

Hybrid hyperinterpolation over general regions

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Abstract

We present an $\ell_2^2 + \ell_1$ -regularized discrete least squares approximation over general regions under assumptions of hyperinterpolation, named hybrid hyperinterpolation. Hybrid hyperinterpolation, using a soft thresholding operator and a filter function to shrink the Fourier coefficients approximated by a high-order quadrature rule of a given continuous function with respect to some orthonormal basis, is a combination of Lasso and filtered hyperinterpolations. Hybrid hyperinterpolation inherits features of them to deal with noisy data once the regularization parameter and the filter function are chosen well. We not only provide L_2 errors in theoretical analysis for hybrid hyperinterpolation to approximate continuous functions with noise and noise-free, but also decompose L_2 errors into three exact computed terms with the aid of a prior regularization parameter choices rule. This rule, making fully use of coefficients of hyperinterpolation to choose a regularization parameter, reveals that L_2 errors for hybrid hyperinterpolation sharply decline and then slowly increase when the sparsity of coefficients ranges from one to large values. Numerical examples show the enhanced performance of hybrid hyperinterpolation when regularization parameters and noise vary. Theoretical L_2 errors bounds are verified in numerical examples on the interval, the unit-disk, the unit-sphere and the unit-cube, the union of disks.

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1 Introduction

Hyperinterpolation, originally introduced by Sloan in the seminal paper [28], is a constructive approximation method for continuous functions over some compact subsets or manifolds. Briefly speaking, hyperinterpolation uses a high-order quadrature rule to approximate a truncated Fourier expansion in a series of orthogonal polynomials for some measure on a given multidimensional domain, and is the unique solution to a discrete least squares approximation, which means it is also a projection ([28, Lemma 5]). It is well known that the least squares approximation usually tackles with the scenario that the noise level is relatively small [19, 21, 22]. Adding a regularization term to the least squares approximation becomes a popular approach to solve the case that the noise level is large. For example, Tikhonov regularized least squares approximation [5] performs well on $[-1, 1]$ to handle noisy data. On the sphere, filtered hyperinterpolation with a certain filter function is equivalent to Tikhonov regularized least squares approximation [2, 5], which is also a constructive polynomial scheme [2, 30]. Tikhonov regularized least squares approximation continuously shrinks all coefficients of hyperinterpolation without dismissing less relevant ones, while filtered hyperinterpolation processes these coefficients in a similar way but dismisses some coefficients related to higher order orthonormal basis exceeding the highest order of polynomial in the polynomial space. Concisely, they do not possess the basis selection ability.

Lasso hyperinterpolation introduced by An and Wu [4] is an ℓ_1 -regularized discrete least square approximation in order to recover functions with noise, which does both continuous shrinkage and automatic basis selection simultaneously. However, Lasso hyperinterpolation could not shrink coefficients in a more flexible manner, that is keeping high relevant coefficients invariant and doubly shrinking less relevant ones. Penalty parameters in Lasso hyperinterpolation are difficult to take effect since they combine with the regularization parameter to shrink coefficients making things more complicated.

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In this paper, we develop a new variant of hyperinterpolation based on filtered hyperinterpolation and Lasso hyperinterpolation — *hybrid hyperinterpolation*, which actually corresponds to an $\ell_2^2 + \ell_1$ -regularized discrete least squares approximation (see Theorem 3.1). Hybrid hyperinterpolation processes coefficients of hyperinterpolation by means of a soft thresholding operator and a filter function. It not only inherits denosing and basis selection abilities of Lasso hyperinterpolation, but also possesses characteristic of filtered hyperinterpolation remaining high relevant coefficients of hyperinterpolation invariant and shrinking less relevant ones; it performs better in function recoveries than Lasso hyperinterpolation and filtered hyperinterpolation over some general regions, and is more robust in denosing when regularization parameters vary.

An useful regularization parameter choices rule is cherished, such as Hesse, Sloan and Womersely once studied prior parameter choice strategies in [21]. In this paper, we propose a prior regularization parameter choices rule (see Theorem 5.1) to fully make use of coefficients of hyperinterpolation to choose parameters. Firstly, this rule gives an explicit upper bound of the sparsity of coefficients of hybrid hyperinterpolation. Secondly, L_2 errors for hybrid hyperinterpolation can be decomposed into three exact computed terms under this rule. One of the three terms is a constant once the quadrature rule and the test function are determined. As for the rest two terms, one is non-positive and first quickly drop and then slowly decrease if the sparsity increases from one to other values. The other is related to the noise and takes effect when the sparsity becomes large. Thus L_2 errors for hybrid hyperinterpolation first sharply decline and then gradually increase when the sparsity varies from one. Our numerical examples show that L_2 errors computed by this decomposition are almost the same as that computed directly.

An interesting numerical example over the union of disks (see section 6.5) is considered, in which the polynomial orthonormal basis can not be given explicitly by a simply expression and is obtained in numerical ways. This example shows that variants of hyperinterpolation could also contribute to denosing over some more complicated compact regions like it does on the interval, the unit-disk, the unit-sphere and the unit-cube [3, 4]. Hyperinterpolation on the square [12], on the disk [18], in the cube [13], on the sphere [14, 23, 26, 27], over nonstandard planar regions [31] and spherical triangles [32] were also considered in the past.

The rest of the paper is organized as follows. In section 2, the backgrounds of hyperinterpolation and its variants are reviewed. In section 3, we show that how hyperinterpolation combines a soft thresholding operator with a filter function to become the hybrid hyperinterpolation. L_2 errors analysis for hybrid hyperinterpolation to approximate continuous functions with noise and noise-free are derived in section 4. In section 5, we focus on a prior regularization parameter choices rule and the decomposition of L_2 errors. Numerical examples on the interval, the unit-disk, the unit-sphere and the unit-cube, the union of disks are specifically presented in section 6.

2 Backgrounds of hyperinterpolation

Let Ω be a compact set of \mathbb{R}^s with a positive measure ω . Suppose Ω has finite measure with respect to $d\omega$, that is,

$$\int_{\Omega} d\omega = V < \infty.$$

We denote by $L^2(\Omega)$ the Hilbert space of square-integrable functions on Ω with the L^2 inner product

$$\langle f, g \rangle = \int_{\Omega} fg d\omega, \quad \forall f, g \in L^2(\Omega), \quad (2.1)$$

and the induced norm $\|f\|_2 := \langle f, f \rangle^{1/2}$. Let $\mathbb{P}_L(\Omega) \subset L^2(\Omega)$ be the linear space of polynomials with total degree not strictly greater than L , restricted to Ω , and let $d = \dim(\mathbb{P}_L(\Omega))$ be the dimension of $\mathbb{P}_L(\Omega)$.

Next we define an orthonormal basis of $\mathbb{P}_L(\Omega)$

$$\{\Phi_{\ell} | \ell = 1, \dots, d\} \subset \mathbb{P}_L(\Omega) \quad (2.2)$$

in the sense of

$$\langle \Phi_{\ell}, \Phi_{\ell'} \rangle = \delta_{\ell\ell'}, \quad \forall 1 \leq \ell, \ell' \leq d. \quad (2.3)$$

The $L^2(\Omega)$ -orthogonal projection $\mathcal{T}_L : L^2(\Omega) \rightarrow \mathbb{P}_L(\Omega)$ can be uniquely defined by

$$\mathcal{T}_L f := \sum_{\ell=1}^d \hat{f}_\ell \Phi_\ell = \sum_{\ell=1}^d \langle f, \Phi_\ell \rangle \Phi_\ell, \quad \forall f \in L^2(\Omega) \quad (2.4)$$

where $\{\hat{f}_\ell\}_{\ell=1}^d$ are the Fourier coefficients

$$\hat{f}_\ell := \langle f, \Phi_\ell \rangle = \int_{\Omega} f \Phi_\ell d\omega, \quad \forall \ell = 1, \dots, d.$$

In order to evaluate numerically the scalar product in (2.4), it is fundamental to consider an N -point quadrature rule of PI-type (Positive weights and Interior nodes), i.e.,

$$\sum_{j=1}^N w_j g(\mathbf{x}_j) \approx \int_{\Omega} g d\omega, \quad \forall g \in \mathcal{C}(\Omega), \quad (2.5)$$

where the quadrature points $\{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ belong to Ω and the corresponding quadrature weights $\{w_1, \dots, w_N\}$ are positive. Furthermore we say that (2.5) has algebraic degree of exactness δ if

$$\sum_{j=1}^N w_j p(\mathbf{x}_j) = \int_{\Omega} p d\omega, \quad \forall p \in \mathbb{P}_\delta(\Omega). \quad (2.6)$$

With the help of a quadrature rule (2.5) with algebraic degree of exactness $\delta = 2L$, we can introduce a “discrete inner product” on $\mathcal{C}(\Omega)$ [28] by

$$\langle f, g \rangle_N := \sum_{j=1}^N w_j f(\mathbf{x}_j) g(\mathbf{x}_j), \quad \forall f, g \in \mathcal{C}(\Omega), \quad (2.7)$$

corresponding to the $L^2(\Omega)$ -inner product (2.1). For any $p, q \in \mathbb{P}_L(\Omega)$, the product pq is a polynomial in $\mathbb{P}_{2L}(\Omega)$. Therefore it follows from the quadrature exactness of (2.6) for polynomials of degree at most $2L$ that

$$\langle p, q \rangle_N = \langle p, q \rangle = \int_{\Omega} pq d\omega, \quad \forall p, q \in \mathbb{P}_L(\Omega). \quad (2.8)$$

In 1995, I.H. Sloan introduced in [28] the *hyperinterpolation operator* $\mathcal{L}_L : \mathcal{C}(\Omega) \rightarrow \mathbb{P}_L(\Omega)$ as

$$\mathcal{L}_L f := \sum_{\ell=1}^d \langle f, \Phi_\ell \rangle_N \Phi_\ell, \quad \forall f \in \mathcal{C}(\Omega), \quad (2.9)$$

where the hyperinterpolant $\mathcal{L}_L f$ is defined to be the projection of f onto $\mathbb{P}_L(\Omega)$ obtained by replacing the $L^2(\Omega)$ -inner products (2.1) in the $L^2(\Omega)$ -orthogonal projection $\mathcal{T}_L f$ by the discrete inner products (2.7).

To explore one of its important features, we introduce the following discrete least squares approximation problem

$$\min_{p \in \mathbb{P}_L(\Omega)} \left\{ \frac{1}{2} \sum_{j=1}^n w_j [p(\mathbf{x}_j) - f(\mathbf{x}_j)]^2 \right\} \quad (2.10)$$

with $p(\mathbf{x}) = \sum_{\ell=1}^d \alpha_\ell \Phi_\ell(\mathbf{x}) \in \mathbb{P}_L(\Omega)$, or equivalently

$$\min_{\boldsymbol{\alpha} \in \mathbb{R}^d} \frac{1}{2} \|\mathbf{W}^{1/2}(\mathbf{A}\boldsymbol{\alpha} - \mathbf{f})\|_2^2, \quad (2.11)$$

where $\mathbf{W} = \text{diag}(w_1, \dots, w_N)$ is the quadrature weights matrix, $\mathbf{A} = (\Phi_\ell(\mathbf{x}_j))_{j\ell} \in \mathbb{R}^{N \times d}$ is the sampling matrix, $\boldsymbol{\alpha} = [\alpha_1, \dots, \alpha_d]^T \in \mathbb{R}^d$ and $\mathbf{f} = [f(\mathbf{x}_1), \dots, f(\mathbf{x}_N)]^T \in \mathbb{R}^N$ are two column vectors (recall $\mathbf{x}_j \in \mathbb{R}^s$). Sloan in [28] first revealed that the relation between the hyperinterpolant $\mathcal{L}_L f$ and the best discrete least squares approximation (weighted by quadrature weights) of f at the quadrature points. More precisely he proved the following important result:

Lemma 2.1 (Lemma 5 in [28]) *Given $f \in \mathcal{C}(\Omega)$, let $\mathcal{L}_L f \in \mathbb{P}_L(\Omega)$ be defined by (2.9), where the discrete scalar product $\langle f, g \rangle_N$ in (2.7) is defined by an N -point quadrature rule of PI-type in Ω with algebraic degree of exactness $2L$. Then $\mathcal{L}_L f$ is the unique solution to the approximation problem (2.10).*

2.1 Filtered hyperinterpolation

Filtered hyperinterpolation was introduced by I.H.Sloan and R.S.Womersley on the unit sphere [30], all coefficients of the hyperinterpolant are filtered by a filter function $h(x)$. To this purpose, we introduce a *filter function* $h \in \mathcal{C}[0, +\infty)$ that satisfies

$$h(x) = \begin{cases} 1, & \text{for } x \in [0, 1/2], \\ 0, & \text{for } x \in [1, \infty). \end{cases}$$

Depending on the behaviour of h in $[1/2, 1]$, one can define many filters. Many examples were described in [29]. In this paper, we only consider the filtered function $h(x)$ defined by

$$h(x) = \begin{cases} 1, & x \in [0, \frac{1}{2}], \\ \sin^2(\pi x), & x \in [\frac{1}{2}, 1]. \end{cases} \quad (2.12)$$

In what follows, we denote by $[\cdot]$ the floor function. Once a filter has been chosen, one can introduce the filtered hyperinterpolation as follows:

Definition 2.1 ([30]) *Suppose that the discrete scalar product $\langle f, g \rangle_N$ in (2.7) is determined by an N -point quadrature rule of PI-type in Ω with algebraic degree of exactness $L-1 + \lfloor L/2 \rfloor$. The filtered hyperinterpolant $\mathcal{F}_{L,N}f \in \mathbb{P}_{L-1}(\Omega)$ of f is defined as*

$$\mathcal{F}_{L,N}f := \sum_{\ell=1}^d h\left(\frac{\deg \Phi_\ell}{L}\right) \langle f, \Phi_\ell \rangle_N \Phi_\ell. \quad (2.13)$$

where $d = \dim(\mathbb{P}_L(\Omega))$ and h is a filter function defined by (2.12).

For simplicity, we use h_ℓ instead of $h(\deg \Phi_\ell/L)$ in this paper. Notice that by (2.13), depending on the fact $\text{supp}(\overline{h}) = [0, 1]$, with respect to the classical hyperinterpolation, one can achieve more *sparsity* in the polynomial coefficients in view of the term h_ℓ , i.e. some less relevant discrete Fourier coefficients are dismissed. Indeed, if $\deg \Phi_\ell \geq L$ then $h_\ell = 0$.

In view of the assumption on the algebraic degree of precision, from (2.13), we easily have $\mathcal{F}_{L,N}f = f$ for all $f \in \mathbb{P}_{\lfloor L/2 \rfloor}(\Omega)$.

Recently, it was shown in [24] that (distributed) filtered hyperinterpolation can reduce weak noise on spherical functions.

2.2 Lasso hyperinterpolation

An alternative to filtered hyperinterpolation is the so called *Lasso hyperinterpolation* [4] with the intention of denoising and feature selection by dismissing less relevant discrete Fourier coefficients.

Definition 2.2 ([17]) *The soft thresholding operator is defined as*

$$\mathcal{S}_k(a) := \max(0, a - k) + \min(0, a + k),$$

where $k \geq 0$.

Alternatively, we can define $\mathcal{S}_k(a)$ as follows

$$\mathcal{S}_k(a) = \begin{cases} a + k, & \text{if } a < -k, \\ 0, & \text{if } -k \leq a \leq k, \\ a - k, & \text{if } a > k. \end{cases} \quad (2.14)$$

Suppose that the discrete scalar product $\langle f, g \rangle_N$ in (2.7) is determined by an N -point quadrature rule of PI-type in Ω with algebraic degree of exactness $2L$. Then the *Lasso hyperinterpolation* of f onto $\mathbb{P}_L(\Omega)$ is defined as

$$\mathcal{L}_L^\lambda f := \sum_{\ell=1}^d \mathcal{S}_{\lambda \mu_\ell}(\langle f, \Phi_\ell \rangle_N) \Phi_\ell \quad (2.15)$$

where $\lambda > 0$ is the regularization parameter and $\{\mu_\ell\}_{\ell=1}^d$ is a set of positive penalty parameters.

The effect of the soft threshold operator is such that if $|\langle f, \Phi_\ell \rangle_N| \leq \lambda \mu_\ell$ then $\mathcal{S}_{\lambda \mu_\ell}(\langle f, \Phi_\ell \rangle_N) = 0$. Now we introduce the ℓ_1 -regularized least squares problem

$$\min_{p_\lambda \in \mathbb{P}_L(\Omega)} \left\{ \frac{1}{2} \sum_{j=1}^N w_j [p_\lambda(\mathbf{x}_j) - f(\mathbf{x}_j)]^2 + \lambda \sum_{\ell=1}^d \mu_\ell |\gamma_\ell^\lambda| \right\} \quad (2.16)$$

with $p_\lambda(\mathbf{x}) = \sum_{\ell=1}^d \gamma_\ell^\lambda \Phi_\ell(\mathbf{x}) \in \mathbb{P}_L(\Omega)$, or equivalently

$$\min_{\boldsymbol{\gamma}^\lambda \in \mathbb{R}^d} \frac{1}{2} \|\mathbf{W}^{1/2}(\mathbf{A}\boldsymbol{\gamma}^\lambda - \mathbf{f})\|_2^2 + \lambda \|\mathbf{R}_1 \boldsymbol{\gamma}^\lambda\|_1, \quad \lambda > 0, \quad (2.17)$$

where $\boldsymbol{\gamma}^\lambda = [\gamma_1^\lambda, \dots, \gamma_d^\lambda]^\top \in \mathbb{R}^d$ and $\mathbf{R}_1 = \text{diag}(\mu_1, \dots, \mu_d)$. In this paper, the subscript and superscript in p_λ and $\boldsymbol{\gamma}^\lambda$ mean that their concrete forms are related to the regularization parameter λ . The following result holds.

Theorem 2.1 (Theorem 3.4 in [4]) *Let $\mathcal{L}_L^\lambda f \in \mathbb{P}_L(\Omega)$ be defined by (2.15), and adopt conditions of Lemma 2.1. Then $\mathcal{L}_L^\lambda f$ is the solution to the regularized least squares approximation problem (2.16).*

As observed in [4], Lasso hyperinterpolation is not invariant under a change of basis and differently, from the previous approaches, is not in general a projection to a certain polynomial space.

It has been shown how Lasso hyperinterpolation operator \mathcal{L}_L^λ can reduce some stability estimates with respect to \mathcal{L}_L in [4, Theorem 4.4] as well as the error related to noise in [4, Theorem 4.6]. Numerical experiments on compact domains Ω , as interval, unit disk, sphere and cube have compared the effectiveness of this approach with respect to the previous techniques.

2.3 Hard thresholding hyperinterpolation

Definition 2.3 ([17]) *The hard thresholding operator*

$$\eta_H(a, k) := \begin{cases} a, & \text{if } |a| > k, \\ 0, & \text{if } |a| \leq k. \end{cases}$$

Suppose that the discrete scalar product $\langle f, g \rangle_N$ in (2.7) is determined by an N -point quadrature rule of PI-type in Ω with algebraic degree of exactness $2L$. Then the *hard thresholding hyperinterpolation* of f onto $\mathbb{P}_L(\Omega)$ is defined as

$$\mathcal{E}_L^\lambda f := \sum_{\ell=1}^d \eta_H(\langle f, \Phi_\ell \rangle_N, \lambda) \Phi_\ell, \quad (2.18)$$

where $\lambda > 0$ is the regularization parameter.

The effect of the hard thresholding operator is such that if $|\langle f, \Phi_\ell \rangle_N| \leq \lambda$ then $\eta_H(\langle f, \Phi_\ell \rangle_N, \lambda) = 0$.

Now we introduce the ℓ_0 -regularized least squares problem

$$\min_{p_\lambda \in \mathbb{P}_L(\Omega)} \left\{ \sum_{j=1}^N w_j [p_\lambda(\mathbf{x}_j) - f(\mathbf{x}_j)]^2 + \lambda^2 \sum_{\ell=1}^d |\nu_\ell^\lambda|_0 \right\} \quad (2.19)$$

with $p_\lambda(\mathbf{x}) = \sum_{\ell=1}^d \nu_\ell^\lambda \Phi_\ell(\mathbf{x}) \in \mathbb{P}_L(\Omega)$, or equivalently

$$\min_{\boldsymbol{\nu}^\lambda \in \mathbb{R}^d} \|\mathbf{W}^{1/2}(\mathbf{A}\boldsymbol{\nu}^\lambda - \mathbf{f})\|_2^2 + \lambda^2 \|\boldsymbol{\nu}^\lambda\|_0, \quad \lambda > 0, \quad (2.20)$$

where $\boldsymbol{\nu}^\lambda = [\nu_1^\lambda, \dots, \nu_d^\lambda]^\top \in \mathbb{R}^d$ and $|\nu_\ell|_0$ denotes the ℓ_0 -norm in one-dimension,

$$\forall \nu_\ell \in \mathbb{R}, \quad |\nu_\ell|_0 = \begin{cases} 0, & \text{if } \nu_\ell = 0, \\ 1, & \text{if } \nu_\ell \neq 0. \end{cases}$$

The following result holds.

Theorem 2.2 (Theorem 3.1 in [3]) *Let $\mathcal{E}_L^\lambda f \in \mathbb{P}_L(\Omega)$ be defined by (2.18), and adopt conditions of Lemma 2.1. Then $\mathcal{E}_L^\lambda f$ is the solution to the regularized least squares approximation problem (2.19).*

3 Hybrid hyperinterpolation

In the following, we propose a novel scheme named *hybrid hyperinterpolation*, i.e., a composition of filtered hyperinterpolation and Lasso hyperinterpolation. Indeed, hybrid hyperinterpolant corresponds to an $\ell_2^2 + \ell_1$ -regularized least approximation problem.

For $\lambda > 0$, $\mu_1, \dots, \mu_d > 0$, consider the following $\ell_2^2 + \ell_1$ -regularized least squares problem

$$\min_{p_\lambda \in \mathbb{P}_L(\Omega)} \left\{ \frac{1}{2} \sum_{j=1}^N w_j [p_\lambda(\mathbf{x}_j) - f(\mathbf{x}_j)]^2 + \frac{1}{2} \sum_{j=1}^N w_j [\mathcal{R}_L p_\lambda(\mathbf{x}_j)]^2 + \lambda \sum_{\ell=1}^d \mu_\ell |\beta_\ell^\lambda| \right\}, \quad (3.1)$$

where $p_\lambda(\mathbf{x}) = \sum_{\ell=1}^d \beta_\ell^\lambda \Phi_\ell(\mathbf{x}) \in \mathbb{P}_L(\Omega)$ and

$$\mathcal{R}_L p_\lambda(\mathbf{x}) = \sum_{\ell=1}^d b_\ell \langle \Phi_\ell, p_\lambda \rangle_N \Phi_\ell = \sum_{\ell=1}^d b_\ell \beta_\ell^\lambda \Phi_\ell(\mathbf{x}) \quad (3.2)$$

with

$$b_\ell = \begin{cases} 0, & \frac{\deg \Phi_\ell}{L} \in [0, \frac{1}{2}], \\ \sqrt{\frac{1}{h_\ell} - 1}, & \frac{\deg \Phi_\ell}{L} \in [\frac{1}{2}, 1). \end{cases} \quad (3.3)$$

in which h is a function suitable for filtered hyperinterpolation defined by (2.12). Note that in (3.3) we have excluded $\deg \Phi_\ell / L = 1$ because we would have $b_\ell = \infty$ and hence $\beta_\ell^\lambda = 0$ in (3.1) in this case.

It is not difficult to see that (3.1) is equivalent to

$$\min_{\beta^\lambda \in \mathbb{R}^d} \left\{ \frac{1}{2} \|\mathbf{W}^{1/2}(\mathbf{A}\beta^\lambda - \mathbf{f})\|_2^2 + \frac{1}{2} \|\mathbf{W}^{1/2}\mathbf{R}_2\beta^\lambda\|_2^2 + \lambda \|\mathbf{R}_1\beta^\lambda\|_1 \right\}, \quad (3.4)$$

where $\beta^\lambda = [\beta_1^\lambda, \dots, \beta_d^\lambda]^\top \in \mathbb{R}^d$ and $\mathbf{R}_1 = \text{diag}(\mu_1, \dots, \mu_d)$, $\mathbf{R}_2 = \mathbf{A}\mathbf{B} \in \mathbb{R}^{N \times d}$ with $\mathbf{A} = (\Phi_\ell(\mathbf{x}_j))_{j\ell} \in \mathbb{R}^{N \times d}$ and $\mathbf{B} = \text{diag}(b_1, \dots, b_d)$, $\mathbf{W} = \text{diag}(w_1, \dots, w_N)$.

We define the *hybrid hyperinterpolation* as follows.

Definition 3.1 (Hybrid hyperinterpolation) *Let $\Omega \subset \mathbb{R}^s$ be a compact domain and $f \in \mathcal{C}(\Omega)$. Suppose that the discrete scalar product $\langle f, g \rangle_N$ in (2.7) is determined by an N -point quadrature rule of PI-type in Ω with algebraic degree of exactness $2L$. The hybrid hyperinterpolation of f onto $\mathbb{P}_L(\Omega)$ is defined as*

$$\mathcal{H}_L^\lambda f := \sum_{\ell=1}^d h\left(\frac{\deg \Phi_\ell}{L}\right) \mathcal{S}_{\lambda\mu_\ell}(\langle f, \Phi_\ell \rangle_N) \Phi_\ell, \quad \lambda > 0, \quad (3.5)$$

where $h(\cdot)$ is a filter function defined by (2.12), $\{\mathcal{S}_{\lambda\mu_\ell}(\cdot)\}_{\ell=1}^d$ are soft thresholding operators and $\mu_1, \dots, \mu_d > 0$.

Then we obtain the following important result.

Theorem 3.1 *Let $\Omega \subset \mathbb{R}^s$ be a compact domain and $f \in \mathcal{C}(\Omega)$. Suppose that the discrete scalar product $\langle f, g \rangle_N$ in (2.7) is determined by an N -point quadrature rule of PI-type in Ω with algebraic degree of exactness $2L$. Next let $\mu_1, \dots, \mu_d > 0$, \mathcal{R}_L as in (3.2) and $\mathcal{H}_L^\lambda f \in \mathbb{P}_L(\Omega)$ be defined by (3.5). Then $\mathcal{H}_L^\lambda f$ is the solution to the regularized least squares approximation problem (3.1).*

Proof. Since the problem (2.11) is a strictly convex problem, the stationary point of the objective in (2.11) with respect to α leads to the first-order condition

$$\mathbf{A}^\top \mathbf{W} \mathbf{A} \alpha - \mathbf{A}^\top \mathbf{W} \mathbf{f} = \mathbf{0}. \quad (3.6)$$

Note that $\mathbf{A}^\top \mathbf{W} \mathbf{A} = \mathbf{I}_d$, where $\mathbf{I}_d \in \mathbb{R}^{d \times d}$ is the identity matrix, since

$$[\mathbf{A}^\top \mathbf{W} \mathbf{A}]_{ik} = \sum_{j=1}^N w_j \Phi_i(\mathbf{x}_j) \Phi_k(\mathbf{x}_j) = \langle \Phi_i, \Phi_k \rangle_N = \langle \Phi_i, \Phi_k \rangle = \delta_{ik}, \quad 1 \leq i, k \leq d, \quad (3.7)$$

By (3.6) and (3.7), we have $\boldsymbol{\alpha} = \mathbf{A}^T \mathbf{W} \mathbf{f}$. Then $\boldsymbol{\beta}^\lambda = [\beta_1^\lambda, \dots, \beta_d^\lambda]^T$ is a solution to the problem (3.4) if and only if

$$\mathbf{0} \in \mathbf{A}^T \mathbf{W} \mathbf{A} \boldsymbol{\beta}^\lambda - \mathbf{A}^T \mathbf{W} \mathbf{f} + \lambda \partial(\|\mathbf{R}_1 \boldsymbol{\beta}^\lambda\|_1) + \mathbf{B}^T \mathbf{A}^T \mathbf{W} \mathbf{A} \mathbf{B} \boldsymbol{\beta}^\lambda, \quad (3.8)$$

where $\partial(\cdot)$ denotes the subdifferential [11, Definition 3.2].

Thus the first-order condition (3.8) amounts to d one-dimensional problems

$$0 \in \beta_\ell^\lambda - \alpha_\ell + \lambda \mu_\ell \partial(|\beta_\ell^\lambda|) + b_\ell^2 \beta_\ell^\lambda, \quad \forall \ell = 1, \dots, d, \quad (3.9)$$

where

$$\partial(|\beta_\ell^\lambda|) = \begin{cases} 1, & \text{if } \beta_\ell^\lambda > 0, \\ \in [-1, 1], & \text{if } \beta_\ell^\lambda = 0, \\ -1, & \text{if } \beta_\ell^\lambda < 0. \end{cases}$$

Let $\boldsymbol{\beta}^\lambda = [\beta_1^\lambda, \dots, \beta_d^\lambda]^T$ be the optimal solution to the problem (3.4). Then we have

$$\beta_\ell^\lambda = \frac{1}{1 + b_\ell^2} [\alpha_\ell - \lambda \mu_\ell \partial(|\beta_\ell^\lambda|)], \quad \forall \ell = 1, \dots, d.$$

In fact, there are three cases we need to consider:

- (i) If $\alpha_\ell > \lambda \mu_\ell$, then $\alpha_\ell - \lambda \mu_\ell \partial(|\beta_\ell^\lambda|) > 0$; thus, $\beta_\ell^\lambda > 0$, yielding $\partial(|\beta_\ell^\lambda|) = 1$, and then $\beta_\ell^\lambda = (\alpha_\ell - \lambda \mu_\ell) / (1 + b_\ell^2) > 0$.
- (ii) If $\alpha_\ell < -\lambda \mu_\ell$, then $\alpha_\ell - \lambda \mu_\ell \partial(|\beta_\ell^\lambda|) < 0$; thus, $\beta_\ell^\lambda < 0$, yielding $\partial(|\beta_\ell^\lambda|) = -1$, and then $\beta_\ell^\lambda = (\alpha_\ell + \lambda \mu_\ell) / (1 + b_\ell^2) < 0$.
- (iii) If $-\lambda \mu_\ell \leq \alpha_\ell \leq \lambda \mu_\ell$, then $\beta_\ell^\lambda > 0$ means that $\beta_\ell^\lambda = (\alpha_\ell - \lambda \mu_\ell) / (1 + b_\ell^2) < 0$, and $\beta_\ell^\lambda < 0$ means that $\beta_\ell^\lambda = (\alpha_\ell + \lambda \mu_\ell) / (1 + b_\ell^2) > 0$. Thus, we get two contradictions and $\beta_\ell^\lambda = 0$.

Recall that $\alpha_\ell = \langle f, \Phi_\ell \rangle_N$ and b_ℓ defined by (3.3) for all $\ell = 1, \dots, d$, we deduce that the polynomial constructed with coefficients

$$\beta_\ell^\lambda = \frac{\mathcal{S}_{\lambda \mu_\ell}(\langle f, \Phi_\ell \rangle_N)}{1 + b_\ell^2} = h_\ell \mathcal{S}_{\lambda \mu_\ell}(\langle f, \Phi_\ell \rangle_N) \quad (3.10)$$

is hybrid hyperinterpolation $\mathcal{H}_L^\lambda f$. □

Remark 3.1 It is clear that hybrid hyperinterpolation $\mathcal{H}_L^\lambda f$ can be regarded as the composite of filtered hyperinterpolation $\mathcal{F}_{L,N} f$ and Lasso hyperinterpolation $\mathcal{L}_L^\lambda f$, that is,

$$\mathcal{H}_L^\lambda f = \mathcal{F}_{L,N}(\mathcal{L}_L^\lambda f).$$

Observe that, since we suppose that $h : [0, 1] \rightarrow [0, 1]$, necessarily

$$|h_\ell \langle f, \Phi_\ell \rangle_N| \leq |\langle f, \Phi_\ell \rangle_N|,$$

the commutative property between $\mathcal{F}_{L,N} f$ and $\mathcal{L}_L^\lambda f$ does not hold in general, i.e., it may be

$$\mathcal{F}_{L,N}(\mathcal{L}_L^\lambda f) \neq \mathcal{L}_L^\lambda(\mathcal{F}_{L,N} f).$$

4 Error analysis in the $L^2(\Omega)$ -norm

Lemma 4.1 (Theorem 1 in [28]) Adopt conditions of Lemma 2.1. Then

$$\|\mathcal{L}_L f\|_2 \leq V^{1/2} \|f\|_\infty, \quad (4.1)$$

and

$$\|\mathcal{L}_L f - f\|_2 \leq 2V^{1/2} E_L(f). \quad (4.2)$$

Thus $\|\mathcal{L}_L f - f\|_2 \rightarrow 0$ as $L \rightarrow \infty$.

Under assumptions of Theorem 3.1, let $\alpha_\ell = \langle f, \Phi_\ell \rangle_N = \sum_{j=1}^N w_j f(\mathbf{x}_j) \Phi_\ell(\mathbf{x}_j)$ and introduce the operator

$$K(f) := \sum_{\ell=1}^d \{h_\ell \mathcal{S}_{\lambda\mu_\ell}(\alpha_\ell) \cdot \alpha_\ell - [h_\ell \mathcal{S}_{\lambda\mu_\ell}(\alpha_\ell)]^2\} \geq 0. \quad (4.3)$$

Note that the positiveness of $K(f)$ is a direct consequence of the fact that

$$\begin{cases} \alpha_\ell < h_\ell \mathcal{S}_{\lambda\mu_\ell}(\alpha_\ell) \leq 0, & \text{if } \alpha_\ell < 0, \\ 0 \leq h_\ell \mathcal{S}_{\lambda\mu_\ell}(\alpha_\ell) \leq \alpha_\ell, & \text{if } \alpha_\ell \geq 0. \end{cases} \quad (4.4)$$

that easily implies that each argument of the sum on the right-hand side of (4.4) is non-negative. If λ is sufficiently large, that is,

$$\lambda \geq \max_{\ell=1, \dots, d} \frac{|\alpha_\ell|}{\mu_\ell},$$

then $K(f) = 0$. With the help of $K(f)$, the following lemma holds:

Lemma 4.2 *Under the conditions of Theorem 3.1,*

- (a) $\langle f - \mathcal{H}_L^\lambda f, \mathcal{H}_L^\lambda f \rangle_N = K(f)$;
- (b) $\langle \mathcal{H}_L^\lambda f, \mathcal{H}_L^\lambda f \rangle_N + \langle f - \mathcal{H}_L^\lambda f, f - \mathcal{H}_L^\lambda f \rangle_N = \langle f, f \rangle_N - 2K(f)$;
- (c) $K(f) \leq \langle f, f \rangle_N / 2$;
- (d) $\langle \mathcal{H}_L^\lambda f, \mathcal{H}_L^\lambda f \rangle_N \leq \langle f, f \rangle_N - 2K(f)$,

Proof. (a) We obtain

$$\begin{aligned} & \langle f - \mathcal{H}_L^\lambda f, \mathcal{H}_L^\lambda f \rangle_N \\ &= \left\langle f - \sum_{\ell=1}^d h_\ell \mathcal{S}_{\lambda\mu_\ell}(\alpha_\ell) \Phi_\ell, \sum_{k=1}^d h_k \mathcal{S}_{\lambda\mu_k}(\alpha_k) \Phi_k \right\rangle_N, \\ &= \sum_{k=1}^d h_k \mathcal{S}_{\lambda\mu_k}(\alpha_k) \left\langle f - \sum_{\ell=1}^d h_\ell \mathcal{S}_{\lambda\mu_\ell}(\alpha_\ell) \Phi_\ell, \Phi_k \right\rangle_N, \end{aligned}$$

and

$$\begin{aligned} & \left\langle f - \sum_{\ell=1}^d h_\ell \mathcal{S}_{\lambda\mu_\ell}(\alpha_\ell) \Phi_\ell, \Phi_k \right\rangle_N \\ &= \langle f, \Phi_k \rangle_N - \left\langle \sum_{\ell=1}^d h_\ell \mathcal{S}_{\lambda\mu_\ell}(\alpha_\ell) \Phi_\ell, \Phi_k \right\rangle_N, \\ &= \alpha_k - h_k \mathcal{S}_{\lambda\mu_k}(\alpha_k). \end{aligned}$$

Hence, we have

$$\begin{aligned} & \sum_{k=1}^d h_k \mathcal{S}_{\lambda\mu_k}(\alpha_k) \left\langle f - \sum_{\ell=1}^d h_\ell \mathcal{S}_{\lambda\mu_\ell}(\alpha_\ell) \Phi_\ell, \Phi_k \right\rangle_N \\ &= \sum_{k=1}^d h_k \mathcal{S}_{\lambda\mu_k}(\alpha_k) \cdot [\alpha_k - h_k \mathcal{S}_{\lambda\mu_k}(\alpha_k)], \\ &= \sum_{k=1}^d \{h_k \mathcal{S}_{\lambda\mu_k}(\alpha_k) \alpha_k - [h_k \mathcal{S}_{\lambda\mu_k}(\alpha_k)]^2\}, \\ &= K(f). \end{aligned}$$

(b) From $\langle \mathcal{H}_L^\lambda f, \mathcal{H}_L^\lambda f \rangle_N = \langle f, \mathcal{H}_L^\lambda f \rangle_N - K(f)$ and

$$\langle f - \mathcal{H}_L^\lambda f, f - \mathcal{H}_L^\lambda f \rangle_N = \langle f, f \rangle_N - 2 \langle f, \mathcal{H}_L^\lambda f \rangle_N + \langle \mathcal{H}_L^\lambda f, \mathcal{H}_L^\lambda f \rangle_N,$$

we immediately have

$$\langle \mathcal{H}_L^\lambda f, \mathcal{H}_L^\lambda f \rangle_N + \langle f - \mathcal{H}_L^\lambda f, f - \mathcal{H}_L^\lambda f \rangle_N = \langle f, f \rangle_N - 2K(f). \quad (4.5)$$

(c) From equality (4.5) and the non-negativeness of $\langle g, g \rangle_N$ for any $g \in \mathcal{C}(\Omega)$, we obtain $\langle f, f \rangle_N - 2K(f) \geq 0$, which means that $K(f) \leq \langle f, f \rangle_N / 2$.

(d) From equality (4.5) and the non-negativeness of $\langle f - \mathcal{H}_L^\lambda f, f - \mathcal{H}_L^\lambda f \rangle_N$, we immediately have $\langle \mathcal{H}_L^\lambda f, \mathcal{H}_L^\lambda f \rangle_N \leq \langle f, f \rangle_N - 2K(f)$. \square

The following result describes $L^2(\Omega)$ -norm of hybrid hyperinterpolation operator and the $L^2(\Omega)$ -error of hybrid hyperinterpolation with noise-free.

Theorem 4.1 *Adopt conditions of Theorem 3.1. Then there exists $\tau_1 < 1$, which relies on f and is inversely related to $K(f)$ such that*

$$\|\mathcal{H}_L^\lambda f\|_2 \leq \tau_1 V^{1/2} \|f\|_\infty, \quad (4.6)$$

and

$$\|\mathcal{H}_L^\lambda f - f\|_2 \leq \|\mathcal{H}_L^\lambda f - \mathcal{L}_L f\|_2 + 2V^{1/2} E_L(f). \quad (4.7)$$

Thus

$$\lim_{L \rightarrow \infty} \lim_{\lambda \rightarrow 0} \|\mathcal{H}_L^\lambda f - f\|_2 = 0.$$

Proof. For any $f \in \mathcal{C}(\Omega)$, we get that $\mathcal{H}_L^\lambda f \in \mathbb{P}_L(\Omega)$. By Lemma 4.2 (d), we have

$$\begin{aligned} \|\mathcal{H}_L^\lambda f\|_2^2 &= \langle \mathcal{H}_L^\lambda f, \mathcal{H}_L^\lambda f \rangle = \langle \mathcal{H}_L^\lambda f, \mathcal{H}_L^\lambda f \rangle_N \leq \langle f, f \rangle_N - 2K(f), \\ &= \sum_{j=1}^N w_j (f(\mathbf{x}_j))^2 - 2K(f) \leq \sum_{j=1}^N w_j \|f\|_\infty^2 - 2K(f), \\ &= V \|f\|_\infty^2 - 2K(f). \end{aligned}$$

Since $\|\mathcal{H}_L^\lambda f\|_2 \leq \sqrt{V \|f\|_\infty^2 - 2K(f)}$ and $K(f) > 0$, there exist $\tau_1 = \tau_1(K(f)) < 1$, which is inversely related to $K(f)$ such that

$$\|\mathcal{H}_L^\lambda f\|_2 \leq \sqrt{V \|f\|_\infty^2 - 2K(f)} = \tau_1 V^{1/2} \|f\|_\infty.$$

With the aid of Lemma 4.1, we have

$$\begin{aligned} \|\mathcal{H}_L^\lambda f - f\|_2 &\leq \|\mathcal{H}_L^\lambda f - \mathcal{L}_L f\|_2 + \|\mathcal{L}_L f - f\|_2, \\ &\leq \|\mathcal{H}_L^\lambda f - \mathcal{L}_L f\|_2 + 2V^{1/2} E_L(f). \end{aligned}$$

Since

$$\lim_{\lambda \rightarrow 0} \|\mathcal{H}_L^\lambda f - \mathcal{L}_L f\|_2 = 0, \quad \text{and} \quad \lim_{L \rightarrow \infty} \|\mathcal{L}_L f - f\|_2 = 0,$$

then we obtain

$$\lim_{L \rightarrow \infty} \lim_{\lambda \rightarrow 0} \|\mathcal{H}_L^\lambda f - f\|_2 = 0. \quad \square$$

Remark 4.1 *Recall the operator norm (4.1) and the error bound (4.2) for \mathcal{L}_L , the upper bound of the operator norm of \mathcal{H}_L^λ is lower than that of \mathcal{L}_L , but the error bound for hybrid hyperinterpolation introduces a term $\|\mathcal{H}_L^\lambda f - \mathcal{L}_L f\|_2$. Thus we will not give priority to hybrid hyperinterpolation to reconstruct the test function with noise-free.*

The following result describes the $L^2(\Omega)$ -error of hybrid hyperinterpolation with noise.

Theorem 4.2 *Adopt conditions of Theorem 3.1 and assume $f^\epsilon \in \mathcal{C}(\Omega)$ is a noisy version of f . Then*

$$\|\mathcal{H}_L^\lambda f^\epsilon - f\|_2 \leq \|\mathcal{H}_L^\lambda f^\epsilon - \mathcal{L}_L f^\epsilon\|_2 + 2V^{1/2}E_L(f^\epsilon) + V^{1/2}\|f^\epsilon - f\|_\infty. \quad (4.8)$$

Thus

$$\lim_{L \rightarrow \infty} \lim_{\lambda \rightarrow 0} \|\mathcal{H}_L^\lambda f^\epsilon - f\|_2 = V^{1/2}\|f - f^\epsilon\|_\infty.$$

Proof. With the aid of Lemma 4.1, we have

$$\begin{aligned} \|\mathcal{L}f^\epsilon - f\|_2 &= \|\mathcal{L}f^\epsilon - f^\epsilon + f^\epsilon - f\|_2, \\ &\leq \|\mathcal{L}f^\epsilon - f^\epsilon\|_2 + \|f^\epsilon - f\|_2, \\ &\leq 2V^{1/2}E_L(f^\epsilon) + V^{1/2}\|f^\epsilon - f\|_\infty, \end{aligned}$$

where the second term on the right side in the third row follows from the Cauchy-Schwarz inequality, which ensures $\|g\|_2 = \sqrt{\langle g, g \rangle} \leq \|g\|_\infty \sqrt{\langle \mathbf{1}, \mathbf{1} \rangle} = V^{1/2}\|g\|_\infty$ for all $g \in \mathcal{C}(\Omega)$. Then we have

$$\begin{aligned} \|\mathcal{H}_L^\lambda f^\epsilon - f\|_2 &\leq \|\mathcal{H}_L^\lambda f^\epsilon - \mathcal{L}_L f^\epsilon\|_2 + \|\mathcal{L}_L f^\epsilon - f\|_2, \\ &\leq \|\mathcal{H}_L^\lambda f^\epsilon - \mathcal{L}_L f^\epsilon\|_2 + 2V^{1/2}E_L(f^\epsilon) + V^{1/2}\|f^\epsilon - f\|_\infty. \end{aligned}$$

From the fact

$$\lim_{\lambda \rightarrow 0} \|\mathcal{H}_L^\lambda f^\epsilon - \mathcal{L}_L f^\epsilon\|_2 = 0 \quad \text{and} \quad \lim_{L \rightarrow \infty} E_L(f^\epsilon) = 0,$$

we obtain

$$\lim_{L \rightarrow \infty} \lim_{\lambda \rightarrow 0} \|\mathcal{H}_L^\lambda f^\epsilon - f\|_2 = V^{1/2}\|f - f^\epsilon\|_\infty. \quad \square$$

Remark 4.2 *Although $\lim_{L \rightarrow \infty} \|\mathcal{L}f^\epsilon - f\|_2 = V^{1/2}\|f^\epsilon - f\|_\infty$, we consider hybrid hyperinterpolation as an effective tool to recover the test function with noise since it possesses the basis selection ability and doubly shrinks less relevant coefficients.*

5 Prior regularization parameter choices rule

In the setting of sparse recovery, the regularization parameter directly decides to what extent the sparsity of coefficients of hybrid hyperinterpolation. It is natural to list coefficients of hyperinterpolation in descending order by absolute values, and then choose the k -th one as the regularization parameter passing to the soft thresholding operator to realise basis selections. Based on this idea, we propose an useful regularization parameter choices rule which fully makes use of coefficients of hyperinterpolation

$$\mathcal{L}_L f^\epsilon = \sum_{\ell=1}^d \langle f^\epsilon, \Phi_\ell \rangle_N \Phi_\ell = \sum_{\ell=1}^d \alpha_\ell \Phi_\ell.$$

Before we give the result, we first state that $\{\Phi_\ell | \ell = 1, \dots, d\}$ is an orthonormal basis of $\mathbb{P}_L(\Omega)$ with respect to (2.8). Once the quadrature rule is determined, we immediately have the sampling matrix $\mathbf{A} = (\Phi_\ell(\mathbf{x}_j))_{j\ell} \in \mathbb{R}^{N \times d}$ and corresponding quadrature weights matrix $\mathbf{W} = \text{diag}(w_1, \dots, w_N) \in \mathbb{R}^{N \times N}$.

Theorem 5.1 *Adopt conditions of Theorem 3.1. Let $f^\epsilon, f \in \mathcal{C}(\Omega)$, $f^\epsilon - f = \epsilon$ and $\{\mu_\ell\}_{\ell=1}^d$ all be 1. Let $\boldsymbol{\epsilon} = [\epsilon_1, \dots, \epsilon_N]^T \in \mathbb{R}^N$ and $\mathbf{f}^\epsilon = [f^\epsilon(\mathbf{x}_1), \dots, f^\epsilon(\mathbf{x}_N)]^T \in \mathbb{R}^N$. Let $\boldsymbol{\alpha} = [\alpha_1, \dots, \alpha_d]^T \in \mathbb{R}^d$ with $\alpha_\ell = \langle f^\epsilon, \Phi_\ell \rangle_N$, and $\boldsymbol{\zeta} = [\zeta_1, \dots, \zeta_d]^T \in \mathbb{R}^d$ with ζ_s being the absolute value of the element in $\boldsymbol{\alpha}$ such that $\zeta_1 \geq \dots \geq \zeta_d$ for $s = 1, \dots, d$. Let*

$$J(\mathbf{z}) := \sum_{\ell=1}^d (z_\ell^2 - 2z_\ell \alpha_\ell) \quad \text{and} \quad H(\mathbf{z}) := 2 \sum_{\ell=1}^d z_\ell \sum_{j=1}^N w_j \epsilon_j \Phi_\ell(\mathbf{x}_j) \quad (5.1)$$

for $\mathbf{z} = [z_1, \dots, z_d]^T \in \mathbb{R}^d$. If the regularization parameters in (2.16), (2.19) and (3.1) are chosen by $\lambda^* = \lambda(s) := \zeta_s$ for $s \in \{1, \dots, d\}$, then

(a) for hybrid hyperinterpolation

$$\|\mathbf{W}^{1/2}(\mathbf{A}\boldsymbol{\beta}^{\lambda(s)} - \mathbf{f})\|_2^2 = J(\boldsymbol{\beta}^{\lambda(s)}) + H(\boldsymbol{\beta}^{\lambda(s)}) + \|\mathbf{W}^{1/2}\mathbf{f}\|_2^2, \quad (5.2)$$

and the value of first term in the right-hand side of (5.2) is non-positive and decreases when s increases, and $\|\boldsymbol{\beta}^{\lambda(s)}\|_0 \leq s - 1$;

(b) for Lasso hyperinterpolation

$$\|\mathbf{W}^{1/2}(\mathbf{A}\boldsymbol{\gamma}^{\lambda(s)} - \mathbf{f})\|_2^2 = J(\boldsymbol{\gamma}^{\lambda(s)}) + H(\boldsymbol{\gamma}^{\lambda(s)}) + \|\mathbf{W}^{1/2}\mathbf{f}\|_2^2, \quad (5.3)$$

and the value of first term in the right-hand side of (5.3) is non-positive and decreases when s increases, and $\|\boldsymbol{\gamma}^{\lambda(s)}\|_0 \leq s - 1$;

(c) for hard thresholding hyperinterpolation

$$\|\mathbf{W}^{1/2}(\mathbf{A}\boldsymbol{\nu}^{\lambda(s)} - \mathbf{f})\|_2^2 = J(\boldsymbol{\nu}^{\lambda(s)}) + H(\boldsymbol{\nu}^{\lambda(s)}) + \|\mathbf{W}^{1/2}\mathbf{f}\|_2^2, \quad (5.4)$$

and the value of first term in the right-hand side of (5.4) is non-positive and decreases when s increases, and $\|\boldsymbol{\nu}^{\lambda(s)}\|_0 \leq s - 1$.

(d) the following inequalities hold

$$J(\boldsymbol{\nu}^{\lambda(s)}) \leq J(\boldsymbol{\gamma}^{\lambda(s)}) \leq J(\boldsymbol{\beta}^{\lambda(s)}) \leq 0. \quad (5.5)$$

Proof. In the following, we only prove (a) and (d) because proofs in (b) and (c) are very similar to (a).

(a) For hybrid hyperinterpolation, since $\mathbf{A}^T\mathbf{W}\mathbf{A} = \mathbf{I}_d$, we easily have

$$\left\langle \mathbf{W}^{1/2}\mathbf{A}\boldsymbol{\beta}^{\lambda(s)}, \mathbf{W}^{1/2}\mathbf{A}\boldsymbol{\beta}^{\lambda(s)} \right\rangle = \sum_{\ell=1}^d \left(\beta_{\ell}^{\lambda(s)} \right)^2.$$

Next, since $\boldsymbol{\alpha} = \mathbf{A}^T\mathbf{W}\mathbf{f}^{\epsilon}$, necessarily $\langle \boldsymbol{\beta}^{\lambda(s)}, \mathbf{A}^T\mathbf{W}\mathbf{f}^{\epsilon} \rangle = \sum_{\ell=1}^d \beta_{\ell}^{\lambda(s)} \alpha_{\ell}$.

Thus

$$\begin{aligned} & \|\mathbf{W}^{1/2}(\mathbf{A}\boldsymbol{\beta}^{\lambda(s)} - \mathbf{f})\|_2^2 \\ &= \left\langle \mathbf{W}^{1/2}\mathbf{A}\boldsymbol{\beta}^{\lambda(s)}, \mathbf{W}^{1/2}\mathbf{A}\boldsymbol{\beta}^{\lambda(s)} \right\rangle - 2 \left\langle \mathbf{W}^{1/2}\mathbf{A}\boldsymbol{\beta}^{\lambda(s)}, \mathbf{W}^{1/2}\mathbf{f} \right\rangle + \left\langle \mathbf{W}^{1/2}\mathbf{f}, \mathbf{W}^{1/2}\mathbf{f} \right\rangle, \\ &= \sum_{\ell=1}^d \left(\beta_{\ell}^{\lambda(s)} \right)^2 - 2 \left\langle \boldsymbol{\beta}^{\lambda(s)}, \mathbf{A}^T\mathbf{W}(\mathbf{f} + \boldsymbol{\epsilon} - \boldsymbol{\epsilon}) \right\rangle + \sum_{j=1}^N w_j f^2(\mathbf{x}_j), \\ &= \sum_{\ell=1}^d \left(\beta_{\ell}^{\lambda(s)} \right)^2 - 2 \left\langle \boldsymbol{\beta}^{\lambda(s)}, \mathbf{A}^T\mathbf{W}\mathbf{f}^{\epsilon} \right\rangle + 2 \left\langle \boldsymbol{\beta}^{\lambda(s)}, \mathbf{A}^T\mathbf{W}\boldsymbol{\epsilon} \right\rangle + \sum_{j=1}^N w_j f^2(\mathbf{x}_j), \\ &= \sum_{\ell=1}^d \left[\left(\beta_{\ell}^{\lambda(s)} \right)^2 - 2\beta_{\ell}^{\lambda(s)} \alpha_{\ell} \right] + 2 \left\langle \boldsymbol{\beta}^{\lambda(s)}, \mathbf{A}^T\mathbf{W}\boldsymbol{\epsilon} \right\rangle + \sum_{j=1}^N w_j f^2(\mathbf{x}_j), \\ &= \sum_{\ell=1}^d \left[\left(\beta_{\ell}^{\lambda(s)} \right)^2 - 2\beta_{\ell}^{\lambda(s)} \alpha_{\ell} \right] + 2 \sum_{\ell=1}^d \beta_{\ell}^{\lambda(s)} \sum_{j=1}^N w_j \epsilon_j \Phi_{\ell}(\mathbf{x}_j) + \sum_{j=1}^N w_j f^2(\mathbf{x}_j), \\ &= J(\boldsymbol{\beta}^{\lambda(s)}) + H(\boldsymbol{\beta}^{\lambda(s)}) + \|\mathbf{W}^{1/2}\mathbf{f}\|_2^2. \end{aligned}$$

By definition of hybrid hyperinterpolation (3.5), we have

$$\beta_{\ell}^{\lambda(s)} = h_{\ell} \mathcal{S}_{\lambda(s)}(\alpha_{\ell}), \quad \forall \ell = 1, \dots, d,$$

where $h_{\ell} = h(\deg \Phi_{\ell}/L)$ and $h(\cdot)$ is defined by (2.12) and the range of h is $[0, 1]$.

- If $\beta_\ell > 0$, then $\beta_\ell = h_\ell(\alpha_\ell - \lambda(s))$ with $\alpha_\ell > 0$ and since $h \in [0, 1]$ it is not difficult to prove that

$$\begin{aligned}
 (\beta_\ell^{\lambda(s)})^2 - 2\beta_\ell^{\lambda(s)}\alpha_\ell &= h_\ell^2(\alpha_\ell - \lambda(s))^2 - 2h_\ell(\alpha_\ell - \lambda(s))\alpha_\ell \\
 &\leq h_\ell(\alpha_\ell - \lambda(s))^2 - 2h_\ell(\alpha_\ell - \lambda(s))\alpha_\ell \\
 &\leq h_\ell(\alpha_\ell - \lambda(s))(\alpha_\ell - \lambda(s) - 2\alpha_\ell) \\
 &= h_\ell(\alpha_\ell - \lambda(s))(-\alpha_\ell - \lambda(s)) \\
 &= h_\ell(\lambda^2(s) - \alpha_\ell^2) \leq 0.
 \end{aligned}$$

- If $\beta_\ell < 0$, then $\beta_\ell = h_\ell(\alpha_\ell + \lambda(s))$ with $\alpha_\ell < 0$ and since $h_\ell \in [0, 1]$, we get

$$\begin{aligned}
 (\beta_\ell^{\lambda(s)})^2 - 2\beta_\ell^{\lambda(s)}\alpha_\ell &= h_\ell^2(\alpha_\ell + \lambda(s))^2 - 2h_\ell(\alpha_\ell + \lambda(s))\alpha_\ell \\
 &\leq h_\ell(\alpha_\ell + \lambda(s))^2 - 2h_\ell(\alpha_\ell + \lambda(s))\alpha_\ell \\
 &\leq h_\ell(\alpha_\ell + \lambda(s))(\alpha_\ell + \lambda(s) - 2\alpha_\ell) \\
 &= h_\ell(\alpha_\ell + \lambda(s))(\lambda(s) - \alpha_\ell) \\
 &= h_\ell(\lambda^2(s) - \alpha_\ell^2) \leq 0.
 \end{aligned}$$

Since $\lambda(s)$ is the s -th largest element in ζ and by Definition 2.2 of the soft thresholding operator

$$\|\beta^{\lambda(s)}\|_0 = \#\{\ell : h_\ell \mathcal{S}_{\lambda(s)}(\alpha_\ell) \neq 0\},$$

where $\#\$ denotes the cardinal number of the corresponding set, it is obvious that $\|\beta^{\lambda(s)}\|_0 \leq s - 1$ and

$$\sum_{\ell=1}^d \left[(\beta_\ell^{\lambda(s)})^2 - 2\beta_\ell^{\lambda(s)}\alpha_\ell \right] \leq 0$$

decreases when s increases.

- (d) Recall that nonzeros $\beta_\ell^{\lambda(s)}$, $\gamma_\ell^{\lambda(s)}$, $\nu_\ell^{\lambda(s)}$ and α_ℓ have the same sign, and

$$|\beta_\ell^{\lambda(s)}| \leq |\gamma_\ell^{\lambda(s)}| \leq |\nu_\ell^{\lambda(s)}| \leq |\alpha_\ell|$$

for $\ell = 1, \dots, d$. Then we obtain

$$\begin{aligned}
 &\sum_{\ell=1}^d \left[(\gamma_\ell^{\lambda(s)})^2 - 2\gamma_\ell^{\lambda(s)}\alpha_\ell \right] - \sum_{\ell=1}^d \left[(\beta_\ell^{\lambda(s)})^2 - 2\beta_\ell^{\lambda(s)}\alpha_\ell \right] \\
 &= \sum_{\ell=1}^d \left[(\gamma_\ell^{\lambda(s)} - \beta_\ell^{\lambda(s)})(\gamma_\ell^{\lambda(s)} + \beta_\ell^{\lambda(s)} - 2\alpha_\ell) \right] \leq 0,
 \end{aligned}$$

and

$$\begin{aligned}
 &\sum_{\ell=1}^d \left[(\nu_\ell^{\lambda(s)})^2 - 2\nu_\ell^{\lambda(s)}\alpha_\ell \right] - \sum_{\ell=1}^d \left[(\gamma_\ell^{\lambda(s)})^2 - 2\gamma_\ell^{\lambda(s)}\alpha_\ell \right] \\
 &= \sum_{\ell=1}^d \left[(\nu_\ell^{\lambda(s)} - \gamma_\ell^{\lambda(s)})(\nu_\ell^{\lambda(s)} + \gamma_\ell^{\lambda(s)} - 2\alpha_\ell) \right] \leq 0.
 \end{aligned}$$

□

Remark 5.1 Recall the Riemann-Lebesgue lemma, most coefficients of hyperinterpolation tend to zero except those corresponding to polynomial basis with low order.

Remark 5.2 Parameters chosen by Theorem 5.1 lead values of non-zero coefficients β_ℓ^λ , γ_ℓ^λ and ν_ℓ^λ fast changing from zero and then gradually approaching α_ℓ when the sparsity ranges from one to large values. This fact indicates that values of the term $J(\cdot)$ in (5.2), (5.3) and (5.4) will first quickly drop and then slowly decrease if the sparsity increases from one to other values. Similarly, the term $H(\cdot)$ in (5.2), (5.3) and (5.4) scarcely effects L_2 errors when the sparsity is small. This result will be supported by some numerical experiments in section 6.

Remark 5.3 From (5.2) and (5.3) and the existence of filtered function defined by (2.12), hybrid hyperinterpolation performs very close to Lasso hyperinterpolation when the sparsity is small. For example, we could see differences between hybrid hyperinterpolation and Lasso hyperinterpolation when the sparsity is relatively large.

6 Numerical experiments

In this section we test qualities of the classical hyperinterpolation as well as the variants mentioned above, proposing some experiments similar to those specified in [4]. In particular we will take as Ω , the interval $[-1, 1]$, the disk $B(\mathbf{0}, 1)$, the unit-sphere $\mathbb{S}_2 \subset \mathbb{R}^3$, the unit-cube $[-1, 1]^3$, but also the case of a bivariate domain defined as union of disks. The main interest for the latter is that, differently for the other examples, an orthonormal basis is not theoretically available and must be computed numerically.

In the five regions mentioned above, we have examined *Tikhonov*, *Filtered*, *Lasso*, *Hybrid*, *Hard thresholding* as well as the classical *Hyperinterpolation* to make a contrast. Concerning the regularization parameters $\lambda^* = \lambda(k)$, as written in Theorem 5.1, we sort in descending order the absolute value of the hyperinterpolation coefficients and take the k -th one. This value is used for the Tikhonov regularization, the Lasso, hard thresholding and hybrid hyperinterpolations. Besides, all the penalty parameters $\{\mu_\ell\}_{\ell=1}^d$ in Lasso and hybrid hyperinterpolations are set to be 1.

In the case of filtered hyperinterpolation we have used as $h : [0, 1] \rightarrow [0, 1]$ the function

$$h(x) = \begin{cases} 1, & x \in [0, \frac{1}{2}], \\ \sin^2(\pi x), & x \in [\frac{1}{2}, 1]. \end{cases}$$

Depending on the numerical experiments, we have added noise to the evaluation of a function f on the N nodes $\{\mathbf{x}_j\}_{j=1}^N$. In particular we considered

- *Gaussian noise* $\mathcal{N}(0, \sigma^2)$ from a normal distribution with mean 0 and standard deviation σ , implemented via the Matlab command

`sigma*randn(N,1);`

- *impulse noise* $\mathcal{J}(a)$ that takes a uniformly distributed random values in $[-a, a]$ with probability 1/2 by means of the Matlab command

`a*(1-2*rand(N,1))*binornd(1,0.5).`

since `binornd(1,0.5)` returns an array of random numbers chosen from a binomial distribution with parameters 1 and 1/2.

In the following five numerical examples, we will take two ways to compute L_2 errors. Specifically, the first one used for Tikhonov, filtered, Lasso, hybrid, hard thresholding as well the classical hyperinterpolation is the traditional way defined by

$$\|f - p_L\|_2 \approx \|f - p_L\|_{2,w},$$

where p_L is the generic hyperinterpolant on the perturbed data $\{(\mathbf{x}_j, f(\mathbf{x}_j) + \epsilon_j)\}_{j=1}^N$ and $\|g\|_{2,w} = \sqrt{\langle g, g \rangle_N}$. Alternatively, in the case of Lasso, hybrid and hard thresholding hyperinterpolations, one can compute the L_2 errors as defined by Theorem 5.1.

All Matlab codes used in these experiments are available at a GITHUB homepage <https://github.com/alvisesommariva/Hyper23>. All tests were performed by Matlab R2022a, Update 4.

6.1 The interval $[-1, 1]$

As the first domain, we set $\Omega = [-1, 1]$. In this case, taking into account the Legendre measure, one can consider the relative orthonormal polynomial basis on $\mathbb{P}_L(\Omega)$ defined as $\Phi_k(x) = \sqrt{\frac{2k-1}{2}}\varphi_{k-1}(x)$, with $k = 1, \dots, n+1$, where

$$\begin{cases} (k+1)\varphi_{k+1}(x) = (2k+1)x\varphi_k(x) - k\varphi_{k-1}(x), & k = 1, 2, \dots, \\ \varphi_0(x) \equiv 1, & \varphi_1(x) = x. \end{cases}$$

Since this family of polynomials is triangular, it is obvious that $\deg(\Phi_k) = k - 1$, knowledge fundamental in the computation of filtered hyperinterpolants.

Next, it is well-known that the Gauss-Legendre rule with nodes $\{\mathbf{x}_j\}_{j=1}^N$ and weights $\{w_j\}_{j=1}^N$, $N = L + 1$, has algebraic degree of precision $2L + 1$ and thus for any $f, g \in \mathbb{P}_L(\Omega)$ we have

$$\int_{-1}^1 f(x)g(x)dx = \langle f, g \rangle = \langle f, g \rangle_N := \sum_{j=1}^N w_j f(x_j)g(x_j).$$

As the numerical experiment, we examine the reconstruction of the Gaussian function $f(x) = \exp(-x^2)$, to which we have added Gaussian noise. This means that having at hand the perturbed data $\{(\mathbf{x}_j, f(\mathbf{x}_j) + \epsilon_j)\}_{j=1}^N$, where $\{\mathbf{x}_j\}_{j=1}^N$ are the quadrature nodes and $\epsilon_j \in \mathcal{N}(0, \sigma^2)$, we wish to approximate f by a polynomial of degree at most $L = N - 1$.

In the battery of 100 numerical tests, we have set $L = 250$ and $N = 251$ and varied the noise.

In Figure 1 (a) we display the an average evaluation of functions J and H in Theorem 5.1. On the upper part of the picture there are the plots of H , while on the lower one those of J . It is clear that the term $H(\cdot)$ will not take effect until the sparsity becomes large and the term $J(\cdot)$ mainly decide the trends of L_2 errors.

In Figure 1 (b) we exhibit the behaviour of $\mathcal{H}_L^\lambda f$, $\mathcal{L}_L^\lambda f$, $\mathcal{E}_L^\lambda f$ (i.e., respectively the hybrid, Lasso and hard thresholding hyperinterpolants) when $\sigma = 0, 4$ and, following the notation of Theorem 5.1, $\lambda^* \in \{\lambda(1), \lambda(2), \dots, \lambda(100)\}$. The results show a better ratio between the sparsity and L_2 errors by the new hybrid technique. Next, hard thresholding hyperinterpolation for a certain λ^* has a very good performance, but it is highly sensitive to a change of such parameter, rapidly deteriorating.

The quality of the new hybrid hyperinterpolant is also manifested in Table 6.1, where the results are in general better then those of the other competitors for the same λ .

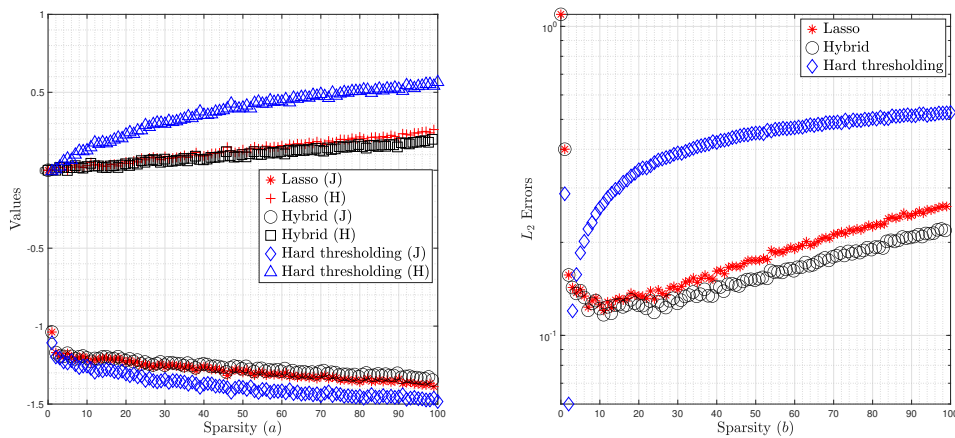


Figure 1: Approximate $f(x) = \exp(-x^2)$ perturbed with Gaussian noise ($\sigma = 0.4$), over $[-1, 1]$, via hybrid hyperinterpolation $\mathcal{H}_L^\lambda f$, Lasso hyperinterpolation $\mathcal{L}_L^\lambda f$, hard thresholding hyperinterpolation $\mathcal{E}_L^\lambda f$, with $L = 250$. (a): the averaged evaluation of the functions J and H defined in Theorem 5.1 over 100 tests. In particular we considered as parameters, $\lambda^* = \lambda(s)$, $s = 1, \dots, 100$. (b): the relation between the sparsity and L_2 errors.

6.2 The unit-disk $B(\mathbf{0}, 1)$

In this section we set $\Omega = B(\mathbf{0}, 1) := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 = 1\}$, i.e. the unit-disk. As orthonormal basis of $\mathbb{P}_L(B(\mathbf{0}, 1))$ we consider the triangular family of ridge polynomials $\{\Phi_k\}_{k=1}^d$, $d = \binom{L+2}{2} = (L+1)(L+2)/2$ that were introduced by Logan and Shepp [25]. The fact that such basis is triangular

	$\sigma = 0.4$ and λ^* takes				$\lambda^* = \lambda(10) \approx 7.7e - 02$ and σ takes			
	$\lambda(5)$	$\lambda(10)$	$\lambda(50)$	$\lambda(100)$	0.1	0.2	0.3	0.4
<i>Tikhonov</i>	0.5230	0.5301	0.5393	0.5437	0.1397	0.2738	0.4077	0.5286
<i>Filtered</i>	0.4662	0.4664	0.4656	0.4643	0.1169	0.2327	0.3538	0.4672
<i>Lasso</i>	0.1369	0.1270	0.1738	0.2591	0.0390	0.0693	0.1051	0.1249
<i>Hybrid</i>	0.1364	0.1234	0.1517	0.2174	0.0386	0.0681	0.1036	0.1223
<i>Hard</i>	0.1806	0.2553	0.4481	0.5204	0.0620	0.1307	0.1961	0.2557
<i>Hyperint.</i>	0.5622	0.5645	0.5618	0.5593	0.1407	0.2817	0.4259	0.5635
$\ \beta\ _0$	4.0	9.0	48.9	98.5	8.9	9.0	9.0	9.0

Table 1: Average approximation errors in the 2-norm and the sparsity of hybrid hyperinterpolation coefficients of noisy versions of $f(x) = \exp(-x^2)$ over $[-1, 1]$ via various variants of hyperinterpolation, with different values of λ^* and different standard deviation σ of Gaussian noise added on.

allows to determine easily the total degree of each polynomial Φ_k , a key-point for the application of filtered hyperinterpolation.

Though on the disk many rules with low cardinality N are known for mild degrees of precision, we consider here, as in [4], a classical formula with algebraic degree of exactness $2L$ obtained first with respect to numerically the integral on the $B(\mathbf{0}, 1)$ in polar coordinates and then integrating on the radial direction via a $L + 1$ points Gauss-Legendre rule with nodes $\{r_j\}_{j=1}^{L+1}$ and positive weights $\{w_j\}_{j=1}^{L+1}$, while using a trapezoidal rule with $2L + 1$ equispaced points on the azimuthal direction. This gives a formula with nodes $\{\mathbf{x}_{j,m}\}_{j,m}$ and weights $\{w_{j,m}\}_{j,m}$ defined as

$$\begin{aligned} \mathbf{x}_{j,m} &= \left(r_j \cos\left(\frac{2\pi m}{2L+1}\right), r_j \sin\left(\frac{2\pi m}{2L+1}\right) \right), \\ w_{j,m} &= \frac{2w_j r_j}{2L+1}, \end{aligned} \quad (6.1)$$

in which $j = 1, \dots, L + 1$, $m = 0, \dots, 2L$. Setting $N = (L + 1)(2L + 1)$, we thus have that for any $f, g \in \mathbb{P}_L(B(\mathbf{0}, 1))$

$$\langle f, g \rangle = \langle f, g \rangle_N := \sum_{j=1}^N w_j f(\mathbf{x}_j) g(\mathbf{x}_j).$$

As in [4] we compare various forms of hyperinterpolation on the test function $f(\mathbf{x}) = (1 - x_1^2 - x_2^2) \exp(x_1 \cos(x_2))$, where $\mathbf{x} = (x_1, x_2)$. Such f is the true solution of a nonlinear Poisson equation analysed in [10], which was solved by hyperinterpolation-based spectral methods in [18].

In the numerical experiments we set $L = 16$ and consider the rule mentioned above, in which $N = (L + 1)(2L + 1) = 561$. Each evaluation of f at nodes $\{\mathbf{x}_j\}_{j=1}^N$ is contaminated, for some fixed a , via some single impulse noise $\mathcal{J}(a)$ with uniformly distributed random values in $[-a, a]$ and probability $1/2$. We considered a battery of 100 tests, varying the impulse noise. Similarly to the case of the interval, in the tables we display for each approach the average errors in the 2-norm of the various type of hyperinterpolants, evaluating these quantities by the discrete norm $\|g\|_{2,w} := \sqrt{\langle g, g \rangle_{5151}} \approx \|g\|_2$, i.e. a rule with algebraic degree of exactness equal to 100. We also report the average *sparsity* of the hybrid hyperinterpolant.

Similarly to the interval case, numerical experiments show again the advantage of the hybrid approach, with respect to Lasso and hard thresholding hyperinterpolation, providing a favorable ratio between sparsity and L_2 errors. This is well illustrated in Figure 2 (b). Again, hard thresholding hyperinterpolation has a very good performance in the neighbourhood of a certain λ^* , but it is highly sensitive to a change of such parameter, rapidly deteriorating. The hybrid approach preserves the quality of the Lasso hyperinterpolation, reducing the sparsity.

6.3 The unit-sphere \mathbb{S}^2

As third domain we set as Ω the unit-sphere, i.e. $\mathbb{S}^2 = \{\mathbf{x} : \|\mathbf{x}\|_2 = 1\}$ (see e.g. [20]). Between the most studied families of quadrature rules on \mathbb{S}^2 with respect to the surface measure $d\omega$ (see e.g. [20]) there are the so called *spherical t -designs*, introduced by Delsarte, Goethals and Seidel in 1977

	$a = 0.5$ and λ^* takes				$\lambda^* = \lambda(50) \approx 4.5e - 02$ and a takes			
	$\lambda(5)$	$\lambda(10)$	$\lambda(50)$	$\lambda(100)$	0.05	0.1	0.4	0.8
<i>Tikhonov</i>	0.4870	0.4912	0.5098	0.5240	0.0530	0.1054	0.4074	0.7974
<i>Filtered</i>	0.3654	0.3672	0.3658	0.3677	0.0372	0.0733	0.2904	0.5829
<i>Lasso</i>	0.5713	0.3051	0.2600	0.3682	0.0300	0.0580	0.2064	0.3968
<i>Hybrid</i>	0.5713	0.3046	0.2202	0.2661	0.0274	0.0513	0.1783	0.3225
<i>Hard</i>	0.2433	0.2570	0.4696	0.5275	0.0468	0.0942	0.3697	0.7612
<i>Hyperint.</i>	0.5357	0.5282	0.5308	0.5345	0.0530	0.1060	0.4194	0.8532
$\ \beta\ _0$	4.0	8.8	44.0	88.5	44.8	44.8	44.7	43.4

Table 2: Average approximation errors in the 2-norm and the sparsity of hybrid hyperinterpolation coefficients of noisy versions of $f(x_1, x_2) = (1 - (x_1^2 + x_2^2)) \exp(x_1 \cos(x_2))$ over the unit-disk $B(\mathbf{0}, 1)$ via various variants of hyperinterpolation, with different values of λ , considering a single-impulse noise relatively to the levels a .

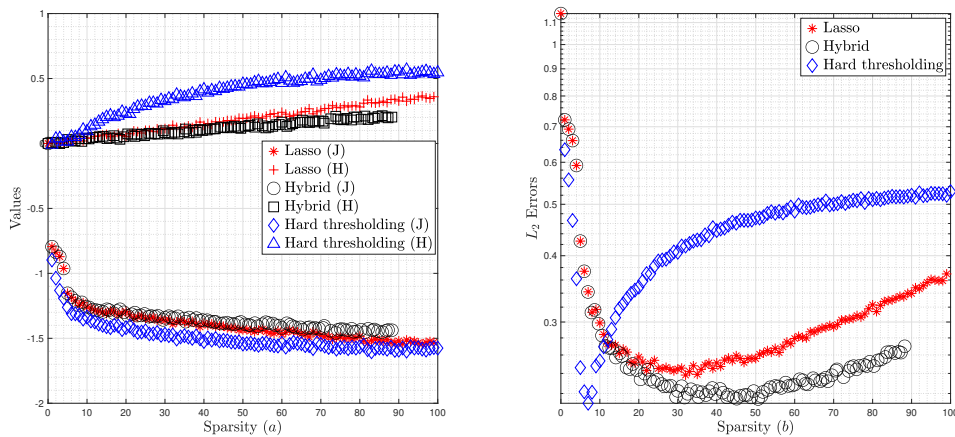


Figure 2: Approximate $f(x_1, x_2) = (1 - (x_1^2 + x_2^2)) \exp(x_1 \cos(x_2))$ perturbed by impulsive noise ($a = 0.5$), over the unit-disk $B(\mathbf{0}, 1)$, via hybrid hyperinterpolation $\mathcal{H}_L^\lambda f$, Lasso hyperinterpolation $\mathcal{L}_L^\lambda f$, hard thresholding hyperinterpolation $\mathcal{E}_L^\lambda f$. (a): the averaged evaluation of the functions J and H defined in Theorem 5.1 over 100 tests. In particular we considered as parameters, $\lambda^* = \lambda(s)$, $s = 1, \dots, 100$. (b): the relation between sparsity and L_2 errors.

	$a = 0.02, \sigma = 0.02$ and λ^* takes				$\lambda^* = \lambda(50) \approx 5e - 03, a = 0.02,$ and σ takes			
	$\lambda(5)$	$\lambda(50)$	$\lambda(100)$	$\lambda(200)$	0.015	0.02	0.025	0.03
<i>Tikhonov</i>	0.1209	0.0638	0.0608	0.0594	0.0534	0.0643	0.0756	0.0897
<i>Filtered</i>	0.0406	0.0403	0.0397	0.0404	0.0338	0.0411	0.0482	0.0577
<i>Lasso</i>	0.0574	0.0225	0.0288	0.0470	0.0201	0.0229	0.0270	0.0324
<i>Hybrid</i>	0.0574	0.0198	0.0218	0.0325	0.0179	0.0201	0.0239	0.0284
<i>Hard</i>	0.0290	0.0462	0.0542	0.0591	0.0388	0.0467	0.0551	0.0655
<i>Hyperint.</i>	0.0600	0.0594	0.0588	0.0593	0.0495	0.0598	0.0706	0.0838
$\ \beta\ _0$	4.0	43.9	87.6	175.0	44.2	43.4	43.5	43.9

Table 3: Average approximation errors in the 2-norm and the sparsity of hybrid hyperinterpolation coefficients of noisy versions of $f(x) = \frac{1}{3} \sum_{i=1}^6 \Phi_2(\|\mathbf{z}_i - \mathbf{x}\|_2)$ over the unit-sphere \mathbb{S}_2 via various variants of hyperinterpolation, with different values of λ , impulse noise $a = 0.02$, standard deviations σ of Gaussian noise added on, summed to a single impulse noise relatively to the level $a = 0.02$.

(see, e.g. the pioneering work [16] and for recent advances [35] and references therein). A pointset $\{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset \mathbb{S}^2$ is a spherical t -design if it satisfies

$$\frac{1}{N} \sum_{j=1}^N p(\mathbf{x}_j) = \int_{\mathbb{S}^2} p(\mathbf{x}) d\omega(\mathbf{x}), \quad \forall p \in \mathbb{P}_t(\mathbb{S}^2).$$

In this work we shall consider in particular an *efficient spherical design* \mathcal{X}_N , between those proposed in [35]. In our tests, since we intend to compute several type of hyperinterpolants of total degree at most $L = 15$, it is necessary to adopt a rule with algebraic degree of exactness equal to $2L = 30$, consisting of $N = 482$ points. We thus have that for any $f, g \in \mathbb{P}_L(\mathbb{S}^2)$

$$\int_{\mathbb{S}^2} f(\mathbf{x})g(\mathbf{x})d\omega(\mathbf{x}) = \langle f, g \rangle = \langle f, g \rangle_{961} := \sum_{j=1}^{961} w_j f(\mathbf{x}_j)g(\mathbf{x}_j).$$

As triangular and orthonormal polynomial basis $\{p_k\}_{k=1, \dots, \dots}$, we adopt the so called *spherical harmonics* [9]. Similarly to the case of the unit-disk $B(\mathbf{0}, 1)$, the fact the basis is triangular allows to determine easily the total degree of each polynomial p_k , making possible the application of filtered hyperinterpolation.

Following [1], we have considered the function

$$f(\mathbf{x}) = \frac{1}{3} \sum_{i=1}^6 \Phi_2(\|\mathbf{z}_i - \mathbf{x}\|_2),$$

where $\Phi_2(r) := \tilde{\Phi}_2(r/\delta_2)$, in which $\tilde{\Phi}_2$ is the C^6 compactly supported of minimal degree Wendland function defined as

$$\tilde{\Phi}_2(r) := (\max\{1 - r, 0\})^6 (35r^2 + 18r + 3)$$

and $\delta_2 = \frac{9\Gamma(5/2)}{2\Gamma(3)}$. We have performed 100 tests in which we have perturbed f by adding Gaussian noise $\mathcal{N}(0, \sigma^2)$ with standard deviation σ and single impulse noise $\mathcal{J}(a)$ relatively to the level $a = 0.02$, illustrating our results in Table 3.

We display for each approach, the average errors in the 2-norm of the various type of hyperinterpolants, evaluating these quantities by the discrete norm $\|g\|_{2,w} := \sqrt{\langle g, g \rangle_{1302}} \approx \|g\|_2$, in which $\langle g, g \rangle_{1302}$ is defined by an efficient spherical design \mathcal{X}_{1302} with degree of precision equal to 50.

Consistently with tests in previous domains, it is remarkable the quality of the new hyperinterpolant, with a general good ratio between the sparsity and L_2 error for all the λ^* taken into account.

6.4 The unit-cube $[-1, 1]^3$

As fourth domain we consider the unit-cube $\Omega = [-1, 1]^3 \subset \mathbb{R}^3$. Having in mind the computation of the hyperinterpolants, it is important to define a scalar product of two functions $f, g \in C(\Omega)$ with

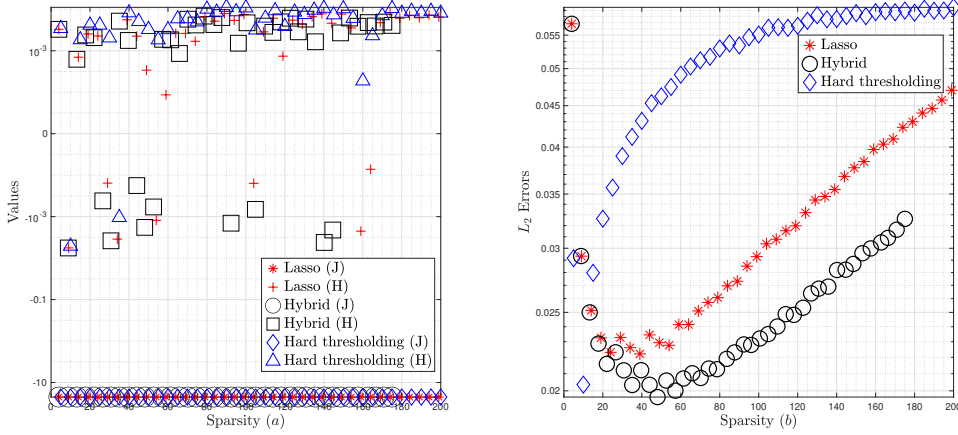


Figure 3: Approximate $f(x) = \frac{1}{3} \sum_{i=1}^6 \Phi_2(\|\mathbf{z}_i - \mathbf{x}\|_2)$, perturbed by impulse noise $a = 0.02$ and Gaussian noise ($\sigma = 0.02$), over the unit-sphere \mathbb{S}_2 , via hybrid hyperinterpolation $\mathcal{H}_L^\lambda f$, Lasso hyperinterpolation $\mathcal{L}_L^\lambda f$, hard thresholding hyperinterpolation $\mathcal{E}_L^\lambda f$. (a): the averaged evaluation of the functions J and H defined in Theorem 5.1 over 100 tests. In particular we considered as parameters, $\lambda^* = \lambda(s)$, $s = 5, 10, 15, \dots, 200$. (b): the relation between the sparsity and L_2 errors.

respect to a certain measure, and here we adopt

$$\langle f, g \rangle = \int_{[-1,1]^3} f(\mathbf{x})g(\mathbf{x})w(\mathbf{x})d\mathbf{x}, \quad w(\mathbf{x}) = \frac{1}{\pi^3} \prod_{i=1}^3 \frac{1}{\sqrt{1-x_i^2}},$$

where as usual $\mathbf{x} = (x_1, x_2, x_3)$. It can be easily verified that $V = \int_{[-1,1]^3} w(\mathbf{x})d\mathbf{x} = 1$.

A well-known orthogonal basis $\{\Phi_\ell\}_\ell$, with respect to the weight function $w(\mathbf{x})$ on $[-1, 1]^3$, consists of the tensor product of Chebyshev polynomials with total degree at most L , that is

$$\tilde{T}_{\ell_1}(x_1)\tilde{T}_{\ell_2}(x_2)\tilde{T}_{\ell_3}(x_3), \quad \ell = (\ell_i)_{i=1,2,3}, \quad \ell_1 + \ell_2 + \ell_3 \leq L,$$

where $\tilde{T}_k(t) = \sqrt{2} \cos(k \arccos(t))$, for $k > 0$, $\tilde{T}_0(t) = 1$, $t \in [-1, 1]$. Also in this case, adopting a suitable ordering, the basis is triangular, and the degree of each polynomial Φ_ℓ can be easily determined.

As for the quadrature rule χ_{2L} on the unit-cube, with respect to such weight function, with algebraic degree of exactness $2L$, we use that introduced in [15], whose cardinality is $N_{2L} \approx \frac{(L+2)^3}{4}$. Since it can be seen that the rule is not minimal, the hyperinterpolant will not be in general interpolant in the pointset determined by the nodes.

This formula is determined as follows. Let $C_L = \{\cos(k\pi/L)\}_{k=0}^L$ be the set of $L+1$ Chebyshev-Lobatto points and let C_{L+1}^E, C_{L+1}^O be respectively the restriction of C_{L+1} to even and odd indices. Indeed, for any quadrature node ξ , the corresponding weight w_ξ is

$$w_\xi := \frac{4}{(L+1)^3} \begin{cases} 1, & \text{if } \xi \text{ is an interior point,} \\ 1/2, & \text{if } \xi \text{ is a face point,} \\ 1/4, & \text{if } \xi \text{ is an edge point,} \\ 1/8, & \text{if } \xi \text{ is a vertex point.} \end{cases}$$

Consequently, if $f, g \in \mathbb{P}_L(\Omega)$, then $\langle f, g \rangle = \langle f, g \rangle_{N_{2L}}$, where $\langle f, g \rangle_{N_{2L}}$ is the discrete scalar product defined by the such quadrature rule with algebraic degree of exactness $2L$.

As observed in [15], setting

$$F(\xi) = F(\xi_1, \xi_2, \xi_3) = \begin{cases} w_\xi f(\xi), & \xi \in \chi_{2L} \\ 0, & \xi \in (C_{L+1} \times C_{L+1} \times C_{L+1}) \setminus \chi_{2L} \end{cases}$$

	$\sigma = 0.2$ and λ^* takes				$\lambda^* = \lambda(50) \approx 1.3e - 02$ and σ takes			
	$\lambda(5)$	$\lambda(10)$	$\lambda(50)$	$\lambda(100)$	0.05	0.1	0.4	0.8
<i>Tikhonov</i>	0.2376	0.2429	0.2426	0.2441	0.0408	0.0813	0.1620	0.3202
<i>Filtered</i>	0.1441	0.1446	0.1452	0.1436	0.0241	0.0480	0.0964	0.1917
<i>Lasso</i>	0.0779	0.0444	0.0474	0.1484	0.0117	0.0190	0.0315	0.0580
<i>Hybrid</i>	0.0779	0.0417	0.0413	0.0882	0.0115	0.0185	0.0300	0.0545
<i>Hard</i>	0.0421	0.1020	0.1329	0.2392	0.0173	0.0341	0.0684	0.1370
<i>Hyperint.</i>	0.2442	0.2460	0.2453	0.2449	0.0409	0.0816	0.1634	0.3257
$\ \beta\ _0$	4.0	42.7	86.1	868.2	44.9	43.4	43.5	43.1

Table 4: Average approximation errors in the 2-norm and the sparsity of hybrid hyperinterpolation coefficients of noisy versions of $f(x) = \exp(-1/(x^2 + y^2 + z^2))$ over the unit-cube $[-1, 1]^3$ via various variants of hyperinterpolation, with different values of λ and of the standard deviations σ defining the Gaussian noise.

the hyperinterpolation coefficients α_ℓ are

$$\alpha_\ell = \gamma_\ell \sum_{i=0}^{L+1} \left(\sum_{j=0}^{L+1} \left(\sum_{k=0}^{L+1} F_{ijk} \cos \frac{k\ell_1\pi}{L+1} \right) \cos \frac{j\ell_2\pi}{L+1} \right) \cos \frac{i\ell_3\pi}{L+1},$$

where $F_{ijk} = F \left(\cos \frac{i\pi}{L+1}, \cos \frac{j\pi}{L+1}, \cos \frac{k\pi}{L+1} \right)$, $i, j, k \in \{0, 1, \dots, L+1\}$ and

$$\gamma_\ell = \prod_{s=1}^3 \gamma_{\ell_s}, \quad \gamma_{\ell_s} = \begin{cases} \sqrt{2}, & \ell_s > 0, \\ 1, & \ell_s = 0, \end{cases} \quad s = 1, 2, 3.$$

In view of this peculiar structure, a fast computation of hyperinterpolation coefficients is feasible via FFT.

In our numerical examples, we examine the case of the function

$$f(x, y, z) = \exp(-1/(x^2 + y^2 + z^2))$$

contaminated by Gaussian noise with various values of standard deviation σ . The various families of hyperinterpolants have total degree at most $L = 20$, computed by means of rules with algebraic degree exactness 40, based on 2662 cubature points. In each of these experiments, we run 100 test and finally take the average of the errors in the 2-norm, that have been estimated by the square root of the discrete scalar product defined by an aforementioned quadrature rule with algebraic degree of exactness 40 that has cardinality 4941.

6.5 Union of disks

In this section we suppose Ω is the union of M disks in general with different centers C_k and radii r_k , i.e. $\Omega = \cup_{k=1}^M B(C_k, r_k) \subset \mathbb{R}^2$. In [33], it was introduced an algorithm that determine low-cardinality cubature rules of PI-type such that

$$\int_{\Omega} p(\mathbf{x}) dx_1 dx_2 = \sum_{k=1}^{N_n} w_k p(\mathbf{x}_k), \quad p \in \mathbb{P}_n(\Omega) \quad (6.2)$$

where n is the algebraic degree of exactness fixed by the user. These rules have the property that $N_n \leq d = \dim(\mathbb{P}_n)$.

Thus, having in mind to provide hyperinterpolants of degree L we will adopt a rule with degree $2L$ and introduce the scalar product

$$\langle f, g \rangle = \langle f, g \rangle_N = \sum_{k=1}^N w_k f(\mathbf{x}_k) g(\mathbf{x}_k), \quad f, g \in \mathbb{P}_L(\Omega) \quad (6.3)$$

where $N = N_{2L}$.

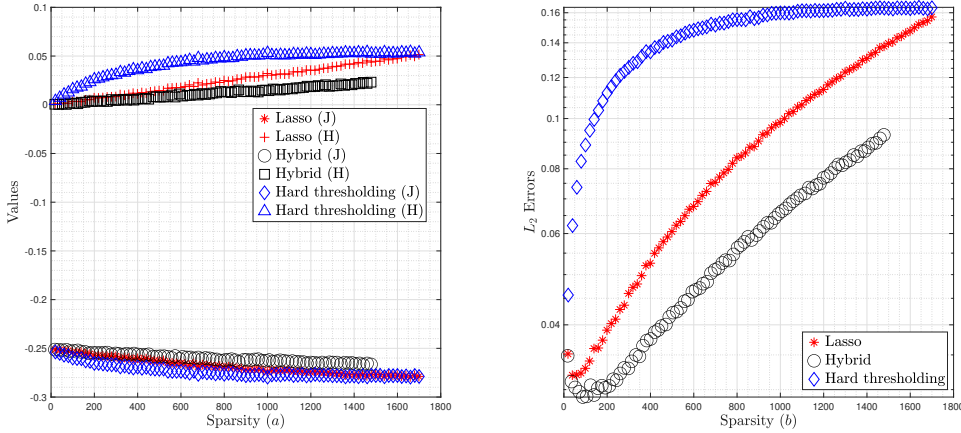


Figure 4: Approximate $f(x_1, x_2, x_3) = \exp(-1/(x_1^2 + x_2^2 + x_3^2))$, perturbed by Gaussian noise ($\sigma = 0.2$), over the unit-cube $[-1, 1]^3$, via hybrid hyperinterpolation $\mathcal{H}_L^\lambda f$, Lasso hyperinterpolation $\mathcal{L}_L^\lambda f$, hard thresholding hyperinterpolation $\mathcal{E}_L^\lambda f$. (a): the averaged evaluation of the functions J and H defined in Theorem 5.1 over 100 tests. In particular we considered as parameters, $\lambda^* = \lambda(s)$, $s = 20, 40, 60, \dots, 1700$. (b): the relation between the sparsity and L_2 errors.

Differently from the previous examples (where Ω was the unit-interval, the unit-disk, the unit-sphere and the unit-cube), the polynomial orthonormal basis of $\mathbb{P}_L(\Omega)$ in general is not known explicitly and must be computed numerically, following the arguments exposed in [31]. In detail, let $\{\varphi_k\}_{k=1}^d$ be a triangular polynomial basis of \mathbb{P}_n and $\mathbf{V}_\varphi = (\varphi_j(\mathbf{x}_i)) \in \mathbb{R}^{N \times d}$ be the Vandermonde matrix at the cubature nodes of a formula (6.2) of PI-type with algebraic degree of exactness $2L$, as in [33]. Let $\sqrt{\mathbf{W}} \in \mathbb{R}^{d \times d}$ the diagonal matrix whose entries are $\sqrt{\mathbf{W}}_{i,i} = \sqrt{w_i}$, $i = 1, \dots, N$. Under these assumptions it can be shown that \mathbf{V}_φ has full-rank and we can apply QR factorization $\sqrt{\mathbf{W}}\mathbf{V}_\varphi = \mathbf{Q}\mathbf{R}$ with $\mathbf{Q} \in \mathbb{R}^{N \times d}$ orthogonal matrix and $\mathbf{R} \in \mathbb{R}^{d \times d}$ an upper triangular non-singular matrix. This easily entails that the polynomial basis $(\Phi_1, \dots, \Phi_d) = (\varphi_1, \dots, \varphi_d)\mathbf{R}^{-1}$ is orthonormal with respect to the scalar product $\langle f, g \rangle_N$ and consequently by (6.3) to that of $L_2(\Omega)$. A fundamental aspect is that since \mathbf{R}^{-1} is an upper triangular non-singular matrix, then the basis (Φ_1, \dots, Φ_d) is also triangular and $\deg(\Phi_k) = \deg(\varphi_k)$, $k = 1, \dots, d$. This result is fundamental to apply filtered hyperinterpolation over $\Omega = \cup_{k=1}^M B(C_k, r_k)$. In our numerical experiments, as triangular basis $\{\varphi_k\}_{k=1}^d$, $d = (L+1)(L+2)/2$, of $\mathbb{P}_L(\Omega)$ we considered the total-degree product Chebyshev basis of the smallest Cartesian rectangle $[a_1, b_1] \times [a_2, b_2]$ containing Ω , with the algebraic degree of exactness lexicographical ordering.

As for the domain, we set $\Omega = \Omega_1^{(r_1)} \cup \Omega_2^{(r_2)}$, where $\Omega_j^{(r_j)}$, $j = 1, 2$, is the union of 19 disks with centers $P_k^{(r_j)} = (r_j \cos(\theta_k), r_j \sin(\theta_k))$, where $\theta_k = 2k\pi/19$, $k = 0, \dots, 18$ and radius equal to $r_j/4$, with $r_1 = 2$ and $r_2 = 4$; notice that the set is the disconnected union of two multiply connected unions (see Figure 5).

In order to define all the hyperinterpolants at $L = 15$ the required formula must have algebraic degree of exactness $2L$ and consists of $N = 496$ nodes that is equal to the dimension of the polynomial space \mathbb{P}_{30} .

Next, we consider contaminated evaluations of

$$f(\mathbf{x}) = (1 - x_1^2 - x_2^2) \exp(x_1 \cos(x_2)),$$

where $\mathbf{x} = (x_1, x_2)$, at the nodes $\{\mathbf{x}_j\}_{j=1}^N$, contaminated by Gaussian noise with various values of standard deviation σ

We replicated the experiments on a battery of 100 different tests, varying the Gaussian noise. In the tables we report for each approach the average errors in the 2-norm of the various type of

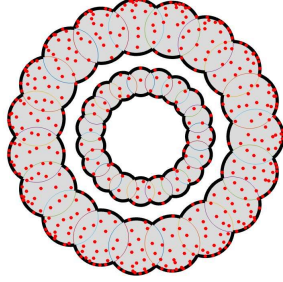


Figure 5: The domain $\Omega = \Omega^{(r_1)} \cup \Omega_2^{(r_2)}$ in which we perform our tests. We represent in red the $N = 496$ nodes of the cubature rule for algebraic degree of exactness=30.

	$\sigma = 0.075$ and λ^* takes				$\lambda^* = \lambda(10) \approx 1.5e - 02$ and σ takes			
	$\lambda(10)$	$\lambda(20)$	$\lambda(40)$	$\lambda(80)$	0.05	0.1	0.4	0.8
<i>Tikhonov</i>	0.0723	0.0705	0.0700	0.0719	0.0484	0.0979	0.3602	0.6811
<i>Filtered</i>	0.0495	0.0495	0.0488	0.0491	0.0335	0.0674	0.2688	0.5204
<i>Lasso</i>	0.0488	0.0379	0.0358	0.0469	0.0373	0.0628	0.1889	0.3325
<i>Hybrid</i>	0.0488	0.0369	0.0317	0.0344	0.0373	0.0627	0.1879	0.3286
<i>Hard</i>	0.0352	0.0495	0.0606	0.0706	0.0224	0.0479	0.1982	0.4050
<i>Hyperint.</i>	0.0722	0.0707	0.0702	0.0721	0.0479	0.0980	0.3819	0.7654
$\ \beta\ _0$	8.8	17.7	35.4	69.7	9.0	8.8	8.7	8.7

Table 5: Average approximation errors in the 2-norm and the sparsity of hybrid hyperinterpolation coefficients of noisy versions of $f(x_1, x_2) = (1 - (x_1^2 + x_2^2)) \exp(x_1 \cos(x_2))$ over the domain Ω depicted in Figure 5 (consisting of union of disks), contaminated with Gaussian noise.

hyperinterpolants, evaluating these quantities by the discrete norm $\|g\|_{2,w} := \sqrt{\langle g, g \rangle_{861}} \approx \|g\|_2$, i.e. a rule with algebraic degree of exactness 40. We also report the average *sparsity* of the hybrid hyperinterpolant.

Numerical tests show again the favourable ratio between the sparsity and L_2 errors of the hybrid hyperinterpolants.

7 Conclusions and outlook

In this work, we have shown that hybrid hyperinterpolation is an effective and robust approach to recover noisy functions over general regions. Specifically, it is a combination of Lasso and filtered hyperinterpolations, and possesses denosing and basis selection abilities and doubly shrinks less relevant coefficients of hyperinterpolation. On the other hand, we propose a prior regularization parameter choices rule (Theorem 5.1) which fully utilizes the information of coefficients of hyperinterpolation. This rule also contributes to decomposing L_2 errors for hybrid hyperinterpolation into three exact computed terms. Our numerical examples have shown that hybrid hyperinterpolation performs enhanced and robust denosing effects over interval, unit-disk, unit-sphere, unit-cube and union of disks, and verify that L_2 errors computed by three exact computed terms are almost the same as that computed directly. It is the first time we apply the variants of hyperinterpolation to a more complicated region — union of disks in which an orthonormal basis is not theoretically available and must be computed numerically.

There are several avenues for further investigation. First, recall that hybrid hyperinterpolation actually corresponds to an $\ell^2 + \ell_1$ -regularized discrete least squares approximation. It would be interesting to consider other regularized terms. For example, the springback penalty ($\|x\|_1 - \alpha/2\|x\|_2^2$ with $\alpha > 0$ being a model parameter) recently proposed in [8] has a good performance in robust signal recovery.

Second, we note that the filter function $h(\cdot)$ have many different forms, and we choose a specific one defined by (2.12). The existence of the soft thresholding operator shrinks all coefficients of hyperinterpolants first, and the filter function may shrink these coefficients again. One may find an

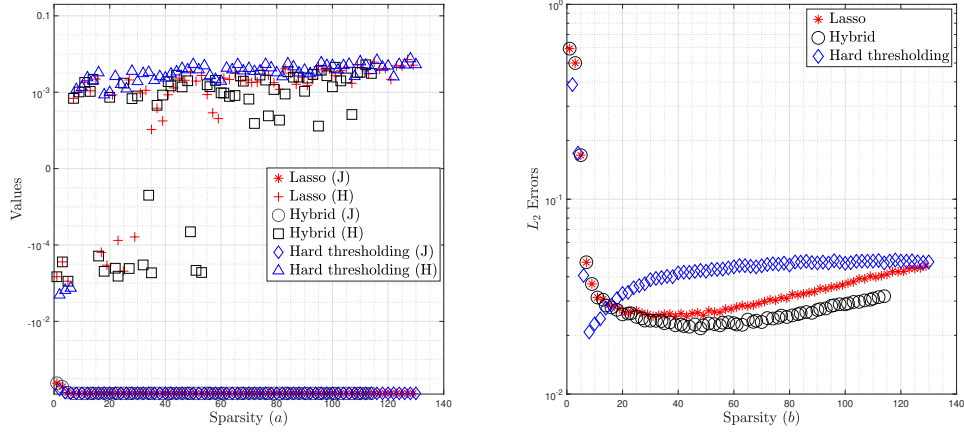


Figure 6: Approximate $f(x_1, x_2) = (1 - (x_1^2 + x_2^2)) \exp(x_1 \cos(x_2))$, perturbed by Gaussian noise ($\sigma = 0.075$), over the union of disks depicted in Figure 5, via hybrid hyperinterpolation $\mathcal{H}_L^\lambda f$, Lasso hyperinterpolation $\mathcal{L}_L^\lambda f$, hard thresholding hyperinterpolation $\mathcal{E}_L^\lambda f$ and $L = 15$. (a): the averaged evaluation of the functions J and H defined in Theorem 5.1 over 100 tests. In particular we considered as parameters, $\lambda^* = \lambda(s)$, $s = 2, 4, 6, \dots, 130$. (b): the relation between the sparsity and L_2 errors.

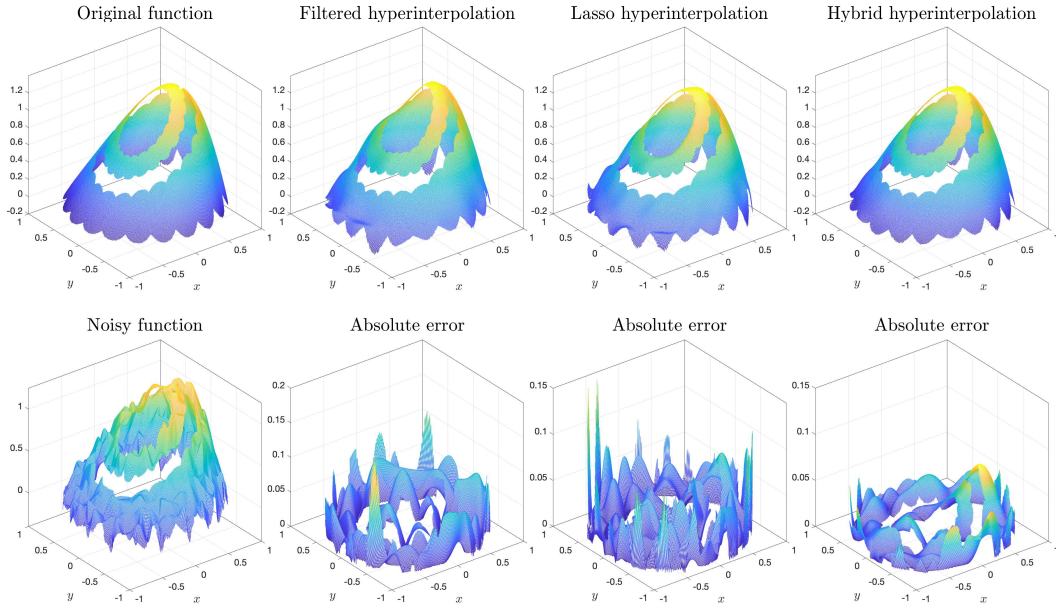


Figure 7: Approximate $f(x_1, x_2) = (1 - (x_1^2 + x_2^2)) \exp(x_1 \cos(x_2))$, perturbed by Gaussian noise ($\sigma = 0.075$), over the union of disks depicted in Figure 5, via filtered hyperinterpolation $\mathcal{F}_{L,N} f$, Lasso hyperinterpolation $\mathcal{L}_L^\lambda f$, hybrid hyperinterpolation $\mathcal{H}_L^\lambda f$, for $\lambda = 0.00725$ and $L = 15$. On the top, the function f without noise, the reconstruction of f via $\mathcal{F}_{L,N} f$, $\mathcal{L}_L^\lambda f$, $\mathcal{H}_L^\lambda f$. Below, the noise, and the absolute errors on a fine mesh by $\mathcal{F}_{L,N} f$, $\mathcal{L}_L^\lambda f$, $\mathcal{H}_L^\lambda f$.

appropriate filter function to make up the shrinkage caused by the soft thresholding operator on these high relevant coefficients and to doubly shrink less relevant ones.

Third, we construct hyperinterpolation and its variants of degree n with a positive-weight quadrature rule with exactness degree $2n$. We find that the required exactness degree $2n$ in hyperinterpolation can be relaxed to $n+k$ with $0 < k \leq n$, and the L_2 norm of the exactness-relaxing hyperinterpolation operator is also bounded by a constant independent of n with the help of Marcinkiewicz–Zygmund inequality [7]. The other important thing revealed in [6] is that the construction of hyperinterpolation on the sphere only need satisfy the Marcinkiewicz–Zygmund property, i.e., hyperinterpolation can be constructed by a positive-weight quadrature rule (not necessarily with quadrature exactness $2n$). There are still a lot work can be done for hyperinterpolation and its variants under this new scheme.

Fourth and finally, we note that there are some interesting quadrature formulas, for example, a low cardinality PI-type algebraic cubature rule of degree n , with at most $(n+1)(n+2)/2$ nodes, over curvilinear polygons defined by piecewise rational functions. One may intend to apply variants of hyperinterpolation to these more complicated regions with some new cubature rules [34].

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