# Newton and interior-point methods for (constrained) nonconvex-nonconcave minmax optimization with stability and instability guarantees

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## Abstract

We address the problem of finding a local solution to a nonconvex-nonconcave minmax optimization using Newton type methods, including interior-point ones. We modify the Hessian matrix of these methods such that, at each step, the modified Newton update direction can be seen as the solution to a quadratic program that locally approximates the minmax problem. Moreover, we show that by selecting the modification in an appropriate way, the only stable equilibrium points of the algorithm's iterations are local minmax points. As a consequence, the algorithm can only converge towards an equilibrium point if such point is a local minmax, and it will escape if the point is not a local minmax. Using numerical examples, we show that the computation time of our algorithm scales roughly linearly with the number of nonzero elements in the Hessian.

**Keywords:** minmax optimization, robust optimization, Newton method, interior-point method, local minmax

# 1 Introduction

In minmax optimization, one minimizes a cost function which is itself obtained from the maximization of a scalar function. Minmax optimization is a powerful modeling framework, generally used to guarantee robustness to an adversarial parameter such as accounting for disturbances in model predictive control [1, 2], security related problems [3, 4], or training neural networks to be robust to adversarial attacks [5]. It can also be used as a framework to model more general problem such as sampling from unknown distributions using generative adversarial networks [6], reformulating stochastic programming as minmax optimization [7–9], or producing robustness of a stochastic program with respect to the probability distribution [10]. Minmax optimization is also known as minimax or robust optimization.

Finding a global minmax point for nonconvex-nonconcave problems is generally difficult, and one has to settle for finding a local minmax point. Surprisingly, only recently a first definition of unconstrained local minmax was proposed in [11], and the definition of constrained local minmax in [12].

It is widely accepted that the application of Newton-like methods to the minimization of nonconvex functions requires the modification of the Hessian matrix through the addition of a matrix, typically a multiple of the identity. An initial contribution of our paper is the observation that by selecting this additive term so that a quadratic local approximation to the cost function becomes convex has two important consequences. First it guarantees progress towards a solution, in the sense that the function decreases with each Newton step – this result is well known [13, Chapter 3.4]. In addition, we show that the same additive term also guarantees that the set of locally asymptotically stable equilibrium points of the Newton iteration is precisely the set of strict local minimum of the optimization. This guarantees that convergence to an equilibrium point necessarily implies convergence to a local minima. We also show that, in the case of constrained minimization, it is possible to analyze interior-point methods as a quadratic approximation which can also be appropriately modified. These results (presented in Section 2) directly motivate the design of novel Newton-type algorithms for minmax optimizations

The Newton-type algorithms proposed in this paper are motivated by a quadratic local approximation to the optimization criteria to which we add terms to make it have a finite minmax solution (without necessarily becoming convex-concave). Any additive terms that guarantee this are said to satisfy the Local Quadratic Approximation Condition (LQAC). We show that contrary to minimization, such modification does not lead to Newton-type iterations with desired stability: a local minmax can be unstable and an equilibrium point that is not a local minmax can be stable. Our minmax result shows that additional conditions are needed to guarantee that every locally asymptotically stable equilibrium point of a Newton-type iteration is a local minmax. This is property is fundamental because, in the one hand, any modification to the Newton method should not impair the algorithm's capacity to converge towards a local minmax, and in the other hand, it essentially guarantees that

the algorithm cannot converge towards an equilibrium point if such point it not a local minmax. To simplify the presentation, we first present this result in Section 3.1 for unconstrained minmax and then extend it in Section 3.2 to interior-point methods for constrained minmax.

The conditions described above to establish the equivalence between local minmax and local asymptotic stability of the equilibria to a Newton-type iteration are directly used to construct a numerical algorithm to find local minmax. By construction, when this algorithm converges to an equilibrium point, its is guaranteed to obtain a local minmax. It is important to clarify that this result fall shy of guaranteeing *global* asymptotic convergence to a local minmax, as the algorithm could simply never converge.

Using numerical examples, we show that by using an appropriate implementation of the LDLt decomposition, the numerical complexity increases roughly with the number of nonzero entries of the Hessian. This is important for problems with stage costs and constraints, such as robust Model Predictive Control, where the number of nonzero entries of the Hessian tend to increase linearly with the number of stages. Using the results of this paper, we have created a solver for minmax optimization and included it in the solvers of `TensCalc`[1] [14]; this solver was used to generate the numerical results we present.

### Notation:

The set of real numbers is denoted by $\mathbb{R}$. Given a vector $v \in \mathbb{R}^n$, its transpose is denoted by $v'$. The operation $\operatorname{diag}(v)$ creates a matrix with diagonal elements $v$ and off-diagonal elements 0. The matrix $I$ is the identity, $\mathbf{1}$ is the matrix of ones and $\mathbf{0}$ the matrix of zeros; their sizes will be provided as subscripts whenever it is not clear from context. If a matrix $A$ only has real eigenvalues, we denote by $\lambda_{min}(A)$ and $\lambda_{max}(A)$ its smallest and largest eigenvalues. The inertia of $A$ is denoted by $\mathbf{in}(A)$, and is a 3-tuple with the number of positive, negative and zero eigenvalues of $A$.

Consider a differentiable function $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^p$. The Jacobian (or gradient if $p = 1$) at a point $(\bar{x}, \bar{y})$ according to the $x$ variable is a matrix of size $n \times p$ and is denoted by $\nabla_x f(\bar{x}, \bar{y})$, and analogously for the variable $y$. When $p = 1$ and $f(\cdot)$ is twice differentiable, we use the notation $\nabla_{yx} f(\bar{x}, \bar{y}) := \nabla_y (\nabla_x f)(\bar{x}, \bar{y})$ which has sizes $m \times n$. We use analogous definition for $\nabla_{xy} f(\bar{x}, \bar{y})$, $\nabla_{xx} f(\bar{x}, \bar{y})$ and $\nabla_{yy} f(\bar{x}, \bar{y})$.

## 1.1 Literature Review

Traditionally, robust optimization focused on the convex-concave case, with two main methods. The first, robust reformulation, uses results from convex analysis to reformulate the minmax optimization as a counterpart minimization problem which has the same solution as the original problem [15–17]. The second, cutting-set methods, solves a sequence of minimization where the

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[1] https://github.com/hespanha/tenscalc

constraint of each minimization is based on subdividing the inner maximization [18]. The robust reformulation is problem specific, while the cutting-set approach requires solving many exact maximization which might not be feasible in large scale.

Motivated by some of the shortcomings of these methods and the necessities of machine learning, research on minmax optimization started to shift towards the study of methods based on variations of gradient descent-ascent. The results tend to focus on providing convergence complexity given different convexity/concavity assumptions on the target function. In multistep gradient descent ascent, also know as unrolled or GDmax, the minimizer is updated by a single gradient descent whereas the maximizer is updated by several gradient ascent steps that aim to approximately find the maximum [11, 19, 20]. In single step, the minimizer and maximizer are updated at each iteration, generally combined with some other features such as different step sizes, momentum or extra gradient [21–25]

In recent years, researchers have also started to work on algorithms that use second order derivatives to determine the directions. These algorithm, in their major part have not attracted as much attention as first order methods. In the Learning with Opponent Learning Awareness (LOLA), the minimizer anticipates the play of the maximizer using the Jacobian of the maximizer's gradient [26, 27]. In competitive gradient descent, both minimizer and maximizer use the cross derivative of the Hessian to compute their direction [28]. In follow the ridge, the gradient ascent step is corrected by a term that avoids a drift away from local maxima [29]. In the total gradient descent-ascent, similarly to LOLA, the descent direction is computed by taking to total derivative of a function which anticipates the maximizer's response to the minimizer [30]. Finally, the complete Newton borrows ideas from follow the ridge and total gradient to obtain a Newton method which prioritizes steps towards local minmax [31]. These three last algorithms are shown to only converge towards local minmax under some conditions, but in none of them it is addressed the issue of how to adjust the Hessian far away from a local minmax point.

Recently, some second order methods have been proposed for the nonconvex-strongly-concave case, where the Hessian is modified such that it is invertible and that the minimizer update is a descent direction of the objective function at its maximum. They either use cubic regularization [32, 33] or randomly perturb the Hessian [34]. Because of some of the assumptions these work make, most important the strong-concavity of the objective function with respect to the maximizer, they are able to establish complexity analysis and guarantee. It is also worth mention that these algorithms are all multistep based, meaning they (approximately) solve the maximization between each update of the minimizer, whereas our algorithm updates both the minimizer and the maximizer simultaneously.

# 2 Minimization

Let $f : \mathcal{X} \to \mathbb{R}$ be a twice continuously differentiable cost function defined in a set $\mathcal{X} \subset \mathbb{R}^{n_x}$, and consider the minimization problem

$$\min_{x \in \mathcal{X}} f(x). \tag{1}$$

We recall that a point $x^*$ is called a local minimum of $f(\cdot)$ if there exist $\delta > 0$ such that $f(x^*) \leq f(x)$ for all $x \in \{x \in \mathcal{X} : \|x - x^*\| < \delta\}$.

## 2.1 Unconstrained minimization

Let $\mathcal{X} = \mathbb{R}^{n_x}$, which is referred to as unconstrained minimization in the literature. If $f(\cdot)$ is twice continuously differentiable in a neighborhood of a point $x$ and $\boldsymbol{\nabla}_x f(x) = \mathbf{0}$ and $\boldsymbol{\nabla}_{xx} f(x) \succ 0$, then $x$ is a local minimum of $f(\cdot)$ [13, Chapter 2].

For unconstrained minimization, closely related to local minima, is the concept of descent direction. A vector $d_x \in \mathbb{R}^n$ is a *descent* direction of $f(\cdot)$ at a point $x$ if $\exists \alpha_0 > 0$ such that $\forall \alpha \in (0, \alpha_0]$, $f(x + \alpha d_x) < f(x)$. It is straightforward to establish that there are no descent direction at a local minimum. For continuously differentiable functions, one can further show that a given vector $d_x \in \mathbb{R}^n$ is a descent direction if an only if $d'_x \boldsymbol{\nabla}_x f(x) < 0$ [13].

In a widely used modification of Newton's method for unconstrained nonconvex optimization [13, Chapter 3.4], a descent direction $d_x$ is obtained by solving the following local quadratic approximation to (1)

$$
\begin{aligned}
d_x &= \arg \min_{\bar{d}_x} f(x) + \boldsymbol{\nabla}_x f(x)' \bar{d}_x + \frac{1}{2} \bar{d}_x (\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x) I) \bar{d}_x \\
&= -(\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x) I)^{-1} \boldsymbol{\nabla}_x f(x)
\end{aligned}
\tag{2}
$$

with $\epsilon_x(x) \geq 0$ chosen such that $(\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x) I)$ is positive definite. For twice differentiable strongly convex functions we can choose $\epsilon_x(x) = 0$ and this corresponds to the classical Newton's methods. However, when $f(\cdot)$ is not convex, the minimization in (2) is only well defined if $\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x) I$ is positive definite, which requires selecting a strictly positive value for $\epsilon_x(x)$, leading to a perturbed Newton's method. Regardless of whether or not $f(\cdot)$ is convex, the positive definiteness of $\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x) I$ guarantees that $d'_x \boldsymbol{\nabla}_x f(x) = -\boldsymbol{\nabla}_x f(x)(\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x) I) \boldsymbol{\nabla}_x f(x) < 0$ and therefore $d_x$ is a descent direction at $x$. The corresponding Newton iteration to obtain a local minimum is then given by

$$x^+ = x + d_x = x - (\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x) I)^{-1} \boldsymbol{\nabla}_x f(x) \tag{3}$$

where we use the notation $x^+$ to designate the value of $x$ at the next iteration.

The following result establishes that the positive definiteness of $\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x)I$ not only guarantees that $d_x$ is a descent direction, but also that every locally asymptotically stable (LAS) equilibrium point of the Newton iteration (3) is a strict local minimum.

**Theorem 1** (Stability of modified Newton method for minimization) *Let $x$ be an equilibrium point in the sense that $\boldsymbol{\nabla}_x f(x) = \mathbf{0}$. Assume that $\boldsymbol{\nabla}_{xx} f(x)$ is invertible and that $\epsilon_x(\cdot)$ and $\boldsymbol{\nabla}_{xx} f(\cdot)$ are differentiable on a neighborhood around $x$. Let $\epsilon_x(x)$ be such that $\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x)I \succ 0$. Then if:*

*i) $x$ is a local minimum, then it is a LAS equilibrium point of (3).*
*ii) $x$ is not a local minimum, then it is an unstable equilibrium point of (3).*

The theorem's first implication is that if the modified Newton iteration starts sufficiently close to a strict local minimum, it will converge asymptotically fast to it. One could think that it would always be preferable to have $\epsilon_x(x) = 0$ if $\boldsymbol{\nabla}_{xx} f(x) \succ 0$, in which case not only stability can be trivially obtained, but also one can show that the Newton method converges to a local minimum superlinearly [13, Theorem 3.6]. However, in practice, there a situations for which this is not true. A typical case happens if the smallest eigenvalue of $\boldsymbol{\nabla}_{xx} f(x)$ is positive but very small, which might bring numerical issues when computing the Newton step $\boldsymbol{\nabla}_{xx} f(x)^{-1} \boldsymbol{\nabla}_x f(x)$. This issue can fixed by taking $\epsilon_x(x) > 0$, and Theorem 1 guarantees that doing so will not impair (at least locally) the algorithm's capacity to converge towards a local minimum.

The theorem's second implication is, in a way, even more relevant than the first one. The regular Newton's method (meaning, with $\epsilon_x(x) = 0$) is infamously known to be attracted to any point that satisfies $\boldsymbol{\nabla}_x f(x) = \mathbf{0}$, regardless of whether it is a local minimum, a saddle point, or a local maximum. What Theorem 1 is essentially saying is that the modified Newton is only attracted to local minima, and that any other equilibrium point repels the iteration. In essence, this means that the modified Newton's method cannot converge towards a point that is not a local minimum, thus fixing one of the biggest drawbacks of the regular Newton's method.

*Proof of Theorem 1* From our assumptionn that $\boldsymbol{\nabla}_{xx} f(x)$ is invertible, $x$ is a local minimum if and only if $\boldsymbol{\nabla}_{xx} f(x) \succ 0$. This comes from the second order necessary condition for minimization [13, Chapter 2].

The first step in our analysis is to calculate the Jacobian of $(\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x)I)^{-1} \boldsymbol{\nabla}_x f(x)$ that appears in (3) at an equilibrium point $x$. This is given by

$$\boldsymbol{\nabla}_x \Big( (\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x)I)^{-1} \boldsymbol{\nabla}_x f(x) \Big) = (\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x)I)^{-1} \boldsymbol{\nabla}_{xx} f(x) +$$

$$\sum_{i=1}^{N} \boldsymbol{\nabla}_x [(\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x)I)^{-1}]_i \boldsymbol{\nabla}_x f(x)^{(i)}$$

where $\boldsymbol{\nabla}_x f(x)^{(i)}$ is the i$^{\text{th}}$ element of $\boldsymbol{\nabla}_x f(x)$ and $[(\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x)I)^{-1}]_i$ is the i$^{\text{th}}$ column of $(\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x)I)^{-1}$. Since $(\boldsymbol{\nabla}_{xx} f(x) + \epsilon_x(x)I)$ is positive definite,

$\nabla_x[(\nabla_{xx} f(x) + \epsilon_x(x)I)^{-1}]_i$ is well defined and since $x$ is an equilibrium point, $\nabla_x f(x)^{(i)} = 0$ for $i \in \{1 \dots N\}$ and therefore the Jacobian of right-hand side of (3) is given by

$$\nabla_x \left( x - (\nabla_{xx} f(x) + \epsilon_x(x)I)^{-1} \nabla_x f(x) \right) = I - (\nabla_{xx} f(x) + \epsilon_x(x)I)^{-1} \nabla_{xx} f(x). \quad (4)$$

The main argument of the proof is based on the following result. Let $v$ be an eigenvector associated to an eigenvalue $\rho$ of (4). Then

$$\left( I - (\nabla_{xx} f(x) + \epsilon_x(x)I)^{-1} \nabla_{xx} f(x) \right) v = \rho v$$

$$\Leftrightarrow (1 - \rho)v = (\nabla_{xx} f(x) + \epsilon_x(x)I)^{-1} \nabla_{xx} f(x) v$$

$$\Leftrightarrow \left( \rho \nabla_{xx} f(x) + (\rho - 1)\epsilon_x(x)I \right) v = \mathbf{0} \quad (5)$$

Therefore, $\rho$ is an eigenvalue of (4) if and only if $\rho \nabla_{xx} f(x) + (\rho - 1)\epsilon_x(x)I$ is singular.

We remind the reader that given a dynamical system, if the system's dynamic equation is continuously differentiable, a point is a LAS equilibrium point if all the eigenvalues of the linearized system are inside the unit circle. Conversely, if at least one of the eigenvalues of the linearized system is outside the unit circle, then the system is unstable [35, Chapter 8].

From (5), $\rho = 0$ is an eigenvalue if and only if $\epsilon_x(x) = 0$, which, by construction, can only happen if $x$ is a local minimum, in which case $x$ is a LAS equilibrium point of (3), as expected.

For $\rho \neq 0$, let us rewrite this expression as $\nabla_{xx} f(x) + \mu \epsilon_x(x)I$ with $\mu := 1 - 1/\rho$. We conclude that $x$ is a LAS equilibrium point of (3) if $\nabla_{xx} f(x) + \mu \epsilon_x(x)$ is nonsingular $\forall \mu \in [0, 2]$. Conversely, $x$ is an unstable equilibrium point of (3) if $\nabla_{xx} f(x) + \mu \epsilon_x(x)$ is singular for some $\mu \in [0, 2]$.

If $x$ is a local minimum, then $\lambda_{min}(\nabla_{xx} f(x)) > 0$. As $\epsilon_x(x) > 0$, we conclude that $\lambda_{min}(\nabla_{xx} f(x) + \mu \epsilon_x(x)I) > 0$ for every $\mu \geq 0$ and therefore $x$ is a LAS equilibrium point of (3). Conversely, if $x$ is not a local minimum then $\lambda_{min}(\nabla_{xx} f(x)) < 0$. By construction of $\epsilon_x(x)$, we have that $\lambda_{min}(\nabla_{xx} f(x) + \mu \epsilon_x(x)I) > 0$, which, by continuity of the eigenvalue, implies $\exists \mu \in (0, 1)$ such that $\lambda_{min}(\nabla_{xx} f(x) + \mu \epsilon_x(x)I) = 0$. Therefore $x$ is an unstable equilibrium point of (3). $\qquad \square$

## 2.2 Constrained minimization

In order to make the presentation of the results of the rest of the paper more clear, it is useful to consider the case with more general constraint with the minimization set $\mathcal{X}$ involving equality and inequality constraints of the form

$$\mathcal{X} = \{ x \in \mathbb{R}^n : G_x(x) = \mathbf{0}, F_x(x) \leq \mathbf{0} \}$$

where the functions $G_x : \mathbb{R}^{n_x} \to \mathbb{R}^{l_x}$ and $F_x : \mathbb{R}^{n_x} \to \mathbb{R}^{m_x}$ are all twice continuously differentiable. It will be convenient for the development of interior-point methods to use slack variables and rewrite (1) as

$$\min_{x, s_x : G_x(x) = \mathbf{0}, F_x(x) + s_x = \mathbf{0}, s_x \geq 0} f(x). \quad (6)$$

where $s_x \in \mathbb{R}^{m_x}$.

Similar to what we have in the unconstrained minimization, we want a second order conditions to determine whether a point is a local minimum. Consider the function

$$L(z) = f(x) + \nu'_x G_x(x) + \lambda'_x(F_x(x) + s_x),$$

where we use the shorthand notation $z := (x, s_x, \nu_x, \lambda_x)$. $L(z)$ is essentially the Lagrangian of (6). In order to present the second order conditions, we need to define two concepts, the linear independence constraint qualification and strict complementarity [13, Definitions 12.4 and 12.5].

**Definition 1** (LICQ and strict complementarity) Let the sets of active inequality constraints for the minimization be defined by

$$\mathcal{A}_x(x) = \{i : F_x^{(i)}(x) = 0, i = 1, \ldots, m_x\} \tag{7}$$

where $F_x^{(i)}(x)$ denote the i$^{\text{th}}$ element of $F_x(x)$. Then:

• The linear independence constraint qualification (LICQ) is said to hold at $z$ if the vectors in the set

$$\{\boldsymbol{\nabla}_x G_x^{(i)}(x), i = 1, \ldots, l_x\} \bigcup \{\boldsymbol{\nabla}_x F_x^{(i)}(x), i \in \mathcal{A}_x(x)\}$$

are linearly independent.
• Strict complementarity is said to hold at $x$ if $\lambda_x^{(i)} > 0 \; \forall i \in \mathcal{A}_x(x)$

We have almost all the ingredients to present the second order condition for constrained minimization. For unconstrained minimization, a sufficient condition for a point $x$ to be a local minimum is that $\boldsymbol{\nabla}_x f(x) = 0$ and $\boldsymbol{\nabla}_{xx} f(x) \succ 0$. If it were not for the inequality constraints in (6), we would be able to state the second order conditions using gradients and Hessians of $L(z)$. The inequality constraints make the statement a bit more complicated. The role of the gradient will be played by

$$g(z, b) := \begin{bmatrix} \boldsymbol{\nabla}_x L(z) \\ \lambda_x \odot s_x - b\mathbf{1} \\ G_x(x) \\ F_x(x) + s_x \end{bmatrix} \tag{8}$$

with $\odot$ denoting the element wise Hadamard product of two vectors and $b \geq 0$ the barrier parameter (its role will be explained shortly). The role of $\boldsymbol{\nabla}_{xx} f(x)$ in the unconstrained minimization will be played by the matrix

$$H_{zz} f(z) = \begin{bmatrix} \boldsymbol{\nabla}_{xx} L(z) & 0 & \boldsymbol{\nabla}_x G_x(x) & \boldsymbol{\nabla}_x F_x(x) \\ 0 & \text{diag}(\lambda_x) & 0 & \text{diag}(s_x^{1/2}) \\ \boldsymbol{\nabla}_x G_x(x)' & 0 & 0 & 0 \\ \boldsymbol{\nabla}_x F_x(x)' & \text{diag}(s_x^{1/2}) & 0 & 0 \end{bmatrix}. \tag{9}$$

We also remind the reader that the inertia $\mathbf{in}(A)$ of a symmetric matrix $A$ is a 3-tuple with the number of positive, negative and zero eigenvalues of $A$.

**Proposition 1** (Constrained second order sufficient conditions) *Let $z$ be an equilibrium point in the sense that $g(z,0) = \mathbf{0}$ with $\lambda_x, s_x \geq \mathbf{0}$. If the LICQ and strict complementarity hold at $z$ and*

$$\mathbf{in}(H_{zz}f(z)) = (n_x + m_x, l_x + m_x, 0) \tag{10}$$

*then $x$ is a local minimum of* (6).

While this result is relatively well known, we present its proof in Appendix A. The proof also makes it easier to understand the proof of the second order conditions for constrained minmax optimization.

### 2.2.1 Interior-point method

[2] Let $d_z := (d_x, d_s, d_\nu, d_\lambda)$ be the update direction for $z$, which will play an equivalent role to $d_x$ in the unconstrained case. A basic interior-point method finds a candidate solution to (6) using the iterations

$$z^+ = z + \alpha d_z = z - \alpha \boldsymbol{\nabla}_z g(z,b)'^{-1} g(z,b) \tag{11}$$

where the barrier parameter $b$ is slowly decreased to 0, so that $z$ converges to a root of $g(z,0) = \mathbf{0}$ while $\alpha \in (0,1]$ is chosen at each step such that the feasibility condition $\lambda_x, s_x > \mathbf{0}$ hold [13, Chapter 19]. This basic interior-point has similar limitation as a (non-modified) Newton method for unconstrained minimization: it might not converge towards a local minimum and $\boldsymbol{\nabla}_z g(z,b)$ might not be invertible. Similar to what we have done in the unconstrained case, we can modify this basic interior-point method such that the update direction $d_z$ is obtained from a quadratic program that locally approximates (6). The rest of this section will be spent mostly constructing such quadratic program.

Let us start with $\mathcal{X}$ described only by equality constraints (*i.e.*, no $F_x(x)$ and no $s_x$), in which case $L(z) = f(x) + \nu_x' G_x(x)$. Consider the optimization

$$\min_{\bar{d}_x : G_x(x) + \boldsymbol{\nabla}_x G_x(x)' \bar{d}_x = \mathbf{0}} L(z) + \bar{d}_x' \boldsymbol{\nabla}_x L(z) + \frac{1}{2} \bar{d}_x' (\boldsymbol{\nabla}_{xx} L(z) + \epsilon_x(z) I) \bar{d}_x, \tag{12}$$

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[2] The method we present is sometimes referred to as primal-dual interior-point method, to distinguish it from the barrier interior-point method.

which locally approximates (6) around $(x, \nu_x)$ [3]. If $\boldsymbol{\nabla}_x G_x(x)$ is full column rank, we can choose $\epsilon_x(z)$ large enough such that the solution of (12) is well defined and unique. To show that, let us look at (12) as an optimization in its own right. Let $\bar{d}_\nu$ be the Lagrange multiplier and define the function $\bar{g}(\bar{d}_x, \bar{d}_\nu)$ which is the function $g(z, b)$ defined in (8) but now for problem (12):

$$\bar{g}(\bar{d}_x, \bar{d}_\nu) := \begin{bmatrix} \boldsymbol{\nabla}_x L(z) + (\boldsymbol{\nabla}_{xx} L(z) + \epsilon_x(z)I)\bar{d}_x + \boldsymbol{\nabla}_x G_x(x)\bar{d}_\lambda \\ G_x(x) + \boldsymbol{\nabla}_x G_x(x)'\bar{d}_x \end{bmatrix}. \tag{13}$$

So if one takes any $\epsilon_x(z) \geq 0$ large enough such that

$$\mathbf{in}\left( \begin{bmatrix} \boldsymbol{\nabla}_{xx} L(z) + \epsilon_x(z) & \boldsymbol{\nabla}_x G_x(x) \\ \boldsymbol{\nabla}_x G_x(x)' & 0 \end{bmatrix} \right) = (n_x, l_x, 0), \tag{14}$$

then we guarantee that any point $\bar{d}_x, \bar{d}_\nu$ that satisfies $\bar{g}(\bar{d}_x, \bar{d}_\nu) = \mathbf{0}$ will be a strict local minimum of (12) (see Proposition 1). Moreover, this choice of $\epsilon_x(z)$ also guarantees that (12) is a strongly convex quadratic program, which, with the fact that $\boldsymbol{\nabla}_x G_x(x)$ is full column rank, means that the solution $(\bar{d}_x, \bar{d}_\nu)$ is *unique*. Therefore, we will take the update directions $(d_x, d_\nu)$ to be the solution $(\bar{d}_x, \bar{d}_\nu)$. Moreover, with some algebra, one can show that the solution to (12) is given by

$$\begin{bmatrix} d_x \\ d_\nu \end{bmatrix} = -\begin{bmatrix} \boldsymbol{\nabla}_{xx} L(z) + \epsilon_x(z) & \boldsymbol{\nabla}_x G_x(x) \\ \boldsymbol{\nabla}_x G_x(x)' & 0 \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{\nabla}_x L(z) \\ G_x(x) \end{bmatrix}$$
$$= -(\boldsymbol{\nabla}_z g(x, b) + \mathrm{diag}([\epsilon_x(z)\mathbf{1}_{n_x}, \mathbf{0}_{l_x}]))^{-1} g(x, b).$$

Let us now address the case in which there there are inequality constraints. Constructing a quadratic program equivalent to (12) would not be enough because it would not take into account the constraint $s_x \geq 0$. To address this, let us start by relaxing the inequality constraint from (6) and including it in the cost as the barrier function $-b\mathbf{1}' \log(s_x)$ (the $\log(\cdot)$ is element wise).

$$\min_{x, s_x : G_x(x) = \mathbf{0}, F_x(x) + s_x = \mathbf{0}} f(x) - b\mathbf{1}' \log(s_x). \tag{15}$$

This is a relaxation because $-b\mathbf{1}' \log(s_x)$ only accepts $s \geq 0$ and goes to $+\infty$ if $s_x \to 0$. The optimization (15) only has equality constraints, so similar to what we did in (12), let us construct a local second order approximation of (15) around $z$:

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[3] Notice that we use the second order linearization of the Lagrangian $L(z)$ as the cost function in (12), not the one of $f(x)$. The justification is that, if $x^*$ is a local minimum of (6) with associated Lagrange multiplier $\nu^*$, then $x^*$ is also a local minimum of

$$\min_{x : G_x(x) = \mathbf{0}} f(x) + \nu_x^{*\,\prime} G_x(x).$$

Evidently, $\nu_x^*$ is not know in advance, so instead one uses the value of $\nu_x$ at the current iteration, which leads to the local approximation (12).

$$\min_{\substack{\bar{d}_x, \bar{d}_s: \\ G_x(x) + \mathbf{\nabla}_x G_x(x)' \bar{d}_x = \mathbf{0}, \\ F_x(x) + s_x + \mathbf{\nabla}_x F_x(x)' \bar{d}_x + \bar{d}_s = \mathbf{0}}} L(z) - b\mathbf{1}' \log(s_x) + \bar{d}_x' \mathbf{\nabla}_x L(z) + \bar{d}_s'(\lambda_x - b\mathbf{1} \oslash s_x)$$

$$+ \frac{1}{2}\bar{d}_x'(\mathbf{\nabla}_{xx}L(z) + \epsilon_x(z)I)\bar{d}_x + \frac{1}{2}\bar{d}_s' \operatorname{diag}(\lambda_x \oslash s_x)\bar{d}_s \quad (16)$$

where $\oslash$ designates the element wise division of $\mathbf{1}$ by $s_x$. Equation (16) is not exactly a second order approximation because instead of using as quadratic term for $\bar{d}_s$ the matrix $b \operatorname{diag}(s_x)^{-2}$ (which is the actual matrix given by second order approximation of $-b\mathbf{1}' \log(s_x + d_s)$ around $s_x$), we used the matrix $\operatorname{diag}(\lambda_x \oslash s_x)$. This is a relatively well known substitutions for interior-point methods, and is what makes it be a primal-dual interior-point method instead of a barrier interior-point method. The technical justification is that, if we were at a point such that $g(z, b) = \mathbf{0}$, the two would be equivalent as $\lambda_x \odot s_x - b\mathbf{1} = \mathbf{0}$. In practice, it has been observed that this modified linearization tends to perform better because it provides directions $d_s$ that also take into account the current value of $\lambda_x$ in the quadratic form, which helps to get a direction $d_z$ that does no violate the constraints $\lambda_x, s_x > 0$ [13, Chapter 19.3].

Because (16) is a quadratic program with linear equality constraints, just as it was the case for (12), we can use the exact same reasoning to choose $\epsilon_x(z)$. Let us define the matrices

$$J_{zz}f(z) = \begin{bmatrix} \mathbf{\nabla}_{xx}L(z) & 0 & \mathbf{\nabla}_x G_x(x) & \mathbf{\nabla}_x F_x(x) \\ 0 & \operatorname{diag}(\lambda_x \oslash s_x) & 0 & I \\ \mathbf{\nabla}_x G_x(x)' & 0 & 0 & 0 \\ \mathbf{\nabla}_x F_x(x)' & I & 0 & 0 \end{bmatrix} \quad (17)$$

and $E(z) := \operatorname{diag}(\epsilon_x(z)\mathbf{1}_{n_x}, \mathbf{0}_{m_x+l_x+m_x})$. If $\epsilon_x(z)$ is chosen large enough such that $\mathbf{in}(J_{zz} + E(z)) = (n_x + m_x, l_x + m_x, 0)$, then the solution $(\bar{d}_x, \bar{d}_s)$ of (16) and associated Lagrange multipliers $(\bar{d}_\nu, \bar{d}_\lambda)$ are unique. With some algebra, one could show that the solution of (16) is

$$d_z = -(J_{zz}f(z) + E(z))^{-1}S^{-1}g(z, b)$$
$$= -(\mathbf{\nabla}_z g(z, b)' + E(z))^{-1}g(z, b)$$

where $S := \operatorname{diag}(\mathbf{1}_{n_x}, s_x, \mathbf{1}_{l_x+m_x})$. Putting it all together, the modified interior-point is governed by the equation

$$z^+ = z + \alpha d_z = z - \alpha(\mathbf{\nabla}_z g(z, b)' + E(z))^{-1}g(z, b). \quad (18)$$

Conveniently, because we used $\operatorname{diag}(\lambda_x \oslash s_x)$ for the second order linearization of the barrier, when $\epsilon_x(x) = 0$, we recover the basic interior-point method from (11). We refer to [13, Chapter 19] for a complete description of an algorithm using (18), including a strategy to decrease the barrier parameter $b$. Alternatively, we describe such strategy in Section 4 for the minmax optimization case.

We can now state a result connecting the stability/instability of any equilibrium point of the modified interior-point method to such point being or not a local minimum. The theorem says essentially the same thing as Theorem 1: On the one hand, even if $\mathbf{in}(J_{zz}f(z)) = (n_x + m_x, l_x + m_x, 0)$, taking $\epsilon_x(z) > 0$ will not impair the algorithm's capacity to converge towards a local minimum; this can be useful, for instance, if $\mathbf{in}(J_{zz}f(z))$ has an eigenvalue close to 0. On the other hand, using the modified interior-point method essentially guarantees that the algorithm can only converge towards and equilibrium point if such point is a local minimum, thus fixing the issue of interior-point methods being attracted to any equilibrium point, regardless of whether such point is a local minimum.

**Theorem 2** (Stability of modified interior-point method for constrained minimization) *Let $\alpha = 1$ and $(z, b)$ with $b > 0$, be an equilibrium point in the sense that $g(z, b) = \mathbf{0}$. Assume LICQ and strict complementarity hold at $z$, that $J_{zz}f(z)$ is invertible, and that $J_{zz}f(\cdot)$ and $\epsilon_x(\cdot)$ are differentiable on a neighborhood around $z$. Let $\epsilon_x(z)$ be such that $\mathbf{in}(J_{zz} + E(z)) = (n_x + m_x, l_x + m_x, 0)$. Then if:*

*i) $z$ is a local minimum, then it is a LAS equilibrium point of* (18).
*ii) $z$ is not a local minimum, then it is an unstable equilibrium point of* (18).

*Proof sketch* First, using the same arguments as in the proof of Theorem 1, we conclude that the Jacobian of the dynamic system (18) around a point $z$ for which $g(z, b) = \mathbf{0}$ is

$$I - \alpha \Big( J_{zz}f(z) + E(z) \Big)^{-1} S^{-1} \mathbf{\nabla}_z g(z, b)' = I - \alpha \Big( J_{zz}f(z) + E(z) \Big)^{-1} J_{zz}f(z) \tag{19}$$

Second, it is straightforward to check that $H_{zz}f(z) = S^{1/2}J_{zz}f(z)S^{1/2}$ which, using Sylvester's law of inertia [36, Theorem 1.5], means that $\mathbf{in}(H_{zz}f(z)) = \mathbf{in}(J_{zz}f(z))$. This means that one can check the second order conditions in (10) by using $J_{zz}f(z)$.

Let us define the matrice

$$R(\mu) = Z_x(z)' \begin{bmatrix} \mathbf{\nabla}_{xx}L(z) + \mu\epsilon_x(z)I & 0 \\ 0 & \mathrm{diag}(\lambda_x \oslash s_x) \end{bmatrix} Z_x(z)$$

where $Z_x(z) \in \mathbb{R}^{n_x + m_x, n_x - l_x}$ is a matrix with full column rank such that

$$\begin{bmatrix} \mathbf{\nabla}_x G_x(x)' & \mathbf{0} \\ \mathbf{\nabla}_x F_x(x)' & I \end{bmatrix} Z_x(z) = \mathbf{0}. \tag{20}$$

Using the same arguments as in the proof of Proposition 1, we conclude that

$$\mathbf{in}(J_{zz}f(z) + E(z)) = \mathbf{in}(R(\mu)) + (l_x + m_x, l_x + m_x),$$

which implies that $\mathbf{in}(J_{zz}f(z)+E(z)) = (n_x+m_x, l_x+m_x)$ is equivalent to $R(1) \succ 0$ and that the second order sufficient condition is equivalent to $R(0) \succ 0$. This means that the rest of the theorem's proof is analogous to the one of Theorem 1, but instead of looking at the sign of the smallest eigenvalue of $\mathbf{\nabla}_{xx}f(x) + \mu\epsilon_x(z)I$, one looks at the sign of the smallest eigenvalue of the matrix $R(\mu)$.

If $z$ is a local minimum, then $\lambda_{min}(R(0)) > 0$. As $\epsilon_x(z) \geq 0$, we conclude that $\lambda_{min}(R(\mu)) > 0$ for every $\mu \geq 0$ and therefore $z$ is a LAS equilibrium point of (11).

Conversely, if $z$ is not a local minimum, $\lambda_{min}(R(0)) < 0$. By construction, $\epsilon_x(z)$ is such that $\lambda_{min}(R(1)) > 0$, therefore, by continuity of the eigenvalue, there is a $\mu \in (0,1)$ such that $\lambda_{min}(R(\mu)) = 0$ and therefore $z$ is an unstable equilibrium point of (11)

$\square$

# 3 Minmax optimization

Consider the minmax optimization problem

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}(x)} f(x,y) \tag{21}$$

where $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}$ is a twice continuously differentiable objective function, $\mathcal{X} \subset \mathbb{R}^{n_x}$ is the feasible set for $x$ and $\mathcal{Y} : \mathcal{X} \rightrightarrows \mathbb{R}^{n_y}$ is a set-valued map that defines an $x$ dependent feasible set for $y$; we do not make any convexity or concavity assumption on $f(\cdot)$, $\mathcal{X}$ and $\mathcal{Y}(\cdot)$. A solution $(x^*, y^*)$ to (21) is called a global minmax and satisfies

$$f(x^*, y) \le f(x^*, y^*) \le \max_{\tilde{y} \in \mathcal{Y}(x)} f(x, \tilde{y}) \qquad \forall (x,y) \in \mathcal{X} \times \mathcal{Y}(x^*).$$

A point $(x^*, y^*)$ is said to be a local minmax of (21) if there exist a constant $\delta_0 > 0$ and a positive function $h(\cdot)$ satisfying $h(\delta) \to 0$ as $\delta \to 0$, such that for every $\delta \in (0, \delta_0]$ and for every $(x,y) \in \{x \in \mathcal{X} : \|x - x^*\| \le \delta\} \times \{y \in \mathcal{Y}(x^*) : \|y - y^*\| \le h(\delta)\}$ we have

$$f(x^*, y) \le f(x^*, y^*) \le \max_{\tilde{y} \in \mathcal{Y}(x) : \|\tilde{y} - y^*\| \le h(\delta)} f(x, \tilde{y})$$

[11, 12]. Inspired by the properties of the modified Newton and interior-point methods for minimization in Section 2, we want to develop a Newton-type iterative algorithm of the form

$$\begin{bmatrix} x^+ \\ y^+ \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d_x \\ d_y \end{bmatrix}. \tag{22}$$

where $d_x$ and $d_y$ satisfy the following properties:

P1: At each time step, $(d_x, d_y)$ is obtained from the solution of a quadratic program that locally approximates (21) and therefore $(x^+, y^+)$ can be seen as an improvement over $(x,y)$ .

P2: The set of locally asymptotic stable equilibrium points of (22) coincides with the set of local minmax of (23).

## 3.1 Unconstrained minmax

We start by considering the case where $\mathcal{X} = \mathbb{R}^{n_x}$ and $\mathcal{Y}(\cdot) = \mathbb{R}^{n_y}$ such that (21) simplifies to

$$\min_{x \in \mathbb{R}^{n_x}} \max_{y \in \mathbb{R}^{n_y}} f(x, y). \tag{23}$$

For this case, [11] establishes second order sufficient conditions to determine if a point $(x, y)$ is a local minmax which can be stated in terms of the inertia of the matrix

$$\boldsymbol{\nabla}_{zz} f(x, y) := \begin{bmatrix} \boldsymbol{\nabla}_{xx} f(x, y) & \boldsymbol{\nabla}_{xy} f(x, y) \\ \boldsymbol{\nabla}_{yx} f(x, y) & \boldsymbol{\nabla}_{yy} f(x, y) \end{bmatrix}.$$

We recall that the inertia $\mathbf{in}(A)$ of a symmetric matrix $A$ is a 3-tuple with the number of positive, negative and zero eigenvalues of $A$.

**Proposition 2** (Unconstrained second order sufficient condition) *Let $(x, y)$ be an equilibrium point in the sense that $\boldsymbol{\nabla}_x f(x, y) = 0$ and $\boldsymbol{\nabla}_y f(x, y) = 0$. If*

$$\boldsymbol{in}(\boldsymbol{\nabla}_{yy} f(x, y)) = (0, n_y, 0) \ \text{ and } \ \boldsymbol{in}(\boldsymbol{\nabla}_{zz} f(x, y)) = (n_x, n_y, 0) \tag{24}$$

*then $(x, y)$ is a local minmax.*

The second order conditions in [11] are:

$$\mathbf{in}(\boldsymbol{\nabla}_{yy} f(x, y)) = (0, n_y, 0) \text{ and}$$
$$\mathbf{in}(\boldsymbol{\nabla}_{xx} f(x, y) - \boldsymbol{\nabla}_{xy} f(x, y) \boldsymbol{\nabla}_{yy} f(x, y)^{-1} \boldsymbol{\nabla}_{yx} f(x, y)) = (n_x, 0, 0),$$

which turn out to be equivalent to the inertia conditions in Proposition 2 in view of Haynsworth inertia additivity formula [36, Theorem 1.6].

For the property P1 the Newton direction $(d_x, d_y)$ for (22) could be obtained by solving the following local quadratic approximation to (23)

$$\min_{\bar{d}_x} \max_{\bar{d}_y} f(x, y) + \boldsymbol{\nabla}_x f(x, y)' \bar{d}_x + \boldsymbol{\nabla}_y f(x, y)' \bar{d}_y + \bar{d}_x' \boldsymbol{\nabla}_{xy} f(x, y) \bar{d}_y$$

$$+ \frac{1}{2} \bar{d}_x' \Big( \boldsymbol{\nabla}_{xx} f(x, y) + \epsilon_x(x, y) I \Big) \bar{d}_x + \frac{1}{2} \bar{d}_y' \Big( \boldsymbol{\nabla}_{yy} f(x, y) - \epsilon_y(x, y) I \Big) \bar{d}_y \tag{25}$$

with $\epsilon_x(\cdot)$ and $\epsilon_y(\cdot)$ chosen so that the minmax problem in (25) has a unique solution, which means that the inner (quadratic) maximization must be strictly concave and that the outer (quadratic) minimization of the maximized function must be strictly convex, which turns out to be precisely the second order sufficient conditions in Proposition 2, applied to the approximation in (25), which can be explicitly written as follows:

$$\begin{aligned} \mathbf{in}\left(\boldsymbol{\nabla}_{yy} f(x, y) - \epsilon_y(x, y) I\right) &= (0, n_y, 0) \text{ and} \\ \mathbf{in}\left(\boldsymbol{\nabla}_{zz} f(x, y) + E(x, y)\right) &= (n_x, n_y, 0) \end{aligned} \tag{LQAC}$$

where $E(x,y) = \text{diag}(\epsilon_x(x,y)\mathbf{1}_{n_x}, -\epsilon_y(x,y)\mathbf{1}_{n_y})$. We call these condition the Local Quadratic Approximation Condition (LQAC). It is straightforward to show that the Newton iterations (22) with $(d_x, d_y)$ obtained from the solution to (25) is given by

$$\begin{bmatrix} x^+ \\ y^+ \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d_x \\ d_y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} - \left( \boldsymbol{\nabla}_{zz} f(x,y) + E(x,y) \right)^{-1} \begin{bmatrix} \boldsymbol{\nabla}_x f(x,y) \\ \boldsymbol{\nabla}_y f(x,y) \end{bmatrix}. \quad (26)$$

For the property P2, we need all locally asymptotically stable equilibrium points of (25) to be local minmax of (23). For the minimization in Section 2, simply selecting $\epsilon_x(\cdot)$ such that the local quadratic approximation (2) has a well defined minimum suffices to guarantee that the only equilibrium points that are LAS for the Newton iterations (3) are strict local minima (Theorem 1). However, now the (LQAC) does not suffice to guarantee that P2 holds, as the two counter examples bellow show.

*Example 1* Consider $f(x,y) = -1.5x^2 - 4xy + y^2$ for which the unique equilibrium point $x = y = 0$ is not a local minmax point. Take $\epsilon_y(0,0) = 4$ and $\epsilon_x(0,0) = 0$ which satisfy (LQAC). The Jacobian of the dynamics is

$$I - \left( \begin{bmatrix} 3 & -4 \\ -4 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 3 & -4 \\ -4 & 2 \end{bmatrix} \approx \begin{bmatrix} 0 & 0.72 \\ 0 & 0.54 \end{bmatrix}$$

which has eigenvalues approximately equal to $(0, 0.54)$. Therefore $(0,0)$ is a LAS equilibrium point of (26) even though it is not a local minmax point.

*Example 2* Consider $f(x,y) := -0.25x^2 + xy - 0.5y^2$, for which the unique equilibrium point $x = y = 0$ is a local minmax point. Take $\epsilon_y(0,0) = 3$ and $\epsilon_x(0,0) = 0.2$ which satisfy (LQAC). The Jacobian of the dynamics is

$$I - \left( \begin{bmatrix} -0.5 & 1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 0.2 & 0 \\ 0 & -3 \end{bmatrix} \right)^{-1} \begin{bmatrix} -0.5 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -4 & 15 \\ -1 & 4.5 \end{bmatrix},$$

for which the eigenvalues are 2 and $-1.5$. Therefore $(0,0)$ is an unstable equilibrium point of (26) even though it is a local minmax point.

The main contribution of this section is a set of sufficient conditions that, in addition to (LQAC), guarantee the P2 holds.

**Theorem 3** (Stability of modified Newton method for unconstrained minmax) *Let $(x,y)$ be an equilibrium point in the sense that $\boldsymbol{\nabla}_x f(x,y) = \mathbf{0}$, $\boldsymbol{\nabla}_y f(x,y) = \mathbf{0}$. Assume that $\boldsymbol{\nabla}_{zz} f(x,y)$ and $\boldsymbol{\nabla}_{yy} f(x,y)$ are invertible and that $\epsilon_x(\cdot)$, $\epsilon_y(\cdot)$, and $\boldsymbol{\nabla}_{zz} f(\cdot)$ are differentiable on a neighborhood around $(x,y)$. Let $\epsilon_x(x,y)$ and $\epsilon_y(x,y)$ be non-negative and such that the (LQAC) hold. There exist $\bar{\epsilon}$ that depends only on $\boldsymbol{\nabla}_{zz} f(x,y)$ and $\epsilon_y(x,y)$ such that if $\epsilon_x(x,y) \geq \bar{\epsilon}$, then it is guaranteed that if:*

*i)* $(x,y)$ *is a local minmax, then it is a LAS equilibrium point of (26).*
*ii)* $(x,y)$ *is not a local minmax, then it is an unstable equilibrium point of (26).*

The theorem's first implication is that $\epsilon_x(x, y)$ needs to be large enough in order to guarantee that the modified Newton method *can* converge towards a local minmax; equation (29) in the proof determines how much "large enough" is. If this is satisfied, then if the modified Newton method starts sufficiently close to a local minmax, it will converge asymptotically fast to it. A particular consequence of (29), is that if $\epsilon_y(x, y) = 0$, then stability will not be impaired by any $\epsilon_x(x, y) \geq 0$. Of course, in general, if a point $(x, y)$ is such that $\mathbf{in}(\nabla_{yy}f(x, y)) = (0, n_y, 0)$ and $\mathbf{in}(\nabla_{zz}f(x, y)) = (n_x, n_y, 0)$, a natural choice is to have $\epsilon_y(x, y) = \epsilon_x(x, y) = 0$. This not only guarantees stability, but also that if $(x, y)$ is close enough to a local minmax, the Newton method will converge superlinearly towards the solution. However, there are situations in which it might be preferable to take $\epsilon_y(x, y) \neq 0$, for instance if $\nabla_{zz}f(x, y)$ has eigenvalues too close to 0. In this case, one should be sure that $\epsilon_x(x, y)$ satisfies (29), otherwise the Newton method might no longer be able to converge towards a local minmax.

The theorem's second implication is that $\epsilon_x(x, y)$ needs to be large enough in order to guarantee that the modified Newton's method *cannot* converge towards an equilibrium point that is not a local minmax. Without this guarantee, the Newton method is attracted towards any equilibrium point, regardless of whether it is a local minmax. The proof also establishes that the (LQAC) is not sufficient to guarantee the desired instability property whenever $\mathbf{in}(\nabla_{yy}f(x, y)) \neq (0, n_y, 0)$ but $\mathbf{in}(\nabla_{zz}f(x, y)) = (n_x, n_y, 0)$ (the reader can verify that this is what happens in Example 1). Intuitively, what happens is that because $\mathbf{in}(\nabla_{zz}f(x, y)) = \mathbf{in}(\nabla_{zz}f(x, y) + E(x, y))$, it is more challenging to generate instability. This is why in this case $\epsilon_x(x, y)$ needs to be higher in order to guarantee instability. Notice that this analysis implies that whenever $f(x, \cdot)$ is strongly concave $\forall x \in \mathbb{R}^{n_x}$, the (LQAC) are sufficient to guarantee instability.

*Proof of Theorem 3* The fact that the (LQAC) can always be satisfied is straightforward: as $\nabla_{zz}f(x, y)$ is differentiable, its eigenvalues are bounded and can be made have the desired inertia by taking sufficiently large (but finite) values of $\epsilon_x(x, y)$ and $\epsilon_y(x, y)$. Moreover, from our assumption that $\nabla_{zz}f(x, y)$ and $\nabla_{yy}f(x, y)$ are invertible, $(x, y)$ is a local minmax point if and only if $(x, y)$ satisfy the second order sufficient in (24); this is implied by the second order necessary conditions for local minmax in [11].

Using the same reasoning as in Theorem 1, as the (LQAC) hold then $(\nabla_{zz}f(x, y) + E(x, y))$ is nonsingular and the Jacobian of the dynamical system (26) at $(x, y)$ is

$$I - (\nabla_{zz}f(x, y) + E(x, y))^{-1}\nabla_{zz}f(x, y). \tag{27}$$

Therefore, we can also use the same reasoning as in the proof of Theorem 1 to conclude that $(x, y)$ is a LAS equilibrium point of (26) if $\nabla_{zz}f(x, y) + \mu E(x, y)$ is nonsingular $\forall \mu \in [0, 2]$. Conversely, $(x, y)$ is an unstable equilibrium point of (26) if $\nabla_{zz}f(x, y) + \mu E(x, y)$ is singular for some $\mu \in (0, 2)$.

For the rest of the proof, it will be useful to have defined the function

$$R(\mu) = \nabla_{xx}f(x, y) - \nabla_{xy}f(x, y)(\nabla_{yy}f(x, y) - \mu\epsilon_y(x, y)I)^{-1}\nabla_{yx}f(x, y) + \mu\epsilon_x I \tag{28}$$

and to drop the inputs $(x, y)$ from the expressions in order to shorten them.

Let us start by proving the statement for the case when $(x, y)$ is a local minmax, in which case the (LQAC) hold with $\epsilon_y = \epsilon_x = 0$. We will prove that if

$$\epsilon_x \geq \bar{\epsilon} := \lambda_{min}(\epsilon_y \nabla_{xy} f \nabla_{yy} f^{-2} \nabla_{yx} f). \tag{29}$$

then $(x, y)$ is a LAS equilibrium point of (26). To prove it, we will show (29) ensures that $\nabla_{zz} f + \mu E$ is nonsingular $\forall \, \mu \geq 0$. First, as $\nabla_{yy} f \prec 0$, $\mu \geq 0$, and $\epsilon_y \geq 0$, we have $\nabla_{yy} f - \mu \epsilon_y I \prec 0$ and is thus nonsingular. Second, let us show that the condition (29) implies that for any vector $v$

$$\min_{\mu \in [0,2]} v' R(\mu) v = v' R(0) v. \tag{30}$$

Taking the derivative of $v' R(\mu) v$ with respect to $\mu$ we obtain

$$v' \Big( \epsilon_x I - \epsilon_y \nabla_{xy} f (\nabla_{yy} f - \mu \epsilon_y I)^{-2} \nabla_{yx} f \Big) v \succ v' \Big( \epsilon_x I - \epsilon_y \nabla_{xy} f \nabla_{yy} f^{-2} \nabla_{yx} f \Big) v$$

in which we use the the fact that $\nabla_{yy} f^{-2} \succeq (\nabla_{yy} f - \mu \epsilon_y I)^{-2}$ for all $\mu \geq 0$ as $\nabla_{yy} f \prec 0$, and $\epsilon_y \geq 0$. Therefore, if (29) holds, the derivative of $v' R(\mu) v$ with respect to $\mu$ is non-negative, thus the cost does not decrease with $\mu$, which implies that the minimum is obtained for $\mu = 0$, which proves (30). Therefore if $\epsilon_x$ and $\epsilon_y$ are chosen to satisfy (29), then $\forall \mu \in [0, 2]$ it holds that $R(\mu) \succeq R(0) \succ 0I$, where the second inequality comes from the second order sufficient conditions for unconstrained minmax (24). As neither $\nabla_{yy} f - \mu \epsilon_y I \prec 0$ nor $R(\mu)$ are singular for $\mu \in [0, 2]$, Haynsworth inertia additivity formula [36, Theorem 1.6] implies that $\nabla_{zz} f + \mu E$ is nonsingular $\forall \mu \in [0, 2]$, and therefore $(x, y)$ is a LAS equilibrium point of (26).

Now the second part, let us prove the statement for the case in which $(x, y)$ is not a local minmax. We will be looking for an $\bar{\epsilon}$ such that if $\epsilon_x > \bar{\epsilon}$ then

$$\nabla_{zz} f + \mu \, \text{diag}(\epsilon_x \mathbf{1}_{n_x}, -\epsilon_y \mathbf{1}_{n_y}) = \nabla_{zz} f + \mu E \tag{31}$$

is singular for some $\mu \in (0, 1)$, which in turn guarantees that $(x, y)$ is an unstable equilibrium point of (26) (see discussion in the beginning of the proof).

If $\mathbf{in}(\nabla_{zz} f) \neq (n_x, n_y, 0)$, then $\bar{\epsilon}$ is equal to any value of $\epsilon_x$ such that (LQAC) hold. The proof is straightforward: If $\mathbf{in}(\nabla_{zz} f) \neq (n_x, n_y, 0)$ and $\mathbf{in}(\nabla_{zz} f + E) = (n_x, n_y, 0)$ (from the (LQAC)), then, by continuity of the eigenvalue $\exists \mu \in (0, 1)$ such that $\nabla_{zz} f + \mu E$ is singular.

If $\mathbf{in}(\nabla_{zz} f) = (n_x, n_y, 0)$ but $\mathbf{in}(\nabla_{yy} f) \neq (0, n_y, 0)$, then the value of $\bar{\epsilon}$ from the paragraph above is not enough to guarantee that $(x, y)$ is an unstable equilibrium point. However, it is possible to guarantee instability. The proof is the following.

Let $\mu^*$ be the largest $\mu \in (0, 1)$ such that $\nabla_{yy} f - \mu \epsilon_y I$ is singular. We know that this point exists because, on the one hand, by assumption $\nabla_{yy} f$ is invertible (and therefore $\mu^* > 0$), and on the other hand, we know that $\nabla_{yy} f \not\prec 0$ and that $\nabla_{yy} f - \epsilon_y I \prec 0$ by construction (and therefore $\mu^* < 1$).

Now take any $\bar{\mu} \in (0, \mu^*)$ such that $\nabla_{yy} f - \bar{\mu} \epsilon_y I$ is invertible (there are uncountably many). Suppose there exists $\bar{\epsilon}$ such that for any $\epsilon_x \geq \bar{\epsilon}$, the (LQAC) hold and $\mathbf{in}(\nabla_{zz} f + \bar{\mu} E) \neq (n_x, n_y, 0)$. If such $\bar{\epsilon}$ exists, then, by the continuity of the eigenvalues, if $\mathbf{in}(\nabla_{zz} f + \bar{\mu} E) \neq (n_x, n_y, 0)$ this means that $\nabla_{zz} f + \mu E$ is singular for some $\mu \in (0, \bar{\mu}]$.

So, to conclude the proof, we just need to show the existance of such $\bar{\epsilon}$. Take any $\epsilon_x$ such that $\mathbf{in}(\nabla_{zz} f + \bar{\mu} E) = (n_x, n_y, 0)$ (otherwise the proof is tautological). From Haynsworth inertia additivity formula, we have that

$$\mathbf{in}(\nabla_{zz} f + \bar{\mu} E) = \mathbf{in}(R(\bar{\mu})) + \mathbf{in}(\nabla_{yy} f - \bar{\mu} \epsilon_y I)$$

with $\mathbf{in}(R(\bar{\mu})) = (n_x - k, k, 0)$ and $\mathbf{in}(\boldsymbol{\nabla}_{yy}f - \bar{\mu}\epsilon_y I) = (k, n_y - k, 0)$ for some $k \in \{1, \ldots, \min(n_x, n_y)\}$. On the one hand, it is straightforward to establish that $\exists \bar{\epsilon}_1$ such that if $\epsilon_x \geq \bar{\epsilon}_1$, then $\mathbf{in}(R(\bar{\mu})) \neq (n_x - k, k, 0)$, which means that $\mathbf{in}(\boldsymbol{\nabla}_{zz}f + \bar{\mu}E) \neq (n_x, n_y, 0)$. On the other hand, $\exists \bar{\epsilon}_2$ such that if $\epsilon_x \geq \bar{\epsilon}_2$, then $\mathbf{in}(\boldsymbol{\nabla}_{zz}f + \mu E) = (n_x, n_y, 0)$. Therefore, we can define $\bar{\epsilon} = \max(\bar{\epsilon}_1, \bar{\epsilon}_2)$, which concludes the proof

$\square$

## 3.2 Constrained minmax

We now consider the case with more general constraint sets involving equality and inequality constraints of the form

$$
\begin{aligned}
\mathcal{X} &= \{x \in \mathbb{R}^{n_x} : G_x(x) = \mathbf{0}, F_x(x) \leq \mathbf{0}\} \quad \text{and} \\
\mathcal{Y}(x) &= \{y \in \mathbb{R}^{n_y} : G_y(x, y) = \mathbf{0}, F_y(x, y) \leq \mathbf{0}\}
\end{aligned}
\tag{32}
$$

where the functions $G_x : \mathbb{R}^{n_x} \to \mathbb{R}^{l_x}$, $F_x : \mathbb{R}^{n_x} \to \mathbb{R}^{m_x}$, $G_y : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{l_y}$ and $F_y : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{m_y}$ are all twice continuously differentiable. Similar to what we did in Section 2.2, it will be convenient for the development of the interior-point method to use slack variables and rewrite the constrained minmax (21) as

$$
\min_{x, s_x : G_x(x) = \mathbf{0}, F_x(x) + s_x = \mathbf{0}, s_x \geq \mathbf{0}} \quad \max_{y, s_y : G_y(x, y) = \mathbf{0}, F_y(x, y) + s_y = \mathbf{0}, s_y \geq \mathbf{0}} f(x, y).
\tag{33}
$$

where $s_x \in \mathbb{R}^{m_x}$ and $s_y \in \mathbb{R}^{m_y}$.

Similar to what we have done in the unconstrained case, we want to present second order conditions to determine if a point is a constrained local minmax. In order to do so, we need to extend some fundamental concepts of constrained minimization to constrained minmax optimization. The function

$$
L(z) := f(x, y) + \nu_x' G_x(x) + \lambda_x'(F_x(x) + s_x) + \nu_y' G_y(x, y) - \lambda_y'(F_y(x, y) + s_y),
$$

will play an equivalent role as the Lagrangian with $(\nu_x, \nu_y, \lambda_x, \lambda_y)$ as the equivalent of Lagrange multipliers; we use the shorthand notation $z = (x, s_x, y, s_y, \nu_y, \lambda_y, \nu_x, \lambda_x)$. Furthermore, the definition of linear independence constraint qualifications (LICQ) and the strict complementarity for minmax optimization are:

**Definition 2** (LICQ and strict complementarity for minmax) Let the sets of active inequality constraints for the minimization and maximization be defined, respectively, by

$$
\begin{aligned}
\mathcal{A}_x(x) &= \{i : F_x^{(i)}(x) = 0, i = 1, \ldots, m_x\} \text{ and} \\
\mathcal{A}_y(x, y) &= \{i : F_y^{(i)}(x, y) = 0, i = 1, \ldots, m_y\}
\end{aligned}
\tag{34}
$$

where $F_y^{(i)}(x, y)$ and $F_x^{(i)}(x)$ denote the i$^{\text{th}}$ element of $F_y(x, y)$ and $F_x(x)$. Then:

- The linear independence constraint qualification (LICQ) is said to hold at $z$ if the vectors in the sets

$$\{\boldsymbol{\nabla}_x G_x^{(i)}(x), i = 1, \ldots, l_x\} \bigcup \{\boldsymbol{\nabla}_x F_x^{(i)}(x), i \in \mathcal{A}_x(x)\} \text{ and}$$

$$\{\boldsymbol{\nabla}_y G_y^{(i)}(x,y), i = 1, \ldots, l_y\} \bigcup \{\boldsymbol{\nabla}_y F_y^{(i)}(x,y), i \in \mathcal{A}_y(x,y)\}$$

  are linearly independent.
- Strict complementarity is said to hold at $z$ if $\lambda_y^{(i)} > 0 \ \forall i \in \mathcal{A}_y(x,y)$ and $\lambda_x^{(i)} > 0 \ \forall i \in \mathcal{A}_x(x)$

We have almost all the ingredients to present the second order conditions for constrained minimization. For the unconstrained minmax optimization, the second order condition in Proposition 2 required that gradients ($\boldsymbol{\nabla}_x f(x,y)$ and $\boldsymbol{\nabla}_y f(x,y)$) were equal to zero and that Hessians ($\boldsymbol{\nabla}_{zz} f(x,y)$ and $\boldsymbol{\nabla}_{yy} f(x,y)$) had a particular inertia. Analogously to what was the case for the constrained minimization in Section 2.2, if it were not for the inequality constraints in (32), we would be able to state the second order conditions using gradients and Hessians of $L(z)$. The inequality constraints make the statement a bit more complicated. The role of the gradient will be played by

$$g(z,b) := \begin{bmatrix} \boldsymbol{\nabla}_x L(z) \\ \lambda_x \odot s_x - b\mathbf{1} \\ \boldsymbol{\nabla}_y L(z) \\ -\lambda_y \odot s_y + b\mathbf{1} \\ G_y(x,y) \\ -F_y(x,y) - s_y \\ G_x(x) \\ F_x(x) + s_x \end{bmatrix}$$

where $\odot$ denotes the element wise Hadamard product of two vectors and $b \geq 0$ the barrier parameter, which is the extension to minmax of the function $g(\cdot)$ defined in (8) for the minimization. The role of $\boldsymbol{\nabla}_{yy} f(x,y)$ will be played by

$$H_{yy} f(z) = \begin{bmatrix} \boldsymbol{\nabla}_{yy} L(z) & \mathbf{0} & \boldsymbol{\nabla}_y G_y(x,y) & -\boldsymbol{\nabla}_y F_y(x,y) \\ \mathbf{0} & -\operatorname{diag}(\lambda_y) & \mathbf{0} & -\operatorname{diag}(s_y^{1/2}) \\ \boldsymbol{\nabla}_y G_y(x,y)' & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\boldsymbol{\nabla}_y F_y(x,y)' & -\operatorname{diag}(s_y^{1/2}) & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (35a)$$

while the role of $\boldsymbol{\nabla}_{zz} f(x,y)$ will be played by

$$H_{zz} f(z) = \begin{bmatrix} H_{xx} f(z) & H_{xy} f(z) & H_{x\lambda} f(z) \\ H_{xy} f(z)' & H_{yy} f(z) & \mathbf{0} \\ H_{x\lambda} f(z)' & \mathbf{0} & \mathbf{0} \end{bmatrix} \qquad (35b)$$

with blocks defined by

$$
H_{xy}f(z) = \begin{bmatrix} \boldsymbol{\nabla}_{xy}L(z) & \mathbf{0} & \boldsymbol{\nabla}_x G_y(x,y) & -\boldsymbol{\nabla}_x F_y(x,y) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}
$$

$$
H_{xx}f(z) = \begin{bmatrix} \boldsymbol{\nabla}_{xx}L(z) & \mathbf{0} \\ \mathbf{0} & \operatorname{diag}(\lambda_x) \end{bmatrix} \quad H_{x\lambda}f(z) = \begin{bmatrix} \boldsymbol{\nabla}_x G_x(x) & \boldsymbol{\nabla}_x F_x(x) \\ \mathbf{0} & \operatorname{diag}(s_x^{1/2}) \end{bmatrix} \tag{35c}
$$

**Proposition 3** (Constrained second order sufficient conditions) *Let $z$ be an equilibrium point in the sense that $g(z,0) = \mathbf{0}$ with $\lambda_y, \lambda_x, s_y, s_x \geq \mathbf{0}$. If the LICQ and strict complementarity hold at $z$ and*

$$
\begin{aligned}
\boldsymbol{in}(H_{yy}f(z)) &= (l_y + m_y, n_y + m_y, 0) \text{ and} \\
\boldsymbol{in}(H_{zz}f(z)) &= (n_x + m_x + l_y + m_y, l_x + m_x + n_y + m_y, 0)
\end{aligned} \tag{36}
$$

*then $(x,y)$ is a local minmax of* (21).

The conditions for Proposition 3 are slightly stricter than the ones in [12] as we require strict complementarity and LICQ both for the max and the min. However, our conditions allow us to verify whether a point is a local minmax using the inertia, instead of having to compute solution cones. We prove that given these stricter assumptions our conditions are equivalent to those in [12] in Appendix A.

### 3.2.1 Interior-point method

Let $d_z = (d_x, d_{s_x}, d_y, d_{s_y}, d_{\nu_y}, d_{\lambda_y}, d_{\nu_x}, d_{\lambda_x})$ be a shorthand notation to designate the update direction of the variables $z = (x, s_x, y, s_y, \nu_y, \lambda_y, \nu_x, \lambda_x)$. Similar to the basic interior-point method introduced in Section 2.2, a basic interior-point method for minmax finds a candidate solution to (33) using the iterations

$$
z^+ = z + \alpha d_z = z - \alpha \boldsymbol{\nabla}_z g(z,b)'^{-1} g(z,b) \tag{37}
$$

where the barrier parameter $b$ is slowly decreased to 0, so that $z$ converges to a root of $g(z,0) = \mathbf{0}$ while $\alpha \in (0,1]$ is chosen at each step such that the feasibility conditions $\lambda_y, \lambda_x, s_y, s_x > 0$ hold. We want to modify this basic interior-point so it satisfies the properties P1 and P2.

For the property P1, $d_z$ needs to be obtained from the solution of a quadratic program that locally approximates (33). Using equivalent arguments as in the development of the quadratic program (16) for the constrained minimization in Section 2.2, we obtain the objective function should be

$$
\begin{aligned}
K(d_x, d_{s_x}, d_y, d_{s_y}) &= L(z) + \boldsymbol{\nabla}_x L(z)' d_x + (\lambda_x - b\mathbf{1} \oslash s_x)' d_{s_x} + \boldsymbol{\nabla}_y L(z)' d_y \\
&\quad - (\lambda_y - b\mathbf{1} \oslash s_y)' d_{s_y} + d_x' \boldsymbol{\nabla}_{xy} L(z) d_y + \frac{1}{2} d_x'(\boldsymbol{\nabla}_{xx} L(z) + \epsilon_x(z)I) d_x \\
&\quad + \frac{1}{2} d_{s_x}' \operatorname{diag}(\lambda_x \oslash s_x) d_{s_x} + \frac{1}{2} d_y'(\boldsymbol{\nabla}_{yy} L(z) - \epsilon_y(z)I) d_y - \frac{1}{2} d_{s_y}' \operatorname{diag}(\lambda_y \oslash s_y) d_{s_y},
\end{aligned}
$$

where $\epsilon_x(z) \geq 0$ and $\epsilon_y(z) \geq 0$ are scalar and $\oslash$ designates the element wise division of two vectors. The feasible sets $d\mathcal{X}$ for $(d_x, d_{s_x})$ and the set-valued map that defines a feasible set $d\mathcal{Y}(d_x)$ for $(d_y, d_{s_y})$ are obtained from the first order linearization of the functions in $\mathcal{X}$ and $\mathcal{Y}(d_y)$ and are given by

$$d\mathcal{X} = \{(d_x, d_{s_x}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{m_x} : G_x(x) + \boldsymbol{\nabla}_x G_x(x)' d_x = \mathbf{0},$$
$$F_x(x) + s_x + \boldsymbol{\nabla}_x F_x(x)' d_x + d_{s_x} = \mathbf{0}\}$$
$$d\mathcal{Y}(d_x) = \{(d_y, d_{s_y}) \in \mathbb{R}^{n_y} \times \mathbb{R}^{m_y} : G_y(x,y) + \boldsymbol{\nabla}_x G_y(x,y)' d_x + \boldsymbol{\nabla}_y G_y(x,y)' d_y$$
$$= \mathbf{0}, F_y(x,y) + s_y + \boldsymbol{\nabla}_x F_y(x,y)' d_x + \boldsymbol{\nabla}_y F_y(x,y)' d_y + d_{s_y} = \mathbf{0}\}.$$

If $\boldsymbol{\nabla}_x G_x(x)$ and $\boldsymbol{\nabla}_y G_y(x,y)$ have linearly independent columns, we propose to obtain $(d_x, d_{s_x}, d_y, d_{s_y})$ as the optimizers and $(d_{\nu_y}, d_{\lambda_y}, d_{\nu_x}, d_{\lambda_x})$ the associated Lagrange multipliers of the minmax optimization

$$\min_{\bar{d}_x, \bar{d}_{s_x} \in d\mathcal{X}} \quad \max_{\bar{d}_y, \bar{d}_{s_y} \in d\mathcal{Y}(\bar{d}_x)} K(\bar{d}_x, \bar{d}_{s_x}, \bar{d}_y, \bar{d}_{s_y}) \tag{38}$$

where $\epsilon_x(z)$ and $\epsilon_y(z)$ are chosen such that the solution to (38) is unique. We can apply to (38) the second order condition from Proposition 3 and obtain that $\epsilon_x(z)$ and $\epsilon_y(z)$ need to be chosen to satisfy

$$\mathbf{in}(J_{yy}f(z) - E_y(z)) = (l_y + m_y, n_y + m_y, 0) \text{ and}$$
$$\mathbf{in}(J_{zz}f(z) + E(z)) = (n_x + m_x + l_y + m_y, l_x + m_x + n_y + m_y, 0) \tag{ConsLQAC}$$

where $E_y(z) := \text{diag}(\epsilon_y(z)\mathbf{1}_{n_y}, \mathbf{0}_{l_y + 2m_y})$ and $E(z) := \text{diag}(\epsilon_x(z)\mathbf{1}_{n_x}, \mathbf{0}_{m_x}, -\epsilon_y(z)\mathbf{1}_{n_y}, \mathbf{0}_{l_y + 2m_y + l_x + m_x})$; $J_{zz}f(z)$ is the equivalent of the matrix defined in (35b) for the problem (38) and can be shown to be equal to

$$J_{zz}f(z) = S^{-1/2} H_{zz}f(z) S^{-1/2} = S^{-1} \boldsymbol{\nabla}_z g(z,b)'. \tag{39}$$

with $S = \text{diag}(\mathbf{1}_{n_x}, s_x, \mathbf{1}_{n_y}, s_y, \mathbf{1}_{l_y + m_y + l_x + m_x})$; $J_{yy}f(z)$ is the equivalent partition of $J_{zz}f(z)$ as $H_{yy}(z)$ is of $H_{zz}(z)$. We will call these conditions the Constrained Local Quadratic Approximation Conditions (ConsLQAC). In this case, it is straightforward to show that modifying the basic interior-point iterations in (37) by taking $d_z$ from the solution of (38) leads to the iterations

$$z^+ = z + \alpha d_z = z - \alpha (J_{zz}f(z) + E(z))^{-1} S^{-1} g(z,b). \tag{40}$$

Analogously to the unconstrained case, choosing $\epsilon_x(z)$ and $\epsilon_y(z)$ such that the (ConsLQAC) hold is not sufficient to guarantee the desired stability/instability, and we need to develop equivalent conditions as those of Theorem 3 (see Section 4 for such an example).

For the property P2, we can now state a result connecting the stability/instability of any equilibrium point of the modified interior-point method to such point being or not a local minmax. The theorem and its proof say

essentially the same thing as Theorem 3: On the one hand, as long as $\epsilon_x(x)$ is large enough, taking $\epsilon_y(z) > 0$ will not impair the algorithm's capacity to converge towards a local minmax; this can be useful, for instance, if $\mathbf{in}(J_{zz})$ has an eigenvalue close to 0. On the other hand, in order to guarantee that the modified interior-point method cannot converge towards an equilibrium point that is not local minmax, the (ConsLQAC) are sufficient only whenever $\mathbf{in}(J_{zz}f(z)) \neq (n_x + m_x + l_y + m_y, l_x + m_x + n_y + m_y, 0)$. Otherwise, additional conditions on $\epsilon_x(z)$ are require to achieve property P2.

**Theorem 4** (Stability of modified interior-point method for constrained minmax) *Let $\alpha = 1$ and $(z, b)$ with $b > 0$, be an equilibrium point in the sense that $g(z, b) = \mathbf{0}$. Assume that (ConsLQAC) and the LICQ hold at $z$, that $J_{zz}f(z)$ and $J_{yy}f(z)$ are invertible, and that $\epsilon_x(\cdot)$, $\epsilon_y(\cdot)$ and $J_{zz}f(\cdot)$ are differentiable on a neighborhood around $z$. Let $\epsilon_x(z)$ and $\epsilon_y(z)$ be non-negative and such that the (ConsLQAC) hold. There exist $\bar{\epsilon}$ that depends only on $J_{zz}f(z)$ and $\epsilon_y(z)$ such that if $\epsilon_x(z) \geq \bar{\epsilon}$, then it is guaranteed that if:*

*i) $z$ is a local minmax, then it is a LAS equilibrium point of (40).*
*ii) $z$ is not a local minmax, then it is an unstable equilibrium point of (40).*

*Proof* Let us define the partitions, $J_{xx}f(z)$, $J_{yx}f(z)$, and $J_{x\lambda}f(z)$ of $J_{zz}f(z)$ analogously to the partitions $H_{xx}f(z)$, $H_{yx}f(z)$, and $H_{x\lambda}f(z)$ of $H_{zz}f(z)$.

Using the same arguments as in the proof of Theorem 1, we conclude that the Jacobian of the dynamic system (40) around a point $z$ such that $g(z, b) = \mathbf{0}$ is

$$I - \alpha\Big(J_{zz}f(z) + E(z)\Big)^{-1}S^{-1}\boldsymbol{\nabla}_z g(z, b)' = I - \alpha\Big(J_{zz}f(z) + E(z)\Big)^{-1}J_{zz}f(z) \quad (41)$$

Moreover from (39) we have that $\mathbf{in}(H_{zz}f(z)) = \mathbf{in}(S^{1/2}J_{zz}f(z)S^{1/2})$. Using Sylvester's law of inertia [36, Theorem 1.5], this simplifies to $\mathbf{in}(H_{zz}f(z)) = \mathbf{in}(J_{zz}f(z))$. If a point $z$ is such that $g(z, b) = \mathbf{0}$, then one can check (36) using $J_{zz}f(z)$ and $J_{yy}f(z)$.

Let us define the matrices

$$R_y(\mu) = Z_y(z)' \begin{bmatrix} \boldsymbol{\nabla}_{yy}L(z) - \epsilon(z)\mu I & \mathbf{0} \\ \mathbf{0} & -\operatorname{diag}(\lambda_y \oslash s_y) \end{bmatrix} Z_y(z) \quad (42a)$$

$$R_x(\mu) = Z_x(z)' \Big( J_{xx}f(z) - J_{xy}f(z)(J_{yy}f(z) - \mu E_y(z))^{-1}J_{yx}f(x, y) + \mu E_x(z) \Big) Z_x(z) \quad (42b)$$

where $Z_y(z) \in \mathbb{R}^{n_y + m_y, n_y - l_y}$ and $Z_x(z) \in \mathbb{R}^{n_x + m_x, n_x - l_x}$ are any full column rank matrices such that

$$\begin{bmatrix} \boldsymbol{\nabla}_y G_y(x, y) & -\boldsymbol{\nabla}_y F_y(x, y) \\ -I & 0 \end{bmatrix} Z_y(z) = \mathbf{0} \quad \text{and} \quad J_{x\lambda}f(z)' Z_x(z) = \mathbf{0}. \quad (43)$$

Using the same reasoning as in the proof of Proposition 3 one can conclude that

$$\mathbf{in}(J_{yy}f(z) - \mu E_y(z)) = \mathbf{in}(R_y(\mu)) + (l_y + m_y, l_y + m_y, 0)$$

$$\mathbf{in}(J_{zz}f(z) + \mu E(z)) = \mathbf{in}(R_x(\mu)) + \mathbf{in}(J_{yy}f(z) - \mu E_y(z)) + (l_x + m_x, l_x + m_x, 0),$$

which implies that the (ConsLQAC) can be stated as

$$R_y(1) \prec \mathbf{0} \quad \text{and} \quad R_x(1) \succ \mathbf{0}.$$

This means that the exact same arguments used in the proof of the unconstrained minmax in Theorem 3 can be used for the constrained case. More specifically, each arguments with

$$\boldsymbol{\nabla}_{yy} f(x,y) - \epsilon_y(x,y)\mu I$$

and

$$\boldsymbol{\nabla}_{xx} f(x,y) - \boldsymbol{\nabla}_{xy} f(x,y)(\boldsymbol{\nabla}_{yy} f(x,y) - \mu\epsilon_y(x,y)I)^{-1}\boldsymbol{\nabla}_{yx} f(x,y) + \mu\epsilon_x(x,y)I.$$

has an analogous statement with $R_y(\mu)$ and $R_x(\mu)$, respectively. For the sake of completeness, we highlight the main points of the analogy.

First, when $z$ is such that (36) holds, the sufficient condition for $z$ to be a LAS equilibrium point of (40) is that

$$\boldsymbol{\nabla}_\mu R_x(0) = Z_x(z)' \Big( E_x(z) - J_{xy} f(z) J_{yy} f(z)^{-1} E_y(z) J_{yy} f(z)^{-1} J_{yx} f(z) \Big) Z_x(z) \succeq 0. \tag{44}$$

The only extra argument needed is to show that condition (44) is always feasible for some $\epsilon_x(z)$ large enough. This is not evident as the matrix

$$M := -J_{xy} f(z) J_{yy} f(z)^{-1} E_y(z) J_{yy} f(z)^{-1} J_{yx} f(z)$$

has size $(n_x + m_x) \times (n_x + m_x)$ while $E_x(z)$ only has $n_x$ nonzero elements in the diagonal. However, because of the structural zeros in $J_{xy} f(z)$ and $E_y(z)$, one can verify with some algebraic manipulation that $\operatorname{rank}(M) := r \leq \min(n_x, n_y)$. Let $\Lambda$ be the matrix with eigenvalues of $M$ in decreasing order and $V$ its associated eigenvectors such that $M = V\Lambda V'$. We can partition $V$ into $V_1$ of size $(r,r)$ associated to the nonzero eigenvalues of $M$ and $V_2 = I_{n_x+m_x-r}$. This partition means that $E_x(z) = V' E_x(z) V$, which means on can conclude that

$$\boldsymbol{\nabla}_\mu R_x(\mu) = Z_x(z)' V' \Big( E_x(z) + \Lambda \Big) V Z_x(z),$$

which implies that one can always take $\epsilon_x$ large enough such that for each negative diagonal entries of $\Lambda$, the equivalent diagonal element of $(E_x(z) + \Lambda)$ is positive.

Now the second part, let us prove the statement when $z$ is such that the second order conditions in (36) do not hold. On the one hand, using the same analysis as in the proof of Theorem 3, we conclude that the (ConsLQAC) are sufficient to guarantee that $z$ is an unstable equilibrium point of (40) if $\mathbf{in}(J_{zz} f(z)) \neq (n_x + m_x + l_y + m_y, l_x + m_x + n_y + m_y, 0)$. On the other hand, if $\mathbf{in}(J_{zz} f(z)) = (n_x + m_x + l_y + m_y, l_x + m_x + n_y + m_y, 0)$, than we can guarantee that by taking $\epsilon_x$ sufficiently large, there is a $\mu \in (0,1)$ such that $\mathbf{in}(J_{zz} f(z) + \mu E) \neq (n_x + m_x + l_y + m_y, l_x + m_x + n_y + m_y, 0)$, which means that $z$ is an unstable equilibrium point of (40). This concludes the proof. □

# 4 Algorithmic development and numerical examples

The following algorithm combines the result of the previous section to propose a method for selecting $\epsilon_x(z)$ and $\epsilon_y(z)$ that satisfies the (ConsLQAC) and guarantees the stability properties of Theorem 4. We only state the algorithm for the constrained case, its specialization to the unconstrained case

is straightforward. In order to keep the algorithm more simple and to highlight the instability property, we will choose $\epsilon_y(z) = \epsilon_x(z) = 0$ whenever the (ConsLQAC) can be satisfied with $\epsilon_y(z) = \epsilon_x(z) = 0$.

---

**Algorithm 1** Interior-point method for minmax

---

**Require:** An initial point $z = (x, s_x, y, s_y, \nu_y, \lambda_y, \nu_x, \lambda_x)$, an initial barrier parameter value $b$, a barrier reduction factor $\sigma \in (0, 1)$, and a stopping accuracy $\delta_s$.

1: **while** $\|g(z, b)\|_\infty > \delta_s$ **do**
2:     **while** $\|g(z, b)\|_\infty > b$ **do**
3:         **if** (ConsLQAC) can be satisfied with $\epsilon_y(z) = \epsilon_x(z) = 0$ **then**
4:             Take $\epsilon_y(z) = \epsilon_x(z) = 0$
5:         **else**
6:             Increase $\epsilon_y(z)$ until

$$\mathbf{in}(J_{yy}f(z) - E_y(z)) = (l_y + m_y, n_y + m_y, 0)$$

7:             Increase $\epsilon_x(z)$ until

$$\mathbf{in}(J_{zz}f(z) + E(z)) = (n_x + m_x + l_y + m_y, l_x + m_x + n_y + m_y, 0)$$

8:             **if** $\mathbf{in}(J_{zz}f(z)) = (n_x + m_x + l_y + m_y, l_x + m_x + n_y + m_y, 0)$ **then**
9:                 Increase $\epsilon_x(z)$ until, for some value of $\mu \in (0, 1)$,

$$\mathbf{in}(J_{zz}f(z) + \mu E(z)) \neq (n_x + m_x + l_y + m_y, l_x + m_x + n_y + m_y, 0)$$

10:             **end if**
11:         **end if**
12:         Compute a new $z$ using the equation

$$z \leftarrow z - \alpha\Big(J_{zz}f(z) + E(z)\Big)^{-1} S^{-1} g(z, b)$$

where $\alpha \in (0, 1]$ is selected such that the feasibility conditions $\lambda_y, \lambda_x, s_y, s_x > \mathbf{0}$ hold.
13:     **end while**
14:     $b \leftarrow \sigma b$
15: **end while**

---

**Proposition 4** (Convergence to feasible local minmax points) *Suppose that Algorithm 1 generates an infinite sequence of iterates $\{z_k\}$ (i.e., $\delta_s = 0$) and that $\{b_k\} \to 0$. Then all limit points $\hat{z}$ of $\{z_k\}$ are feasible. Furthermore, if the LICQ and strict complementarity condition hold at a given limit point $\hat{z}$, then $g(\hat{z}, 0) = \mathbf{0}$.*

The proof is analogous to the proof of Theorem 19.1 in [13]. The crucial point of this algorithm is that at each step, $(\epsilon_x, \epsilon_y)$ is selected using the results from Theorem 4, which guarantees that the only stable equilibrium points of (40) are local minmax points. Because a dynamical system will not, in general, converge towards an equilibrium point that is not stable, we can conclude that **any limit point $\hat{z}$ will most likely be a local minmax point**.

The key aspect of the algorithm is that it finds $(\epsilon_x, \epsilon_y)$ that satisfies the condition of Theorem 4 by only computing the inertia of matrices. The inertia of a matrix can be efficiently computed using either the LBLt or LDLt decomposition, as we further detail in the following remark.

*Remark 1* (Computing the inertia) It is not necessary to actually compute the eigenvalues of $J_{zz}f(z)$ in order to determine the inertia. A first option is to use the lower-triangular-block-lower-triangular-transpose (LBLt) decomposition [13, Appendix A], which decomposes $J_{zz}f(z)$ into the product $LBL'$ where $L$ is a lower triangular matrix and $B$ a block diagonal one, the inertia of $B$ is the same as the inertia of $J_{zz}f(z)$.

Let $\Gamma = \text{diag}(\gamma \mathbf{1}_{n_x+m_x}, -\gamma \mathbf{1}_{n_y+m_y}, \gamma \mathbf{1}_{l_y+m_y}, -\gamma \mathbf{1}_{l_x+m_x})$, with $\gamma$ a small positive number. A second approach is to use the lower-triangular-diagonal-lower-triangular-transpose (LDLt) decomposition, to decompose $J_{zz}f(z) + \Gamma$ into the product $LDL'$ where $L$ is a lower triangular matrix and $D$ is a diagonal matrix; the inertia of $D$, which is given by the number of positive, negative and zero elements of the diagonal of $D$, gives the inertia of $J_{zz}f(z)+\Gamma$. The matrix $\Gamma$ introduces a distortion in the inertia but it helps to stabilize the computation of the LDLt decomposition, which tends to be faster than the LBLt decomposition. This is the approach we use in our implementation; it has been studied in interior-point algorithms for minimization and the distortion introduced by $\Gamma$ tends to be compensated by a better numerical algorithm [37, 38]. □

## 4.1 Benchmark example for unconstrained minmax

Consider the following functions

$$f_1(x,y) = 2x^2 - y^2 + 4xy + 4/3y^3 - 1/4y^4$$
$$f_2(x,y) = (4x^2 - (y - 3x + 0.05x^3)^2 - 0.1y^4)\exp(-0.01(x^2 + y^2))$$
$$f_3(x,y) = (x - 0.5)(y - 0.5) + \exp(-(x - 0.25)^2 - (y - 0.75)^2)$$

which have been used as examples in [25, 29, 39] respectively. We have chosen these functions because, as we will show, they illustrate some interesting behaviors.

Our goal is to compare the performance of Algorithm 1 to the performance of two well established algorithms. On the one hand, we look at the performance of a "pure" Newton algorithm, *i.e.,* using $\epsilon_x(z) = \epsilon_y(z) = 0$. On the

|  | Pure Newton | | | GDA | | | Algorithm 1 | | |
|---|---|---|---|---|---|---|---|---|---|
|  | Cnvg | Cnvg mm | Iter | Cnvg | Cnvg mm | Iter | Cnvg | Cnvg mm | Iter |
| $f_1$ | 1000 | 1000 | 4.1 | 1000 | 1000 | 485 | 1000 | 1000 | 5.7 |
| $f_2$ | 1000 | 665 | 7.3 | 976 | 976 | 18195 | 996 | 996 | 8.1 |
| $f_3$ | 954 | 485 | 4.8 | 373 | 373 | 40936 | 709 | 709 | 7.1 |

**Table 1**: Comparing the performance of Pure Newton's method, Gradient Descent Ascent and Algorithm 1

.

other hand, we look into the convergence of a Gradient Descent Ascent , *i.e.,*

$$x^+ = x - \alpha_x \boldsymbol{\nabla}_x f(x, y)$$
$$y^+ = y + \alpha_y \boldsymbol{\nabla}_y f(x, y)$$

where $\alpha_x$ and $\alpha_y$ are constant and different for each problem; we did our best to select the best values $\alpha_x$ and $\alpha_y$ for each problem.

Each algorithm is initialized 1000 times, using the same initialization for the three of them each time. We compare their convergence properties according to three criteria: the number of times the algorithm converged (Cnvg), the number of times it converged to a local minmax point (Cnvg mm) and the average number of iterations to converge to a local minmax point (Iter). The algorithm is terminated when the infinity norm of the gradient is smaller than $\delta_s = 10^{-5}$ and we declare that they did not converge if it has not terminated in less than 500 iterations for the pure Newton and Algorithm 1, and 50 000 for GDA. The result of the comparison is displayed in Table 1. We can make the following observations from this comparison.

- The pure Newton algorithm has good overall convergence for the three problems, but it also tends to often converge towards an equilibrium point that is not a local minmax problems. On the other hand, the pure Newton converges to a local minmax in less iterations than the other two methods. While this is expected when comparing the GDA, it might not be clear why it is the case when compared to Algorithm 1. We believe the most likely reason is that by taking $\epsilon_x$ and $\epsilon_y$ different than 0, it requires some more iterations to converge towards a local minmax.
- The GDA algorithm seems to enjoy the property of always converging towards a local minmax, and except for $f_3(\cdot)$, it has good rate of convergence. However, GDA takes an exceptionally long number of iterations to converge. This is somehow expected from the fact that it is a first order method, and it is partially compensated by each iteration being more simple to compute. However, one must keep in mind that none of this takes into account the time that needs to be spent adjusting the step sizes until a good convergence rate can be obtained.
- At last, Algorithm 1 is across the board the algorithm with better convergence towards local minmax, and it does so in the smallest number of iterations. While this was somehow expected from the theory, the biggest
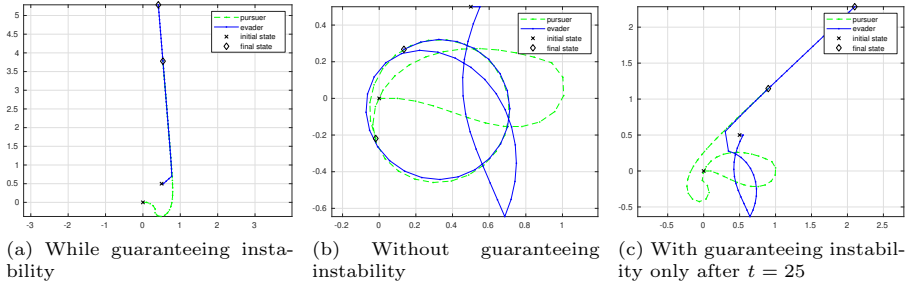
(a) While guaranteeing insta-
bility

(b) Without guaranteeing
instability

(c) With guaranteeing instabil-
ity only after $t = 25$

**Fig. 1**: Trajectory for Homicidal Chauffeur problem with and without guaranteeing instability at equilibrium points that are not a local minmax.

take away is that including $\epsilon_x(z)$ and $\epsilon_y(z)$ does not decrease too much the number of steps to achieve convergence.

## 4.2 The homicidal chauffeur example for constrained minmax

In the homicidal chauffeur problem, a pursuer driving a car is trying to hit a pedestrian, who (understandably) is trying to evade it. The pursuer is modeled as a discrete time Dubins's vehicle with equations

$$
x_p^+ = \begin{bmatrix} x_p^{(1)} + v\cos x_p^{(3)} \\ x_p^{(2)} + v\sin x_p^{(3)} \\ x_p^{(3)} + u \end{bmatrix} =: \phi_p(x_p, u)
$$

where $x_p^{(i)}$ designates the i$^{\text{th}}$ element of the vector $x_p$, $v$ is a constant forward speed and $u$ is the steering, over which the driver has control. The pedestrian is modeled by the accumulator

$$
x_e^+ = x_e + d =: \phi_e(x_e, d)
$$

where $d$ is the velocity vector. Given a time horizon $T$, and initial positions $x_e(t)$ and $x_p(t)$, we want to solve

$$
\min_{U \in \mathcal{U}} \ \max_{D \in \mathcal{D}} \sum_{i=0}^{T-1} \left\| x_p^{(1,2)}(t+i+1) - x_e(t+i+1) \right\|_2^2 + \gamma_u u(t+i)^2 - \gamma_d \|d(t+i)\|_2^2
$$

(45)

where $x_p^{(1,2)}$ designates the first and second elements of the vector $x_p$; $\gamma_u$ and $\gamma_d$ are positive weights; and $U$, $\mathcal{U}$, $D$ and $\mathcal{D}$ are defined for $i = 0, \ldots, T-1$

$$
U := u(t+i), x_p(t+i+1)
$$

$$\mathcal{U} := \{u(t+i), x_p(t+i+1): \ |u(t+i)| \le u_{max},$$
$$x_p(t+i+1) = \phi_p\big(x_p(t+i), u(t+i)\big)\}$$
$$D := d(t+i), x_e(t+i+1)$$
$$\mathcal{D} := \{d(t+i), x_e(t+i+1): \ \|d(t+i)\|_2 \le d_{max},$$
$$x_e(t+i+1) = \phi_e\big(x_e(t+i), d(t+i)\big)\}.$$

Instead of explicitly computing the solution of the trajectory of the pursuer and evaders, we are implicitly computing them by setting the dynamics as equality constraints; we will show shortly that this has an important impact on the scalability of the algorithm.

Each player is controlled using Model Predictive Control (MPC), meaning that at each time step $t$ we solve (45) obtaining controls $u(t)$ and $d(t)$, which are then used to control the system for the next time step.

### The importance of guaranteeing instability

It is natural to ask whether it is important to enforce the instability guarantee, specially in the case where the (ConsLQAC) is not enough to enforce it. In Figure 1 we show what can happen if they are not enforced. We take the homicidal chauffeur problem with a horizon of $T = 20$ and we run the MPC control for $t = 1, \ldots, 50$. In one case we enforce the instability guarantee, meaning that we use line 8 of Algorithm 1, on the second case we only enforce the (ConsLQAC), and on the third case we only enforce the instability guarantees after $t = 25$. In all cases, we start the system with the exact same initial conditions.

In the first case, the evader (which is the maximizer), is able to find a control that allows it to get further from the pursuer. The average cost for all the time steps ($t = 1, \ldots, 50$) ends up being around 0.2. In the second case, the solver keeps being attracted towards a point that is not a local minmax (and more precisely, not a local maximum), which means that the evader is not capable of escaping the pursuer; as a consequence, the average cost for all the time steps ends up being around 0.05, which is lower, as expected. Finally, in the third case, at $t = 25$ the solver starts to be able to converge towards a local minmax, and the evader is able to escape from the pursuer.

This example illustrates how crucial it is to enforce instability. By doing it, we guarantee that the algorithm can only converge towards an equilibrium point that is a local minmax, and this can completely change the numerical solution.

### Exploiting sparsity

Instead of setting the dynamics as equality constraints in (45), one could simply find the solution of the trajectory equation at each time step. This means to explicitly calculate $x_p(t+i+1) = \phi_p\Big(\phi_p\big(\ldots, u(t+i-1)\big), u(t+i)\Big)$. In the MPC literature, this is known as the sequential approach, versus the simultaneous approach we used in (45) [40, Chapter 8.1.3]. We want to study
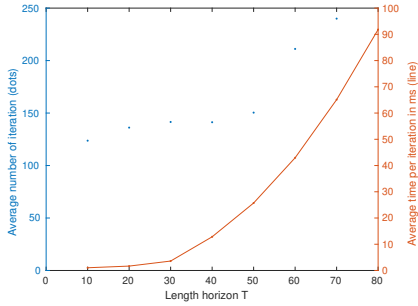
(a) Computational scaling for solving homicidal chauffeur per horizon length
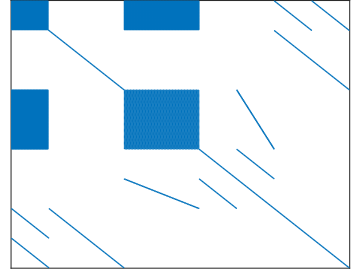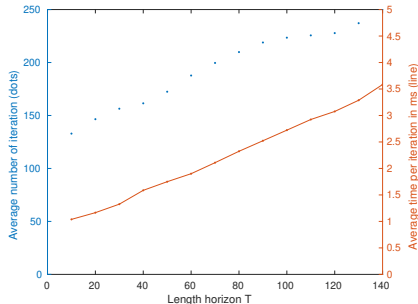


(b) Structural sparsity pattern of $J_{zz}f(z)$

**Fig. 2**: Scaling of homicidal chauffeur with horizon length and sparsity pattern of the Hessian when using the sequential approach

the scalability of the algorithm by enlarging the horizon $T$, both when using the sequential and the simultaneous approaches.

The sequential approach solves an optimization problem in a smaller space state, because it only needs to solve the optimization for $u(t), \ldots, u(t+T)$ and $d(t), \ldots, d(t+T)$ and it does not have to handle equality constraints. However, as we can see from the sparsity pattern in Figure 2b, the Hessian is rather dense, with large parts of it containing nonzero entries. As it can be seen in Figure 2a, the algorithm scales rather poorly as the horizon length (and hence, the number of variables) increases; it no longer converges reliably after $T = 80$.

The simultaneous approach on the other hand solves the optimization problem in a much larger space state, because not only it needs to also solve for $u(t), \ldots, u(t+T)$ and $d(t), \ldots, d(t+T)$, but also for $x_p(t), \ldots, x_p(t+T)$ and $x_e(t), \ldots, x_e(t+T)$ and it also needs to handle equality constraints. Fortunately, as we can see from the sparsity pattern in Figure 3b, most of the entries in the Hessian are actually structurally zero (meaning they are always zero). `TensCalc`'s implementation of the LDLt factorization exploits sparsity patterns and scales roughly in $O(T)$, which makes it substantially more efficient than standard LDLt decomposition, which scales in $O(T^3)$ [13, Appendix A]. At each step of Algorithm 1, most of the time is spent computing the LDLt decomposition, either for adjusting $\epsilon_x$ and $\epsilon_y$ or to invert $H_{zz}f(z)$. As a consequence, we can see in Figure 3a that both the number of iterations necessary to solve the optimization as well as the time per iteration scale roughly linear, the first being multiplied by about 1.7 while the second by 3.5 while the horizon length $T$ is multiplied by roughly 30.

*Remark 2* (Minmax problems with shared dynamics) In the homicidal chauffeur, the control of the pursuer does not impact the *dynamics* of the evader, and vice versa. This is why in (45) the dynamics can be set as equality constraints independently for the min and for the max.

(a) Computational scaling for solving homicidal chauffeur per horizon length
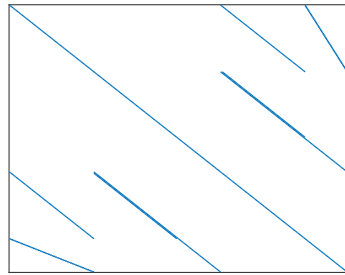


(b) Structural sparsity pattern of $J_{zz}f(z)$

**Fig. 3**: Scaling of homicidal chauffeur with horizon length and sparsity pattern of the Hessian

Now consider the problem

$$x^+ = f(x, u, d)$$

where $u$ is the control and $d$ is the disturbance and one wants to minimize a cost function $V(x(1), \ldots, x(T), u(0), \ldots, u(T-1))$ given the worst disturbance $d(1), \ldots, d(T)$. Because both the control and the disturbances influence the dynamics, we need to include the dynamics as equality constraints for the *maximization*, leading to the optimization problem

$$\min_{u(i) \in \mathcal{U}, i=0, \ldots, T-1} \max_{\substack{d(i) \in \mathcal{D}, x(i+1), i=0, \ldots, T-1: \\ x(i+1)=f(x(i), u(i), d(i))}} V\Big(x(1), \ldots, x(T), u(0), \ldots, u(T-1)\Big)$$

where $\mathcal{U}, \mathcal{D}$ are the feasible sets for the control and disturbances. It is important to notice that $x$ just acts as a latent/dummy variable that allows us to avoid solving the trajectory equation. Setting it as a maximization variable does not changes the result as $x$ is always exactly determined by the value of $u$ and $d$. It does, however, improves the numerical efficiency of the algorithm as now the Hessian matrices are sparse and their LDL decomposition can be efficiently computed.    □

# 5  Conclusion

The main contribution of this article is the construction of Newton and interior-point algorithm for nonconvex-nonconcave minmax optimization that can only converge towards an equilibrium point if such point is a local minmax. We established this results by modifying the Hessian matrices such that the update steps can be seen as the solution of quadratic programs that locally approximate the minmax problem. While our results are only local, using numerical simulations we see that the algorithm is able to make progress towards a solution even if it does not start close to it. We also illustrated using numerical examples how important it is to have a formulation of the minmax problem such that the Hessian matrix is sparse.

The main future direction would be to develop non-local convergence results. We believe that the best approach to obtain such results would be to develop a type of Armijo rule which could be used to obtain similar results to those from minimization. Developing filters and merit function could also play an important role in coming up with ways to improve the algorithm's convergence.

# Appendix A    Second order sufficient conditions for constrained minimization and minmax optimization

## A.1   Proof of Proposition 1 (constrained minimization)

The first step is to show that $g(z,0) = \mathbf{0}$ is equivalent to the Karush–Kuhn–Tucker (KKT) conditions [13, Chapter 12]. Consider the "full" Lagrangian $\tilde{L}(x, s_x, \nu_x, \lambda_x, \tau_x) = f(x) + \nu_x' G_x(x) + \lambda_x'(F_x(x) + s_x) - \tau_x' s_x$ for the optimization (6). The KKT condition would then be that

$$\begin{bmatrix} \boldsymbol{\nabla}_x \tilde{L}(x, s_x, \nu_x, \lambda_x, \tau_x) \\ \boldsymbol{\nabla}_{s_x} \tilde{L}(x, s_x, \nu_x, \lambda_x, \tau_x) = \lambda_x - \tau_x \\ G_x(x) \\ F_x(x) + s_x \\ \tau_x \odot s_x \end{bmatrix} = \mathbf{0} \tag{A1}$$

and $s_x, \tau_x \geq \mathbf{0}$. The second equation can be used to substitute $\tau_x$ by $\lambda_x$, which gives the equality $g(z,0) = \mathbf{0}$.

Now the second order sufficient conditions. Let us start by rewriting the minimization (1) but instead of using as slack variables $s_x$ with the constraint $s_x \geq 0$, using the slack variable $w_x \odot w_x$ (where $\odot$ is the element wise product):

$$\min_{x, w_x : G_x(x) = \mathbf{0}, F_x(x) + w_x \odot w_x = \mathbf{0}} f(x). \tag{A2}$$

Consider now the solution cone

$$\mathcal{C}_x(z) := \{(d_x, d_w) \in \mathbb{R}^{n_x + m_x} \backslash \{\mathbf{0}\} : \boldsymbol{\nabla}_x G_x(x)' d_x = \mathbf{0}, \\ \boldsymbol{\nabla}_x F_x(x) d_x + 2 \operatorname{diag}(w_x) d_w = \mathbf{0}\}$$

Let $(x, w_x, \nu_x, \lambda_x)$ be a point such that the KKT conditions for (A2) hold. As, by assumption, the LICQ and strict complementarity conditions hold, if

$$\begin{bmatrix} d_x \\ d_w \end{bmatrix}' \begin{bmatrix} \boldsymbol{\nabla}_{xx} L(z) & \mathbf{0} \\ \mathbf{0} & 2 \operatorname{diag}(\lambda_x) \end{bmatrix} \begin{bmatrix} d_x \\ d_w \end{bmatrix}' > 0 \ \forall (d_x, d_w) \in \mathcal{C}_x(z) \tag{A3}$$

then $(x, w_x, \nu_x, \lambda_x)$ is a local minimum of (A2). The proof can be found in [13, Theorem 12.5].

We now need to prove that (A3) is equivalent to the condition (10) from the proposition. Because the LICQ and strict complementarity hold, the set $\mathcal{C}_x(z)$ is given by the null space (a.k.a. the kernel) of the matrix

$$\tilde{H}_{x\lambda}f(z) = \begin{bmatrix} \boldsymbol{\nabla}_x G_x(x) & \boldsymbol{\nabla}_x F_x(x) \\ 0 & 2\operatorname{diag}(w_x). \end{bmatrix} \tag{A4}$$

This result can be found in [13, Chapter 12.5], in the subsection "Second-order conditions and projected Hessian". Let $Z_x \in \mathbb{R}^{n_x+m_x, n_x+m_x-m_x-l_x}$ be a matrix with full column rank such that $\tilde{H}_{x\lambda}f(z)'\, Z_x = \mathbf{0}$. Then, the condition (A3) can be rewritten as

$$Z_x' \begin{bmatrix} \boldsymbol{\nabla}_{xx}L(z) & \mathbf{0} \\ \mathbf{0} & 2\operatorname{diag}(\lambda_x) \end{bmatrix} Z_x \succ 0$$

which is equivalent to say that

$$\mathbf{in}\left( Z_x' \begin{bmatrix} \boldsymbol{\nabla}_{xx}L(z) & \mathbf{0} \\ \mathbf{0} & 2\operatorname{diag}(\lambda_x) \end{bmatrix} Z_x \right) = (n_x - l_x, 0, 0)$$

Now consider the matrix

$$\tilde{H}_{zz}f(z) = \begin{bmatrix} \boldsymbol{\nabla}_{xx}L(z) & \mathbf{0} & \boldsymbol{\nabla}_x G_x(x) & \boldsymbol{\nabla}_x F_x(x) \\ \mathbf{0} & 2\operatorname{diag}(\lambda_x) & \mathbf{0} & 2\operatorname{diag}(w) \\ \boldsymbol{\nabla}_x G_x(x)' & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\nabla}_x F_x(x)' & 2\operatorname{diag}(w) & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{A5}$$

As the LICQ conditions hold, according to [13, Theorem 16.3]

$$\mathbf{in}(\tilde{H}_{zz}f(z)) = \mathbf{in}\left( Z_x' \begin{bmatrix} \boldsymbol{\nabla}_{xx}L(z) & \mathbf{0} \\ \mathbf{0} & 2\operatorname{diag}(\lambda_x) \end{bmatrix} Z_x \right) + (l_x + m_x, l_x + m_x, 0).$$

Therefore (A3) holds if and only if $\mathbf{in}(\tilde{H}_{zz}f) = (n_x + m_x, l_x + m_x, 0)$.

We have almost finished the proof, we now just need to prove that $\mathbf{in}(\tilde{H}_{zz}f(z)) = \mathbf{in}(H_{zz}f(z))$. Using the equality condition $F_x(x) + w_x \odot w_x = 0$, we obtain the relation $w_x = (-F_x(x))^{1/2} = s_x^{1/2}$. If we substitute back this result in $\tilde{H}_{zz}f(z)$ we almost have that $\tilde{H}_{zz}f(z)$ is equal to $H_{zz}f(z)$ except for the 2 in front of $\operatorname{diag}(\lambda_x)$ and $\operatorname{diag}(s^{1/2})$. Take the matrix $\Xi$ defined by

$$\Xi = \operatorname{diag}([\mathbf{1}_{n_x}, [a^{(1)}, a^{(2)}, ..., a^{(m_x)}], \mathbf{1}_{l_x+m_x}])$$

where

$$a^{(i)} = \begin{cases} \frac{1}{2} & \text{if } \lambda_x^{(i)} = 0 \text{ and } s_x^{(i)} \neq 0 \\ \frac{1}{\sqrt{2}} & \text{if } \lambda_x^{(i)} \neq 0 \text{ and } s_x^{(i)} = 0 \end{cases}$$

with $\lambda_x^{(i)} = 0$ and $s_x^{(i)}$ denoting the $i^{\text{th}}$ elements of $\lambda_x$ and $s_x$. Then $\Xi \tilde{H}_{zz} f(z) \Xi = H_{zz} f(z)$ which, according to Sylvester's law of inertia [36, Theorem 1.5], implies that $\text{inertia}(\tilde{H}_{zz}f(z)) = \text{inertia}(H_{zz}f(z))$, which finishes the proof.     $\square$

## A.2  Proof of Proposition 3 (constrained minmax optimization)

First, using the exact same reasoning as in the proof of Proposition 1, one can show that $g(z,0) = 0$ is equivalent to the first order necessary condition in [12].

Similarly to what we did in the proof of Proposition 1, let us start by rewriting the constrained minmax optimization (33) using the slack variables $w \odot w$:

$$\min_{x,w_x:G_x(x)=\mathbf{0},F_x(x)+w_x\odot w_x=\mathbf{0}} \quad \max_{y,w_y:G_y(x,y)=\mathbf{0},F_y(x,y)+w_y\odot w_y=\mathbf{0}} f(x,y).$$

Consider the solution cones

$$\mathcal{C}_y(z) := \{(d_y,d_{w_y}) \in \mathbb{R}^{n_y+m_y}\backslash\{\mathbf{0}\} : \boldsymbol{\nabla}_y G_y(x,y)d_y = \mathbf{0},$$
$$\boldsymbol{\nabla}_y F(x,y)d_y + 2\,\text{diag}(w_y)d_{w_y} = \mathbf{0}\}$$

and

$$\mathcal{C}_x(z) := \{(d_x,d_{w_x}) \in \mathbb{R}^{n_x+m_x}\backslash\{\mathbf{0}\} : \boldsymbol{\nabla}_x G_x(x)'d_x = \mathbf{0}$$
$$\boldsymbol{\nabla}_x F_x(x)d_x + 2\,\text{diag}(w_x)d_{w_x} = \mathbf{0}\}$$

Let $z$ be a point such that $g(z,0) = \mathbf{0}$. As, by assumption, the LICQ and strict complementarity hold, if

$$\begin{bmatrix} d_y \\ d_{w_y} \end{bmatrix}' \begin{bmatrix} \boldsymbol{\nabla}_{yy}L(z) & \mathbf{0} \\ \mathbf{0} & -2\,\text{diag}(\lambda_y) \end{bmatrix} \begin{bmatrix} d_y \\ d_{w_y} \end{bmatrix} < 0 \ \forall \ (d_y,d_{w_y}) \in \mathcal{C}_y(z) \qquad \text{(A6a)}$$

and

$$\begin{bmatrix} d_x \\ d_{w_x} \end{bmatrix}' \left( H_{xx}L(z) - H_{xy}f(z)H_{yy}f(z)^{-1}H_{xy}f(z)' \right) \begin{bmatrix} d_x \\ d_{w_x} \end{bmatrix} > 0 \ \forall \ (d_x,d_{w_x}) \in \mathcal{C}_x(z)$$
$$\text{(A6b)}$$

then $(x,w_x,\nu_x,\lambda_x)$ is a local minimum of (A2). The proof can be found in [12, Theorem 3.2].

The proof between the equivalence of the condition (A6a) and $\mathbf{in}(H_{yy}f(z)) = (l_y + m_y, n_y + m_y, 0)$ is almost identical to the proof of Proposition (1).

The condition on the inertia of $\mathbf{in}(H_{zz}f(z))$ require some more development. In an analogous way to the proof of Proposition (1), let $Z_x$ be a

matrix with full column rank such that $H_{x\lambda}f(z)'\,Z_x = \mathbf{0}$. Then the sufficient conditions (A6b) for the reformulated outer minimization is

$$Z_x'\Big(H_{xx}f(z) - H_{yx}f(z)'H_{yy}f(z)^{-1}H_{yx}f(z)\Big)Z_x \succ 0. \tag{A7}$$

We want now to define a new partition of $H_{zz}f(z)$ which we will use to finish the proof. Consider the matrices

$$\bar{H}_{zz}f(z) = \begin{bmatrix} H_{xx}f(z) & H_{xy}f(z) \\ H_{xy}f(z)' & H_{yy}f(z) \end{bmatrix} \quad \text{and} \quad \bar{H}_{x\lambda}f(z) = \begin{bmatrix} H_{x\lambda}f(z) \\ \mathbf{0}_{n_y+m_y+l_y+m_y,l_x+m_x} \end{bmatrix}.$$

such that

$$H_{zz}f(z) = \begin{bmatrix} \bar{H}_{zz}f(z) & \bar{H}_{x\lambda}f(z) \\ \bar{H}_{x\lambda}f(z)' & \mathbf{0}_{l_x+m_x} \end{bmatrix}$$

Let the matrix

$$\bar{Z}_x := \begin{bmatrix} Z_x & \mathbf{0}_{n_x+m_x,n_y+m_y+l_y+m_y} \\ \mathbf{0}_{n_y+m_y+l_y+m_y,n_x-l_x} & I_{n_y+m_y+l_y+m_y}. \end{bmatrix}$$

One can show that $\bar{Z}_x$ is full column rank and such that $\bar{H}_{x\lambda}f(z)'\,\bar{Z}_x = \mathbf{0}$. Therefore if we apply [13, Theorem 16.3] to $H_{zz}f(z)$ (with the new partitioning) gives

$$\mathbf{in}(H_{zz}f(z)) = \mathbf{in}\left(\bar{Z}_x'\bar{H}_{zz}f(z)\bar{Z}_x\right) + (l_x+m_x,l_x+m_x,0)$$

In turn, $\mathbf{in}\left(\bar{Z}_x'\bar{H}_{zz}f(z)\bar{Z}_x\right)$ can be simplified using Haynsworth inertia additivity formula [36, Theorem 1.6]:

$$\mathbf{in}\left(\bar{Z}_x'\bar{H}_{zz}f(z)\bar{Z}_x\right)$$
$$= \mathbf{in}\left(\begin{bmatrix} Z_x'H_{xx}f(z)Z_x & Z_x'H_{xy}f(z) \\ H_{xy}f(z)'Z_x & H_{yy}f(z) \end{bmatrix}\right)$$
$$= \mathbf{in}\left(Z_x'\Big(H_{xx}f(z) - H_{xy}f(z)H_{yy}f(z)^{-1}H_{xy}f(z)'\Big)Z_x\right) + \mathbf{in}(H_{yy}f(z)).$$

Therefore, if (A6a) holds, (A6b) is equivalent to

$$\mathbf{in}(H_{zz}f(z)) = (n_x-l_x,0,0) + (l_y+m_y,n_y+m_y,0) + (l_x+m_x,l_x+m_x,0)$$

which finishes the proof. $\qquad\square$

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