

Faster Gradient-Free Algorithms for Nonsmooth Nonconvex Stochastic Optimization

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Abstract

We consider the optimization problem of the form $\min_{x \in \mathbb{R}^d} f(x) \triangleq \mathbb{E}_\xi[F(x; \xi)]$, where the component $F(x; \xi)$ is L -mean-squared Lipschitz but possibly nonconvex and nonsmooth. The recently proposed gradient-free method requires at most $\mathcal{O}(L^4 d^{3/2} \epsilon^{-4} + \Delta L^3 d^{3/2} \delta^{-1} \epsilon^{-4})$ stochastic zeroth-order oracle complexity to find a (δ, ϵ) -Goldstein stationary point of objective function, where $\Delta = f(x_0) - \inf_{x \in \mathbb{R}^d} f(x)$ and x_0 is the initial point of the algorithm. This paper proposes a more efficient algorithm using stochastic recursive gradient estimator, which improves the complexity to $\mathcal{O}(L^3 d^{3/2} \epsilon^{-3} + \Delta L^2 d^{3/2} \delta^{-1} \epsilon^{-3})$.

1 Introduction

We study the stochastic optimization problem

$$\min_{x \in \mathbb{R}^d} f(x) \triangleq \mathbb{E}_\xi[F(x; \xi)], \quad (1)$$

where the stochastic component $F(x; \xi)$, indexed by random variable ξ , is possibly nonconvex and nonsmooth. We focus on tackling the problem with Lipschitz continuous objective, which arises in many popular applications including simulation optimization [17, 34], deep neural networks [4, 15, 33, 48], statistical learning [11, 31, 49, 50, 52], reinforcement learning [5, 21, 30, 41], financial risk minimization [40] and supply chain management [10].

The Clarke subdifferential [6] for Lipschitz continuous function is a natural extension of gradient for smooth function and subdifferential for convex function. Unfortunately, the hard instances suggest that finding a (near) ϵ -stationary point in terms of Clarke subdifferential is computationally intractable [23, 51]. Zhang et al. [51] addressed this issue by proposing the notion of (δ, ϵ) -Goldstein stationarity as a valid criterion for non-asymptotic convergence analysis in nonsmooth nonconvex optimization, which considers the convex hull of all Clarke subdifferential at points in the δ -ball neighbour [16]. They also proposed the stochastic interpolated normalized gradient descent method (SINGD) for finding a (δ, ϵ) -Goldstein stationary point of Hadamard directionally differentiable objective, which has a stochastic first-order oracle complexity of $\mathcal{O}(\Delta L^3 \delta^{-1} \epsilon^{-4})$, where L is the Lipschitz constant of the objective, $\Delta = f(x_0) - \inf_{x \in \mathbb{R}^d} f(x)$ and x_0 is the initial point of the algorithm. In the case where both exact function value and gradient oracle are available, Davis and Drusvyatskiy [7] proposed the perturbed stochastic interpolated normalized gradient descent (PINGD) to relax the Hadamard directionally differentiable assumption. They showed that an exact oracle complexity of $\tilde{\mathcal{O}}(\Delta L d \delta^{-1} \epsilon^{-2})$ can be obtained by a cutting plane algorithm. Later, Tian et al. [43] proposed the perturbed stochastic interpolated normalized gradient descent (PSINGD) for stochastic settings. The lower bounds for this task have also been studied [22, 23, 24, 42].

For many real applications [4, 10, 17, 30, 34, 40, 48], accessing the first-order oracle may be extremely expensive or even impossible. The randomized smoothing method is a well-known approach to design zeroth-order algorithms [12, 14, 20, 29, 36, 44]. It approximates the first-order oracle using the difference of function values. Most of the existing works on zeroth-order optimization focus on convex problems [9, 36, 39, 44] and smooth nonconvex problems [12, 14, 20, 29, 36]. Recently, Lin et al. [28] considered the nonsmooth nonconvex

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Table 1: We present the complexities of finding a (δ, ϵ) -Goldstein stationary point for d -dimensional L -Lipschitz objective under both deterministic and stochastic settings, where we denote $\Delta = f(x_0) - f^*$ and x_0 is the initial point of the algorithm. We remark for “*” that the algorithms proposed by Zhang et al. [51] use the non-standard first-order oracle called the Hadamard directional derivative.

Setting	Method	Oracle	Reference	Complexity
Stochastic	1st*	SINGD	Zhang et al. [51]	$\tilde{\mathcal{O}}\left(\frac{\Delta L^3}{\delta \epsilon^4}\right)$
	1st	PSINGD	Tian et al. [43]	$\tilde{\mathcal{O}}\left(\frac{\Delta L^3}{\delta \epsilon^4}\right)$
	0th	GFM	Lin et al. [28]	$\mathcal{O}\left(d^{3/2}\left(\frac{L^4}{\epsilon^4} + \frac{\Delta L^3}{\delta \epsilon^4}\right)\right)$
	0th	GFM ⁺	Theorem 3.1	$\mathcal{O}\left(d^{3/2}\left(\frac{L^3}{\epsilon^3} + \frac{\Delta L^2}{\delta \epsilon^3}\right)\right)$
Deterministic	0th & 1st*	INGD	Zhang et al. [51]	$\tilde{\mathcal{O}}\left(\frac{\Delta L^2}{\delta \epsilon^3}\right)$
	0th & 1st	PINGD	Davis and Drusvyatskiy [7]	$\tilde{\mathcal{O}}\left(\frac{\Delta L^2}{\delta \epsilon^3}\right)$
	0th & 1st	cutting plane	Davis and Drusvyatskiy [7]	$\tilde{\mathcal{O}}\left(\frac{\Delta L d}{\delta \epsilon^2}\right)$

case by establishing the relationship between Goldstein subdifferential [16] and randomized smoothing [36, 39]. As a consequence, they showed the gradient-free method (GFM) could find a (δ, ϵ) -Goldstein stationary point within at most $\mathcal{O}(L^4 d^{3/2} \epsilon^{-4} + \Delta L^3 d^{3/2} \delta^{-1} \epsilon^{-4})$ numbers of stochastic zeroth-order oracle calls.

In this paper, we propose an efficient stochastic gradient-free method named GFM⁺ for nonsmooth non-convex stochastic optimization. The algorithm takes the advantage of smoothness arise from the randomized smoothing to construct the stochastic recursive gradient estimators [12, 27, 37] for the smoothed surrogate of the objective. It achieves a stochastic zeroth-order oracle complexity of $\mathcal{O}(L^3 d^{3/2} \epsilon^{-3} + \Delta L^2 d^{3/2} \delta^{-1} \epsilon^{-3})$ for finding a (δ, ϵ) -Goldstein stationary point, improving the dependency both on L and ϵ compared with GFM [28]. Interestingly, in the case of $\delta L \lesssim \Delta$, the oracle complexity of GFM⁺ in terms of L and ϵ is even better than existing stochastic first-order methods [43, 51].¹ We summarize the results of this paper and related works in Table 1, where the deterministic setting refers to the case that exact zeroth-order and first-order oracle is available.

We also extend our results to nonsmooth convex optimization. The optimality of the zeroth-order algorithms to minimize convex function in the measure of function value has been established by Shamir [39], while the goal of finding approximate stationary points is much more challenging [1, 25, 35]. The lower bound provided by Kornowski and Shamir [24] suggests finding a point with small subgradient for Lipschitz convex objective is intractable. Hence, finding an approximate Goldstein stationary point is also a more reasonable task in convex optimization. We propose the two-phase gradient-free methods to take the advantage of the convexity. It shows that GFM⁺ with warm-start strategy could find a (δ, ϵ) -Goldstein stationary point within $\mathcal{O}(L^3 d^{3/2} \epsilon^{-3} + \Delta L^2 d^{4/3} \delta^{-2/3} \epsilon^{-2})$ stochastic zeroth-order oracle complexity. We summarize the results for convex case in Table 2.

¹In the case of $\Delta \lesssim \delta L$, the initial point would have already satisfied the optimality condition $f(x_0) - \inf_{x \in \mathbb{R}^d} f(x) = \mathcal{O}(\epsilon)$, which is shown by taking $\delta = \epsilon$ [7].

Table 2: We present the complexities of finding a (δ, ϵ) -Goldstein stationary point for d -dimensional L -Lipschitz objectives with zeroth-order oracles under the stochastic setting, where we denote $R = \text{dist}(x_0, \mathcal{X}^*)$ and x_0 is the initial point of the algorithm.

Method	Reference	Complexity
GFM	Lin et al. [28]	$\mathcal{O}\left(\frac{d^{3/2}L^4}{\epsilon^4} + \frac{d^{3/2}L^4R}{\delta\epsilon^4}\right)$
WS-GFM	Theorem 3.4	$\mathcal{O}\left(\frac{d^{3/2}L^4}{\epsilon^4} + \frac{d^{4/3}L^{8/3}R^{2/3}}{\delta^{2/3}\epsilon^{4/3}}\right)$
GFM ⁺	Theorem 3.1	$\mathcal{O}\left(\frac{d^{3/2}L^3}{\epsilon^3} + \frac{d^{3/2}L^3R}{\delta\epsilon^3}\right)$
WS-GFM ⁺	Theorem 3.3	$\mathcal{O}\left(\frac{d^{3/2}L^3}{\epsilon^3} + \frac{d^{4/3}L^2R^{2/3}}{\delta^{2/3}\epsilon^2}\right)$

2 Preliminaries

In this section, we introduce the background for nonsmooth nonconvex function and randomized smoothing technique in zeroth-order optimization.

2.1 Notation and Assumptions

Throughout this paper, we use $\|\cdot\|$ to represent the Euclidean norm of a vector. We define $\mathbb{B}_\delta(x) \triangleq \{x \in \mathbb{R}^d : \|x\| = \delta\}$ as the Euclidean ball centered at point x with radius δ . We also define $\mathbb{B} = \mathbb{B}_1(0)$ and $\mathbb{S} \triangleq \{x \in \mathbb{R}^d : \|x\| = 1\}$ as the unit Euclidean ball and sphere centered at origin respectively. We let $\text{conv}\{A\}$ be the convex hull of set A and $\text{dist}(x, A) = \inf_{y \in A} \|x - y\|$ be the distance between vector x and set A .

Following the standard setting of stochastic optimization, we assume the objective f can be expressed as an expectation of some stochastic components.

Assumption 2.1. *We suppose that the objective function has the form of $f(x) = \mathbb{E}_\xi[F(x; \xi)]$, where ξ denotes the random index.*

We focus on the Lipschitz function with finite infimum, which satisfies the following assumptions.

Assumption 2.2. *We suppose that the stochastic component $F(\cdot; \xi) : \mathbb{R}^d \rightarrow \mathbb{R}$ is $L(\xi)$ -Lipschitz for any ξ , i.e. it holds that*

$$\|F(x; \xi) - F(y; \xi)\| \leq L(\xi)\|x - y\| \quad (2)$$

for any $x, y \in \mathbb{R}^d$ and $L(\xi)$ has bounded second-order moment, i.e. there exists some constant $L > 0$ such that

$$\mathbb{E}_\xi[L(\xi)^2] \leq L^2.$$

Assumption 2.3. *We suppose that the objective $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is lower bounded and define*

$$f^* := \inf_{x \in \mathbb{R}^d} f(x).$$

The inequality (2) in Assumption 2.2 implies the mean-squared Lipschitz continuity of $F(x; \xi)$.

Proposition 2.1 (mean-squared continuity). *Under Assumption 2.2, for any $x, y \in \mathbb{R}^d$ it holds that*

$$\mathbb{E}_\xi\|F(x; \xi) - F(y; \xi)\|^2 \leq L^2\|x - y\|^2.$$

All of above assumptions follow the setting of Lin et al. [28]. We remark that Assumption 2.2 is weaker than assuming each stochastic component $F(\cdot; \xi)$ is L -Lipschitz.

2.2 Goldstein Stationary Point

According to Rademacher’s theorem, a Lipschitz function is differentiable almost everywhere. Thus, we can define the Clarke subdifferential [6] as follows.

Definition 2.1 (Clarke subdifferential). *The Clarke subdifferential of Lipschitz function f at point x is defined by*

$$\partial f(x) := \text{conv} \left\{ g : g = \lim_{x_k \rightarrow x} \nabla f(x_k) \right\}.$$

However, finding an ϵ -Clarke stationary point is computationally intractable [51]. Furthermore, finding a point that is δ -close to an ϵ -stationary point may have an inevitable exponential dependence on the problem dimension [23]. As a relaxation, we pursue the (δ, ϵ) -Goldstein stationary point [51], whose definition is based on the following Goldstein subdifferential [16].

Definition 2.2 (Goldstein subdifferential). *Given $\delta > 0$, the Goldstein δ -subdifferential of Lipschitz function f at point x is defined by*

$$\partial_\delta f(x) := \text{conv} \left\{ \bigcup_{y \in \mathbb{B}_\delta(x)} \partial f(y) \right\},$$

where $\partial f(x)$ is the Clarke subdifferential.

The (δ, ϵ) -Goldstein stationary point [51] is formally defined as follows.

Definition 2.3 (approximate Goldstein stationary point). *Given a Lipschitz function f , we say the point x is a (δ, ϵ) -Goldstein stationary point of f if it holds that*

$$\text{dist}(0, \partial_\delta f(x)) \leq \epsilon. \tag{3}$$

Our goal is designing efficient stochastic algorithms to find a (δ, ϵ) -Goldstein stationary point in expectation.

2.3 Randomized smoothing

Recently, Lin et al. [28] established the relationship between uniform smoothing and Goldstein subdifferential. We first present the definition of the smoothed surrogate.

Definition 2.4 (uniform smoothing). *Given a Lipschitz function f , we denote its smoothed surrogate as*

$$f_\delta(x) := \mathbb{E}_{u \sim \mathcal{P}}[f(x + \delta u)],$$

where \mathcal{P} is the uniform distribution on unit ball \mathbb{B} .

The smoothed surrogate has the following properties [28, Proposition 2.3 and Theorem 3.1].

Proposition 2.2. *Suppose that function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is L -Lipschitz, then it holds that:*

- $|f_\delta(\cdot) - f(\cdot)| \leq \delta L$.
- $f_\delta(\cdot)$ is L -Lipschitz.
- $f_\delta(\cdot)$ is differentiable and with $c\sqrt{d}L\delta^{-1}$ -Lipschitz gradient for some numeric constant $c > 0$.
- $\nabla f_\delta(\cdot) \in \partial_\delta f(\cdot)$, where $\partial_\delta f(\cdot)$ is the Goldstein subdifferential.

Flaxman et al. [13] showed that we can obtain an unbiased estimate of $\nabla f_\delta(x)$ by using the two function value evaluations of points sampled on unit sphere \mathbb{S} , which leads to the zeroth-order gradient estimator.

Definition 2.5 (zeroth-order gradient estimator). Given a stochastic component $F(\cdot; \xi) : \mathbb{R}^d \rightarrow \mathbb{R}$, we denote its stochastic zeroth-order gradient estimator at $x \in \mathbb{R}^d$ by:

$$g(x; w, \xi) = \frac{d}{2\delta}(F(x + \delta w; \xi) - F(x - \delta w; \xi))w$$

where $w \in \mathbb{R}^d$ is sampled from a uniform distribution on a unit sphere \mathbb{S} .

We also introduce the mini-batch zeroth-order gradient estimator that plays an important role in the stochastic zeroth-order algorithms.

Definition 2.6 (mini-batch zeroth-order gradient estimator). Let $S = \{(\xi_i, w_i)\}_{i=1}^b$, where vectors $w_1, \dots, w_b \in \mathbb{R}^d$ are i.i.d. sampled from a uniform distribution on unit sphere \mathbb{S} and random indices ξ_1, \dots, ξ_b are i.i.d. We denote the mini-batch zeroth-order gradient estimator of $F(\cdot) : \mathbb{R}^d \rightarrow \mathbb{R}$ in terms of S at $x \in \mathbb{R}^d$ by

$$g(x; S) = \frac{1}{b} \sum_{i=1}^b g(x; w_i, \xi_i).$$

Next we present some properties for zeroth-order gradient estimators.

Proposition 2.3 (Lemma D.1 of Lin et al. [28]). Under Assumption 2.1 and 2.2, it holds that

$$\mathbb{E}_{w, \xi}[g(x; w, \xi)] = \nabla f_\delta(x)$$

and

$$\mathbb{E}_{w, \xi}\|g(x; w, \xi)\|^2 \leq 16\sqrt{2\pi}dL^2.$$

Corollary 2.1. Under Assumption 2.1 and 2.2, it holds that

$$\mathbb{E}_S\|g(x; S) - \nabla f_\delta(x)\|^2 \leq \frac{16\sqrt{2\pi}dL^2}{b}.$$

The following continuity condition of gradient estimator is an essential element for variance reduction.

Proposition 2.4. Under Assumption 2.1 and 2.2, for any $w \in \mathbb{S}$ and $x, y \in \mathbb{R}^d$, it holds that

$$\mathbb{E}_\xi\|g(x; w, \xi) - g(y; w, \xi)\|^2 \leq \frac{d^2 L^2}{\delta^2}\|x - y\|^2.$$

To simplify the presentation, we introduce the following notations:

$$\begin{aligned} L_\delta &= \frac{c\sqrt{d}L}{\delta}, \quad \sigma_\delta^2 = 16\sqrt{2\pi}dL^2, \quad \Delta = f(x_0) - f^* \\ M_\delta &= \frac{dL}{\delta}, \quad f_\delta^* := \inf_{x \in \mathbb{R}^d} f_\delta(x), \quad \Delta_\delta = \Delta + L\delta, \end{aligned} \tag{4}$$

where x_0 is the initial point of the algorithm.

Then the above results can be written as

- $\|\nabla f_\delta(x) - \nabla f_\delta(y)\| \leq L_\delta\|x - y\|$ for any $x, y \in \mathbb{R}^d$, where f_δ follows Definition 2.4;
- $\mathbb{E}_\xi\|g(x; w, \xi) - g(y; w, \xi)\|^2 \leq M_\delta^2\|x - y\|^2$ for any $x, y \in \mathbb{R}^d$, where w and ξ follow Definition 2.5;
- $\mathbb{E}_S\|g(x; S) - \nabla f_\delta(x)\|^2 \leq \sigma_\delta^2/b$ for any $x \in \mathbb{R}^d$, where S and b follow Definition 2.6;
- $f_\delta(x_0) - f_\delta^* \leq \Delta_\delta$.

In Appendix, we also prove the orders of Lipschitz constants $L_\delta = \mathcal{O}(\sqrt{d}L\delta^{-1})$ and $M_\delta = \mathcal{O}(dL\delta^{-1})$ are tight. However, it remains unknown whether the order of σ_δ could be improved.

Algorithm 1 GFM (x_0, η, T)

- 1: **for** $t = 0, 1, \dots, T - 1$
 - 2: sample a random direction $w_t \in \mathbb{B}$ and a random index ξ_t
 - 3: update $x_{t+1} = x_t - \eta g(x_t; w_t, \xi_t)$
 - 4: **end for**
 - 5: **return** x_{out} chosen uniformly from $\{x_t\}_{t=0}^{T-1}$
-

Algorithm 2 GFM⁺ (x_0, η, T, m, b, b')

- 1: $v_0 = g(x_0; S')$
 - 2: **for** $t = 0, 1, \dots, T - 1$
 - 3: **if** $t \bmod m = 0$
 - 4: sample $S' = \{(\xi_i, w_i)\}_{i=1}^{b'}$
 - 5: calculate $v_t = g(x_t; S')$
 - 6: **else**
 - 7: sample $S = \{(\xi_i, w_i)\}_{i=1}^b$
 - 8: calculate $v_t = v_{t-1} + g(x_t; S) - g(x_{t-1}; S)$
 - 9: **end if**
 - 10: update $x_{t+1} = x_t - \eta v_t$
 - 11: **end for**
 - 12: **return** x_{out} chosen uniformly from $\{x_t\}_{t=0}^{T-1}$
-

3 Algorithms and Main Results

This section introduces GFM⁺ for nonsmooth nonconvex stochastic optimization problem (1). We also provide complexity analysis to show the algorithm has better theoretical guarantee than GFM [28].

3.1 The Algorithms

We propose GFM⁺ in Algorithm 2. Different from GFM (Algorithm 1) [28] that uses a vanilla zeroth-order gradient estimator $g(x_t; w_t, \xi_t)$, GFM⁺ approximates $\nabla f_\delta(x_t)$ by recursive gradient estimator v_t with update rule

$$v_t = v_{t-1} + g(x_t; S) - g(x_{t-1}; S).$$

The estimator v_t properly reduces the variance in estimating $\nabla f_\delta(x)$, which leads to a better stochastic zeroth-order oracle upper complexity than GFM.

The variance reduction is widely used to design stochastic first-order and zeroth-order algorithms for smooth nonconvex optimization [12, 18, 20, 29, 37, 38, 45]. The existing variance reduced algorithms for nonsmooth problem [19, 26, 32, 38, 45, 46, 47] require objective function to have the composite structure, which does not include our setting that each stochastic component $F(x; \xi)$ could be nonsmooth.

3.2 Complexity Analysis

This subsection considers the upper bound complexity of proposed GFM⁺. First of all, we present the descent lemma.

Lemma 3.1 (Li et al. [27, Lemma 2]). *Under Assumption 2.1 and 2.2, Algorithm 2 holds that*

$$f_\delta(x_{t+1}) \leq f_\delta(x_t) - \frac{\eta}{2} \|\nabla f_\delta(x_t)\|^2 + \frac{\eta}{2} \|v_t - \nabla f_\delta(x_t)\|^2 - \left(\frac{\eta}{2} - \frac{L_\delta \eta^2}{2} \right) \|v_t\|^2,$$

where L_δ follows the definition in (4).

Secondly, we show the variance bound of stochastic recursive gradient estimators for smooth surrogate $f_\delta(x)$, which is similar to Lemma 1 of Fang et al. [12].

Lemma 3.2 (variance bound). *Under Assumption 2.1, 2.2, for Algorithm 2 it holds that*

$$\mathbb{E}\|v_{t+1} - \nabla f_\delta(x_{t+1})\|^2 \leq \mathbb{E} \left[\|v_t - \nabla f_\delta(x_t)\|^2 + \frac{\eta^2 M_\delta^2}{b} \|v_t\|^2 \right],$$

where M_δ follows the definition in (4).

For convenience, we denote $n_t \triangleq \lceil t/m \rceil$ as the index of epoch such that $(n_t - 1)m \leq t \leq n_t m - 1$. Then Lemma 3.2 leads to the following corollary.

Corollary 3.1. *Under Assumption 2.1 and 2.2, Algorithm 2 holds that*

$$\mathbb{E}\|v_{t+1} - \nabla f_\delta(x_{t+1})\|^2 \leq \frac{\sigma_\delta^2}{b'} + \frac{\eta^2 M_\delta^2}{b} \sum_{i=m(n_t-1)}^t \mathbb{E}\|v_i\|^2.$$

Combing the descent Lemma 3.1 and Corollary 3.1, we obtain the progress of one epoch.

Lemma 3.3 (one epoch progress). *Under Assumption 2.1 and 2.2, Algorithm 2 holds that*

$$\begin{aligned} \mathbb{E}[f_\delta(x_{mn_t})] &\leq f_\delta(x_{m(n_t-1)}) - \frac{\eta}{2} \sum_{i=m(n_t-1)}^{mn_t-1} \mathbb{E}\|\nabla f_\delta(x_i)\|^2 + \frac{m\eta\sigma_\delta^2}{2b'} \\ &\quad - \left(\frac{\eta}{2} - \frac{L_\delta\eta^2}{2} - \frac{m\eta^3 M_\delta^2}{2b} \right) \sum_{i=m(n_t-1)}^{mn_t-1} \mathbb{E}\|v_i\|^2, \end{aligned}$$

where $L_\delta, M_\delta, \sigma_\delta$ follow the definition in (4).

Connecting the progress for all of T epochs leads to the following result.

Corollary 3.2. *Under Assumption 2.1 and 2.2, Algorithm 2 holds that*

$$\mathbb{E}[f_\delta(x_T)] \leq f_\delta(x_0) - \frac{\eta}{2} \sum_{i=0}^{T-1} \mathbb{E}\|\nabla f_\delta(x_i)\|^2 + \underbrace{\frac{T\eta\sigma_\delta^2}{2b'} - \left(\frac{\eta}{2} - \frac{L_\delta\eta^2}{2} - \frac{m\eta^3 M_\delta^2}{2b} \right)}_{(*)} \sum_{i=0}^{T-1} \mathbb{E}\|v_i\|^2. \quad (5)$$

Using Corollary 3.2 with

$$\eta = \frac{\sqrt{b'}}{mM_\delta}, \quad m = \left\lceil \frac{L_\delta\sqrt{b'}}{M_\delta} \right\rceil \quad \text{and} \quad b = \left\lceil \frac{2b'}{m} \right\rceil,$$

we know that the term $(*)$ in inequality (5) is positive and obtain

$$\mathbb{E}[f_\delta(x_T)] \leq f_\delta(x_0) - \frac{\eta}{2} \sum_{i=0}^{T-1} \mathbb{E}\|\nabla f_\delta(x_i)\|^2 + \frac{T\eta\sigma_\delta^2}{2b'}, \quad (6)$$

which means

$$\mathbb{E}\|\nabla f_\delta(x_{\text{out}})\|^2 \leq \frac{2\Delta\delta}{\eta T} + \frac{\sigma_\delta^2}{b'}. \quad (7)$$

Applying inequality (7) with

$$T = \left\lceil \frac{4\Delta\delta}{\eta\epsilon^2} \right\rceil \quad \text{and} \quad b' = \left\lceil \frac{2\sigma_\delta^2}{\epsilon^2} \right\rceil, \quad (8)$$

Algorithm 3 WS-GFM⁺($x_0, \eta_0, T_0, \eta, T, m, b, b'$)

- 1: $x_1 = \text{GFM}(x_0, \eta_0, T_0)$
 - 2: $x_T = \text{GFM}^+(x_1, \eta, T, m, b, b')$
 - 3: **return** x_T
-

we conclude that the output x_{out} is an ϵ -stationary point of the smooth surrogate $f_\delta(\cdot)$ in expectation.

Finally, the relationship $\nabla f_\delta(x_{\text{out}}) \in \partial_\delta f(x_{\text{out}})$ shown in Proposition 2.2 indicates that Algorithm 2 with above parameter settings could output a (δ, ϵ) -Goldstein stationary point of $f(\cdot)$ in expectation, and the total stochastic zeroth-order oracle complexity is

$$T \left(\left\lceil \frac{b'}{m} \right\rceil + 2b \right) = \mathcal{O} \left(d^{3/2} \left(\frac{L^3}{\epsilon^3} + \frac{\Delta L^2}{\delta \epsilon^3} \right) \right).$$

We formally summarize the result of our analysis as follows.

Theorem 3.1 (GFM⁺). *Under Assumption 2.1, 2.2 and 2.3, we run GFM⁺ (Algorithm 2) with*

$$T = \left\lceil \frac{4\Delta_\delta}{\eta \epsilon^2} \right\rceil, \quad b' = \left\lceil \frac{2\sigma_\delta^2}{\epsilon^2} \right\rceil, \quad b = \left\lceil \frac{2b'}{m} \right\rceil, \quad \eta = \frac{\sqrt{b'}}{mM_\delta} \quad \text{and} \quad m = \left\lceil \frac{L_\delta \sqrt{b'}}{M_\delta} \right\rceil,$$

where $L_\delta, \sigma_\delta, M_\delta$ and Δ_δ follows the definition in (4). Then it outputs a (δ, ϵ) -Goldstein stationary point of $f(\cdot)$ in expectation and the total stochastic zeroth-order oracle complexity is at most

$$\mathcal{O} \left(d^{3/2} \left(\frac{L^3}{\epsilon^3} + \frac{\Delta L^2}{\delta \epsilon^3} \right) \right), \tag{9}$$

where $\Delta = f(x_0) - f^*$.

Theorem 3.1 achieves the best known stochastic zeroth-order oracle complexity for finding a (δ, ϵ) -Goldstein stationary point of nonsmooth nonconvex stochastic optimization problem, which improves upon GFM [28] in the dependency of both ϵ and L .

3.3 The Results for Convex Optimization

This subsection extends the idea of GFM⁺ to find (δ, ϵ) -Goldstein stationary points for nonsmooth convex optimization problem. We propose warm-started GFM⁺ (WS-GFM⁺) in Algorithm 3, which initializes GFM⁺ with GFM.

The complexity analysis of WS-GFM⁺ requires the following assumption.

Assumption 3.1. *We suppose that the objective $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is convex and the set $\mathcal{X}^* := \arg \min_{x \in \mathbb{R}^d} f(x)$ is non-empty.*

We remark the assumption of non-empty \mathcal{X}^* is stronger than Assumption 2.3 since the Lipschitzness of f implies

$$f(x_0) - \inf_{x \in \mathbb{R}^d} f(x) \leq L \text{dist}(x_0, \mathcal{X}^*).$$

Therefore, under Assumption 3.1, directly using GFM⁺ (Algorithm 2) requires

$$\mathcal{O} \left(d^{3/2} \left(\frac{L^3}{\epsilon^3} + \frac{L^3 R}{\delta \epsilon^3} \right) \right) \tag{10}$$

iterations to find a (δ, ϵ) -Goldstein stationary point of f , where we denote $R = \text{dist}(x_0, \mathcal{X}^*)$.

Next we show the warm-start strategy can improve the complexity in (10). It is based on the fact that GFM obtains the optimal stochastic zeroth-order oracle complexity in the measure of function value [39].

Algorithm 4 WS-GFM ($x_0, \eta_0, T_0, \eta, T$)

- 1: $x_1 = \text{GFM}(x_0, \eta_0, T_0)$
 - 2: $x_T = \text{GFM}(x_1, \eta, T)$
 - 3: **return** x_T
-

Theorem 3.2 (GFM in function value). *Under Assumption 2.1, 2.2 and 3.1, we run GFM (Algorithm 1) with $T = \lceil 2\sigma_\delta R^2/\zeta \rceil$, $\eta = R/(\sigma_\delta\sqrt{T})$ and $\delta = \zeta/(4L)$ where σ_δ follows definition in (4). Then the output satisfies $\mathbb{E}[f(x_{\text{out}}) - f^*] \leq \zeta$ and the total stochastic zeroth-order oracle complexity is at most $\mathcal{O}(dL^2R^2\zeta^{-2})$, where $R = \text{dist}(x_0, \mathcal{X}^*)$.*

Theorem 3.2 means using the output of GFM as the initialization for GFM^+ can make the term Δ in (9) be small. We denote $\zeta = f(x_1) - f^*$, then the total complexity in WS-GFM^+ is

$$\mathcal{O}\left(\frac{d^{3/2}L^3}{\epsilon^3} + \frac{d^{3/2}L^3\zeta}{\delta\epsilon^3} + \frac{dL^2R^2}{\zeta^2}\right).$$

Then an appropriate choice of ζ leads to the following result.

Theorem 3.3 (WS-GFM^+). *Under Assumption 2.1, 2.2 and 3.1, Algorithm 3 with an appropriate parameter setting can output a (δ, ϵ) -Goldstein stationary point in expectation within the stochastic zeroth-order oracle complexity of*

$$\mathcal{O}\left(\frac{d^{3/2}L^3}{\epsilon^3} + \frac{d^{4/3}L^2R^{2/3}}{\delta^{2/3}\epsilon^2}\right),$$

where $R = \text{dist}(x_0, \mathcal{X}^*)$.

Naturally, we can also use the idea of warm-start to obtain the complexity of GFM [28] for convex case. We present warm-started GFM (WS-GFM) in Algorithm 4 and provide its theoretical guarantee as follows.

Theorem 3.4 (WS-GFM). *Under Assumption 2.1, 2.2 and 3.1, Algorithm 4 with an appropriate parameter setting can output a (δ, ϵ) -Goldstein stationary point in expectation within the stochastic zeroth-order oracle complexity of*

$$\mathcal{O}\left(\frac{d^{3/2}L^4}{\epsilon^4} + \frac{d^{4/3}L^{8/3}R^{2/3}}{\delta^{2/3}\epsilon^{8/3}}\right),$$

where $R = \text{dist}(x_0, \mathcal{X}^*)$.

4 Numerical Experiments

In this section, we conduct the numerical experiments on nonconvex penalized SVM and black-box attack to show the empirical superiority of proposed GFM^+ .

4.1 Nonconvex Penalized SVM

We consider the nonconvex penalized SVM with capped- ℓ_1 regularizer [52]. The model targets to train the binary classifier $x \in \mathbb{R}^d$ on dataset $\{(a_i, b_i)\}_{i=1}^n$, where $a_i \in \mathbb{R}^d$ and $b_i \in \{1, -1\}$ are the feature of the i -th sample and its corresponding label. It is formulated as the following nonsmooth nonconvex problem

$$\min_{x \in \mathbb{R}^d} f(x) \triangleq \frac{1}{n} \sum_{i=1}^n l(b_i a_i^\top x) + r(x),$$

where $l(x) = \max\{1 - x, 0\}$, $r(x) = \lambda \sum_{j=1}^d \min\{|x_j|, \alpha\}$ and $\lambda, \alpha > 0$ are hyperparameters. We take $\lambda = 10^{-5}/n$ and $\alpha = 2$ in our experiments.

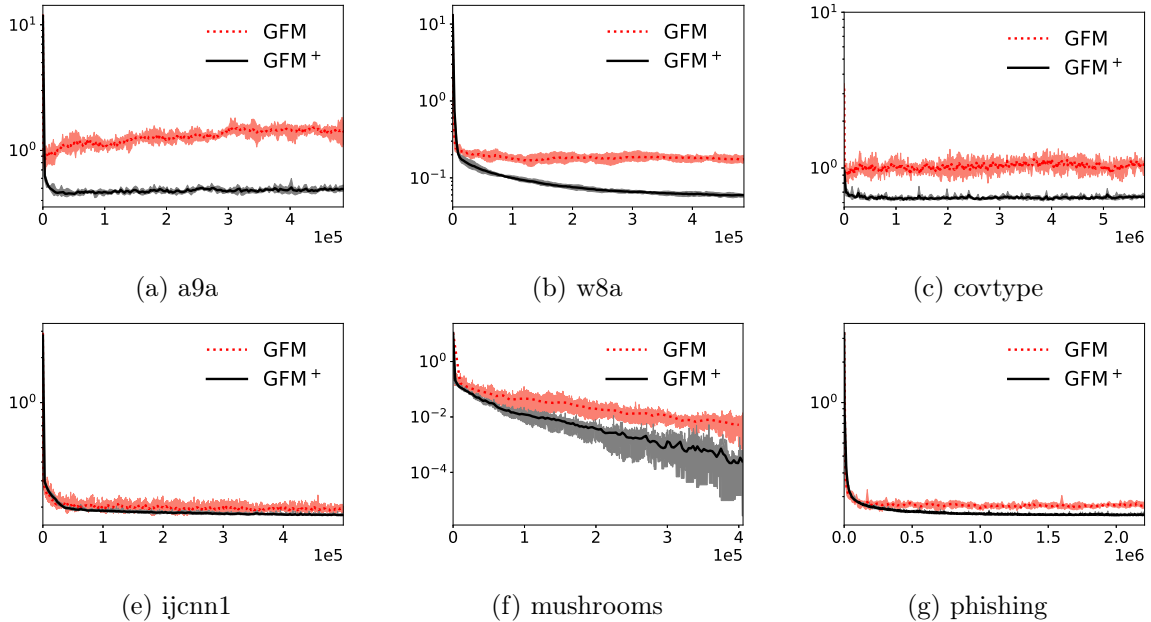


Figure 1: For nonconvex penalized SVM, we present the experimental results for the number of stochastic zeroth-order oracle calls against function value on LIBSVM datasets “a9a”, “w8a”, “covtype”, “ijcn1”, “mushrooms” and “phishing”.

We compare the proposed GFM^+ with GFM [28] on LIBSVM datasets “a9a”, “w8a”, “covtype”, “ijcn1”, “mushrooms” and “phishing” [3]. We set $\delta = 0.001$ and tune the stepsize η from $\{0.1, 0.01, 0.001\}$ for the two algorithms. For GFM^+ , we tune both m and b in $\{1, 10, 100\}$ and set $b' = mb$ by following our theory. We run the algorithms with five different random seeds for each dataset and demonstrate the results in Figure 1. It can be seen that GFM^+ leads to faster convergence and improve the stability in the training.

4.2 Black-Box Attack on CNN

We consider the untargeted black-box adversarial attack on image classification with convolutional neural network (CNN). We present detailed architecture of the network in Appendix I. For a given sample $z \in \mathbb{R}^d$, we aim to find a sample $x \in \mathbb{R}^d$ that is close to z and leads to misclassification. We formulate the problem as the following nonsmooth nonconvex problem [2]:

$$\min_{\|x-z\|_\infty \leq \kappa} \max\{\log[Z(x)]_t - \max_{i \neq t} \log[Z(x)]_i, -\theta\}, \quad (11)$$

where κ is the constraint level of the distortion, t is the class of sample z and $Z(x)$ is the logit layer representation after softmax in the CNN for x such that $[Z(x)]_i$ represents the predicted probability that x belongs to class i . We set $\theta = 4$ and $\kappa = 0.2$ for our experiments. To address the constraint in the problem, we heuristically conduct additional projection step for the update of x in GFM and GFM^+ .

We compare the proposed GFM^+ with GFM [28] on datasets MNIST and Fashion-MNIST. We train the CNN with SGD by 100 epochs with stepsize starting with 0.1 and decaying by 1/2 every 20 epochs. It achieves classification accuracies 98.95% on the MNIST and 91.94% on the Fashion-MNIST. We use GFM^+ and GFM to attack the trained CNNs on all of 10,000 images on testsets and set $\delta = 0.01$. For both GFM and GFM^+ , we tune b' from $\{500, 1000, 2000\}$. For GFM^+ , we additionally tune m from $\{10, 20, 50\}$ and set $b = b'/m$. For both two algorithm, we tune the initial learning rate η in $\{0.5, 0.05, 0.005\}$ and decay by 1/2 if there is no improvement in 10 iterations at one attack. The experiment results are shown in Figure 2 and we can observe that GFM^+ has better performance.

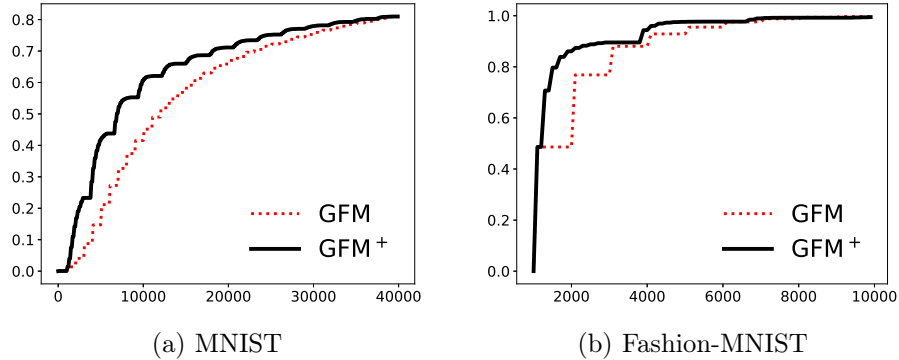


Figure 2: For black-box attack, we present the experimental result for the number of stochastic zeroth-order oracle calls against success attack rate on dataset “MNIST” and “Fashion-MNIST”.

5 Conclusion

This paper proposes GFM^+ for stochastic nonsmooth nonconvex optimization. We prove that the algorithm requires at most $\mathcal{O}(L^3 d^{3/2} \epsilon^{-3} + \Delta L^2 d^{3/2} \delta^{-1} \epsilon^{-3})$ stochastic zeroth-order oracle complexity for finding a (ϵ, δ) -Goldstein stationary point in expectation, which improve the best known result of GFM [28]. The numerical experiments also support our theory. However, the tightness of GFM^+ is still unclear. It is interesting to study the lower bound of finding approximate Goldstein stationary point by zeroth-order algorithms.

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A The Proof of Proposition 2.4

Proof. According to the definition of $g(x; w, \xi)$, we use Young’s inequality and Proposition 2.1 to obtain

$$\begin{aligned} & \mathbb{E}_\xi \|g(x; w, \xi) - g(y; w, \xi)\|^2 \\ & \leq \frac{d^2}{2\delta^2} \mathbb{E}_\xi \|F(x + \delta w; \xi) - F(y + \delta w; \xi)\|^2 + \frac{d^2}{2\delta^2} \mathbb{E}_\xi \|F(x - \delta w; \xi) - F(y - \delta w; \xi)\|^2 \\ & \leq \frac{d^2 L^2}{\delta^2} \|x - y\|^2. \end{aligned}$$

□

B The Tightness of L_δ and M_δ

We formally show the tightness of L_δ and M_δ as follows.

Proposition B.1. *The order of Lipschitz constant $L_\delta = \mathcal{O}(\sqrt{d}L\delta^{-1})$ for the gradient of $f_\delta(\cdot)$ is tight.*

Proof. According to Lemma 10 of Duchi et al. [8], it holds that

$$E_U(f)E_V(f) \geq c'L^2\sqrt{d}$$

for some constant $c' > 0$, where $E_U(f)$ and $E_V(f)$ are defined as

$$E_U(f) = \inf_{\kappa \in \mathbb{R}} \sup_{x \in \mathbb{R}^d} \{|f(x) - f_\delta(x)| \leq \kappa\}$$

and

$$E_V(f) = \inf_{\kappa \in \mathbb{R}} \sup_{x, y \in \mathbb{R}^d} \{\|\nabla f_\delta(x) - \nabla f_\delta(y)\| \leq \kappa\|x - y\|\}.$$

The above lower bound can be taken by the function

$$f(x) = \frac{L}{2}\|x\| + \frac{L}{2} \left| \frac{x^\top y}{\|y\|^2} - \frac{1}{2} \right|.$$

Using Proposition 2.3, we know that $E_U(f) \leq L\delta$ and it immediately implies our claim. □

Proposition B.2. *The order of mean-squared Lipschitz constant $M_\delta = \mathcal{O}(dL\delta^{-1})$ for the stochastic zeroth-order gradient estimator $g(\cdot; w, \xi)$ is tight.*

Proof. We construct a function $f(x) = F(x; \xi) = \frac{L}{\sqrt{d}} \sum_{i=1}^d |x_i|$ for any random vector ξ . It is clear that each $F(x; \xi)$ is L -Lipschitz for any random index ξ by noting:

$$|F(x; \xi) - F(y; \xi)| = \frac{L}{\sqrt{d}} \left| \|x\|_1 - \|y\|_1 \right| \leq \frac{L}{\sqrt{d}} \|x - y\|_1 \leq L \|x - y\|_2.$$

Choosing $x = \delta \mathbf{1}$ and $y = -\delta \mathbf{1}$, we know that $\|x - y\|_2 = 2\sqrt{d}\delta$. Hence, we further know that (a) $\triangleq M_\delta \|x - y\|_2 = 2d^{3/2}L$. Next, we calculate the zeroth order gradient for $w = 1/\sqrt{d} \cdot \mathbf{1}$ and verify that $g(x; w, \xi) = Ld\mathbf{1}$. Similarly, we also have $g(y; w, \xi) = -Ld\mathbf{1}$. Therefore, (b) $\triangleq \|g(x; w, \xi) - g(y; w, \xi)\|_2 = 2d^{3/2}L$. Finally, we complete the proof by noting that term (a) is exactly the same as term (b). □

C The Proof of Lemma 3.1

Proof. We have

$$\begin{aligned}
f_\delta(x_{t+1}) &\leq f_\delta(x_t) + \nabla f_\delta(x_t)^\top (x_{t+1} - x_t) + \frac{L_\delta}{2} \|x_{t+1} - x_t\|^2 \\
&= f_\delta(x_t) - \eta \nabla f_\delta(x_t)^\top v_t + \frac{L_\delta \eta^2}{2} \|v_t\|^2 \\
&= f_\delta(x_t) - \frac{\eta}{2} \|\nabla f_\delta(x_t)\|^2 - \left(\frac{\eta}{2} - \frac{L_\delta \eta^2}{2} \right) \|v_t\|^2 + \frac{\eta}{2} \|v_t - \nabla f_\delta(x_t)\|^2,
\end{aligned}$$

where the first inequality uses the L_δ -smoothness of f_δ in Proposition 2.2; the second line follows the update $x_{t+1} = x_t - \eta v_t$; the last step uses fact $2a^\top b = \|a\|^2 + \|b\|^2 - \|a - b\|^2$ for any $a, b \in \mathbb{R}^d$. \square

D The Proof of Lemma 3.2

Proof. The update of v_{t+1} leads to

$$\begin{aligned}
\mathbb{E}\|v_{t+1} - \nabla f_\delta(x_{t+1})\|^2 &= \mathbb{E}\|v_t - g(x_t; S) + g(x_{t+1}; S) - \nabla f_\delta(x_{t+1})\|^2 \\
&= \mathbb{E}\|v_t - \nabla f_\delta(x_t)\|^2 + \mathbb{E}\|g(x_{t+1}; S) - g(x_t; S) + \nabla f_\delta(x_t) - \nabla f_\delta(x_{t+1})\|^2 \\
&\leq \mathbb{E}\|v_t - \nabla f_\delta(x_t)\|^2 + \frac{1}{b} \mathbb{E}\|g(x_{t+1}; w_1, \xi_1) - g(x_t; w_1, \xi_1)\|^2 \\
&\leq \mathbb{E}\|v_t - \nabla f_\delta(x_t)\|^2 + \frac{\eta^2 M_\delta^2}{b} \mathbb{E}\|v_t\|^2,
\end{aligned}$$

where the first line follows the update rule; the second line use the property of martingale [12, Proposition 1]; the first inequality uses the fact $\mathbb{E}\|\sum_{i=1}^m a_i\|^2 = m\mathbb{E}\|a_1\|^2$ for any i.i.d random vector $a_1, a_2, \dots, a_m \in \mathbb{R}^d$ with zero-mean and Definition 2.6; the second inequality is obtained by Proposition 2.4 and the update $x_{t+1} = x_t - \eta v_t$. \square

E The Proof of Lemma 3.3 and Corollary 3.2

Proof. We just need to plug the result of Corollary 3.1 into Lemma 3.1. \square

F The Proof of Theorem 3.1

Proof. The parameter settings of this theorem guarantee that the term (*) in Corollary 3.2 be positive and we can drop it to obtain inequalities (6) and (7). Using the Jensen's inequality $\mathbb{E}\|\nabla f_\delta(x_{\text{out}})\| \leq \sqrt{\mathbb{E}\|\nabla f_\delta(x_{\text{out}})\|^2}$ and relationship $\nabla f_\delta(x_{\text{out}}) \in \partial_\delta f(x_{\text{out}})$ shown in Proposition 2.2, we know that the output satisfies $\mathbb{E}\text{dist}(0, \partial_\delta f(x)) \leq \epsilon$. The overall stochastic zeroth-order oracle complexity is obtained by plugging definitions in (4) into $T(\lceil b'/m \rceil + 2b)$. \square

G The Proof of Theorem 3.2

Proof. For any $x^* \in \mathcal{X}^*$, it holds that

$$\begin{aligned}
&\mathbb{E}[f_\delta(x_t) - f_\delta(x^*)] \\
&\leq \mathbb{E}[\nabla f_\delta(x_t)^\top (x_t - x^*)] \\
&= \frac{1}{\eta} \mathbb{E}[(x_t - x_{t+1})^\top (x_t - x^*)] \\
&= \frac{1}{2\eta} \mathbb{E}[\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2 + \|x_{t+1} - x_t\|^2]
\end{aligned}$$

Table 3: Summary of datasets used in our SVM experiments

Dataset	n	d
a9a	48,842	123
w8a	64,700	300
covtype	581,012	54
ijcnn1	49,990	22
mushrooms	8,142	112
phishing	11,055	68

$$\leq \frac{1}{2\eta} \mathbb{E} [\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2] + \frac{\eta\sigma_\delta^2}{2},$$

where the first inequality uses the convexity of $f_\delta(\cdot)$ and the second inequality uses Proposition 2.3. Combining the Jensen's inequality, we have

$$\mathbb{E}[f_\delta(x_{\text{out}}) - f_\delta(x^*)] \leq \frac{R^2}{2\eta T} + \frac{\eta\sigma_\delta^2}{2} = \frac{R\sigma_\delta}{2\sqrt{T}}. \quad (12)$$

Since Proposition 2.2 means $\|f - f_\delta\|_\infty \leq L\delta$, the results of (12) implies

$$\mathbb{E}[f(x_{\text{out}}) - f^*] \leq \mathbb{E}[f_\delta(x_{\text{out}}) - f_\delta(x^*)] + 2L\delta \leq \frac{R\sigma_\delta}{2\sqrt{T}} + 2L\delta.$$

□

H Proof of Theorem 3.3 and 3.4

Proof. For Algorithm 4, the complexity is given by

$$\mathcal{O}\left(\frac{d^{3/2}L^4}{\epsilon^4} + \frac{d^{3/2}L^4\zeta}{\delta\epsilon^4} + \frac{dR^2L^2}{\zeta^2}\right).$$

For Algorithm 3, the complexity is given by

$$\mathcal{O}\left(\frac{d^{3/2}L^3}{\epsilon^3} + \frac{d^{3/2}L^3\zeta}{\delta\epsilon^3} + \frac{dR^2L^2}{\zeta^2}\right).$$

Then we tune ζ to achieve the best upper bound using the inequality $3\sqrt[3]{a^2b} \leq 2a\zeta + b/\zeta^2$ for any constant $a > 0, b > 0$. □

I The Details of Experiments

The details of the datasets for the experiments of nonconvex penalized SVM is shown in Table 3. The network used in the experiments of black box attack is shown in Table 4.

Table 4: Network used in the experiments

Layer	Size
Convolution	$5 \times 5 \times 16$
ReLU	-
Max Pooling	2×2
Convolution	$5 \times 5 \times 16$
ReLU	-
Max Pooling	2×2
Linear	3136×128
ReLU	-
Linear	128×10