

# Tropical Geometry, Quantum Affine Algebras, and Scattering Amplitudes

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ABSTRACT. The goal of this paper is to make a connection between tropical geometry, representations of quantum affine algebras, and scattering amplitudes in physics. The connection allows us to study important and difficult questions in these areas:

- (1) We give a systematic construction of prime modules (including prime non-real modules) of quantum affine algebras using tropical geometry. We also introduce new objects which generalize positive tropical Grassmannians.
- (2) We propose a generalization of Grassmannian string integrals in physics, in which the integrand a product indexed by prime modules of a quantum affine algebra. We give a general formula of  $u$ -variables using prime tableaux (corresponding to prime modules of quantum affine algebras of type  $A$ ) and Auslander-Reiten quivers of Grassmannian cluster categories.
- (3) We study limit  $g$ -vectors of cluster algebras. This is another way to obtain prime non-real modules of quantum affine algebras systematically. Using limit  $g$ -vectors, we construct new examples of non-real modules of quantum affine algebras.

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## 1. INTRODUCTION

Quantum groups were introduced independently by Drinfeld [32] and Jimbo [66] around 1985. A quantum affine algebra is a Hopf algebra that is a  $q$ -deformation of the universal enveloping algebra of an affine Lie algebra [28]. Quantum affine algebras have many applications to physics, for example, to the theory of solvable lattice models in quantum statistical mechanics [11, 49], integrable systems [36]. Quantum affine algebras also have many connections to different areas of mathematics, for example, cluster algebras [62], KLR algebras [69], geometric representation theory [85], representations of affine Hecke algebras and  $p$ -adic groups [31, 56, 76].

Let  $\mathfrak{g}$  be a simple Lie algebra over  $\mathbb{C}$  and  $U_q(\widehat{\mathfrak{g}})$  the corresponding quantum affine algebra [28]. Chari and Pressley have classified simple finite dimensional  $U_q(\widehat{\mathfrak{g}})$ -modules. They proved that every simple finite dimensional  $U_q(\widehat{\mathfrak{g}})$ -module corresponds to an  $I$ -tuple of Drinfeld polynomials, where  $I$  is the set of vertices of the Dynkin diagram of  $\mathfrak{g}$ , and equivalently, corresponds to a dominant monomial  $M$  in certain formal variables  $Y_{i,a}$ ,  $i \in I$ ,  $a \in \mathbb{C}^*$ . The simple  $U_q(\widehat{\mathfrak{g}})$ -module corresponding to  $M$  is denoted by  $L(M)$ .

A simple  $U_q(\widehat{\mathfrak{g}})$ -module  $L(M)$  is called prime if it is not isomorphic to  $L(M') \otimes L(M'')$  for any non-trivial modules  $L(M')$ ,  $L(M'')$ , see [31]. When  $\mathfrak{g} = \mathfrak{sl}_2$ , all prime modules of  $U_q(\widehat{\mathfrak{sl}_2})$  are Kirillov-Reshetikhin modules [31]. Kirillov-Reshetikhin modules are simple  $U_q(\widehat{\mathfrak{g}})$ -modules which correspond to dominant monomials of the form  $Y_{i,s} Y_{i,s+2d_i} \cdots Y_{i,s+2rd_i}$ , where  $i \in I$ , and  $d_i$ 's are diagonal entries of a diagonal matrix  $D$  such that  $DC$  is diagonal,  $C$  is the Cartan matrix of  $\mathfrak{g}$  (we choose  $D$  such that  $d_i$ 's are as small as possible). In general, to classify prime modules of  $U_q(\widehat{\mathfrak{g}})$  is an important and difficult problem in representation theory, see for example, [16, 26, 31, 62, 83].

Hernandez and Leclerc [62] made a breakthrough to the problem of constructing prime modules of  $U_q(\widehat{\mathfrak{g}})$  using the theory of cluster algebras [51]. For every simple Lie algebra  $\mathfrak{g}$  over  $\mathbb{C}$ , they constructed a cluster algebra with initial cluster variables given by certain Kirillov-Reshetikhin modules. Using cluster algebras, prime modules can be generated using a procedure called mutation. The prime modules generated in this way are cluster variables. They conjectured that all cluster variables (resp. cluster monomials) are real prime modules (resp. real modules), and all real prime modules (resp. real modules) are cluster variables (resp. cluster monomials). Here a simple  $U_q(\widehat{\mathfrak{g}})$ -module  $L(M)$  is called real if  $L(M) \otimes L(M)$  is still simple [75]. One direction of their conjecture “all cluster variables (resp. cluster monomials) are real prime modules (resp. real modules)” is proved by Qin in [86] and by Kang, Kashiwara, Kim, Oh, and Park in [69, 70, 71, 72]. The other direction “all real prime modules (resp. real modules) are cluster variables (resp. cluster monomials)” of the conjectural is widely open [65]. It is shown in [37] that all prime snake modules of types  $A, B$  are cluster variables and in [39] that all snake modules of types  $A, B$  are cluster monomials.

Recently, Lapid and Minguez [76] classified all real simple  $U_q(\widehat{\mathfrak{sl}}_k)$ -modules (in the language of representations of  $p$ -adic groups) satisfying a certain condition called regular. This classification is surprisingly related to the classification of rationally smooth Schubert varieties in type  $A$  flag varieties. They also gave more conjectures and results in a more recent work [77].

By the results in [69, 70, 71, 72, 86], using the procedure of mutations, one can generate a large family of prime modules (these prime modules are cluster variables). On the other hand, there are many prime modules which are not real (and thus not cluster variables). These non-real modules are also important in applications. For example, it is shown in [7, 34, 60] that non-real prime tableaux (corresponding to non-real prime modules by [23]) determine the so-called square roots and are used to construct algebraic letters in the computations of Feynman integrals in the study of scattering amplitudes in  $\mathcal{N} = 4$  super Yang-Mills theory in physics.

The goal of this paper is to make a connection between tropical geometry, representations of quantum affine algebras, and scattering amplitudes in physics. The connection allows us to study important and difficult questions in these areas: use tropical geometry to construct prime modules (conjecturally we can obtain all prime modules including prime non-real modules) of quantum affine algebras, to propose a generalization of Grassmannian string integrals in physics, and study limit  $g$ -vectors of cluster algebras.

1.1. First consider the case where  $\mathfrak{g}$  is of type  $A$ , i.e.  $\mathfrak{g} = \mathfrak{sl}_k$  for some positive integer  $k$ . Simple modules of  $U_q(\widehat{\mathfrak{sl}}_k)$  correspond to dual canonical basis elements of a quotient  $\mathbb{C}[\text{Gr}(k, n, \sim)]$  of the Grassmannian cluster algebra  $\mathbb{C}[\text{Gr}(k, n)]$  [62, 23]. We define Newton polytopes  $\mathbf{N}_{k,n}^{(d)}$  by using the formula for dual canonical basis elements of  $\mathbb{C}[\text{Gr}(k, n)]$ , [23] as follows. Denote by  $\mathcal{T}_{k,n}^{(0)}$  the set of all one-column tableaux which are cyclic shifts of the one-column tableau with entries  $\{1, 2, \dots, k-2, k\}$ . For  $d \geq 0$ , we define (see Definition 4.1)

$$\mathbf{N}_{k,n}^{(d)} = \text{Newt} \left( \prod_{T \in \mathcal{T}_{k,n}^{(d)}} \text{ch}_T(x_{i,j}) \right),$$

where  $\mathcal{T}_{k,n}^{(d)}$  ( $d \geq 1$ ) is the set of tableaux which correspond to facets of  $\mathbf{N}_{k,n}^{(d-1)}$ ,  $\text{ch}_T(x_{ij})$  is the evaluation of  $\text{ch}_T = \text{ch}(T)$  on the web matrix [96] (see Section 3.3) and  $\text{ch}(T)$  is given in Theorem 5.8 and Definition 5.9 in [23], see Section 3.1.

The facets of the Newton polytope  $\mathbf{N}_{k,n}^{(0)}$  have been classified in recent work of the first author [43]; there are exactly of them  $\binom{n}{k} - n$ , one for each of  $\binom{n}{k} - n$  generalized positive roots.

We give a procedure to construct the highest  $l$ -weights (equivalently, the tableaux corresponding to the highest  $l$ -weight monomials, see [23]) of simple  $U_q(\widehat{\mathfrak{sl}}_k)$ -modules

from facets of  $\mathbf{N}_{k,n}^{(d)}$  explicitly, see Section 6. We conjecture that for every  $k \leq n$  and  $d \geq 0$ , every facet of  $\mathbf{N}_{k,n}^{(d)}$  gives a prime module of  $U_q(\widehat{\mathfrak{sl}}_k)$  and every prime  $U_q(\widehat{\mathfrak{sl}}_k)$ -module corresponds to a facet of the Newton polytope  $\mathbf{N}_{k,n}^{(d)}$  for some  $d$ , see Conjecture 6.1). The procedure gives a systematical way to construct prime modules of  $U_q(\widehat{\mathfrak{sl}}_k)$ .

We show that a module corresponding to a 2-column tableau is prime if and only if the 2-column tableau is the union of two one-column tableaux which are not weakly separated and which are noncrossing, see Theorem 7.3.

We check that all the facets of  $\mathbf{N}_{3,9}^{(1)}$  correspond to prime modules of  $U_q(\widehat{\mathfrak{sl}}_3)$  in Section 8. This gives more evidence of Conjecture 6.1.

The study of Newton polytopes of Laurent expansions of cluster variables was initiated by Sherman and Zelevinsky in their study of rank 2 cluster algebras [99]. It was further developed in [15, 46, 68, 78, 79, 82]. Our definition of Newton polytopes in this paper involve not only cluster variables but also other prime elements in the dual canonical basis of cluster algebras.

We also count the number of prime modules corresponding to 2-column tableaux: for  $k \leq n/2$ , the number of 2-column prime tableaux is  $a_{k,n,2} - b_{k,n}$ , where  $a_{k,n,m} = \prod_{i=1}^k \prod_{j=1}^m \frac{n-i+j}{k+m-i-j+1}$  and  $b_{k,n} = \binom{n}{k} + \sum_{j=1}^k j \binom{n}{k-j, 2j, n-k-j}$ , where  $\binom{n}{a,b,c} = \frac{n!}{a!b!c!}$ , see Proposition 7.4.

We give an explicit conjectural description of a very large family of prime  $U_q(\widehat{\mathfrak{sl}}_k)$ -modules: for any  $k$ -element subsets  $J_1, \dots, J_r$  of  $[n]$  such that each pair of them is noncrossing and not weakly separated. Then  $T = \cup_{i=1}^r T_{J_i}$  is a prime tableau, see Conjecture 8.1. This conjecture is proved in the case of  $r = 2$ , see Theorem 7.3.

We also introduce another version (non-recursive) of Newton polytopes  $\mathbf{N}_{k,n}^{r(d)}$  for Grassmannian cluster algebras, see Definition 4.3.

The normal fans  $\mathcal{N}(\mathbf{N}_{k,n}^{(d)})$  are generalizations of tropical Grassmannians  $\text{Trop}^+ G(k, n)$  (see [96]), see Section 4.2. It would be an interesting and important problem to find a combinatorial model which describes the facets of  $\mathcal{N}(\mathbf{N}_{k,n}^{(d)})$ , and to relate them to scattering amplitudes.

1.2. We generalize the construction in Section 1.1 to general quantum affine algebras  $U_q(\widehat{\mathfrak{g}})$ . For any simple Lie algebra over  $\mathbb{C}$  and  $\ell \geq 1$ , we define a sequence of Newton polytopes  $\mathbf{N}_{\mathfrak{g},\ell}^{(d)}$  ( $d \in \mathbb{Z}_{\geq 0}$ ) recursively. Let  $\mathcal{M}$  be the set of all equivalence classes of fundamental modules of  $U_q(\widehat{\mathfrak{g}})$  in Hernandez and Leclerc's category  $\mathcal{C}_\ell$ . Denote  $\mathcal{M}^{(0)} = \mathcal{M}$  and  $\mathcal{M}^{(d+1)}$  ( $d \geq 0$ ) the collection of equivalence classes of  $U_q(\widehat{\mathfrak{g}})$ -modules which correspond to facets of

$$\mathbf{N}_{\mathfrak{g},\ell}^{(d)} := \text{Newt} \left( \prod_{[L(M)] \in \mathcal{M}^{(d)}} \tilde{\chi}_q(L(M)) \right),$$

where  $\tilde{\chi}_q(L(M))$  is the truncated  $q$ -characters [48] of the  $U_q(\widehat{\mathfrak{g}})$ -module  $L(M)$ , see Section 9. We conjecture that (1) for any  $d \geq 0$ , every facet of  $\mathbf{N}_{\mathfrak{g},\ell}^{(d)}$  correspond to a prime  $U_q(\widehat{\mathfrak{g}})$ -module and (2) every prime  $U_q(\widehat{\mathfrak{g}})$ -module corresponds to a facet of the Newton polytope  $\mathbf{N}_{\mathfrak{g},\ell}^{(d)}$  for some  $d$ , see Conjecture 9.4. We also introduce another version (non-recursive) of Newton polytopes  $\mathbf{N}_{\mathfrak{g},\ell}^{(d)}$  for representations of quantum affine algebras, see Definition 9.5.

1.3. In 1969, Z. Koba and K. Nielsen [73] introduced an integral representation for the Veneziano-type  $n$ -point function, parametrized so that a quantum field theoretic phenomenon known as crossing-symmetry, which asserts that particles are indistinguishable from anti-particles traveling back in time, is manifest. The integrand is expressed as a product of certain cross-ratios, with exponents the kinematic Mandelstam parameters; the cross-ratios are solutions to a particularly combinatorially nice set of binomial algebraic equations, of the form

$$u_{j_1,j_3} + \prod_{\{j_2,j_4\}} u_{j_2,j_4} = 1,$$

where the product is over all pairs  $\{j_2, j_4\}$  such that  $j_1 < j_2 < j_3 < j_4$  up to cyclic rotation. These equations, later rediscovered by Brown [19] in the context of multiple zeta values and moduli spaces, characterize a certain partial compactification of the moduli space  $\mathfrak{M}_{0,n}$  of  $n$  distinct points on the Riemann sphere, which is closely related to the tropical Grassmannian  $\text{Trop}G(2, n)$ , and in our context a certain subset of it, the *positive* tropical Grassmannian  $\text{Trop}^+G(2, n)$ . For more recent work which is important in our context, see [4, 6].

A generalization of the Koba-Nielsen string integral was announced by Arkani-Hamed, Lam and Spradlin [7] using finite-type (Grassmannian) cluster algebras, where type  $A$  corresponds to usual string amplitudes, and developed in detail in [6]. However, Grassmannian cluster algebras for  $\text{Gr}(k, n)$  with  $(k-2)(n-k-2) > 3$  not only have infinitely-many cluster variables, but it turns out that not all physically relevant elements of Lusztig's dual canonical basis can be constructed using a finite sequence of cluster mutations (there are prime non-real elements in the dual canonical basis). This is problematic, in particular, for the cases  $k = 4$  and  $n \geq 8$  which are of interest to amplitudes and it is an important problem to come up with a combinatorial framework.

In the theory of quantum affine algebras, prime modules do not have an analog in the representation theory of simple Lie algebras: a module is prime if it cannot be decomposed nontrivially as the tensor product of two other modules. The fact that representations of quantum affine algebras possess both additive and multiplicative structures appears to be very important in our context.

The main physical contribution of this work is to propose the definition of a Grassmannian stringy integral (as in [4]) with an integrand involving a product which is now in

general infinite; the key point is that the integrand should be indexed by prime tableaux, in which case our results in the rest of the paper will be directly applicable to study physical aspects of the integral. Our proposal is valid for any Grassmannian  $\text{Gr}(k, n)$ , where  $k = 2$  corresponds to the usual Koba-Nielsen string integral. So for  $2 \leq k \leq n - 2$  and every  $d \geq 1$ , we define

$$\mathbf{I}_{k,n}^{(d)} = (\alpha')^a \int_{(\mathbb{R}_{>0}^{n-k-1})^{\times(k-1)}} \left( \prod_{(i,j)} \frac{dx_{i,j}}{x_{i,j}} \right) \left( \prod_T \text{ch}_T^{-\alpha' c_T}(x_{i,j}) \right),$$

where the second product is over all tableaux  $T$  such that the face  $\mathbf{F}_T$  corresponding to  $T$  (see Section 6.4) is a (codimension one) facet of  $\mathbf{N}_{k,n}^{(d-1)}$ ,  $a, \alpha', c_T$  are some parameters, see Section 10.1, Formulas (10.2), (10.3), (10.4), for more details. We point out that the character polynomials  $\text{ch}_T$  are manifestly positive in the interior of the totally nonnegative Grassmannian, see [23, Section 5.3].

We give a general formula of  $u$ -variables using prime tableaux (corresponding to prime modules of quantum affine algebras of type  $A$ ) and Auslander-Reiten quivers of Grassmannian cluster categories  $\text{CM}(B_{k,n})$  [67]. For every mesh

$$\begin{array}{ccc} & T_1 & \\ & \nearrow & \searrow \\ S & \rightarrow & S' \\ & \searrow & \nearrow \\ & T_r & \end{array}$$

in the Auslander-Reiten quiver of  $\text{CM}(B_{k,n})$ , where we label the vertices by tableaux, we define the corresponding  $u$ -variable as

$$u_S = \frac{\prod_{i=1}^r \text{ch}_{T_i}}{\text{ch}_S \text{ch}_{S'}}, \quad (1.1)$$

see Definition 10.5.

We also give a definition of stringy integral for Grassmannian cluster algebras using  $u$ -variables. Denote by  $\text{PSSYT}_{k,n}$  the set of all prime tableaux of rectangular shapes and with  $k$  rows and with entries in  $[n]$ . For  $k \leq n$ , we define

$$\mathbf{I}_{k,n}^{(\infty)} = (\alpha')^a \int_{(\mathbb{R}_{>0}^{n-k-1})^{\times(k-1)}} \prod_{i,j} \frac{dx_{i,j}}{x_{i,j}} \prod_{T \in \text{PSSYT}_{k,n}} (u_T)^{\alpha' U_T},$$

where  $\alpha', U_T$  are some parameters, and  $u_T$  is the  $u$ -variable corresponding to a prime tableau  $T$ , See Definition 10.4. This new integrand involves an infinite product of  $u$ -variables, indexed by prime tableaux.

The  $u$ -equations which we propose involve the Grassmannian cluster category  $\text{CM}(B_{k,n})$  [67] (on the other hand, the  $u$ -equations in [2, 3] use cluster categories of finite type or categories of representations of quivers with relations): for every prime tableau  $T$  in  $\text{SSYT}(k, [n])$ ,

$$u_T + \prod_{T' \in \text{PSSYT}_{k,n}} u_{T'}^{\dim \text{Hom}(M_T, \tau(M_{T'})) + \dim \text{Hom}(M_{T'}, \tau(M_T))} = 1,$$

where  $M_T$  is the indecomposable module in Grassmannian cluster category  $\text{CM}(B_{k,n})$  corresponding to  $T$ , and  $\tau$  is the Auslander-Reiten translation [10, 67]. We conjecture that the  $u$ -variables we constructed are unique solutions of  $u$ -equations.

We also generalize the stringy integral to the setting of general quantum affine algebras and define stringy integrals using prime modules of quantum affine algebras, see Section 10.3.

1.4. Recently, Arkani-Hamed, Lam, Spradlin [7], and Drummond, Foster, Gürdoğan, Kalousios [34], and Henke, Papathanasiou [60], have constructed limit  $g$ -vectors for the Grassmannian cluster algebra  $\mathbb{C}[\text{Gr}(k, n)]$  using infinite sequence of mutations. These limit  $g$ -vectors do not correspond to cluster variables in  $\mathbb{C}[\text{Gr}(k, n)]$  but correspond to prime elements in the dual canonical basis of  $\mathbb{C}[\text{Gr}(k, n)]$ .

Motivated by these works, we define limit  $g$ -vectors for any cluster algebra of infinite type, see Definition 11.2. We say that a facet of a Newton polytope for a quantum affine algebra is a limit facet if the facet corresponds to a  $U_q(\widehat{\mathfrak{g}})$ -module whose  $g$ -vector is a limit  $g$ -vector of the cluster algebra for  $U_q(\widehat{\mathfrak{g}})$ . We say that a  $U_q(\widehat{\mathfrak{g}})$ -module corresponds to a limit  $g$ -vector if the  $g$ -vector of the module is a limit  $g$ -vector of the cluster algebra for  $U_q(\widehat{\mathfrak{g}})$ . We conjecture that every module corresponding to a limit  $g$ -vector is prime and non-real. This is another way to obtain prime non-real  $U_q(\widehat{\mathfrak{g}})$ -modules systematically.

Using limit  $g$ -vectors, we construct new examples of non-real modules of quantum affine algebras. As an example, we prove that the module  $L(Y_{2,-4}Y_{2,0})$  in type  $D_4$  is non-real, see Section 12.3.

1.5. The paper is organized as follows. In Section 2, we recall results of quantum affine algebras and Hernandez-Leclerc's category  $\mathcal{C}_\ell$ . In Section 3, we recall results of Grassmannian cluster algebras and tropical Grassmannians. In Section 4, we define a sequence of Newton polytopes and tropical fans for Grassmannian cluster algebras. In Section 5, we study relations between semistandard Young tableaux of rectangular shapes and generalized root polytopes. In Section 6, we construct prime modules from facets. In Section 7, we explicitly describe 2-column prime tableaux. In Section 8, we give more evidence of Conjecture 6.1 that facets of Newton polytopes correspond to prime modules. In Section 9, we define Newton polytopes and tropical fans for general quantum affine algebras. In Section 10, we generalize Grassmannian string integrals and study  $u$ -equations and  $u$ -variables. In Section 11, we study limit  $g$ -vectors for general cluster algebras. In Section

12, we give an example of prime non-real module of  $U_q(\widehat{\mathfrak{g}})$  when  $\mathfrak{g}$  is of type  $D_4$  using limit  $g$ -vectors. In Section 13, we discuss some future directions of the paper.

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## 2. QUANTUM AFFINE ALGEBRAS AND HERNANDEZ-LECLERC'S CATEGORY $\mathcal{C}_\ell$

In this section, we recall results of quantum affine algebras [28, 48], Hernandez-Leclerc's category  $\mathcal{C}_\ell$  and cluster algebra structure on the Grothendieck ring of  $\mathcal{C}_\ell$  [62].

**2.1. Quantum Affine Algebras.** Let  $\mathfrak{g}$  be a simple finite-dimensional Lie algebra and  $I$  the set of vertices of the Dynkin diagram of  $\mathfrak{g}$ . Denote by  $\{\omega_i : i \in I\}$ ,  $\{\alpha_i : i \in I\}$ ,  $\{\alpha_i^\vee : i \in I\}$  the set of fundamental weights, the set of simple roots, the set of simple coroots, respectively. Denote by  $P$  the integral weight lattice and  $P^+$  the set of dominant weights. The Cartan matrix is  $C = (\alpha_j(\alpha_i^\vee))_{i,j \in I}$ . Let  $D = \text{diag}(d_i : i \in I)$ , where  $d_i$ 's are minimal positive integers such that  $DC$  is symmetric.

Let  $z$  be an indeterminate. The quantum Cartan matrix  $C(z)$  is defined as follows: for  $i \in I$ ,  $C_{i,i}(z) = z^{d_i} + z^{-d_i}$ , and for  $i \neq j \in I$ ,  $C_{ij}(z) = [C_{ij}]_z$ , where  $[m]_z = \frac{z^m - z^{-m}}{z - z^{-1}}$ , see [48].

The *quantum affine algebra*  $U_q(\widehat{\mathfrak{g}})$  is a Hopf algebra that is a  $q$ -deformation of the universal enveloping algebra of  $\widehat{\mathfrak{g}}$  [33, 66]. In this paper, we take  $q$  to be a non-zero complex number which is not a root of unity.

Denote by  $\mathcal{P}$  the free abelian group generated by formal variables  $Y_{i,a}^{\pm 1}$ ,  $i \in I$ ,  $a \in \mathbb{C}^*$ , denote by  $\mathcal{P}^+$  the submonoid of  $\mathcal{P}$  generated by  $Y_{i,a}$ ,  $i \in I$ ,  $a \in \mathbb{C}^*$ . Let  $\mathcal{C}$  denote the monoidal category of finite-dimensional representations of the quantum affine algebra  $U_q(\widehat{\mathfrak{g}})$ .

Any finite dimensional simple object in  $\mathcal{C}$  is a highest  $l$ -weight module with a highest  $l$ -weight  $m \in \mathcal{P}^+$ , denoted by  $L(m)$  (see [29]). The elements in  $\mathcal{P}^+$  are called dominant monomials.

Frenkel and Reshetikhin [48] introduced the  $q$ -character map which is an injective ring morphism  $\chi_q$  from the Grothendieck ring of  $\mathcal{C}$  to  $\mathbb{Z}\mathcal{P} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^*}$ . For a  $U_q(\widehat{\mathfrak{g}})$ -module



$V$ ,  $\chi_q(V)$  encodes the decomposition of  $V$  into common generalized eigenspaces for the action of a large commutative subalgebra of  $U_q(\widehat{\mathfrak{g}})$  (the loop-Cartan subalgebra). These generalized eigenspaces are called  $l$ -weight spaces and generalized eigenvalues are called  $l$ -weights. One can identify  $l$ -weights with monomials in  $\mathcal{P}$  [48]. Then the  $q$ -character of a  $U_q(\widehat{\mathfrak{g}})$ -module  $V$  is given by (see [48])

$$\chi_q(V) = \sum_{m \in \mathcal{P}} \dim(V_m) m \in \mathbb{Z}\mathcal{P},$$

where  $V_m$  is the  $l$ -weight space with  $l$ -weight  $m$ .

Let  $\mathcal{Q}^+$  be the monoid generated by

$$A_{i,a} = Y_{i,aq} Y_{i,aq^{-1}} \left( \prod_{j: C_{ji}=-1} Y_{j,a}^{-1} \right) \left( \prod_{j: C_{ji}=-2} Y_{j,aq}^{-1} Y_{j,aq^{-1}}^{-1} \right) \left( \prod_{j: C_{ji}=-3} Y_{j,aq^2}^{-1} Y_{j,a}^{-1} Y_{j,aq^{-2}}^{-1} \right), \quad (2.1)$$

where  $q_i = q^{d_i}$ ,  $i \in I$ . There is a partial order  $\leq$  on  $\mathcal{P}$  (cf. [48]) defined by  $m' \leq m$  if and only if  $mm'^{-1} \in \mathcal{Q}^+$ .

For  $i \in I$ ,  $a \in \mathbb{C}^\times$ ,  $k \in \mathbb{Z}_{\geq 1}$ , the modules

$$X_{i,a}^{(k)} := L(Y_{i,a} Y_{i,aq^2} \cdots Y_{i,aq^{2k-2}})$$

are called *Kirillov-Reshetikhin modules*. The modules  $X_{i,a}^{(1)} = L(Y_{i,a})$  are called *fundamental modules*.

**2.2. Hernandez-Leclerc's Category  $\mathcal{C}_\ell$  and Truncated  $q$ -characters.** We recall the definition of Hernandez-Leclerc's category  $\mathcal{C}_\ell$  [62].

For integers  $a \leq b$ , we denote  $[a, b] = \{i : a \leq i \leq b\}$  and  $[a] = \{i : 1 \leq i \leq a\}$ .

Let  $\mathfrak{g}$  be a simple Lie algebra over  $\mathbb{C}$  and let  $\mathcal{C}$  be the category of finite-dimensional  $U_q(\widehat{\mathfrak{g}})$ -modules. In [62], [64], Hernandez and Leclerc introduced a full subcategory  $\mathcal{C}_\ell = \mathcal{C}_\ell^{\mathfrak{g}}$  of  $\mathcal{C}$  for every  $\ell \in \mathbb{Z}_{\geq 0}$ . Let  $I$  be the set of vertices of the Dynkin diagram of  $\mathfrak{g}$ . We fix  $a \in \mathbb{C}^*$  and denote  $Y_{i,s} = Y_{i,aq^s}$ ,  $i \in I$ ,  $s \in \mathbb{Z}$ . For  $\ell \in \mathbb{Z}_{\geq 0}$ , denote by  $\mathcal{P}_\ell$  the subgroup of  $\mathcal{P}$  generated by  $Y_{i,\xi(i)-2r}^{\pm 1}$ ,  $i \in I$ ,  $r \in [0, d\ell]$ , where  $d$  is the maximal diagonal element in the diagonal matrix  $D$ , and  $\xi : I \rightarrow \mathbb{Z}$  is a certain function called height function defined in Section 2.2 in [63] and Definition 4.1 in [52]. Denote by  $\mathcal{P}_\ell^+$  the submonoid of  $\mathcal{P}^+$  generated by  $Y_{i,\xi(i)-2r}$ ,  $i \in I$ ,  $r \in [0, d\ell]$ . An object  $V$  in  $\mathcal{C}_\ell$  is a finite-dimensional  $U_q(\widehat{\mathfrak{g}})$ -module which satisfies the condition: for every composition factor  $S$  of  $V$ , the highest  $l$ -weight of  $S$  is a monomial  $\mathcal{P}_\ell^+$ , [62]. Simple modules in  $\mathcal{C}_\ell$  are of the form  $L(M)$  (see [28], [62]), where  $M \in \mathcal{P}_\ell^+$ .

**Remark 2.1.** In [62], fundamental modules in  $\mathcal{C}_\ell$  are  $L(Y_{i,-\xi(i)-2r})$ ,  $i \in I$ ,  $r \in [0, \ell]$ , see Definition 3.1 in [62]. In this paper, we slightly modify  $\mathcal{C}_\ell$  such that the fundamental modules in  $\mathcal{C}_\ell$  are  $L(Y_{i,\xi(i)-2r})$ ,  $i \in I$ ,  $r \in [0, d\ell]$ , where  $d$  is the maximal diagonal element in the diagonal matrix  $D$ .

For a module  $V$  in  $\mathcal{C}_\ell$ , the truncated  $q$ -character  $\tilde{\chi}_q(V)$  is the Laurent polynomial obtained from the  $q$ -character  $\chi_q(V)$  defined in Section 2.1 by removing all monomials which have a factor  $Y_{i,s}$  or  $Y_{i,s}^{-1}$ , where  $Y_{i,s}$  is not in  $\mathcal{P}_\ell^+$ , see [63, 64].

Hernandez and Leclerc constructed a cluster algebra for every category  $\mathcal{C}_\ell$  of every quantum affine algebra  $U_q(\widehat{\mathfrak{g}})$  [62, 64]. The cluster algebra for  $\mathcal{C}_\ell$  of  $U_q(\widehat{\mathfrak{sl}}_k)$  is isomorphic to the cluster algebra for a certain quotient  $\mathbb{C}[\text{Gr}(k, n, \sim)]$  (see Section 3.1) of the Grassmannian cluster algebra  $\mathbb{C}[\text{Gr}(k, n)]$  [23, 62, 93],  $n = k + \ell + 1$ .

### 3. GRASSMANNIAN CLUSTER ALGEBRAS AND TROPICAL GRASSMANNIANS

In this section, we recall results of Grassmannian cluster algebras, [93, 23] and tropical Grassmannians [94, 96].

**3.1. Grassmannian Cluster Algebras and Semistandard Young Tableaux.** For  $k \leq n$ , the Grassmannian  $\text{Gr}(k, n)$  is the set of  $k$ -dimensional subspaces in an  $n$ -dimensional vector space. In this paper, we denote by  $\text{Gr}(k, n)$  (the affine cone over) the Grassmannian of  $k$ -dimensional subspaces in  $\mathbb{C}^n$ , and denote by  $\mathbb{C}[\text{Gr}(k, n)]$  its coordinate ring. This algebra is generated by Plücker coordinates

$$p_{i_1, \dots, i_k}, \quad 1 \leq i_1 < \dots < i_k \leq n.$$

In this paper, we use  $p_J$  for Plücker coordinates and  $P_J$  for its tropical version, where  $J$  is a  $k$ -element subset of  $[n]$ .

It was shown by Scott [93] that the ring  $\mathbb{C}[\text{Gr}(k, n)]$  has a cluster algebra structure. Define  $\mathbb{C}[\text{Gr}(k, n, \sim)]$  to be the quotient of  $\mathbb{C}[\text{Gr}(k, n)]$  by the ideal generated by  $P_{i, \dots, i-k+1} - 1$ ,  $i \in [n - k + 1]$ . In [23], it is shown that the elements in the dual canonical basis of  $\mathbb{C}[\text{Gr}(k, n, \sim)]$  are in bijection with semistandard Young tableaux in  $\text{SSYT}(k, [n], \sim)$ .

A semistandard Young tableau is a Young tableau with weakly increasing rows and strictly increasing columns. For  $k, n \in \mathbb{Z}_{\geq 1}$ , we denote by  $\text{SSYT}(k, [n])$  the set of rectangular semistandard Young tableaux with  $k$  rows and with entries in  $[n]$  (with arbitrarily many columns). The empty tableau is denoted by  $\mathbb{1}$ .

For  $S, T \in \text{SSYT}(k, [n])$ , let  $S \cup T$  be the row-increasing tableau whose  $i$ th row is the union of the  $i$ th rows of  $S$  and  $T$  (as multisets), [23]. It is shown in Section 3 in [23] that  $S \cup T$  is semistandard for any pair of semistandard tableaux  $S, T$ .

We call  $S$  a factor of  $T$ , and write  $S \subset T$ , if the  $i$ th row of  $S$  is contained in that of  $T$  (as multisets), for  $i \in [k]$ . In this case, we define  $\frac{T}{S} = S^{-1}T = TS^{-1}$  to be the row-increasing tableau whose  $i$ th row is obtained by removing that of  $S$  from that of  $T$  (as multisets), for  $i \in [k]$ .

A tableau  $T \in \text{SSYT}(k, [n])$  is trivial if each entry of  $T$  is one less than the entry below it. For any  $T \in \text{SSYT}(k, [n])$ , we denote by  $T_{\text{red}} \subset T$  the semistandard tableau obtained by removing a maximal trivial factor from  $T$ . For a trivial  $T$ , one has  $T_{\text{red}} = \mathbb{1}$ .

Let “ $\sim$ ” be the equivalence relation on  $S, T \in \text{SSYT}(k, [n])$  defined by:  $S \sim T$  if and only if  $S_{\text{red}} = T_{\text{red}}$ . We denote by  $\text{SSYT}(k, [n], \sim)$  the set of  $\sim$ -equivalence classes.

The elements in the dual canonical basis of  $\mathbb{C}[\text{Gr}(k, n, \sim)]$  are in bijection with simple modules in the category  $\mathcal{C}_\ell$  of  $U_q(\widehat{\mathfrak{sl}}_k)$  in Section 3.1, see [62, 23].

A one-column tableau is called a fundamental tableau if its content is  $[i, i+k] \setminus \{r\}$  for  $r \in \{i+1, \dots, i+k-1\}$ . Any tableau in  $\text{SSYT}(k, [n])$  is  $\sim$ -equivalent to a unique semistandard tableau whose columns are fundamental tableaux, see Lemma 3.13 in [23].

By [23, Theorem 5.8], for every  $T \in \text{SSYT}(k, [n])$ , the corresponding element  $\widetilde{\text{ch}}(T)$  in the dual canonical basis of  $\mathbb{C}[\text{Gr}(k, n, \sim)]$  is given by

$$\widetilde{\text{ch}}(T) = \sum_{u \in S_m} (-1)^{\ell(uw_T)} \mathbf{p}_{uw_0, w_T w_0}(1) p_{u; T'} \in \mathbb{C}[\text{Gr}(k, n, \sim)], \quad (3.1)$$

where  $m$  is the number of columns of  $T'$ ,  $T'$  is the tableau whose columns are fundamental tableaux and such that  $T \sim T'$ ,  $p_{u; T'}$  is a certain monomial of Plücker coordinates,  $w_T$  is a certain permutation in  $S_m$ ,  $\mathbf{p}_{u, v}(q)$  is a Kazhdan-Lusztig polynomial, see [23]. Let  $T'' = T' T^{-1}$  and define  $\text{ch}(T) = \frac{1}{p_{T''}} \widetilde{\text{ch}}(T)$ , where  $p_{T''} = p_{T''_1} \cdots p_{T''_r}$ ,  $T''_i$ 's are columns of  $T''$ . We also denote  $\text{ch}_T = \text{ch}(T)$ .

**3.2. Relation between Dominant Monomials and Tableaux.** In Section 2.2, we recalled Hernandez and Leclerc's category  $\mathcal{C}_\ell$ . It is shown in Theorem 3.17 in [23] that in the case of  $\mathfrak{g} = \mathfrak{sl}_k$ , the monoid  $\mathcal{P}_\ell^+$  (we take the height function to be  $\xi(i) = i - 2$ ,  $i \in [k - 1]$ ) of dominant monomials is isomorphic to the monoid of semistandard Young tableaux  $\text{SSYT}(k, [n], \sim)$ ,  $n = k + \ell + 1$ . The correspondence of dominant monomials and tableaux is induced by the following map sending fundamental monomials to fundamental tableaux:

$$Y_{i, s} \mapsto T_{i, s}, \quad (3.2)$$

where  $T_{i, s}$  is a one-column tableau consisting of entries  $\frac{i-s}{2}, \frac{i-s}{2} + 1, \dots, \frac{i-s}{2} + k - i - 1, \frac{i-s}{2} + k - i + 1, \dots, \frac{i-s}{2} + k$ . We denote the monomial corresponding to a tableau  $T$  by  $M_T$  and denote the tableau corresponding to a monomial  $M$  by  $T_M$ . Note that by the definition of  $\mathcal{C}_\ell$  and the choice of the height function  $\xi(i) = i - 2$ ,  $i \in [k - 1]$ , the indices of  $Y_{i, s}$  in the highest  $l$ -weight monomials of simple modules in  $\mathcal{C}_\ell$  satisfy  $i - s \pmod{2} = 0$ .

When computing the monomial corresponding to a given tableau, we first decompose the tableau into a union of fundamental tableaux. Then we send each fundamental tableau to the corresponding fundamental monomial. For example, the tableaux  $[[1, 2, 4, 6], [3, 5, 7, 8]]$  (each list is a column of the tableau),  $[[1, 3, 5, 7], [2, 4, 6, 8]]$  correspond to the modules

$$L(Y_{2, -6} Y_{1, -3} Y_{3, -3} Y_{2, 0}), \quad L(Y_{1, -7} Y_{2, -4} Y_{1, -5} Y_{3, -1} Y_{2, -2} Y_{3, 1}),$$

respectively.

Recall that a simple  $U_q(\widehat{\mathfrak{g}})$ -module  $L(M)$  is called prime if it is not isomorphic to  $L(M') \otimes L(M'')$  for any non-trivial modules  $L(M')$ ,  $L(M'')$  [31]. A simple  $U_q(\widehat{\mathfrak{g}})$ -module  $L(M)$  is called real if  $L(M) \otimes L(M)$  is still simple [75]. We say that a tableau  $T$  is prime (resp. real) if the corresponding  $U_q(\widehat{\mathfrak{sl}}_k)$ -module  $L(M_T)$  is prime (resp. real). The problem of classification of prime  $U_q(\widehat{\mathfrak{sl}}_k)$ -modules in the category  $\mathcal{C}_\ell$  ( $\ell \geq 0$ ) is equivalent to the problem of classification of prime tableaux in  $\text{SSYT}(k, [n])$ ,  $n \geq k + 1$ , [23].

**3.3. Matroid Subdivisions and Tropical Grassmannians.** We recall results of tropical Grassmannians [94, 96] and matroid subdivisions [22, 43].

For integers  $1 \leq k \leq n - 1$ , the hypersimplex  $\Delta_{k,n}$  [53, 100] is the  $k$ th cross-section of a cube,

$$\Delta_{k,n} = \left\{ x \in [0, 1]^n : \sum_{j=1}^n x_j = k \right\}.$$

It is the convex hull of  $\binom{n}{k}$  vectors of the form  $e_J = \sum_{j \in J} e_j$  for  $J \in \binom{[n]}{k}$ , where  $\binom{[n]}{k}$  is the set of  $k$ -element subsets of  $[n]$ .

The following constructions are standard in convex and combinatorial geometry, [40, 55, 59, 74, 95].

**Definition 3.1** ([40, 55, 74]). A matroid polytope  $P$  is a subpolytope of a hypersimplex  $\Delta_{k,n}$ , all of whose edges are edges of  $\Delta_{k,n}$ . A matroid subdivision (also called matroidal subdivision) of  $\Delta_{k,n}$  is a polytopal subdivision  $(P_1, \dots, P_t)$  of  $\Delta_{k,n}$  such that each  $P_1, \dots, P_t$  is a matroid polytope. A matroid subdivision  $(P_1, \dots, P_t)$  is called *regular* if there exists a piecewise linear function on  $\Delta_{k,n}$  whose regions of linearity are exactly the polytopes  $P_i$ .

A matroid polytope such that the defining inequalities are of the form  $x_i + x_{i+1} + \dots + x_j \geq r_{ij}$  for some integers  $r_{ij}$  with  $\{i, i+1, \dots, j\}$  a cyclic interval, is called a positroid polytope, see Section 2 in [80], Proposition 5.6 in [9], and Section 2.2 in [41], where the indices are assumed to be cyclic modulo  $n$ . A positroid subdivision of  $\Delta_{k,n}$  is a matroid subdivision  $(P_1, \dots, P_t)$  of  $\Delta_{k,n}$  such that each  $P_i$  is a positroid polytope, see Section 2 in [80], Section 5 in [9], and Section 2.2 in [41].

Let  $\mathcal{P} = (P_1, \dots, P_t)$  and  $\mathcal{P}' = (P'_1, \dots, P'_t)$  be two matroid subdivisions of  $\Delta_{k,n}$ . It is said that  $\mathcal{P}$  refines  $\mathcal{P}'$  if every maximal cell  $P'_i$  of  $\mathcal{P}'$  is a union of maximal cells of  $\mathcal{P}$ , and it is said that  $\mathcal{P}$  coarsens  $\mathcal{P}'$  if every maximal cell  $P_i$  of  $\mathcal{P}$  is a union of maximal cells of  $\mathcal{P}'$ , see Definition 2.3.8 in [40].

The tropical Grassmannian  $\text{Trop}G(k, n)$ , introduced in [94], parametrizes realizable tropical linear spaces; it is the tropical variety of the Plücker ideal of the Grassmannian  $\text{Gr}(k, n)$ . For general  $(k, n)$  the Plücker ideal contains higher degree generators and to calculate  $\text{Trop}G(k, n)$  quickly becomes a completely intractable problem, but for  $k = 2$ ,  $\text{Trop}G(2, n)$  is completely characterized by the tropicalization of the 3-term tropical

Plücker relations. We present the full definition and then immediately specialize to the so-called positive tropical Grassmannians.

**Definition 3.2** ([94]). Given  $e = (e_1, \dots, e_N) \in \mathbb{Z}_{\geq 0}^N$ , denote  $\mathbf{x}^e = x_1^{e_1} \dots x_N^{e_N}$ . Let  $E \subset \mathbb{Z}_{\geq 0}^N$ . If  $f = \sum_{e \in E} f_e \mathbf{x}^e$  is nonzero, denote by  $\text{Trop}(f)$  the set of all points  $(X_1, \dots, X_N)$  such that for the collection of numbers  $\sum_{i=1}^N e_i X_i$  for  $e$  ranging over  $E$ , the minimum of the collection is achieved at least twice. We say that  $\text{Trop}(f)$  is the tropical hypersurface associated to  $f$ . The *tropical Grassmannian*  $\text{Trop}G(k, n)$  is the intersection of all tropical hypersurfaces  $\text{Trop}(f)$  where  $f$  ranges over all elements in the Plücker ideal.

On the other hand, in [96], Speyer-Williams introduced the *positive* tropical Grassmannian  $\text{Trop}^+G(k, n)$ , which was later shown independently in [8, 97] to be equal to the positive Dressian, which is characterized by the 3-term tropical Plücker relations,

$$\pi_{Lac} + \pi_{Lbd} = \min\{\pi_{Lab} + \pi_{Lcd}, \pi_{Lad} + \pi_{Lbc}\},$$

for each pair  $(L, \{a, b, c, d\}) \in \binom{[n]}{k-2} \times \binom{[n] \setminus L}{4}$  with  $a < b < c < d$ .

Generalized positive roots were defined in [22] and developed in depth in [43] in the context of root polytopes and CEGM scattering amplitudes [21]. We now recall the definition of generalized positive roots<sup>1</sup>.

We use  $\binom{[n]}{k}^{nf}$  to denote the set of  $k$ -element subsets of  $[n]$  which are nonfrozen, i.e., not of the form  $[i, i+k-1]$  up to cyclic shifts.

**Definition 3.3** ([22]). Given any  $J = \{j_1 < \dots < j_k\} \in \binom{[n]}{k}^{nf}$ , the *generalized positive root*  $\gamma_J$  is the linear function on the space  $\mathbb{T}^{k-1, n-k}$ :

$$\gamma_J = \sum_{t=j_1}^{j_2-2} \alpha_{1,t} + \sum_{t=j_2-1}^{j_3-3} \alpha_{2,t} + \dots + \sum_{t=j_{k-1}-(k-2)}^{j_k-k} \alpha_{k-1,t}. \quad (3.3)$$

When there is no risk of confusion we also call the vector  $v_J$  dual to  $\gamma$  a generalized positive root:

$$v_J = \sum_{t=j_1}^{j_2-2} e_{1,t} + \sum_{t=j_2-1}^{j_3-3} e_{2,t} + \dots + \sum_{t=j_{k-1}-(k-2)}^{j_k-k} e_{k-1,t}.$$

Denote

$$X(k, n) = \left\{ g \in \text{Gr}(k, n) : \prod_J p_J(g) \neq 0 \right\} / (\mathbb{C}^*)^n.$$

We construct an embedding

$$(\mathbb{CP}^{n-k-1})^{\times(k-1)} \hookrightarrow X(k, n)$$

<sup>1</sup>The name originates from [43, Theorem 4.3], according to which the their convex hull, the generalized root polytope  $\mathcal{R}_{n-k}^{(k)}$ , admits a flag-unimodular triangulation which specializes to that the triangulation of the type  $A$  root polytope of Gelfand-Graev-Postnikov in the context of hypergeometric systems [54].

of a Cartesian product of projective spaces into  $X(k, n)$ .

Define a  $(k-1) \times (n-k-1)$  polynomial-valued matrix  $M_{k,n}$  with entries  $m_{i,j}(x)$ , with  $(i, j) \in [1, k-1] \times [1, n-k]$ , defined by

$$m_{i,j}(\{x_{a,b} : (a,b) \in [i, k-1] \times [1, j]\}) = (-1)^{k+i} \sum_{1 \leq b_i \leq b_{i+1} \leq \dots \leq b_{k-1} \leq j} x_{i,b_i} x_{i+1,b_{i+1}} \dots x_{k-1,b_{k-1}}.$$

For the embedding  $(\mathbb{C}\mathbb{P}^{n-k-1})^{\times k-1} \hookrightarrow X(k, n)$ , we construct a  $k \times (n-k)$  matrix  $M$  (called web matrix [96]) with  $M_{k,n}$  as its upper right block:

$$M = \begin{bmatrix} 1 & & 0 & m_{1,1} & \dots & m_{1,n-k} \\ & \ddots & & \vdots & \ddots & \vdots \\ & & 1 & m_{k-1,1} & & m_{k-1,n-k} \\ 0 & & & 1 & 1 & \dots & 1 \end{bmatrix}. \quad (3.4)$$

For instance, for rank  $k = 3$  we have

$$M_{3,6} = \begin{bmatrix} x_{1,1}x_{2,1} & x_{1,1}x_{2,12} + x_{1,2}x_{2,2} & x_{1,1}x_{2,123} + x_{1,2}x_{2,23} + x_{1,3}x_{2,3} \\ -x_{2,1} & -x_{2,12} & -x_{2,123} \end{bmatrix}$$

and for the embedding we have

$$M = \begin{bmatrix} 1 & 0 & 0 & x_{1,1}x_{2,1} & x_{1,1}x_{2,12} + x_{1,2}x_{2,2} & x_{1,1}x_{2,123} + x_{1,2}x_{2,23} + x_{1,3}x_{2,3} \\ 0 & 1 & 0 & -x_{2,1} & -x_{2,12} & -x_{2,123} \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

For any  $k$ -subset  $J$  of  $[n]$ , define  $e_J = \sum_{j \in J} e_j$ . Further denote by  $\{e^J : J \in \binom{[n]}{k}\}$  the standard basis of  $\mathbb{R}^{\binom{[n]}{k}}$ .

**Definition 3.4** ([81]). A pair of  $k$ -element subsets  $I, J$  is said to be *weakly separated* if the difference of indicator vectors  $e_I - e_J$  alternates sign at most twice, that is one does not have the pattern  $e_a - e_b + e_c - e_d$  for  $a < b < c < d$ , up to cyclic rotation.

**Definition 3.5** ([88]). A pair  $I = \{i_1 < \dots < i_k\}$ ,  $J = \{j_1 < \dots < j_k\}$  of  $k$ -subsets of  $[n]$  is said to be noncrossing if for each  $1 \leq a < b \leq k$ , either the pair  $\{i_a, i_{a+1}, \dots, i_b\}$ ,  $\{j_a, j_{a+1}, \dots, j_b\}$  is weakly separated, or  $\{i_{a+1}, \dots, i_{b-1}\} \neq \{j_{a+1}, \dots, j_{b-1}\}$ .

**Remark 3.6.** Here it is easy to see that Definition 3.4 is a slight reformulation of the original construction given in [81]. Similarly, Definition 3.5 is clearly equivalent to the one given in [88].

Denote by  $\mathbf{NC}_{k,n}$  the poset of all collections of pairwise noncrossing nonfrozen  $k$ -element subsets of  $[n]$ , ordered by inclusion.

Denote  $\mathbb{T}^{k-1, n-k} = (\mathbb{T}^{n-k})^{\times (k-1)}$ , where  $\mathbb{T}^{n-k} = \mathbb{R}^{n-k} / \mathbb{R}(1, \dots, 1)$ .

#### 4. NEWTON POLYTOPES AND TROPICAL FANS FOR GRASSMANNIAN CLUSTER ALGEBRAS

In this section, we define Newton polytopes and tropical fans for Grassmannian cluster algebras.

**4.1. Newton polytopes for Grassmannian cluster algebras.** In what follows, we give a recursive construction of a collection of Newton polytopes  $\mathbf{N}_{k,n}^{(d)}$  ( $d \in \mathbb{Z}_{\geq 0}$ ), starting from the Planar Kinematics (PK) polytope  $\Pi_{k,n}$  [22], see also [43], which is equal to  $\mathbf{N}_{k,n}^{(0)}$  in the present notation.

For any tableau  $T \in \text{SSYT}(k, [n])$ , we evaluate  $\text{ch}_T$  on the web matrix  $M$  in (3.4) and obtain a polynomial in  $x_{i,j}$ -coordinates. Note that there is a monomial transformation relating the  $x_{i,j}$  coordinates to the so-called  $X$ -coordinates [50] in cluster algebras:  $X_{ij} = \frac{x_{k-i,j}}{x_{k-i,j+1}}$ .

For any tableaux  $T$  with columns  $T_1, \dots, T_r$ , define

$$\gamma_T = \gamma_{T_1} + \dots + \gamma_{T_r}, \quad v_T = v_{T_1} + \dots + v_{T_r}, \quad (4.1)$$

where  $\gamma_{T_i} = \gamma_{J_i}$  and  $v_{T_i} = v_{J_i}$  are defined in Section 3.3 and  $J_i$  is the sorted content of the one-column tableau  $T_i$ .

**Definition 4.1.** Let  $\mathcal{T}_{k,n}^{(0)}$  be the set of all one-column tableaux which are obtained by cyclic shifts of the one-column tableau with entries  $1, 2, \dots, k-1, k+1$ . For  $d \geq 0$ , we define recursively

$$\mathbf{N}_{k,n}^{(d)} = \text{Newt} \left( \prod_{T \in \mathcal{T}_{k,n}^{(d)}} \text{ch}_T(x_{i,j}) \right),$$

where  $\mathcal{T}_{k,n}^{(d+1)}$  is the set of all tableaux which correspond to facets of  $\mathbf{N}_{k,n}^{(d)}$ .

More precisely,

$$\mathcal{T}_{k,n}^{(d+1)} = \left\{ T : \gamma_T \text{ is minimized on a facet of } \mathbf{N}_{k,n}^{(d)} \right\},$$

see Section 6 for details on computing  $\mathcal{T}_{k,n}^{(d)}$ .

In particular, the so-called *Planar Kinematics* (PK) polytope [22], denoted there

$$\Pi_{k,n} = \text{Newt} \left( \prod_{j=1}^n \frac{p_{j,j+1,\dots,j+k-2,k}}{p_{j,j+1,\dots,j+k-2,k-1}} \right),$$

is the same as  $\mathbf{N}_{k,n}^{(0)}$  noting that when we evaluated on the matrix  $M$  all denominators are monomials in the  $x_{i,j}$  coordinates. For example, evaluating on the matrix  $M$  we have

$$\mathbf{N}_{2,6}^{(0)} = \frac{x_{1,12}x_{1,23}x_{1,34}x_{1,1234}}{x_{1,1}x_{1,2}x_{1,3}x_{1,4}}$$

and

$$\mathbf{N}_{3,6}^{(0)} = \frac{(x_{1,1}x_{2,1} + x_{1,1}x_{2,2} + x_{1,2}x_{2,2})(x_{1,2}x_{2,2} + x_{1,2}x_{2,3} + x_{1,3}x_{2,3})(x_{1,123})(x_{2,123})}{x_{1,1}x_{1,2}x_{1,3}x_{2,1}x_{2,2}x_{2,3}},$$

where we abbreviate for example  $x_{1,123} = x_{1,1} + x_{1,2} + x_{1,3}$ .

On the other hand, the polytope  $\mathbf{N}_{k,n}^{(1)}$ , which is closely related<sup>2</sup> to the positive tropical Grassmannian, is given by

$$\mathbf{N}_{k,n}^{(1)} = \text{Newt} \left( \prod_{J \in \binom{[n]}{k}} p_J \right).$$

**Remark 4.2.** We are concerned with the facets of polytopes  $\mathbf{N}_{k,n}^{(0)}, \mathbf{N}_{k,n}^{(1)}, \dots, \mathbf{N}_{k,n}^{(d)}$ . Motivated in part by work of Arkani-Hamed, Frost, Plamondon, Salvatori, and Thomas, [2] on polyhedra modeled on punctured surfaces which they call surfacehedra, having infinitely many Minkowski summands, ultimately ( $d \rightarrow \infty$ ) we are interested in a new object (denoted by  $\mathbf{N}_{k,n}^{(\infty)}$ ) which again has infinitely many Minkowski summands (and infinitely many facets). In our proposal, these Minkowski summands are by construction in bijection with prime tableaux in  $\text{SSYT}(k, [n])$  (equivalently prime modules of the quantum affine algebra in the category  $\mathcal{C}_\ell$ ).

We define another version of Newton polytopes non-recursively. For  $k \leq n$  and  $r \in \mathbb{Z}_{\geq 1}$ , denote by  $\text{SSYT}_{k,n}^r$  the set of all tableaux in  $\text{SSYT}(k, [n])$  with  $r$  or less columns.

**Definition 4.3.** For  $k \leq n$  and  $d \in \mathbb{Z}_{\geq 1}$ , we define

$$\mathbf{N}_{k,n}^{(d)} = \text{Newt} \left( \prod_{T \in \text{SSYT}_{k,n}^d} \text{ch}_T(x_{i,j}) \right).$$

**4.2. Tropical Fans for Grassmannian Cluster Algebras.** Recall that given a polytope  $P$  in a real vector space  $V$ , its normal fan  $\mathcal{N}(P)$  is the polyhedral complex on the dual space  $V^*$ , (closed) faces consist of all linear functionals minimized on a given face of  $P$ . In the following, we describe the normal fan of the Newton polytope  $\mathbf{N}_{k,n}^{(d)}$  defined in Section 4.1.

The evaluation of  $\text{ch}_T = \text{ch}(T)$  on the web matrix  $M$  [96] (see Section 3.3), we obtain a subtraction free polynomial in the  $x_{i,j}$  coordinates. For example, for  $\text{Gr}(2, 5)$ , we have that

$$\begin{aligned} p_{1,2}(M) &= p_{1,3} = p_{1,4} = p_{1,5} = 1, & p_{2,3}(M) &= x_{1,1}, & p_{3,4}(M) &= x_{1,2}, & p_{4,5}(M) &= x_{1,3}, \\ p_{2,4}(M) &= x_{1,1} + x_{1,2}, & p_{2,5}(M) &= x_{1,1} + x_{1,2} + x_{1,3}, & p_{3,5}(M) &= x_{1,2} + x_{1,3}. \end{aligned}$$

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<sup>2</sup>In particular, the normal fan of  $\mathbf{N}_{k,n}^{(1)}$  has the following property: its cones are in bijection with the cones in the positive tropical Grassmannian  $\text{Trop}^+G(k, n)$ .



Recall that we denote by  $\mathcal{T}_{k,n}^{(0)}$  the set of all tableaux obtained by cyclic shifts of the one-column tableau with entries  $1, 2, \dots, k-1, k+1$ . By tropicalizing all  $\text{ch}_T(M)$ ,  $T \in \mathcal{T}_{k,n}^{(0)}$ , we obtain piecewise linear functions in the space of dimension  $(k-1)(n-k-1)$  parametrized by  $y_{i,j}$  ( $y_{i,j}$  is the tropical version of  $x_{i,j}$ ). Such a function is linear on a collection of cones; these cones assemble to define a polyhedral fan. The common refinement of these fans is the normal fan  $\mathcal{N}(\mathbf{N}_{k,n}^{(0)})$  of the Newton polytope  $\mathbf{N}_{k,n}^{(0)}$ . By [43, Corollary 10.5], the set of rays of  $\mathcal{N}(\mathbf{N}_{k,n}^{(0)})$  is given by

$$\left\{ \text{Ray}(v_J) : J \in \binom{[n]}{k} \right\},$$

where  $\text{Ray}(v) = \{cv : c \geq 0\}$  is the ray in the direction of a vector  $v$ .

For  $d \geq 1$ , the normal fan  $\mathcal{N}(\mathbf{N}_{k,n}^{(d)})$  can be constructed from  $\mathcal{N}(\mathbf{N}_{k,n}^{(d-1)})$  as follows. Let  $\mathcal{T}_{k,n}^{(d)}$  be the set of all tableaux corresponding to rays of  $\mathcal{N}(\mathbf{N}_{k,n}^{(d-1)})$ , that is

$$\mathcal{T}_{k,n}^{(d)} = \left\{ T : \text{Ray}(v_T) \text{ is a ray of } \mathcal{N}(\mathbf{N}_{k,n}^{(d-1)}) \right\}.$$

Here  $v_T$  is defined in Equation (4.1) and the construction of tableaux from rays is given in Section 6. Indeed, in Section 6 we construct tableaux from facets of Newton polytopes. The construction of tableaux from rays of normal fans is the same.

**Remark 4.4.** Tropical fans for Grassmannian cluster algebras have been defined in [13, 34], by tropical evaluations of finite sets of cluster variables. The tropical fans  $\mathcal{N}(\mathbf{N}_{k,n}^{(d)})$  defined here use not only cluster variables but also other prime elements in the dual canonical basis of  $\mathbb{C}[\text{Gr}(k, n)]$ .

**4.3. Relation with positive tropical Grassmannians.** Recall that we use  $\binom{[n]}{k}^{nf}$  to denote the set of  $k$ -element subsets of  $[n]$  which are nonfrozen, i.e., not of the form  $[i, i+k-1]$  up to cyclic shifts, and recall that  $\{e^J : J \in \binom{[n]}{k}\}$  is the standard basis of  $\mathbb{R}^{\binom{[n]}{k}}$ . For each  $J \in \binom{[n]}{k}^{nf}$ , recall that  $\mathfrak{h}_J \in \mathbb{R}^{\binom{[n]}{k}}$  [43, 44] is defined by

$$\mathfrak{h}_J = -\frac{1}{n} \sum_{I \in \binom{[n]}{k}} \min \{L_1(e_J - e_I), L_2(e_J - e_I), \dots, L_n(e_J - e_I)\} e^I, \quad (4.2)$$

where

$$L_j(x) = x_{j+1} + 2x_{j+2} + \dots + (n-1)x_{j-1}.$$

The *lineality subspace*  $\text{Lin}_{k,n}$  of  $\mathbb{R}^{\binom{[n]}{k}}$  is defined by

$$\text{Lin}_{k,n} = \text{span} \left\{ \sum_{J \in \binom{[n]}{k}, J \ni j} e^J, j = 1, \dots, n \right\},$$

see [44, Definition 2.1]. Clearly,  $\dim(\text{Lin}_{k,n}) = n$ .

For each  $J = \{j_1, \dots, j_k\} \in \binom{[n]}{k}^{nf}$ , define a cubical collection of  $k$ -element subsets of  $\{1, \dots, n\}$  by

$$\mathcal{U}(J) = \{\{j_1 + t_1, \dots, j_k + t_k\} : t_i \in \{0, 1\}, \text{ and } t_i = 0 \text{ whenever } j_i + 1 \in J\},$$

where addition is modulo  $n$ , see [43].

Denote by  $\omega_J(y)$  [43] the tropical planar cross-ratio

$$\omega_J = \sum_{J' \in \mathcal{U}(J)} (-1)^{k - \#(J' \cap J) + 1} P_{J'}(y),$$

where  $P_{J'}(y) = \text{Trop}(p_{J'})(y)$  is the tropicalization of the Plücker coordinate  $p_{J'}(x)$ , evaluated on the web matrix  $M = (x_{i,j})_{k \times n}$  in Section 3.3.

Denote by  $\mathcal{F}_n^{(k)} : \mathbb{R}^{(k-1) \times (n-k)} \rightarrow \mathbb{R}^{\binom{n}{k}} / \text{Lin}_{k,n}$  the map

$$\mathcal{F}_n^{(k)}(y) = \sum_{J \in \binom{[n]}{k}^{nf}} \omega_J(y) \mathfrak{h}_J. \quad (4.3)$$

The normal fan  $\mathcal{N}(\mathbf{N}_{k,n}^{(1)})$  defined in Section 4.2 has been shown [8, Proposition 11.5] to satisfy the following property: its cones are in bijection with the cones in the positive tropical Grassmannian  $\text{Trop}^+ G(k, n)$ , as defined by Speyer and Williams [96]. In particular, this bijection is achieved via the piecewise-linear map  $\mathcal{F}_n^{(k)}$ , which is equal (modulo a change in parameterization) to the map  $\text{Trop}(\Phi_2)$  defined in [96, Section 4], see also [97].

## 5. SEMISTANDARD YOUNG TABLEAUX AND GENERALIZED ROOT POLYTOPES

In this section, we study relation between semistandard tableaux and generalized root polytopes.

**5.1. Isomorphism of Monoids.** Recall that the set  $\text{SSYT}(k, [n], \sim)$  of all  $\sim$ -equivalence classes of semistandard Young tableaux of rectangular shape with  $k$  rows and with entries in  $[n]$  form a monoid under the multiplication “ $\cup$ ” [23], see Section 3.1. This monoid is isomorphic to the monoid  $\mathcal{P}_\ell^+$  of dominant monomials in  $\mathcal{C}_\ell^{\text{sl}k}$ ,  $n = k + \ell + 1$ .

The vector space  $\mathbb{Z}_{\geq 0}^{(k-1) \times (n-k)}$  form a monoid  $(\mathbb{Z}_{\geq 0}^{(k-1) \times (n-k)}, +)$  generated by  $e_{i,j}$ ,  $i \in [k-1]$ ,  $j \in [n-k]$ , where  $e_{i,j}$ 's form a standard basis of  $\mathbb{Z}_{\geq 0}^{(k-1) \times (n-k)}$  ( $e_{i,j}$ 's also form a standard basis of  $\mathbb{R}^{(k-1) \times (n-k)}$ ).

**Lemma 5.1.** *We have an isomorphism of monoids*

$$(\text{SSYT}(k, [n], \sim), \cup) \rightarrow (\mathbb{Z}_{\geq 0}^{(k-1) \times (n-k)}, +).$$

*Proof.* For  $i \in [k-1]$ ,  $j \in [n-k]$ , denote by  $T_{i,j}$  the fundamental tableau with entries  $[j, j+k] \setminus \{i+j\}$ .

By Lemma 3.13 in [23], every tableau in  $\text{SSYT}(k, [n], \sim)$  is  $\sim$ -equivalent to the union of a set of fundamental tableaux. The isomorphism  $(\text{SSYT}(k, [n], \sim), \cup) \rightarrow (\mathbb{Z}_{\geq 0}^{(k-1) \times (n-k)}, +)$  is induced by  $T_{i,j} \mapsto e_{i,j}$ . The inverse isomorphism is given as follows. Every element  $v$  in  $(\mathbb{Z}_{\geq 0}^{(k-1) \times (n-k)}, +)$  can be written as  $v = \sum_{i,j} c_{i,j} e_{i,j}$  for some positive integers  $c_{i,j}$ . Let  $T_v = \cup_{i,j} T_{i,j}^{\cup c_{i,j}}$ . The inverse isomorphism is given by  $v \mapsto T_v$ .  $\square$

We denote by  $T_v$  the tableau in  $\text{SSYT}(k, [n], \sim)$  corresponding to  $v \in \mathbb{Z}_{\geq 0}^{(k-1) \times (n-k)}$  and denote by  $v_T$  the element in  $\mathbb{Z}_{\geq 0}^{(k-1) \times (n-k)}$  corresponding to  $T \in \text{SSYT}(k, [n], \sim)$ .

**5.2. Generalized Root Polytopes.** For any collection  $\mathcal{J} = \{J_1, \dots, J_m\} \in \mathbf{NC}_{k,n}$  of nonfrozen subsets  $J_i$ , define

$$[\mathcal{J}] = \text{Convex hull}(\{0, v_{J_1}, \dots, v_{J_m}\}).$$

The generalized root polytope  $\mathcal{R}_{n-k}^{(k)}$  is the convex hull of all generalized positive roots  $v_J$ ,

$$\mathcal{R}_{n-k}^{(k)} = \left\{ v_J \in \mathbb{T}^{k-1, n-k} : J \in \binom{[n]}{k}^{nf} \right\},$$

where we remind that  $\mathbb{T}^{k-1, n-k} = (\mathbb{T}^{n-k})^{\times (k-1)}$  and  $\mathbb{T}^{n-k} = \mathbb{R}^{n-k} / \mathbb{R}(1, \dots, 1)$ .

**Theorem 5.2** ([43, Theorem 1.2]). *The set of simplices  $\{[\mathcal{J}] : \mathcal{J} \in \mathbf{NC}_{k,n}\}$  defines a flag, unimodular triangulation of  $\mathcal{R}_{n-k}^{(k)}$ : simplices in the triangulation have equal volume and are in bijection with pairwise noncrossing collections of nonfrozen  $k$ -element subsets. In particular, the set of cones*

$$\mathcal{C}_{\mathcal{J}} = \left\{ \sum_{J \in \mathcal{J}} c_J v_J : c_J > 0 \right\}$$

*assemble to define a complete simplicial in  $\mathbb{T}^{k-1, n-k}$ , and any point in  $\mathbb{T}^{k-1, n-k}$  lies in the relative interior of a unique cone in the fan.*

Now under the isomorphism in Lemma 5.1, one-column tableaux correspond to generalized positive roots; thus, Theorem 5.2 says that any linear combination of generalized positive roots with real coefficients decomposes uniquely as a linear combination, with positive coefficients, indexed by a pairwise noncrossing collection. This means that if we restrict to integer coefficients then the triangulation has a beautiful representation-theoretic interpretation in terms of tableaux! We give a proof in the next subsection of this result for integer coefficients  $c_J$  using representation theory of quantum affine algebras, and study the relation between semistandard Young tableaux and noncrossing tuples.

**Example 5.3.** Let

$$v = -v_{1,5,9} + 2v_{2,6,10} + 3v_{3,7,11} + 4v_{4,8,12}.$$

Then  $v$  has the following noncrossing decomposition using Theorem 5.2:

$$v = v_{1,2,6} + v_{2,8,10} + v_{2,9,10} + 3v_{3,8,10} + 2v_{4,6,10} + 3v_{4,7,10} + v_{7,8,10}$$

**5.3. Semistandard Young Tableaux and Noncrossing Tuples.** First we consider the case of  $k = 2$ .

**Lemma 5.4.** *For every tableau  $T \in \text{SSYT}(2, [n])$ , there is a unique unordered  $m$ -tuple  $(S_1, \dots, S_m)$  of one-column tableaux which are pairwise noncrossing such that  $T = S_1 \cup \dots \cup S_m$ .*

*Proof.* First note that for 2-row one column tableaux  $\begin{smallmatrix} a \\ b \end{smallmatrix}$ ,  $\begin{smallmatrix} a' \\ b' \end{smallmatrix}$ , they are noncrossing if and only if they are weakly separated. If  $b = a + 1$ , then  $\begin{smallmatrix} a \\ b \end{smallmatrix}$  is frozen and it is weakly separated with any 2-row one-column tableau. Let  $T$  be a 2-row one column tableau and let  $T'$  be the tableau obtained from  $T$  by removing all factors of the form  $\begin{smallmatrix} a \\ a+1 \end{smallmatrix}$ . Denote these frozen factors by  $T_1'', \dots, T_t''$ . By Theorem 1.1 in [23],  $T'$  corresponds to a simple  $U_q(\widehat{\mathfrak{sl}}_2)$ -module  $L(M_T) = L(M_{T'})$ . By Sections 4.8, 4.9, 4.11 in [27], every prime  $U_q(\widehat{\mathfrak{sl}}_2)$ -module is a Kirillov-Reshetikhin module and every simple  $U_q(\widehat{\mathfrak{sl}}_2)$ -module is decomposed as a tensor product of Kirillov-Reshetikhin modules (note that evaluation modules of  $U_q(\widehat{\mathfrak{sl}}_2)$  are Kirillov-Reshetikhin modules). Therefore

$$\chi_q(L(M_T)) = \chi_q(L(M_1)) \cdots \chi_q(L(M_r)) \quad (5.1)$$

for some Kirillov-Reshetikhin modules  $L(M_1), \dots, L(M_r)$ . Every Kirillov-Reshetikhin module corresponds to a one-column tableau, see Section 3.3 in [23]. Let  $T_{M_1}, \dots, T_{M_r}$  be the one-column tableaux corresponding to  $L(M_1), \dots, L(M_r)$  respectively. By Equation (5.1), we have that for any  $i, j$ ,  $L(M_i) \otimes L(M_j)$  is simple. Hence by Theorem 1.1 in [81],  $T_{M_i}$  and  $T_{M_j}$  are weakly separated. Therefore  $T = T_{M_1} \cup \dots \cup T_{M_r} \cup T_1'' \cup \dots \cup T_t''$  and any two one-column tableaux in the  $\cup$ -product are weakly separated.  $\square$

**Example 5.5.** Let  $T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 2 & 3 & 3 & 4 \\ \hline 3 & 5 & 5 & 6 & 8 & 9 \\ \hline \end{array}$ . The factors of  $T$  of the form  $\begin{smallmatrix} a \\ a+1 \end{smallmatrix}$  are  $\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$ ,  $\begin{smallmatrix} 4 \\ 5 \end{smallmatrix}$ . Removing these factors, we obtain  $T' = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 3 \\ \hline 5 & 6 & 8 & 9 \\ \hline \end{array}$ . The corresponding  $U_q(\widehat{\mathfrak{sl}}_2)$ -module is

$$L(M_T) = L(Y_{1,-1}Y_{1,-3}^2Y_{1,-5}^4Y_{1,-7}^3Y_{1,-9}^2Y_{1,-11}^2Y_{1,-13}),$$

see Section 3.2. By taking all maximal strings of Kirillov-Reshetikhin modules, we have that

$$\chi_q(L(M_T)) = \chi_q(L(M_1))\chi_q(L(M_2))\chi_q(L(M_3))\chi_q(L(M_4)),$$

where

$$M_1 = Y_{1,-1}Y_{1,-3}\cdots Y_{1,-13}, \quad M_2 = Y_{1,-3}Y_{1,-5}\cdots Y_{1,-11}, \quad M_3 = Y_{1,-5}, \quad M_4 = Y_{1,-5}Y_{1,-7}.$$

The corresponding one-column tableaux are  $\begin{array}{|c|} \hline 1 \\ \hline 9 \\ \hline \end{array}$ ,  $\begin{array}{|c|} \hline 2 \\ \hline 8 \\ \hline \end{array}$ ,  $\begin{array}{|c|} \hline 3 \\ \hline 5 \\ \hline \end{array}$ ,  $\begin{array}{|c|} \hline 3 \\ \hline 6 \\ \hline \end{array}$ , respectively. Therefore the unordered 6-tuple of pairwise noncrossing one-column tableaux corresponding to  $T$  is

$$\left( \begin{array}{|c|} \hline 1 \\ \hline 9 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 8 \\ \hline \end{array}, \begin{array}{|c|} \hline 3 \\ \hline 5 \\ \hline \end{array}, \begin{array}{|c|} \hline 3 \\ \hline 6 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline \end{array} \right).$$

**Lemma 5.6.** *For every tableau  $T \in \text{SSYT}(k, [n])$ , there is a unique unordered  $m$ -tuple  $(S_1, \dots, S_m)$  of one-column tableaux which are pairwise noncrossing such that  $T = S_1 \cup \dots \cup S_m$ .*

*Proof.* We prove by induction on  $k$ . The result is clearly true in the case of  $k = 1$ . The case of  $k = 2$  is proved in Lemma 5.4.

Suppose that  $k \geq 3$  and the result is true for  $\text{SSYT}(k', [n])$  for any  $k' \leq k - 1$ . Let  $T \in \text{SSYT}(k, [n])$ . Let  $T'$  be the sub-tableau of  $T$  consisting of the first  $k - 1$  rows of  $T$ . By induction hypothesis, there there is a unique unordered  $m$ -tuple  $S' = (S'_1, \dots, S'_m)$  of one-column tableaux which are pairwise noncrossing such that  $T' = S'_1 \cup \dots \cup S'_m$ .

Let  $T''$  be the sub-tableau of  $T$  consisting of the last  $k - 1$  rows of  $T$ . By induction hypothesis, there there is a unique unordered  $m$ -tuple  $S'' = (S''_1, \dots, S''_m)$  of one-column tableaux which are pairwise noncrossing such that  $T'' = S''_1 \cup \dots \cup S''_m$ .

Let  $T'''$  be the sub-tableau of  $T$  consisting of the middle  $k - 2$  rows of  $T$ . By induction hypothesis, there there is a unique unordered  $m$ -tuple  $S''' = (S'''_1, \dots, S'''_m)$  of one-column tableaux which are pairwise noncrossing such that  $T''' = S'''_1 \cup \dots \cup S'''_m$ .

We can choose some ordering of  $S', S'', S'''$  such that for each  $j \in [m]$ ,  $S_j''' = S'_j \cap S''_j$ . Let  $S_j = S'_j \cup S''_j$ ,  $j \in [m]$ . Then  $T = S_1 \cup \dots \cup S_m$  and  $S_1, \dots, S_m$  are pairwise noncrossing.  $\square$

**Example 5.7.** Let  $T = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline 7 & 8 \\ \hline \end{array}$ . The unique noncrossing 2-tuple of one-column tableaux

corresponding to  $T$  is  $\left( \begin{array}{|c|} \hline 1 \\ \hline 4 \\ \hline 5 \\ \hline 8 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline 6 \\ \hline 7 \\ \hline \end{array} \right)$ .

## 6. FROM FACETS TO PRIME MODULES

In this section, we describe a procedure to produce a simple  $U_q(\widehat{\mathfrak{sl}}_k)$ -module from every facet of the Newton polytope defined in Section 4 and we conjecture that the obtained simple  $U_q(\widehat{\mathfrak{sl}}_k)$ -module is prime.

**6.1. A Procedure to Produce a Simple  $U_q(\widehat{\mathfrak{sl}}_k)$ -module from a Given Facet.** Adapting the results of [23] (see Section 3.2), it suffices to give a procedure to produce a semistandard Young tableau from a given facet.

The Newton polytope  $\mathbf{N}_{k,n}^{(d)}$  defined in Section 4 is described using certain equations and inequalities in its H-representation (represent the polytope by an intersection half-spaces and hyperplanes). Let  $F$  be a facet of the Newton polytope  $\mathbf{N}_{k,n}^{(d)}$ . The normal vector  $v_F$  of  $F$  is the coefficient vector in one of the inequalities in the H-representation of  $\mathbf{N}_{k,n}^{(d)}$ . If there is an entry of the vector  $v_F$  which is negative, then we add some vectors which are coefficients of the equations in the H-representation of  $\mathbf{N}_{k,n}^{(d)}$  to  $v_F$  such that the resulting vector  $v'_F$  all have non-negative entries. The vector  $v'_F$  can be written as  $v'_F = \sum_{i,j} c_{i,j} e_{i,j}$  for some positive integers  $c_{i,j}$ , where  $e_{i,j}$  is the standard basis of  $\mathbb{R}^{(k-1) \times (n-k)}$ . By Lemma 5.1, each  $e_{i,j}$  corresponds to a fundamental tableau  $T_{i,j}$  which is defined to be the one-column tableau with entries  $[j, j+k] \setminus \{i+j\}$ . The tableau  $T_F$  corresponding to  $F$  is obtained from  $\cup_{i,j} T_{i,j}^{\cup c_{i,j}}$  by removing all frozen factors (if any).

**Conjecture 6.1.** (1) For  $d \in \mathbb{Z}_{\geq 0}$ ,  $k \leq n$ , and every facet  $F$  of the Newton polytope  $\mathbf{N}_{k,n}^{(d)}$ , we have that the corresponding tableau  $T_F$  is prime.  
(2) For  $k \leq n$  and every nonfrozen prime tableau  $T$  in  $\text{SSYT}(k, [n])$ , there is  $d \geq 1$  and a facet  $F$  of  $\mathbf{N}_{k,n}^{(d)}$  such that  $T = T_F$ .

Conjecture 6.1 gives systematic procedure to construct all prime  $U_q(\widehat{\mathfrak{sl}}_k)$ -modules.

We will give an explicit description of all 2-column prime tableaux in Section 7. Namely, we will prove that a 2-column tableau in  $\text{SSYT}(k, [n])$  is prime if and only if it is the union of two one-column tableaux which are noncrossing and not weakly separated, in Section 7.

**6.2. Example:**  $\text{Gr}(3, 6)$ . In the case of  $\mathbb{C}[\text{Gr}(3, 6)]$ , we use the web matrix (see Section 3.3)

$$M = \begin{bmatrix} 1 & 0 & 0 & x_{1,1}x_{2,1} & x_{1,1}x_{2,12} + x_{1,2}x_{2,2} & x_{1,1}x_{2,123} + x_{1,2}x_{2,23} + x_{1,3}x_{2,3} \\ 0 & 1 & 0 & -x_{2,1} & -x_{2,12} & -x_{2,123} \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix},$$

where we abbreviate for example  $x_{2,23} = x_{2,2} + x_{2,3}$ . Evaluating all Plücker coordinates on  $M$  and take their product, we obtain a polynomial  $p$ . The Newton polytope  $\mathbf{N}_{3,6}^{(1)}$  is the Newton polytope defined by the vertices given by the exponents of monomials of  $p$ . The

H-representation of  $\mathbf{N}_{3,6}^{(1)}$  is given by

$$\begin{aligned}
(0, 0, 0, 1, 1, 1) \cdot x - 20 &= 0, & (1, 1, 1, 0, 0, 0) \cdot x - 10 &= 0, & (0, 1, 1, 0, 0, 0) \cdot x - 4 &\geq 0, \\
(0, 0, 1, 0, 0, 0) \cdot x - 1 &\geq 0, & (0, 0, 0, 0, 1, 1) \cdot x - 11 &\geq 0, & (0, 0, 0, 0, 0, 1) \cdot x - 4 &\geq 0, \\
(0, 0, 1, 1, 0, 0) \cdot x - 6 &\geq 0, & (0, 0, 0, 0, 1, 0) \cdot x - 4 &\geq 0, & (0, 0, 0, 1, 0, 0) \cdot x - 4 &\geq 0, \\
(1, 0, 0, 0, 0, 0) \cdot x - 1 &\geq 0, & (1, 0, 0, 0, 1, 0) \cdot x - 6 &\geq 0, & (1, 1, 0, 0, 1, 1) \cdot x - 16 &\geq 0, \\
(1, 1, 0, 0, 0, 0) \cdot x - 4 &\geq 0, & (0, 0, 0, 1, 1, 0) \cdot x - 11 &\geq 0, & (0, 1, 0, 0, 0, 0) \cdot x - 1 &\geq 0, \\
(1, 0, 0, 0, 1, 1) \cdot x - 14 &\geq 0, & (0, 1, 0, 0, 0, 1) \cdot x - 6 &\geq 0, & (1, 1, 0, 0, 0, 1) \cdot x - 11 &\geq 0,
\end{aligned} \tag{6.1}$$

where  $(0, 0, 0, 1, 1, 1) \cdot x$  is the inner product of the vectors  $(0, 0, 0, 1, 1, 1)$  and  $x$ .

Now we compute the tableau corresponding to each facet. For example, for the facet  $F$  with the normal vector  $v_F = (0, 1, 1, 0, 0, 0)$  in the first line of (6.1), we have that  $v_F = e_{1,2} + e_{1,3}$ . The generalized roots  $e_{1,2}$ ,  $e_{1,3}$  corresponds to tableaux  $\begin{array}{|c|} \hline 2 \\ \hline 4 \\ \hline 5 \\ \hline \end{array}$ ,  $\begin{array}{|c|} \hline 3 \\ \hline 5 \\ \hline 6 \\ \hline \end{array}$  respectively.

Removing the frozen factor  $\begin{array}{|c|} \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline \end{array}$  in  $\begin{array}{|c|} \hline 2 \\ \hline 4 \\ \hline 5 \\ \hline \end{array} \cup \begin{array}{|c|} \hline 3 \\ \hline 5 \\ \hline 6 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 4 & 5 \\ \hline 5 & 6 \\ \hline \end{array}$ , we obtain  $T_F = \begin{array}{|c|} \hline 2 \\ \hline 5 \\ \hline 6 \\ \hline \end{array}$ . Similarly, we have the correspondence in Table 1.

Moreover, the two hyperplanes in (6.1) of the Newton polytope correspond to the following generalized roots and frozen tableaux:

$$\begin{aligned}
(0, 0, 0, 1, 1, 1), & \quad \alpha_{2,1} + \alpha_{2,2} + \alpha_{2,3}, \quad [1, 2, 6], \\
(1, 1, 1, 0, 0, 0), & \quad \alpha_{1,1} + \alpha_{1,2} + \alpha_{1,3}, \quad [1, 5, 6].
\end{aligned}$$

The facets of  $\mathbf{N}_{3,6}^{(1)}$  give all the prime modules (non-frozen) in the category  $\mathcal{C}_\ell$ ,  $\ell = 2$ .

By the same computations, we have that  $\mathbf{N}_{3,6}^{(0)}$  has 14 facets. The Newton polytope  $\mathbf{N}_{3,6}^{(1)}$  has 16 facets but is not a simple polytope<sup>3</sup>. The Newton polytope  $\mathbf{N}_{3,6}^{(d)}$  has 16 facets for any  $d \geq 2$  and it is simple. The Newton polytope  $\mathbf{N}_{3,6}^{(d)}$  is the type  $D_4$  associahedron for any  $d \geq 2$ , see Section 6 of [96].

**6.3. Examples:**  $\text{Gr}(3, 8)$ ,  $\text{Gr}(4, 8)$ . In the case of  $\mathbb{C}[\text{Gr}(3, 8)]$ , the PK polytope  $\mathbf{N}_{3,8}^{(0)}$  has  $\binom{8}{3} - 8 = 48$  facets,  $\mathbf{N}_{3,8}^{(1)}$  has 120 facets (see also Proposition 6.2 in [13]),  $\mathbf{N}_{3,8}^{(2)}$  has 128 facets but it is not a simple polytope;  $\mathbf{N}_{3,8}^{(d)}$  has 128 facets and  $\mathbf{N}_{3,8}^{(d)}$  is a simple polytope for any  $d \geq 3$ . This agrees with the fact that the dual canonical basis  $\mathbb{C}[\text{Gr}(3, 8)]$  has 128 prime elements (not including frozen variables).

<sup>3</sup>A polytope  $P$  of dimension  $d$  is simple if every vertex has exactly  $d$  incident edges.

generalized roots	facets, hyperplanes	tableaux	modules
$\gamma_{124} = \alpha_{2,1}$	$(0, 0, 0, 1, 0, 0)$	[124]	$Y_{1,-1}$
$\gamma_{125} = \alpha_{2,1} + \alpha_{2,2}$	$(0, 0, 0, 1, 1, 0)$	[125]	$Y_{1,-3}Y_{1,-1}$
$\gamma_{134} = \alpha_{1,1}$	$(1, 0, 0, 0, 0, 0)$	[134]	$Y_{2,0}$
$\gamma_{135} = \alpha_{1,1} + \alpha_{2,2}$	$(1, 0, 0, 0, 1, 0)$	[135]	$Y_{1,-3}Y_{2,0}$
$\gamma_{136} = \alpha_{1,1} + \alpha_{2,2} + \alpha_{2,3}$	$(1, 0, 0, 0, 1, 1)$	[136]	$Y_{1,-5}Y_{1,-3}Y_{2,0}$
$\gamma_{145} = \alpha_{1,1} + \alpha_{1,2}$	$(1, 1, 0, 0, 0, 0)$	[145]	$Y_{2,-2}Y_{2,0}$
$\gamma_{146} = \alpha_{1,1} + \alpha_{1,2} + \alpha_{2,3}$	$(1, 1, 0, 0, 0, 1)$	[146]	$Y_{1,-5}Y_{2,-2}Y_{2,0}$
$\gamma_{235} = \alpha_{2,2}$	$(0, 0, 0, 0, 1, 0)$	[235]	$Y_{1,-3}$
$\gamma_{236} = \alpha_{2,2} + \alpha_{2,3}$	$(0, 0, 0, 0, 1, 1)$	[236]	$Y_{1,-5}Y_{1,-3}$
$\gamma_{245} = \alpha_{1,2}$	$(0, 1, 0, 0, 0, 0)$	[245]	$Y_{2,-2}$
$\gamma_{246} = \alpha_{1,2} + \alpha_{2,3}$	$(0, 1, 0, 0, 0, 1)$	[246]	$Y_{1,-5}Y_{2,-2}$
$\gamma_{256} = \alpha_{1,2} + \alpha_{1,3}$	$(0, 1, 1, 0, 0, 0)$	[256]	$Y_{2,-4}Y_{2,-2}$
$\gamma_{346} = \alpha_{2,3}$	$(0, 0, 0, 0, 0, 1)$	[346]	$Y_{1,-5}$
$\gamma_{356} = \alpha_{1,3}$	$(0, 0, 1, 0, 0, 0)$	[356]	$Y_{2,-4}$
$\gamma_{124} + \gamma_{356} = \alpha_{1,3} + \alpha_{2,1}$	$(0, 0, 1, 1, 0, 0)$	[[124],[356]]	$Y_{2,-4}Y_{1,-1}$
$\gamma_{145} + \gamma_{236} = \alpha_{1,1} + \alpha_{1,2} + \alpha_{2,2} + \alpha_{2,3}$	$(1, 1, 0, 0, 1, 1)$	[[135],[246]]	$Y_{1,-5}Y_{2,-2}Y_{1,-3}Y_{2,0}$
$\gamma_{126} = \alpha_{2,1} + \alpha_{2,2} + \alpha_{2,3}$	$(0, 0, 0, 1, 1, 1)$	[126]	$Y_{1,-5}Y_{1,-3}Y_{1,-1}$
$\gamma_{156} = \alpha_{1,1} + \alpha_{1,2} + \alpha_{1,3}$	$(1, 1, 1, 0, 0, 0)$	[156]	$Y_{2,-4}Y_{2,-2}Y_{2,0}$

TABLE 1. Correspondence of generalized roots, facets (together with two hyperplanes defining the ambient space of  $\mathbf{N}_{3,6}^{(1)}$ ), tableaux, and dominant monomials of prime modules in the case of  $\text{Gr}(3, 6)$ . In the table, each list in  $[[124], [356]]$  is a column of the tableau.

In the case of  $\mathbb{C}[\text{Gr}(4, 8)]$ ,  $\mathbf{N}_{4,8}^{(0)}$  has  $\binom{8}{4} - 8 = 62$  facets,  $\mathbf{N}_{4,8}^{(1)}$  has 360 facets. Four facets of  $\mathbf{N}_{4,8}^{(1)}$  correspond to prime non-real tableaux

1	2
3	4
5	6
7	8

1	3
2	5
4	7
6	8

1	1	2	3
2	4	4	5
3	6	6	7
5	7	8	8

1	1	2	4
2	3	3	6
4	5	5	7
6	7	8	8

[7, 35, 60, 87].

**6.4. The Face Corresponding to a Tableau.** Take a Newton polytope  $\mathbf{N}_{k,n}^{(d)}$ . We now describe a procedure to produce a face of  $\mathbf{N}_{k,n}^{(d)}$  for a given semistandard tableau.

Let  $T \in \text{SSYT}(k, [n])$ . By Lemma 3.13 in [23], there is a unique semistandard tableau  $T'$  in  $\text{SSYT}(k, [n])$  whose columns  $T'_1, \dots, T'_m$  are fundamental tableaux and  $T \sim T'$ .

For each fundamental tableau  $T_{i,j}$  with entries  $[j, j+k] \setminus \{i+j\}$ , we define  $\alpha(T_{i,j}) = \alpha_{i,j}$ . Let  $\gamma_T = \sum_{j=1}^m \alpha(T'_j)$  and let  $\mathbf{F}_T = \{x \in \mathbf{N}_{k,n}^{(d)} : \gamma_T(y) \geq \gamma_T(x), \forall y \in \mathbf{N}_{k,n}^{(d)}\}$ . Then  $\mathbf{F}_T$  is the



face of  $\mathbf{N}_{k,n}^{(d)}$  such that  $\gamma_T$  is minimized on  $\mathbf{F}_T$ . We say that the tableau  $T$  corresponds to a facet if  $\mathbf{F}_T$  is a facet of  $\mathbf{N}_{k,n}^{(d)}$ .

To compute  $\mathbf{F}_T$ , we compute the monomials in  $p$  which are minimized by  $\gamma_T$ , where  $p$  is the polynomial which defines  $\mathbf{N}_{k,n}^{(d)}$ . The face  $\mathbf{F}_T$  is the face of  $\mathbf{N}_{k,n}^{(d)}$  which is the convex hull of the exponent vectors of these monomials.

Conjecture 6.1 (2) is equivalent to the following conjecture.

**Conjecture 6.2.** *For  $k \leq n$  and any nonfrozen prime tableau  $T \in \text{SSYT}(k, [n])$ , there exists  $d \geq 0$  such that the face  $\mathbf{F}_T$  of  $\mathbf{N}_{k,n}^{(d)}$  has codimension 1.*

We say that two tableaux  $T, T'$  (resp., two simple modules  $L(M), L(M')$ ) are compatible if  $\text{ch}(T)\text{ch}(T') = \text{ch}(T \cup T')$  (resp.,  $\chi_q(L(M))\chi_q(L(M')) = \chi_q(L(MM'))$ ). We give a conjecture about compatibility of two prime tableaux (equivalently, two prime modules).

**Conjecture 6.3.** *Let  $k \leq n$  and  $T, T'$  be two prime tableaux in  $\text{SSYT}(k, [n])$ . Then  $T, T'$  are compatible if and only if there exists  $d \geq 0$ , such that the faces  $\mathbf{F}_T, \mathbf{F}_{T'}$  corresponding to  $T, T'$  are facets and the intersection of  $\mathbf{F}_T, \mathbf{F}_{T'}$  is nonempty.*

**Example 6.4.** Consider a tableau  $T = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 3 \\ \hline 4 & 5 \\ \hline \end{array}$  and  $\mathbf{N}_{3,6}^{(1)}$ . We now check that  $\mathbf{F}_T$  is not a

facet of  $\mathbf{N}_{3,6}^{(1)}$ . The tableau  $T'$  whose columns are fundamental tableaux and such that  $T' \sim T$  is  $\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & 3 \\ \hline 4 & 4 & 5 \\ \hline \end{array}$ . We have that  $\gamma_T = \alpha_{2,1} + \alpha_{1,1} + \alpha_{2,2}$ . We compute the exponents of the

monomials in  $p$  which take the minimal value when applying  $\gamma_T$ , where  $p$  is the polynomial which defines  $\mathbf{N}_{3,6}^{(1)}$ . These exponents define the integer lattice points in  $\mathbf{F}_T$ . The affine span of these points is the face  $\mathbf{F}_T$  and it is of codimension 2. Therefore  $\mathbf{F}_T$  is not a (codimension 1) facet of  $\mathbf{N}_{3,6}^{(1)}$ . Similarly,  $\mathbf{F}_T$  is not a facet of  $\mathbf{N}_{3,6}^{(d)}$  for any  $d \geq 1$ .

On the other hand,  $T$  is non-prime. Indeed, we have that  $\text{ch}(T) = p_{134}p_{125}$ . This agrees with Conjecture 6.2.

We give an example that the facets of two compatible prime tableaux have a nonempty intersection.

**Example 6.5.** In Example 6.4, we see that  $\text{ch}\left(\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 3 \\ \hline 4 & 5 \\ \hline \end{array}\right) = p_{134}p_{125}$ . So the two tableaux

$T = \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}$ ,  $T' = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 5 \\ \hline \end{array}$  are compatible. Both of the faces  $\mathbf{F}_T$  and  $\mathbf{F}_{T'}$  in  $\mathbf{N}_{3,6}^{(1)}$  have codimension

1. The intersection of the facets  $\mathbf{F}_T$  and  $\mathbf{F}_{T'}$  is nonempty and has codimension 2. This example verifies Conjecture 6.3.

## 7. EXPLICIT DESCRIPTION OF 2-COLUMN PRIME TABLEAUX

In this section, we prove that a 2-column tableau is prime if and only if it is the union of two one-column tableaux which are noncrossing and not weakly separated. We also compute the number of 2-column prime tableaux in  $\text{SSYT}(k, [n])$ .

**Lemma 7.1.** *Suppose that  $T_1, T_2$  are 1-column tableaux and they are noncrossing and not weakly separated. Then for any pair of 1-column tableaux  $S_1, S_2$  such that  $S_1 \cup S_2 = T_1 \cup T_2$ , we have that  $S_1, S_2$  are not weakly separated.*

*Proof.* Suppose that  $T_1, T_2$  are 1-column tableaux and they are noncrossing and not weakly separated. By Lemma 5.6, for every pair of 1-column tableaux  $S_1, S_2$  such that  $S_1 \cup S_2 = T_1 \cup T_2$ , either  $\{S_1, S_2\} = \{T_1, T_2\}$  or  $S_1, S_2$  are crossing. If  $\{S_1, S_2\} = \{T_1, T_2\}$ , then  $S_1, S_2$  are not weakly separated.

If  $\{S_1, S_2\} \neq \{T_1, T_2\}$ , then  $S_1, S_2$  are crossing. If there are  $1 \leq a < b \leq k$  such that the sub-tableau of  $S_1$  consisting of the  $a$ th to  $b$ th rows of  $S_1$  and the sub-tableau of  $S_2$  consisting of the  $a$ th to  $b$ th rows of  $S_2$  are not weakly separated, then  $S_1, S_2$  are not weakly separated.

Now suppose that for any  $1 \leq a < b \leq k$ , the sub-tableau of  $S_1$  consisting of the  $a$ th to  $b$ th rows of  $S_1$  and the sub-tableau of  $S_2$  consisting of the  $a$ th to  $b$ th rows of  $S_2$  are weakly separated. This contradicts the fact that  $S_1, S_2$  are crossing.  $\square$

**Example 7.2.** Let  $T_1 = \begin{array}{|c|} \hline 1 \\ \hline 4 \\ \hline 5 \\ \hline 8 \\ \hline \end{array}, T_2 = \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline 6 \\ \hline 7 \\ \hline \end{array}$ . We have that  $T_1, T_2$  are noncrossing and not weakly

separated. All pairs of 1-column tableaux  $S_1, S_2$  such that  $S_1 \cup S_2 = T_1 \cup T_2$  are

$$\left( \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 5 \\ \hline 7 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 4 \\ \hline 6 \\ \hline 8 \\ \hline \end{array} \right), \left( \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 5 \\ \hline 8 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 4 \\ \hline 6 \\ \hline 7 \\ \hline \end{array} \right), \left( \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 6 \\ \hline 7 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 4 \\ \hline 5 \\ \hline 8 \\ \hline \end{array} \right), \left( \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 6 \\ \hline 8 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 4 \\ \hline 5 \\ \hline 7 \\ \hline \end{array} \right), \left( \begin{array}{|c|} \hline 1 \\ \hline 4 \\ \hline 5 \\ \hline 7 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline 6 \\ \hline 8 \\ \hline \end{array} \right), \left( \begin{array}{|c|} \hline 1 \\ \hline 4 \\ \hline 6 \\ \hline 8 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline 5 \\ \hline 7 \\ \hline \end{array} \right), \left( \begin{array}{|c|} \hline 1 \\ \hline 4 \\ \hline 6 \\ \hline 7 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline 5 \\ \hline 8 \\ \hline \end{array} \right), \left( \begin{array}{|c|} \hline 1 \\ \hline 4 \\ \hline 5 \\ \hline 8 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline 6 \\ \hline 7 \\ \hline \end{array} \right).$$

All of these pairs are not weakly separated.

**Theorem 7.3.** *Let  $L(M)$  be a simple  $U_q(\widehat{\mathfrak{sl}}_k)$ -module such that  $T_M$  is a 2-column tableau. Then  $L(M)$  is prime if and only if there are one-column tableaux  $T_1, T_2$  such that  $T_M = T_1 \cup T_2$ , and  $T_1, T_2$  are noncrossing and not weakly separated.*

*Proof.* By [81, Theorem 1.1] and [92, Proposition 3], two quantum Plücker coordinates in the quantum Grassmannian cluster algebra are quasi-commute if and only if the corresponding  $k$ -subsets are weakly separated. This implies that for any  $k$ -element subsets  $J, J'$  of  $[n]$ , the  $U_q(\widehat{\mathfrak{sl}}_k)$ -module  $L(M_J) \otimes L(M_{J'})$  is simple if and only if  $J, J'$  are weakly separated.

By Lemma 5.6, there is a unique pair  $T_1, T_2$  of one-column tableaux  $T_1, T_2$  such that  $T_1, T_2$  are noncrossing and  $T_M = T_1 \cup T_2$ .

Suppose that  $T_1, T_2$  are weakly separated. Then  $L(M_{T_1}) \otimes L(M_{T_2})$  is simple. It follows that  $\chi_q(L(M)) = \chi_q(L(M_{T_1}))\chi_q(L(M_{T_2}))$ . Therefore  $L(M)$  is not prime.

Now suppose that  $T_1, T_2$  are not weakly separated. By Lemma 7.1, for any pair  $T'_1, T'_2$  of 1-column tableaux such that  $T_1 \cup T_2 = T'_1 \cup T'_2$ , we have that  $T'_1, T'_2$  are not weakly separated. Therefore  $\text{ch}(T_M) \neq \text{ch}(T'_1)\text{ch}(T'_2)$ ,  $\chi_q(L(M)) \neq \chi_q(L(M_{T'_1}))\chi_q(L(M_{T'_2}))$ , for any pair of 1-column tableaux  $T'_1, T'_2$  such that  $T_M = T'_1 \cup T'_2$ . Therefore  $L(M)$  and  $T_M$  are prime.  $\square$

Denote  $\binom{n}{a,b,c} = \frac{n!}{a!b!c!}$  and  $I\Delta J = (I \setminus J) \cup (J \setminus I)$  for two sets  $I, J$ .

**Proposition 7.4.** *For  $k \leq n/2$ , the number of 2-column prime tableaux is  $a_{k,n,2} - b_{k,n}$ , where  $a_{k,n,m} = \prod_{i=1}^k \prod_{j=1}^m \frac{n-i+j}{k+m-i-j+1}$  and  $b_{k,n} = \binom{n}{k} + \sum_{j=1}^k j \binom{n}{k-j, 2j, n-k-j}$ .*

*Proof.* The number of semistandard Young tableaux of rectangular shape with  $k$  rows and with entries in  $\{1, \dots, n\}$  and with  $m$  columns is  $a_{k,n,m}$ , see [98].

Assume that  $k \leq n/2$ . If  $I = J$ , then  $I, J$  are weakly separated and there are  $\binom{n}{k}$  choices of  $I = J$ . Now assume that  $I \neq J$ . Denote  $|I - J| = |J - I| = j$ . Since  $|I \cap J| = k - j$ ,  $|I\Delta J| = 2j$ , there are  $\binom{n}{k-j, 2j, n-k-j}$  ways to fix the sets  $I \cap J$  and  $I\Delta J$ .

Since either  $I \setminus J$  or  $J \setminus I$  should be a segment of  $s$  consecutive elements of the  $2j$  elements in  $I\Delta J$ , there are  $2j$  choices of  $I - J$ . Since the pair  $(I, J)$  is unordered, there are  $\frac{1}{2} \sum_{j=1}^k 2j \binom{n}{k-j, 2j, n-k-j}$  choices of weakly separated pairs  $(I, J)$  (unordered) in the case of  $I \neq J$ . It follows that the number of unordered weakly separated pairs among all Plücker coordinates is  $b_{k,n}$ .

Therefore the number of 2-column prime tableaux is  $a_{k,n,2} - b_{k,n}$ .  $\square$

**Remark 7.5.** It is conjectured in [12] that for  $k \leq n/2$ , there are

$$\sum_{r=3}^k \left( \frac{2r}{3} \cdot p_1(r) + 2r \cdot p_2(r) + 4r \cdot p_3(r) \right) \cdot \binom{n}{2r} \binom{n-2r}{k-r}$$

2-column cluster variables in  $\mathbb{C}[\text{Gr}(k, n)]$ , where  $p_i(r)$  is the number of partitions  $r = r_1 + r_2 + r_3$  such that  $r_1, r_2, r_3 \in \mathbb{Z}_{\geq 1}$  and  $|\{r_1, r_2, r_3\}| = i$ . The number  $a_{k,n,2} - b_{k,n}$  in Proposition 7.4 includes prime tableaux which are not cluster variables.

## 8. MORE EVIDENCE OF CONJECTURE 6.1

In this section, we verify that the facets of  $\mathbf{N}_{3,9}^{(1)}$  correspond to prime modules of  $U_q(\widehat{\mathfrak{sl}}_3)$ -modules. This gives more evidence of Conjecture 6.1. We also give an explicit conjectural description of the highest  $l$ -weights of a very large family of prime  $U_q(\widehat{\mathfrak{sl}}_k)$ -modules.

**8.1. Facets of  $\mathbf{N}_{3,9}^{(1)}$  correspond to prime modules.** There are 471 facets of  $\mathbf{N}_{3,9}^{(1)}$  (see also [57, 87]). These facets give 471 tableaux in  $\text{SSYT}(3, [9])$ . We now verify that these tableaux are prime.

Among these tableaux, there are 75 one-column tableaux. These correspond to all non-frozen Plücker coordinates in  $\mathbb{C}[\text{Gr}(3, 9)]$ .

There are 168 two-column tableaux in the 471 tableaux. These tableaux can be obtained from the two prime 2-column tableaux in  $\text{SSYT}(3, [6])$  by replacing  $1 < 2 < \dots < 6$  by  $a_1 < a_2 < \dots < a_6$  ( $a_i \in [9]$ ).

There are 156 tableaux with 3 columns in the 471 tableaux. Totally there are 228 prime tableaux with 3 columns in  $\text{SSYT}(3, [9])$  (this can be seen by translating the results about the number of indecomposable modules in Grassmannian cluster category  $\text{CM}(B_{3,9})$  in [12]). The 156 tableaux are part of them. Up to promotion [89, 90, 91], these 156 tableaux are

1 2 5	1 2 5	1 2 6	1 2 6	1 3 6	1 2 6	1 2 4	1 2 4	1 2 6
3 4 7	3 4 8	3 4 8	3 5 8	4 5 8	4 5 8	3 4 7	3 4 8	3 4 8
5 6 9	5 6 9	5 7 9	6 7 9	6 7 9	6 7 9	5 6 9	5 6 9	6 7 9
1 3 3	1 3 4	1 2 5	1 2 5	1 3 5	1 2 5	1 2 4	1 2 5	1 2 5
2 5 6	2 5 6	3 4 7	3 5 8	4 5 8	4 5 8	3 4 8	3 4 7	3 4 8
4 7 9	4 7 9	6 8 8	6 7 9	6 7 9	6 7 9	5 7 9	6 7 9	6 7 9

The orbit (under promotion) of the last tableau has 3 tableaux. The sizes of the other orbits are 9.

There are 69 tableaux which have 4 columns in the 471 tableaux. Up to promotions, these tableaux are

1 2 4 5	1 2 4 5	1 2 2 5	1 1 3 3	1 1 3 4	1 1 4 4	1 1 3 4	1 1 3 3	1 2 4 5
3 4 7 7	3 4 7 8	3 4 5 8	2 5 6 6	2 5 6 6	2 3 7 7	2 5 6 7	2 4 5 6	3 4 7 8
5 6 8 9	5 6 8 9	6 6 7 9	4 7 8 9	4 7 8 9	5 6 8 9	4 7 8 9	4 7 8 9	6 7 8 9

The orbits of the 6th and 7th tableaux have 3 tableaux, respectively. The sizes of the other orbits are 9. In [24], cluster variables in terms of tableaux up to 10 columns in  $\mathbb{C}[\text{Gr}(3, 9)]$  are computed extensively. We check directly that these 69 tableaux are in the list of cluster variables obtained in [24]. Therefore these 69 tableaux are prime.

There are 3 tableaux which have 5 columns in these 471 tableaux. These three tableaux are promotions of the following tableau

1	1	2	4	5
2	3	4	7	8
5	6	7	8	9

These 3 tableaux are in the list of cluster variables obtained in [24]. Therefore they are prime.



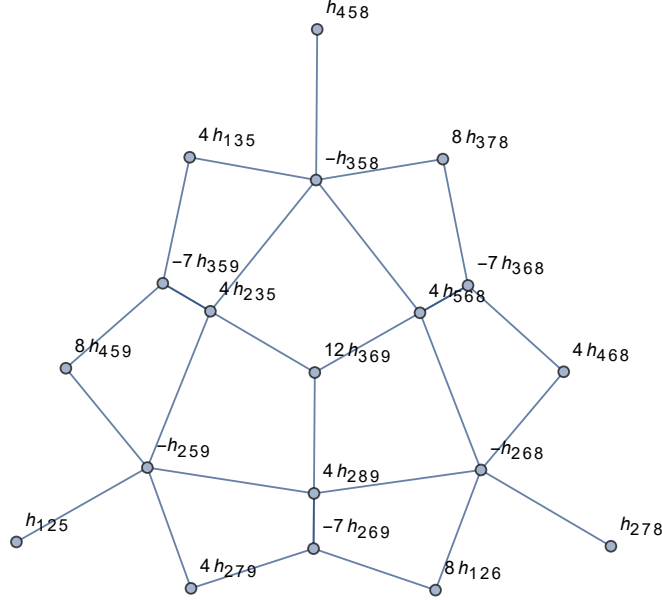


FIGURE 1. Given a tableau  $T$  with 39 columns corresponding to a (conjectural) prime module of  $U_q(\widehat{\mathfrak{sl}}_3)$ , we use the map  $\mathcal{F}_9^{(3)}(v_T)$  to define a positive tropical Plücker vector (8.2), expanded in the  $\mathfrak{h}_{abc}$  basis of  $\mathbb{R}^{\binom{[9]}{3}}$ , see also [42] on matroidal weighted blade arrangements. The induced matroid subdivision is dual to the (unweighted) graph in Figure 2. Prime tableaux with more and more columns may appear with the same diagram but with different (integer) coefficients; however the matroid subdivision will not change.

This tableau is obtained using limit  $g$ -vector method in Section 11. Apply the map  $\mathcal{F}_9^{(3)}$  to  $v_T$  (see the formula (4.3)), we obtain

$$\mathcal{F}_9^{(3)}(v_T) = \mathfrak{h}_{1,2,5} + 8\mathfrak{h}_{1,2,6} + 4\mathfrak{h}_{1,3,5} + 4\mathfrak{h}_{2,3,5} - \mathfrak{h}_{2,5,9} - \mathfrak{h}_{2,6,8} - 7\mathfrak{h}_{2,6,9} + \mathfrak{h}_{2,7,8} + 4\mathfrak{h}_{2,7,9} \quad (8.2)$$

$$+ 4\mathfrak{h}_{2,8,9} - \mathfrak{h}_{3,5,8} - 7\mathfrak{h}_{3,5,9} - 7\mathfrak{h}_{3,6,8} + 12\mathfrak{h}_{3,6,9} + 8\mathfrak{h}_{3,7,8} + \mathfrak{h}_{4,5,8} + 8\mathfrak{h}_{4,5,9} + 4\mathfrak{h}_{4,6,8} + 4\mathfrak{h}_{5,6,8}. \quad (8.3)$$

The linear combination is shown schematically in Figure 1 as a *weighted blade arrangement*, see [41, 44] for more about matroid subdivisions. The induced matroidal subdivision is dual to the unweighted graph in Figure 2.

It would be interesting to study the relation between prime modules and matroid subdivisions in detail. We leave this question to a future work.

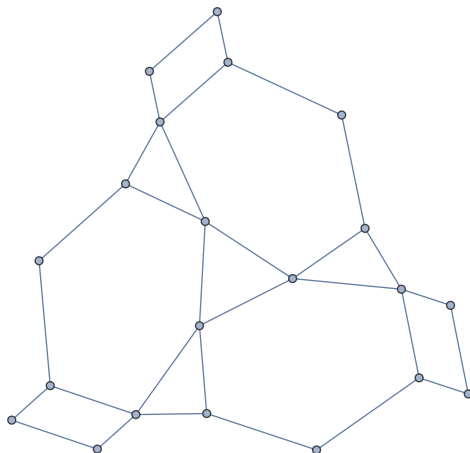


FIGURE 2. Dual graph to the matroid subdivision of  $\Delta_{3,9}$  that is induced by the positive tropical Plücker vector (8.2): place a node at the center of each maximal cell in the subdivision; two nodes are connected by an edge if the corresponding cells share an internal codimension 1 face. This graph is also sometimes called the tight span of the matroid subdivision. Here the subdivision is finest; it has  $\binom{9-2}{3-1} = 21$  maximal cells and cannot be further decomposed into a collection of matroid polytopes.

## 9. NEWTON POLYTOPES AND TROPICAL FANS FOR QUANTUM AFFINE ALGEBRAS

Readers interested in physical applications and stringy integrals of Grassmann type may skip to Section 10.

In this section, for any simple Lie algebra  $\mathfrak{g}$  over  $\mathbb{C}$ , we define Newton polytopes associated to the quantum affine algebra  $U_q(\widehat{\mathfrak{g}})$  using truncated  $q$ -characters of simple  $U_q(\widehat{\mathfrak{g}})$ -modules.

**9.1. Newton Polytopes for Quantum Affine Algebras.** Let  $\mathcal{M}$  be the set of all equivalence classes of fundamental modules of  $U_q(\widehat{\mathfrak{g}})$  in  $\mathcal{C}_\ell$ . For simplicity, we also write an equivalence class  $[L(M)]$  as  $L(M)$ .

**Definition 9.1.** Let  $\mathcal{M}^{(0)} = \mathcal{M}$ . We define recursively

$$\mathbf{N}_{\mathfrak{g},\ell}^{(d)} = \text{Newt} \left( \prod_{L(M) \in \mathcal{M}^{(d)}} \tilde{\chi}_q(L(M)) \right),$$

where  $\mathcal{M}^{(d+1)}$  ( $d \geq 0$ ) is the collection of equivalence classes of simple  $U_q(\widehat{\mathfrak{g}})$ -modules which correspond to facets of  $\mathbf{N}_{\mathfrak{g},\ell}^{(d)}$ .

**Remark 9.2.** The existence of  $\mathbf{N}_{\mathfrak{g},\ell}^{(d)}$  in Definition 9.1 is conjectural. Namely, we do not have a proof that the facets of  $\mathbf{N}_{\mathfrak{g},\ell}^{(d)}$  correspond to simple  $U_q(\widehat{\mathfrak{g}})$ -modules. It would be interesting to give an explicit construction of highest  $l$ -weights of simple  $U_q(\widehat{\mathfrak{g}})$ -modules which correspond to facets of  $\mathbf{N}_{\mathfrak{g},\ell}^{(d)}$ .

We will define another version of Newton polytopes for quantum affine algebras non-recursively in Definition 9.5.

**Remark 9.3.** In type  $A$ , the definition of the Newton polytopes  $\mathbf{N}_{\mathfrak{sl}_{k,\ell}}^{(d)}$  is slightly different from the definition of the Newton polytopes  $\mathbf{N}_{k,n}^{(d)}$  ( $n = k + \ell + 1$ ) in Section 4 for Grassmannian cluster algebras. Here  $\mathbf{N}_{\mathfrak{sl}_{k,\ell}}^{(0)}$  is defined using all fundamental representations of  $U_q(\widehat{\mathfrak{sl}_k})$ . Finite dimensional simple  $U_q(\widehat{\mathfrak{sl}_k})$ -modules in  $\mathcal{C}_\ell$ ,  $n = k + \ell + 1$ , correspond to tableaux in  $\text{SSYT}(k, [n], \sim)$  [23]. In Section 4,  $\mathbf{N}_{k,n}^{(0)}$  is defined using all cyclic shifts of the one-column tableau with entries  $1, 2, \dots, k-1, k+1$ . These one-column tableaux correspond to a set of minimal affinizations of  $U_q(\widehat{\mathfrak{sl}_k})$  [20, 23, 30].

Recall that two simple  $U_q(\widehat{\mathfrak{g}})$ -modules  $L(M), L(M')$  are called compatible if the identity  $\chi_q(L(M))\chi_q(L(M')) = \chi_q(L(MM'))$  holds.

**Conjecture 9.4.** *Let  $\mathfrak{g}$  be a simple Lie algebra over  $\mathbb{C}$  and  $\ell \geq 1$ . We have the following.*

- (1) *For any  $d \geq 0$ , every facet of  $\mathbf{N}_{\mathfrak{g},\ell}^{(d)}$  corresponds to a prime  $U_q(\widehat{\mathfrak{g}})$ -module in  $\mathcal{C}_\ell$ .*
- (2) *For every prime  $U_q(\widehat{\mathfrak{g}})$ -module (nonfrozen)  $L(M)$  in  $\mathcal{C}_\ell$ , there exists  $d \geq 0$  such that  $L(M)$  corresponds to a facet of the polytope  $\mathbf{N}_{\mathfrak{g},\ell}^{(d)}$ .*
- (3) *For any two prime modules in  $\mathcal{C}_\ell$ , they are compatible if and only if there is some  $d \geq 0$  such that there are two facets of  $\mathbf{N}_{\mathfrak{g},\ell}^{(d)}$  corresponding to them and the intersection of these two facets is nonempty.*

We define another version of Newton polytopes for quantum affine algebras non-recursively. For  $d \in \mathbb{Z}_{\geq 1}$ , denote by  $\mathcal{P}_\ell^{+,d}$  the set of all dominant monomials in  $\mathcal{P}_\ell^+$  with degrees less or equal to  $d$ .

**Definition 9.5.** For a simple Lie algebra  $\mathfrak{g}$  over  $\mathbb{C}$ ,  $\ell \geq 1$ , and  $d \in \mathbb{Z}_{\geq 1}$ , we define

$$\mathbf{N}'_{\mathfrak{g},\ell}{}^{(d)} = \text{Newt} \left( \prod_{M \in \mathcal{P}_\ell^{+,d}} \chi_q(L(M)) \right).$$

In the next three subsections, we compute some examples of  $\mathbf{N}_{\mathfrak{g},\ell}^{(d)}$  and  $\mathbf{N}'_{\mathfrak{g},\ell}{}^{(d)}$ .

**9.2. Example:  $\mathfrak{g}$  is of type  $A_1$  and  $\ell = 2$ .** Consider the case of type  $A_1$  and  $\ell = 2$ . The category  $\mathcal{C}_\ell$  has 5 prime modules (not including frozen variables). We choose a height function  $\xi(1) = -1$ , see Section 2.2.



The truncated  $q$ -characters of fundamental modules in  $\mathcal{C}_\ell$  are

$$\tilde{\chi}_q(L(Y_{1,-1})) = Y_{1,-1}, \quad \tilde{\chi}_q(L(Y_{1,-3})) = Y_{1,-3} + Y_{1,-1}^{-1}, \quad \tilde{\chi}_q(L(Y_{1,-5})) = Y_{1,-5} + Y_{1,-3}^{-1}. \quad (9.1)$$

We take the order of the fundamental monomials  $Y_{1,-1}, Y_{1,-3}, Y_{1,-5}$ . Then the Newton polytope  $\mathbf{N}_{sl_2,2}^{(0)}$  is given by the following hyperplanes and half-spaces:

$$\begin{aligned} (1, -1, 1) \cdot x + 0 &= 0, & (0, 1, -1) \cdot x + 0 &\geq 0, & (0, 0, 1) \cdot x + 0 &\geq 0, \\ (0, -1, 1) \cdot x + 1 &\geq 0, & (0, 0, -1) \cdot x + 1 &\geq 0. \end{aligned} \quad (9.2)$$

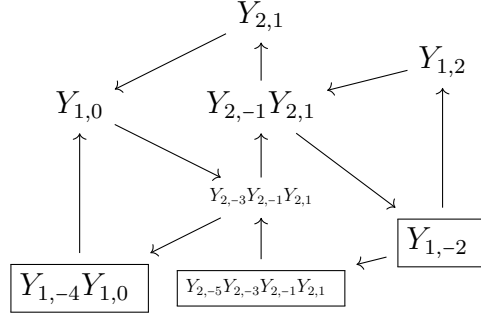
We see that the polytope  $\mathbf{N}_{sl_2,2}^{(0)}$  is a square. The remaining two prime modules (not frozen) in  $\mathcal{C}_\ell$  are  $L(Y_{1,-1}Y_{1,-3})$  and  $L(Y_{1,-3}Y_{1,-5})$ . Their truncated  $q$ -characters are

$$\tilde{\chi}_q(L(Y_{1,-1}Y_{1,-3})) = Y_{1,-1}Y_{1,-3}, \quad \tilde{\chi}_q(L(Y_{1,-3}Y_{1,-5})) = Y_{1,-3}Y_{1,-5} + \frac{Y_{1,-5}}{Y_{1,-1}} + \frac{1}{Y_{1,-1}Y_{1,-3}}. \quad (9.3)$$

The truncated  $q$ -character of  $L(Y_{1,-1}Y_{1,-3})$  is a monomial. Multiply a monomial to the defining polynomial of  $\mathbf{N}_{sl_2,2}^{(0)}$  does not increase the number of facets. Therefore we expect that the module which is not a fundamental module corresponding to one inequality in (9.2) is  $L(Y_{1,-3}Y_{1,-5})$ . The Newton polytope  $\mathbf{N}_{sl_2,2}^{(1)}$  is defined using  $L(Y_{1,-1})$ ,  $L(Y_{1,-3})$ ,  $L(Y_{1,-5})$ ,  $L(Y_{1,-3}Y_{1,-5})$ . The polytope  $\mathbf{N}_{sl_2,2}^{(1)}$  has 5 facets. The Newton polytope  $\mathbf{N}_{sl_2,2}^{(d)}$  also has 5 facets for any  $d \geq 1$ .

**9.3. Example:  $\mathfrak{g}$  is of type  $A_2$ ,  $\ell = 2$ .** In the case of type  $A_2$ ,  $\ell = 2$ , we choose the height function  $\xi(1) = -1$ ,  $\xi(2) = 0$ . There are 18 prime modules (including frozen) in the category  $\mathcal{C}_\ell$ . We have the following truncated  $q$ -characters of prime modules (not including frozen) in  $\mathcal{C}_\ell$ :

$$\begin{aligned} \tilde{\chi}_q(L(Y_{1,-1})) &= Y_{1,-1}, & \tilde{\chi}_q(L(Y_{1,-3})) &= \frac{1}{Y_{2,0}} + \frac{Y_{2,-2}}{Y_{1,-1}} + Y_{1,-3}, & \tilde{\chi}_q(Y_{2,0}) &= Y_{2,0}, \\ \tilde{\chi}_q(L(Y_{1,-5})) &= \frac{1}{Y_{2,-2}} + \frac{Y_{2,-4}}{Y_{1,-3}} + Y_{1,-5}, & \tilde{\chi}_q(Y_{2,-2}) &= \frac{Y_{1,-1}}{Y_{2,0}} + Y_{2,-2}, & \tilde{\chi}_q(L(Y_{2,-2}Y_{2,0})) &= Y_{2,0}Y_{2,-2}, \\ \tilde{\chi}_q(Y_{2,-4}) &= \frac{1}{Y_{1,-1}} + \frac{Y_{1,-3}}{Y_{2,-2}} + Y_{2,-4}, & \tilde{\chi}_q(L(Y_{1,-1}Y_{1,-3})) &= Y_{1,-1}Y_{1,-3}, \\ \tilde{\chi}_q(L(Y_{1,-3}Y_{1,-5})) &= \frac{Y_{1,-5}}{Y_{2,0}} + \frac{1}{Y_{2,0}Y_{2,-2}} + Y_{1,-3}Y_{1,-5} + \frac{Y_{2,-4}}{Y_{2,0}Y_{1,-3}} + \frac{Y_{2,-2}Y_{1,-5}}{Y_{1,-1}} + \frac{Y_{2,-2}Y_{2,-4}}{Y_{1,-1}Y_{1,-3}}, \\ \tilde{\chi}_q(L(Y_{2,-2}Y_{2,-4})) &= Y_{2,-2}Y_{2,-4} + \frac{Y_{1,-1}Y_{2,-4}}{Y_{2,0}} + \frac{Y_{1,-1}Y_{1,-3}}{Y_{2,0}Y_{2,-2}}, & \tilde{\chi}_q(L(Y_{1,-3}Y_{2,0})) &= Y_{2,0}Y_{1,-3} + \frac{Y_{2,0}Y_{2,-2}}{Y_{1,-1}}, \\ \tilde{\chi}_q(L(Y_{2,-2}Y_{1,-5})) &= Y_{2,-2}Y_{1,-5} + \frac{Y_{1,-1}}{Y_{2,0}Y_{2,-2}} + \frac{Y_{1,-1}Y_{1,-5}}{Y_{2,0}} + \frac{Y_{2,-2}Y_{2,-4}}{Y_{1,-3}} + \frac{Y_{1,-1}Y_{2,-4}}{Y_{2,0}Y_{1,-3}}, \end{aligned}$$

FIGURE 3. An initial cluster for  $K_0(\mathcal{C}_\ell^{B_2})$ ,  $\ell = 1$ .

$$\begin{aligned}\tilde{\chi}_q(L(Y_{1,-1}Y_{2,-4})) &= Y_{1,-1}Y_{2,-4} + \frac{Y_{1,-1}Y_{1,-3}}{Y_{2,-2}}, \quad Y_{2,0}Y_{2,-2}Y_{1,-5} + \frac{Y_{2,0}Y_{2,-2}Y_{2,-4}}{Y_{1,-3}}, \\ \tilde{\chi}_q(L(Y_{2,0}Y_{1,-3}Y_{1,-5})) &= Y_{2,0}Y_{1,-3}Y_{1,-5} + \frac{Y_{2,0}Y_{2,-2}Y_{1,-5}}{Y_{1,-1}} + \frac{Y_{2,0}Y_{2,-2}Y_{2,-4}}{Y_{1,-1}Y_{1,-3}},\end{aligned}$$

$$\tilde{\chi}_q(L(Y_{2,0}Y_{1,-3}Y_{2,-2}Y_{1,-5})) = Y_{2,-2}Y_{1,-5} + \frac{Y_{2,-2}Y_{2,-4}}{Y_{1,-3}} + Y_{2,0}Y_{1,-3}Y_{2,-2}Y_{1,-5} + \frac{Y_{2,0}Y_{2,-2}^2Y_{1,-5}}{Y_{1,-1}} + \frac{Y_{2,0}Y_{2,-2}^2Y_{2,-4}}{Y_{1,-1}Y_{1,-3}}.$$

We take the order of the fundamental monomials as  $Y_{1,-1}, Y_{1,-3}, Y_{1,-5}, Y_{2,0}, Y_{2,-2}, Y_{2,-4}$ . Then the Newton polytope  $\mathbf{N}_{sl_3,2}^{(d)}$  ( $d \geq 4$ ) is given by the following hyperplanes and half-spaces:

$$\begin{aligned}(0, 1, -1, -1, 1, 0) \cdot x - 9 &= 0, \quad (1, 0, -1, 0, 1, -1) \cdot x - 8 = 0, \quad (0, 0, -1, 0, 1, -1) \cdot x + 1 \geq 0, \\ (0, 0, 0, 0, 0, 1) \cdot x + 0 &\geq 0, \quad (0, 0, -1, -1, 1, -1) \cdot x + 6 \geq 0, \quad (0, 0, -1, 0, 0, 0) \cdot x + 6 \geq 0, \\ (0, 0, 1, 0, 0, 1) \cdot x - 3 &\geq 0, \quad (0, 0, -1, 0, 0, -1) \cdot x + 9 \geq 0, \quad (0, 0, 1, -1, 0, 1) \cdot x + 2 \geq 0, \\ (0, 0, 1, 1, -1, 1) \cdot x + 0 &\geq 0, \quad (0, 0, 0, 1, 0, 0) \cdot x - 2 \geq 0, \quad (0, 0, 0, 0, 1, -1) \cdot x + 0 \geq 0, \\ (0, 0, 0, 1, -1, 0) \cdot x + 8 &\geq 0, \quad (0, 0, 0, -1, 0, 0) \cdot x + 8 \geq 0, \quad (0, 0, 1, 0, 0, 0) \cdot x + 0 \geq 0, \\ (0, 0, 0, -1, 0, 1) \cdot x + 7 &\geq 0, \quad (0, 0, 0, -1, 1, 0) \cdot x + 2 \geq 0, \quad (0, 0, 0, -1, 1, -1) \cdot x + 3 \geq 0.\end{aligned}$$

On the other hand, there are 18 prime modules (including frozens) in  $\mathcal{C}_2^{sl_3}$ . Therefore there is a one to one correspondence between facets together with hyperplanes defining the ambient space of  $\mathbf{N}_{k,n}^{(d)}$  ( $d \geq 4$ ) and prime modules (including frozens) in  $\mathcal{C}_\ell$ .

**9.4. Example:  $\mathfrak{g}$  is of type  $B_n$  and  $\ell = 1$ .** Consider the case of type  $B_2$  and  $\ell = 1$ . We choose a height function as shown in Figure 3, see [62, 64] for the definition of the cluster algebra associated to  $\mathcal{C}_\ell$ .

There are 25 prime modules (not including the 3 frozen modules) in  $\mathcal{C}_1^{B_2}$  are

$$\begin{aligned} &L(Y_{1,2}), L(Y_{1,0}), L(Y_{1,-2}), L(Y_{1,-4}), L(Y_{2,1}), L(Y_{2,-1}), L(Y_{2,-3}), L(Y_{2,-5}), \\ &L(Y_{1,-4}Y_{2,1}), L(Y_{1,-4}Y_{1,2}), L(Y_{1,0}Y_{2,-5}), L(Y_{1,2}Y_{2,-3}), L(Y_{2,-5}Y_{2,1}), L(Y_{2,-5}Y_{2,-3}), \\ &L(Y_{2,-3}Y_{2,-1}), L(Y_{2,-1}Y_{2,1}), L(Y_{1,-4}Y_{2,-1}Y_{2,1}), L(Y_{1,0}Y_{2,-5}Y_{2,-3}), L(Y_{1,2}Y_{2,-5}Y_{2,-3}), \\ &L(Y_{1,2}Y_{2,-3}Y_{2,-1}), L(Y_{2,-5}Y_{2,-3}Y_{2,-1}), L(Y_{2,-3}Y_{2,-1}Y_{2,1}), L(Y_{1,0}Y_{1,2}Y_{2,-5}Y_{2,-3}), \\ &L(Y_{1,2}Y_{2,-5}Y_{2,-3}Y_{2,-1}), L(Y_{1,2}Y_{2,-5}Y_{2,-3}Y_{2,-3}Y_{2,-1}). \end{aligned}$$

Computing the Newton polytope using the  $q$ -characters of prime modules in  $\mathcal{C}_\ell$ , we obtain a polytope which has 25 facets. This polytope is  $\mathbf{N}_{B_n,1}^{(d)}$  ( $d \geq 5$ ) and it is the type  $D_5$  associahedron.

We expect that for every  $n \in \mathbb{Z}_{\geq 2}$ ,  $\mathbf{N}_{B_n,1}^{(d)}$  ( $d$  is large enough) is the type  $D_{2n+1}$  associahedron.

**9.5. Tropical Fans for Quantum Affine Algebras.** Let  $\mathcal{M}^{(0)} = \mathcal{M}$  be the set of all equivalence classes of fundamental modules of  $U_q(\widehat{\mathfrak{g}})$  in  $\mathcal{C}_\ell$ . By tropicalizing all  $\chi_q(L(M))$ ,  $L(M) \in \mathcal{M}$ , we obtain piecewise linear functions in the space of dimension  $r$  parametrized by  $y_{i,j}$  ( $y_{i,j}$  is the tropical version of  $Y_{i,j}$ ), where  $r$  is the number of fundamental modules in  $\mathcal{C}_\ell$ . Such a function is linear on a collection of cones; these cones assemble to define a polyhedral fan. The common refinement of these fans is the normal fan  $\mathcal{N}(\mathbf{N}_{\mathfrak{g},\ell}^{(0)})$  of the Newton polytope  $\mathbf{N}_{\mathfrak{g},\ell}^{(0)}$ .

For  $d \geq 1$ , let  $\mathcal{M}^{(d)}$  be the set of all equivalence classes of simple modules corresponding to rays of  $\mathcal{N}(\mathbf{N}_{\mathfrak{g},\ell}^{(d-1)})$ . By tropicalizing all  $\chi_q(L(M))$ ,  $L(M) \in \mathcal{M}^{(d)}$ , and using the same procedure as above, we obtain the normal fan  $\mathcal{N}(\mathbf{N}_{\mathfrak{g},\ell}^{(d)})$  of the Newton polytope  $\mathbf{N}_{\mathfrak{g},\ell}^{(d)}$  defined in Section 9.1.

## 10. PHYSICAL MOTIVATION: STRINGY INTEGRALS AND CEGM SCATTERING AMPLITUDES

In this section, we propose a formula which extends the main construction in the work of Arkani-Hamed, He, Lam [4] on so-called Grassmannian string integrals, and Cachazo, Early, Guevara, Mizera (CEGM) [21] on generalized biadjoint scalar amplitudes. Grassmannian string integrals and generalized biadjoint scalar amplitudes are related by taking a certain  $\alpha' \rightarrow 0$  limit of the stringy integral.

**10.1. Stringy Integrals For Grassmannian Cluster Algebras.** From a physical point of view, the central objective of this subsection which we describe here is twofold. First, in this subsection we give an explicit formula for a completion of the stringy integral by making use of *all* of the elements in Lusztig's dual canonical basis of  $\mathbb{C}[\text{Gr}(k, n)]$ ; the rest of the subsection aims to provide a combinatorial framework for the evaluation of a

limit which is standard in physics, the so-called  $\alpha' \rightarrow 0$  limit of the Grassmannian string integral, which is known to be given ([4, Claim 1]) by the CEGM scattering equations formula.

Such calculations are still highly nontrivial, but the formula which we propose removes an enormous amount of redundancy by making use of character polynomials for only *prime* tableaux. It is known that any simple  $U_q(\widehat{\mathfrak{g}})$ -module decomposes as the tensor product of prime simple modules [28]. Therefore the  $q$ -character<sup>4</sup> of any simple  $U_q(\widehat{\mathfrak{g}})$ -module is the product of the  $q$ -characters of its prime factors [48].

Moreover, our formula is essentially nonrecursive using  $\text{ch}_T$  in Theorem 5.8 in [23], and it is more general than possible constructions coming from cluster algebras which use only cluster variables.

Arkani-Hamed, He, and Lam introduced Grassmannian string integrals in [4, Equation (6.11)]:

$$\mathbf{I}_{k,n} = (\alpha')^a \int_{(\mathbb{R}_{>0}^{n-k-1})^{\times(k-1)}} \left( \prod_{(i,j)} \frac{dx_{i,j}}{x_{i,j}} \right) \left( \prod_J p_J^{-\alpha' c_J}(x_{i,j}) \right), \quad (10.1)$$

where  $a = (k-1)(n-k-1)$ ,  $\alpha'$ ,  $c_J$  are some parameters,  $p_J$ 's are Plücker coordinates, the product runs over all  $k$ -element subsets of  $[n]$ .

We emphasize that the original formulations (10.1) in [21] and [4] involved only the finite collection of all Plücker coordinates.

We now define the completion of the Grassmannian string integral, using for the integrand all prime elements in the dual canonical basis of  $\mathbb{C}[\text{Gr}(k, n)]$ .

**Definition 10.1.** For  $2 \leq k \leq n-2$  and every  $d \geq 1$ , we define

$$\mathbf{I}_{k,n}^{(d)} = (\alpha')^a \int_{(\mathbb{R}_{>0}^{n-k-1})^{\times(k-1)}} \left( \prod_{(i,j)} \frac{dx_{i,j}}{x_{i,j}} \right) \left( \prod_T \text{ch}_T^{-\alpha' c_T}(x_{i,j}) \right). \quad (10.2)$$

where the second product is over all tableaux  $T$  such that the face  $\mathbf{F}_T$  corresponding to  $T$  (see Section 6.4) is a (codimension one) facet of  $\mathbf{N}_{k,n}^{(d-1)}$ . Here we abbreviate  $a = (k-1)(n-k-1)$ . Also  $c_T$ ,  $x_{i,j} > 0$  are positive (real) parameters, and  $\alpha'$  is a parameter known in physics as the string tension. The first product is over  $(i, j) \in [1, k-1] \times [1, n-k-1]$ , and we have chosen the normalization where  $x_{i, n-k} = 1$  for all  $i = 1, \dots, k-1$ .

In the integral (10.2) we have conditions under which the integral converges, namely that the parameters  $\alpha_{i,j}$  and  $c_T$  must be chosen such that the origin is in the interior of the Newton polytope, see [4, Claim 1] for details.

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<sup>4</sup>There is a connection between the  $q$ -character of a simple module  $L(M)$  and the polynomial  $\text{ch}_{T_M}$ , where  $T_M$  is the tableau corresponding to  $M$ , see Sections 3.1 and 3.2.

Denote by  $\text{PSSYT}_{k,n}^r \subset \text{SSYT}(k, [n])$  the set of prime tableaux in  $\text{SSYT}(k, [n])$  with  $r$  or less columns and by  $\text{PSSYT}_{k,n} \subset \text{SSYT}(k, [n])$  the set of all prime tableaux in  $\text{SSYT}(k, [n])$ .

It is natural<sup>5</sup> to introduce the  $d \rightarrow \infty$  limit of the Grassmannian string integral (10.2):

$$\mathbf{I}_{k,n}^{(\infty)} = (\alpha')^a \int_{(\mathbb{R}_{>0}^{n-k-1})^{\times(k-1)}} \left( \prod_{(i,j)} \frac{dx_{i,j}}{x_{i,j}} \right) \left( \prod_{T \in \text{PSSYT}_{k,n}} \text{ch}_T^{-\alpha' c_T}(x_{i,j}) \right). \quad (10.3)$$

For finite type cluster algebras, our integrand is finite. However, starting at  $(k, n) = (3, 9)$  the integrand involves an infinite product.

We also introduce another version of the Grassmannian string integral (10.2) using all prime tableaux up to certain columns.

**Definition 10.2.** For  $2 \leq k \leq n - 2$  and  $r \geq 1$ , we define

$$\mathbf{I}_{k,n}^{(r)} = (\alpha')^a \int_{(\mathbb{R}_{>0}^{n-k-1})^{\times(k-1)}} \left( \prod_{(i,j)} \frac{dx_{i,j}}{x_{i,j}} \right) \left( \prod_{T \in \text{PSSYT}_{k,n}^r} \text{ch}_T^{-\alpha' c_T}(x_{i,j}) \right), \quad (10.4)$$

where  $a = (k-1)(n-k-1)$ , and  $\alpha'$ ,  $c_T$  are certain parameters defined in the same way as Definition 10.1.

Note that in the limit  $r \rightarrow \infty$  the integrands for (10.4) and (10.3) coincide. In this way we have a combinatorial construction which relates prime tableaux to stringy integrals, and a geometric interpretation of the set of all prime tableaux in terms of a polytope.

The polynomials  $\text{ch}_T$  that appear in the integrands of (10.2) and (10.4) are in bijection with prime tableaux and can be calculated using (3.2).

An important problem which may help with the evaluation will be investigated in Section 10.2: to rewrite Equation (10.3) in terms of rational functions which are invariant under the torus action, that is the so-called  $u$ -variables [5], and then to calculate the binary relations among them. See also [43] for another physical application of binary relations in the context of CEGM scattering amplitudes.

By [4, Claim 1], the leading order term in the series expansion around  $\alpha' = 0$  has a beautiful interpretation as the volume of a polytope, where the simple poles correspond to facets. The polytope is dual to the Newton polytope  $\mathbf{N}_{k,n}^{(1)}$ . This leading order contribution was formulated originally in [21] by Cachazo, Early, Guevara and Mizera (CEGM) using the scattering equations formalism.

**Remark 10.3.** The stringy integral in Equation (10.3) converges if and only if the origin is in the interior of the Newton polytope, see [4, Claim 1]; however, the  $\alpha' \rightarrow 0$  limit is calculated by the CEGM scattering equations formula [21], which has no such convergence limitation.

<sup>5</sup>See talks by Arkani-Hamed, Frost, Plamondon, Salvatori, and Thomas in [2, 3].

It turns out that the limit  $\alpha' \rightarrow 0$  of the Grassmannian string integral (10.1) coincides with the CEGM scattering equations formula [4]. Let us sketch the CEGM formula, referring to [21] for details.

First we define a scattering potential function

$$\mathcal{S}_{k,n}^{(d=1)} = \sum_J \log(p_J) \mathfrak{s}_J,$$

where  $p_J$  is the maximal  $k \times k$  minor with column set  $J = \{j_1, \dots, j_k\}$ , and the *Mandelstam variables*  $\mathfrak{s}_J$  are coordinate functions on the *kinematic space*

$$\mathcal{K}(k, n) = \left\{ (\mathfrak{s}_J) \in \mathbb{R}^{\binom{n}{k}} : \sum_{J: J \ni i} \mathfrak{s}_J = 0, i = 1, \dots, n \right\}.$$

Then [21] defined the (planar) generalized biadjoint scalar amplitude

$$m_n^{(k)} = \sum_{c \in \text{crit}(\mathcal{S}_{k,n}^{(1)})} \frac{1}{\det' \Phi} \left( \prod_{j=1}^n \frac{1}{p_{j,j+1,\dots,j+k-1}(c)} \right)^2,$$

where the sum is over all critical points  $c$  of  $\mathcal{S}_{k,n}^{(1)}$ , and where  $\det' \Phi$  is the so-called reduced Hessian determinant (see [21, Equation 2.4] for details). For example,

$$m_4^{(2)} = \frac{1}{\mathfrak{s}_{12}} + \frac{1}{\mathfrak{s}_{23}}$$

and

$$m_5^{(2)} = \frac{1}{\mathfrak{s}_{12}\mathfrak{s}_{34}} + \frac{1}{\mathfrak{s}_{23}\mathfrak{s}_{45}} + \frac{1}{\mathfrak{s}_{34}\mathfrak{s}_{15}} + \frac{1}{\mathfrak{s}_{12}\mathfrak{s}_{45}} + \frac{1}{\mathfrak{s}_{23}\mathfrak{s}_{15}}.$$

In general, Cachazo-He-Yuan [25] introduced a compact formula for biadjoint scalar amplitudes (as well as amplitudes for many other Quantum Field Theories).

The  $k \geq 3$  analog was discovered by Cachazo-Early-Guevara-Mizera (CEGM); they have been the subject of intensive study since their introduction [21].

A second expression for the leading order in the expansion around  $\alpha' \rightarrow 0$  is a compact expression involving piecewise-linear functions,

$$\lim_{\alpha' \rightarrow 0} \mathbf{I}_{k,n}^{(\infty)} = \int_{\mathbb{T}^{k-1, n-k}} \exp \left( - \sum_{T \in \text{PSSYT}_{k,n}} \mathfrak{s}_T \text{ch}_T^{\text{Trop}}(y_{i,j}) \right) dy_{i,j},$$

where  $\text{ch}_T^{\text{Trop}}(y_{i,j})$  is the usual tropicalization of the character polynomial  $\text{ch}_T(x_{i,j})$ , and where  $\mathbb{T}^{k-1, n-k} = (\mathbb{T}^{n-k})^{\times(k-1)}$  and  $\mathbb{T}^{n-k} = \mathbb{R}^{n-k} / \mathbb{R}(1, \dots, 1)$ .

The fundamental tropical integral of this form was defined first in [22], called there the *global Schwinger parametrization* of Feynman diagrams. In this work we propose a generalization of the integrand which includes all prime elements in Lusztig's dual canonical basis, as parameterized by prime tableaux.

Clearly there are many questions about this integral which we leave to future work. One of these is highly nontrivial:

- To evaluate the tropical limit, one has to either compute an infinite Minkowski sum, or else find a way to evaluate the CEGM formula for a scattering potential that involves an infinite summation indexed by prime tableaux.

**10.2.  $u$ -equations and  $u$ -variables.** In what follows, building on [4, Section 6.2], we propose a system of so-called  $u$ -variables for the (infinite) Grassmannian string integral  $\mathbf{I}_{k,n}^{(\infty)}$  defined in Equation (10.3). The motivation is to make the integrand manifestly compatible with the singularities of the function obtained by taking the  $\alpha' \rightarrow 0$  limit. To construct the infinite integrand is an important problem [6, Section 12.3]. The second step, to characterize the binary relations among the  $u$ -variables, will be considered in future work.

The new integrand is reorganized as a product of cross-ratios  $u_T$  on the Grassmannian  $\text{Gr}(k, n)$ , the so-called  $u$ -variables, one for each prime tableau  $T$ . The  $u$ -variables [5] have been defined for finite-type cluster algebras arising from  $\text{Gr}(2, n)$  [6] (and see the original work of Koba-Nielsen [73] in the physics literature), but for general Grassmannians  $\text{Gr}(k, n)$  the cluster algebras are of infinite type and new methods are required [4]. The main idea of our solution is to construct  $u$ -variables for Grassmannian string integrals as ratios of characters  $\text{ch}_T$  of prime tableaux  $T$  (equivalently, as ratios of  $q$ -characters of prime modules of the quantum affine algebra  $U_q(\widehat{\mathfrak{sl}}_k)$ ). In this way, our proposal for an integrand with infinite product of  $u$ -variables which satisfy certain binary-type identities, as has been explored in the finite type case in [5].

We first formulate our proposal and then we label the  $u$ -variables and  $u$ -equations for  $\mathbf{I}_{3,6}^{(2)}$  (in our notation) in [4] to using prime tableaux.

**Definition 10.4.** For  $k \leq n$ , we define

$$\mathbf{I}_{k,n}^{(\infty)} = (\alpha')^a \int_{(\mathbb{R}_{>0}^{n-k-1})^{\times(k-1)}} \prod_{i,j} \frac{dx_{i,j}}{x_{i,j}} \prod_{T \in \text{PSSYT}_{k,n}} (u_T)^{\alpha' U_T}, \quad (10.5)$$

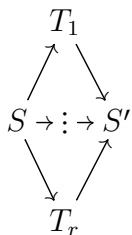
where  $\alpha'$ ,  $U_T$  are some parameters, and  $u_T$  is the  $u$ -variable corresponding to a prime tableau  $T$  which is defined in (10.6).

Jensen, King, and Su [67] introduced an additive categorification of Grassmannian cluster algebras using a category  $\text{CM}(B_{k,n})$  of Cohen-Macaulay  $B_{k,n}$ -modules, where  $B_{k,n}$  is a certain quotient of the complete path algebra of a certain quiver. According to their result, there is a one to one correspondence between cluster variables  $\mathbb{C}[\text{Gr}(k, n)]$  and reachable (meaning that the module can be obtained by mutations) rigid indecomposable modules in  $\text{CM}(B_{k,n})$ . On the other hand, cluster variables of  $\mathbb{C}[\text{Gr}(k, n)]$  are in one to one correspondence with reachable (meaning that the tableau can be obtained by

mutations) prime real tableaux in  $\text{SSYT}(k, [n])$ . Therefore there is a one to one correspondence between reachable rigid indecomposable modules in  $\text{CM}(B_{k,n})$  and reachable prime real tableaux in  $\text{SSYT}(k, [n])$ . In particular, in finite type cases, there is a one to one correspondence between indecomposable modules (in finite type, all indecomposable modules are rigid) in  $\text{CM}(B_{k,n})$  and prime tableaux in  $\text{SSYT}(k, [n])$  (in finite type, all prime tableau are real).

We conjecture that in general, there is a one to one correspondence between indecomposable modules in  $\text{CM}(B_{k,n})$  and prime tableaux in  $\text{SSYT}(k, [n])$ . Denote by  $M_T$  the indecomposable module in  $\text{CM}(B_{k,n})$  corresponding to a prime tableau  $T$ . We can label the Auslander-Reiten quiver by prime tableaux instead of indecomposable modules, see Figure 4.

**Definition 10.5.** For every mesh



in the Auslander-Reiten quiver of  $\text{CM}(B_{k,n})$ , we define the corresponding  $u$ -variable as

$$u_S = \frac{\prod_{i=1}^r \text{ch}_{T_i}}{\text{ch}_S \text{ch}_{S'}}. \quad (10.6)$$

Here we label the  $u$ -variables by semistandard Young tableaux rather than noncrossing tuples. The mesh can be degenerate. For example, in Figure 4,  $u_{126} = \frac{p_{136}}{p_{126}}$ .

General  $u$ -equations have been introduced in [3]. In [3],  $u$ -equations are defined in the setting of representations of quiver with relations and cluster categories of finite type. In this paper, we work in the setting of Grassmannian cluster categories. The  $u$ -equations are: for every prime tableau  $T$  in  $\text{SSYT}(k, [n])$ ,

$$u_T + \prod_{T' \in \text{PSSYT}_{k,n}} u_{T'}^{\dim \text{Hom}(M_T, \tau(M_{T'})) + \dim \text{Hom}(M_{T'}, \tau(M_T))} = 1, \quad (10.7)$$

where  $M_T$  is the indecomposable  $B_{k,n}$ -module corresponding to  $T$ , and  $\tau$  is the Auslander-Reiten translation [10, 67].

**Conjecture 10.6.** *The  $u$ -variables are unique solutions of the  $u$ -equations (10.7).*

We give an example to explain Conjecture 10.6.



**Example 10.7.** In the case of  $\mathbb{C}[\text{Gr}(3, 6)]$ , the  $u$ -equations are

$$\begin{aligned} u_{124} + u_{135}u_{136}u_{235}u_{236}u_{356}u_{135,246} &= 1, \\ u_{125} + u_{136}u_{346}u_{135,246}u_{246}u_{236}u_{146} &= 1, \\ u_{135} + u_{135,246}u_{246}^2u_{124,356}u_{245}u_{346}u_{236}u_{146}u_{256}u_{124} &= 1, \\ u_{124,356} + u_{135}u_{136}u_{145}u_{146}u_{235}u_{236}u_{245}u_{246}u_{135,246}^2 &= 1, \end{aligned}$$

and their cyclic shifts. Note that the cyclic shifts of the indices of all Plücker coordinates in  $\text{ch}_T$  corresponds to promotions of  $T$ , [90].

The solutions of the  $u$ -equations can be read from the Auslander-Reiten quiver. We have

$$\begin{aligned} u_{126} &= \frac{p_{136}}{p_{126}}, \quad u_{345} = \frac{p_{346}}{p_{345}}, \quad u_{125} = \frac{p_{126}p_{135}}{p_{125}p_{136}}, \quad u_{136} = \frac{\text{ch}_{135,246}}{p_{136}p_{245}}, \quad u_{245} = \frac{p_{345}p_{246}}{p_{245}p_{346}}, \\ u_{346} &= \frac{\text{ch}_{124,356}}{p_{346}p_{125}}, \quad u_{124,356} = \frac{p_{125}p_{134}p_{356}}{\text{ch}_{124,356}p_{135}}, \quad u_{134} = \frac{p_{135}p_{234}}{p_{134}p_{235}}, \quad u_{135} = \frac{p_{136}p_{145}p_{235}}{p_{135}\text{ch}_{135,246}}, \\ u_{235} &= \frac{\text{ch}_{135,246}}{p_{235}p_{146}}, \quad u_{135,246} = \frac{p_{146}p_{245}p_{236}}{\text{ch}_{135,246}p_{246}}, \quad u_{146} = \frac{p_{246}p_{156}}{p_{146}p_{256}}, \quad u_{246} = \frac{p_{346}p_{256}p_{124}}{p_{246}\text{ch}_{124,356}}, \\ u_{256} &= \frac{\text{ch}_{124,356}}{p_{256}p_{134}}, \quad u_{234} = \frac{p_{235}}{p_{234}}, \quad u_{156} = \frac{p_{256}}{p_{156}}, \quad u_{356} = \frac{p_{135}p_{456}}{p_{356}p_{145}}, \quad u_{145} = \frac{\text{ch}_{135,246}}{p_{145}p_{236}}, \\ u_{236} &= \frac{p_{246}p_{123}}{p_{236}p_{124}}, \quad u_{124} = \frac{\text{ch}_{124,356}}{p_{124}p_{356}}, \quad u_{456} = \frac{p_{145}}{p_{456}}, \quad u_{123} = \frac{p_{124}}{p_{123}}, \end{aligned}$$

where we use  $\text{ch}_{T_1, \dots, T_r}$  to denote  $\text{ch}_T$ , and  $T_i$ 's are columns of  $T$ . Here  $\text{ch}_{124,356} = p_{124}p_{356} - p_{123}p_{456}$ , and  $\text{ch}_{135,246} = p_{145}p_{236} - p_{123}p_{456}$ . We checked that the  $u$ -variables satisfy  $u$ -equations directly. The solution agrees with Section 9.3 of [4].

The same computations can be done for other finite type cases. The Auslander-Reiten quivers for Grassmannian cluster categories  $\text{CM}(B_{k,n})$  of finite type have been computed in [67] (the vertices are labelled by Cohen-Macaulay modules in  $\text{CM}(B_{k,n})$ ) and [38] (the vertices are labelled by tableaux).

When  $\mathbb{C}[\text{Gr}(k, n)]$  is of infinite type, the Auslander-Reiten quiver of the Grassmannian cluster category  $\text{CM}(B_{k,n})$  has infinitely many components. We will study Conjecture 10.6 about the  $u$ -equations and their solutions in the future.

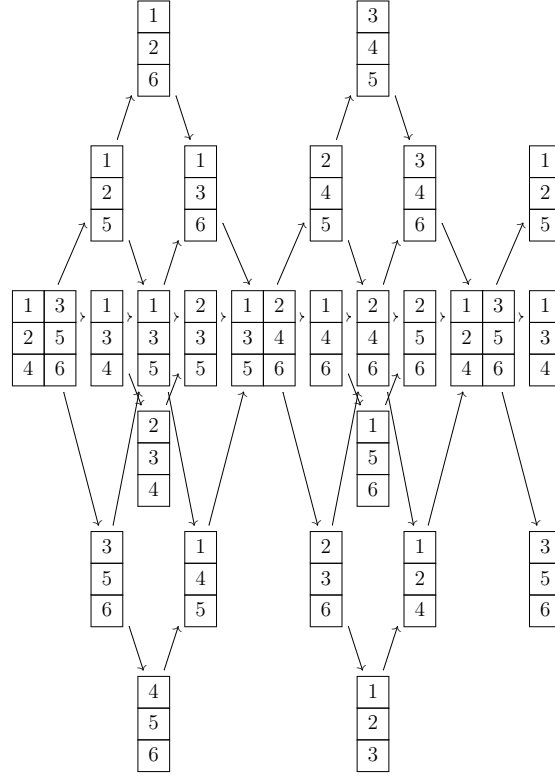


FIGURE 4. The Auslander-Reiten quiver for  $\text{CM}(B_{3,6})$  with vertices labelled by tableaux.

**10.3. Stringy Integrals For Quantum Affine Algebras.** We generalize the stringy integrals in Section 10.1 to the setting for any quantum affine algebra as follows.

Let  $\mathfrak{g}$  be a simple Lie algebra over  $\mathbb{C}$  and  $\ell \geq 0$ . Recall that  $\hat{I} = \{(i, s) : i \in I, s = \xi(i) - 2d_i r, r \in [0, \ell]\}$ , where  $\xi : I \rightarrow \mathbb{Z}$  is a chosen height function, see Section 2.

**Definition 10.8.** For every simple Lie algebra  $\mathfrak{g}$  over  $\mathbb{C}$ ,  $\ell \geq 1$ , and  $d \geq 1$ , we define

$$\mathbf{I}_{\mathfrak{g}, \ell}^{(d)} = (\alpha')^{|\hat{I}|} \int_{\mathbb{R}_{>0}^{|\hat{I}|}} \left( \prod_{(i,s) \in \hat{I}} \frac{dY_{i,s}}{Y_{i,s}} \right) \left( \prod_M \chi_q(L(M))^{-\alpha' c_M} \right), \quad (10.8)$$

where the product is over all dominant monomials  $M$  such that the modules  $L(M)$  correspond to facets of  $\mathbf{N}_{\mathfrak{g}, \ell}^{(d-1)}$ , and  $\alpha', c_M$  are some parameters.

We also define another version of stringy integrals for quantum affine algebras.

**Definition 10.9.** For every simple Lie algebra  $\mathfrak{g}$  over  $\mathbb{C}$ ,  $\ell \geq 1$ , and  $d \geq 1$ , we define

$$\mathbf{I}'_{\mathfrak{g},\ell}{}^{(d)} = (\alpha')^{|\hat{I}|} \int_{\mathbb{R}_{>0}^{|\hat{I}|}} \left( \prod_{(i,s) \in \hat{I}} \frac{dY_{i,s}}{Y_{i,s}} \right) \left( \prod_M \chi_q(L(M))^{-\alpha' c_M} \right), \quad (10.9)$$

where the product is over all dominant monomials  $M$  in  $\mathcal{P}_\ell^+$  of degree less or equal to  $d$ , and  $\alpha'$ ,  $c_M$  are some parameters.

We also define stringy integrals for quantum affine algebras in the case when  $d \rightarrow \infty$  as follows.

**Definition 10.10.** For every simple Lie algebra  $\mathfrak{g}$  over  $\mathbb{C}$  and  $\ell \geq 1$ , we define

$$\mathbf{I}_{\mathfrak{g},\ell}^{(\infty)} = (\alpha')^{|\hat{I}|} \int_{\mathbb{R}_{>0}^{|\hat{I}|}} \left( \prod_{(i,s) \in \hat{I}} \frac{dY_{i,s}}{Y_{i,s}} \right) \left( \prod_M \chi_q(L(M))^{-\alpha' c_M} \right), \quad (10.10)$$

where the product is over all dominant monomials  $M$  such that  $L(M)$ 's are prime modules in  $\mathcal{C}_\ell$ , and  $\alpha'$ ,  $c_M$  are some parameters.

We hope that the stringy integrals for quantum affine algebras will have applications to physics.

## 11. LIMIT $g$ -VECTORS, LIMIT FACETS, AND PRIME NON-REAL MODULES

In this section, we study prime non-real modules of quantum affine algebras using limit  $g$ -vectors.

**11.1. Limit  $g$ -vectors and Limit Facets.** By results in [64, Section 5.2.2], see also [39, Section 2.6], [23, Section 7], we have that for any simple  $U_q(\widehat{\mathfrak{g}})$ -module  $L(M)$ , its  $g$ -vector is obtained as follows. The dominant monomial  $M$  can be written as  $M = \prod_{i,s} Y_{i,s}^{a_{i,s}}$  for some non-negative integers  $a_{i,s}$ , where the product runs over all fundamental modules  $L(Y_{i,s})$  in  $\mathcal{C}_\ell$ . On the other hand,  $M$  can also be written as  $M = \prod_{M_i} M_i^{g_i}$  for some integers  $g_i$ , where the product runs over all initial cluster variables and frozen variables  $L(M_i)$  in  $\mathcal{C}_\ell$ . The  $g_i$ 's are the unique solution of  $\prod_{i,s} Y_{i,s}^{a_{i,s}} = \prod_{M_i} M_i^{g_i}$ . With a chosen order,  $g_i$ 's form the  $g$ -vector of  $L(M)$ . A simple module  $L(M)$  is determined by its  $g$ -vector uniquely.

In the case of Grassmannian cluster algebras, in Section 7 of [23], it is shown that given any tableau, one can recover its  $g$ -vector as follows. Any tableau  $T \in \text{SSYT}(k, [n])$  can be written uniquely as  $S_1^{e_1} \cup \dots \cup S_m^{e_m}$  for some integers  $e_1, \dots, e_m \in \mathbb{Z}$ , where  $S_1, \dots, S_m$  are the tableaux in the initial cluster (we choose an order of the initial cluster variables). The vector  $(e_1, \dots, e_m)$  is the  $g$ -vector of  $T$ . In Sections 6 and 9, we conjecture that facets of certain Newton polytopes correspond to prime  $U_q(\widehat{\mathfrak{g}})$ -modules (hence facets correspond to certain  $g$ -vectors).

**Remark 11.1.** In this paper, every element in the dual canonical basis of  $K_0(\mathcal{C}_\ell)$  and  $\mathbb{C}[\text{Gr}(k, n)]$  has a  $g$ -vector in the above sense even if it is not a cluster monomial.

It is observed in [34, 61] that some prime non-real elements in the dual canonical basis of  $\mathbb{C}[\text{Gr}(k, n)]$  which can be computed using limit  $g$ -vectors (these limit  $g$ -vectors are called limit rays). We generalize the concept of limit  $g$ -vectors to any cluster algebra in the following.

For a vector  $v = (v_1, \dots, v_m)$  in  $\mathbb{R}^m$ , denote its  $l^2$ -norm by  $\|v\| = \sqrt{\sum_{i=1}^m |v_i|^2}$ .

**Definition 11.2.** For a cluster algebra  $\mathcal{A}$  of infinite type of rank  $m$ , we say that a sequence of  $g$ -vectors  $g_1, g_2, \dots$  of  $\mathcal{A}$  has a limit  $g$  if the greatest common factor of entries of  $g$  is 1 and for every  $\epsilon > 0$ , there is a positive integer  $N$  such that for every  $j \geq N$ , there is some positive real number  $c_j$  such that  $\|c_j g - g_j\| < \epsilon$ .

**Definition 11.3.** Let  $\mathfrak{g}$  be a simple Lie algebra over  $\mathbb{C}$  and let  $\ell \in \mathbb{Z}_{\geq 1}$ ,  $d \in \mathbb{Z}_{\geq 0}$ . We say that a facet of a Newton polytope  $\mathbf{N}_{\mathfrak{g}, \ell}^{(d)}$  defined in Section 9 for a quantum affine algebra is a limit facet if the facet corresponds to a module whose  $g$ -vector is a limit  $g$ -vector of a sequence of  $g$ -vectors of modules obtained by a sequence of mutations of the corresponding cluster algebra.

We conjecture that every simple module corresponding to a limit  $g$ -vector is prime non-real.

**Conjecture 11.4.** *Let  $\mathfrak{g}$  be a simple Lie algebra over  $\mathbb{C}$  and let  $\ell \in \mathbb{Z}_{\geq 1}$ . If a simple module  $L(M)$  in  $\mathcal{C}_\ell$  corresponds to a limit  $g$ -vector of the cluster algebra corresponding to  $\mathcal{C}_\ell$ , then  $L(M)$  is prime non-real.*

We explain an example of limit  $g$ -vector in the case of  $\mathbb{C}[\text{Gr}(4, 8)]$ . We fix the order

$$\begin{aligned} & [1, 2, 3, 5], [1, 2, 4, 5], [1, 3, 4, 5], [1, 2, 3, 6], [1, 2, 5, 6], [1, 4, 5, 6], \\ & [1, 2, 3, 7], [1, 2, 6, 7], [1, 5, 6, 7], [1, 2, 3, 4], [2, 3, 4, 5], [3, 4, 5, 6], \\ & [4, 5, 6, 7], [5, 6, 7, 8], [1, 2, 3, 8], [1, 2, 7, 8], [1, 6, 7, 8] \end{aligned}$$

of the initial cluster, where each list corresponds to a Plücker coordinate. Mutate at the vertices in Figure 5 alternatively where the tableaux  $\begin{array}{|c|} \hline 4 \\ \hline 6 \\ \hline 7 \\ \hline 8 \\ \hline \end{array}$  and  $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 4 & 7 \\ \hline 6 & 8 \\ \hline \end{array}$  resides, we obtain the

following  $g$ -vectors

$$\begin{aligned} & (-1, 1, 0, 1, -1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0), \\ & (-2, 2, 0, 2, -1, -1, 0, -1, 1, 0, 0, 2, 0, 0, 0, 2, 0), \\ & (-3, 3, 0, 3, -1, -2, 0, -2, 2, 0, 0, 3, 0, 0, 0, 3, 0), \\ & (-4, 4, 0, 4, -1, -3, 0, -3, 3, 0, 0, 4, 0, 0, 0, 4, 0), \dots \end{aligned}$$

When the mutation step  $r$  is large enough, the  $g$ -vector we obtain is

$$(-r, r, 0, r, -1, -r + 1, 0, -r + 1, r - 1, 0, 0, r, 0, 0, 0, r, 0).$$

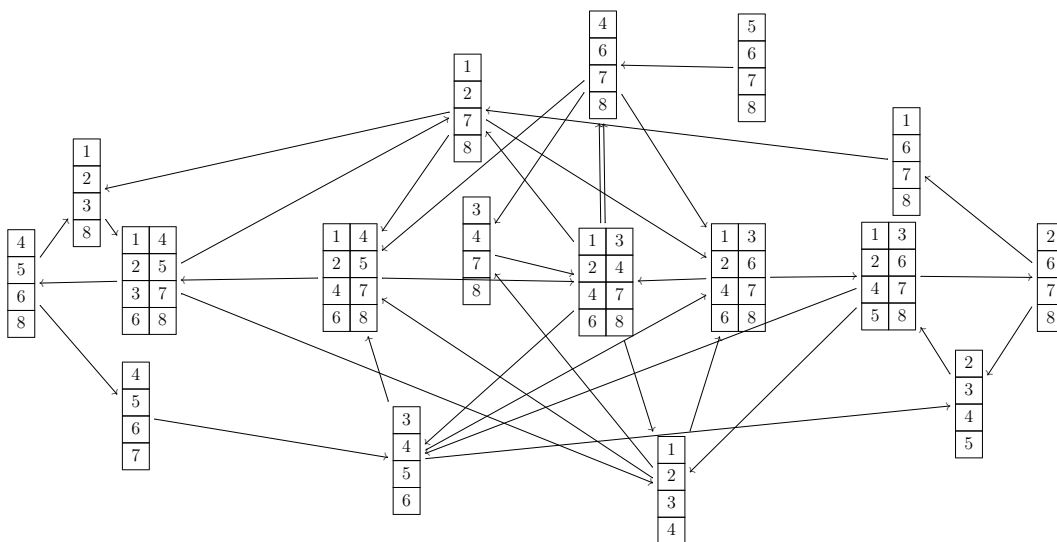


FIGURE 5. Mutate alternatively at the vertices of the double arrow, we obtain a limit  $g$ -vector  $(-1, 1, 0, 1, 0, -1, 0, -1, 1, 0, 0, 1, 0, 0, 0, 1, 0)$ . This  $g$ -vector corresponds to the prime non-real tableau  $[[1, 2, 4, 6], [3, 5, 7, 8]]$  with columns 1, 2, 4, 6 and 3, 5, 7, 8.

The limit  $g$ -vector of the sequence is

$$(-1, 1, 0, 1, 0, -1, 0, -1, 1, 0, 0, 1, 0, 0, 0, 1, 0).$$

This limit  $g$ -vector corresponds to the prime non-real tableau

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 7 \\ \hline 6 & 8 \\ \hline \end{array}$$

correspond to the non-real module  $L(Y_{2,0}Y_{1,-3}Y_{3,-3}Y_{2,-6})$ . Up to shift the second indices, this module is the non-real module found in Section 13.6 in [62].

**11.2. Limit  $g$ -vectors in  $\mathbb{C}[\text{Gr}(3, 9)]$  and  $\mathbb{C}[\text{Gr}(4, 8)]$ .** Recall that we say that a tableau in  $\text{SSYT}(k, [n])$  has rank  $r$  if the tableau has  $r$  columns and an element in  $\mathbb{C}[\text{Gr}(k, n)]$  has rank  $r$  if the tableau corresponding to it has  $r$  columns. We say that a  $g$ -vector has rank  $r$  is the tableaux corresponding to it has rank  $r$ .

We compute cluster variables and limit  $g$ -vectors for  $\mathbb{C}[\text{Gr}(k, n)]$  in terms of tableaux up to certain numbers of columns. That is, at each step of mutation, if we obtain some tableau with columns more than some number  $m$ , we mutate this vertex again. In this way, we can collect only tableaux (cluster variables) with columns less or equal to  $m$ .

We compute limit  $g$ -vectors for  $\mathbb{C}[\text{Gr}(3, 9)]$  and  $\mathbb{C}[\text{Gr}(4, 8)]$  up to rank 56 (the corresponding tableaux have at most 56 columns). The sequence of numbers of rank  $r$  ( $r \geq 1$ )

limit  $g$ -vectors for  $\mathbb{C}[\text{Gr}(3, 9)]$  (conjecturally, prime non-real tableaux in  $\text{SSYT}(3, [9])$ ) is

$$0, 0, 3, 0, 0, 3, 0, 0, 6, 0, 0, 6, 0, 0, 12, 0, 0, 6, 0, 0, 18, 0, 0, 12, 0, 0, 12, 0, 0, 12, \\ 0, 0, 18, 0, 0, 12, 0, 0, 30, 0, 0, 12, 0, 0, 36, 0, 0, 18, 0, 0, 24, 0, 0, 24, \dots$$

The sequence of numbers of rank  $r$  ( $r \geq 1$ ) limit  $g$ -vectors for  $\mathbb{C}[\text{Gr}(4, 8)]$  (conjecturally, prime non-real tableaux in  $\text{SSYT}(4, [8])$ ) is

$$0, 2, 0, 2, 0, 4, 0, 4, 0, 8, 0, 4, 0, 12, 0, 8, 0, 12, 0, 8, 0, 12, 0, 8, 0, \\ 20, 0, 8, 0, 24, 0, 12, 0, 16, 0, 16, 0, 32, 0, 12, 0, 36, 0, 16, 0, 24, 0, 20, 0, 44, \dots$$

Based on the computations, we have the following interesting conjecture. Denote by  $\phi(m)$  the Euler totient function which counts the numbers less or equal to  $m$  and prime to  $m$ .

**Conjecture 11.5.** *The number of rank  $r$  ( $r \geq 1$ ) prime non-real tableaux in  $\text{SSYT}(3, [9])$  is*

$$f_{3,9,r} = \begin{cases} 0, & r \pmod{3} = i, \quad i \in \{1, 2\}, \\ 3\phi(r/3), & r \pmod{3} = 0. \end{cases}$$

*The number of rank  $r$  ( $r \geq 1$ ) prime non-real tableaux in  $\text{SSYT}(4, [8])$  is*

$$f_{4,8,r} = \begin{cases} 0, & r \pmod{2} = 1, \\ 2\phi(r/2), & r \pmod{2} = 0. \end{cases}$$

**11.3. Limit  $g$ -vectors for  $\text{Gr}(4, 9)$ .** Using the algorithm in Theorem 1.1 in [18], we find that there are 18 two-column prime non-real tableaux and 252 three-column prime non-real tableaux in  $\text{SSYT}(4, [9])$ . We also checked that by computer that they all correspond to limit  $g$ -vectors.

The 18 prime non-real tableaux are obtained from the two prime non-real tableaux

1	3	1	2
2	5	3	4
4	7	5	6
6	8	7	8

by replacing  $1 < 2 < \dots < 8$  by  $a_1 < \dots < a_8 \in [9]$ .

Up to promotion, the 252 three-column prime non-real tableaux in  $\text{SSYT}(4, [9])$  are

$\begin{array}{ c c c } \hline 1 & 2 & 3 \\ \hline 2 & 5 & 6 \\ \hline 4 & 7 & 8 \\ \hline 7 & 8 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 1 & 2 \\ \hline 3 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \hline 8 & 9 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 1 & 3 \\ \hline 2 & 4 & 5 \\ \hline 3 & 6 & 7 \\ \hline 6 & 8 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 1 & 4 \\ \hline 2 & 3 & 6 \\ \hline 4 & 5 & 8 \\ \hline 7 & 8 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 2 & 3 \\ \hline 2 & 5 & 6 \\ \hline 4 & 7 & 8 \\ \hline 7 & 9 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 1 & 4 \\ \hline 2 & 3 & 6 \\ \hline 4 & 5 & 8 \\ \hline 7 & 9 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 2 & 2 \\ \hline 3 & 4 & 5 \\ \hline 6 & 6 & 7 \\ \hline 8 & 8 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 1 & 2 \\ \hline 3 & 4 & 5 \\ \hline 6 & 7 & 8 \\ \hline 8 & 9 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 1 & 3 \\ \hline 2 & 5 & 6 \\ \hline 4 & 7 & 8 \\ \hline 6 & 8 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 1 & 2 \\ \hline 3 & 4 & 5 \\ \hline 6 & 7 & 8 \\ \hline 8 & 9 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 1 & 3 \\ \hline 2 & 5 & 6 \\ \hline 4 & 7 & 8 \\ \hline 6 & 8 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 2 & 4 \\ \hline 3 & 4 & 6 \\ \hline 5 & 5 & 8 \\ \hline 7 & 9 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 2 & 2 \\ \hline 3 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \hline 8 & 9 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 1 & 3 \\ \hline 2 & 4 & 5 \\ \hline 3 & 7 & 8 \\ \hline 6 & 9 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 2 & 4 \\ \hline 2 & 3 & 6 \\ \hline 4 & 5 & 8 \\ \hline 7 & 8 & 9 \\ \hline \end{array}$
$\begin{array}{ c c c } \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 6 & 7 & 8 \\ \hline 8 & 9 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 1 & 2 \\ \hline 3 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \hline 7 & 8 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 2 & 2 \\ \hline 3 & 4 & 5 \\ \hline 6 & 7 & 8 \\ \hline 8 & 9 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 2 & 2 \\ \hline 3 & 4 & 5 \\ \hline 5 & 6 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 2 & 3 \\ \hline 2 & 4 & 5 \\ \hline 3 & 7 & 8 \\ \hline 6 & 9 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 2 & 4 \\ \hline 3 & 3 & 5 \\ \hline 4 & 5 & 8 \\ \hline 6 & 7 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 3 & 3 \\ \hline 2 & 5 & 6 \\ \hline 4 & 7 & 8 \\ \hline 6 & 9 & 9 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 1 & 3 \\ \hline 2 & 3 & 6 \\ \hline 4 & 5 & 8 \\ \hline 7 & 8 & 9 \\ \hline \end{array}$							

On the other hand, it is stated in [61, Section 5.3] that there are  $g$ -vectors (they call them rays) that are neither cluster variables nor limit  $g$ -vectors (limit  $g$ -vectors are called limit rays in [61]). This means that there are prime non-real modules which cannot be obtained by computing limit  $g$ -vectors.

## 12. LIMIT $g$ -VECTORS FOR TYPE $D_n$ QUANTUM AFFINE ALGEBRAS

In a recent paper [17], Brito and Chari found for the first time some examples of non-real  $U_q(\widehat{\mathfrak{g}})$ -modules when  $\mathfrak{g}$  is of type  $D_4$  which do not arise from a type  $A_n$  module. One non-real module they found is  $L(Y_{1,-11}Y_{1,-9}Y_{2,-6}Y_{1,-3}Y_{1,-1})$  (up to shifting the indices  $s$  in  $Y_{i,s}$ ). In Section 11, we see that limit  $g$ -vectors can be used to produce non-real modules. In this section, we construct one explicit example of non-real module of type  $D_4$  using a limit  $g$ -vector.

**12.1. The type  $D_4$  module  $L(Y_{2,-4}Y_{2,0})$  corresponds to a limit  $g$ -vector.** Let  $\mathfrak{g}$  be of type  $D_4$  and  $\ell = 2$ . An initial cluster for the cluster algebra  $K_0(\mathcal{C}_2^{D_4})$  is shown in Figure 6.

The module  $L(Y_{2,-4}Y_{2,0})$  in type  $D_4$  corresponds to a limit  $g$ -vector. This limit  $g$ -vector is obtained as follows. We label the vertices of the initial quiver as shown in Figure 6. After mutations at the vertices 1, 6, 8, we obtain a quiver with a double arrow from the vertex 3 to the vertex 4, see the quiver on the right hand side in Figure 6.

We take the order of initial cluster variables as:

$$L(Y_{1,-1}), L(Y_{1,-3}Y_{1,-1}), L(Y_{2,0}), L(Y_{2,-2}Y_{2,0}), L(Y_{3,1}), L(Y_{3,-1}Y_{3,1}), L(Y_{4,1}), \\ L(Y_{4,-1}Y_{4,1}), L(Y_{1,-5}Y_{1,-3}Y_{1,-1}), L(Y_{2,-4}Y_{2,-2}Y_{2,0}), L(Y_{3,-3}Y_{3,-1}Y_{3,1}), L(Y_{4,-3}Y_{4,-1}Y_{4,1}).$$

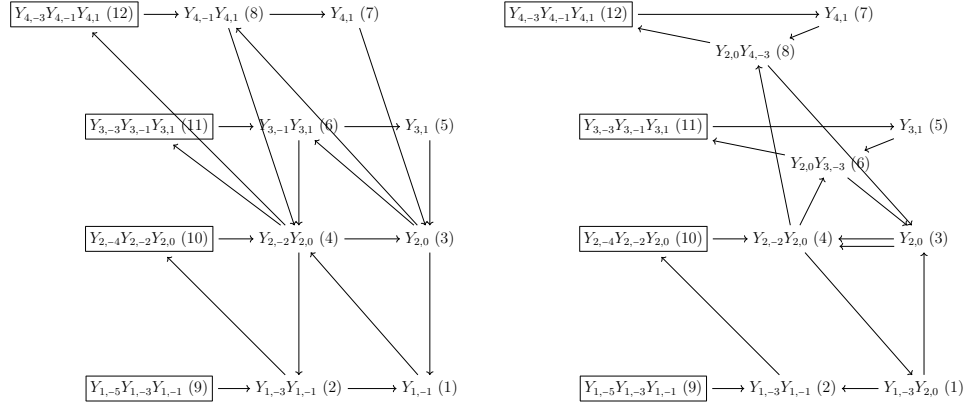


FIGURE 6. The left hand side is an initial cluster of  $K_0(\mathcal{C}_2^{D_4})$ . The numbers in the brackets are labels of the vertices. The right hand side is the quiver obtained from the initial quiver by mutating at the vertices 1, 6, 8. The arrow from vertex 3 to 4 is a double arrow.

Now we mutate the vertices 3, 4 alternatively and obtain the  $g$ -vectors:

$$\begin{aligned} & (0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0), & (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0), \\ & (-1, 1, 2, 0, 0, -1, 0, -1, 0, 0, 1, 1), & (-1, 1, 3, -1, 0, -1, 0, -1, 0, 1, 1, 1), \\ & (-1, 1, 4, -2, 0, -1, 0, -1, 0, 2, 1, 1), & (-1, 1, r+2, -r, 0, -1, 0, -1, 0, r, 1, 1), \quad r \geq 3. \end{aligned}$$

The limit  $g$ -vector for this mutation sequence is  $(0, 0, 1, -1, 0, 0, 0, 0, 1, 0, 0)$ . This is the  $g$ -vector of the module  $L(Y_{2,-4}Y_{2,0})$ .

We will verify that the module  $L(Y_{2,-4}Y_{2,0})$  in type  $D_4$  is prime non-real by using  $(q, t)$ -characters [84, 85] (see also [63, 14]) below. Note that the module  $L(Y_{2,-4}Y_{2,0})$  in type  $A_n$  ( $n \geq 2$ ) is real (it is a snake module [37]).

We first recall the results of [84, 85] about  $(q, t)$ -characters.

**12.2.  $(q, t)$ -characters.** Let  $C(z)$  be the quantum Cartan matrix of  $\mathfrak{g}$  [48] and let  $\tilde{C}(z) = (\tilde{C}_{ij}(z))$  be the inverse of  $C(z)$ , see Section 2. The entries of  $(\tilde{C}_{ij}(z))$  have power series expressions in  $z$  of the form [63]  $\tilde{C}_{ij}(z) = \sum_{m \geq 1} \tilde{C}_{ij}(m)z^m$ . Nakajima [84, 85] introduced  $(q, t)$ -characters of  $U_q(\widehat{\mathfrak{g}})$ -modules which are  $t$ -deformations of  $q$ -characters. Let  $K_t(\mathcal{C}_\ell)$  be the  $t$ -deformation of the Grothendieck ring  $K_0(\mathcal{C}_\ell)$  [63, 14]. Let  $\hat{I}$  be the set of vertices of the initial quiver of the cluster algebra  $K_0(\mathcal{C}_\ell)$ . Denote by  $\mathbf{Y}_t$  the  $\mathbb{Z}[t^{\pm 1}]$ -algebra generated by  $Y_{i,p}^{\pm 1}$ ,  $(i, p) \in \hat{I}$ , subject to the relations ([84, 85], see also [14, 63]):

$$Y_{i,p} * Y_{j,s} = t^{N(i,p;j,s)} Y_{j,s} * Y_{i,p},$$



where we use Nakajima's convention [84, 85] (in type ADE,  $d_i = 1$ ):

$$N(i, p; j, s) = 2(\tilde{C}_{ij}(s - p - d_i) - \tilde{C}_{ij}(p - s - d_i)).$$

For any family  $\{u_{i,p} \in \mathbb{Z} : (i, p) \in \hat{I}\}$ , denote

$$\prod_{(i,p) \in \hat{I}} Y_{i,p}^{u_{i,p}} = t^{-\frac{1}{2} \sum_{(i,p) < (j,s)} u_{i,p} u_{j,s} N(i,p;j,s)} \overrightarrow{\star}_{(i,p) \in \hat{I}} Y_{i,p}^{u_{i,p}}.$$

The expression on the right hand side of the above equation does not depend on the order of  $Y_{i,p}$ 's and so  $\prod_{(i,p) \in \hat{I}} Y_{i,p}^{u_{i,p}}$  is well-defined. The monomial  $\prod_{(i,p) \in \hat{I}} Y_{i,p}^{u_{i,p}}$  is called a commutative monomial [84, 63]. For a dominant monomial  $m = \prod_{(i,p) \in \hat{I}} Y_{i,p}^{u_{i,p}(m)}$ , ([84, 85], see also [63, 14]) the standard module  $M(m)$  is the tensor product of the fundamental modules corresponding to each of the factors in  $m$  in a particular order. In this paper, we choose the order as  $Y_{i,s} < Y_{j,t}$  if and only if  $s < t$ . The truncated  $(q, t)$ -character of  $M(m)$  is given by

$$[M(m)]_t = t^{\alpha(m)} \overrightarrow{\star}_{p \in \mathbb{Z}} \prod_{i \in I} \tilde{\chi}_{q,t}(L(Y_{i,p}))^{*u_{i,p}(m)},$$

where  $\alpha(m)$  is the integer such that  $m$  occurs with multiplicity one in the expansion of  $[M(m)]_t$  on the basis of the commutative monomials of  $\mathbf{Y}_t$  and the product  $\overrightarrow{\star}_{p \in \mathbb{Z}}$  is taken as increasing order. Since  $\tilde{\chi}_{q,t}(L(Y_{i,p}))$  and  $\tilde{\chi}_{q,t}(L(Y_{i,p'}))$  commute for any  $p, p'$ , the above expression is well-defined.

In [84, 85], a  $\mathbb{Z}$ -algebra anti-automorphism of  $\mathbf{Y}_t$  called bar-involution is defined by:  $t \mapsto t^{-1}$ ,  $Y_{i,p} \mapsto Y_{i,p}$ ,  $(i, p) \in \hat{I}$ .

For a simple module  $L(m)$ , denote by  $[L(m)]_t$  its  $(q, t)$ -character. The following theorem by Nakajima [84, 85] gives an algorithm to compute (truncated)  $(q, t)$ -characters of a simple  $U_q(\hat{\mathfrak{g}})$ -module: for every dominant monomial  $m \in \mathcal{P}_\ell^+$ , there is a unique element  $[L(m)]_t$  of  $K_t(\mathcal{C}_\ell)$  such that

- $\overline{[L(m)]_t} = [L(m)]_t$ ,
- $[L(m)]_t \in [M(m)]_t + \sum_{m' < m} t^{-1} \mathbb{Z}[t^{-1}][M(m')]_t$ .

This result is generalized to non-simply-laced types in [58, 52].

**12.3. The Type  $D_4$  Module  $L(Y_{2,-4}Y_{2,0})$  is Prime Non-real.** The quantum Cartan

matrix in type  $D_4$  is  $\begin{pmatrix} \frac{z^2+1}{z} & -1 & 0 & 0 \\ -1 & \frac{z^2+1}{z} & -1 & -1 \\ 0 & -1 & \frac{z^2+1}{z} & 0 \\ 0 & -1 & 0 & \frac{z^2+1}{z} \end{pmatrix}$ . In the following, we also write  $m_1 m_2^{-1}$  as  $\frac{m_1}{m_2}$  for

two dominant monomials  $m_1, m_2$ . By modified Frenkel-Mukhin algorithm [47, 84, 85], we

have that  $\tilde{\chi}_{q,t}(L(Y_{2,0})) = Y_{2,0}$ ,

$$\begin{aligned} \tilde{\chi}_{q,t}(Y_{2,-4}) = & Y_{2,-4} + \frac{Y_{1,-3}Y_{3,-3}Y_{4,-3}}{Y_{2,-2}} + \left(t + \frac{1}{t}\right) \frac{Y_{2,-2}}{Y_{2,0}} + \frac{Y_{1,-1}Y_{1,-3}}{Y_{2,0}} + \frac{Y_{1,-3}Y_{3,-3}}{Y_{4,-1}} + \frac{Y_{4,-3}Y_{1,-3}}{Y_{3,-1}} + \frac{Y_{1,-1}Y_{3,-1}Y_{4,-1}}{Y_{2,0}^2} \\ & + \frac{Y_{3,-3}Y_{3,-1}}{Y_{2,0}} + \frac{Y_{4,-1}Y_{4,-3}}{Y_{2,0}} + \frac{Y_{4,-3}Y_{3,-3}}{Y_{1,-1}} + \frac{Y_{1,-3}Y_{2,-2}}{Y_{4,-1}Y_{3,-1}} + \frac{Y_{1,-1}Y_{3,-1}}{Y_{2,0}Y_{4,1}} + \frac{Y_{1,-1}Y_{4,-1}}{Y_{2,0}Y_{3,1}} + \frac{Y_{3,-3}}{Y_{3,1}} + \frac{Y_{4,-3}}{Y_{4,1}} \\ & + \frac{Y_{4,-3}Y_{2,-2}}{Y_{1,-1}Y_{3,-1}} + \frac{Y_{1,-1}}{Y_{4,1}Y_{3,1}} + \frac{Y_{2,-2}}{Y_{4,-1}Y_{4,1}} + \frac{Y_{2,-2}}{Y_{3,-1}Y_{3,1}} + \frac{Y_{2,-2}^2}{Y_{1,-1}Y_{3,-1}Y_{4,-1}} + \frac{Y_{2,-2}Y_{3,-3}}{Y_{1,-1}Y_{4,-1}}, \end{aligned}$$

where the monomials on the right hand side are commutative monomials. Therefore  $\tilde{\chi}_{q,t}(L(Y_{2,-4})) * \tilde{\chi}_{q,t}(L(Y_{2,0})) = p_1 + tp_2 + t^2p_3$ , where

$$p_1 = \frac{Y_{1,-1}Y_{3,-1}Y_{4,-1}}{Y_{2,0}} + \frac{Y_{1,-1}Y_{3,-1}}{Y_{4,1}} + \frac{Y_{1,-1}Y_{4,-1}}{Y_{3,1}} + \frac{Y_{2,0}Y_{1,-1}}{Y_{4,1}Y_{3,1}} + Y_{2,-2},$$

$$p_2 = Y_{1,-1}Y_{1,-3} + Y_{3,-1}Y_{3,-3} + Y_{4,-1}Y_{4,-3} + \frac{Y_{3,-3}Y_{2,0}}{Y_{3,1}} + \frac{Y_{4,-3}Y_{2,0}}{Y_{4,1}} + \frac{Y_{2,0}Y_{2,-2}}{Y_{4,-1}Y_{4,1}} + \frac{Y_{2,0}Y_{2,-2}}{Y_{3,-1}Y_{3,1}},$$

$$\begin{aligned} p_3 = & \frac{Y_{1,-3}Y_{3,-3}Y_{4,-3}Y_{2,0}}{Y_{2,-2}} + Y_{2,-2} + \frac{Y_{1,-3}Y_{3,-3}Y_{2,0}}{Y_{4,-1}} + \frac{Y_{1,-3}Y_{4,-3}Y_{2,0}}{Y_{3,-1}} + Y_{2,-4}Y_{2,0} + \frac{Y_{3,-3}Y_{4,-3}Y_{2,0}}{Y_{1,-1}} + \frac{Y_{2,-2}Y_{1,-3}Y_{2,0}}{Y_{3,-1}Y_{4,-1}} \\ & + \frac{Y_{2,-2}Y_{3,-3}Y_{2,0}}{Y_{1,-1}Y_{4,-1}} + \frac{Y_{2,-2}Y_{4,-3}Y_{2,0}}{Y_{1,-1}Y_{3,-1}} + \frac{Y_{2,-2}^2Y_{2,0}}{Y_{1,-1}Y_{3,-1}Y_{4,-1}}. \end{aligned}$$

It follows that  $\tilde{\chi}_{q,t}(L(Y_{2,-4}Y_{2,0})) = p_3$ . Since  $\tilde{\chi}_q(L(Y_{2,-4}Y_{2,0})) \neq \tilde{\chi}_q(L(Y_{2,-4}))\tilde{\chi}_q(L(Y_{2,0}))$ , we have that  $L(Y_{2,-4}Y_{2,0})$  is prime.

By computing  $\tilde{\chi}_{q,t}(L(Y_{2,-4}Y_{2,0})) * \tilde{\chi}_{q,t}(L(Y_{2,-4}Y_{2,0}))$ , we found that the dominant monomials in  $\tilde{\chi}_{q,t}(L(Y_{2,-4}Y_{2,0})) * \tilde{\chi}_{q,t}(L(Y_{2,-4}Y_{2,0}))$  are:

$$Y_{2,-4}^2Y_{2,0}^2, Y_{1,-3}Y_{2,0}Y_{3,-3}Y_{4,-3}, Y_{2,-2}^2, 2Y_{2,-4}Y_{2,-2}Y_{2,0}, \quad (12.1)$$

where  $2Y_{2,-4}Y_{2,-2}Y_{2,0}$  means that the monomial  $Y_{2,-4}Y_{2,-2}Y_{2,0}$  appears two times. Therefore in the decomposition

$$\tilde{\chi}_{q,t}(L(Y_{2,-4}Y_{2,0})) * \tilde{\chi}_{q,t}(L(Y_{2,-4}Y_{2,0})) = \sum_i f_i(t)\tilde{\chi}_{q,t}(L(m_i)), \quad (12.2)$$

where  $f_i(t)$  is a polynomial in  $t$ , we have that every  $m_i$  can only be chosen from the monomials in (12.1).

By computing  $\tilde{\chi}_{q,t}(L(Y_{1,-3})) * \tilde{\chi}_{q,t}(L(Y_{3,-3})) * \tilde{\chi}_{q,t}(L(Y_{4,-3})) * \tilde{\chi}_{q,t}(L(Y_{2,0}))$ , we obtain that

$$\begin{aligned} \tilde{\chi}_{q,t}(L(Y_{1,-3}Y_{2,0}Y_{3,-3}Y_{4,-3})) &= Y_{1,-3}Y_{2,0}Y_{3,-3}Y_{4,-3} + (t + \frac{1}{t})Y_{2,-2}^2 + Y_{1,-3}Y_{1,-1}Y_{2,-2} + \frac{Y_{1,-3}Y_{2,-2}Y_{2,0}Y_{3,-3}}{Y_{4,-1}} \\ &+ \frac{Y_{1,-3}Y_{2,-2}Y_{2,0}Y_{4,-3}}{Y_{3,-1}} + \frac{Y_{1,-1}Y_{2,-2}Y_{3,-1}Y_{4,-1}}{Y_{2,0}} + Y_{2,-2}Y_{3,-3}Y_{3,-1} + Y_{2,-2}Y_{4,-3}Y_{4,-1} + \frac{Y_{2,-2}Y_{2,0}Y_{3,-3}Y_{4,-3}}{Y_{1,-1}} \\ &+ \frac{Y_{1,-3}Y_{2,-2}^2Y_{2,0}}{Y_{3,-1}Y_{4,-1}} + \frac{Y_{1,-1}Y_{2,-2}^2Y_{3,-1}}{Y_{4,1}} + \frac{Y_{1,-1}Y_{2,-2}Y_{4,-1}}{Y_{3,1}} + \frac{Y_{2,-2}Y_{2,0}Y_{3,-3}}{Y_{3,1}} + \frac{Y_{2,-2}Y_{2,0}Y_{4,-3}}{Y_{4,1}} + \frac{Y_{2,-2}^2Y_{2,0}Y_{3,-3}}{Y_{1,-1}Y_{4,-1}} \\ &+ \frac{Y_{2,-2}^2Y_{2,0}}{Y_{4,-1}Y_{4,1}} + \frac{Y_{2,-2}^2Y_{2,0}}{Y_{3,-1}Y_{3,1}} + \frac{Y_{2,-2}^3Y_{2,0}}{Y_{1,-1}Y_{3,-1}Y_{4,-1}} + \frac{Y_{2,-2}^2Y_{2,0}Y_{4,-3}}{Y_{1,-1}Y_{3,-1}} + \frac{Y_{1,-1}Y_{2,-2}Y_{2,0}}{Y_{4,1}Y_{3,1}}. \end{aligned}$$

We checked that there is a monomial  $\frac{Y_{1,-1}Y_{2,-2}Y_{3,-1}}{Y_{4,1}}$  appearing in  $\tilde{\chi}_{q,t}(L(Y_{1,-3}Y_{2,0}Y_{3,-3}Y_{4,-3}))$  but not appearing in  $\tilde{\chi}_{q,t}(L(Y_{2,-4}Y_{2,0})) * \tilde{\chi}_{q,t}(L(Y_{2,-4}Y_{2,0}))$ . Therefore any  $m_i$  on the right hand side of (12.2) cannot be  $Y_{1,-3}Y_{2,0}Y_{3,-3}Y_{4,-3}$ . Similarly, any  $m_i$  on the right hand side of (12.2) cannot be  $Y_{2,-2}^2$ . Therefore the only possible dominant monomial appearing on the right hand side of (12.2) are  $Y_{2,-4}^2Y_{2,0}^2$  and  $Y_{2,-4}Y_{2,-2}Y_{2,0}$ .

By computing  $f = \tilde{\chi}_{q,t}(L(Y_{2,-4})) * \tilde{\chi}_{q,t}(L(Y_{2,-4})) * \tilde{\chi}_{q,t}(L(Y_{2,0})) * \tilde{\chi}_{q,t}(L(Y_{2,0}))$  and checking the coefficient of  $\frac{1}{t^8}f$ , we find that the monomial  $Y_{2,-4}Y_{2,-2}Y_{2,0}$  appears in the truncated  $(q, t)$ -character of  $L(Y_{2,-4}^2Y_{2,0}^2)$  exactly one time. Since the monomial  $Y_{2,-4}Y_{2,-2}Y_{2,0}$  appears two times in  $\tilde{\chi}_{q,t}(Y_{2,-4}Y_{2,0}) * \tilde{\chi}_{q,t}(Y_{2,-4}Y_{2,0})$ , we have that

$$\tilde{\chi}_{q,t}(Y_{2,-4}Y_{2,0}) * \tilde{\chi}_{q,t}(Y_{2,-4}Y_{2,0}) = \tilde{\chi}_{q,t}(L(Y_{2,-4}^2Y_{2,0}^2)) + \tilde{\chi}_{q,t}(L(Y_{2,-4}Y_{2,-2}Y_{2,0})),$$

and  $\tilde{\chi}_{q,t}(L(Y_{2,-4}^2Y_{2,0}^2)) = p_1 + (t + \frac{1}{t})p_2 + (t^2 + \frac{1}{t^2})p_3 + (t^3 + \frac{1}{t^3})p_4$ , where

$$\begin{aligned} p_1 &= Y_{2,-4}^2Y_{2,0}^2 + \frac{Y_{1,-3}^2Y_{4,-3}^2Y_{2,0}^2}{Y_{3,-1}^2} + 2\frac{Y_{1,-3}Y_{4,-3}^2Y_{3,-3}Y_{2,0}^2}{Y_{1,-1}Y_{3,-1}} + 2\frac{Y_{1,-3}^2Y_{3,-3}Y_{4,-3}Y_{2,0}^2}{Y_{4,-1}Y_{3,-1}} + \frac{Y_{2,-2}^2Y_{4,-3}^2Y_{2,0}^2}{Y_{1,-1}^2Y_{3,-1}^2} \\ &+ \frac{Y_{2,-2}^4Y_{2,0}^2}{Y_{1,-1}^2Y_{3,-1}^2Y_{4,-1}^2} + \frac{Y_{2,-2}^2Y_{3,-3}^2Y_{2,0}^2}{Y_{1,-1}^2Y_{4,-1}^2} + \frac{Y_{1,-3}^2Y_{3,-3}^2Y_{4,-3}^2Y_{2,0}^2}{Y_{2,-2}^2} + \frac{Y_{1,-3}^2Y_{2,-2}^2Y_{2,0}^2}{Y_{3,-1}^2Y_{4,-1}^2} + 2\frac{Y_{2,-2}^2Y_{1,-3}Y_{3,-3}Y_{2,0}^2}{Y_{1,-1}Y_{3,-1}Y_{4,-1}^2} \\ &+ 2\frac{Y_{2,-2}^2Y_{1,-3}Y_{4,-3}Y_{2,0}^2}{Y_{3,-1}^2Y_{4,-1}Y_{1,-1}} + 2\frac{Y_{1,-3}Y_{3,-3}^2Y_{4,-3}Y_{2,0}^2}{Y_{1,-1}Y_{4,-1}} + \frac{Y_{1,-3}^2Y_{3,-3}^2Y_{2,0}^2}{Y_{4,-1}^2} + \frac{Y_{3,-3}^2Y_{4,-3}^2Y_{2,0}^2}{Y_{1,-1}^2} + Y_{2,-2}^2 \\ &+ 2\frac{Y_{3,-3}Y_{4,-3}Y_{2,-2}^2Y_{2,0}^2}{Y_{4,-1}Y_{3,-1}Y_{1,-1}^2} + Y_{2,-2}Y_{2,-4}Y_{2,0}, \end{aligned}$$

$$\begin{aligned}
p_2 = & \frac{Y_{2,-2}^2 Y_{4,-3} Y_{2,0}}{Y_{1,-1} Y_{3,-1}} + \frac{Y_{2,-2}^3 Y_{2,0}}{Y_{1,-1} Y_{3,-1} Y_{4,-1}} + \frac{Y_{2,-2} Y_{4,-3} Y_{2,-4} Y_{2,0}^2}{Y_{1,-1} Y_{3,-1}} + \frac{Y_{1,-3}^2 Y_{3,-3} Y_{2,-2} Y_{2,0}^2}{Y_{4,-1}^2 Y_{3,-1}} + \frac{Y_{3,-3}^2 Y_{4,-3}^2 Y_{1,-3} Y_{2,0}^2}{Y_{1,-1} Y_{2,-2}} \\
& + \frac{Y_{2,-2}^3 Y_{1,-3} Y_{2,0}^2}{Y_{1,-1} Y_{3,-1}^2 Y_{4,-1}^2} + \frac{Y_{2,-2}^3 Y_{4,-3} Y_{2,0}^2}{Y_{1,-1}^2 Y_{3,-1}^2 Y_{4,-1}} + \frac{Y_{2,-2} Y_{3,-3} Y_{2,-4} Y_{2,0}^2}{Y_{1,-1} Y_{4,-1}} + \frac{Y_{1,-3}^2 Y_{3,-3}^2 Y_{4,-3} Y_{2,0}^2}{Y_{4,-1} Y_{2,-2}} + \frac{Y_{2,-2}^3 Y_{3,-3} Y_{2,0}^2}{Y_{1,-1}^2 Y_{4,-1}^2 Y_{3,-1}} \\
& + \frac{Y_{1,-3} Y_{3,-3}^2 Y_{2,-2} Y_{2,0}^2}{Y_{4,-1}^2 Y_{1,-1}} + \frac{Y_{2,0} Y_{1,-3} Y_{2,-2} Y_{3,-3}}{Y_{4,-1}} + \frac{Y_{2,-2}^2 Y_{1,-3} Y_{2,0}}{Y_{4,-1} Y_{3,-1}} + \frac{Y_{2,-2}^2 Y_{3,-3} Y_{2,0}}{Y_{1,-1} Y_{4,-1}} + \frac{Y_{1,-3}^2 Y_{2,-2} Y_{4,-3} Y_{2,0}^2}{Y_{3,-1}^2 Y_{4,-1}} \\
& + \frac{Y_{2,-4} Y_{1,-3} Y_{4,-3} Y_{2,0}^2}{Y_{3,-1}} + \frac{Y_{1,-3} Y_{3,-3} Y_{2,-4} Y_{2,0}^2}{Y_{4,-1}} + \frac{Y_{2,-4} Y_{3,-3} Y_{4,-3} Y_{2,0}^2}{Y_{1,-1}} + Y_{1,-3} Y_{3,-3} Y_{4,-3} Y_{2,0} + \frac{Y_{1,-3} Y_{3,-3} Y_{4,-3} Y_{2,-4} Y_{2,0}^2}{Y_{2,-2}} \\
& + \frac{Y_{1,-3}^2 Y_{3,-3} Y_{4,-3}^2 Y_{2,0}^2}{Y_{2,-2} Y_{3,-1}} + \frac{Y_{2,-2}^2 Y_{2,-4} Y_{2,0}^2}{Y_{1,-1} Y_{3,-1} Y_{4,-1}} + \frac{Y_{2,-4} Y_{1,-3} Y_{2,-2} Y_{2,0}^2}{Y_{4,-1} Y_{3,-1}} + \frac{Y_{2,-2} Y_{4,-3}^2 Y_{3,-3} Y_{2,0}^2}{Y_{1,-1}^2 Y_{3,-1}} + \frac{Y_{2,-2} Y_{3,-3}^2 Y_{4,-3} Y_{2,0}^2}{Y_{1,-1}^2 Y_{4,-1}} \\
& + \frac{Y_{1,-3} Y_{4,-3}^2 Y_{2,-2} Y_{2,0}^2}{Y_{1,-1} Y_{3,-1}^2} + \frac{Y_{1,-3} Y_{4,-3} Y_{2,-2} Y_{2,0}}{Y_{3,-1}} + \frac{Y_{3,-3} Y_{4,-3} Y_{2,-2} Y_{2,0}}{Y_{1,-1}} + 3 \frac{Y_{1,-3} Y_{2,-2} Y_{3,-3} Y_{4,-3} Y_{2,0}^2}{Y_{1,-1} Y_{3,-1} Y_{4,-1}}, \\
p_3 = & \frac{Y_{2,-2}^2 Y_{1,-3} Y_{3,-3} Y_{2,0}^2}{Y_{1,-1} Y_{3,-1} Y_{4,-1}^2} + \frac{Y_{2,-2}^2 Y_{1,-3} Y_{4,-3} Y_{2,0}^2}{Y_{3,-1}^2 Y_{4,-1} Y_{1,-1}} + \frac{Y_{3,-3} Y_{4,-3} Y_{2,-2}^2 Y_{2,0}^2}{Y_{4,-1} Y_{3,-1} Y_{1,-1}^2} + \frac{Y_{1,-3} Y_{4,-3}^2 Y_{3,-3} Y_{2,0}^2}{Y_{1,-1} Y_{3,-1}} + \frac{Y_{1,-3}^2 Y_{3,-3} Y_{4,-3} Y_{2,0}^2}{Y_{4,-1} Y_{3,-1}} \\
& + \frac{Y_{1,-3} Y_{3,-3}^2 Y_{4,-3} Y_{2,0}^2}{Y_{1,-1} Y_{4,-1}}, \\
p_4 = & \frac{Y_{1,-3} Y_{2,-2} Y_{3,-3} Y_{4,-3} Y_{2,0}^2}{Y_{1,-1} Y_{3,-1} Y_{4,-1}}.
\end{aligned}$$

Therefore the module  $L(Y_{2,-4} Y_{2,0})$  in type  $D_4$  is non-real.

### 13. DISCUSSION

In this work we make a connection among tropical geometry, representation theory of quantum affine algebras, and scattering amplitudes in physics.

In mathematical side, we introduce a sequence of Newton polytopes and in the case of  $U_q(\widehat{\mathfrak{sl}_k})$ , we construct explicitly simple modules from given facets of a Newton polytope. We conjecture that the obtained simple modules are prime modules, see Conjecture 6.1.

Representations of quantum affine algebras can be also applied to study questions in tropical geometry. For example, in Section 8.3, we construct matroid subdivisions from prime modules of quantum affine algebras.

On physics side, we generalize the Grassmannian string integral to the setting that the integrand is the infinite product of prime elements in the dual canonical basis of  $\mathbb{C}[\text{Gr}(k, n)]$ , see Section 10, and more generally the infinite product of the  $q$ -characters of all prime modules in the category  $\mathcal{C}_\ell$  of a quantum affine algebra, see Section 10.3.

We also define the so called  $u$ -variables for every prime element in the dual canonical basis of  $\mathbb{C}[\text{Gr}(k, n)]$  and we conjecture that the  $u$ -variables are unique solutions of  $u$ -equations, see Section 10.2. The  $u$ -equations are important in the study of scattering amplitudes in physics, see [4].

Our work raises many related questions. On mathematical side, it is important to give an explicit construction of dominant monomials corresponding to facets of Newton polytopes defined in Section 9 and prove that every prime module in the category  $\mathcal{C}_\ell$  corresponds to a facet of  $\mathbf{N}_{\mathfrak{g},\ell}^{(d)}$  for some  $d$ , see Conjecture 9.4. It is also important to study compatibility of prime modules using Newton polytopes, see Conjecture 9.4.

On physics side, it is important to compute explicitly  $u$ -equations and  $u$ -variables and verify that  $u$ -variables are unique solutions of  $u$ -equations, see Conjecture 10.6.

In the simplest examples, the Newton polytopes for representations of quantum affine algebras defined in Section 9 are associahedra. It would be very interesting to study the relation between the Newton polytopes for representations of quantum affine algebras and the surfacehedra defined in [2].

Finally, let us discuss in some detail an exciting potential research direction which was beyond the scope of the present work to pursue. In [43],  $u$ -equations were introduced in order to define a certain generalized worldsheet associahedron, related to the moduli space of  $n$  points in  $\mathbb{P}^{k-1}$ . A parameterization of the solution to the  $u$ -equations was conjectured when  $k = 3, 4$ ; the details are being worked out in [45]. What is striking is that these  $u$ -equations are manifestly finite; there is a Newton polytope which is (conjecturally) simple with a face lattice which is anti-isomorphic to the noncrossing complex  $\mathbf{NC}_{k,n}$ . In particular there are finitely many  $u$ -variables and finitely many  $u$ -equations. It would be very interesting to determine if this can be explained in the context of the present paper. Does the generalized worldsheet associahedron relate to the solution to the  $u$ -equations which we propose in the present work? If so, which prime tableaux are involved? We leave such fascinating questions to future work.

## REFERENCES

- [1] N. Arkani-Hamed, Y. Bai, and T. Lam, *Positive Geometries and Canonical Forms*, JHEP 11 (2017), 039.
- [2] N. Arkani-Hamed, H. Frost, P.-G. Plamondon, G. Salvatori, H. Thomas, *Surfacehedra*, a talk in Mini-Workshop: Scattering Amplitudes, Cluster Algebras, and Positive Geometries (hybrid meeting), Mathematisches Forschungsinstitut Oberwolfach, Report No. 57/2021, DOI: 10.4171/OWR/2021/57, 2021.
- [3] N. Arkani-Hamed, H. Frost, P.-G. Plamondon, G. Salvatori, H. Thomas, *Solving  $u$ -equations with quiver representations*, a talk in Mini-Workshop: Scattering Amplitudes, Cluster Algebras, and Positive Geometries (hybrid meeting), Mathematisches Forschungsinstitut Oberwolfach, Report No. 57/2021, DOI: 10.4171/OWR/2021/57, 2021.
- [4] N. Arkani-Hamed, S. He, and T. Lam. *Stringy canonical forms*, Journal of High Energy Physics **2021**, no. 2 (2021), 1–62.
- [5] N. Arkani-Hamed, S. He, T. Lam, and H. Thomas, *Binary geometries, generalized particles and strings, and cluster algebras*, arXiv preprint arXiv:1912.11764 (2019).
- [6] N. Arkani-Hamed, S. He, and T. Lam, *Cluster configuration spaces of finite type*, Symmetry, Integrability and Geometry: Methods and Applications, SIGMA **17** (2021), 092.

- [7] N. Arkani-Hamed, T. Lam, and M. Spradlin, *Non-perturbative geometries for planar  $N = 4$  SYM amplitudes*, J. High Energ. Phys. **2021**, 65 (2021).
- [8] N. Arkani-Hamed, T. Lam, and M. Spradlin, *Positive configuration space*, Commun. Math. Phys. **384** (2021), 909–954.
- [9] F. Ardila, F. Rincón, and L. Williams, *Positroids and non-crossing partitions*, Transactions of the American Mathematical Society, Vol. **368** (2016), no. 1, 337–363.
- [10] M. Auslander, I. Reiten, *Representation theory of Artin algebras. III. Almost split sequences*, Comm. Algebra **3** (1975), 239–294.
- [11] R. J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1989. Reprint of the 1982 original.
- [12] K. Baur, D. Bogdanic, A. Garcia Elsener, J.-R. Li, *Rigid indecomposable modules in Grassmannian cluster categories*, arXiv:2011.09227.
- [13] D. Bendle, J. Boehm, Y. Ren, B. Schröter, *Parallel Computation of tropical varieties, their positive part, and tropical Grassmannians*, arXiv:2003.13752.
- [14] L. Bittmann, *A quantum cluster algebra approach to representations of simply laced quantum affine algebras*, Math. Z. **298**, 1449–1485 (2021).
- [15] S. Brodsky and C. Stump, *Towards a uniform subword complex description of acyclic finite type cluster algebras*, Algebraic Combinatorics, volume **1** (2018) no. 4, 545–572.
- [16] M. Brito, V. Chari, and A. Moura, *Demazure modules of level two and prime representations of quantum affine  $\mathfrak{sl}_{n+1}$* , Journal of the Institute of Mathematics of Jussieu, **17** (2018), no. 1, 75–105.
- [17] M. Brito and V. Chari, *Higher order Kirillov-Reshetikhin modules, Imaginary modules and Monoidal Categorification for  $U_q(A_n^{(1)})$* , arXiv:2207.11731.
- [18] L. Bittmann, J.-R. Li, *On the simplicity of the tensor product of two simple modules of quantum affine algebras*, arXiv:2203.17268.
- [19] F. Brown, *Multiple zeta values and periods of moduli spaces  $\overline{\mathfrak{M}}_{0,n}$* , In Annales scientifiques de l'École normale supérieure, vol. **42** (2009), no. 3, 371–489.
- [20] V. Chari, *Minimal affinizations of representations of quantum groups: the rank 2 case*, Publ. Res. Inst. Math. Sci. **31** (1995), no. 5, 873–911.
- [21] F. Cachazo, N. Early, A. Guevara, and S. Mizera. *Scattering equations: from projective spaces to tropical grassmannians*, Journal of High Energy Physics **2019**, no. 6 (2019): 39.
- [22] F. Cachazo and N. Early, *Planar kinematics: cyclic fixed points, mirror superpotential,  $k$ -dimensional Catalan numbers, and root polytopes*, arXiv preprint arXiv:2010.09708 (2020).
- [23] W. Chang, B. Duan, C. Fraser, J.-R. Li. *Quantum affine algebras and Grassmannians*, Math. Z. **296** (2020), 1539–1583.
- [24] M.-W. Cheung, P.-P. Dechant, Y.-H. He, E. Heyes, E. Hirst, J.-R. Li, *Clustering Cluster Algebras with Clusters*, arXiv:2212.09771.
- [25] F. Cachazo, S. He, and E.Y. Yuan, *Scattering equations and Kawai-Lewellen-Tye orthogonality*, Phys. Rev. D **90**, 065001, 2014.
- [26] V. Chari, A. Moura, and C. Young, *Prime representations from a homological perspective*, Math. Z. **274**, 613–645 (2013).
- [27] V. Chari and A. Pressley, *Quantum affine algebras*, Comm. Math. Phys. **142** (1991), 261–283.
- [28] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1994. xvi+651 pp.
- [29] V. Chari and A. Pressley, *Quantum affine algebras and their representations*, in Representations of groups (Banff, AB, 1994), **16** of CMS Conf. Proc. Amer. Math. Soc., Providence, RI, 1995.

- [30] V. Chari and A. Pressley, *Minimal affinizations of representations of quantum groups: the simply-laced case*, Journal of Algebra **184** (1996), 1–30.
- [31] V. Chari, A. Pressley, *Factorization of representations of quantum affine algebras*, Modular interfaces, (Riverside CA 1995), AMS/IP Stud. Adv. Math., **4** (1997), 33–40.
- [32] V. G. Drinfeld, *Hopf algebra and the Yang-Baxter equation*, Soviet Math. Dokl. (English translation) **32** (1985), 254–258.
- [33] V. Drinfeld, *A new realization of Yangians and of quantum affine algebras*, Dokl. Akad. Nauk SSSR, **296** (1987), no.1, 13–17.
- [34] J. Drummond, J. Foster, Ö. Gürdoğan, and C. Kalousios, *Tropical Grassmannians, cluster algebras and scattering amplitudes*, J. High Energ. Phys. 2020, 146 (2020).
- [35] J. Drummond, J. Foster, Ö. Gürdoğan, and C. Kalousios, *Algebraic singularities of scattering amplitudes from tropical geometry*, J. High Energ. Phys. **2021**, 2 (2021).
- [36] B. Davies, O. Foda, M. Jimbo, T. Miwa, A. Nakayashiki, *Diagonalization of the XXZ Hamiltonian by vertex operators*, Commun. Math. Phys. **151** (1993), 89–153.
- [37] B. Duan, J.-R. Li, and Y.-F. Luo, *Cluster algebras and snake modules*, Journal of Algebra, **519** (2019), 325–377.
- [38] B. Duan, L. Lamberti, J.-R. Li, *Combinatorial model for  $m$ -cluster categories in type  $E$* , arXiv:1911.12042.
- [39] B. Duan and R. Schiffler, *A geometric  $q$ -character formula for snake modules*, J. Lond. Math. Soc. **102** (2) (2020), 846–878.
- [40] J. A. De Loera, J. Rambau, and F. Santos, *Triangulations: Structures for Algorithms and Applications*, Algorithms and Computation in Mathematics **25**, Springer, 2010.
- [41] N. Early, *From weakly separated collections to matroid subdivisions*, Combinatorial Theory, **2** (2), 2022.
- [42] N. Early, *Weighted blade arrangements and the positive tropical Grassmannian*, arXiv preprint arXiv:2005.12305 (2020).
- [43] N. Early, *Planarity in generalized scattering amplitudes: PK polytope, generalized root systems and worldsheet associahedra*, arXiv preprint arXiv:2106.07142 (2021).
- [44] N. Early, *Factorization for Generalized Biadjoint Scalar Amplitudes via Matroid Subdivisions*, arXiv:2211.16623.
- [45] N. Early and H. Thomas, in preparation.
- [46] Jiarui Fei, *Combinatorics of  $F$ -polynomials*, International Mathematics Research Notices, 2022, rnab365.
- [47] E. Frenkel and E. Mukhin, *Combinatorics of  $q$ -characters of finite-dimensional representations of quantum affine algebras*, Commun. Math. Phys. **216**, 23–57 (2001).
- [48] E. Frenkel and N. Reshetikhin, *The  $q$ -characters of representations of quantum affine algebras and deformations of  $W$ -algebras*, Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998), pp. 163–205. Contemp. Math., vol. **248**. American Mathematical Society, Providence (1999).
- [49] I.B. Frenkel and N.Y. Reshetikhin, *Quantum affine algebras and holonomic difference equations*, Commun. Math. Phys. **146** (1992), 1–60.
- [50] V. V. Fock and A. B. Goncharov, *Cluster ensembles, quantization and the dilogarithm*, Annales scientifiques de l'École Normale Supérieure, Serie 4, Volume **42** (2009) no. 6, 865–930.
- [51] S. Fomin, A. Zelevinsky, *Cluster algebras I: foundations*, J. Amer. Math. Soc., **15** (2) (2002), 497–529.

- [52] R. Fujita, D. Hernandez, S.-j. Oh, and H. Oya, *Isomorphisms among quantum Grothendieck rings and propagation of positivity*, Journal für die reine und angewandte Mathematik (Crelles Journal), vol. **2022**, no. 785, 2022, 117–185.
- [53] A.M. Gabriélov, I.M. Gel’fand, and M.V. Losik, *Combinatorial computation of characteristic classes*, Funct. Anal. Its Appl. **9**, 103–115 (1975).
- [54] I. M. Gelfand, M. Graev, and A. Postnikov, *Combinatorics of hypergeometric functions associated with positive roots*, In The Arnold-Gelfand mathematical seminars, pp. 205–221. Birkhauser Boston, 1997.
- [55] I. Gelfand, M. Goresky, R. MacPherson, and V. Serganova, *Combinatorial geometries, convex polyhedra and Schubert cells*, Adv. in Math. **63** (1987), 301–316.
- [56] M. Gurevich, *On restriction of unitarizable representations of general linear groups and the non-generic local Gan–Gross–Prasad conjecture*, J. Eur. Math. Soc. **24** (2022), no. 1, 265–302.
- [57] S. He, L. Ren, and Y. Zhang, *Notes on polytopes, amplitudes and boundary configurations for Grassmannian string integrals*, J. High Energ. Phys. **2020**, 140 (2020).
- [58] D. Hernandez, *Algebraic approach to  $q$ ,  $t$ -characters*, Adv. Math., **187** (1) (2004), 1–52.
- [59] S. Herrmann and M. Joswig, *Splitting polytopes*, Münster J. Math., **1** (2008), 109–141.
- [60] N. Henke, G. Papathanasiou, *How tropical are seven- and eight-particle amplitudes?* J. High Energ. Phys. **2020**, 5 (2020).
- [61] N. Henke, G. Papathanasiou, *Singularities of eight- and nine-particle amplitudes from cluster algebras and tropical geometry*, J. High Energ. Phys. **2021**, 7 (2021).
- [62] D. Hernandez and B. Leclerc, *Cluster algebras and quantum affine algebras*, Duke Math. J., **154**(2) (2010), 265–341.
- [63] D. Hernandez and B. Leclerc, *Quantum Grothendieck rings and derived Hall algebras*, Journal für die reine und angewandte Mathematik (Crelles Journal), vol. **2015**, no. 701, (2015) 77–126.
- [64] D. Hernandez and B. Leclerc, *A cluster algebra approach to  $q$ -characters of Kirillov–Reshetikhin modules*, J. Eur. Math. Soc. **18** (2016), no. 5, 1113–1159.
- [65] D. Hernandez and B. Leclerc, *Quantum affine algebras and cluster algebras*, In: Greenstein, J., Hernandez, D., Misra, K.C., Senesi, P. (eds) Interactions of Quantum Affine Algebras with Cluster Algebras, Current Algebras and Categorification. Progress in Mathematics, vol **337**. Birkhäuser, Cham, 2021.
- [66] M. Jimbo, *A  $q$ -difference analogue of  $U(\mathfrak{g})$  and the Yang-Baxter equation*, Lett. Math. Phys., **10** (1985), no. 1, 63–69.
- [67] B.T. Jensen, A. King, X.P. Su, *A categorification of Grassmannian cluster algebras*, Proc. Lond. Math. Soc. (3) **113** (2016), no. 2, 185–212.
- [68] A. Kalman, *Newton polytopes of cluster variables of type  $A_n$* , 26th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2014), 2014, Chicago, United States, 137–148, [ff10.46298/dmtcs.2387ff](https://arxiv.org/abs/10.46298/dmtcs.2387ff).
- [69] S.-J. Kang, M. Kashiwara, M. Kim, and S.-j. Oh, *Monoideal categorification of cluster algebras*, J. Amer. Math. Soc. **31** (2018), no. 2, 349–426.
- [70] M. Kashiwara, M. Kim, S.-j. Oh, and E. Park, *Monoideal categorification and quantum affine algebras*, Compositio Mathematica, **156** (5) (2020), 1039–1077.
- [71] M. Kashiwara, M. Kim, S.-j. Oh, and E. Park, *Monoideal categorification and quantum affine algebras II*, arXiv:2103.10067.
- [72] M. Kashiwara, M. Kim, S.-j. Oh, and E. Park, *Cluster algebra structures on module categories over quantum affine algebras*, Proc. London Math. Soc. **124** (2022), 301–372.



- [73] Z. Koba, H. B. Nielsen, *Reaction amplitude for  $n$ -mesons: A generalization of the Veneziano-Bardakçi-Ruegg-Virasoro model*, Nuclear Physics B, **10** (1969), 633–665.
- [74] L. Lafforgue, *Chirurgie des Grassmanniennes*, CRM Monograph Series, vol. **19**, American Mathematical Society, Providence, RI, 2003.
- [75] B. Leclerc, *Imaginary vectors in the dual canonical basis of  $U_q(n)$* , Transform Groups **8** (2003), no. 1, 95–104.
- [76] E. Lapid and A. Mínguez, *Geometric conditions for  $\square$ -irreducibility of certain representations of the general linear group over a non-archimedean local field*, Adv. Math. **339** (2018), 113–190.
- [77] E. Lapid and A. Mínguez, *Conjectures and results about parabolic induction of representations of  $GL_n(F)$* , Invent. math. **222**, 695–747 (2020).
- [78] K. Lee, L. Li, and R. Schiffler, *Newton polytopes of rank 3 cluster variables*, Algebraic Combinatorics, volume **3** (2020) no. 6, 1293–1330.
- [79] F. Li and J. Pan, *Recurrence formula, positivity and polytope basis in cluster algebras via Newton polytopes*, arXiv:2201.01440.
- [80] T. Lam and A. Postnikov, *Polypositroids*, arXiv:2010.07120, 2020.
- [81] B. Leclerc and A. Zelevinsky, *Quasicommuting families of quantum Plücker coordinates*, In Kirillov’s seminar on representation theory, volume **181** of Amer. Math. Soc. Transl. Ser. 2, pages 85–108. Amer. Math. Soc., Providence, RI, 1998.
- [82] A. Mattoo and M. Sherman-Bennett, *Saturation of Newton polytopes of type A and D cluster variables*, combinatorial theory **2** (2) (2022), # 6.
- [83] A. Moura, C. Silva, *On the primality of totally ordered  $q$ -factorization graphs*, arXiv:2204.10439, 2022.
- [84] H. Nakajima,  *$t$ -analogs of  $q$ -characters of Kirillov–Reshetikhin modules of quantum affine algebras*, Represent. Theory **7**, 259–274 (2003).
- [85] H. Nakajima, *Quiver varieties and  $t$ -analogs of  $q$ -characters of quantum affine algebras*, Ann. Math. (2) **160**(3), 1057–1097 (2004).
- [86] F. Qin, *Triangular bases in quantum cluster algebras and monoidal categorification conjectures*, Duke Math. J. **166** (2017), no. 12, 2337–2442.
- [87] L. Ren, M. Spradlin, A. Volovich, *Symbol alphabets from tensor diagrams*, J. High Energ. Phys. **2021**, 79 (2021).
- [88] F. Santos, C. Stump, and V. Welker, *Noncrossing sets and a Grassmann associahedron*, In Forum of Mathematics, Sigma (Vol. **5**), Cambridge University Press, 2017.
- [89] M. P. Schützenberger, *Quelques remarques sur une construction de Schensted*, Math. Scand. **12** (1963), 117–128.
- [90] M. P. Schützenberger, *Promotion des morphismes d’ensembles ordonnés*, Discrete Math. **2** (1972), 73–94.
- [91] M. P. Schützenberger, *La correspondance de Robinson*, in Combinatoire et représentation du groupe symétrique, Lecture Notes in Math., Vol. **579**, Springer-Verlag, Berlin-New York, 1977, 59–113.
- [92] J. Scott, *Quasi-commuting families of quantum minors*, Journal of Algebra, **290** (1), 204–220, 2005.
- [93] J. Scott, *Grassmannians and cluster algebras*, Proceedings of the London Mathematical Society, **92**(3) (2006), 345–380.
- [94] D. Speyer and B. Sturmfels, *The tropical Grassmannian*, **4** (2004), no. 3, 389–411.
- [95] D.E. Speyer, *Tropical linear spaces*, SIAM J. Discrete Math. **22** (2008), no. 4, 1527–1558.

- [96] D. Speyer and L. Williams, *The Tropical Totally Positive Grassmannian*, J Algebr Comb **22** (2005), 189–210.
- [97] D. Speyer and L. Williams, *The positive Dressian equals the positive tropical Grassmannian*, Transactions of the American Mathematical Society, Series B, **8**, no. 11 (2021), 330–353.
- [98] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge Studies in Advanced Mathematics **62**, Cambridge University Press, Cambridge, 1999.
- [99] P. Sherman and A. Zelevinsky, *Positivity and canonical bases in rank 2 cluster algebras of finite and affine types*, Mosc. Math. J. **4** (2004), no. 4, 947–974.
- [100] G. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, Springer, New York/Berlin (1995).

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