

Joint Feedback Loop for Spend and Return-On-Spend Constraints

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Abstract

Budget pacing is a popular service that has been offered by major internet advertising platforms since their inception. Budget pacing systems seek to optimize advertiser returns subject to budget constraints through smooth spending of advertiser budgets. In the past few years, autobidding products that provide real-time bidding as a service to advertisers have seen a prominent rise in adoption. A popular autobidding strategy is value maximization subject to return-on-spend (ROS) constraints. For historical or business reasons, the algorithms that govern these two services, namely budget pacing and RoS pacing, are not necessarily always a single unified and coordinated entity that optimizes a global objective subject to both constraints. The purpose of this work is to study the benefits of coordinating budget and RoS pacing services from an empirical and theoretical perspective.

We compare (a) a sequential algorithm that first constructs the advertiser’s ROS-pacing bid and then lowers that bid for budget pacing, with (b) the optimal joint algorithm that optimizes advertiser returns subject to both budget and ROS constraints. We establish the superiority of joint optimization both theoretically as well as empirically based on data from a large advertising platform. In the process, we identify a third algorithm with minimal interaction between services that retains the theoretical properties of the joint optimization algorithm and performs almost as well empirically as the joint optimization algorithm. This algorithm eases the transition from a sequential to a fully joint implementation by minimizing the amount of interaction between the two services.

1 Introduction

Internet advertisers purchase advertising opportunities by bidding in real-time auctions, and, to control their expenditures, it is common for advertisers to set budgets for their campaigns [Goo, b; Fac, b; Twi]. Budget pacing is a popular service offered by most advertising platforms that allows advertisers to specify their budgets and then optimizes advertiser bids in real-time to maximize advertisers’ return subject to the spend being at most the budget. In the past few years, thanks to the increasing availability of ROS-related metrics, and the vastly improved conversion prediction models, autobidding products have seen a prominent rise in adoption [Fac, a; Goo, a]. These are tools that provide value-optimizing real-time bidding subject to return-on-spend (RoS) constraints (on top of the existing budget constraints) as a service to advertisers. Autobidding takes as input high-level advertiser goals like the target cost per conversion or acquisition of an advertiser and places real-time bids on a per-query basis to optimize advertiser returns.

The algorithms that govern budget and RoS pacing, namely value-optimization subject to budget and RoS constraints, are not necessarily always a unified entity that optimizes a global objective. These services

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are often managed by different business units within the same organization or by different organizations altogether (many third-party demand-side platforms offer autobidding services), which results in different algorithms separately choosing/modifying advertisers’ bids. This is not surprising in light of the meaningful gap between the times at which these products gained traction, with budget pacing systems having been standard and popular much earlier. As a result, even if the objectives of both services are aligned, the presence of budget and RoS constraints can introduce inefficiencies in the bidding process. How do the separate and joint pacing services compare? Systematically answering this question, with both theoretical analysis and empirical studies, is the focus of this work.

1.1 Pacing Services

Pacing services are online algorithms that adaptively adjust advertisers’ bids based on auction feedback to maximize certain objectives while satisfying different constraint. Nowadays, a popular paradigm in internet advertising markets is that of *value maximization* [Fac, a; Goo, a]. Unlike the usual quasilinear utility model, where the bidder seeks to maximize the difference between their value and payment, the bidder’s stated objective in autobidding/budgeting products is to maximize their overall value (e.g., the number of conversions or conversion value) while respecting their budget and RoS constraints. For example, a bidder could ask to maximize the total number of conversions they get, subject to spending at most \$1000 and not paying more than \$5 per conversion. Figure 1a illustrates a *joint optimization pacing service*, which takes as input the advertiser’s budget and RoS target, and then automatically bids on behalf of the advertiser in the platform’s auction. Importantly, the pacing services maintain a feedback loop that monitors the real-time spend and conversions from the auction and uses this information to adjust bids.

As we discussed, in many cases, the budget and pacing services maintain separate feedback loops. For historical reasons, budget pacing services are offered by platforms themselves, and ROS pacing services are built on top of them (they are either offered by the same advertising platform or third parties). In Figure 1b we illustrate a typical *sequential pacing service* in which the ROS pacing services feeds bids to the budget pacing service, which, in turn, bids in the platform’s auction. Each service consumes the spend and conversion feedback from the auction to adjust bids dynamically. The benefit of the sequential optimization architecture is decentralization, i.e., it could operate separate modules for budget pacing and ROS pacing.

We also consider a third decentralized architecture (Figure 1c), which we call the *min pacing service*. Rather than organizing the pacing services sequentially, they are organized in parallel. For each auction, the bid is obtained by taking the minimum of the bids generated by the two systems. While more generally one can think about other reduction operations of the two pacing systems’ bids, as we show in this work, the min optimization already performs quite well and approaches the performance of the joint optimization while maintaining some of the benefits of decentralization.

1.2 Our Results

We compare all three algorithms described above, both theoretically and empirically. We next overview the algorithmic implementations of the pacing services, the empirical evaluation, and our theoretical analysis. Our findings consistently establish the superiority of joint optimization both mathematically and empirically.

Algorithmic implementation. In this work, we consider *uniform bidding policies* (which were first proposed and analyzed in [Feldman et al., 2007]) that multiplicatively scale advertisers’ values, which are usually generated using advanced machine learning prediction algorithms [McMahan et al., 2013; He et al., 2014; Zhou et al., 2018; Juan et al., 2016; Lu et al., 2017]. Uniform bidding is appealing for its simplicity, can be shown to be optimal in many settings, and is extensively used in practice [Aggarwal et al., 2019]. The bid multiplier k of the uniform bidding policy is adjusted in real-time using a feedback loop. While many choices are possible for the feedback loop, in this work we consider Lagrangian dual algorithms, which are the work-horse algorithms of budget pacing [Balseiro and Mirrokni, 2022]. At a high level, these algorithms

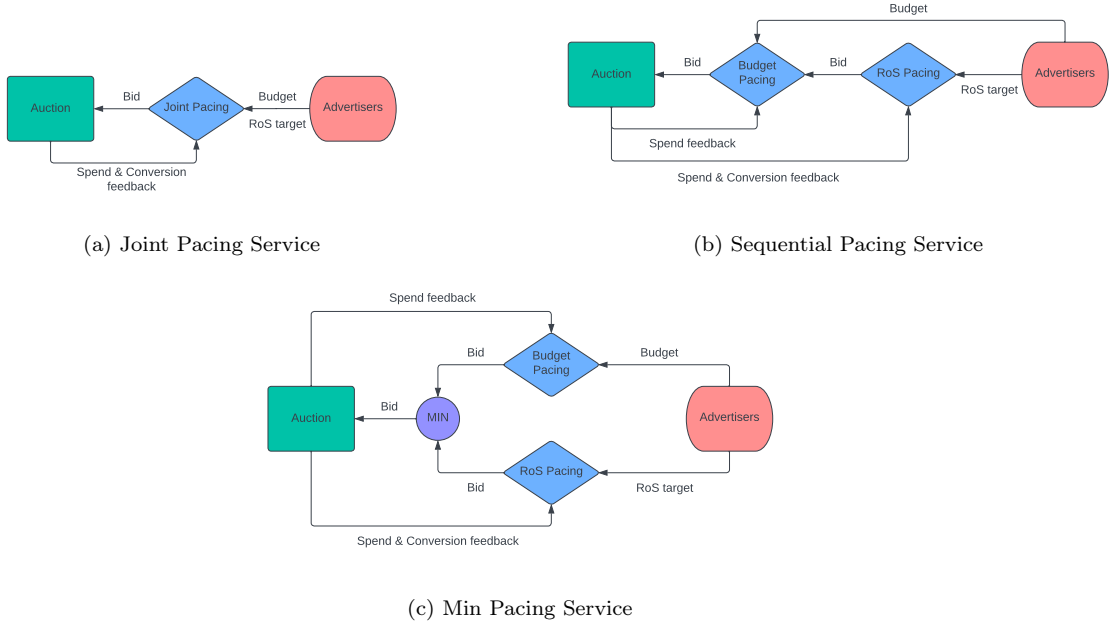


Figure 1: Three Different Pacing Services for Budget and ROS Constraints

introduce a dual variable for each constraint and then adjust these dual variables dynamically using a first-order algorithm. The final bid multiplier is calculated using these dual variables. Dual-based algorithms have strong performance guarantees and have been shown to subsume PID controllers—one of the most popular feedback controllers used in practice [Tashman et al., 2020; Zhang et al., 2016; Smirnov et al., 2016; Yang et al., 2019; Ye et al., 2020; Balseiro et al., 2022c]. Therefore, we believe the algorithms studied in this paper are representative of those used by pacing services in practice. We provide more details on the concrete algorithmic implementation in Section 2.

Empirical evaluation. Section 3 explains in detail our evaluation methodology, including how we construct our semi-synthetic dataset, how we obtain the different quantities in our optimization formulation (1) based on real auction data. Here we give a high-level summary of our result. The objective of the algorithms is to maximize conversion value subject to budget and ROS constraints. In our simulations, as we explain in Section 3 we enforce a hard stop once the budget constraint is violated, but we do not enforce a hard stop for the ROS constraint. This is aligned with practice as budget constraints are usually enforced more strictly than ROS constraints. As a result, we cannot compare conversion values directly because some algorithms might produce solutions that are infeasible, i.e., they could violate the ROS constraints. Therefore, we evaluate the different algorithms as follows. For each algorithm, we determine for each percentual level $z\%$ violation of the ROS constraint, the total conversion value obtained by the algorithm over all the campaigns that violated the constraint by at most $z\%$. By comparing these quantities, we can obtain the following critical insight: what percentage of ROS constraint violation does the naive sequential optimization need, to obtain the same value as the joint optimization does, at say 1% constraint violation, or the min optimization does, at say 5% constraint violation. Such plots are shown in Figure 5b. Similar plots, but instead focusing on the number of campaigns that violate the ROS constraint by $z\%$ is portrayed in Figure 5a.

The high level summary is quite evident from these figures: *the naive sequential optimization needs to violate the ROS constraint by a very significant percentage to approach anywhere near the joint optimization, while the min optimization approaches the joint optimization at a much smaller percentage of ROS constraint violation. Moreover, in sequential pacing, the feedback loops of budget and pacing can lead to unstable*

dynamics. Our findings suggest avoiding the sequential implementation despite its simplicity and appeal, and point towards having the two feedback loops either operating in a centralized manner, or at least communicating meaningfully.

Theoretical evaluation. For our theoretical evaluation, we focus on a continuous time model, as opposed to the discrete-time model for empirical evaluation. Continuous-time models are widely used to analyze the behavior of discrete-time algorithms because they simplify the analysis and yield qualitatively similar insights. Each pacing algorithm we study induces a continuous-time dynamical system. We analyze these dynamical systems, particularly seeking to answer four questions: (a) whether the multiplier k converges to a stationary point, (b) whether the stationary point is unique, (c) whether the value it converges to is optimal and (d) whether the dynamics are stable. Our results are summarized in Table 2. In particular, we show that sequential pacing is not guaranteed to have a stationary point, making the other three questions irrelevant. For joint and min pacing, we answer all four questions in the affirmative using dynamical systems theory.

Overall, our work has implications for the design and operation of pacing services. Our findings suggest that the lack of coordination of sequential pacing can lead to suboptimal and unstable outcomes. Advertising should, whenever possible, adopt algorithms that have some level of coordination between budget and ROS pacing.

1.3 Related Work

Traditional auction theory in microeconomics studies maximizing objectives such as welfare, revenue and gains from trade in the presence of buyer(s) with quasilinear utility, namely, a utility of $v - p$ where v is the value derived and p be the payment. In this work, we adopt a different behavioral model, namely, one where advertisers maximize their value, subject to constraints on the return-on-spend (ROS) and total budget. As mentioned earlier, the significant rise in the adoption of autobidding algorithms in the past few years [Fac, a; Goo, a] motivates the study of this model.

Optimal bidding algorithm for a single value-maximizing bidder with budget and/or ROS constraints. Aggarwal et al. [2019] initiated the study of value-maximizing bidders (value maximizers for short) subject to quite general constraints on value and cost. In particular, their model includes budget and ROS constraints. They show how the uniform bidding strategy is optimal if and only if the underlying auction is truthful (where truthfulness is defined from the point-of-view of a quasilinear bidder). Closest to our work is Feng et al. [2022] who study the advertiser’s value maximization problem in the presence of both budget and ROS constraints in an online repeated auction setting. They show that a close variant of what we call the joint pacing algorithm in this work achieves a $O(\sqrt{T} \log T)$ regret while respecting both the budget and RoS constraints. Their algorithm computes the bid as a function of the two Lagrange multipliers exactly as in Equation (4). A slight difference is that our Algorithm 1 uses dual projected subgradient descent to update the Lagrange multipliers, while theirs uses dual projected subgradient descent for the budget constraint multiplier and a multiplicative update for the ROS constraint multiplier.

Welfare in equilibrium among value maximizers. While the description so far, and also our work, focuses on a single bidder’s optimal bidding problem, the equilibrium under the presence of multiple value maximizing bidders has also been a very active area recently. Aggarwal et al. [2019] show how the VCG mechanism, which is welfare maximizing with quasilinear utility maximizers, can achieve, in the worst case, only a fraction $\frac{1}{2}$ of the optimal social welfare. Recent work by Mehta [2022] shows how randomization can improve the efficiency beyond the $\frac{1}{2}$ guaranteed by VCG, by establishing a POA of 1.89 for 2 bidders and how the POA is unimprovable beyond 2 even with randomized mechanisms when $n \rightarrow \infty$. Liaw et al. [2022] study whether non-truthfulness can improve the POA beyond 2 and show that this is not possible with a deterministic mechanism. But with the combined power of randomization and non-truthful mechanisms, they show how a randomized first-price auction can improve the POA to 1.8 for two bidders, but again show

it is unimprovable beyond 2 when the number of bidders is large. Departing from the no information case studied by the above referenced papers, recent works by [Balseiro et al. \[2021a\]](#); [Deng et al. \[2021\]](#) show how to improve the efficiency under equilibrium beyond $\frac{1}{2}$ by adding boosts and reserves respectively, based on additional information from machine learned advice.

Revenue-optimal auction for value maximizers with budget and/or ROS constraints. Much like the design of optimal auctions for utility-maximizing bidders [[Myerson, 1981](#)], a recent line of work has focused on the design of revenue optimal mechanisms for value maximizers. [Balseiro et al. \[2021b\]](#); [Li et al. \[2020a\]](#) initiate this line of work, studying the revenue optimal mechanism in the presence of RoS constraints, but no budget constraints, under various information structures regarding whether or not the value is private, whether or not the advertiser specified target is private. [Balseiro et al. \[2022a\]](#) extend this work to include budget constraints for advertisers, and consider the information structure where value is public, so are advertiser budgets, but advertiser specified target is private.

Optimal bidding algorithm for a single utility maximizing bidder with & without budget constraint. While works dealing with budget and ROS constraints in the presence of value maximizers have already been discussed, there has been a long line of work on doing the same for utility maximizers, but usually with just budget constraints. When values and competing bids are drawn from i.i.d. distributions, [Balseiro and Gur \[2019\]](#) show that the dual subgradient descent algorithm gives the optimal $O(\sqrt{T})$ regret, and in the adversarial setting they show that it obtains the optimal asymptotic competitive ratio, namely, B/T divided by the maximum value. [Zhou et al. \[2008\]](#) also study pacing in the adversarial setting and give an optimal competitive ratio, but one that is differently parameterized compared to [Balseiro and Gur \[2019\]](#). [Kumar et al. \[2022\]](#) study an episodic setting and show how to compute per-period target expenditures based on estimating the probability density based on samples, and ultimately pace based on these target expenditures. On similar lines [Jiang et al. \[2020\]](#) also show how to obtain the optimal \sqrt{T} regret in a non-stationary setting by first learning the probability distributions and then computing target expenditures based on those, using $T \log T$ samples per distribution. Our paper is also loosely related with the rich literature about *Learning to bid in repeated auctions* [Borgs et al. \[2007\]](#); [Weed et al. \[2016\]](#); [Feng et al. \[2018\]](#); [Balseiro et al. \[2019\]](#); [Han et al. \[2020\]](#), in which the existing papers usually abstract this problem as contextual bandits and do not incorporate budget or ROS constraints into them.

Equilibrium among budget-pacing strategies of utility maximizers. There is a line of work studying equilibrium outcomes of budget pacing agents interacting with each other. We refer the reader to [[Gaitonde et al., 2022](#); [Fikioris and Tardos, 2022](#); [Chen et al., 2021](#); [Conitzer et al., 2022](#); [Balseiro et al., 2015](#)] and the references therein for more on this topic. Interestingly, these papers show that uniform bidding is also optimal in the presence of budget constraints. Also, [Balseiro et al. \[2017\]](#) perform a comprehensive study of different common budget-pacing strategies and compare the system equilibrium in terms of their welfare, platform revenue, and advertiser utility.

Online resource allocation problems. The budget pacing problem discussed in the preceding paragraphs is known to be a special case of online resource allocation problems, which have a long line of work. Most of the literature on this topic has focused on the i.i.d. input model or the slightly more general random permutation model. [Devanur and Hayes \[2009\]](#) introduce a training-based algorithm that learns the optimal dual variables from a batch of initial requests and then uses those to assign the rest of the requests. They show how to obtain a $O(T^{2/3})$ regret for the budgeted allocation problem (also known as the adwords problem) in the random permutation model. [Feldman et al. \[2010\]](#) obtain a $O(T^{2/3})$ regret for more general linear packing problems in the random permutation model. [Agrawal et al. \[2014\]](#) obtain an improved $O(\sqrt{T})$ regret by repeatedly solving for the optimal dual variables at geometrically increasing time lengths. The algorithm of [Kesselheim et al. \[2014\]](#) further solves a linear program at every step and apart from $O(\sqrt{T})$, also obtain the optimal dependence on the number of resources. [Devanur et al. \[2019\]](#) consider more general online packing and covering LPs, but in the i.i.d. model and obtain a $O(\sqrt{T})$ regret with the optimal dependence

on the number of resources. Their algorithm does not need to solve auxiliary linear programs if given an estimate of OPT. [Gupta and Molinaro, 2014; Agrawal and Devanur, 2015; Balseiro et al., 2022b] make the formal connection between dual descent algorithms and online resource allocation, and show how one can use dual descent algorithms as a black box to obtain a $O(\sqrt{T})$ regret. In particular, [Balseiro et al., 2022b; Li et al., 2020b] present simple algorithms that do not require solving auxiliary optimization problems.

2 The setup

In this section, we define a formal model for budget and ROS constraint pacing. We consider a single bidder who participates in T repeated auctions. The bidder derives a value of $v_t \in [0, 1]$ from getting allocated in auction $t = 1, \dots, T$. Upon submitting a bid of b_t , the bidder gets an allocation of $x_t(b_t)$ and an expected payment of $p_t(b_t)$. I.e., $x_t : \mathcal{R}_{\geq 0} \rightarrow \mathcal{R}_{\geq 0}$, and $p_t : \mathcal{R}_{\geq 0} \rightarrow \mathcal{R}_{\geq 0}$ are the allocation and payment functions respectively. The triple (v_t, x_t, p_t) is drawn i.i.d. every round from an unknown distribution. At the beginning of round t , the bidder has knowledge of the value v_t and the historical information of past auctions to decide on a bid, b_t . Denote $\delta_t = (x_t(b_t), p_t(b_t))$ to represent the outcome of the auction at round t . At the end of round t , the bidder observes δ_t . Thus the historical information at the beginning of round t is $h_t = \{(v_s, \delta_s)\}_{s \leq t-1}$.

The optimization objective The advertiser is a value-maximizer and seeks to maximize the overall value while respecting the budget of B dollars and the ROS constraint. Formally, the bidder’s optimization problem is stated as follows:

$$\begin{aligned} & \underset{b_t: t=1, \dots, T}{\text{maximize}} && \sum_{t=1}^T v_t \cdot x_t(b_t) \\ & \text{subject to} && \sum_{t=1}^T p_t(b_t) \leq \sum_{t=1}^T v_t \cdot x_t(b_t), \\ & && \sum_{t=1}^T p_t(b_t) \leq B \end{aligned} \tag{1}$$

The first constraint is the ROS constraint, which states that for every dollar spent, there is at least a dollar of value.¹ The second constraint is the budget constraint. We define the per-round budget by $\rho := B/T$. In round t the bidder bids $b_t = \pi_t(v_t, h_t)$. The function $\pi_t(\cdot, \cdot)$ could be randomized.

Truthful auctions, nontruthful auctions, uniform bidding policy. We restrict attention to a uniform bidding policy in which bids are obtained by multiplicatively adjusting values. More formally, in a uniform bidding policy, one computes a bid multiplier k_t independently of the current value v_t , and the bid submitted is $b_t = k_t \cdot v_t$. If the underlying auction is truthful for quasi-linear utility maximizers², Aggarwal et al. [Aggarwal et al. 2019] showed that the optimal bidding algorithm for problem (1) is indeed a uniform bidding policy, and hence the restriction to uniform bidding is without loss of generality. If the underlying auction is non-truthful, the restriction to uniform bidding can be made without loss if the buyer has access to an optimizer $g_t(v)$ that computes the optimal bid to submit in a one-shot auction for any given true value³ v . In this case, bidding $b_t = g_t(k_t \cdot v)$ would be optimal for the bidder due to the revelation principle.

2.1 The Bidding Algorithms

Despite the simplicity and appeal of uniform bidding, computing the optimal multiplier k_t in uniform bidding requires knowledge of the entire set of $\{v_t, x_t, p_t\}_{t=1..T}$, while information is only revealed in an online manner in our setting. Thus, to approach the performance of uniform bidding policy in an online setting, a standard

¹More generally, one can have the constraint to state that for every dollar spent, there is at least τ dollars of value. But without loss of generality, one can set $\tau = 1$. The update to the bidding formula as a function of τ is quite straightforward, and we skip this here to avoid carrying the notational clutter of τ everywhere.

²An auction is truthful if the allocation function $x_t(b_t)$ is weakly monotonically increasing, and the payment function satisfies $p_t(b_t) = p_t(0) + b_t x_t(b_t) - \int_0^{b_t} x_t(z) dz$.

³If the bidder had access to $x_t(\cdot)$ and $p_t(\cdot)$ before placing the bid at time t , the optimizer is $g_t(v) \in \operatorname{argmax}_b \{v \cdot x_t(b) - p_t(b)\}$.

technique is to dualize the constraints and look at the Lagrangian dual of the problem. Notice that (1) is a constraint optimization problem, and a powerful tool for constraint optimization is Lagrangian duality. We introduce dual variables $\mu \geq 0$ for the budget constraint and λ for the RoS constraint. The Lagrangian dual of the problem (1) is

$$\underset{\lambda \geq 0, \mu \geq 0}{\text{minimize}} \quad \underset{b_t: t=1, \dots, T}{\text{maximize}} \left\{ T\rho\mu + \sum_{t=1}^T \left((1 + \lambda) \cdot v_t \cdot x_t(b_t) - (\mu + \lambda) \cdot p_t(b_t) \right) \right\}. \quad (2)$$

Starting with an arbitrarily initialized dual variables, we update the duals via dual projected subgradient descent (or more generally dual mirror descent) and at each time t , compute the multiplier k_t as a function of the current set of duals. That is, the bidder sets a bid of

$$b_t = k_t \cdot v_t \quad (3)$$

where k_t is computed using the Lagrangian multipliers λ_t and μ_t of the ROS and budget constraints, respectively. The Lagrangian dual variables are updated using dual projected subgradient descent.

Joint Pacing From the mathematical program in (1) it is clear that the optimal solution cannot afford to handle the ROS and budget constraints separately. Thus, the algorithm implementing the optimal solution should perform a joint or a centralized optimization. Indeed, the multiplier k_t is a function of both dual variables, and the algorithm updates both dual variables at every step. We now discuss how to derive the optimal bidding multiplier k_t when the underlying auction is truthful for quasi-linear utility maximizers. Note that the Lagrangian dual problem (2) becomes separable over time after dualizing the constraints. Therefore, at time t , assuming that the dual variables are μ_t and λ_t , the optimal bid is

$$\begin{aligned} b_t &= \underset{b_t}{\text{argmax}} \{ (1 + \lambda_t) \cdot v_t \cdot x_t(b_t) - (\mu_t + \lambda_t) p_t(b_t) \} \\ &= \underset{b_t}{\text{argmax}} \left\{ \frac{1 + \lambda_t}{\mu_t + \lambda_t} \cdot v_t \cdot x_t(b_t) - p_t(b_t) \right\} = \frac{1 + \lambda_t}{\mu_t + \lambda_t} \cdot v_t, \end{aligned} \quad (4)$$

where the second equation follows from extracting the factor $(\mu_t + \lambda_t)$ and the last because the bidder's problem is equivalent to that of bidding in a truthful auction when the value is $(1 + \lambda_t)/(\mu_t + \lambda_t)v_t$. In other words $k_t = (1 + \lambda_t)/(\mu_t + \lambda_t)$. Note that k_t is multiplicatively *inseparable* across λ_t and μ_t .

The dual variables are updated using feedback loops based on the auction feedback that have natural self-correcting features that prevent constraint violations. For example, in the case of the budget constraint, the feedback loop in (6) seeks to equate the actual spend of the auction $p_t(b_t)$ with the per-round budget ρ to satisfy the budget constraint (whenever this constraint is binding). Intuitively, the actual spend being consistently larger than the per-round budget would result in violations of the budget constraint. Therefore, when the actual spend is larger than the per-round budget, the feedback loop (6) reduces the dual variable μ_{t+1} . Smaller dual variables lead to larger bids by (4), which, in turn, results in lower payments. Mathematically, the Lagrangian multipliers variables λ_t and μ_t are updated through dual projected subgradient descent. We refer the reader to [Balseiro and Gur \[2019\]](#); [Feng et al. \[2022\]](#) for more details. [Feng et al. \[2022\]](#) show that this algorithm obtains near-optimal regret $O(\sqrt{T})$, where regret is the difference between the offline optimal total value and the bidding policy's total value.

Sequential Pacing. If one were to consider the problem (1) with just the budget constraint, the bidding policy (from Lagrangian duality with the ROS dual variable $\lambda_t = 0$) would be to bid $b_t = v_t/\mu_t$, with the dual variable μ_t alone getting updated as in Algorithm 1. Similarly, if one were to consider the problem (1) with just the ROS constraint, the bidding policy (from Lagrangian duality with budget dual variable $\mu_t = 0$) would be to bid $b_t = v_t \cdot \frac{1 + \lambda_t}{\lambda_t}$, with the dual variable λ_t alone updated as in Algorithm 1. Given the historical context mentioned earlier, budget pacing systems have been around for longer than ROS pacing

Algorithm 1: Joint updates for λ and μ through dual projected subgradient descent

Initialize: Initial dual variables $\lambda_1 = 1$, $\mu_1 = 0$, total initial budget $B_1 := \rho T$, gradient descent step-sizes α and η ;

for $t = 1, 2, \dots, T$ **do**

Observe the value v_t , and set the bid

$$b_t = \min \left\{ \frac{1 + \lambda_t}{\mu_t + \lambda_t} \cdot v_t, B_t \right\}.$$

Update the dual variable of the ROS constraint

$$\lambda_{t+1} := \text{Proj}_{\lambda \geq 0} \left(\lambda_t - \alpha \cdot (v_t \cdot x_t(b_t) - p_t(b_t)) \right). \quad (5)$$

Update the dual variable of the budget constraint as

$$\mu_{t+1} := \text{Proj}_{\mu \geq 0} \left(\mu_t - \eta \cdot (\rho - p_t(b_t)) \right). \quad (6)$$

Update the leftover budget $B_{t+1} = B_t - p_t(b_t)$;

optimization. Therefore, it is not unexpected to have separate servers handling the feedback loops of the budget and ROS constraints and the final bid constructed in a sequential manner, namely,

$$b_t = \frac{1 + \lambda_t}{\lambda_t} \cdot \frac{1}{\mu_t} \cdot v_t. \quad (7)$$

In other words, the ROS constraint pacing service determines an intermediary bid $\widehat{b}_t = (1 + \lambda_t)/\lambda_t \cdot v_t$ which is fed to the budget service and, in turn, the budget pacing service operates on the scaled bid \widehat{b}_t to get the final bid of \widehat{b}_t/μ_t . While not optimal, this implementation has the benefit of being decentralized, i.e., it could operate separate servers for budget pacing and ROS pacing, that (a) only communicate the temporary bid \widehat{b}_t and (b) could update their respective variables at different frequencies.

Min Pacing. If the transition from sequential to joint optimization proves prohibitively expensive in the short term for organizational or engineering reasons, we propose and study another decentralized optimization, that we call the min pacing service. Rather than applying the bid-lowering operations of the two pacing systems sequentially, we take the minimum of the bids generated by the two systems:

$$b_t = \min \left\{ \frac{1 + \lambda_t}{\lambda_t} \cdot v_t, \frac{1}{\mu_t} \cdot v_t \right\}. \quad (8)$$

The corresponding dual variables are updated as in Algorithm 1. The min pacing service operates in parallel instead of sequentially and also requires minimum coordination between budgeting and ROS pacing.

3 Empirical Study

We empirically evaluate the three algorithms discussed in Section 2.1. For confidentiality and advertiser privacy reasons, we evaluate their performance on a semi-synthetic dataset based on actual online advertising auctions. In particular, we focus on advertising campaigns from an online advertising platform that use a bidding product which is captured by our optimization formulation (1). More specifically, an advertiser bids (and therefore also pays) for clicks, i.e., submits bids for cost-per-click, and the objective is to maximize expected acquisitions (e.g. site visits, calls, conversions) with constraints on total spend being below an

input budget and average cost per acquisition below an input target cost ($tcpa$). In our formulation (1), this corresponds to:

1. The value v_t is equal to $tcpa \cdot pconv_t$, where $pconv_t$ is the probability of a conversion conditioned on a click (note both $tcpa$ and $pconv_t$ are taken to be independent of the bid; while it is obvious for $tcpa$ to be independent of the bid, $pconv$'s independence is supported by empirical studies [Varian \[2009\]](#));
2. The allocation $x_t(b_t)$ is the number of clicks won by the advertiser at a bid of b_t ;
3. The payment $p_t(b_t)$ is the cost of the clicks won at a bid of b_t .

In the rest of this section, we will discuss our dataset construction, the evaluation framework, and present our empirical results.

3.1 Semi-synthetic Dataset

Since we study the stochastic setting where the functions $x_t(\cdot), p_t(\cdot)$ are drawn i.i.d. from some distribution, our dataset consists of a set of generative models. The parameters of the generative model for any given (actual) advertising campaign we study are derived from the performance of that campaign in the (actual) auction. We now discuss in more detail the generative model itself and how we pick the parameters for each campaign.

Bidding Landscape. To see how the auction performance of a campaign determines its model parameters in the generative model, it is useful to begin with the notion of a bidding landscape. For each campaign C , we construct a bidding landscape as a function from bids to the (predicted) number of clicks and cost. This is done first at a per-query level using auction simulation. In more detail, for an ad opportunity (a.k.a. query) q where campaign C is eligible to show its advertisement, we look at the logged bids of all the other campaigns participating in the auction for this query q , and simulate the auction for any bid b of C to know if/where C 's ad would be shown. This gives us the predicted number of clicks and cost per click corresponding to any particular bid b , and we refer to them as $click_{C,q}(b)$ and $cost_{C,q}(b)$. In our model, we use the actual (i.e., advertiser submitted) target cost per acquisition of C as $tcpa(C)$, and the logged average predicted conversion probability generated by the production machine-learning model as $pconv_q(C)$. For a query q , bids are given by $b = k \cdot v_q$ where $v_q = tcpa(C) \cdot pconv_q(C)$ is the value of the query.

We aggregate these single-query landscape functions to get C 's daily bidding landscape by summing up the respective functions over all the queries in a day, e.g., $click_C(k) = \sum_q click_{C,q}(k \cdot v_q)$, and $cost_C(k) = \sum_q cost_{C,q}(k \cdot v_q)$. Note that these functions are non-decreasing in k . The per-query bidding landscapes are inherently step functions represented by the various bid thresholds that makes C 's ad to be displayed at various positions (or not displayed at all). While the aggregated landscapes are already smoother than the per-query landscapes, we further smooth the aggregated landscapes by linearly interpolating between consecutive thresholds. See [Figure 2](#) for an example of the aggregated daily bidding landscape of an ad campaign.⁴

Generative Model. We will use a i.i.d. stochastic model to generate the $x_t(b_t)$ and $p_t(b_t)$ in iteration t as a function of bid b_t (as discussed earlier, we slightly abuse notation to use b_t to be the multiplier to $tcpa \cdot pconv$). We use a Poisson distribution for $x_t(b_t)$ (i.e. number of clicks in an iteration at a bid b_t). With parameter λ , its probability mass function is $f(x; \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$. The parameter λ is the expected number of clicks in an iteration, and we set it using the bidding landscape. In particular, for the model corresponding to a campaign C , the expected number of clicks at bid b_t in an iteration would be $\lambda_C(b_t) = click_C(b_t)/T$ where T is the total number of iterations in a day. In our empirical evaluation, we pick $T = 144$, which translates to each iteration being a 10-minute period, i.e., the dual variables of the algorithms are updated

⁴We normalize the values of click, value and cost in all the plots of this section, so the quantities shown do not represent real traffic or revenue.

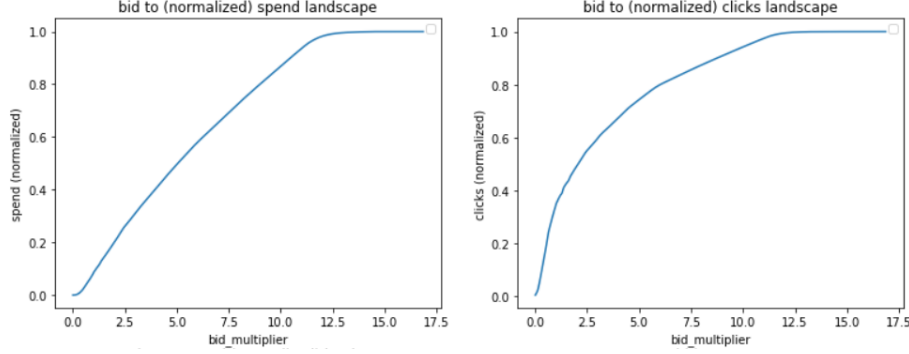


Figure 2: The bidding landscape of an example campaign. The x-axis is the bid (as a multiplier to value), and the y-axis are the daily cost and number of clicks (all normalized to be in $[0, 1]$) respectively.

every 10 minutes instead of after every auction. We also derive from the bidding landscape a cost-per-click $cpc_C(b_t) = \frac{cost_C(b_t)}{click_C(b_t)}$.

To summarize, the value, click and cost at bid b_t are as follows:

$$v_t = tpa(C) \cdot pconv_t(C), \quad x_t(b_t) \sim \text{Poisson}(\lambda_C(b_t)), \quad p_t(b_t) = x_t(b_t) \cdot cpc(b_t) \cdot noise_p$$

where we introduce i.i.d. non-negative multiplicative noise $noise_p$ with expected value 1 to the cost. In our evaluation, we use a Gaussian distribution centered at 1 with standard deviation 0.1 and truncated to be within $[0, 2]$ (so it's non-negative and has expected value 1). Also, when empirically evaluating the tCPA campaigns, the conversion rates $pconv_t(C)$ are drawn from a Gaussian distribution centered at the average $pconv$ of the campaign with a standard deviation of 0.1 and truncated to be in $[0, 2]$.

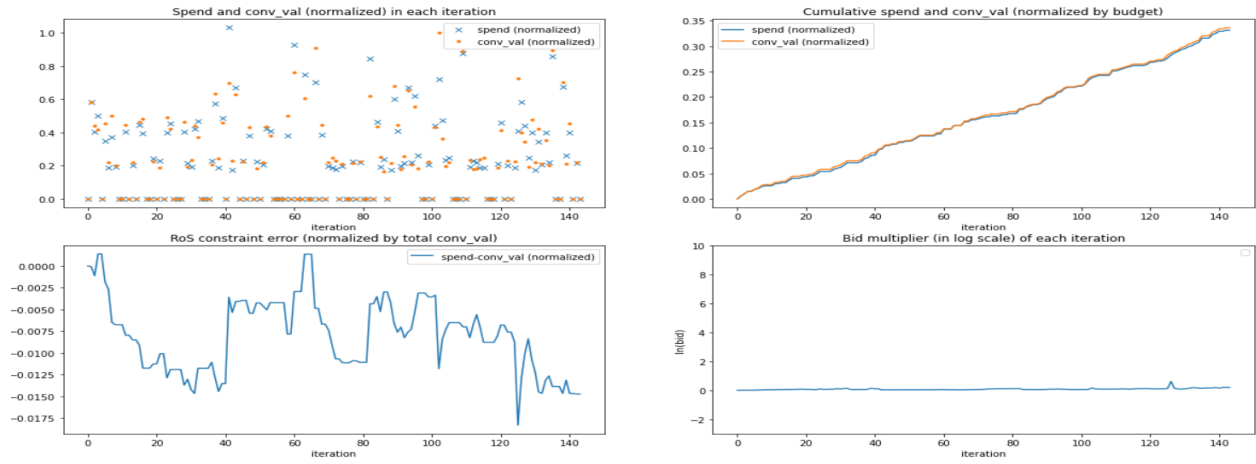
3.2 Evaluation Setup

We randomly select 10,000 campaigns and construct their bidding landscapes as discussed. For each campaign, we set the budget constraint (i.e. ρT in (1)) using its actual daily budget B . We divide the day into 10-minute periods and use $T = 144$. To evaluate an algorithm, in each iteration, after the algorithm gives the bid it wants to submit, we compute the allocation and payment x_t, p_t using our generative model to get the number of clicks and cost of that iteration, and let the algorithm update the bid for the next iteration. We sum up the total cost and value through all T iterations. For the budget constraint, we follow the common practice to always strictly enforce it as follows: if in an iteration the generated cost is larger than the remaining budget, we modify that iteration's cost and value both to be 0. We do not enforce the ROS constraint strictly⁵, but of course, measuring how much the different algorithms violate the ROS constraint is an important aspect of this study and will be discussed here. In Figure 3 we visualize the simulation of the joint pacing algorithm and sequential pacing algorithm on an example campaign.

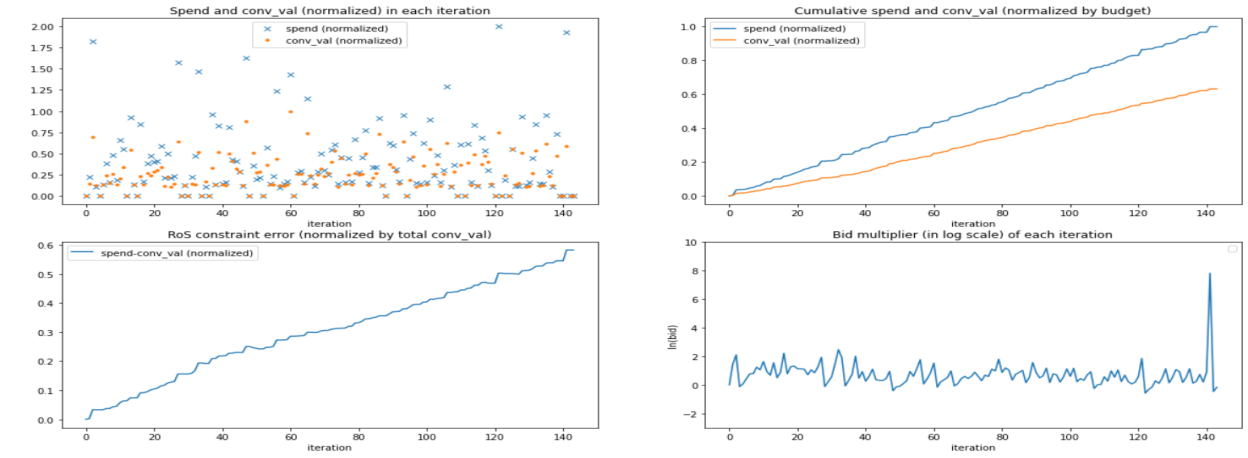
In our empirical evaluation, for each campaign, we simulate an algorithm 10 times to take the average total $spend$ and total $conv_val$ or conversion value as the result of the algorithm on that campaign. For each algorithm, we take the 10,000 $(spend, conv_val)$ pairs from all the campaigns, and arrange them into buckets based on the relative ROS constraint error⁶ $\max(0, spend/conv_val - 1)$. For each bucket, we sum up the $conv_val$ of all the campaigns in it. Moreover, for each algorithm, we do a grid search over the step-sizes

⁵Note that it is always possible for an ROS constraint to be temporarily violated after t rounds, but in the $t + 1$ -th round it could become satisfied because of a really high value query coming through at low cost. Therefore it is suboptimal to stop serving right after ROS constraint gets violated in a round. This is not the case for budget constraint: once violated, it always remains violated because cumulative spend is monotonically increasing.

⁶ROS constraint states that $spend \leq conv_val$. So a constraint violation would imply $spend > conv_val$, i.e., $spend/conv_val - 1 > 0$.



(a) Joint Pacing Algorithm



(b) Sequential Pacing Algorithm

Figure 3: Simulation of the joint bidding (top) and sequential bidding algorithms (bottom) on an example campaign. We plot the per-iteration *value* and *cost* (normalized so *value* $\in [0, 1]$), cumulative *value* and *cost* (normalized by budget), cumulative ROS error (as *cost* - *value* normalized by total *value*), and bids *k* (in log scale).

used in the dual variables’ updates. Each pair of step-sizes (one for each dual variable) is evaluated over the entire dataset, and for each algorithm we pick the best pair of step-sizes according to the total *conv_val* in the bucket of zero ROS constraint error. We compare the results associated with the best step-sizes for each algorithm.

Benchmark. For each campaign, our benchmark is the fluid relaxation of (1), but restricted to uniform bidding, i.e., $b_t = k \cdot v_t$ for all t . Formally, the benchmark is given by

$$\begin{aligned} & \underset{k \geq 0}{\text{maximize}} && \sum_{t=1}^T \mathbb{E}[v_t \cdot x_t(k \cdot v_t)] \\ & \text{subject to} && \sum_{t=1}^T \mathbb{E}[p_t(k \cdot v_t)] \leq \sum_{t=1}^T \mathbb{E}[v_t x_t(k \cdot v_t)], \\ & && \sum_{t=1}^T \mathbb{E}[p_t(k \cdot v_t)] \leq \rho T. \end{aligned} \tag{9}$$

It is easy to see that in the stochastic i.i.d. model, the optimal value of (9) is an upper bound on the expectation of the ex-post optimal value. In our generative model, by design we have

$$\begin{aligned} \text{conv_val}_C(k) &= \mathbb{E}[v_t x_t(b_t)] = \frac{\text{tcpa}(C)}{T} \cdot \mathbb{E}[\text{pconv}_t(C) \cdot \text{click}_C(k \cdot \text{tcpa}(C) \cdot \text{pconv}_t(C))], \\ \text{spend}_C(k) &= \mathbb{E}[p_t(b_t)] = \mathbb{E}\left[x_t(b_t) \frac{\text{cost}_C(b_t)}{\text{click}_C(b_t)}\right] \mathbb{E}[\text{noise}_p] = \frac{1}{T} \mathbb{E}[\text{cost}_C(k \cdot \text{tcpa}(C) \cdot \text{pconv}_t(C))], \end{aligned}$$

where the expectation is taken with respect to the distribution of conversion probabilities of the different queries.

Our benchmark for campaign C in (9) becomes

$$\begin{aligned} & \underset{k \geq 0}{\text{maximize}} && \text{conv_val}_C(k) \\ & \text{subject to} && \text{spend}_C(k) \leq \text{conv_val}_C(k), \\ & && \text{spend}_C(k) \leq \rho. \end{aligned} \tag{10}$$

In our experiments, we approximate $\text{conv_val}_C(k)$ and $\text{spend}_C(k)$ by performing a certainty equivalent approximation in which we replace random quantities (i.e., the predicted conversion probabilities) by their expected values. We solve the above optimization problem on the bidding landscape functions by finding the largest bid multiplier k^* such that $\text{spend}_C(k^*)$ is below C ’s budget and $\text{conv_val}_C(k^*) \geq \text{spend}_C(k^*)$. Such multiplier k^* is easy to find using a line search since our landscape functions are all monotone in k . Furthermore, the restriction to uniform bidding in (9) is without loss of generality when the $\text{conv_val}_C(k)$ versus $\text{spend}_C(k)$ function is concave, which qualitatively holds in our data (e.g. Figure 4).

We use $\text{conv_val}_C(k^*)$ as computed above as the benchmark for C (see Figure 4 for examples). Note this captures the expected optimal solution, but algorithms running on the generative model of C may achieve better ex-post value than the benchmark due to the stochasticity of the model. We add up the expected optimal value over all campaigns as the overall benchmark. Figure 4 shows the pairs of spend and conversion value levels that can be achieved by varying the bidding multiplier k for a typical campaign. The achievable curves $(\text{spend}_C(k), \text{conv_val}_C(k))_{k \geq 0}$ lie in \mathbb{R}_+^2 , start at the origin for $k = 0$, increase along both axis as the bid multiplier increases, and end at $k \rightarrow \infty$. For each algorithm, we look at the cumulative total value achieved by the algorithm through the ROS violation buckets. That is, for the bucket of at most $z\%$ relative error in the ROS constraint, we get the total value over all campaigns such that the algorithm has a relative ROS violation of at most $z\%$. The cumulative total value over ROS error buckets gives us the picture of how an algorithm performs with respect to both the optimization objective and the constraints.

3.3 Result

We show the performance evaluations of the three algorithms in Table 1, where each column is associated with a particular error bound, and we show the cumulative fraction of campaigns (top) and cumulative total

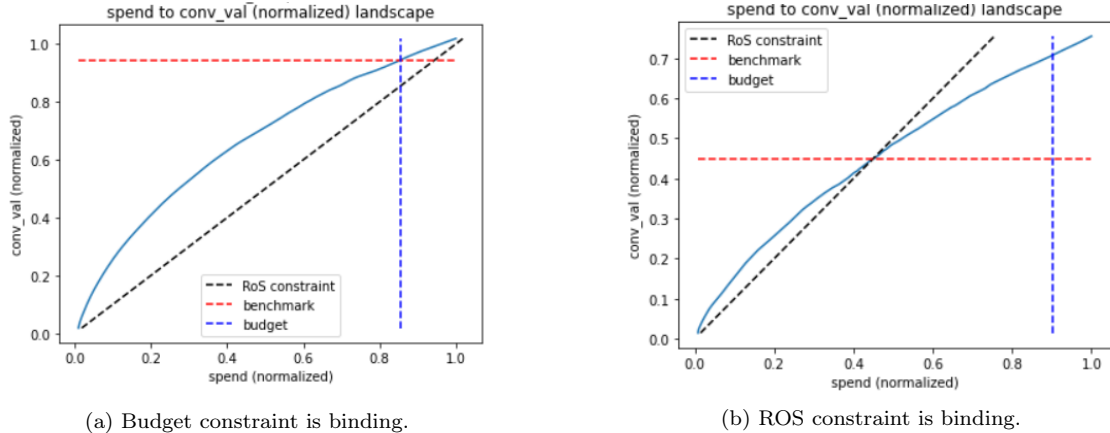


Figure 4: The expected optimal of an example campaign. The achievable curve (solid blue) delineates the pairs of spend-conversion value pairs that can be achieved by different bidding multipliers. The black diagonal dotted line captures the ROS constraint (feasible pairs should lie above this line), the blue vertical dotted line captures the budget constraint (feasible pairs should lie to the left of this line). We prove in Corollary 1 that the optimal operating point is the smallest of the intersection points of the achievable curve with one of the constraints and is shown using the red horizontal dotted line. In (a) the budget constraint is binding, while in (b) the ROS constraint is binding.

value of campaigns (bottom) with relative ROS error up to the bound in each column. We normalize the quantities in the table: for value we normalize by our benchmark, i.e. sum of expected opt for all campaigns, and for number of campaigns we normalized by the total number 10000. We look at the results both in terms of how well the algorithms respect the ROS constraint, and also the optimization objective of value maximization.

ROS constraint. The joint pacing algorithm performs the best: 62% of the campaigns satisfy the ROS constraint exactly, and a reasonably large fraction 80% of campaigns finish with at most 20% relative ROS error. The min pacing algorithm performs clearly worse than the joint pacing algorithm as fewer campaigns finish with low relative ROS error, although a large enough fraction (80%) still finishes with at most 20% error. The sequential pacing algorithm performs poorly in obeying the ROS constraint: only around 11% of campaigns satisfy the ROS constraint, and in Figure 5(a) we can see a considerable fraction $> 20\%$ of campaigns have spend more than twice the conversion value, i.e. $< 80\%$ of campaigns in the relative ROS error ≤ 1.0 bucket.

Value maximization. The joint pacing algorithm also does fairly well at approximating the benchmark value. Recall our benchmark on each campaign should be fairly close to the expected optimal value of the fluid relaxation where both the ROS constraint and budget constraint are satisfied on expectation, so it is roughly an upper bound on the expected offline or hindsight optimal. The joint pacing algorithm achieves a very large fraction of the benchmark with fairly small ROS error, e.g., the campaigns with at most 15% relative ROS error in total get 81% of the total benchmark conversion values over all campaigns, and the total conversion value of all campaigns is about 88% of the total benchmark. The min pacing algorithm performs reasonably well and in total achieves 94% of the benchmark. Note this is slightly larger than the 88% for the joint pacing algorithm, but this comes at the cost of having larger ROS error on a relatively large fraction of campaigns compared to the joint pacing algorithm. Similarly, the sequential pacing algorithm achieves 144% of the benchmark when summed over all campaigns, but a majority of the campaigns finish with very large ROS error, while the benchmark satisfies the constraints on expectation. If we look at the cumulative value achieved by campaigns finishing with no ROS error, both the min bidding and sequential pacing algorithms get considerably smaller total value compared to the joint pacing algorithm.

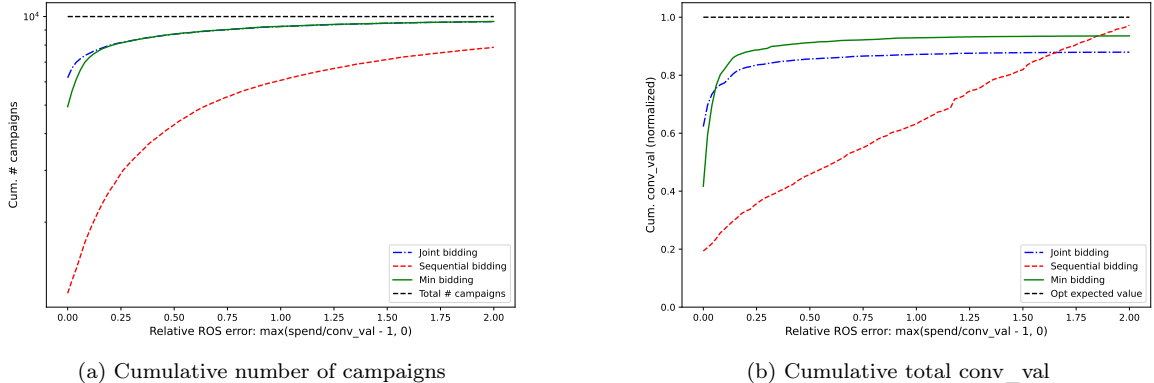


Figure 5: The cumulative number of campaigns and total conv_val for each algorithm over the ROS relative error buckets.

Stability and convergence We observe that the trajectory of bidding multipliers generated by the joint and min pacing algorithms converge to the optimal solution of the benchmark (10). Figure 3 shows a representative campaign for which the ROS constraint is binding in the benchmark (but the budget constraint is not). After a small learning phase, the joint pacing algorithm converges to the optimal multiplier of around $k^* \approx 1$. The return-on-spend constraint is mostly obeyed and the total spend is smaller than the budget.

For the sequential pacing algorithm, however, we do not observe the convergence of bid multipliers. In Figure 3, it can be seen that the bid multipliers generated by the sequential pacing algorithm for the same campaign are highly unstable. Moreover, the ROS constraint is violated by a significant amount and the budget is exactly depleted by the end of the horizon. Interestingly, the behavior of the sequential pacing algorithm is driven by conflicting feedback loops between the budget and ROS pacing services. Recall that, at optimality, only the ROS constraint should bind. Initially, as the ROS pacing service detects a violation of the ROS constraint, it starts increasing its dual variable λ_t to satisfy the constraint. This results in a smaller bid multiplier k_t and reduced spend. The budget pacing service, however, believing that the budget constraint is not binding reacts to the lower spend by decreasing its dual variable μ_t , which in turn, results in a higher multiplier. These two opposing feedback loops generate unstable dynamics and one constraint ends up being violated. Similar behaviors are observed across campaigns even when the budget constraint is binding.

Table 1: Cumulative fraction of campaigns and total conversion (normalized by total benchmark) over the ROS relative error buckets.

		Relative Constraint Violation											
Alg		(\leq)0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	∞
Frac. of Campaigns	Joint	0.62	0.71	0.75	0.78	0.80	0.82	0.83	0.84	0.85	0.86	0.87	1.00
	Min.	0.49	0.64	0.72	0.77	0.80	0.81	0.83	0.84	0.85	0.86	0.87	1.00
Seq.	Seq.	0.11	0.15	0.18	0.22	0.26	0.29	0.32	0.35	0.38	0.40	0.43	1.00
	Cum. Total Value	Joint	0.62	0.75	0.77	0.81	0.83	0.84	0.84	0.85	0.85	0.86	0.88
Min.	Min.	0.42	0.73	0.82	0.86	0.88	0.89	0.89	0.89	0.90	0.91	0.91	0.94
	Seq.	0.19	0.23	0.27	0.30	0.33	0.36	0.38	0.40	0.42	0.44	0.46	1.44

4 Theoretical Results

In this section, we consider a continuous-time approximation of the three algorithms mentioned in the previous sections, and study their asymptotic behaviors. Compared to the previous section, there are two major differences: (1) We consider the continuous-time limit in this section, while the dynamics in the previous section are discrete-time; (2) We assume the multipliers are updated using the expected gradients in this section, while noisy stochastic gradients are used in the previous section. Continuous-time are widely

used to analyze the behavior of discrete-time algorithms (citations) because they simplify the analysis and yield qualitatively similar insights. It is genuinely the case that the discrete-time algorithms have the same topological behaviors as their continuous-time counterparts as long as the step-size is small enough and the continuous-time dynamic is stable and non-degenerate [Su et al. \[2014\]](#); [Lu \[2022\]](#).

More specifically, we consider the continuous-time fluid model with uniform bidding (9). Let $k(t)$ be the uniform multiplier at time t . We denote

$$\mathit{spend}(k) = \mathbb{E}_v[\rho(k \cdot v)] , \quad \mathit{conv_val}(k) = \mathbb{E}_v[v \cdot x(k \cdot v)]$$

as the expected spend and the expected conversion value with multiplier k . The primal problem is (10). The dual problem becomes $\min_{\mu \geq 0, \lambda \geq 0} D(\mu, \lambda)$ where

$$D(\mu, \lambda) := \underset{k \geq 0}{\text{maximize}} \{ (1 + \lambda)\mathit{conv_val}(k) + \rho \cdot \mu - (\lambda + \mu)\mathit{spend}(k) \} ,$$

is the dual function. We utilize projected gradient descent to update the dual variables $\mu(t)$ and $\lambda(t)$, which in turn determines the multiplier by the policies. The continuous limit of the dynamic for $\mu(t)$ and $\lambda(t)$ follows from the following differential inclusion (this specific differential inclusion is also called projected gradient flow):

$$\begin{aligned} \dot{\mu}(t) &\in -\eta(\rho - \mathit{spend}(k(t)) + \partial \mathbb{1}_{\mathbb{R}^+}(\mu(t))) \\ \dot{\lambda}(t) &\in -\alpha(\mathit{conv_val}(k(t)) - \mathit{spend}(k(t)) + \partial \mathbb{1}_{\mathbb{R}^+}(\lambda(t))) , \end{aligned} \tag{11}$$

where $\mathbb{1}_{\mathbb{R}^+}$ is the indicator function onto the non-negative orthant, and $\partial \mathbb{1}_{\mathbb{R}^+}(a) = \{0\}$ if $a > 0$ and $\partial \mathbb{1}_{\mathbb{R}^+}(a) = \mathbb{R}^+$ if $a = 0$ is its differential set at point a . More specifically, the differential inclusion (11) can be specified as

$$\begin{aligned} \dot{\mu}(t) &= \begin{cases} 0 & \text{if } \mu(t) = 0, \rho - \mathit{spend}(k(t)) > 0, \\ -\eta \cdot (\rho - \mathit{spend}(k(t))) & \text{otherwise,} \end{cases} \\ \dot{\lambda}(t) &= \begin{cases} 0 & \text{if } \lambda(t) = 0, \mathit{conv_val}(k(t)) - \mathit{spend}(k(t)) > 0, \\ -\alpha \cdot (\mathit{conv_val}(k(t)) - \mathit{spend}(k(t))) & \text{otherwise.} \end{cases} \end{aligned}$$

Intuitively, in the continuous-time dynamics, the projection operator only impacts the evolution of the dual variables when these are zero, i.e., they lie at the boundary non-negative real line. Moreover, when dual variables are zero, they can move into the interior if the gradient is negative and stays at the boundary if the gradient is positive. The bidding multipliers for the three algorithms are given by

$$k_{\text{JOINT}}(t) = \frac{1 + \lambda(t)}{\mu(t) + \lambda(t)} \tag{12}$$

$$k_{\text{SEQ}}(t) = \frac{1 + \lambda(t)}{\lambda(t)} \frac{1}{\mu(t)} \tag{13}$$

$$k_{\text{MIN}}(t) = \min \left(\frac{1 + \lambda(t)}{\lambda(t)}, \frac{1}{\mu(t)} \right) . \tag{14}$$

We are interested in understanding the behavior of the continuous-time dynamic system induced by the different algorithms. In particular, for each different algorithm, whether $k_{\text{ALG}}(t)$ converges to a stationary point as $t \rightarrow \infty$ (where $k_{\text{ALG}} \in \{k_{\text{JOINT}}, k_{\text{SEQ}}, k_{\text{MIN}}\}$), we seek to understand whether the limiting bidding multiplier is optimal, and whether the dynamics are stable. More formally, the stationary point of the differential inclusion (11) is defined as:

Definition 1. *The stationary points $(\lambda_{\text{ALG}}^*, \mu_{\text{ALG}}^*, k_{\text{ALG}}^*)$ of the differential inclusion (11) satisfy*

$$\begin{aligned} \mathit{spend}(k_{\text{ALG}}^*) \leq \rho &\perp \mu_{\text{ALG}}^* \geq 0, \\ \mathit{spend}(k_{\text{ALG}}^*) \leq \mathit{conv_val}(k_{\text{ALG}}^*) &\perp \lambda_{\text{ALG}}^* \geq 0, \end{aligned} \tag{15}$$

where the complementary condition \perp means that one of the two inequalities should bind, and k_{ALG}^* is given by one of the three corresponding bidding strategies specified in (12)-(14) with λ_{ALG}^* and μ_{ALG}^* .

	Existence	Uniqueness	Optimality	Stability
Joint	✓	✓	✓	✓
Sequential	×	N/A	N/A	N/A
Min	✓	✓	✓	✓

Table 2: Summary of the behaviors of different policies.

Throughout this section, we have two assumptions:

Assumption 1. *We assume that the functions $\text{spend}(k)$ and $\text{conv_val}(k)$ are continuous in k . Moreover, the following hold:*

1. *There exists a unique $k^{\text{BUDGET}} > 0$ such that $\text{spend}(k^{\text{BUDGET}}) = \rho$ and $\lim_{k \rightarrow \infty} \text{spend}(k) > \rho$.*
2. *There exists a unique k^{ROS} such that $\text{spend}(k^{\text{ROS}}) = \text{conv_val}(k^{\text{ROS}})$, $\text{spend}(0) = \text{conv_val}(0) = 0$, $\lim_{k \rightarrow \infty} \text{spend}(k) - \text{conv_val}(k) > 0$ and a small enough ϵ such that $\text{spend}(\epsilon) - \text{conv_val}(\epsilon) < 0$.*

As a direct consequence of the Assumption 1 we obtain that the budget and RoS constraint satisfy a single-crossing property: they cross the k -axis once and from below.

Remark 1. *It holds by utilizing the continuity of function spend and conv_val and Assumption 1 that*

1. *For any $0 \leq k < k^{\text{BUDGET}}$, we have $\text{spend}(k^{\text{BUDGET}}) < \rho$ and for any $k > k^{\text{BUDGET}}$, we have $\text{spend}(k^{\text{BUDGET}}) > \rho$.*
2. *For any $0 < k < k^{\text{ROS}}$, we have $\text{spend}(k) - \text{conv_val}(k) < 0$ and for any $k > k^{\text{ROS}}$, we have $\text{spend}(k) - \text{conv_val}(k) > 0$.*

Assumption 2. *The problem is non-degenerate, namely, $k^{\text{BUDGET}} \neq k^{\text{ROS}}$.*

The non-degenerate assumption guarantees that only one of the budget constraint or the ROS constraint can be binding for the uniform bidding policy. In practice, the data comes from a random process, and the budget ρ and λ are given by the advertiser. Notice that the degenerate case stays in a lower dimension manifold, thus it is very likely that the non-degenerate assumption holds.

Table 2 summaries the existence, uniqueness and optimality of the stationary points for the three dynamics, as well as whether the dynamic converges to the stationary point. In the rest of this section, we present detailed analyses of these results. We begin with the joint pacing algorithm.

Theorem 1. *Consider the joint pacing algorithm given by (11) and (12). Then it holds that:*

1. **(Existence and Uniqueness).** *There exists a unique stationary point $(\lambda_{\text{JOINT}}^*, \mu_{\text{JOINT}}^*, k_{\text{JOINT}}^*)$ of the dynamics.*
2. **(Optimality).** *The bidding multiplier k_{JOINT}^* of the stationary point is an optimal multiplier for the problem.*
3. **(Stability).** *The dynamics converge to the unique stationary point from any initial solution.*

Proof. We prove each part at a time.

Part 1 (Existence and Uniqueness) Suppose $(\mu_{\text{JOINT}}^*, \lambda_{\text{JOINT}}^*, k_{\text{JOINT}}^*)$ is a stationary point to the joint dynamics (11)-(12), then we have by definition that

- $\rho = \text{spend}(k_{\text{JOINT}}^*)$ or $\mu_{\text{JOINT}}^* = 0$ and $\rho - \text{spend}(k_{\text{JOINT}}^*) > 0$;
- $\text{conv_val}(k_{\text{JOINT}}^*) = \text{spend}(k_{\text{JOINT}}^*)$ or $\lambda_{\text{JOINT}}^* = 0$ and $\text{conv_val}(k_{\text{JOINT}}^*) - \text{spend}(k_{\text{JOINT}}^*) > 0$.

It is easy to see that $(0, 0)$ is not a stationary point for the dynamic, otherwise, we have $k_{\text{JOINT}}^* = \infty$ from (12), and $\text{conv_val}(k_{\text{JOINT}}^*) - \text{spend}(k_{\text{JOINT}}^*) < 0$, violating (2). Therefore, it must be the case that $k_{\text{JOINT}}^* = k^{\text{BUDGET}}$ or $k_{\text{JOINT}}^* = k^{\text{ROS}}$, namely, one of the two constraints must hold at the optimal solution.

If $k_{\text{JOINT}}^* = k^{\text{BUDGET}}$, i.e., the budget constraint binds, then $\lambda_{\text{JOINT}}^* = 0$ and $\text{conv_val}(k^{\text{BUDGET}}) - \text{spend}(k^{\text{BUDGET}}) > 0$, thus it must be the case that $k^{\text{BUDGET}} < k^{\text{ROS}}$ by utilizing Remark 1 (2). In this case, we have $\mu_{\text{JOINT}}^* = \frac{1}{k^{\text{BUDGET}}}$ by (12). If $k_{\text{JOINT}}^* = k^{\text{ROS}}$, i.e., the ROS constraint binds, then $\mu_{\text{JOINT}}^* = 0$ and $\rho - \text{spend}(k^{\text{ROS}}) > 0$, thus it must be the case that $k^{\text{ROS}} < k^{\text{BUDGET}}$ by utilizing Remark 1 (1). In this case, we have $\lambda_{\text{JOINT}}^* = \frac{1}{1 - k^{\text{ROS}}}$ by (12).

The above argument gives raise to a formula of the unique $(\mu_{\text{JOINT}}^*, \lambda_{\text{JOINT}}^*)$.

Part 2 (Optimality) By the design of the pacing algorithm, we have $k(t) = \arg \max_k (1 + \lambda) \text{conv_val}(k) + \rho - (\lambda + \mu) \text{spend}(k)$, thus $\rho - \text{spend}(k(t)) \in \partial_{\mu} D(\mu(t), \lambda(t))$ and $\text{conv_val}(k(t)) - \text{spend}(k(t)) \in \partial_{\lambda} D(\mu(t), \lambda(t))$. By the convexity of $D(\mu, \lambda)$ and the definition of $(\mu_{\text{JOINT}}^*, \lambda_{\text{JOINT}}^*)$, we know that $(\mu_{\text{JOINT}}^*, \lambda_{\text{JOINT}}^*)$ is an optimal solution to the dual problem. Notice that uniform bidding is an optimal policy, and the multiplier given by the optimal dual variables is $k_{\text{JOINT}}^* = \frac{1 + \lambda_{\text{JOINT}}^*}{\mu_{\text{JOINT}}^* + \lambda_{\text{JOINT}}^*}$, which thus is an optimal multiplier for the model.

Part 3 (Stability) We prove the result by showing that the dual function is a Lyapunov function. Notice that the dynamics follow from the gradient flow of the dual function. It holds that

$$\frac{d}{dt} D(\mu(t), \lambda(t)) = \partial_{\mu} D(\mu(t), \lambda(t)) \cdot \dot{\mu}(t) + \partial_{\lambda} D(\mu(t), \lambda(t)) \cdot \dot{\lambda}(t) = - \left(\frac{1}{\eta} \dot{\mu}(t)^2 + \frac{1}{\alpha} \dot{\lambda}(t)^2 \right) \leq 0.$$

Thus $D(\mu(t), \lambda(t))$ monotonically decays. Let R be the diameter of the level set $\{(\mu, \lambda) : D(\mu(0), \lambda(0))\}$. Denote $(\mu_0, \lambda_0) \in \arg \min_{\mu \geq 0, \lambda \geq 0} D(\mu, \lambda)$, and $D^* = \min_{\mu, \lambda \geq 0} D(\mu, \lambda)$. Then we know that $\|(\mu(t), \lambda(t)) - (\mu_0, \lambda_0)\| \leq R$. Furthermore, by convexity of the dual function D , we have

$$D(\mu(t), \lambda(t)) - D^* \leq \left(\frac{1}{\eta} |\dot{\mu}(t)| + \frac{1}{\alpha} |\dot{\lambda}(t)| \right) R.$$

Therefore, it holds that

$$\frac{d}{dt} (D(\mu(t), \lambda(t)) - D^*) \leq - \frac{\min\{\eta, \alpha\}}{R^2} (D(\mu(t), \lambda(t)) - D^*)^2.$$

This guarantees that $\lim_{t \rightarrow \infty} D(\mu(t), \lambda(t)) - D^* = 0$, whereby $-\left(\frac{1}{\eta} \dot{\mu}(t)^2 + \frac{1}{\alpha} \dot{\lambda}(t)^2\right) \rightarrow 0$, which shows the convergence of the dynamic to a stationary point. \square

A direct consequence of Theorem 1 is that the optimal bidding multiplier is the minimum of k^{BUDGET} and k^{ROS} , that is, the multipliers that make the budget and ROS constraint of the advertisers bind, respectively. Therefore, the optimal bid multiplier is the smallest of the intersection points of the achievable curve with the budget and ROS constraints (see Figure 4). The proof is in Appendix A.

Corollary 1. *We have $k_{\text{JOINT}}^* = \min\{k^{\text{BUDGET}}, k^{\text{ROS}}\}$ is an optimal multiplier.*

The next result shows that sequential pacing has no stationary points. This is aligned with the results from our empirical study, where we observed that the multipliers generated by the pacing algorithm were unstable. A proof is available in Appendix A.

Theorem 2. *The sequential pacing algorithm given by (11) and (13) has no stationary point.*

Finally, we show that min pacing enjoys similar properties as joint pacing. Proving stability of dynamics is challenging as we do not have access to a Lyapunov function as in the joint pacing case. We prove stability by analyzing the phase portrait and showing that the stationary point is a global attractor. A proof is available in Appendix A.

Theorem 3. *Consider the min pacing algorithm given by (11) and (14). Then, it holds that:*

1. (**Existence and Uniqueness**). *There exists a unique stationary point of the dynamics.*
2. (**Optimality**). *The bidding multiplier k_{MIN} of the stationary point is an optimal multiplier for the problem.*
3. (**Stability**). *The dynamics converge to the unique stationary point from any initial solution.*

5 Concluding Remarks

Advertisers seeking to maximize their value subject to both budget and ROS constraints has become mainstream in the past few years. For historical and other reasons explained in the paper, the systems that provide budget-pacing ROS-pacing services do not always operate as a unified entity that optimizes a global objective. In this work, we study the benefit of running a joint pacing algorithm, that paces both budget and ROS. Using theoretical analysis as well as empirical studies, we show that there are significant advantages to operating a joint pacing algorithm. Our findings suggest that to the extent possible, advertisers should adopt algorithms that have some level of coordination between budget and ROS pacing. An interesting and challenging research direction is to develop tools to analyze and establish regret bounds for the decentralized pacing algorithms.

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A Proof of Missing Results

A.1 Proof of Corollary 1

The proof of Theorem 1, part 1 shows that at the stationary point we have $k_{\text{JOINT}}^* = k^{\text{BUDGET}}$ and $k^{\text{BUDGET}} < k^{\text{ROS}}$ or $k_{\text{JOINT}}^* = k^{\text{ROS}}$ and $k^{\text{ROS}} < k^{\text{BUDGET}}$. This shows that $k_{\text{JOINT}}^* = \min\{k^{\text{BUDGET}}, k^{\text{ROS}}\}$. Part 2 shows that the stationary point k_{JOINT}^* is an optimal multiplier. Combining the above arguments, we have $k_{\text{JOINT}}^* = \min\{k^{\text{BUDGET}}, k^{\text{ROS}}\}$ is the optimal multiplier. \square

A.2 Proof of Theorem 2

Suppose $(\mu_{\text{SEQ}}^*, \lambda_{\text{SEQ}}^*, k_{\text{SEQ}}^*)$ is a stationary point to the sequential pacing's dynamics (11)-(13). The same argument as Theorem 1, part 1 yields that either $\mu_{\text{SEQ}}^* = 0$ or $\lambda_{\text{SEQ}}^* = 0$.

Suppose $\mu_{\text{SEQ}}^* = 0$ and λ_{SEQ}^* takes a finite value, then we have $k_{\text{SEQ}}^* = \frac{1 + \lambda_{\text{SEQ}}^*}{\lambda_{\text{SEQ}}^* \mu_{\text{SEQ}}^*} = \infty$. Furthermore, we know that $\lim_{k \rightarrow \infty} \text{conv_val}(k) < \text{spend}(k)$, thus we have $\dot{\lambda} > 0$ at the stationary point, which contradicts with the fact that $(\mu_{\text{SEQ}}^*, \lambda_{\text{SEQ}}^*, k_{\text{SEQ}}^*)$ is a stationary point to the dynamic.

Suppose $\lambda_{\text{SEQ}}^* = 0$ and μ_{SEQ}^* takes a finite value, then we still have $k_{\text{SEQ}}^* = \frac{1 + \lambda_{\text{SEQ}}^*}{\lambda_{\text{SEQ}}^* \mu_{\text{SEQ}}^*} = \infty$. Furthermore, we know that $\lim_{k \rightarrow \infty} \text{spend}(k) > \rho$, thus $\dot{\mu} > 0$ at the stationary point, which contradicts with the fact that $(\mu_{\text{SEQ}}^*, \lambda_{\text{SEQ}}^*, k_{\text{SEQ}}^*)$ is a stationary point to the dynamics.

As a result, there is no finite stationary point to the dynamics. \square

A.3 Proof of Theorem 3

We prove each part at a time.

Part 1 (Existence and Uniqueness) Suppose $(\mu_{\text{MIN}}^*, \lambda_{\text{MIN}}^*, k_{\text{MIN}}^*)$ is a stationary point to the MIN dynamic (11)(14). The same argument as in Theorem 1, part 1 yields that either $\mu_{\text{MIN}}^* = 0$ or $\lambda_{\text{MIN}}^* = 0$, and they cannot both be 0. As a result, it must be the case that $k_{\text{MIN}}^* = k^{\text{BUDGET}}$ or $k_{\text{MIN}}^* = k^{\text{ROS}}$, namely, one of the two constraints must hold at the stationary solution.

If $k_{\text{MIN}}^* = k^{\text{BUDGET}}$, i.e., the budget constraint binds, then $\lambda_{\text{MIN}}^* = 0$ and $\text{conv_val}(k^{\text{BUDGET}}) - \text{spend}(k^{\text{BUDGET}}) > 0$, thus it must be the case that $k^{\text{BUDGET}} < k^{\text{ROS}}$ by utilizing Remark 1, part 2. In this case, we have $\mu_{\text{MIN}}^* = \frac{1}{k^{\text{BUDGET}}}$ by (14).

If $k_{\text{MIN}}^* = k^{\text{ROS}}$, i.e., the ROS constraint binds, then $\mu_{\text{MIN}}^* = 0$ and $\rho - \text{spend}(k^{\text{ROS}}) > 0$, thus it must be the case that $k^{\text{ROS}} < k^{\text{BUDGET}}$ by utilizing Remark 1, part 1. In this case, we have $\lambda_{\text{MIN}}^* = \frac{1}{1 - k^{\text{ROS}}}$ by (14).

This showcases the existence and the uniqueness of a stationary point of the min pacing algorithm.

Part 2 (Optimality) It follows from part 1 that $k_{\text{MIN}}^* = \min\{k^{\text{BUDGET}}, k^{\text{ROS}}\}$, which is optimal by Corollary 1.

Part 3 (Stability) It follows from Assumption 1 and 2 that $\text{conv_val}(k) > \text{spend}(k)$ for $k < k^{\text{ROS}}$ and $\text{conv_val}(k) < \text{spend}(k)$ for $k > k^{\text{ROS}}$. Furthermore, denote $\mu_1 = \frac{1}{k^{\text{BUDGET}}}$, $\mu_2 = \frac{1}{k^{\text{ROS}}}$, $\lambda_1 = \frac{1}{k^{\text{BUDGET}} - 1}$, $\lambda_2 = \frac{1}{k^{\text{ROS}} - 1}$. We consider two cases depending on whether k^{BUDGET} or k^{ROS} is smaller.

Case 1: $k^{\text{BUDGET}} > k^{\text{ROS}}$. In this case, we have $k_{\text{MIN}}^* = k^{\text{ROS}}$, and the unique stationary point is given by $\mu_{\text{MIN}}^* = 0$ and $\lambda_{\text{MIN}}^* = \frac{1}{k^{\text{ROS}} - 1}$.

The whole space can be split into three regions (see Figure 6a):

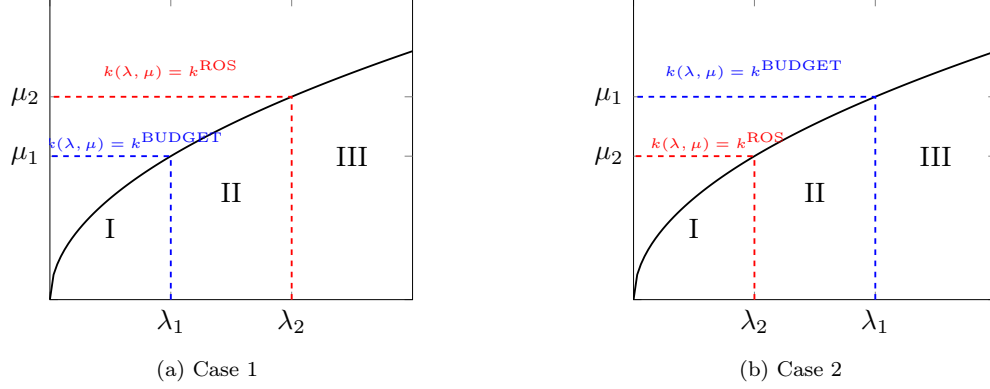


Figure 6: Illustration of the two cases for the MIN dynamic.

Region I: $k > k^{\text{BUDGET}}$. This region corresponds to $\{(\mu, \lambda) : \mu < \mu_1, \lambda < \lambda_1\}$. In this region, we have $\rho < \text{spend}(k)$ and $\text{conv_val}(k) < \text{spend}(k)$, thus $\dot{\mu} > 0$ and $\dot{\lambda} > 0$.

Region II: $k^{\text{ROS}} < k < k^{\text{BUDGET}}$. This region corresponds to $\{(\mu, \lambda) : \mu < \mu_2, \lambda < \lambda_2\}$ subtracting region I. In this region, we have $\rho > \text{spend}(k)$ and $\text{conv_val}(k) < \text{spend}(k)$, thus $\dot{\mu} < 0$ and $\dot{\lambda} > 0$.

Region III: $k > k^{\text{ROS}}$. This region corresponds to the complementary set of region I and II. In this region, we have $\rho > \text{spend}(k)$ and $\text{conv_val}(k) > \text{spend}(k)$, thus $\dot{\mu} < 0$ and $\dot{\lambda} < 0$.

Now we are ready to show that $(\mu(t), \lambda(t)) \rightarrow (\mu_{\text{MIN}}^*, \lambda_{\text{MIN}}^*)$.

First, we claim that for any initial solution $\mu(0), \lambda(0)$, there exists t_1 such that it holds for all $t > t_1$ that $\mu(t) \leq \hat{\mu} := \frac{1}{2}(\mu_1 + \mu_2)$. This is because once $\mu(t) \leq \hat{\mu}$, $\mu(t)$ would never go above $\hat{\mu}$ due to the dynamic in the region II and III. So we just need to consider the first time $\mu(t) \leq \hat{\mu}$. Notice that for all (μ, λ) such that $\mu \geq \mu_1$, there exists δ_1 such that we have $\dot{\mu} < \delta_1 < 0$. Thus, we just need to choose $t_1 = \frac{1}{|\delta_1|}((\mu(0) - \mu_1)^+)$.

Second, we claim there exists $t_2 > t_1$ such that for $t > t_2$, we have $\mu(t) \leq \hat{\mu}$ and $\lambda(t) \geq \hat{\lambda} := \frac{1}{2}(\lambda_1 + \lambda_2)$. This is because after t_1 , $\mu(t) \leq \hat{\mu}$. Thus, once $\lambda(t) \geq \hat{\lambda}$, $\lambda(t)$ would never go below $\hat{\lambda}$ due to the dynamic in the region I and II. So we just need to consider the first time $\lambda(t) \geq \hat{\lambda}$. Notice that for all (μ, λ) such that $\mu \leq \mu_1, \lambda \leq \hat{\lambda}$, there exists δ_2 such that we have $\dot{\lambda} \geq \delta_2 > 0$. Thus, we just need to choose $t_2 = t_1 + \frac{1}{\delta_2}((\hat{\lambda} - \lambda(t_1))^+)$.

Third, we claim there exists $t_3 > t_2$ such that for $t > t_3$, we have $\mu(t) = 0$ and $\lambda(t) \geq \hat{\lambda}$. This is because after t_2 , $\mu(t) \leq \hat{\mu}, \lambda(t) \geq \hat{\lambda}$. Thus, there exists δ_3 such that $\dot{\mu} \leq \delta_3 < 0$. Thus, we just need to choose $t_3 = t_2 + \frac{1}{|\delta_3|}(\hat{\mu})$.

Eventually, we restrict the dynamic onto the segment $\mu = 0, \lambda \geq \hat{\lambda}$, in which case $\lambda(t) \rightarrow \lambda_2$. This shows that the dynamic globally converges to $(0, \lambda_2)$, the unique stationary point.

Case 2: $k^{\text{BUDGET}} < k^{\text{ROS}}$. This case is exactly symmetric to Case 1 by flipping μ and λ (see Figure 6b). \square