = 1, 2, ..., m)$ 2: Elicit $L = (l_{ij})_{n \times n}$ where $l_{ij} \in S$ 3: Construct $A = (a_{ij})_{n \times n}$ such that $A = f_{(Saaty)}(L)$ 4: Calculate w such that $A \times w = \lambda_{max} \times w$ 5: Construct $W = (w_{ij})_{n \times n}$ where $w_{ij} = \frac{w_i}{w_i}$ 6: Initialize $f_{(ind)}^k = 0$ for k = 1, 2, ..., m7: $f_{(ind)}^{rac{m+1}{2}} = 1$ 8: for $k = \frac{m+1}{2} + 1, ..., m$ do 9: Initialize $T_k = \{\}$ for $\forall i, j$ of L do 10: if $l_{ij} = S_k$ then 11: $T_{k} = T_{k} \cup (i, j)$ $T_{k} = max \left\{ \frac{\sum_{(i,j) \in T_{k}}^{w_{ij}}}{|T_{k}|}, f_{(ind)}^{k-1} \right\} \forall (i,j) \in T_{k}$ 12:13: 14: for $k = 1, 2, ..., \frac{m-1}{2}$ do 15: $f_{(ind)}^k = \left(f_{(ind)}^{\frac{m+1}{2}+k}\right)^{-1}$ 16: $f_{(ind)} \rightarrow Individualized Scale$

The Algorithm 1 takes preferences in the form of linguistic labels as input and generates a personalized numerical scale $f_{(ind)}^k$ for k = 1, 2, ..., m. S is a tuple representing linguistic labels and L is elicited such that $l_{ij} \in S$ for i, j = 1, 2, ..., n. As part of the process, L is converted to A using the Saaty scale so that $A = f_{(Saaty)}(L)$. After calculating the eigenvector w corresponding to the maximum eigenvalue (λ_{max}) of A, a theoretical pairwise comparison matrix W is constructed. Recall that in the Saaty scale, we have 17 linguistic labels, where S_9 corresponds to "equals", i.e., $f_{(Saaty)}^9 = 1$. Algorithm 1 first determines an individualized scale for linguistic labels S_9 to S_{17} . For the remaining linguistic labels (i.e., S_1 to S_8), the reciprocal property is observed. Line 8 sets the linguistic label corresponding to "equal" to have a value of 1. From lines 9 to 12, we determine the (i, j) pairs, where the entry of the L matrix is S_k or $S_{(m-k+1)}$ (i.e., the reciprocal counterpart of S_k) and construct a set of these ordered pairs referred to as T_k . Line 13 averages the values of the (i, j)entries of the matrix W for all ordered pairs that are in set T_k . The maximum operator in this line ensures the monotonicity of the scale. Lines 14 and 15 ensure that the scale is reciprocal.



Figure 3: A Basic Algorithm for Generating Individualized Numerical Scale (BAGINS)

3.1. Illustrative Example

Recall the illustrative example presented in subsection 2.1, a hypothetical linguistic pairwise comparison matrix elicited from the decision maker (i.e., matrix L) is provided by Equation 8. Suppose that the numerical counterparts of the linguistic labels that the decision maker utilizes while providing the preferences are represented by the inherent scale which is presented in Table 4. The corresponding A matrix constructed from the Saaty scale is given in Equation 9. The eigenvector corresponding to the maximum eigenvalue of A referred to as the *calculated* priority vector is determined as w = [0.4329, 0.2671, 0.1975, 0.0669, 0.0356] and the theoretical pairwise comparison matrix W constructed from this *calculated* priority vector is given by Equation 10. This theoretical matrix W leads to the construction of individualized scales i.e., the numerical counterpart of the linguistic labels (S_i) .

For example, l_{12} and l_{23} corresponds to S_{10} in matrix L. The corresponding numbers in matrix W (w_{12} and w_{23}) are 1.62 and 1.35. Therefore, the average of these two numbers is determined as the numerical counterpart of the linguistic label S_{10} i.e., $S_{10} = 1.49$. The complete individualized numerical scale obtained from this process is provided in Table 4. Using this scale, a new individualized NPCM is determined as follows:

$$A_{ind} = \begin{pmatrix} 1.00 & 1.49 & 2.19 & 6.47 & 12.14 \\ 0.67 & 1.00 & 1.49 & 4.76 & 5.22 \\ 0.46 & 0.67 & 1.00 & 5.22 & 4.76 \\ 0.15 & 0.21 & 0.19 & 1.00 & 2.19 \\ 0.08 & 0.19 & 0.21 & 0.46 & 1.00 \end{pmatrix}$$
(12)

The eigenvector corresponding to the maximum eigenvalue of the matrix A_{ind} is now referred to as the *calculated* priority vector. This vector is calculated as w = [0.4215, 0.2665, 0.2130, 0.0603, 0.0387]. Various GCIV measures that are presented in Section 2 will be used as the performance metric to compare different scales. For this illustrative example, the GCIV measures are tabulated in Table 3.

Table 3: Generalized Compatibility Index Values									
Scales/GCIV	GCIV-AV	GCIV-AW	GCIV-VW						
Saaty scale	1.1174	1.0687	1.0443						
Inherent scale	1.0501	1.0272	1.0229						
Individualized	1.0426	1.0175	1.0245						

Table 4:	Numerical	scales	used	to	construct	NP	CM
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Linguistic Label	Saaty Scale	Inherent Scale	Individualized Scale
$S_1 = \text{extremely less important}$	0.11	0.08	0.08
$S_2 =$ very, very strongly less important	0.13	0.11	0.08
$S_3 = $ demonstratedly less important	0.14	0.13	0.15
$S_4 = $ strongly plus less important	0.17	0.17	0.19
$S_5 = \text{strongly less important}$	0.20	0.20	0.21
$S_6 = $ moderately plus less important	0.25	0.40	0.46
$S_7 = $ moderately less important	0.33	0.50	0.46
$S_8 =$ weakly less important	0.50	0.67	0.67
$S_9 = $ equally important	1.00	1.00	1.00
S_{10} = weakly more important	2.00	1.50	1.49
$S_{11} = $ moderately more important	3.00	2.00	2.19
S_{12} = moderately plus more important	4.00	2.50	2.19
$S_{13} = $ strongly more important	5.00	5.00	4.76
$S_{14} = $ strongly plus more important	6.00	6.00	5.22
$S_{15} = $ demonstratedly more important	7.00	8.00	6.47
$S_{16} = $ very; very strongly more important	8.00	9.00	12.14
S_{17} = extremely more important	9.00	12.00	12.14

3.2. Motivation Behind BAGINS

Earlier research on scale individualization (Dong et al. (2013), Zhou et al. (2018)) aims to reduce the inconsistency of the numerical pairwise comparison matrix where the focus is on deciding the individualized scale that enhances the transitivity relations in the matrix A, i.e., $a_{ij} \times a_{jk} = a_{ik} \forall i, j, k$. BAGINS takes an alternative approach where the objective is choosing the individualized scales with the aim that the compatibility between matrix A and matrix W (recall Equation 3) is improved so that:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \cdot \frac{w_j}{w_i} = n^2 \tag{13}$$

Note that the sum in Equation 13 holds iff $a_{ij} = \frac{w_i}{w_j}$ (Saaty, 1994), i.e., GCIV-AW=1. The proof is simple when one considers that due to the reciprocal structure of the left hand side term in Equation 13, it can be represented as $\frac{n^2}{2}$ pairs of terms of the convex form $x + \frac{1}{x}$ each of which has a minimum value of 2. Therefore, the minimum value of the double summation is attained when $a_{ij} = \frac{w_i}{w_j}$, i.e., the compatibility is equal to 1.

As a result of this observation, BAGINS targets a proxy optimization problem that aims to minimize the following objective function:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} - \frac{w_i}{w_j})^2 \tag{14}$$

Equation 14 aims to minimize the sum of the squared deviations between the a_{ij} and $\frac{w_i}{w_j}$, which in turn improves the compatibility between matrix A and matrix W. Note that the sample mean is the value that minimizes the sum of the squared deviations. This is taken into consideration in BAGINS at Line 13 which averages the values of the (i, j) entries of the matrix W for all ordered pairs that are in set T_k .

Targeting the compatibility as opposed to the transitivity is what sets BAGINS apart from the previous scale

individualization algorithms. In that aspect, BAGINS objective is similar to the objective of the LLSM (Crawford & Williams, 1985) algorithm which was proposed as one of the earliest alternatives to Saaty's eigenvector approach for deriving the priority vector. The objective of LLSM is as follows:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} [\ln a_{ij} - (\ln w_i - \ln w_j)]^2$$
(15)

LLSM's decision variables are the weights (i.e., $w_i \forall i$) and assume Saaty's fixed scale as the gold standard while converting the linguistic pairwise comparison matrix into the numerical pairwise comparison matrix (i.e., determining the a_{ij}). On the other hand, BAGINS objective is determining the individualized scale that will be used to determine the numerical pairwise comparison matrix (i.e., determining the a_{ij}) so that the compatibility of the matrix A and the TNPCM derived from it (i.e., W) is minimized. That is to say, LLSM determines the priority vector and assumes the fixed Saaty scale, while BAGINS determines the individualized scale and assumes the priority vectors determined by Saaty's eigenvector approach.

4. Research Methodology

In this study, we will evaluate the performance of three methods based on a synthetically created *Numerical Dataset* as well as two *Empirical Datasets*. The methods and the corresponding nomenclature used in this research for each of the three methods are tabulated in Table 5.

Table 5: Nomenclature of methods

Method	Abbreviation
Original Method based on linear scale proposed by Thomas L. Saaty (1977)	Saaty
Individualization using the heuristic proposed in this study	BAGINS
Individualization based on Nonlinear Programming (NLP) model proposed by Dong et al. (2013)	NLP

The details of the numerical and empirical studies are provided in the following subsections.

4.1. Numerical Dataset

A numerical study is conducted through a dataset of linguistic pairwise comparison matrices. For this purpose, a methodology is developed that imitates the process of an expert providing pairwise comparisons. In practice, pairwise comparisons are elicited from the experts in the form of linguistic labels. It is assumed that during this elicitation process, they utilize a certain weight vector, an inherent scale (e.g., similar to the one provided in Table 4) and provide preferences in the form of pairwise comparisons. Due to various reasons such as the inconsistency of the expert, the vagueness associated with the natural language, and the dubious nature of the scale used to quantify these linguistic labels, the resulting NPCM is an inconsistent matrix. Therefore, in order to mimic this elicitation process, which also includes inconsistency, a numerical dataset is generated which consists of NPCMs of four different matrix sizes i.e., n = 3, 7, 11, 15 and three levels of inconsistencies i.e., Low, Medium, High.

A random normalized vector w_1, w_2, \dots, w_n is generated that corresponds to the weight vector of an expert evaluating *n* criteria. Using this random weight vector and Equation 16, a fully consistent pairwise comparison matrix is generated.

$$W = \begin{pmatrix} w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \cdots & w_2/w_n \\ \vdots & \vdots & \ddots & \vdots \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{pmatrix}$$
(16)

As previously stated, pairwise comparison matrices in real life applications almost always consist of inherent inconsistency. Therefore, to represent real elicitation process, inconsistency is synthetically added into pairwise comparison matrices using a parameter $\beta \in \{0, 0.2, 0.4, 0.6, 0.8, 1.0\}$. For each number in the matrix a_{ij} representing pairwise comparison, an interval $[a \ b]$ is created, such that $a = w_i/w_j - \beta \times w_i/w_j$ and $b = w_i/w_j + \beta \times w_i/w_j$. From this interval $[a \ b]$, a number x_{ij} is randomly chosen and replaced with the corresponding a_{ij} in the matrix. The reciprocal property of the pairwise comparison matrices is preserved during this process of adding inconsistency.

Note that due to the randomness in the methodology employed to incorporate inconsistency, a numerically generated pairwise comparison matrix can have a different level of inconsistency than initially intended through the β parameter. In order to address this issue, the real inconsistency of matrices is calculated through CR = CI/RI, and then this CR measure is used for the classification of matrices on different levels of inconsistency. A CR value between 0 - 0.03 corresponds to *low level* inconsistent matrices. Similarly, a CR value between 0.03-0.06 corresponds to *medium level* inconsistent matrices. While all matrices with CR between 0.06-0.1 are classified as a *high level* inconsistent matrices. Any matrix with a CR value ≥ 0.1 is regarded as not sufficiently consistent (Saaty, 1977) and therefore such matrices are discarded from the data set.

Next, an inverse scale function and Saaty scale of 1-9 is used to transform this data set of inconsistent NPCM into LPCM. Weight vectors generated at the start of this process are considered as true weights and numerically generated LPCMs are considered as preferences elicited from the expert. This procedure is summarized in Algorithm 1.

Algorithm 2 Generate Synthetic Pairwise Comparison Matrices

1: n=3 2: while $n \leq 15$ do True Weights \leftarrow Generate n Random Weight which sums upto 1 3: $W = \{w_i / w_j\}$ 4: $\beta = 0$ 5: while $\beta \leq 1$ do 6: $NPCM = Add \ Inconsistency(NPCM)$ 7: $LPCM = f^{-1}(NPCM)$ 8: $\beta = \beta + 0.2$ 9: n = n + 410: 11: FinalDataset = sort(LPCM)

The initial data set generated through Algorithm 1 is composed of 4,800 pairwise comparison matrices (i.e., 4 number of matrix sizes \times 6 number of β parameter \times 200 replications). However, due to the uneven distribution of the *low-level* and *high-level* inconsistent matrices, the dataset is reclassified based on the *CR* value (instead of β). After reclassification, the initial data set of 4,800 matrices is reduced to a total of 900 matrices (i.e., 4 number of matrix sizes \times 3 consistency levels \times 75 replications) which are *evenly* distributed on all experimental parameters. For each combination of *n* and *CR*, 75 replications were made and thus *final data set* is composed of 900 (= 4 \times 3 \times 75) matrices.

4.2. Empirical Dataset

We conducted two experiments for which true priority vectors can be measured due to the inherent natural scales in the experiments. The first experiment deals with visual observations in which participants were shown two images at a time and asked to provide pairwise comparisons in the form of linguistic labels such as *"Extremely Dense"*, *"Moderately Dense"* etc. For this experiment, nine different images (Figure 4) were employed. In the second experiment consisting of nine different bottles, participants were asked to weigh two bottles at a time and provide their preferences in the form of linguistic labels such as *"Extremely Heavy"*, *"Moderately Heavy"*, etc. Bottles were covered with black paint so that visual observations did not impact their preferences.

Figure 4: Visual Experiment to seek pairwise comparisons of different densities

The participants in both studies were undergraduate students enrolled at Sabanci University, Istanbul, Turkey. In the visual experiment, there were 164 participants; in the mass experiment, 154 participants. For both experiments, a total of 36 comparisons were done by each student and there was no specific ordering of the comparisons. For their participation in the experiments, students received 2 bonus points for their course grades. To ensure their focus during the experiments, an additional 1 bonus point was promised for the top performing participants in terms of their consistency index value.

Both experiments were approved by the university ethics committee and written consent of the students was obtained prior to participation in the study. The average time required for the visual experiment and the mass experiment was approximately 15 minutes and 20 minutes, respectively per participant. Since both experiments had an underlying natural scale (i.e., for the visual experiment, the number of dots; for the mass experiment, grams), it was possible to assess the *true weights* and corresponding weight vector for both experiments as tabulated in Table 6.

	0	
Number of Dots	Mass of Bottles (Grams)	Weight Vector
10	50	0.0222
20	100	0.0444
30	150	0.0667
40	200	0.0889
50	250	0.1111
60	300	0.1333
70	350	0.1556
80	400	0.1778
90	450	0.2000

Table 6: True normalized weight vector for visual and mass experiment

For researchers and practitioners working in similar domains, the numerical and empirical datasets of pairwise comparison matrices along with necessary Matlab codes and documentation are made available online at (Ahmed, 2022).

5. Results and Discussion

In this section, we present and discuss the results of both the numerical and empirical studies. We considered all three Generic Compatibility Index Values that were introduced in Section 2 (i.e., GCIV-AW, GCIV-VW, GCIV-AV) as the performance metrics of the methods that are considered in this study. Recall that GCIV-AW is the original CIV, which was introduced by Saaty and interrelated with the CI metric as we have discussed earlier. On the other hand, both for the numerical study and the empirical studies we also have the true priority vectors. Therefore, it is also possible to determine the GCIV-VW and GCIV-AV while comparing the performance of the three methods that are tabulated in Table 5.

5.1. Results of the Numerical Study

Table 7 tabulates the mean GCIV-AW, GCIV-VW and GCIV-AV for different matrix sizes. Results demonstrate that the mean CIV between the priority vector and the corresponding NPCM (i.e., GCIV-AW), is minimum for the NLP for all matrix sizes (i.e., n = 3, 7, 11, and 15). The results from the LSD post hoc test (Table 8) indicate that these differences are statistically significant ($P \leq 0.05$). This result is not surprising due to the objective function of the NLP. The NPCMs constructed from the LP model is highly consistent and therefore the priority vector calculated from these matrices would easily yield minimum CIV when compared with the corresponding NPCM. On the other hand, BAGINS also outperforms Saaty in this metric and the difference is statistically significant for all matrix sizes.

Mean GCIV-VW, which addresses the deviation of the calculated priority vector from the true priority vector is tabulated in Table 7. For the smallest matrix size (i.e., n = 3), all of the three methods perform similarly and there are no statistically significant differences among them (Table 8). However, for larger matrix sizes (i.e., n = 7, 11, and 15), the NLP method is significantly outperformed by the other two methods (i.e., Saaty and BAGINS). There isn't a statistically significant difference between Saaty and BAGINS when n = 7. Saaty significantly outperforms BAGINS only for large matrix sizes (i.e., n = 11 and 15). That is to say, in terms of the GCIV-VW metric, Saaty is the best performing method and NLP is the worst performing method for different matrix sizes.

The third comparison is in terms of the mean GCIV-AV which demonstrates the deviation between the true priority vector and the matrix constructed from a particular scale. Again for the smallest matrix size (i.e., n = 3) there is no statistically significant difference among the three methods. Table 8 reveals that as the size of the matrix is increased (i.e., n = 7, 11, and 15), the BAGINS outperforms NLP and the differences in the mean GCIV-AV are statistically significant ($P \le 0.05$). BAGINS also outperforms Saaty statistically significantly for n = 11 and 15).

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Performance Metric	Method	Ν	Mean $(n = 3)$	Mean $(n = 7)$	Mean $(n = 11)$	Mean $(n = 15)$
	NLP	225	1.00217	1.00789	1.01048	1.01411
GCIV-AW	Saaty	225	1.01936	1.05830	1.07163	1.07942
	BAGINS	225	1.00697	1.02591	1.02945	1.03133
	NLP	225	1.07531	1.07959	1.08061	1.08200
GCIV-VW	Saaty	225	1.05852	1.02675	1.01916	1.01597
	BAGINS	225	1.05850	1.03210	1.03015	1.03154
	NLP	225	1.07904	1.08796	1.09188	1.09685
GCIV-AV	Saaty	225	1.08010	1.08579	1.09105	1.09529
	BAGINS	225	1.06602	1.05869	1.06041	1.06348

Table 7: GCIV-AW, GCIV-VW, GCIV-AV for different matrix sizes. (Bold figures represents the best results)

Table 8: Post Hoc LSD Test for different matrix sizes.

Performance	Method	Method	Mean Dif	f. _{Cim}	Mean Diff.	C:	Mean Diff.	C:	Mean Diff.	C:
Metric	(I)	(J)	(I-J) $n = 3$	Sig.	(I-J) $n = 7$	Sig.	(I-J) $n = 11$	Sig.	(I-J) $n = 15$	Sig.
	NLP	Saaty	-0.01719	0.000	-0.05041	0.000	-0.06115	0.000	-0.06531	0.000
GCIV-AW	NLP	BAGINS	S -0.00480	0.000	-0.01802	0.000	-0.01896	0.000	-0.01722	0.000
	Saaty	BAGINS	50.01238	0.000	0.03239	0.000	0.04218	0.000	0.04809	0.000
	NLP	Saaty	0.01679	0.246	0.05284	0.000	0.06145	0.000	0.06603	0.000
GCIV-VW	NLP	BAGINS	50.01681	0.225	0.04749	0.000	0.05046	0.000	0.05046	0.000
	Saaty	BAGINS	S 0.00002	1.000	-0.00534	0.095	-0.01099	0.000	-0.01556	0.000
	NLP	Saaty	-0.00106	0.996	0.00217	0.877	0.00083	0.980	0.00156	0.933
GCIV-AV	NLP	BAGINS	50.01301	0.462	0.02927	0.000	0.03147	0.000	0.03338	0.000
	Saaty	BAGINS	S0.01408	0.528	0.02711	0.095	0.03064	0.000	0.03181	0.000

Table 9 tabulates mean GCIV-AW, GCIV-VW and GCIV-AV at different levels of inconsistencies. Mean CIV between the priority vector and the corresponding NPCM (GVIV-AW) is minimum for NLP at all of the three levels of inconsistency similar to the previous analysis. Table 10 indicates that these differences are statistically significant ($P \leq 0.05$).

Deviation of the calculated priority vector from the true priority vector (GCIV-VW) is minimum for Saaty and BAGINS and results from Table 10 depict that both these methods significantly outperform NLP at different levels on inconsistency. In terms of GCIV-VW, the difference between Saaty and BAGINS is not statistically significant for different levels of inconsistencies.

For GCIV-AV, which addresses the deviation between the true priority vector and the matrix constructed from a particular scale, the BAGINS method provides better results as compared to the other two methods. At low levels of inconsistency, both BAGINS and Saaty significantly outperform NLP, at medium inconsistency levels, BAGINS significantly outperforms the other two methods, while at higher levels of inconsistency, BAGINS and NLP significantly outperform Saaty.

Performance Metric	Method	Ν	Mean $(CR = low)$	Mean $(CR = medium)$	Mean $(CR = high)$
	NLP	225	1.00219	1.00877	1.01503
GCIV-AW	Saaty	225	1.01597	1.05621	1.09935
	BAGINS	225	1.01115	1.02660	1.03250
	NLP	225	1.07851	1.07782	1.08180
GCIV-VW	Saaty	225	1.01088	1.03183	1.04760
	BAGINS	225	1.01347	1.04052	1.06023
	NLP	225	1.08066	1.08817	1.09796
GCIV-AV	Saaty	225	1.02677	1.08886	1.14854
	BAGINS	225	1.02456	1.06776	1.09412

Table 9: GCIV-AW, GCIV-VW, GCIV-AV for different levels of inconsistency (Bold figures represents the best results)

Table 10: Post Hoc LSD Test for different levels of inconsistency - (GCIV-AW)

Performance	Method	Method	Mean Diff. (I-J)	C:	Mean Diff. (I-J)	Q:	Mean Diff. (I-J)	C:	
Metric	(I)	(J)	CR = low	Sig.	CR = medium	Sig.	CR = high	Sig.	
	NLP	Saaty	-0.01378	0.000	-0.04744	0.000	-0.08431	0.000	
GCIV-AW	NLP	BAGINS	-0.00896	0.000	-0.01783	0.000	-0.01747	0.000	
	Saaty	BAGINS	0.00483	0.000	0.02961	0.000	0.06684	0.000	
	NLP	Saaty	0.06764	0.000	0.04599	0.000	0.03420	0.000	
GCIV-VW	NLP	BAGINS	0.06505	0.000	0.03729	0.000	0.02158	0.000	
	Saaty	BAGINS	-0.00259	0.517	-0.00869	0.470	-0.01263	0.079	
	NLP	Saaty	0.05389	0.000	-0.00069	0.995	-0.05057	0.000	
GCIV-AV	NLP	BAGINS	0.05610	0.000	0.02041	0.009	0.00384	0.683	
	Saaty	BAGINS	0.00221	0.627	0.02110	0.014	0.05442	0.000	

The results from the numerical study demonstrate that the two methods that are based on individualized scales (i.e., NLP or BAGINS) outperforms the conventional Saaty method which is based on fixed scale in terms of GCIV-AW and GCIV-AV metrics. The differences are always statistically significant regardless of the matrix sizes and the levels of inconsistencies in the case of GCIV-AW. On the other hand, for the case of GCIV-AV the results are statistically significant for larger matrix sizes (i.e., n = 11 and 15) and when there is *Medium* or *High* inconsistency in the pairwise comparison matrices.

Among the two methods that are based on individualization NLP outperforms BAGINS for GCIV-AW as expected due to its objective function. However for the other two metrics (i.e., GCIV-VW and GCIV-AV), BAGINS outperforms NLP on the average regardless of the matrix sizes and inconsistency levels. In terms of GCIV-VW, BAGINS outperforms NLP statistically significantly for all cases. On the other hand, for GCIV-AV, there is no statistically significant difference between the algorithms only for the smallest matrix size (i.e., n = 3) and *High* level of inconsistency. For the remaining cases, BAGINS outperforms NLP statistically significantly.

5.2. Results of the Empirical Study

Numerical studies based on synthetic data are highly valuable in order to analyze the performance of alternative algorithms for different experimental conditions (e.g., different matrix sizes, different levels of inconsistency). Compared to a numerical study, an empirical study is constrained by the conditions of a particular context, thus, the conclusions. Having said that, even though empirical studies represent a particular instance of a bigger picture, they are *real* and science aims to give meaning to reality. Thus, empirical studies are invaluable for the development of the theory of AHP as indicated by various researchers (e.g., Brunelli (2018), Cavallo et al. (2019), Sato & Tan (2022)).

With this motivation, we wanted to compare the performance of the three alternative methods, namely, NLP, Saaty and BAGINS with two empirical studies (Visual Experiment and Mass Experiment). We again used GCIV-AW, GCIV-VW and GCIV-AV as three performance metrics in the analysis. The means of the performance metrics for both of the experiments are tabulated in Table 11. The results imply that in both of the experiments the methods that are based on individualized scales outperform Saaty's method which is based on the fixed scale in both of the experiments. The difference between the performance of NLP and Saaty for all three metrics is statistically significant in both experiments. On the other hand, BAGINS outperforms Saaty in terms of GCIV-AW for both of the experiments statistically significantly. For the other two metrics (namely, GCIV-VW and GCIV-AV) the differences are not statistically significant for two methods.

Even though we didn't have the chance to analyze the performance of the algorithms in terms of the matrix sizes due to the design of the experiments, we conducted further analysis in terms of the level of inconsistencies. The results of this analysis are provided in Table 14 for the Visual Experiment and Table 15 for the Mass Experiment. The LSD Post-Hoc Test results are tabulated in Table 16 and Table 17 for the Visual and Mass Experiments, respectively.

The results indicate that NLP outperforms the other two algorithms statistically significantly for all three metrics in both of the experiments when the inconsistency levels are *High*. This is also the case when the inconsistency levels are *Medium* with the exception of the GCIV-VW metric for the Visual Experiment (in this case there is no statistically significant difference between NLP and Saaty). On the other hand, when the inconsistency levels are *Low* the results are blurred. NLP outperforms the other two algorithms only for the GCIV-AW metric in both of the experiments. Particularly for the Visual Experiment Saaty outperforms NLP on the average (i.e., the difference is not statistically significant with ($P \leq 0.05$)) and in terms of GCIV-AV metric BAGINS outperforms both algorithms statistically significantly.

	1		1	/
		Visual Experiment		
Model	Ν	GCIV-AW	GCIV-VW	GCIV-AV
NLP	164	1.02785	1.12557	1.15548
Saaty	164	1.07727	1.15077	1.22999
BAGINS	164	1.04858	1.16318	1.21772
		Mass Experiment		
NLP	154	1.04640	1.09973	1.14823
Saaty	154	1.10109	1.14718	1.25588
BAGINS	154	1.07269	1.14751	1.22790

Table 11: Descriptive Statistics (Bold figures represents the best results)

 Table 12: LSD Post-Hoc Test (Visual Experiment)

Method	Method	GCIV-AW		GCIV-VW		GCIV-AV	
(I)	(J)	Mean Diff. (I-J)	Sig.	Mean Diff. (I-J)	Sig.	Mean Diff. (I-J)	Sig.
NLP	Saaty	-0.04942	0.000	-0.02520	0.004	-0.07452	0.000
NLP	BAGINS	-0.02074	0.000	-0.03761	0.000	-0.06225	0.000
Saaty	BAGINS	0.02868	0.000	-0.01240	0.415	0.01227	0.590

Table 13: LSD Post-Hoc Test (Mass Experiment)

Method	Method	GCIV-AW	GCIV-AW GCIV-VW GCIV-AV			GCIV-AV	
(I)	(\mathbf{J})	Mean Diff. (I-J)	Sig.	Mean Diff. (I-J)	Sig.	Mean Diff. (I-J)	Sig.
NLP	Saaty	-0.05469	0.000	-0.04746	0.000	-0.10765	0.000
NLP	BAGINS	-0.02629	0.000	-0.04779	0.000	-0.07966	0.000
Saaty	BAGINS	0.02840	0.000	-0.00033	0.999	0.02799	0.060

Table 14: Descriptive Statistics for different levels of inconsistency (Visual Experiment) (Bold figures represents the best results)

Performance Metric	Model	Ν	CR=low	Ν	CR=medium	Ν	CR=High
	NLP	19	1.01238	94	1.02104	33	1.03149
GCIV-AW	Saaty	19	1.03055	94	1.05783	33	1.09362
	BAGINS	19	1.02173	94	1.03547	33	1.05721
	NLP	19	1.14716	94	1.11738	33	1.12571
GCIV-VW	Saaty	19	1.11687	94	1.13615	33	1.17516
	BAGINS	19	1.11996	94	1.14311	33	1.19300
	NLP	19	1.16059	94	1.13988	33	1.15853
GCIV-AV	Saaty	19	1.14956	94	1.19526	33	1.26969
	BAGINS	19	1.14503	94	1.18343	33	1.25630

Table 15: Descriptive Statistics for different levels of inconsistency (Mass Experiment) (Bold figures represents the best results)

Performance Metric	Model	Ν	CR=low	Ν	CR=medium	Ν	CR=High
	NLP	7	1.01343	63	1.02753	51	1.04322
GCIV-AW	Saaty	7	1.03447	63	1.06003	51	1.09972
	BAGINS	7	1.02370	63	1.04059	51	1.06762
	NLP	7	1.09956	63	1.10267	51	1.09012
GCIV-VW	Saaty	7	1.11979	63	1.13541	51	1.14604
	BAGINS	7	1.12289	63	1.13120	51	1.14154
	NLP	7	1.11353	63	1.13275	51	1.13565
GCIV-AV	Saaty	7	1.15687	63	1.20062	51	1.25567
	BAGINS	7	1.14974	63	1.17802	51	1.21954

Table 16: LSD Post-Hoc Test for different levels of inconsistency (Visual Experiment)

	Method	Method	CR=low	CR=medium			CR=high	
Performance Metric	(I)	(J)	Mean Diff. (I-J)	Sig.	Mean Diff. (I-J)	Sig.	Mean Diff. (I-J)	Sig.
	NLP	Saaty	-0.01816	0.000	-0.03679	0.000	-0.06212	0.000
GCIV-AW	NLP	BAGINS	-0.00935	0.003	-0.01443	0.000	-0.02572	0.000
	Saaty	BAGINS	0.00882	0.000	0.02236	0.000	0.03640	0.000
	NLP	Saaty	0.03029	0.232	-0.01877	0.134	-0.04945	0.017
GCIV-VW	NLP	BAGINS	0.02720	0.327	-0.02573	0.041	-0.06729	0.004
	Saaty	BAGINS	-0.00309	0.986	-0.00696	0.800	-0.01784	0.670
	NLP	Saaty	0.01103	0.822	-0.05537	0.000	-0.11116	0.000
GCIV-AV	NLP	BAGINS	0.01556	0.700	-0.04355	0.000	-0.09777	0.000
	Saaty	BAGINS	0.00453	0.973	0.01183	0.538	0.01339	0.785

Table 17: LSD Post-Hoc Test for different levels of inconsistency (Mass Experiment)

	Method	Method	CR=low	CR=medium			CR=high	
Performance Metric	(I)	(J)	Mean Diff. (I-J)	Sig.	Mean Diff. (I-J)	Sig.	Mean Diff. (I-J)	Sig.
	NLP	Saaty	-0.02104	0.000	-0.03250	0.000	-0.05650	0.000
GCIV-AW	NLP	BAGINS	-0.01027	0.012	-0.01306	0.000	-0.02440	0.000
	Saaty	BAGINS	0.01077	0.005	0.01945	0.000	0.03210	0.000
	NLP	Saaty	-0.02023	0.700	-0.03274	0.000	-0.05592	0.000
GCIV-VW	NLP	BAGINS	-0.02333	0.697	-0.02853	0.004	-0.05142	0.000
	Saaty	BAGINS	-0.00310	0.994	0.00421	0.881	0.00450	0.926
	NLP	Saaty	-0.04334	0.228	-0.06787	0.000	-0.12003	0.000
GCIV-AV	NLP	BAGINS	-0.03621	0.407	-0.04528	0.000	-0.08389	0.000
	Saaty	BAGINS	0.00713	0.969	0.02260	0.030	0.03613	0.019

6. Concluding Remarks and Future Research

Most decision-making environments contain qualitative information in the form of verbal phrases and the quantification of such phrases has remained a contentious issue in the literature. With the advance of digital

transformation in all aspects of our daily lives, human-machine teaming is becoming *normal*, thus, accurate quantification of such verbal phrases is of critical importance while eliciting judgments (e.g., probability, weights, etc.) from human experts and/or decision-makers.

Being the most popular method in the realm of MCDM, AHP always received excessive attention from both researchers and practitioners. Since the early days that it was introduced, one of the major controversies regarding to AHP was the *arbitrary scale* that was utilized while transforming the linguistic comparison matrices to the numerical comparison matrices which are used to determine the true priority vectors of the decision-makers. Therefore, many alternative *static* scales are proposed in the literature. Recently, the individualization of the scale that is used in the process has been receiving attention from researchers. In this study, we have introduced BAGINS, a simple, easy-to-learn, and quantitatively less demanding heuristic that performs comparable with other alternatives.

Our contribution in this study is threefold. First of all, we have developed an experimental setup that can be used not only for the problem in our focus (i.e., the individualization of the scales) but also for many other researchers that are contributing to the theory of AHP in other aspects. The developed framework starts with the *true weight vector* in order to construct the numerical pairwise comparison matrices which are used to determine the priority vectors. As part of this framework, two new performance metrics are introduced to the literature (i.e., GCIV-AV and GCIV-VW) alongside with the commonly used CIV (i.e., GCIV-AW). GCIV-AV addresses the compatibility with the numerical pairwise comparison matrix and the theoretical numerical pairwise comparison matrix that is constructed from the true priority vectors. On the other hand, GCIV-VW addresses the compatibility between the true priority vector and the calculated priority vector.

The second contribution of this study is the BAGINS that can be used to generate individualized scales. Unlike the existing algorithms that are proposed earlier for this purpose, BAGINS doesn't target the inconsistency of the numerical comparison matrix but aims to choose the individualized scales that would improve the compatibility of the numerical comparison matrix and the theoretical numerical comparison matrix.

In order to measure the performance of BAGINS, a numerical study and two experimental analyses were conducted. Both the numerical and empirical data sets, with necessary Matlab codes and documentation are made available online for the use of researchers and practitioners. We sincerely believe the benefit of open data and consider it as the third contribution of this research.

The numerical study demonstrated that methods based on the individualized scales outperform the conventional Saaty method with fixed scale statistically significantly in 11 out of the 14 experimental conditions in terms of the newly introduced GCIV-AV metric and the conventional GCIV-AW metrics (and for the remaining three experimental conditions the individualized scales outperforms the fixed sale *on the average*). On the other hand, among the two methods that are based on individualized scales, NLP statistically significantly outperforms BAGINS in terms of the GCIV-AW metric, and BAGINS statistically significantly outperforms NLP in terms of the GCIV-VW metric for all of the seven experimental conditions, and in terms of the GCIV-AV, this was the case for five out of the seven experimental conditions.

One of the interesting observations from the numerical study which might worth further analysis is the performance of Saaty with fixed scale in terms of the GCIV-VW metric. As depicted earlier in Figure 2, in terms of the compatibility between the theoretical true weights and the numerical pairwise comparison matrix, the fixed scale based Saaty is not performing as good as the individualized scaled based approaches. This is also the case for the compatibility between the numerical pairwise comparison matrix and the theoretical numerical matrix constructed from the calculated priority vector. However, when one considers the process end-to-end, Saaty outperforms the other two methods. That is to say, it somehow straightens out the bad performances in the intermediary two steps (i.e., Step 1: Representation of the true weights with the numerical pairwise comparison matrix). We left this observation as a possible research question.

Some other future research questions are regarding the improvement of the BAGINS. As is, it targets a proxy optimization problem that minimizes a squared deviation as explained earlier. Another option might

be targeting the absolute deviation. A more interesting question would be changing the method that is used to determine the priority vector. As is, BAGINs use Saaty's conventional eigenvalue-eigenvector approach. However one might also consider other methods (e.g., LLSM, Mean of Normalized Values Heuristic, etc.). It is also possible to consider the recursive version of BAGINS in which after the determination of the individualized scale, the priority vector can be recalculated and the whole process is repeated until a termination criterion. The convergence of the recursive BAGINS is also another possible research problem that might be addressed in the future. Finally, additional *empirical* studies would benefit not only this research but also other research that contributes to the theory of AHP (e.g., new algorithms to determine priority vectors, new performance metrics for evaluation of different approaches, new static and/or individualized scales, etc.).

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