

On Split-State Quantum Tamper Detection and Non-Malleability

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Abstract

Tamper-detection codes (TDCs) and non-malleable codes (NMCs) are now fundamental objects at the intersection of cryptography and coding theory. Both of these primitives represent natural relaxations of error-correcting codes and offer related security guarantees in adversarial settings where error correction is impossible. While in a TDC, the decoder is tasked with either recovering the original message or rejecting it, in an NMC, the decoder is additionally allowed to output a completely unrelated message.

In this work, we study quantum analogs of one of the most well-studied adversarial tampering models: the so-called split-state tampering model. In the t -split-state model, the codeword (or code-state) is divided into t shares, and each share is tampered with “locally”. Previous research has primarily focused on settings where the adversaries’ local quantum operations are assisted by an unbounded amount of pre-shared entanglement, while the code remains unentangled, either classical or separable.

We construct quantum TDCs and NMCs in several natural quantum analogs of the split-state model, including local (unentangled) operations (LO), local operations and classical communication (LOCC), as well as a ‘bounded storage model’ where the adversaries are limited to a finite amount of pre-shared entanglement (LO_a). It is impossible to achieve tamper detection in any of these models using classical codes. Similarly, non-malleability against LOCC is unattainable. Nevertheless, we demonstrate that the situation changes significantly when our code-states are multipartite *entangled* states. In particular:

- We construct rate $1/11$ quantum TDCs in the 3-split-state bounded storage model.
- We construct quantum NMCs for single-bit messages in the 2-split-state LOCC model.

As our flagship application, we show how to construct quantum secret sharing schemes for classical messages with similar tamper-resilience guarantees:

- Ramp tamper-detecting secret sharing schemes in the LO model for classical messages.
- Threshold non-malleable secret sharing schemes in the LOCC model for classical messages.

We complement our results by establishing connections between quantum TDCs, NMCs and quantum encryption schemes. We leverage these connections to prove singleton-type bounds on the capacity of certain families of quantum non-malleable codes in the t -split-state model.

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1 Introduction

The phenomenon of quantum data hiding [TDL00, DLT02, EW02] is a fundamental feature of the non-locality of quantum states. In a quantum data hiding scheme, a message (say, a bit b) is encoded into a bipartite state and handed to two physically separate parties, such that Alice and Bob can't discern the message if they only had access to local quantum operations as well as unbounded *classical* communication. However, the data is perfectly retrievable if they perform an entangled measurement on their state. Data hiding is a signature of the non-classicality of quantum systems, and has received ample interest from the quantum information community due to its connections with the study of quantum correlations and resources (see, e.g. [CLM⁺12]). In this work, we ask:

What if Alice and Bob's goal wasn't to guess the bit b , but instead, to flip it?

We abstract this question into a coding-theoretic setting through the following “tampering experiment”: A sender encodes a message (either classical or quantum) into a public coding scheme (Enc, Dec), where Enc is defined on $t \geq 2$ different registers, or “shares”. A collection of t parties is then allowed to locally tamper with all of the shares using a channel Λ , aided perhaps by some restricted resource such as classical communication and/or additional pre-shared entanglement. The shares are then sent to a receiver who attempts to decode, resulting in an “effective channel” $\text{Dec} \circ \Lambda \circ \text{Enc}$ acting on the initial message.

If the messages and ciphertexts were classical strings (and no classical communication were allowed), this setup would correspond to the well-studied “split-state” tampering model introduced by Liu and Lysyanskaya [LL12]. In this model, the codeword is divided into t parts, and t different adversaries tamper with each part independently, transforming (c_1, c_2, \dots, c_t) into $(f_1(c_1), f_2(c_2), \dots, f_t(c_t))$. It is not difficult to see that error correction is impossible in this setting. In fact, even *tamper-detection codes* [JW15], where the receiver either recovers the original message or aborts (in case they detect the adversary), are infeasible in the classical setting. After all, since (Enc, Dec) are public (there is no secret key), they are always subject to adversaries who can replace the ciphertext with a valid codeword of a pre-agreed message. Nevertheless, the closely related goal known as *non-malleability* is possible.

In a non-malleable code (NMC) [DPW18], in addition to the original message, the decoder is allowed to output a completely unrelated message. In particular, if the message is a number m , there should be no way to change it to $m' = m + 1$ with any non-negligible bias. Although their guarantees may seem a bit arbitrary at first, NMCs are motivated by applications to tamper-resilient hardware, and have found numerous connections to cryptography and coding theory, including secret sharing [GK18a], bit commitments [GPR16], randomness extractors [CG16], etc. Consequently, the study of NMCs under stronger tampering models and with other desiderata like high coding rate and small security error have been the subject of extensive research. However, only recently have adversaries, and coding schemes, with *quantum capabilities* been considered [ABJ22, Ber23, BGJR23, BBJ23].

Quantum Analogs of Split-State Tampering. Inspired by the study of non-local quantum correlations, Aggarwal, Boddu, and Jain [ABJ22] designed classical t -split-state NMCs secure against adversaries which could use an unbounded amount of entanglement to tamper with the codeword. In a natural generalization, [BGJR23] introduced a notion of non-malleability for quantum states, and presented constructions in the 2-split-state model against entangled adversaries. Analogous to their classical counterparts, the effective channel $\text{Dec} \circ \Lambda \circ \text{Enc}$ of a quantum NMC is near a convex combination of the identity map (preserving external entanglement) and a *depolarizing channel*.

Motivated by the study of quantum resources, in this work we design quantum codes for several *resource-restricted* quantum analogs to the split-state tampering model (see Figure 1), including adversaries with access to finite amounts of shared entanglement and/or classical communication. We point out that due to a quantum analog of the previously mentioned “substitution attack”, tamper detection is still impossible against adversaries with unlimited pre-shared entanglement. However, no such impossibility stands in the other models. After all, the codespace may be highly entangled, but the adversaries are not.

- LO^t : Local quantum operations on t registers.
 - LOCC^t : Local quantum operations and (unbounded) classical communication between t registers.
 - LO_*^t : Local quantum operations on t registers with arbitrary pre-shared entanglement.
- If $\leq a$ qubits of pre-shared entanglement are allowed, we call these channels LO_a^t and LOCC_a^t .

Figure 1: A glossary of the split-state tampering models studied in this work.

In this work we design codes in these quantum analogs of the split-state model with two new security guarantees, namely, tamper detection against LO, and non-malleability against LOCC. Furthermore, our conclusions are robust to a finite amount of pre-shared entanglement between the parties, up to a constant fraction of the blocklength n (in qubits) of the code. This is near-optimal, in the sense that neither guarantee is possible in the presence of n qubits of pre-shared entanglement.¹

We complement our results in two directions. First, the flagship application of our techniques (and most technical part of our work) lies in designing threshold secret sharing schemes with the same tamper-resilient guarantees. To do so, we combine our new codes with techniques from a long literature of non-malleable and leakage-resilient secret-sharing schemes [GK18a, GK18b, ADN⁺19, CKOS22]. Second, we prove connections between TDCs, NMCs, and quantum encryption schemes, and use them to prove singleton-type bounds on the capacity of certain families of quantum NMCs and TDCs.

1.1 Summary of contributions

1.1.1 Code Constructions

We present two main code constructions: an NMC against LOCC_a^2 , and a TDC against LO_a^3 . Despite the distinct tampering models and techniques, our constructions are tied together in a concise conceptual message: In order to flip a random encoded bit with high enough bias, the adversaries must be using a large amount $a = \Omega(n)$ of pre-shared entanglement.

Non-Malleability against LOCC. We begin with a simple and illustrative example, a NMC for single-bit messages, secure against LOCC^2 . Dziembowski, Kazana, and Obremski [DKO13] proved that NMCs for single-bit messages have a rather clean and concise equivalent definition: a code is non-malleable against a function family \mathcal{F} with error ε iff “it is hard to negate a random bit” with bias $\leq \varepsilon$:

$$\forall f \in \mathcal{F} : \mathbb{P}_{B \leftarrow \{0,1\}} [\text{Dec} \circ f \circ \text{Enc}(B) = \neg B] \leq \frac{1}{2} + \varepsilon. \quad (1)$$

Suppose, to encode a bit $B = 0$, Alice and Bob share n EPR pairs $\Phi^{\otimes n}$. To encode $B = 1$, they share two n qubit maximally mixed states. This code has near perfect correctness, and is certainly *not* data hiding². Moreover, it is completely trivial to flip an encoding of 0 to that of 1. However, since $\text{Enc}(1)$ is unentangled, but $\text{Enc}(0)$ is maximally entangled, it is impossible to flip 1 to 0 just using LOCC. Consequently,

Theorem 1.1. *For every $a, n \in \mathbb{N}$, there exists an efficient non-malleable code for single-bit messages secure against LOCC_a^2 , of blocklength $2n$ qubits, and error $O(2^{a-n})$.*

Unfortunately, NMCs in the 2-split model with higher rate are significantly more challenging to construct, and remain an open problem in the LOCC model. We complement our results in this setting by constructing

¹If the size of the largest share is $\alpha \cdot n$ qubits, then using $2 \cdot (1 - \alpha) \cdot n$ pre-shared EPR pairs and classical communication the t parties could teleport the smaller shares back and forth [BBC⁺93].

²Since Alice and Bob could simply measure their sides, and predict $b = 0$ if they are equal. It is instructive to compare this to classical 2-split NMCs, which are well-known to be 2-out-of-2 secret sharing (each party can’t individually guess the message) [ADKO15]. In contrast, in LOCC NMCs, the parties are able to learn the message.

NMCs with inverse-polynomial rate in the 4-split model LOCC⁴. As we discuss shortly, our 4-split constructions are based on a combination of data hiding schemes and classical 2-split NMCs, and play an important role in our applications to secret sharing.

Tamper Detection against LO. The example above hints at a key feature of the tamper-resilience of entangled code-states. Our main result furthers this intuition by showing that *tamper detection*³ is possible in the unentangled split-state model LO^t, even with a constant number of shares $t \geq 3$. Previously, this result was only known in the regime of an asymptotically large number t of shares [Ber23].

Theorem 1.2. *For every $n \in \mathbb{N}$, $\gamma \in (0, \frac{1}{20})$, there exists an efficient quantum tamper-detection code against LO _{$\Theta(\frac{1}{n})$} ³, of blocklength n , rate $\frac{1}{11} - \gamma$, and error $2^{-n^{\Omega(1)}}$.*

In fact, arguably one of the main messages of this work is that tamper detection arises naturally as a consequence of non-malleability against unentangled adversaries: We show that any classical or quantum non-malleable code in the split-state model LO^t directly implies a quantum tamper-detection code against LO^{t+1}, with one extra share/split. Along the way, and to instantiate our results, we present a construction of a quantum NMC against LO² with constant rate and inverse exponential error, building on recent techniques by [BGJR23] with a near-optimal construction of classical 2-split-state NMCs by [Li23a].

Our codes offer interesting conclusions from a resource-theoretic perspective, but at face value may not seem too applicable to other adversarial settings. After all, neither data hiding schemes, nor our TDCs or NMCs, offer any security when composed in parallel.⁴ Nevertheless, as the main application (and technical challenge) in our work, we show how to augment secret sharing schemes with the same tamper-resilience guarantees as above.

1.1.2 Applications to Tamper-Resilient Secret Sharing

In a secret sharing scheme, a dealer encodes a secret into p shares and distributes them among p parties, which satisfy an *access structure* \mathcal{A} . An access structure is a family of subsets of $[p]$, where the shares of any (unauthorized) subset of parties $T \notin \mathcal{A}$ don't reveal any information about the secret, but the shares of any (authorized) subset $T \in \mathcal{A}$ are enough to uniquely reconstruct it. Of particular interest are t -out-of- p threshold secret sharing schemes, where the authorized subsets are all those of size $\geq t$, while those of size $< t$ are un-authorized.

Non-malleable secret sharing schemes (NMSSs), introduced by Goyal and Kumar [GK18a], simultaneously strengthen both secret sharing schemes and non-malleable codes. In an NMSSs, even if all the p parties tamper with each share independently or under some restricted joint tampering model, the tampered shares of an *authorized set* are always enough to either recover the secret, or a completely unrelated secret. The design of NMSSs directly from split-state NMCs has seen a flurry of recent interest [GK18a, ADN⁺19, BS19, CKOS22] - including recent extensions to quantum non-malleable secret sharing [BGJR23].

Here, we show how to encode classical messages into multipartite quantum states, offering threshold-type secret sharing guarantees, together with tamper detection against unentangled adversaries/non-malleability against LOCC:

Theorem 1.3. *For every $p, t \geq 5$, there exists a t -out-of- p quantum secret sharing scheme for k bit secrets which is non-malleable against LOCC, on shares of size $k \cdot \text{poly}(p)$ qubits and error $2^{-(kp)^{\Omega(1)}}$.*

Our “tamper-detecting” secret sharing schemes are in fact *ramp* secret sharing scheme, a standard generalization of threshold secret sharing where the privacy and robust reconstruction thresholds differ:

³Recall that tamper-detection entails the receiver either recovers the original message or rejects with high probability, strictly strengthening non-malleability. We refer the reader to Section 1.3 for formal definitions.

⁴For instance, our TDCs against local adversaries LO³ involve entangled code-states. Given two TDCs, shared among the same set of parties, can the entanglement in one of them be used to break the security of the other?

Theorem 1.4. *For every p, t s.t. $4 \leq t \leq p - 2$, there exists a t -out-of- p quantum secret sharing scheme for k bit secrets, of share size $k \cdot \text{poly}(p)$ qubits, where even if all the shares are tampered with an LO^p channel, an authorized set of size $\geq (t + 2)$ either reconstructs the secret or rejects with error $2^{-(kp)^{\Omega(1)}}$.*

Both of our secret sharing schemes are inspired by a well-known result by [ADN⁺19], who showed how to compile generic secret sharing schemes into leakage-resilient and non-malleable secret sharing schemes. At a very high level, their approach to NMSS can be viewed as “concatenating” secret sharing schemes with 2-split NMCs. Here, we revisit their compiler by combining it with our new (3-split) tamper-detection code, and LOCC data hiding schemes. As previously mentioned, proving the security of our compilers is the main challenge in this work and requires a number of new ideas, including relying on new quantum secure forms of leakage-resilient secret sharing schemes (see Section 1.3).

1.1.3 Connections to Quantum Encryption Schemes

The notion of tamper detection studied in this work closely resembles the guarantees of quantum authentication schemes (QAS) [BCG⁺01, ABOE08, BW16, GYZ16, HLM16]. In a QAS, a sender and receiver (who share a secret key), are able to communicate a private quantum state while detecting arbitrary man-in-the-middle attacks. A remarkable feature of *quantum* authentication schemes, in contrast to their classical counterparts, is that they encrypt the message from the view of adversaries who do not know the key.⁵ Here, we similarly show that split-state tamper-detection codes must encrypt their shares:

Theorem 1.5. *The reduced density matrix (marginal) of each share of quantum tamper-detection code against LO^t with error ε , is $4\sqrt{\varepsilon}$ close to a state which doesn’t depend on the message.*

Neither classical authentication schemes nor non-malleable codes in the split-state model (necessarily) encrypt their shares⁶. Although in the body we show the above implies partial conclusions for quantum non-malleable codes, the question of whether they encrypt their shares remains open. Nevertheless, we find Theorem 1.5 to be a useful tool in characterizing the capacity of both NMCs and TDCs. By generalizing (and simplifying) classical information-theoretic arguments by Cheraghchi and Guruswami [CG13a], we prove strong singleton-type upper bounds on the rate of certain families of quantum non-malleable codes in the split-state model. In particular, those which are *separable*, unentangled across the shares.

Theorem 1.6. *Fix $n \in \mathbb{N}$ and $\delta \geq 4 \cdot \log n/n$. Any separable state quantum non-malleable code against LO^t of blocklength n qubits and rate $\geq 1 - \frac{1}{t} + \delta$, must have error $\varepsilon_{\text{NM}} = \Omega(\delta^2)$.*

Unfortunately, our techniques fall short of proving the capacity of *entangled* non-malleable codes. Nevertheless, prior to this work, the only known constructions of quantum non-malleable codes were separable [BGJR23, BBJ23], and thereby we find Theorem 1.6 to be a valuable addition.

1.2 Related Work

Comparison to Prior Work. [ABJ22] studied *quantum secure* versions of classical non-malleable codes in the split-state model, where the adversaries have access to pre-shared entanglement. [BGJR23] introduced quantum non-malleable codes in the entangled split-state model LO_* , and constructed codes with inverse-polynomial rate in the 2-split-state model (which is the hardest). [BBJ23], leveraging constructions of quantum secure non-malleable *randomness encoders*, showed how to construct constant rate codes in the entangled 3-split-state model.

Closely related to our results on tamper detection is work by [Ber23], who introduced the notion of a Pauli Manipulation Detection code, in a quantum analog to the well-studied classical algebraic manipulation

⁵This is since an adversary that could distinguish between the code-states of two known orthogonal messages, $|x\rangle$ and $|y\rangle$, is able to map between their superpositions $(|x\rangle \pm |y\rangle)/\sqrt{2}$ [BCG⁺01, AAS20, GJMZ22].

⁶2-split-state non-malleable codes do encrypt their shares [ADKO15], however, 3-split codes not necessarily. Indeed, encoding a message into a 2-split-state code and placing a copy of the message (in the clear) into the third register is non-malleable in the 3 split model, but trivially doesn’t encrypt its shares.

detection (AMD) codes [CDF⁺08]. By combining these codes with stabilizer codes, he constructed tamper-detection codes in a “qubit-wise” setting, a model which can be understood as the unentangled n -split-state model LO^n for asymptotically large n (and each share is a single qubit). However, to achieve tamper detection (and high rate), [Ber23] relied on strong quantum circuit lower bounds for stabilizer codes [AN20]. In contrast, as we will see shortly, we simply leverage Bell basis measurements.

Work by	Rate	Messages	Adversary	NM	TD
[ABJ22]	$\frac{1}{\text{poly}(n)}$	classical	LO_*^2	Yes	No
[BBJ23]	1/3	classical	LO_*^3	Yes	No
[BBJ23]	1/11	quantum	LO_*^3	Yes	No
[BGJR23]	$\frac{1}{\text{poly}(n)}$	quantum	LO_*^2	Yes	No
[Ber23]	≈ 1	quantum	LO^n	Yes	Yes
This Work	1/11	quantum	$\text{LO}_{(\infty, \infty, \Omega(n))}^3$	Yes	Yes
This Work	1/ n	classical	LOCC^2	Yes	No
This Work	1/ $n^{o(1)}$	classical	LOCC^4	Yes	No
This Work	$\Omega(1)$	quantum	LO^2	Yes	No

Table 1: Comparison between known explicit constructions of split-state NMCs and TDCs. Here, n denotes the blocklength.

Other Classical Non-Malleable Codes and Secret Sharing Schemes. NMCs were originally introduced by Dziembowski, Pietrzak and Wichs [DPW18], in the context of algebraic and side-channel attacks to tamper-resilient hardware. In the past decade, the split-state and closely related tampering models for NMCs and secret sharing schemes have witnessed a flurry of work [LL12, DKO13, CG16, CG14, ADL18, CGL20, Li15, Li17, GK18a, GK18b, ADN⁺19, BS19, FV19, Li19, AO20, BFO⁺20, BFV21, GSZ21, AKO⁺22, CKOS22, Li23a], culminating in recent explicit constructions of classical split-state NMCs of explicit constant rate 1/3 [AKO⁺22] and constructions with (small) constant rate but also smaller error [Li23a]. [CG13a] proved that the capacity of non-malleable codes in the t split-state model approaches $1 - \frac{1}{t}$.

Split-state NMCs and related notions have also found numerous applications to other cryptographic tasks, such as non-malleable commitments [GPR16], secure message transmission [GK18a, GK18b], and non-malleable signatures [ADN⁺19, SV19, CKOS22], in addition to secret sharing.

Data Hiding Schemes and Leakage Resilience. Peres and Wootters [PW91] are credited with the first observations of data hiding features of bipartite quantum states. They showed that one could encode classical random variables into mixtures of bipartite product states, such that two parties restricted to LOCC operations would learn significantly less about the message than parties with generic entangling operations. This led to a series of results constructing and characterizing collections of states with different hiding attributes [MP00, TDL00, DLT01, DHT02, HLSW03, MWW08, CLM⁺12, LPW17], as well as studying connections to secret sharing [HLS04, WXC⁺17, ÇGLZR23], with still with fascinating open questions [Win17].

Most related to our work is the recent result by [ÇGLZR23], who designed cryptographic primitives (e.g. public key encryption, digital signatures, secret sharing schemes, etc.) with quantum secret keys, for classical messages, which are leakage-resilient to unbounded LOCC. In particular, their secret sharing schemes are robust only to non-adaptive (one-way) leakage, albeit have polynomially smaller share size compared to ours.

Non-Malleable Quantum Encryption. Other notions of non-malleability for quantum messages have previously been studied in the context of quantum encryption schemes. Ambainis, Bouda, and Winter

[ABW09], and later Alagic and Majenz [AM17], introduced related notions of quantum non-malleable encryption schemes which (informally) can be understood as a keyed versions of the NMCs we study here. [AM17] showed that in the keyed setting, non-malleability actually implies encryption in a strong sense. We view the connections we raise to encryption schemes as analogs to these results in a coding-theoretic setting.

1.3 Our Techniques

We begin in Section 1.3.1 by presenting a reduction from quantum non-malleability to tamper-detection, which showcases the intuition behind many of the techniques in our work. We proceed in Section 1.3.2 by describing how to construct 3-split tamper-detection codes in the bounded storage model from classical, quantum secure non-malleable codes (our result in Theorem 1.2).

In the remainder of this subsection, we present the intuition behind our constructions of secret sharing schemes. In Section 1.3.3, we describe our constructions of tamper-detection secret sharing schemes (Theorem 1.4) based on our 3-split TDCs. Finally, in Section 1.3.4, we describe our secret sharing schemes which are non-malleable against LOCC (Theorem 1.3).

1.3.1 Quantum Non-Malleability implies Tamper Detection

A key ingredient in our work is a recently introduced notion of *quantum non-malleable codes* by [BGJR23]. Informally, their definition stipulates that the “effective channel” is near a convex combination of the identity channel and a depolarizing channel. However, either preserving, or completely breaking entanglement with any side-information which doesn’t go through the channel (a property known as *decoupling*).

Definition 1.1 (Quantum Non-Malleable Codes [BGJR23]). *A pair of CPTP channels (Enc, Dec) is a quantum non-malleable code against a tampering channel Λ with error ε if the “effective channel” satisfies*

$$\forall \psi \in \mathcal{D}(\mathcal{H}_M \otimes \mathcal{H}_{\hat{M}}) : (\text{Dec} \circ \Lambda \circ \text{Enc} \otimes \mathbb{I}_{\hat{M}})(\psi_{M\hat{M}}) \approx_\varepsilon p \cdot \psi_{M\hat{M}} + (1 - p) \cdot \sigma \otimes \psi_{\hat{M}}, \quad (2)$$

where $p \in [0, 1]$ and $\sigma \in \mathcal{D}(\mathcal{H}_M)$ depend only on Λ , and are independent of ψ .

If instead of the fixed state σ , the decoder rejected by outputting a flag $\sigma = \perp$, then we would refer to (Enc, Dec) as a *tamper-detection code* for Λ . The starting point of our work notes that this *decoupling* nature of non-malleability in the quantum setting, implies strong tamper detection guarantees when the tampering channel Λ has limited entanglement.

Tamper Detection in the unentangled Split-State Model. We design a compiler which converts a quantum non-malleable code $(\text{Enc}_{\text{NM}}, \text{Dec}_{\text{NM}})$ for the unentangled t -split model, LO^t , into a tamper-detection code for the unentangled $(t + 1)$ -split model LO^{t+1} (with one extra share):

$\text{Enc}(\psi)$: We pad ψ using λ halves of EPR pairs $\Phi^{\otimes \lambda}$ (on a bipartite register $E\hat{E}$), and encode both the message and half of the EPR pairs into the non-malleable code. The remaining EPR halves \hat{E} are placed in an additional $(t + 1)$ st share:

$$\text{Enc}(\psi) = [\text{Enc}_{M,E}^{\text{NM}} \otimes \mathbb{I}_{\hat{E}}](\psi \otimes \Phi_{E\hat{E}}^{\otimes \lambda}). \quad (3)$$

Dec : First, decode the non-malleable code using Dec^{NM} , resulting in registers $M'E'$, and then measure E' and the $(t + 1)$ st share in the Bell basis. If both steps accept, we output M' , and if not, we reject.

Since the adversaries in LO^{t+1} are unentangled, no matter how they individually tamper their shares, after applying Dec_{NM} the decoder always either receives ψ back, or a *product state* with the $(t + 1)$ st share. However, the Bell basis measurement is guaranteed to reject product states⁷, and therefore can be used to

⁷A Bell basis measurement which accepts only if we have projected onto $\Phi^{\otimes \lambda}$, rejects product states and low Schmidt-Rank states with negligible error in λ .

verify (and reject) if the adversaries have tampered with their shares. We defer formal details to [Section 3](#).

Applications. While extremely simple, we find that our reduction above singles out an important property of quantum non-malleability which we leverage in our more elaborate constructions. Here we briefly highlight some of its applications:

1. Explicit constructions of tamper-detection codes against LO^3 with constant rate and inverse-exponential error, by instantiating the compiler above with a quantum non-malleable code in the unentangled 2-split model (LO^2) with the same parameters (constructed in [Appendix B](#), see [Section 3](#) for details).
2. We show that our reduction is surprisingly robust to the *bounded-storage* model LO_a^{t+1} . To do so, we introduce a notion of *augmented* quantum NMCs ([Definition 2.15](#)), which may be of independent interest.
3. We use this characterization to prove an upper bound on the rate of separable-state quantum TDCs and NMCs, following techniques by [\[CG13a\]](#) on the capacity of split-state NMCs. We present these ideas in [Section 9](#) and [Section 10](#).

Unfortunately, our reduction only takes us so far. The reason is two-fold: First, to construct tamper-detection codes in the bounded storage model, we require quantum non-malleable codes against entangled adversaries. However, constructing high-rate codes in the *entangled* 2-split model remains an open problem - the codes in [\[BGJR23\]](#) have only inverse-polynomial rate. Moreover, it remains quite unclear whether this compiler can even be used to construct secret sharing schemes with tamper detection guarantees. In the next subsection, we detail how to construct tamper-detection codes directly from classical non-malleable codes, which (partially) addresses both these questions.

1.3.2 Tamper-Detection Codes in the Bounded Storage Model

In this subsection, we overview our explicit constructions of tamper-detection codes in the 3-split ‘bounded storage’ model LO_a^3 , with constant rate and inverse-exponential error. Our approach closely follows recent constructions of split-state quantum codes by [\[Ber23, BGJR23, BBJ23\]](#), who observed that a *non-malleable secret key* could be used to achieve non-malleability for quantum states. Here we briefly overview their approach and highlight our modifications.⁸

The 3-Split Quantum Non-Malleable Codes by [\[BGJR23, BBJ23\]](#) combined:

- (a) $(\text{Enc}_{\text{NM}}, \text{Dec}_{\text{NM}})$: A quantum secure (augmented) 2-split non-malleable code.
- (b) A family of unitary 2-designs $\{C_R\}_{R \in K}$ ⁹.

To encode a message ψ , their encoding channel

1. Samples a uniformly random key R and encodes it into the non-malleable code, $(X, Y) = \text{Enc}_{\text{NM}}(R)$
2. Authenticates ψ : $C_R \psi C_R^\dagger$ (on some register Q), using the family of 2-designs keyed by R .

Two of the three parties hold the classical registers X, Y , and the last holds Q : (Q, X, Y) .

⁸The approach undertaken here resembles a framework for constructing quantum secret sharing schemes which are resilient to restricted tampering models. In this framework, a classical key is used to encrypt the state, and then the key is “hidden” back into the code, using a classical code/secret-sharing scheme which is robust to the same tampering model. Similar approaches have been studied for robust secret sharing [\[CGS05, HP17, BGG22\]](#) and data hiding schemes [\[DHT02\]](#).

⁹Here we point out that any Quantum Authentication Scheme (QAS) would suffice [\[BCG⁺01, HLM16, BW16, GYZ16\]](#), as would a non-malleable *randomness encoder* in place of a non-malleable code [\[KOS18, BBJ23\]](#). For background on these components, refer to [Section 2](#).

The non-malleability of this scheme against local adversaries with shared entanglement, LO_*^3 , relies crucially on the *augmented* security of the quantum-proof non-malleable code. Informally, suppose the two shares X, Y are tampered to X', Y' using shared entanglement, and W denotes any quantum side-information held by one of the adversaries. By definition, the resulting joint distribution (state) over the original key, the recovered key $R' = \text{Dec}_{\text{NM}}(X', Y')$, and the *side-information*, is a convex combination of the original key and a fixed, independent key:

$$R, R', W \approx p \cdot U_{R=R'} \otimes W + (1 - p) \cdot U_R \otimes (R', W), \quad (4)$$

where $p \in [0, 1]$ is independent of R , and $U_{R=R'}$ denotes the recovered key is the same as the original and uniformly random.

It turns out that the non-malleability of the secret key, suffices to achieve non-malleability for the quantum message. In the first case above, the key is recovered and is independent of the side information. By the properties of the two-design unitaries, the message is authenticated [ABOE08], and thereby we either recover it, or reject. In the second case, the key is lost to the decoder. Thereby, from the decoder’s perspective, it receives a message scrambled by a random unitary C_R , and therefore ‘looks’ encrypted and independent of the original message.

Inspired by our reductions, we differ from the outline above in only one detail: we encrypt the message state ψ together with λ halves of EPR pairs. The remaining EPR halves are handed to one of the other two adversaries, together with one of the NMC splits. In detail,

Our Construction of 3-Split Tamper-Detection Codes is defined by an encoding channel $\text{Enc}_{\text{TD}}^\lambda$, parametrized by an integer λ , which:

1. Prepares λ EPR pairs, on quantum registers E, \hat{E} .
2. Samples a uniformly random key R and encodes it into the 2-split non-malleable code $(X, Y) = \text{Enc}_{\text{NM}}(R)$
3. Encrypts E together with the message ψ , using a family of 2-design unitaries keyed by R . Let the resulting register be Q .

The final 3 shares are $(Q, Y\hat{E}, X)$.

By combining the properties of the quantum secure non-malleable code with a Bell basis measurement (on states of low Schmidt rank), we prove that the construction above is a secure tamper-detection code against an even stronger ‘bounded storage’ adversarial model, which we refer to as $\text{LO}_{(a,*,*)}^3$. In $\text{LO}_{(a,*,*)}^3$, the 3 adversaries hold a pre-shared entangled state on registers A_1, A_2, A_3 , where the 1st party (the one holding the register Q) is limited to $|A_1| \leq a$ qubits, but the other two may be unbounded. As we will see shortly, this model plays a particularly important role in our constructions of tamper-detecting secret sharing schemes.

1.3.3 Tamper-Detecting Secret Sharing Schemes against LO

We dedicate this section to an overview of our construction of ramp secret sharing schemes (for classical messages) that detect local tampering. At a very high level, our secret sharing schemes are based on the concatenation of classical secret sharing schemes, with our tamper-detection codes. The key challenge lies in reasoning about the security of our tamper-detection codes when composed in parallel, which we address by combining them with extra leakage-resilient properties of the underlying classical scheme.

The outline of our construction, as well as our constructions of non-malleable secret sharing schemes for LOCC, lies in a compiler by [ADN+19]. Their compiler takes secret sharing schemes for arbitrary access structures as input, and produces either leakage-resilient or non-malleable secret sharing schemes for the same access structure. We begin with a brief description:

The Compiler by [ADN⁺19]. To encode a message m , they begin by

1. Encoding it into the base secret sharing scheme $(M_1, \dots, M_p) \leftarrow \text{Share}(m)$.
2. Then, each share M_i is encoded into some (randomized) *bipartite* coding scheme (Enc, Dec) , to obtain two shares L_i and R_i .
3. For each $i \in [n]$, the new compiled shares amount to giving L_i to the i -th party, and a copy of R_i to every other party. At the end of this procedure, the i -th party has a compiled share S_i , where

$$S_i = (R_1, \dots, R_{i-1}, L_i, R_{i+1}, \dots, R_p) \tag{5}$$

The natural decoding algorithm takes all pairs of shares (L_i, R_i) within an authorized set of parties, recovers $M_i \leftarrow \text{Dec}(L_i, R_i)$, and then uses the reconstruction algorithm for the secret sharing scheme to recover the message m . By instantiating this compiler using different choices of the underlying bipartite coding scheme (strong seeded extractors or non-malleable extractors), they construct secret sharing schemes satisfying different properties (leakage resilience or non-malleability).

Our Construction attempts a tripartite version of the compiler by [ADN⁺19]. We combine

- (a) $(\text{Enc}_{\text{TDC}}^\lambda, \text{Dec}_{\text{TDC}}^\lambda)$, our 3-split-state TDC in the bounded storage model.
- (b) $(\text{Share}, \text{Rec})$, a classical t -out-of- p secret sharing scheme, which is additionally leakage-resilient against a certain local quantum leakage model.

On input a classical message m , we similarly begin by

1. Encoding it into the base secret sharing scheme $(M_1, \dots, M_p) \leftarrow \text{Share}(m)$.
2. Then, every triplet of parties $a < b < c \in [n]$ encodes a copy of their shares M_a, M_b, M_c into a certain 3-out-of-3 “gadget”, $\text{Enc}_\Delta(M_a, M_b, M_c)$, which is then re-distributed between the three parties.

The most technical part of our work is to find a choice of Enc_Δ which guarantees a local version of tamper detection for the shares in the gadget, and can be combined with the leakage-resilience of the underlying secret sharing scheme to ensure tamper detection of the actual message. We motivate this challenge and our choice of Enc_Δ below.

Weak Tamper Detection and the Selective Bot Problem. The most natural tripartite candidate for Enc_Δ would directly be our tamper-detection code. That is, M_a, M_b, M_c are jointly encoded into $\text{Enc}_{\text{TDC}}(M_a || M_b || M_c)$, resulting in 3 (entangled) quantum registers A, B, C , which are handed to parties a, b, c respectively.

One can readily verify that this construction already offers a certain “weak” tamper detection guarantee. If any three parties a, b, c , were to tamper with their shares, then by the properties of Enc_{TDC} ¹⁰, one can show that the *marginal* distribution over the recovered shares M'_a, M'_b, M'_c is near a convex combination of the original shares M_a, M_b, M_c , or rejection. Similarly, given tampered shares of any authorized subset of parties $T \subset [p]$, after decoding all the copies of Enc_{TDC} in T , by a standard union bound we either reject or recover all the honest shares of T . By the correctness of the secret sharing scheme, this implies we either recover the message, or reject, $\text{Dec} \circ \Lambda \circ \text{Enc}(m) \in \{m, \perp\}$ with high probability.

¹⁰Technically, we additionally require Enc_{TDC} to satisfy a strong form of 3-out-of-3 secret sharing, where any two shares are maximally mixed.

The reason this code only offers “weak” tamper detection is that *the probability the decoder rejects may depend on the message* (See [Definition 2.22](#) for a formal definition). That is, whether the gadgets in T reject may correlate with all the shares of T , and therefore the message itself, violating tamper detection/non-malleability. This “selective bot” problem is a fundamental challenge in the design of non-malleable secret sharing schemes [[GK18a](#)], and we combine two new ideas to overcome it.

A Multipartite Encoder with staggered Entanglement. One might hope that instead of simply relying on the privacy of the underlying secret sharing scheme, one could leverage stronger properties to simplify the tampering process (to some other effective tampering on the classical shares). While this is indeed roughly the approach we undertake, the main obstacle lies in that to implement $\text{Enc}_{\text{T D}}(M_a || M_b || M_c)$, one of the parties $i \in \{a, b, c\}$ will contain a register with an encryption of all three shares. Any simulation¹¹ of $\text{Enc}_{\text{T D}}$ acting directly on the shares would then seem to require communication, which is too strong to impose on the classical secret sharing scheme.

Our first idea is to instead use independent tamper-detection codes $\text{Enc}_{\text{T D}}^{\lambda_i}(M_i)$ for each $i \in \{a, b, c\}$, using different numbers λ_i of “trap” EPR pairs $(E_i \hat{E}_i)$. This new gadget Enc_{Δ} can be implemented using only *local knowledge* of m_i and pre-shared entanglement between the parties, but no communication.

$\text{Enc}_{\Delta}(M_a, M_b, M_c)$: Each party $i \in \{a, b, c\}$,

1. Encodes their share into $\text{Enc}_{\text{T D}}^{\lambda_i}(M_i)$, resulting in registers $(Q_i, Y_i \hat{E}_i, X_i)$.
2. Party i keeps the register Q_i , hands X_i to party $i - 1$, and both Y_i, \hat{E}_i to party $i + 1$.

The decoding channel Dec_{Δ} simply runs $\text{Dec}_{\text{T D}}^{\lambda_i}$ on the 3 TDCs, and rejects if anyone of them does.

We find it illustrative to imagine parties a, b, c in a triangle, where each party i hands the EPR halves \hat{E}_i to the party immediately to their left in the ordering $c > b > a$. As we will see shortly, the locality in this scheme plays an important role in addressing the selective bot problem, however, it also comes at a cost. Suppose we tamper with $\text{Enc}_{\Delta}(M_a, M_b, M_c)$ using a channel in LO^3 . Since Enc_{Δ} is comprised of 3 copies of $\text{Enc}_{\text{T D}}$ “in parallel”, we apriori have no global guarantee on all 3 of them.

Nevertheless, we prove that we can recover *two out of the three shares* by picking an increasing sequence of EPR pairs, say, $\lambda_a, \lambda_b, \lambda_c = \lambda, 2 \cdot \lambda, 3 \cdot \lambda$ (where $a < b < c$). This follows by carefully inspecting the marginal distribution on any single recovered share M'_i . We observe that the effective tampering channel on the i th tamper-detection code can be simulated by a channel in the bounded storage model $\text{LO}^3_{(\lambda_{i-1}, *, *)}$: where the other two copies of $\text{Enc}_{\text{T D}}$ act as pre-shared entanglement which can aid the adversaries.¹² Since $\text{Enc}_{\text{T D}}^{\lambda_i}$ is secure against $\text{LO}_{(\lambda_{i-1}, *, *)}$ whenever $\lambda_i > \lambda_{i-1}$, we acquire weak tamper detection for both M_b and M_c (but not M_a).¹³

We point out that this is the main technical reason for which we relax our model to ramp secret sharing (instead of threshold): our decoder will have essentially no guarantees on the smallest share of T . Nevertheless, by the same union bound, all the other shares of T except for the smallest are “weak” tamper-detected. If we assume $|T| \geq t + 1$, we acquire weak tamper detection for the message as well.

Correlations between Triangles and Leakage-Resilient Secret Sharing. To conclude our construction, we show how to leverage leakage-resilient secret sharing to ensure the probability the decoder rejects doesn’t depend on the message. Together with the weak tamper detection guarantee inherited by the outer tamper-detection codes, this implies the secret sharing scheme is actually (strongly) tamper detecting.

¹¹By simulation, we mean that the joint distribution over the recovered shares is the same as that of the quantum process.

¹²Recall from [Section 1.3.2](#) that $\text{LO}^3_{(e, *, *)}$ corresponds to the set of 3-split-state adversaries aided by a pre-shared entangled state where only one of the parties is limited to e qubits, but the other two may have unbounded sized registers.

¹³This idea resembles [[GK18a](#)] approach to non-malleable secret sharing by combining secret sharing schemes with different privacy parameters to acquire one-sided independence.

At a high level, we devise a *bipartite* decoder which captures the following scenario. Suppose we partition an authorized subset $T = A \cup B$ into disjoint, unauthorized sets of size ≥ 3 . After independent tampering on the shares of T , suppose we use Dec_Δ to decode all the triplets a, b, c contained entirely in A or B , but we completely ignore the gadgets which cross between A and B . What is the probability any of the gadgets rejects? Can it correlate with the message m ?

It turns out this scenario is cleanly captured by a model of quantum leakage-resilience between A and B on the underlying classical secret sharing scheme. Note that the amount of shared entanglement between A, B comprises $\mu = O(p^3 \cdot \lambda)$ qubits, due to all the EPR pairs in the triplets of $A \cup B$. Using the locality of the gadget Enc_Δ , we claim that the joint measurement performed to decide whether A or B reject in our compiler (a binary outcome POVM), can be simulated directly on the underlying classical secret sharing scheme by a μ qubit “leakage channel” from A to B .

In slightly more detail, this leakage channel comprises the following experiment on the classical secret sharing scheme. The parties in A are allowed to jointly process their classical shares M_A into a (small) $\mu \ll |M_A|$ qubit quantum state σ_{M_A} , which is then sent to the parties in B . In turn, the parties in B are allowed to jointly use their own shares M_B , together with the leakage σ_{M_A} , to attempt to guess the message. If B can’t distinguish between any two messages m^0, m^1 with bias more than ε_{LR} , that is,

$$M_B^0, \sigma_{M_A^0} \approx_{\varepsilon_{\text{LR}}} M_B^1, \sigma_{M_A^1} \text{ where } M_A^b, M_B^b = (\text{Share}_{\text{LRSS}}(m^b))_{A \cup B}, \quad (6)$$

then we say the secret sharing scheme is leakage-resilient (to this bipartite quantum leakage model) with error ε_{LR} (see [Definition 2.18](#)). In the context of our compiler, if we choose the underlying secret sharing scheme to be such an LRSS, we conclude that the probability A, B reject is (roughly) ε_{LR} close to independent of the message. Together with the weak tamper detection guarantee, we all but conclude the proof.

It only remains to construct such classical leakage-resilient secret sharing schemes. Unfortunately, this bipartite quantum leakage model is slightly non-standard, and to the best of our knowledge, there are no “off-the-shelf” solutions in the literature. This is due in part to the joint nature of the leakage model, but also to its security against quantum adversaries. In [Appendix C](#), we show how to construct leakage-resilient secret sharing schemes in this model following simple modifications to a classical LRSS compiler by [\[CKOS22\]](#).

1.3.4 Non-Malleable Secret-Sharing against LOCC

In this subsection, we present an overview of our constructions of non-malleable secret sharing schemes for classical messages, secure against LOCC. However, to introduce our proof techniques, we begin by presenting a much simpler 4-split-state non-malleable code secure against LOCC (see [Figure 2a](#)).

Non-Malleable Codes in the 4-Split-State Model. To devise our codes against LOCC^4 , we combine:

- (a) $(\text{Enc}_{\text{LOCC}}, \text{Dec}_{\text{LOCC}})$, a family of bipartite LOCC data hiding schemes (See [Definition 2.25](#)).
- (b) $(\text{Enc}_{\text{NM}}, \text{Dec}_{\text{NM}})$, a classical 2-split-state non-malleable code.

To encode a message m ,

1. We first share it into two classical shares using the non-malleable code $(L, R) \leftarrow \text{Enc}_{\text{NM}}(m)$.
2. Then, we further encode L into the bipartite data hiding scheme supported on quantum registers (L_1, L_2) , and similarly R into (R_1, R_2) . The result is a quantum state on 4 registers, (L_1, L_2, R_1, R_2) :

$$\text{Enc}^4(m) = \text{Enc}_{\text{LOCC}}^{\otimes 2} \circ \text{Enc}_{\text{NM}}(m) \quad (7)$$

Our decoder is relatively straightforward: We first decode the “inner” data hiding schemes, and then decode the “outer” non-malleable code. Now, suppose we tamper with the shares using a quantum channel

$\Lambda \in \text{LOCC}^4$. To prove the non-malleability of our construction, it suffices to show that the distribution over recovered messages $M' \leftarrow \text{Dec} \circ \Lambda \circ \text{Enc}(m)$, can be simulated by a split-state tampering channel $\Lambda' \in \text{LO}^2$ acting directly on the underlying 2-split-state code¹⁴, a technique known as a *non-malleable reduction*.

Simulating the Communication Transcript with Shared Randomness. In general, since the density matrix describing the code-state $\text{Enc}^4(m)$ is a separable state across the left/right cut, after LOCC tampering, it remains separable. What this implies is that the post-tampered state on each side of the cut can only depend on the original classical share L (or R) on that side, in addition to the entire classical communication transcript $C \in \{0, 1\}^*$ of the LOCC protocol. If we were to then condition on L, R and the transcript C , the distribution over the recovered classical shares \tilde{L}, \tilde{R} is conditionally independent:

$$p(\tilde{L}, \tilde{R} | L, R, C) = p(\tilde{L} | L, C) \times p(\tilde{R} | R, C) \quad (8)$$

However, a priori, C may carry information about L, R . To conclude, we leverage the data hiding guarantee to prove the distribution over the transcript C is nearly independent of L, R . If so, then it can be sampled using shared randomness between the two splits, and subsequently one can sample \tilde{L}, \tilde{R} to complete the non-malleable reduction.

We are now in a position to describe our application to non-malleable secret sharing schemes against LOCC^p :

Non-Malleable Secret Sharing against LOCC. We combine:

- (a) $(\text{Enc}_{\text{LOCC}}, \text{Dec}_{\text{LOCC}})$, a family of bipartite LOCC data hiding schemes. Moreover, we assume Enc_{LOCC} is separable.
- (b) $(\text{Share}_{\text{NMSS}}, \text{Rec}_{\text{NMSS}})$, a classical t -out-of- p non-malleable secret sharing scheme secure against *joint tampering* (Definition 2.21).

The outline of our construction is similar to the compiler by [ADN⁺19]. To encode a message m ,

1. We begin by secret-sharing it into the classical scheme: $(M_1, \dots, M_p) \leftarrow \text{Share}_{\text{NMSS}}(m)$.
2. Then, each share M_i is encoded into the bipartite data hiding scheme. In fact, we create $p - 1$ copies of $\text{Enc}_{\text{LOCC}}(M_i)$ on bipartite registers $(L_{i,j}, R_{i,j}), \forall j \in [p] \setminus \{i\}$.

The final i th share, a quantum register S_i , collects all the “left” halves $L_{i,j}$ and “right” halves $R_{j,i}$:

$$S_i = (L_{i,1}, L_{i,2}, \dots, L_{i,i-1}, L_{i,i+1}, \dots, L_{i,p}, R_{1,i}, \dots, R_{i-1,i}, R_{i+1,i}, \dots, R_{p,i}) \quad (9)$$

Separable State Data Hiding Schemes and their Composability. A key challenge in establishing the security of the [ADN⁺19] compiler in the LOCC setting lies in the composability of LOCC data hiding schemes. That is, suppose two parties A, B are handed shares of two code-states $\text{Enc}_{\text{LOCC}}(x), \text{Enc}_{\text{LOCC}}(y)$ of two different data hiding schemes. Can the possession of a (possibly entangled) copy of $\text{Enc}_{\text{LOCC}}(y)$ assist A, B in distinguishing x ?

This question of a composable definition of data hiding schemes was posed in [HP07], and to the best of our knowledge remains largely unstudied. Nevertheless, [DLT02] presented a proof (credited to Wooters), that if the data hiding code-states were in fact separable states, then their distinguishability would be additive. Informally, this is since any LOCC channel which distinguishes x using $\text{Enc}_{\text{LOCC}}(y)$ as “advice” can be simulated using LOCC directly on $\text{Enc}_{\text{LOCC}}(x)$. Shortly thereafter, [EW02] presented a construction of bipartite LOCC data hiding schemes with separable state encodings, which we use in our secret sharing schemes.

¹⁴In particular, an independent convex combination over pairs of deterministic functions (f, g) .

To warm up our proof techniques, we show that already by instantiating the compiler with any threshold secret sharing scheme ($\text{Share}_{\text{SS}}, \text{Rec}_{\text{SS}}$) and a separable data hiding scheme, the resulting code is a “LOCC secret sharing scheme”. That is, even if an unauthorized subset T is allowed arbitrary quantum communication to other parties within T , as well as unbounded classical communication between all the p parties, they would not be able to distinguish the message.

A Multipartite Decoder and a Non-Malleable Reduction to Joint Tampering. To prove the non-malleability of our compiler, we require stronger properties of the underlying secret sharing scheme. In particular, to decode any of the classical shares (M_1, \dots, M_p) , note that we require joint, entangling operations across the shares, which immediately breaks standard “individual” tampering models (like LO).

Instead, inspired by our tamper-detecting secret sharing schemes, we devise a *multipartite* decoder. That is, upon receiving any authorized subset of parties $T \subset [p]$, we begin by partitioning T into disjoint pairs and triplets of parties: $T = T_1 \cup T_2 \cdots T_{\lfloor \frac{|T|}{2} \rfloor}$. Then, we decode the data hiding schemes which are entirely contained within each T_j , but completely ignore the schemes between T_j and any other $T_{j'}$.

The advantage in this multipartite decoder lies in that - conditioned on the classical transcript $c \in \{0, 1\}^*$ of the LOCC tampering channel - the post-tampered shares M'_{T_j} within each partition j are independent of the shares of the other partitions. This is analogous to our 4-split construction, where the transcript “mediates” the tampering channel, and moreover we similarly prove that the transcript itself can be treated as shared randomness. Put together, the distribution over recovered shares M'_{T_j} is close to a convex combination of (deterministic) tampering functions acting only on the shares M_{T_j} within each partition T_j , which is precisely a (mild) form of joint tampering on the classical secret sharing scheme. We use a construction of NMSS secure against such a joint tampering model by [GK18a] to instantiate our result.

1.4 Discussion

We dedicate this section to a discussion on the directions and open questions raised in our work.

Is tamper detection possible against LOCC? Our techniques fall short of constructing tamper-detection codes against adversaries who are allowed classical communication, even though they are limited to an unentangled form of tampering. This is by and large due to the lack of composability of (entangled) LOCC data hiding protocols, together with the fact that we require entanglement to achieve tamper detection.

Optimal tamper detection in the bounded storage model. Our TDCs against LO_a^3 achieve tamper detection even against adversaries whose amount of pre-shared entanglement is at most a constant fraction of the blocklength n . Can one design codes robust against $(1 - o(1)) \cdot n$ qubits of pre-shared entanglement, approaching the non-malleability threshold?

Is tamper detection against LO possible in the 2-split model? Based on our constructions for LO, there are a number of “natural” 2-split candidates which combine classical 2-split non-malleable codes with entanglement across the cut. Nevertheless, they all seem to fall short of tamper detection, due in part to adversaries which correlate their attack on the entangled state with their attack *on both of the shares* of the non-malleable code.

Stronger secret sharing schemes. Can one extend our results to quantum secret sharing schemes, where the messages are quantum states? Extending [GK18a] or [BS19] approach to tamper-detecting secret sharing schemes for quantum messages seems to require completely new ideas, including at the very least a definition of leakage resilient quantum secret sharing.

Can one strengthen our results to threshold secret sharing, or arbitrary access structures? What about decreasing the share sizes (currently polynomially larger than the message)?

What is the capacity of entangled non-malleable codes? Can one generalize our results on the

capacity of separable state quantum non-malleable codes? In [Section 10](#), we point out the key bottlenecks in extending our techniques.

1.4.1 Paper Organization

We organize the rest of this work as follows.

- In [Section 2](#), we present the necessary background in quantum information, as well as basic definitions of tampering channels, tamper-detection and non-malleable codes, secret sharing schemes, and LOCC data hiding schemes.
- In [Section 3](#), we present our reductions from t -split non-malleability to $(t + 1)$ -split tamper detection.
- In [Section 4](#), we present the explicit construction of tamper-detection codes in the bounded storage model of [Theorem 1.2](#). In [Section 5](#), we present the construction of ramp secret sharing schemes which detect unentangled tampering of [Theorem 1.4](#).
- In [Section 6](#), we formalize the construction of a 2-split-state NMC for single-bit messages secure against LOCC of [Theorem 1.1](#). In [Section 7](#), we present the construction of NMCs in the 4-split-state model against LOCC.
- In [Section 8](#), we present the non-malleable secret sharing schemes against LOCC of [Theorem 1.3](#).
- Finally, in [Section 9](#), we discuss connections between tamper-resilient quantum codes and quantum encryption schemes, followed by our proof of the capacity of separable state non-malleable codes in [Section 10](#) ([Theorem 1.5](#) and [Theorem 1.6](#)).

Our appendix is organized as follows:

- In [Appendix A](#), we present relevant facts and background on Pauli and Clifford operators as well as unitary 2-designs.
- In [Appendix B](#), we revisit and simplify [[BGJR23](#)]'s analysis of 2 split non-malleable codes when the adversaries are unentangled, which we use to instantiate our reductions.
- In [Appendix C](#), we present our construction of secret sharing schemes resilient to quantum leakage, based on a compiler by [[CKOS22](#)], which we use in [Section 5](#).

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2 Preliminaries

In this section we cover some important basic prerequisites from quantum information theory (in [Section 2.1](#)), as well as basic definitions of codes and secret sharing schemes. We refer the reader familiar with quantum states and channels directly to [Section 2.2](#). In the remainder of this section, we define Clifford authentication schemes ([Section 2.2](#)), quantum analogs of the split-state model ([Section 2.3](#)), non-malleable and tamper-detection codes ([Section 2.4](#)), secret sharing schemes ([Section 2.5](#)), and data hiding schemes ([Section 2.6](#)).

2.1 Quantum and Classical Information Theory

The notation and major part of this subsection is Verbatim taken from [\[BGJR23\]](#).

2.1.1 Basic General Notation

We denote sets by uppercase calligraphic letters such as \mathcal{X} and use uppercase roman letters such as X and Y for both random variables and quantum registers. The distinction will be clear from context. We denote the uniform distribution over $\{0, 1\}^d$ by U_d . For a *random variable* $X \in \mathcal{X}$, we use X to denote both the random variable and its distribution, whenever it is clear from context. We use $x \leftarrow X$ to denote that x is drawn according to X , and, for a finite set \mathcal{X} , we use $x \leftarrow \mathcal{X}$ to denote that x is drawn uniformly at random from \mathcal{X} . For two random variables X, Y we use $X \otimes Y$ to denote their product distribution. We call random variables X, Y , *copies* of each other iff $\mathbb{P}[X = Y] = 1$.

2.1.2 Quantum States and Registers

Consider a finite-dimensional Hilbert space \mathcal{H} endowed with an inner-product $\langle \cdot, \cdot \rangle$ (we only consider finite-dimensional Hilbert spaces). A quantum state (or a density matrix or a state) is a positive semi-definite operator on \mathcal{H} with trace 1. It is called *pure* if and only if its rank is 1. Let $|\psi\rangle$ be a unit vector on \mathcal{H} , that is $\langle \psi, \psi \rangle = 1$. With some abuse of notation, we let ψ represent both the state and the density matrix $|\psi\rangle\langle\psi|$.

A *quantum register* A is associated with some Hilbert space \mathcal{H}_A . Define $|A| := \log(\dim(\mathcal{H}_A))$ the size (in qubits) of A . The identity operator on \mathcal{H}_A is denoted \mathbb{I}_A . Let U_A denote the maximally mixed state in \mathcal{H}_A . For a sequence of registers A_1, \dots, A_n and a set $T \subseteq [n]$, we define the projection according to T as $A_T = (A_i)_{i \in T}$. Let $\mathcal{L}(\mathcal{H}_A)$ represent the set of all linear operators on the Hilbert space \mathcal{H}_A , and $\mathcal{D}(\mathcal{H}_A)$ the set of all quantum states on the Hilbert space \mathcal{H}_A . The state ρ with subscript A indicates that $\rho_A \in \mathcal{D}(\mathcal{H}_A)$.

Composition of two registers A and B , denoted AB , is associated with the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. For two quantum states $\rho \in \mathcal{D}(\mathcal{H}_A)$ and $\sigma \in \mathcal{D}(\mathcal{H}_B)$, $\rho \otimes \sigma \in \mathcal{D}(\mathcal{H}_{AB})$ represents the tensor product (*Kronecker product*) of ρ and σ . Let $\rho_{AB} \in \mathcal{D}(\mathcal{H}_{AB})$. Define

$$\rho_B \stackrel{\text{def}}{=} \text{tr}_A \rho_{AB} \stackrel{\text{def}}{=} \sum_i (\langle i| \otimes \mathbb{I}_B) \rho_{AB} (|i\rangle \otimes \mathbb{I}_B),$$

where $\{|i\rangle\}_i$ is an orthonormal basis for the Hilbert space \mathcal{H}_A . The state $\rho_B \in \mathcal{D}(\mathcal{H}_B)$ is referred to as the marginal state or partial trace of ρ_{AB} on the register B . Given $\rho_A \in \mathcal{D}(\mathcal{H}_A)$, a *purification* of ρ_A is a pure state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_{AB})$ such that $\text{tr}_B \rho_{AB} = \rho_A$. Purifications of quantum states are not unique, however, given two registers A, B of same dimension and orthonormal bases $\{|i\rangle_A\}$ and $\{|i\rangle_B\}$ over \mathcal{H}_A and \mathcal{H}_B respectively, the *canonical purification* of a quantum state ρ_A is a pure state $\rho_{AB} \stackrel{\text{def}}{=} (\rho_A^{\frac{1}{2}} \otimes \mathbb{I}_B) (\sum_i |i\rangle_A |i\rangle_B)$.

2.1.3 Quantum Measurements, Channels and Instruments

A *Hermitian* operator $H : \mathcal{H}_A \rightarrow \mathcal{H}_A$ is such that $H = H^\dagger$. A projector $\Pi \in \mathcal{L}(\mathcal{H}_A)$ is a Hermitian operator such that $\Pi^2 = \Pi$. A *unitary* operator $V_A : \mathcal{H}_A \rightarrow \mathcal{H}_A$ is such that $V_A^\dagger V_A = V_A V_A^\dagger = \mathbb{I}_A$. The set of all unitary operators on \mathcal{H}_A is denoted by $\mathcal{U}(\mathcal{H}_A)$. A positive operator-valued measure (*POVM*) $\{M_i\}_i$ is a collection of Hermitian operators where $0 \leq M_i \leq \mathbb{I}$ and $\sum_i M_i = \mathbb{I}$. Each M_i is associated to a measurement

outcome i , where outcome i occurs with probability $\text{tr}[M_i\rho]$ when measuring state ρ . We use the shorthand $\bar{M} \stackrel{\text{def}}{=} \mathbb{I} - M$, and M to represent $M \otimes \mathbb{I}$, whenever \mathbb{I} is clear from context.

A quantum map $\mathcal{E} : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_B)$ is a completely positive and trace preserving (CPTP) linear map. A CPTP map \mathcal{E} is described by the Kraus operators $\{K_i : \mathcal{H}_A \rightarrow \mathcal{H}_B\}_i$ such that $\mathcal{E}(\rho) = \sum_i K_i \rho K_i^\dagger$ but $\sum_i K_i^\dagger K_i = \mathbb{I}_A$. A (trace non-increasing) CP map \mathcal{E} is similarly described by the Kraus operators $\{K_i\}_i$ but $\sum_i K_i^\dagger K_i < \mathbb{I}_A$. A (discrete) quantum instrument \mathcal{J} is a family of CP maps $(\mathcal{E}_j : j \in \Theta)$, where Θ is some finite or countable set such that $\sum_{j \in \Theta} \mathcal{E}_j$ is trace preserving. Moreover, a quantum instrument defines the quantum-classical CPTP map $\sum_j |j\rangle\langle j| \otimes \mathcal{E}_j(\cdot)$. Finally, we use the notation $\mathcal{N} \circ \mathcal{M}$ to denote the composition of maps \mathcal{N}, \mathcal{M} .

2.1.4 Norms, Distances, and Divergences

This section collects definitions of some important quantum information-theoretic quantities and related useful properties.

Definition 2.1 (Schatten p -norm). *For $p \geq 1$ and a matrix A , the Schatten p -norm of A , denoted by $\|A\|_p$, is defined as $\|A\|_p \stackrel{\text{def}}{=} (\text{tr}(A^\dagger A)^{\frac{p}{2}})^{\frac{1}{p}}$.*

Definition 2.2 (Trace distance). *The trace distance between two states ρ and σ is given by $\|\rho - \sigma\|_1$. We write $\rho \approx_\varepsilon \sigma$ if $\|\rho - \sigma\|_1 \leq \varepsilon$.*

For the facts stated below without citation, we refer the reader to standard textbooks [NC00, Wat18]. The following facts state some basic properties of trace distance.

Fact 2.1 (Data-processing inequality/Monotonicity of trace distance, Proposition 3.8 [Tom15]). *Let ρ, σ be states and \mathcal{E} be a (trace non-increasing) CP map. Then, $\|\mathcal{E}(\rho) - \mathcal{E}(\sigma)\|_1 \leq \|\rho - \sigma\|_1$, which is saturated whenever \mathcal{E} is a CPTP map corresponding to an isometry.*

Fact 2.2. *Let ρ, σ be states. Let Π be a projector. Then,*

$$\text{tr}(\Pi\rho) \left\| \frac{\Pi\rho\Pi}{\text{tr}(\Pi\rho)} - \frac{\Pi\sigma\Pi}{\text{tr}(\Pi\sigma)} \right\|_1 \leq \|\rho - \sigma\|_1.$$

Fact 2.3. *Let ρ, σ be states such that $\rho = \sum_x p_x \rho^x$, $\sigma = \sum_x p_x \sigma^x$, $\{\rho^x, \sigma^x\}_x$ are states and $\sum_x p_x = 1$. Then,*

$$\|\rho - \sigma\|_1 \leq \sum_x p_x \|\rho^x - \sigma^x\|_1.$$

Definition 2.3 (Fidelity). *The fidelity between two states ρ and σ , denoted by $F(\rho, \sigma)$, is defined as*

$$F(\rho, \sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1.$$

Fact 2.4 (Fuchs-van de Graaf inequalities [FvdG06] (see also [Wat11, Theorem 4.10])). *Let ρ, σ be two states. Then,*

$$1 - F(\rho, \sigma) \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho, \sigma)^2}$$

2.1.5 Quantum-Classical, Separable, and Low Schmidt Rank States

This section collects definitions of certain registers and operations on them.

Definition 2.4. *Let \mathcal{X} be a set. A classical-quantum (c-q) state ρ_{XE} is of the form*

$$\rho_{XE} = \sum_{x \in \mathcal{X}} p(x) |x\rangle\langle x| \otimes \rho_E^x,$$

where ρ_E^x are density matrices. Whenever it is clear from context, we identify the random variable X with the register X via $\mathbb{P}[X = x] = p(x)$.

Definition 2.5 (Conditioning). *Let*

$$\rho_{XE} = \sum_{x \in \{0,1\}^n} p(x) |x\rangle \langle x| \otimes \rho_E^x,$$

be a c - q state. For an event $\mathcal{S} \subseteq \{0,1\}^n$, define

$$\mathbb{P}[\mathcal{S}]_\rho \stackrel{\text{def}}{=} \sum_{x \in \mathcal{S}} p(x) \quad \text{and} \quad (\rho|X \in \mathcal{S}) \stackrel{\text{def}}{=} \frac{1}{\mathbb{P}[\mathcal{S}]_\rho} \sum_{x \in \mathcal{S}} p(x) |x\rangle \langle x| \otimes \rho_E^x.$$

We sometimes shorthand $(\rho|X \in \mathcal{S})$ as $(\rho|\mathcal{S})$ when the register X is clear from context.

Entangled bi- or multi-partite quantum states are those which are not separable.

Definition 2.6. *A bipartite mixed state ρ is said to be separable if it can be written as a convex combination over product states,*

$$\rho = \sum_i p_i \cdot \sigma_i \otimes \tau_i \tag{10}$$

Where $\{\sigma_i, \tau_i\}_i$ are collections of density matrices, and $p_i \in \mathbb{R}^+$ s.t. $\sum_i p_i = 1$.

Entanglement can't be created across multi-partite quantum states using only local operations (CP maps) and classical communication, and thereby separable states remain separable under LOCC. The notion of a Schmidt rank extends this behavior to states with only a finite amount of entanglement across its partitions. For a bipartite pure state $|\psi\rangle$ in $\mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, we write its Schmidt decomposition as:

$$|\psi\rangle = \sum_{i=1}^R \sqrt{\lambda_i} |a_i\rangle \otimes |b_i\rangle \tag{11}$$

Where $\{|a_i\rangle\}_{i \in [R]}$ and $\{|b_i\rangle\}_{i \in [R]}$ define collections of orthonormal vectors. The integer R corresponds to the Schmidt rank of ψ , and is a measure of how entangled ψ is across the bipartition. Moreover, $R \leq |A|, |B|$. [TH99] introduced a notion of Schmidt number for density matrices, generalizing the definition for pure states.

Definition 2.7 (The Schmidt Number, [TH99]). *A bipartite density matrix ρ has Schmidt number R if:*

1. *For any decomposition of $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$, at least one of the vectors with $p_i > 0$, $|\psi_i\rangle$ has Schmidt rank at least R .*
2. *There exists a decomposition of ρ with all vectors $\{|\psi_i\rangle\}$ of Schmidt rank at most R .*

In other words, the Schmidt number of a mixed state ρ is the minimum (over all possible decompositions of ρ) of the maximum Schmidt rank in the decomposition. [TH99] proved that much like the the Schmidt rank for pure states, the mixed state Schmidt number cannot increase under unentangled operations.

Proposition 2.1 ([TH99]). *The Schmidt number of a density matrix cannot increase under local quantum operations and classical communication (LOCC).*

2.2 Clifford Authentication Schemes

Roughly speaking, a quantum authentication scheme (QAS) enables two parties who share a secret key, to reliably communicate an encoded quantum state over a noisy channel with the following guarantee: If the state is untampered, the receiver accepts, whereas if the state was altered by the channel, the receiver rejects [BCG⁺01, ABOE08].

[ABOE08] showed how to construct QASs from unitary 2-designs, and in particular *the Clifford group*.¹⁵ To encode a message state ψ , [ABOE08] apply a uniformly random Clifford unitary C to ψ , which later is decoded by applying C^\dagger (potentially after adversarial tampering). The description of C is used as a shared secret key between sender and receiver, and is unknown to the adversary. [ABOE08] showed this scheme either recovers the message, or completely scrambles it, by appealing to Pauli randomization properties of the Clifford group. Here, we briefly summarize two of these properties, and describe a key-efficient family of unitary 2-designs by [CLLW16].

The most basic property is that conjugating any state by a random Pauli (or Clifford), encrypts it:

Fact 2.5 (Pauli’s are 1-designs). *Let $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, and $\mathsf{P}(\mathcal{H}_A)$ be the Pauli group on \mathcal{H}_A . Then,*

$$\frac{1}{|\mathsf{P}(\mathcal{H}_A)|} \sum_{Q \in \mathsf{P}(\mathcal{H}_A)} (Q \otimes \mathbb{I}) \rho_{AB} (Q^\dagger \otimes \mathbb{I}) = U_A \otimes \rho_B.$$

The second property we require is known as the “Clifford Twirl” property. Informally, applying a uniformly random Clifford operator (by conjugation) maps any Pauli Q to a uniformly random non-identity Pauli operator. [ABOE08] used this property to argue the security of their authentication scheme after tampering. We actually require a (standard) strengthening of the Clifford twirl property in the presence of side-information, which we depict visually in Figure 4 and prove in Appendix A for completeness:

Lemma 2.2 (Clifford Twirl with Side Information). *Consider Figure 4. Let R denote a uniformly random key of $\log |\mathcal{C}(\mathcal{H}_A)|$ bits, where $\mathcal{C}(\mathcal{H}_A)$ denotes the Clifford group on \mathcal{H}_A . Let $\Lambda : \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_E) \rightarrow \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_E)$ be an arbitrary CPTP map on A, E . Then, there exists CP maps $\Phi_1, \Phi_2 : \mathcal{L}(\mathcal{H}_E) \rightarrow \mathcal{L}(\mathcal{H}_E)$ acting only on register E , depending only on Λ , such that*

$$\rho_{\hat{A}AE} = \frac{1}{|\mathcal{C}(\mathcal{H}_A)|} \sum_{C \in \mathcal{C}(\mathcal{H}_A)} C^\dagger \circ \Lambda \circ C(\psi_{\hat{A}AE}) \approx \frac{2}{2^{2|A|}-1} \Phi_1(\psi_{\hat{A}AE}) + (\Phi_2(\psi_{\hat{A}E}) \otimes U_A),$$

$\forall \psi_{\hat{A}AE} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_{\hat{A}} \otimes \mathcal{H}_E)$. Moreover, $\Phi_1 + \Phi_2$ is CPTP.

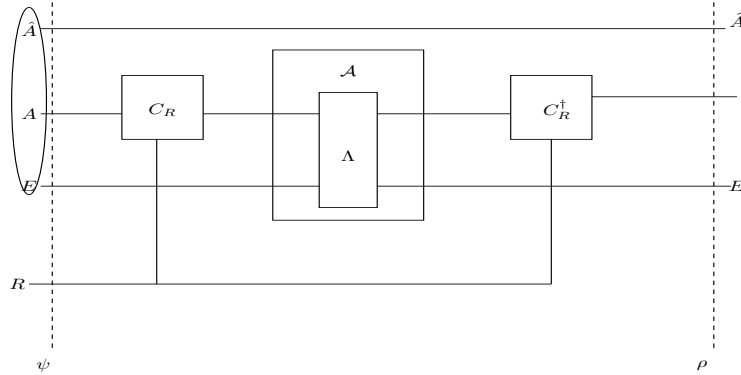


Figure 4: Clifford Twirling with Side Information.

In this work, following [BGJR23], we leverage a particular key-efficient (near-linear) construction of 2-designs by Cleve, Leung, Liu and Wang [CLLW16]:

Fact 2.6 (Subgroup of the Clifford group [CLLW16]). *There exists a subgroup $\mathcal{SC}(\mathcal{H}_A)$ of the Clifford group $\mathcal{C}(\mathcal{H}_A)$ such that given any non-identity Pauli operators $P, Q \in \mathsf{P}(\mathcal{H}_A)$ we have that*

$$|\{C \in \mathcal{SC}(\mathcal{H}_A) | C^\dagger P C = Q\}| = \frac{|\mathcal{SC}(\mathcal{H}_A)|}{|\mathsf{P}(\mathcal{H}_A)| - 1} \quad \text{and} \quad |\mathcal{SC}(\mathcal{H}_A)| = 2^{5|A|} - 2^{3|A|}.$$

¹⁵The Clifford group is the collection of unitaries that map Pauli matrices to Pauli matrices (up to a phase). We refer the reader to Appendix A for a more comprehensive definitions of Paulis, Cliffords and their properties.

Moreover, there exists a procedure **Samp** which given as input a uniformly random string $R \leftarrow \{0, 1\}^{5|A|}$ outputs in time $\text{poly}(|A|)$ a Clifford operator $C_R \in \mathcal{SC}(\mathcal{H}_A)$ drawn from a distribution $2 \cdot 2^{-2|A|}$ close to the uniform distribution over $\mathcal{SC}(\mathcal{H}_A)$.

We remark that both properties above are true as stated even for unitaries chosen at random from $\mathcal{SC}(\mathcal{H}_A)$ using **Samp**.

2.3 Formal Definitions of Split-State Tampering Models

In the absence of a secret key, it is impossible to offer tamper-resilience against arbitrary tampering channels. In this subsection, we formalize the various models of split-state tampering that we consider in this work. Suppose t parties share a quantum state defined on $\mathcal{D}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_t)$. We begin by introducing the simplest family of tampering channels, where the split-state adversaries are restricted to unentangled operations:

Definition 2.8 (LO^t : Unentangled t -Split Model). *These are t -split quantum adversaries without shared entanglement, comprised of tensor product channels $\Lambda = \Lambda_1 \otimes \cdots \otimes \Lambda_t$ where each $\Lambda_i : \mathcal{L}(\mathcal{H}_i) \rightarrow \mathcal{L}(\mathcal{H}_i)$ is a CPTP map.*

We refer to this model as the unentangled t -split model, or simply as the unentangled split-state model. Note that if each channel Λ_i were a deterministic function f_i and the ciphertext were classical, then we would recover the classical model [LL12]. Moreover, we remark that by a standard averaging argument, codes which are secure against this model also capture security in the presence of shared randomness.

Only recently have adversaries with *quantum capabilities* and *shared entanglement* been considered. The works of [ABJ22, BGJR23, BBJ23] generalized the definition above to adversaries with an unbounded amount of pre-shared entanglement, a model we refer to as LO_*^t .

Definition 2.9 (LO_*^t : Entangled t -Split Model). *These are t -split quantum adversaries with unbounded shared entanglement, comprised of an arbitrary multi-partite ancilla state $\psi \in \mathcal{D}(\mathcal{H}_{E_1} \otimes \cdots \otimes \mathcal{H}_{E_t})$, and a tensor product channel $(\Lambda_1 \otimes \cdots \otimes \Lambda_t)$ where each $\Lambda_i : \mathcal{L}(\mathcal{H}_i \otimes \mathcal{H}_{E_i}) \rightarrow \mathcal{L}(\mathcal{H}_i \otimes \mathcal{H}_{E_i})$ is a CPTP map.*

For a visual representation of this model with $t = 3$, refer to Figure 7. Recall that tamper detection is impossible in LO_*^t , due to the standard substitution attack, where the adversaries store in ϕ a pre-shared copy of the encoding a fixed message. This impossibility motivates the next model, which we refer to as the “bounded storage model”, where we impose restrictions on the amount of pre-shared entanglement between the parties.

Definition 2.10 ($\text{LO}_{(e_1, e_2, \dots, e_t)}^t$: Bounded Storage Model). *These are t -split quantum adversaries with bounded shared entanglement, where the ancilla state $\psi \in \mathcal{D}(\mathcal{H}_{E_1} \otimes \cdots \otimes \mathcal{H}_{E_t})$, satisfies $|E_i| \leq e_i \forall i \in [t]$.*

A visual representation of this bounded storage model with $t = 3$ can be found in Figure 7.

Remark 2.1. *We note to the reader that LO^t is same as $\text{LO}_{(e_1, e_2, \dots, e_t)}^t$ for $e_i = 0$ for every $i \in [t]$, and LO_*^t can be understood as the limit of $e_i \rightarrow \infty$ for every $i \in [t]$.*

To conclude this subsection, the last model of tampering channels we consider is to allow the adversaries local quantum operations and classical communication. It is notoriously hard to characterize the precise mathematical structure of the multi-partite CPTP maps which are implementable using LOCC [CLM⁺12]. This is in part due to the non-locality of information enabled by the classical communication, but also due to the “feedback” that arises from measurement outcomes at some point in time effecting the operations performed in the future¹⁶. Fortunately, for our purposes, a coarse characterization in terms of separable channels, and the transcript of the classical communication in the LOCC protocol, will suffice:

¹⁶We refer the reader to [CLM⁺12] for a review of LOCC protocols.

Definition 2.11 (LOCC^t). *These are t -split quantum adversaries with local quantum operations and classical communication. Any CPTP channel $\Lambda \in \text{LOCC}^t$ can be expressed as a linear combination of tensor products of CP maps,*

$$\Lambda(\sigma) = \sum_{c \in \{0,1\}^*} (\mathcal{E}_1^c \otimes \mathcal{E}_2^c \otimes \cdots \otimes \mathcal{E}_t^c)(\sigma) \quad \forall \sigma \in \mathcal{D}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_t), \quad (12)$$

where $c \in \{0,1\}^*$ denotes the classical transcript of the communication protocol implementing Λ , and each $\mathcal{E}_i^c : \mathcal{L}(\mathcal{H}_i) \rightarrow \mathcal{L}(\mathcal{H}_i)$.

The description in [Definition 2.11](#) is simply formalizing the following process: The first party performs some quantum instrument on their share, and broadcasts the classical outcome j . Conditioned on j , the resulting (unnormalized) state is the outcome of the CP map $(\mathcal{E}_1^j \otimes \mathbb{I})(\sigma)$. Then, j can be used by the other parties to infer a next operation, and so on. Indexing the CP maps by the classical transcript $c \in \{0,1\}^*$ provides a coarse description of Λ .

2.4 Non-Malleable Codes and Tamper-Detection Codes

We dedicate this section to define classical and quantum non-malleable codes and tamper-detection codes.

2.4.1 Classical Non-Malleable Codes

Consider the following tampering experiment. We are given some message m , encode it into some coding scheme $c \leftarrow \text{Enc}(m)$, tamper with it using some function $f : c' = f(c)$, and finally decode it to $\text{Dec}(f(c)) = m'$. If Enc were an error correcting code robust against f , then our goal would be to recover $m' = m$. If Enc were an error (or *tamper*) detecting code robust against f , we would like either $m' = m$ or we reject, $m' = \perp$. Unfortunately, error correction or detection isn't possible for every f (for instance, constant functions).

Nevertheless, there are cases where the notion of non-malleability can be much more versatile. Dziembowski, Pietrzak, and Wichs formalized a coding-theoretic notion of non-malleability using a simulation paradigm: m' can be simulated by a distribution that depends only on the adversarial function f (and not the message):

Definition 2.12 (Non-Malleable Codes, adapted from [\[DPW18\]](#)). *A pair of (randomized) algorithms $(\text{Enc} : \{0,1\}^k \rightarrow \{0,1\}^n, \text{Dec} : \{0,1\}^n \rightarrow \{0,1\}^k)$, is an (ε, δ) non-malleable code with respect to a family of functions $\mathcal{F} \subset \{f : \{0,1\}^n \rightarrow \{0,1\}^n\}$, if the following properties hold:*

1. **Correctness:** For every message $m \in \{0,1\}^k$, $\mathbb{P}[\text{Dec} \circ \text{Enc}(m) \neq m] \leq \delta$.
2. **Non-Malleability:** For every $f \in \mathcal{F}$ there is a constant $p_f \in [0,1]$ and a random variable D_f on $\{0,1\}^k$ which is independent of the message, such that

$$\forall m \in \{0,1\}^k : \text{Dec} \circ f \circ \text{Enc}(m) \approx_\varepsilon p_f \cdot m + (1 - p_f) \cdot D_f \quad (13)$$

In other words, the outcome of the effective channel on the message is a convex combination of the original message or a fixed distribution.¹⁷ When the code is perfectly correct or its correctness is otherwise clear from context, we simply say non-malleable with error ε . We point out that [\[DKO13\]](#) showed that NMCs with a single bit message could be defined in an alternative, arguably much simpler way: A classical code is non-malleable if and only if “it is hard to negate the encoded bit with functions from \mathcal{F} ”. That is,

Theorem 2.3. *A classical code (Enc, Dec) is non-malleable with error ε iff for all $f \in \mathcal{F}$,*

$$\mathbb{P}_{b \leftarrow \{0,1\}} [\text{Dec} \circ f \circ \text{Enc}(b) = \neg b] \leq \frac{1}{2} + \varepsilon. \quad (14)$$

¹⁷Non-malleability is originally defined in terms of a simulator which is allowed to output a special symbol *same** which is later replaced by the encoded m . The reader may be more accustomed to this notion, nevertheless, it is well-known to be equivalent to the above.

Note that flipping the encoded bit with probability $\frac{1}{2}$ is often trivial: It suffices to swap the message with a copy of an encoding of a random bit. What the above ensures is that (Enc, Dec) is non-malleable iff no $f \in \mathcal{F}$ can do even marginally better.

Remark 2.2. *When designing NMCs for classical messages encoded into quantum states, we change the definitions above only slightly. If the Hilbert space \mathcal{H} denotes the codespace, we modify the image and domain of Enc, Dec to be $\mathcal{D}(\mathcal{H})$, and allow the adversary to tamper with the code using channels $\Lambda : \mathcal{D}(\mathcal{H}) \rightarrow \mathcal{D}(\mathcal{H})$.*

As previously introduced, the classical t -split-state model corresponds to the the collection of individual tampering functions $\{(f_1, f_2, \dots, f_t)\}$ on the shares (for a visual representation of this model with $t = 3$, refer to [Figure 5](#)).

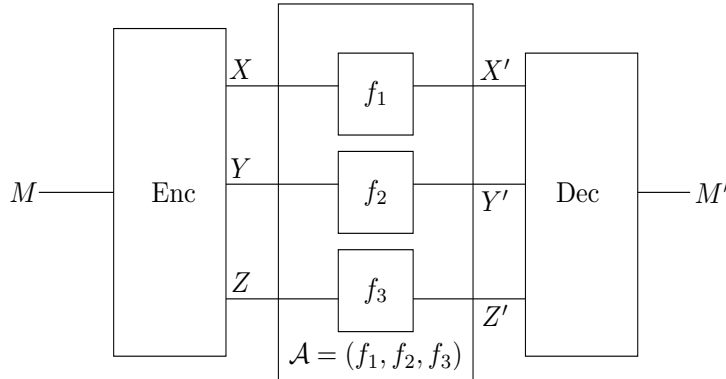


Figure 5: t -split-state tampering model for $t = 3$.

The notion of an *augmented* non-malleable code naturally extends the split-state model, in that the tampered message m' is simulatable even *together* with one of the shares of the NMC:

Definition 2.13 (Augmented NMCs). *Let (Enc, Dec) be a non-malleable code in the t -split-state model, and let X_i denote the random variable corresponding to the i th share. Then, we refer to the code as an augmented non-malleable code with error ε if there exists $i \in [t]$ satisfying: For all t -split-state tampering functions f , there exists $p_f \in [0, 1]$ and two distributions \mathcal{X}_f over $\{0, 1\}^{|X_i|}$ and \mathcal{D}_f over $\{0, 1\}^k \times \{0, 1\}^{|X_i|}$, such that*

$$(m', X_i) \approx_\varepsilon p_f \cdot (m, \mathcal{X}_f) + (1 - p_f) \cdot \mathcal{D}_f. \quad (15)$$

Where $p_f, \mathcal{D}_f, \mathcal{X}_f$ only depend on the function f .

To instantiate our compilers, we require constructions of classical non-malleable codes in the 2-split-state model (both with and without augmentation). Here we use an existential (inefficient) result by [\[CG13a\]](#) on capacity-achieving codes, and a recent near-optimal efficient construction by [\[Li23b\]](#).

Theorem 2.4 ([\[Li23b\]](#)). *For any $n \in \mathbb{N}$ there exists an efficient augmented non-malleable code in the 2-split-state model, which has block length $2n$, rate $k/(2n) = \Omega(1)$, perfect correctness and error $2^{-\Omega(n)}$.*

Theorem 2.5 ([\[CG13a\]](#), Theorem 3.1). *For every sufficiently large block-length $n \in \mathbb{N}$ there exist non-malleable codes in the 2-split-state model of rate $1/2 - o(1)$, perfect correctness and error $2^{-\Omega(n)}$.*

2.4.2 Quantum Secure Non-Malleable Codes and Extractors

To construct our tamper-detection codes in the bounded storage model, we actually require classical split-state non-malleable codes which are *quantum secure*, that is, robust to split-state adversaries who are allowed pre-shared entanglement LO_* [\[ABJ22\]](#). While it is not too conceptually challenging to modify [Definition 2.12](#) to address quantum adversaries, unfortunately, constructions of high-rate quantum secure non-malleable

codes remains an open problem. Instead, in order to optimize the rate of our results, we appeal to fact that we don't actually require the non-malleability of a worst case message in our codes, just that of a random secret key (see [Section 1.3.2](#)). For this purpose, [\[BBJ23, BGJR23\]](#) leverage the notion of a *quantum secure, 2-source non-malleable extractor*, and we follow suit.

The connection between 2-source non-malleable extractors and codes was first made explicit by [\[CG13b\]](#). Informally, a 2-source extractor nmExt takes as input two some-what random sources X and Y , and outputs a near-uniformly random string $R = \text{nmExt}(X, Y)$ (which will be our secret key). What makes its non-malleable is that R is still near uniformly random even conditioned on any $R' = \text{nmExt}(f(X), g(Y))$, which is the outcome of independent tampering functions f, g on the sources X, Y . In our work we make use of a construction of *quantum secure* 2-source non-malleable extractors by [\[BBJ23\]](#):

Fact 2.7 (Quantum Secure 2-Source Non-malleable Extractor [\[BBJ23\]](#)). *Consider the split-state tampering experiment in [Figure 6](#) with a split-state tampering adversary $\mathcal{A} = (U, V, |\psi\rangle_{E_1 E_2})$. Based on this figure, define $p_{\text{same}} = \mathbb{P}[(X, Y) = (X', Y')]_{\hat{\rho}}$ and the conditioned quantum states*

$$\rho^{\text{same}} = (\text{nmExt} \otimes \text{nmExt})(\hat{\rho}|(X, Y) = (X', Y')) \text{ and } \rho^{\text{tamp}} = (\text{nmExt} \otimes \text{nmExt})(\hat{\rho}|(X, Y) \neq (X', Y')).$$

Then, for any $n \geq n_0$ and constant $\delta > 0$, there exists an explicit function $\text{nmExt} : \{0, 1\}^n \times \{0, 1\}^{\delta n} \rightarrow \{0, 1\}^r$ with $r = (1/2 - \delta) \cdot n$ such that for independent sources $X \leftarrow \{0, 1\}^n$ and $Y \leftarrow \{0, 1\}^{\delta n}$ and any such split-state tampering adversary $\mathcal{A} = (U, V, |\psi\rangle_{E_1 E_2})$, nmExt satisfies

1. *Strong Extraction:* $\|\text{nmExt}(X, Y)X - U_r \otimes U_n\|_1 \leq \varepsilon$ and $\|\text{nmExt}(X, Y)Y - U_r \otimes U_{\delta n}\|_1 \leq \varepsilon$,
2. *Augmented Non-Malleability:* $p_{\text{same}}\|\rho_{R'E_2'}^{\text{same}} - U_r \otimes \rho_{E_2'}^{\text{same}}\|_1 + (1 - p_{\text{same}})\|\rho_{RR'E_2'}^{\text{tamp}} - U_r \otimes \rho_{R'E_2'}^{\text{tamp}}\|_1 \leq \varepsilon$,

with $\varepsilon = 2^{-n^{\Omega_\delta(1)}}$. Furthermore, $\text{nmExt}(x, y)$ can be computed in time $\text{poly}(n)$.

Item 1 in [Fact 2.7](#) should be the familiar 2-source (strong) extraction guarantee. Item 2 however, captures the *augmented* non-malleability of nmExt . If the tampering attack doesn't change X and Y , then R should be close to uniformly random even together with any updated side-entanglement E_2' held by one of the adversaries. Conversely, if the tampering attack changed X and Y , then R should be independent of the tampered output $R' = \text{nmExt}(X', Y')$, jointly with E_2' .

We remark that the role of *augmented* non-malleability is more subtle in the quantum secure model. In the classical setting ([Definition 2.13](#)), the tampering functions are deterministic, and therefore the left share X (before tampering) uniquely determines the entire "view" of the left adversary after tampering. In contrast, in the quantum secure model, the entangled state that the split-state adversaries share may a priori collapse and correlate with *both* post-tampered shares X', Y' - thus not immediately captured by just X .

2.4.3 Quantum Non-Malleable Codes and Augmented Non-Malleable Codes

We now recollect the definition of NMCs from [\[BGJR23\]](#) for quantum messages, referred to as *quantum non-malleable codes*. For concreteness we state a definition for the $\text{LO}_{(e_1, e_2, e_3)}^3$ adversary model, which can be generalized to any adversary model easily.

Let σ_M be an arbitrary state in a message register M and $\sigma_{M\hat{M}}$ be its canonical purification. Our coding scheme is given by an encoding Completely Positive Trace-Preserving (CPTP) map $\text{Enc} : \mathcal{L}(\mathcal{H}_M) \rightarrow \mathcal{L}(\mathcal{H}_X \otimes \mathcal{H}_Y \otimes \mathcal{H}_Z)$ and a decoding CPTP map $\text{Dec} : \mathcal{L}(\mathcal{H}_X \otimes \mathcal{H}_Y \otimes \mathcal{H}_Z) \rightarrow \mathcal{L}(\mathcal{H}_M)$. Here, $\mathcal{H}_X, \mathcal{H}_Y, \mathcal{H}_Z$ denote the Hilbert spaces for the three shares. The most basic property we require of this coding scheme (Enc, Dec) is correctness, including preserving entanglement with external systems:

$$\text{Dec}(\text{Enc}(\sigma_{M\hat{M}})) = \sigma_{M\hat{M}}.$$

Now, suppose we tamper with the code using a tampering adversary \mathcal{A} in $\text{LO}_{(e_1, e_2, e_3)}^3$. Recall from [Definition 2.10](#) that such adversaries are specified by three tampering maps, $U : \mathcal{L}(\mathcal{H}_X \otimes \mathcal{H}_{E_1}) \rightarrow \mathcal{L}(\mathcal{H}_{X'} \otimes \mathcal{H}_{E_1'})$, $V : \mathcal{L}(\mathcal{H}_Y \otimes \mathcal{H}_{E_2}) \rightarrow \mathcal{L}(\mathcal{H}_{Y'} \otimes \mathcal{H}_{E_2'})$ and $W : \mathcal{L}(\mathcal{H}_Z \otimes \mathcal{H}_{E_3}) \rightarrow \mathcal{L}(\mathcal{H}_{Z'} \otimes \mathcal{H}_{E_3'})$ along with a quantum

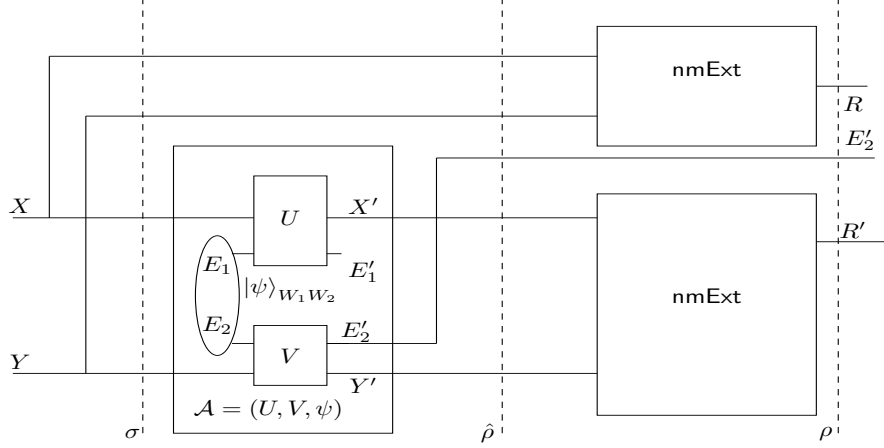


Figure 6: Tampering experiment for quantum secure 2-source non-malleable extractors.

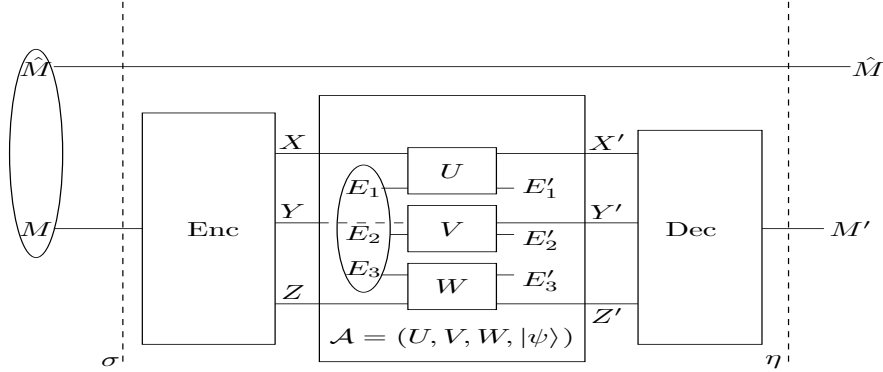


Figure 7: $\text{LO}_{(e_1, \dots, e_t)}^t$: t -split tampering model with shared entanglement for $t = 3$. The shared entanglement ψ is stored in registers E_1, E_2 and E_3 such that $|E_i| \leq e_i$ for $i \in [3]$.

state $|\psi\rangle_{E_1 E_2 E_3}$ such that $|E_i| \leq e_i, \forall i \in [3]$ which captures the pre-shared entanglement between the non-communicating quantum tampering adversaries. After the tampering channel, the decoding procedure Dec is applied to the corrupted codeword $\tau_{X'Y'Z'}$ and stored in register M' . Figure 7 presents a diagram of this tampering model. Let

$$\eta = \text{Dec}((U \otimes V \otimes W) (\text{Enc}(\sigma_{M\hat{M}}) \otimes |\psi\rangle\langle\psi|))$$

be the final state after applying the tampering adversary \mathcal{A} followed by the decoding procedure. The non-malleability of the coding scheme (Enc, Dec) is defined as follows.

Definition 2.14 (Quantum Non-Malleable Code against $\text{LO}_{(e_1, e_2, e_3)}^3$ [BGJR23]). *See Figure 7 for the tampering experiment. We say that the coding scheme (Enc, Dec) is an ε -secure quantum non-malleable code against $\text{LO}_{(e_1, e_2, e_3)}^3$ if for every $\text{LO}_{(e_1, e_2, e_3)}^3$ tampering adversary $\mathcal{A} = (U, V, W, |\psi\rangle_{E_1 E_2 E_3})$ and for every quantum message σ_M (with canonical purification $\sigma_{M\hat{M}}$) it holds that*

$$\eta_{M'\hat{M}} \approx_\varepsilon p \mathcal{A} \sigma_{M\hat{M}} + (1 - p) \gamma_M^{\mathcal{A}} \otimes \sigma_{\hat{M}}, \quad (16)$$

where $p_{\mathcal{A}} \in [0, 1]$ and $\gamma_M^{\mathcal{A}}$, depend only on the tampering adversary $\mathcal{A} = (U, V, W, |\psi\rangle_{E_1 E_2 E_3})$.

An important restriction to the definition above is known as ‘‘average-case’’ non-malleability:

Remark 2.3 ([BGJR23]). If Equation (16) is only guaranteed to hold if σ_M is the maximally mixed state, then (Enc, Dec) is referred to as an average-case ε -secure quantum non-malleable code.

In our reductions, we require guarantees on the correlations between the recovered message, and the registers of individual split-state adversaries (even after tampering). We introduce the following notion of augmented quantum non-malleable codes, in an analog to its classical counterpart (Definition 2.13).

Definition 2.15 (Augmented Quantum Non-Malleable Code against $\text{LO}_{(e_1, e_2, e_3)}^3$). See Figure 7 for the tampering experiment. We say that the coding scheme (Enc, Dec) is an ε -secure augmented quantum non-malleable code against $\text{LO}_{(e_1, e_2, e_3)}^3$ if there exists $i \in [3]$ such that for every $\text{LO}_{(e_1, e_2, e_3)}^3$ tampering adversary $\mathcal{A} = (U, V, W, |\psi\rangle_{E_1 E_2 E_3})$ and message σ_M (with canonical purification $\sigma_{M\hat{M}}$), it holds that

$$\eta_{M' \hat{M} E'_i} \approx_\varepsilon p_{\mathcal{A}} \sigma_{M\hat{M}} \otimes \zeta_{E'_i}^{\mathcal{A}} + (1 - p_{\mathcal{A}}) \gamma_{M' E'_i}^{\mathcal{A}} \otimes \sigma_{\hat{M}}, \quad (17)$$

where $p_{\mathcal{A}} \in [0, 1]$, $\gamma_{M' E'_i}^{\mathcal{A}}$ and $\zeta_{E'_i}^{\mathcal{A}}$ depend only on the tampering adversary $\mathcal{A} = (U, V, W, |\psi\rangle_{E_1 E_2 E_3})$.

In other words, the i th adversaries “view” E'_i is simulatable jointly with the message.

2.4.4 Quantum Tamper-Detection Codes

Finally, we formally define the notion of quantum tamper-detection codes. Following the previous subsection, for concreteness we present an explicit definition for $\text{LO}_{(e_1, e_2, \dots, e_t)}^t$ tampering adversaries with $t = 3$, which can be easily generalized to the different tampering models of Section 2.3.

Definition 2.16 (Quantum Tamper-Detection Code against $\text{LO}_{(e_1, e_2, e_3)}^3$). See Figure 7 for the tampering experiment. We say that the coding scheme (Enc, Dec) is an ε -secure quantum tamper-detection code against $\text{LO}_{(e_1, e_2, e_3)}^3$ if for every $\text{LO}_{(e_1, e_2, e_3)}^3$ adversary $\mathcal{A} = (U, V, W, |\psi\rangle_{E_1 E_2 E_3})$ and for every quantum message σ_M (with canonical purification $\sigma_{M\hat{M}}$) it holds that

$$\eta_{M' \hat{M}} \approx_\varepsilon p_{\mathcal{A}} \sigma_{M\hat{M}} + (1 - p_{\mathcal{A}}) \perp_{M'} \otimes \sigma_{\hat{M}}, \quad (18)$$

where $p_{\mathcal{A}} \in [0, 1]$ depend only on the tampering adversary \mathcal{A} and $\perp_{M'}$ denotes a special abort symbol stored in register M' to denote tamper detection.

2.5 Secret Sharing Schemes and their Variants

In this subsection we present basic definitions of secret sharing schemes, in addition to known constructions in the literature that we require to instantiate our compilers. We begin in Section 2.5.1 by introducing threshold secret sharing, proceeded by leakage-resilient secret sharing in Section 2.5.2, and non-malleable secret sharing in Section 2.5.3. Finally, in Section 2.5.4 we formally introduce the tamper detecting secret sharing schemes we construct in this work.

2.5.1 Threshold Secret Sharing Schemes

Informally, in a t -out-of- p secret sharing scheme, any t honest shares suffice to reconstruct the secret, but no $t - 1$ shares offer any information about it.

Definition 2.17 ($(p, t, \varepsilon_{\text{priv}}, \varepsilon_c)$ -Secret Sharing Scheme). Let \mathcal{M} be a finite set of secrets, where $|\mathcal{M}| \geq 2$. Let $[p] = \{1, 2, \dots, p\}$ be a set of identities (indices) of p parties. A sharing channel Enc with domain of secrets \mathcal{M} is a $(p, t, \varepsilon_{\text{priv}}, \varepsilon_c)$ threshold secret sharing scheme if the following two properties hold:

1. **Correctness:** the secret can be reconstructed by any set of parties $T \subset [p]$, $|T| \geq t$. That is, for every such T , there exists a reconstruction channel Dec_T such that

$$\forall m \in \mathcal{M} : \mathbb{P}[\text{Dec}_T \circ \text{Enc}(m)_T \neq m] \leq \varepsilon_c \quad (19)$$

2. **Statistical Privacy:** Any collusion of $|T| \leq t - 1$ parties has “almost” no information about the underlying secret. That is, for every distinguisher D with binary output:

$$\forall m_0, m_1 \in \mathcal{M} : |\mathbb{P}[D(\text{Enc}(m_0)_T) = 1] - \mathbb{P}[D(\text{Enc}(m_1)_T) = 1]| \leq \varepsilon_{\text{priv}} \quad (20)$$

We point out that we use the same syntactic definition of secret sharing for schemes which hide classical messages into classical or quantum ciphers. However, when the cipher is classical, we refer to the encoding/decoding channels as **Share** and **Rec** (reconstruction) functions, as opposed to quantum channels (**Enc**, **Dec**). The block-length of the secret sharing scheme is the total qubit + bit-length of all the shares, and its rate as the ratio of message length (in bits) to block-length.

To instantiate our constructions, we use Shamir’s standard threshold secret sharing scheme:

Fact 2.8 ([Sha79]). *For any number of parties p and threshold t such that $t \leq p$, there exists a t -out-of- p secret sharing scheme (**Share**, **Rec**) for classical messages of length b with share size at most $\max(p, b)$, where both the sharing and reconstruction procedures run in time $\text{poly}(p, b)$.*

2.5.2 Leakage-Resilient Secret Sharing Schemes

Integral in our constructions are classical secret sharing schemes which offer privacy guarantees even if unauthorized subsets are allowed to leak information about their shares to each other in an attempt to distinguish the message. Moreover, we depart from standard classical leakage models in that the leaked information itself could be a quantum state, even if the shares are classical. The definition of leakage-resilient secret sharing formalizes this idea as follows:

Definition 2.18 (Leakage-Resilient Secret Sharing). *Let (**Share**, **Rec**) be a secret sharing scheme with randomized sharing function $\text{Share} : \mathcal{M} \rightarrow \{\{0, 1\}^l\}^p$, and let \mathcal{F} be a family of leakage channels. Then **Share** is said to be $(\mathcal{F}, \varepsilon_{lr})$ leakage-resilient if, for every channel $\Lambda \in \mathcal{F}$,*

$$\forall m_0, m_1 \in \mathcal{M} : \Lambda(\text{Share}(m_0)) \approx_{\varepsilon_{lr}} \Lambda(\text{Share}(m_1)) \quad (21)$$

As an example, the standard local leakage model in the context of classical secret sharing schemes allows bounded leakage queries $\{f_i : \{0, 1\}^l \rightarrow \{0, 1\}^\mu\}_{i \in K}$, on each share corresponding to an arbitrary set of indices $K \subset [p]$, and further allows full share queries corresponding to an unauthorized subset $T \subset [p]$:

$$\text{Leak}(\text{Share}(m)) = \text{Share}(m)_T, \{f_i(\text{Share}(m)_i)\}_{i \in K}. \quad (22)$$

In our constructions, we unfortunately require a slight modification to this setting, where the leakage parties K are an unauthorized subset, but are allowed to *jointly* leak a small quantum state (dependent on their shares) to T . We formally introduce this model as follows:

Definition 2.19 (Quantum k -Local Leakage Model). *For any integer sizes p, t, k and leakage length (in qubits) μ , we define the (p, t, k, μ) -local leakage model to be the collection of channels specified by*

$$\mathcal{F}_{k, \mu}^{p, t} = \left\{ (T, K, \Lambda) : T, K \subset [p], |T| < t, |K| \leq k, \text{ and } \Lambda : \{0, 1\}^{l \cdot |K|} \rightarrow \mathcal{L}(\mathcal{H}) \right\}, \quad (23)$$

where $\log \dim(\mathcal{H}) = \mu$. A leakage query $(T, K, \Lambda) \in \mathcal{F}_{k, \mu}^{p, t}$ on a secret m is the density matrix:

$$(\mathbb{I}_T \otimes \Lambda_K)(\text{Share}(m)_{T \cup K}) \quad (24)$$

In other words, the parties in K perform a quantum channel on their (classical) shares and send the μ qubit output state to T . To the extent of our knowledge, in the literature we do not know of LRSS constructions in this model (even with classical leakage). However, in [Appendix C](#) we show that simple modifications to a construction of LRSS against local leakage by [\[CKOS22\]](#) provides such guarantees.

Theorem 2.6 ([Theorem C.1](#), restatement). *For every $\mu, k, t, p, l \in \mathbb{N}$ such that $k < t, t + k \leq p$, and $p \leq l$, there exists an $(p, t, 0, 0)$ threshold secret sharing scheme on messages of l bits and shares of size $l + \mu + o(l, \mu)$ bits, which is perfectly correct and private and $p \cdot 2^{-\tilde{\Omega}(\sqrt{\frac{l+\mu}{p}})}$ leakage-resilient against $\mathcal{F}_{k, \mu}^{p, t}$.*

2.5.3 Non-Malleable Secret Sharing Schemes

[GK18a] introduced the notion of a non-malleable secret sharing scheme (NMSS). An NMSS is a secret sharing scheme which is robust to certain types of tampering on the shares.

Definition 2.20 (($p, t, \varepsilon_{\text{priv}}, \varepsilon_c, \varepsilon_{\text{NM}}$) Non-Malleable Secret Sharing Scheme). *Let (Enc, Dec) be a $(p, t, \varepsilon_{\text{priv}}, \varepsilon_c)$ secret sharing scheme, and let \mathcal{F} be a family of tampering channels. Then, we refer to (Enc, Dec) as a $(p, t, \varepsilon_{\text{priv}}, \varepsilon_c, \varepsilon_{\text{NM}})$ non-malleable secret sharing scheme against \mathcal{F} if, for all $\Lambda \in \mathcal{F}$ and authorized subset $T \subset [p]$ of size $|T| \geq t$, there exists $p_\Lambda \in [0, 1]$ and a distribution \mathcal{D}_Λ such that*

$$\forall m \in \mathcal{M}, \quad \text{Dec}_T \circ \Lambda \circ \text{Enc}(m) \approx_{\varepsilon_{\text{NM}}} p_\Lambda \cdot m + (1 - p_\Lambda) \cdot \mathcal{D}_\Lambda \quad (25)$$

In other words, the recovered secret after tampering is a convex combination of the original secret, or a fixed distribution over secrets. We leverage a construction of non-malleable secret sharing schemes against *joint tampering functions* by [GK18a].

Definition 2.21 (Joint Tampering Functions $\mathcal{F}_{p,t}^{\text{joint}}$). *For any t -out-of- p secret sharing scheme, the adversary chooses any t -out-of- p shares to obtain an authorized set T , partitions the set $T = A \cup B$ into any two non-empty disjoint subsets which have different cardinalities, and jointly tampers with shares in each of the two subsets arbitrarily and independently using deterministic functions f_A, f_B .*

Theorem 2.7 ([GK18a]). *For every number of parties p , threshold $t \geq 3$, and sufficiently large block-length b , there exists a $(p, t, 0, 0, 2^{-b^{\Omega(1)}})$ – NMSS against $\mathcal{F}_{p,t}^{\text{joint}}$ of rate $p^{-O(1)}$.*

2.5.4 Tamper Detecting Secret Sharing Schemes

Below we formalize two tamper detection properties for the secret sharing schemes that we consider in this section. The first of the two is known as “weak” tamper detection, and informally can be understood as the ability to *either* recover the message or reject after tampering, with high probability. However, the probability that either of these two events occurs may depend on m :

Definition 2.22 (Weak Tamper-Detecting Secret Sharing Schemes). *Let a family of CPTP maps (Enc, Dec) be a $(p, t, \varepsilon_{\text{priv}}, \varepsilon_c)$ secret sharing scheme over k bit messages and \mathcal{F} a family of tampering channels. We refer to (Enc, Dec) as $(\mathcal{F}, r, \varepsilon_{\text{TD}})$ weak tamper-detecting if, for every subset $R \subset [p]$ of size $|R| \geq r$ and tampering channel $\Lambda \in \mathcal{F}$,*

$$\forall m \in \{0, 1\}^k, \quad \mathbb{P}[\text{Dec}_R \circ \Lambda \circ \text{Enc}(m) \notin \{m, \perp\}] \leq \varepsilon_{\text{TD}} \quad (26)$$

Note that in the above, we implicitly consider *ramp* secret sharing schemes, where the threshold for privacy, reconstruction under honest shares, and reconstruction under tampered shares, are different. In a (generic) tamper-detecting secret sharing scheme, we stipulate that the distribution over the recovered message is near convex combination of the original m and \perp , which doesn’t depend on m :

Definition 2.23 (Tamper-Detecting Ramp Secret Sharing Schemes). *Let a family of CPTP maps (Enc, Dec) be a $(p, t, \varepsilon_{\text{priv}}, \varepsilon_c)$ secret sharing scheme over k bit messages and \mathcal{F} a family of tampering channels. We refer to (Enc, Dec) as $(\mathcal{F}, r, \varepsilon_{\text{TD}})$ tamper-detecting if, for every subset $R \subset [p]$ of size $|R| \geq r$ and tampering channel $\Lambda \in \mathcal{F}$, there exists a constant $p_\Lambda \in [0, 1]$ such that*

$$\forall m \in \{0, 1\}^k, \quad \text{Dec}_R \circ \Lambda \circ \text{Enc}(m) \approx_{\varepsilon_{\text{TD}}} p_\Lambda \cdot m + (1 - p_\Lambda) \cdot \perp \quad (27)$$

2.6 Bipartite and Multipartite LOCC Data Hiding Schemes

In this subsection we present basic definitions of multi-partite quantum measurements restricted to local operations and quantum communication, as well as formal definitions of quantum data hiding schemes.

2.6.1 Bipartite Data Hiding

Definition 2.24. Let ρ, σ be two apriori equiprobable bipartite mixed states, and define the distance $\|\rho - \sigma\|_{\text{LOCC}} = 4 \cdot (p - \frac{1}{2})$, where p is the probability of correctly distinguishing the states, maximised over all decision procedures that can be implemented using only local operations and classical communication (LOCC).

Remark 2.4. Notions of distance based on restricted classes of measurements define induced norms on operators (below) only when the set of measurements \mathbb{M} is a closed, convex subset of the operator interval $\{M : 0 \leq M \leq \mathbb{I}\}$ containing \mathbb{I} and such that $\mathbb{I} - M \in \mathbb{M}$. See [MW14] for a detailed description.

$$\|X\|_{\mathbb{M}} = \max_{M \in \mathbb{M}} \text{Tr}(2M - \mathbb{I})X \quad (28)$$

The set of LOCC measurements is *not* topologically closed [CLM⁺12], and therefore the distance in Definition 2.24 is not a norm. Nevertheless, it does satisfy the triangle inequality.

Definition 2.25 (LOCC Data Hiding Schemes [TDL00, HLSW03]). A bipartite (ε, δ) -LOCC data hiding scheme is a pair of quantum channels (Enc, Dec) where $\text{Enc} : \{0, 1\}^k \rightarrow \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ and $\text{Dec} : \mathcal{L}(\mathcal{H} \otimes \mathcal{H}) \rightarrow \{0, 1\}^k$ satisfying:

1. **Correctness:** $\forall m \in \{0, 1\}^k : \mathbb{P}[\text{Dec} \circ \text{Enc}(m) \neq m] \leq \delta$.
2. **Security:** $\forall m_0, m_1 \in \{0, 1\}^k : \|\text{Enc}(m_0) - \text{Enc}(m_1)\|_{\text{LOCC}} \leq \varepsilon$

To instantiate our code construction, we use two pre-existing constructions of bipartite data hiding schemes. The first of which is an inefficient, high-rate result by [HLSW03]:

Theorem 2.8 ([HLSW03]). For every sufficiently large integer n , there exists a bipartite $(2^{-\Omega(\frac{n}{\log n})}, 2^{-\Omega(\frac{n}{\log n})})$ -LOCC data hiding scheme encoding $n/2 - o(n)$ bits into n qubits.

And the second is an efficient construction by [EW02], which furthermore is *separable*.

Theorem 2.9 ([EW02]). For every $k, \lambda = \Omega(\log k) \in \mathbb{N}$, there exists a 2-out-of-2 $(2^{-\Omega(\lambda)}, 2^{-\Omega(\lambda)})$ LOCC data hiding scheme encoding k bits into $n = 2 \cdot k \cdot \lambda^2$ qubits. Moreover, the data hiding states are separable states.

2.6.2 Data Hiding with an Access Structure

The study of multi-partite data hiding schemes, endowed with some access structure \mathcal{A} , can be traced to [EW02, DHT02, HLS04]. Recall an access structure \mathcal{A} is simply a family of subsets of $[p]$, where p denotes the number of parties/registers in the data hiding scheme. A subset of parties $T \in \mathcal{A}$ implies that the parties in T , using quantum communication between each other, can recover the secret.

However, if $T \notin \mathcal{A}$, then the data hiding scheme offers a notion of privacy which strengthens the standard secret-sharing definition: no unauthorized party T can learn information about the secret even if they are allowed arbitrary classical communication with their complement $[p] \setminus T$, in addition to quantum communication within T . Here, the parties in $[p] \setminus T$ are restricted to LOCC themselves.

Definition 2.26. Fix a number p of parties and a subset $T \subset [p]$ of them. We refer to the set of CPTP channels LOCC_T^p as the set of operations implementable using local operations and quantum communication within T , as well as local operations and classical communication between T , and each individual $u \in [p] \setminus T$. When no quantum communication is allowed, we refer to the set of channels as LOCC^p .

Following Section 7, given any two apriori equiprobable density matrices ρ, σ supported on an p partite Hilbert space, we denote $\frac{1}{2}\|\rho - \sigma\|_{\text{LOCC}_T^p}$ as the optimal bias in distinguishing between the two using binary outcome distinguishers $\Lambda \in \text{LOCC}_T^p$.

Definition 2.27 ($(p, t, \varepsilon_{\text{LOCC}}, \varepsilon_c)$ -LOCC Secret Sharing Scheme). A pair of channels (Enc, Dec) is an $(p, t, \varepsilon_{\text{LOCC}}, \varepsilon_c)$ -LOCC secret sharing scheme if it satisfies

1. **Correctness:** The secret can be reconstructed by any set of parties $T \subset [p], |T| \geq t$ using quantum communication within T . That is, for every T there exists a reconstruction channel Dec_T such that for all messages m :

$$\mathbb{P}[\text{Dec}_T \circ \text{Enc}_T(m) \neq m] \leq \varepsilon_c \quad (29)$$

2. **Security:** Any collusion of $|T| \leq t-1$ parties has “almost” no information about the underlying secret, even if the remaining $[p] \setminus T$ parties can aid them using arbitrary classical communication. That is, for every pair of messages m_0, m_1 ,

$$\|\text{Enc}(m_0) - \text{Enc}(m_1)\|_{\text{Locc}_T^p} \leq \varepsilon_{\text{Locc}}. \quad (30)$$

3 From Non-Malleability to Tamper Detection against LO

We dedicate this section to our two black-box reductions that construct tamper-detection codes (TDCs) from non-malleable codes (NMCs) in the split state model. We refer the reader to [Section 2.3](#) for formal definitions of the relevant tampering models, namely the unentangled t -split-state model LO^t , and the bounded storage model LO_a^t . Our first result shows that any non-malleable code in the unentangled split-state model implies a tamper-detection code in the unentangled split-state model, but with one extra share.

Theorem 3.1. *Let the coding scheme $(\text{Enc}^t, \text{Dec}^t)$ be a quantum NMC against LO^t , with message length k qubits, blocklength n , and error ε . Then, for every $0 < \lambda < k$, there exists a quantum TDC against tampering adversaries LO^{t+1} of message length $k - \lambda$, blocklength $n + \lambda$, and error $\varepsilon + 2^{1-\lambda}$.*

Our second result shows the reduction above can be made robust against adversaries with a bounded amount of entanglement, highlighting that tamper detection is not solely an artifact of the unentangled model. The challenge in extending our ideas lies in the fact that quantum NMCs are, a priori, not composable. Their definition offers no guarantees on any entanglement that the adversaries may have with the quantum message, both before or after tampering. Consequently, adversaries with shared entanglement may correlate their attack on the additional $(t + 1)$ st share, with their attack on the NMC.

To address this challenge, we introduce the concept of an ‘‘augmented’’ quantum NMC, drawing an analogy to their classical counterparts. In a standard split-state NMC, the decoded message is either perfectly recovered or entirely fixed. In an augmented NMC, the decoded message, in conjunction with the left share, is either recovered and remains independent or are jointly fixed. We formalized this notion in the quantum setting, accounting for any pre-shared entanglement, in [Definition 2.15](#). We can now state our reduction in the ‘‘bounded storage model’’, denoted as $\text{LO}_{(e_1, \dots, e_t)}^t$.

Theorem 3.2. *Let the coding scheme $(\text{Enc}^t, \text{Dec}^t)$ be an quantum NMC against $\text{LO}_{(e_1, e_2, \dots, e_t)}^t$, with message length k qubits, blocklength n , and error ε , which is augmented at the first share.*

Then, for every $0 \leq \lambda < k$ and $0 \leq a \leq \min(\lambda, e_1)$ there exists a quantum TDC against $\text{LO}_{(e_1 - a, \dots, e_t, e_{t+1} = a)}^{(t+1)}$, of message length $k - \lambda$, blocklength $n + \lambda$, and error $\varepsilon + 2^{1+a-\lambda}$.

In particular, if $(\text{Enc}^t, \text{Dec}^t)$ is secure against arbitrary pre-shared entanglement LO_*^t , then the result is secure against $\text{LO}_{(*, *, \dots, a)}^t$. We defer instantiations of the reductions above, as well as explicit constructions of the augmented quantum NMCs to [Appendix B](#).

3.1 Quantum Tamper-Detection Codes against LO

Let the coding scheme $(\text{Enc}^t, \text{Dec}^t)$ be a quantum non-malleable code (NMC) against LO^t , with message length k qubits, blocklength n , and error ε . In [Algorithm 1](#), for every integer $0 < \lambda < k$, we define a quantum tamper-detection code (Enc, Dec) against LO^{t+1} :

Algorithm 1: Enc: A quantum TDC defined on $t + 1$ registers.

Input: Any input state σ_M (with canonical purification register \hat{M}) such that $|M| = k - \lambda$ and an integer λ .

- 1: Prepare λ EPR pairs, $\Phi^{\otimes \lambda} \equiv \Phi_{E\hat{E}}$, on a bipartite register E, \hat{E} .
- 2: Encode σ_M and the EPR halves in register E into the quantum NMC, Enc^t .
- 3: Output the t -splits, and the remaining EPR halves in \hat{E} in the $(t + 1)$ st split.

$$\text{Enc}(\sigma_M) = ((\text{Enc}^t)_{ME} \otimes \mathbb{I}_{\hat{E}}) (\sigma_M \otimes \Phi_{E\hat{E}}) \tag{31}$$

Algorithm 2: Dec.

Input: Any code-state $\Lambda \circ \text{Enc}(\sigma_M)$ on $(t+1)$ shares, tampered by a channel $\Lambda \in \text{LO}^{t+1}$.

- 1: Decode the quantum NMC in the first t shares using Dec^t , resulting in registers M' and E' .
 - 2: Perform an EPR test on registers $E'\hat{E}$. This entails a binary outcome measurement with two possible outcomes: $\Pi = \Phi_{E\hat{E}}, \bar{\Pi} = \mathbb{I}_{E\hat{E}} - \Phi_{E\hat{E}}$.
 - 3: An acceptance decision is made if the outcome is $\Pi = \Phi_{E\hat{E}}$. In this case, the operation Rep (stands for 'Replace') outputs the register M' .
 - 4: If the outcome is $\bar{\Pi}$, the operation Rep outputs a special symbol \perp in register M' .
-

3.1.1 Proof of Theorem 3.1

Let us denote the tampering adversary $\text{LO}^{(t+1)}$ as $\mathcal{A} = (\Lambda_1, \Lambda_2, \dots, \Lambda_{t+1})$, and its restriction to the first t shares as $\mathcal{A}^t = (\Lambda_1, \Lambda_2, \dots, \Lambda_t)$. We depict the tampering process of \mathcal{A} on our code (Enc, Dec) in Figures 8 and 9. These two figures describe the same tampering experiment, but only differ in that in Figure 9, we have delayed the action of Λ_{t+1} to introduce an intermediate state τ . τ represents the state of the system after \mathcal{A}^t has tampered with the first t shares and we have decoded the non-malleable code, but before the action of Λ_{t+1} .

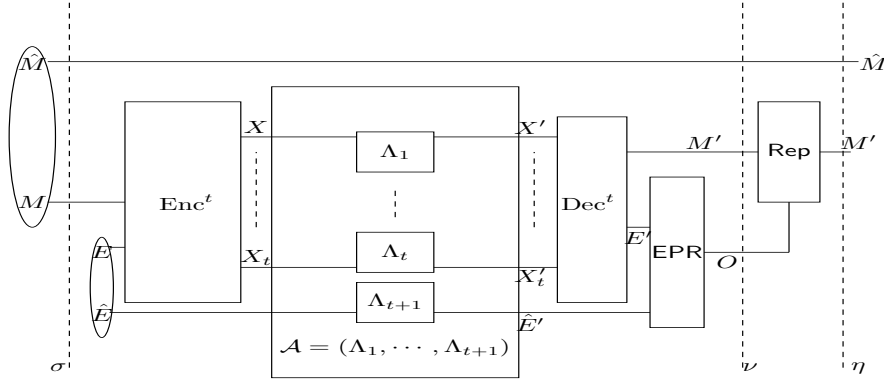


Figure 8: Quantum TDC against LO^{t+1} .

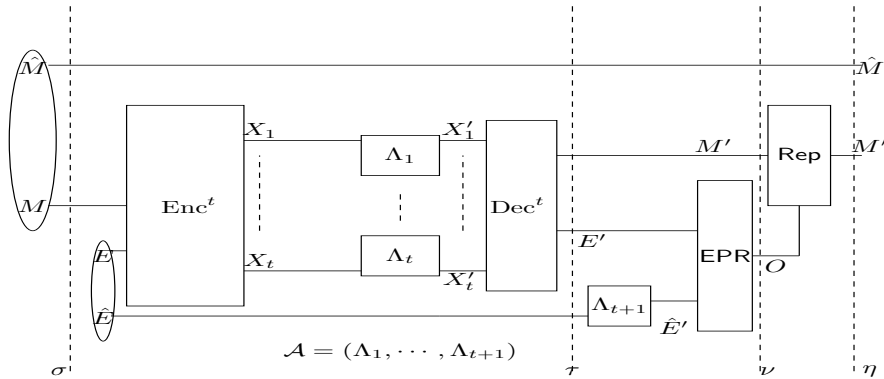


Figure 9: Analysis of quantum TDC against LO^{t+1} .

Since the state τ is derived after tampering against LO^t , the security of the quantum NMC against LO^t guarantees:

$$\tau_{\hat{M}'E'\hat{E}} \approx_{\varepsilon} p_{\mathcal{A}^t} \sigma_{\hat{M}\hat{M}} \otimes \Phi_{E\hat{E}} + (1 - p_{\mathcal{A}^t}) \gamma_{M'E'} \otimes \sigma_{\hat{M}} \otimes \Phi_{\hat{E}},$$

where $(p_{\mathcal{A}^t}, \gamma_{M'E'})$ depend only on tampering adversary \mathcal{A}^t . The data processing inequality [Fact 2.1](#) now tells us that the Bell basis measurement in EPR is either completely independent of the the message M , or acts on a product state:

$$\nu_{\hat{M}M'O} \approx_{\varepsilon} p_{\mathcal{A}^t} \sigma_{\hat{M}M} \otimes \text{EPR}(\Lambda_{t+1}(\Phi_{E\hat{E}})) + (1 - p_{\mathcal{A}^t}) \sigma_{\hat{M}} \otimes \text{EPR}(\gamma_{M'E'} \otimes \Lambda_{t+1}(\Phi_{\hat{E}})). \quad (32)$$

To proceed, we state two claims, [Claim 3.3](#) and [Claim 3.4](#), which enable us to analyze the two terms in the convex combination above. In [Claim 3.3](#), we point out the (trivial) observation that if the non-malleable code recovers the message $\sigma_{\hat{M}M}$, then the Bell basis measurement is independent of σ :

Claim 3.3. $\text{EPR}(\Lambda_{t+1}(\Phi_{E\hat{E}})) = p_{\Lambda_{t+1}} |1\rangle\langle 1| + (1 - p_{\Lambda_{t+1}}) |0\rangle\langle 0|$, where $p_{\Lambda_{t+1}}$, depends only on Λ_{t+1} .

In [Claim 3.4](#), we formalize the fact that the Bell basis measurement rejects product states with high probability. We state its proof in [Section 3.1.2](#).

Claim 3.4. $\text{EPR}(\gamma_{M'E'} \otimes \Lambda_{t+1}(\Phi_{\hat{E}})) \approx_{2 \cdot 2^{-\lambda}} \zeta_{M'} \otimes |0\rangle\langle 0|$, for some density matrix $\zeta_{M'}$.

By combining [Claim 3.3](#), [Claim 3.4](#) and [Equation \(32\)](#), we are guaranteed the decoder outputs a convex combination of the original message $\sigma_{\hat{M}M}$ or the Bell basis measurement rejects:

$$\nu_{\hat{M}M'O} \approx_{\varepsilon+2 \cdot 2^{-\lambda}} p_{\mathcal{A}^t} p_{\Lambda_{t+1}} \sigma_{\hat{M}M} \otimes |1\rangle\langle 1| + p_{\mathcal{A}^t} (1 - p_{\Lambda_{t+1}}) \sigma_{\hat{M}M} \otimes |0\rangle\langle 0| + (1 - p_{\mathcal{A}^t}) \sigma_{\hat{M}} \otimes \zeta_{M'} \otimes |0\rangle\langle 0|.$$

We can now conclude using the data processing [Fact 2.1](#):

$$\text{Dec} \circ \Lambda \circ \text{Enc}(\sigma_{\hat{M}M}) = \eta_{\hat{M}M'} \approx_{\varepsilon+2 \cdot 2^{-\lambda}} p_{\mathcal{A}^t} p_{\Lambda_{t+1}} \sigma_{\hat{M}M} + (1 - p_{\mathcal{A}^t} p_{\Lambda_{t+1}}) \sigma_{\hat{M}} \otimes \perp_{M'},$$

which implies [Theorem 3.1](#), since the probability $p_{\mathcal{A}^t} p_{\Lambda_{t+1}}$ depends only on the tampering adversary \mathcal{A} .

3.1.2 Proof of [Claim 3.4](#)

We point out that $\phi_{M'E'\hat{E}} = \gamma_{M'E'} \otimes \Lambda_{t+1}(\Phi_{\hat{E}})$ is a product state, and therefore can be written as a convex combination of pure states of rank 1 across the cut $(M'E')/\hat{E}$. Using [Claim 3.5](#) below, we prove $\text{EPR}(\phi_{M'E'\hat{E}})$ outputs outcome 0 with probability at least $1 - 2^{-\lambda}$.

Claim 3.5. Let $\Phi_{E\hat{E}} \equiv \Phi^{\otimes \lambda}$ denote λ EPR pairs on a bipartite register E, \hat{E} . Let $\tau_{E\hat{E}}$ be a pure state of Schmidt rank R . Then, measuring τ with the binary measurement $\{\Phi_{E\hat{E}}, \mathbb{I}_{E\hat{E}} - \Phi_{E\hat{E}}\}$ outputs the $\Phi_{E\hat{E}}$ with negligible probability, i.e.,

$$\text{Tr}[\Phi^{\otimes \lambda} \tau] \leq R \cdot 2^{-\lambda}.$$

Similarly, if $\tau_{E\hat{E}}$ were a mixed state with Schmidt number R , then $\text{Tr}[\Phi^{\otimes \lambda} \tau] \leq R \cdot 2^{-\lambda}$.

Recall that the Schmidt rank/number ([Definition 2.7](#)) is non-increasing under local operations, and so if $\phi_{M'E'\hat{E}}$ has rank 1 across the cut $(M'E')/\hat{E}$, the RDM $\phi_{E'\hat{E}}$ has Schmidt number 1. [Claim 3.4](#) follows.

Proof. [of [Claim 3.5](#)] Let us first consider a pure state $|\phi\rangle$ of rank R , and let $|\phi\rangle = \sum_{i=1}^R \alpha_i |u_i\rangle_E \otimes |v_i\rangle_{\hat{E}}$, $\Phi_{E\hat{E}} \equiv |\Phi\rangle^{\otimes \lambda} = 2^{-\lambda/2} \sum_j |j\rangle_E \otimes |j\rangle_{\hat{E}}$ define Schmidt decompositions. Then, by the triangle and the Cauchy-Schwartz inequalities:

$$\text{Tr}[\Phi^{\otimes \lambda} \phi]^{1/2} = |\langle \Phi |^{\otimes \lambda} \phi\rangle| \leq \sum_i^R |\alpha_i| \cdot |\langle \Phi |^{\otimes \lambda} |u_i\rangle \otimes |v_i\rangle| \leq \quad (33)$$

$$\leq \max_{|u\rangle \otimes |v\rangle} |\langle \Phi |^{\otimes \lambda} |u\rangle \otimes |v\rangle| \cdot R^{1/2} \cdot \left(\sum_i |\alpha_i|^2 \right)^{1/2} \leq R^{1/2} \cdot \max_{|u\rangle \otimes |v\rangle} |\langle \Phi |^{\otimes \lambda} |u\rangle \otimes |v\rangle| \quad (34)$$

In turn,

$$|\langle \Phi |^{\otimes \lambda} |u\rangle \otimes |v\rangle| = 2^{-\lambda/2} \cdot \left| \sum_j \langle j|u\rangle \cdot \langle j|v\rangle \right| \leq 2^{-\lambda/2} \left(\sum_j |\langle j|u\rangle|^2 \right)^{1/2} \left(\sum_j |\langle j|v\rangle|^2 \right)^{1/2} \leq 2^{-\lambda/2}, \quad (35)$$

which gives us the desired bound. If τ is a mixed state of Schmidt number R , then it can be written as a convex combination of pure states of Schmidt rank $\leq R$, which concludes the claim. \square

3.2 Augmented Non-Malleable Codes and the Bounded Storage Model

To extend our reduction to the bounded storage model, the sole modification to our construction lies in picking an *augmented* non-malleable code $(\text{Enc}_t, \text{Dec}_t)$ against $\text{LO}_{(e_1, e_2, \dots, e_t)}^t$ tampering adversaries.

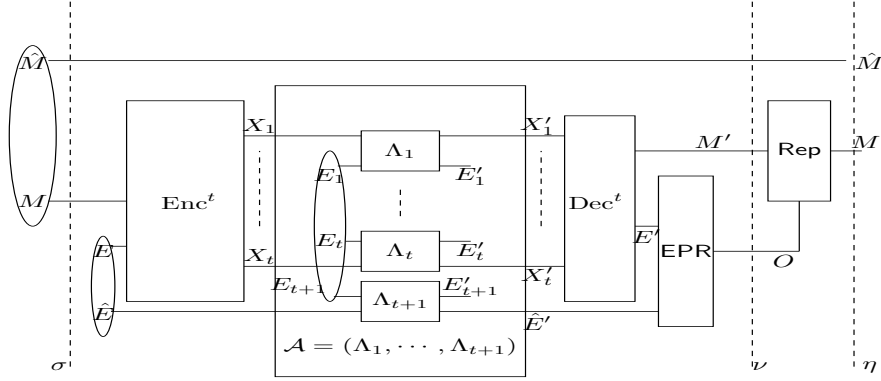


Figure 10: Quantum tamper-detection code against $\text{LO}_{(e_1-a, e_2, \dots, e_{t+1}=a)}^{t+1}$.

Let us denote a tampering adversary $\text{LO}_{(e_1-a, e_2, \dots, e_{t+1}=a)}^{(t+1)}$ as $\mathcal{A} = (\Lambda_1, \Lambda_2, \dots, \Lambda_{t+1}, |\psi\rangle\langle\psi|_{E_1 E_2 \dots E_{t+1}})$, and its “truncation” to the first t shares as $\mathcal{A}^t = (\Lambda_1, \Lambda_2, \dots, \Lambda_t, |\psi\rangle\langle\psi|_{E_1 \dots E_{t+1}})$. In Figure 10, we depict the tampering experiment on the code in the bounded storage model. We are now in a position to prove our reduction.

3.2.1 Proof of Theorem 3.2

To begin our proof, we similarly view the tampering process in Figure 10 through the equivalent process in Figure 11. Note that they only differ in that in Figure 11 we have delayed the action of Λ_{t+1} to consider an intermediate state τ . Informally τ amounts to the system after \mathcal{A}^t and the non-malleable code has been decoded, but before Λ_{t+1} .

By bundling together the (augmented) first register E_1 with the entanglement in the new register E_{t+1} , note that $|E_1 E_{t+1}| \leq a + (e_1 - a) = e_1$. The security of the augmented quantum non-malleable code against $\text{LO}_{(e_1, e_2, \dots, e_t)}^t$ ensures the following about state τ :

$$\tau_{\hat{M} M' E' \hat{E} E_{t+1} E'_1} \approx_\varepsilon p_{\mathcal{A}^t} \sigma_{\hat{M} M'} \otimes \Phi_{E \hat{E}} \otimes \zeta_{E_{t+1} E'_1} + (1 - p_{\mathcal{A}^t}) \gamma_{M' E' E_{t+1} E'_1} \otimes \sigma_{\hat{M}} \otimes \Phi_{\hat{E}},$$

where $p_{\mathcal{A}^t}$, $\gamma_{M' E' E_{t+1} E'_1}$ and $\zeta_{E_{t+1} E'_1}$ depend only on adversary \mathcal{A}^t . Using data processing Fact 2.1, we have

$$\nu_{\hat{M} M' O} \approx_\varepsilon p_{\mathcal{A}^t} \sigma_{\hat{M} M'} \otimes \text{EPR}(\text{Tr}_{E'_{t+1}} (\Lambda_{t+1} (\Phi_{E \hat{E}} \otimes \zeta_{E_{t+1}}))) + (1 - p_{\mathcal{A}^t}) \sigma_{\hat{M}} \otimes \text{EPR}(\text{Tr}_{E'_{t+1}} (\Lambda_{t+1} (\gamma_{M' E' E_{t+1}} \otimes \Phi_{\hat{E}}))). \quad (36)$$

To proceed, we state two claims, Claim 3.6 and Claim 3.7, which enable us to analyze the two terms in the convex combination above. In Claim 3.6, we point out the (trivial) observation that if the non-malleable code recovers the message $\sigma_{\hat{M} M'}$, then the Bell basis measurement is independent of σ :

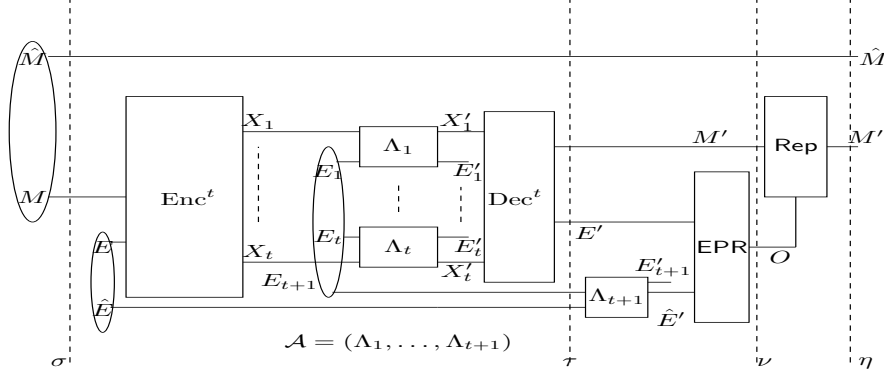


Figure 11: Analysis of Quantum tamper-detection code against $\text{LO}_{(e_1, e_2, \dots, e_{t+1}=a)}^{t+1}$.

Claim 3.6. $\text{EPR}(\text{Tr}_{E'_{t+1}}(\Lambda_{t+1}(\Phi_{E\hat{E}} \otimes \zeta_{E_{t+1}}))) = p_{t+1} |1\rangle\langle 1| + (1 - p_{t+1}) |0\rangle\langle 0|$, where p_{t+1} , depends only on $(\Lambda_{t+1}, \zeta_{E_{t+1}})$.

In turn, [Claim 3.7](#) formalizes that the Bell basis measurement rejects states of low Schmidt rank.

Claim 3.7. $\text{EPR}(\text{Tr}_{E'_{t+1}}(\Lambda_{t+1}(\gamma_{M'E'E_{t+1}} \otimes \Phi_{\hat{E}}))) \approx_{2^{1+a-\lambda}} \kappa_{M'} \otimes |0\rangle\langle 0|$, where $\kappa_{M'}$ is some density matrix, so long as the size of the $(t+1)$ st register is $|E_{t+1}| \leq a$.

Once again by combining [Claim 3.6](#), [Claim 3.7](#) and [Equation \(36\)](#), we are guaranteed the decoder outputs a convex combination of the original message $\sigma_{\hat{M}M}$ or the Bell basis measurement rejects:

$$\nu_{\hat{M}M'O} \approx_{\varepsilon+2^{1+a-\lambda}} p_{\mathcal{A}^t} p_{t+1} \sigma_{\hat{M}M} \otimes |1\rangle\langle 1| + p_{\mathcal{A}^t} (1 - p_{t+1}) \sigma_{\hat{M}M} \otimes |0\rangle\langle 0| + (1 - p_{\mathcal{A}^t}) \sigma_{\hat{M}} \otimes \kappa_{M'} \otimes |0\rangle\langle 0|.$$

Again by the data processing [Fact 2.1](#),

$$\eta_{\hat{M}M'} \approx_{\varepsilon+2^{1+a-\lambda}} p_{\mathcal{A}^t} p_{t+1} \sigma_{\hat{M}M} + (1 - p_{\mathcal{A}^t} p_{t+1}) \sigma_{\hat{M}} \otimes \perp_{M'}.$$

Which implies the desired theorem if we note that $p_{\mathcal{A}^t} p_{t+1}$ depends only on tampering adversary \mathcal{A} . This follows since p_{t+1} depends only on $(\Lambda_{t+1}, \zeta_{E_{t+1}})$ and $\zeta_{E_{t+1}}$ depends only on \mathcal{A}^t . It only remains to prove [Claim 3.7](#):

Proof. [of [Claim 3.7](#)] Our goal is to prove the Bell basis measurement on the state $\text{Tr}_{E'_{t+1}}(\Lambda_{t+1}(\gamma_{M'E'E_{t+1}} \otimes \Phi_{\hat{E}}))$ rejects with high probability. Note that the Schmidt number of this state across the (E', E_{t+1}) cut is at most the Schmidt number of $\gamma_{M'E'E_{t+1}} \otimes \Phi_{\hat{E}}$ across the $(M'E', E_{t+1}\hat{E})$, since the partial trace and Λ_{t+1} are local operations, and local operations do not increase Schmidt number [Proposition 2.1](#).

However, the Schmidt number of $\gamma_{M'E'E_{t+1}} \otimes \Phi_{\hat{E}}$ across the cut $(M'E', E_{t+1}\hat{E})$ is at most the Schmidt number of $\gamma_{M'E'E_{t+1}}$ across the $(M'E', E_{t+1})$ cut, which is in turn $\leq 2^a$ since the size of $|E_{t+1}| \leq a$. Applying [Claim 3.5](#) gives us the desired bound. \square

4 Tamper Detection in the Bounded Storage Model

In this section, we present and analyze our quantum tamper detection code in the 3-split-state model $\text{LO}_{(e_1, e_2, e_3)}^3$, where the adversaries are allowed a finite pre-shared quantum state to assist in tampering with their shares of the code (see [Definition 2.10](#)). In particular, we show security even when e_1, e_2 are both unbounded, that is, only the entanglement of one of the parties is restricted.

Theorem 4.1 ([Theorem 1.2](#), restatement). *For every integer k , $a \leq \lambda$, there exists an efficient tamper-detection code against $\text{LO}_{(a, *, *)}^3$ for k qubit messages with error $2^{-(k+\lambda)\Omega(1)} + 6 \cdot 2^{a-\lambda}$ and rate $(11 + 12\frac{\lambda}{k})^{-1}$.*

Our construction combines quantum secure non-malleable extractors and families of unitary 2 designs. We refer the reader to [Section 2.4.2](#) and [Section 2.2](#) respectively for formal definitions of the ingredients. We dedicate [Section 4.1](#) for a description of our construction, and [Section 4.2](#) for its analysis.

4.1 Code Construction

4.1.1 Ingredients

Let $\delta, \delta' > 0$ be constants, $k = (1/2 - \delta - 5\delta')n/5$, and $\lambda = \delta' \cdot n$. We combine

1. $\text{nmExt} : \{0, 1\}^n \times \{0, 1\}^{\delta \cdot n} \rightarrow \{0, 1\}^{(1/2 - \delta)n}$, a quantum secure two source non-malleable extractor with error $\varepsilon_{\text{nmExt}} = 2^{-n^{\Omega_\delta(1)}}$ from [Fact 2.7](#).
2. The family of 2-design unitaries C_R from [Fact 2.6](#).

4.1.2 Our candidate TDC

Our candidate construction of quantum TDC against $\text{LO}_{(e_1=\infty, e_2=\infty, e_3)}^3$ is given in [Figure 12](#) along with the tampering process. We describe it explicitly in the following [Algorithm 3](#). The decoding scheme, denoted as Dec, for the quantum TDC operates as in [Algorithm 4](#).

Algorithm 3: Enc: Quantum TDC against $\text{LO}_{(e_1=\infty, e_2=\infty, e_3)}^3$ (see [Figure 12](#)).

Input: A k qubit quantum message σ_M (with canonical purification \hat{M}).

- 1: Sample classical registers X, Y uniformly and independently of size $n, \delta n$ respectively. Evaluate $R = \text{nmExt}(X, Y)$.
 - 2: Prepare $\lambda = \delta' n$ EPR pairs, $\Phi^{\otimes \lambda}$, on a bipartite register E, \hat{E} .
 - 3: Consider X as the first share and (Y, E) as the second share.
 - 4: Let C_R be the Clifford unitary picked using sampling process [Samp](#) in [Fact 2.6](#). Apply C_R on registers (\hat{E}, M) to generate Z . This is possible since $|R| = (1/2 - \delta)n = 5(k + \delta'n)$ from our choice of parameters. Consider Z as the third share.
 - 5: Output shares $X, (Y, E), Z$.
-

4.2 Analysis

We represent the “tampering experiment” in [Figure 12](#), where a message is encoded into Enc, tampered with a channel in the bounded storage model, and then decoded.

Algorithm 4: Dec:

Input: Any tampered code-state $\Lambda \circ \text{Enc}(\sigma_M)$ on 3 shares (X, YE, Z) .

- 1: The decoder first computes $R' = \text{nmExt}(X', Y')$.
 - 2: Subsequently $C_{R'}^\dagger$ is applied on register Z' to obtain \hat{E}', M' .
 - 3: Perform an EPR test on registers $E' \hat{E}'$. This entails a binary outcome projective measurement $(\Pi = \Phi_{E\hat{E}}, \bar{\Pi} = \mathbb{I}_{E\hat{E}} - \Phi_{E\hat{E}})$.
 - 4: An acceptance decision is made if the outcome is $\Pi = \Phi_{E\hat{E}}$. In this case, the “replace” operation Rep outputs the register M' .
 - 5: If the outcome is not Π , the operation Rep outputs a special symbol \perp in register M' .
-

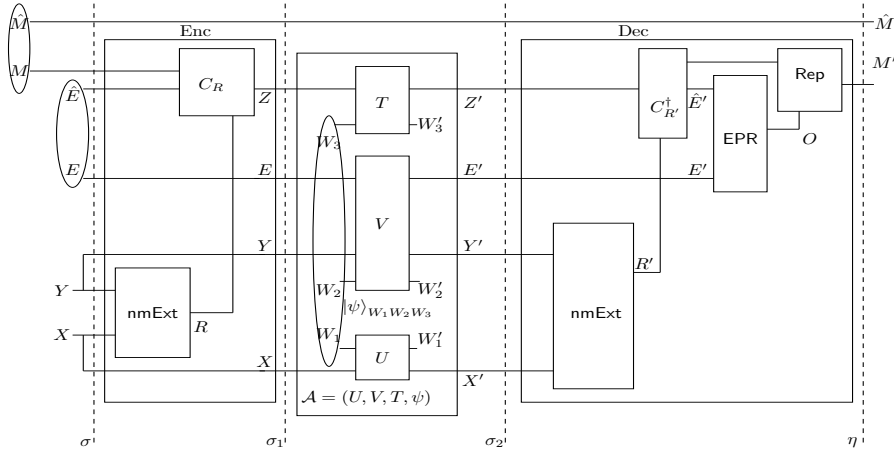


Figure 12: A quantum TDC against $\text{LO}_{(e_1=\infty, e_2=\infty, e_3)}^3$ along with tampering process.

In this section, we prove two key results about our code construction. Our main result is that the code described above is a secure tamper-detection code in the bounded storage model:

Theorem 4.2 (Tamper Detection). *Consider Figure 12. (Enc, Dec) as shown in Figure 12 is a quantum TDC against $\text{LO}_{(\infty, \infty, e_3)}^3$ with error $\varepsilon' \leq 2\varepsilon_{\text{nmExt}} + 6 \cdot 2^{e_3 - \lambda}$.*

The proof of which we defer to Section 4.2.1. By instantiating Theorem 4.2 with the ingredients above, we obtain the following explicit construction.

Corollary 4.3. *For any constant $0 < \gamma < 1/12$, there exists an efficient quantum TDC of blocklength n and rate $\frac{1-12\gamma}{11}$ secure against $\text{LO}_{(\infty, \infty, \gamma \cdot n)}^3$ with error $2^{-n^{\Omega(1)}}$.*

We further prove that our construction inherits quite strong secret sharing properties, which we extensively leverage in our future applications to ramp secret sharing.

Theorem 4.4 (3-out-of-3 Secret Sharing). *(Enc, Dec) from Theorem 4.2 is also a 3-out-of-3 secret sharing scheme with error ε' . In fact, any two shares of the quantum TDC are ε' -close to the maximally mixed state.*

Proof. Note that $\text{Enc}(\cdot)$ in Figure 12 first samples an independent (X, Y) and then generates $R = \text{nmExt}(X, Y)$. It also independently prepares $\lambda = \delta'n$ EPR pairs $\Phi_{E\hat{E}}$. It follows from the strong-extraction property of nmExt (see Fact 2.7) that

$$RX \approx_{\varepsilon_{\text{nmExt}}} U_R \otimes U_X \quad ; \quad RY \approx_{\varepsilon_{\text{nmExt}}} U_R \otimes U_Y.$$

Recall that the three shares are $(X, YE, Z = C_R(\hat{E}M)C_R^\dagger)$. Thus for every message $\sigma_{\hat{M}\hat{M}}$, using the fact that Cliffords are 1-Designs (Fact 2.5), and the approximate sampler Samp from Fact 2.6, we have

$$(\text{Enc}(\sigma))_{\hat{M}ZX} \approx_{\varepsilon_{\text{nmExt}}+2^{1-\lambda}} \sigma_{\hat{M}} \otimes U_Z \otimes U_X \quad ; \quad (\text{Enc}(\sigma))_{\hat{M}ZYE} \approx_{\varepsilon_{\text{nmExt}}+2^{1-\lambda}} \sigma_{\hat{M}} \otimes U_Z \otimes U_Y \otimes U_E. \quad (37)$$

Since $|Z| > \lambda = \delta'n$. Moreover, since (X, Y) are sampled independently, we also have

$$(\text{Enc}(\sigma))_{\hat{M}XYE} = \sigma_{\hat{M}} \otimes U_X \otimes U_Y \otimes U_E. \quad (38)$$

□

4.2.1 Proof of Theorem 4.2

To show that (Enc, Dec) is an ε' -secure quantum TDC, it suffices to show that for every $\mathcal{A} = (U, V, T, \psi_{W_1W_2W_3})$ it holds that (in Figure 12)

$$\eta_{\hat{M}M'} \approx_{\varepsilon'} p_{\mathcal{A}} \sigma_{\hat{M}M} + (1 - p_{\mathcal{A}})(\sigma_{\hat{M}} \otimes \perp_{M'}), \quad (39)$$

where $p_{\mathcal{A}}$ depends only on the tampering adversary \mathcal{A} . To do so, we proceed similarly to Section 3, where we “delayed” the tampering channel on the extra $(t+1)$ st share. In Figure 13 below, we represent the same tampering experiment as Figure 12, except for the delayed action of the tampering map T . We show that Equation (39) holds in Figure 13, which completes the proof.

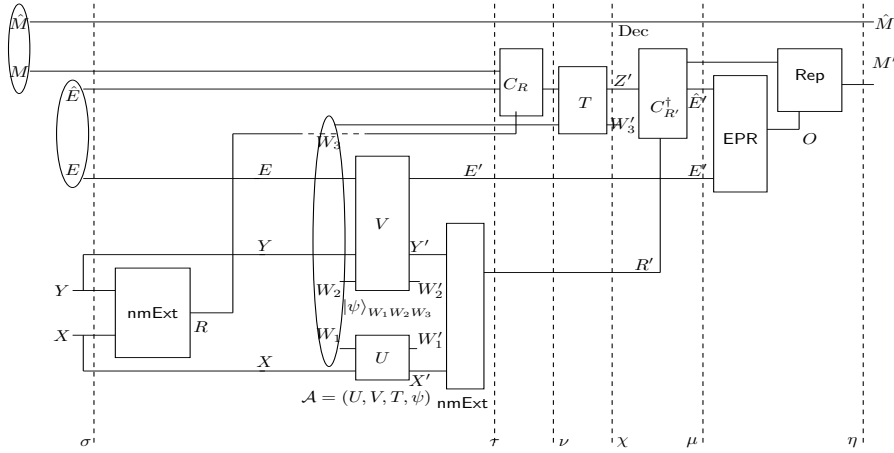


Figure 13: Analysis of quantum TDC against $\text{LO}_{(e_1=\infty, e_2=\infty, e_3)}^3$.

Proof. Consider the state τ in Figure 13. Note $\tau_{\hat{M}M} \equiv \sigma_{\hat{M}M}$ is a pure state (thus independent of other registers in τ) and

$$\tau = (\text{nmExt}_{X'Y'} \otimes \text{nmExt}_{XY}) ((U \otimes V)(\sigma \otimes |\psi\rangle\langle\psi|_{W_1W_2W_3})).$$

Our analysis will proceed by cases, depending on whether the X and Y registers are modified by the tampering experiment in Figure 13 (i.e., $XY \neq X'Y'$) or not. To this end, we consider two different conditionings of τ based on these two cases. Let τ^1 be the state if the tampering adversary ensures $(XY = X'Y')$ and τ^0 be the state conditioned on $(XY \neq X'Y')$.

Using Fact 2.7, state τ can be written as convex combination of two states τ^1 and τ^0 such that:

$$(\tau)_{RR'W_2'W_3'E'\hat{E}M\hat{M}} = p_{\text{same}}(\tau^1)_{RR'W_2'W_3'E'\hat{E}M\hat{M}} + (1 - p_{\text{same}})(\tau^0)_{RR'W_2'W_3'E'\hat{E}M\hat{M}}, \quad (40)$$

where p_{same} depends only on the adversary $\mathcal{A}' = (U, V, \psi_{W_1W_2W_3})$. In case $(XY = X'Y')$, the key is recovered $\Pr(R = R')_{\tau^1} = 1$. Fact 2.7 guarantees the non-malleability of the secret key R :

$$p_{\text{same}} \|(\tau^1)_{RR'W_2'W_3'E'\hat{E}M\hat{M}} - U_{|R|} \otimes (\tau^1)_{W_2'W_3'E'\hat{E}M\hat{M}}\|_1 + (1 - p_{\text{same}}) \|(\tau^0)_{RR'W_2'W_3'E'\hat{E}M\hat{M}} - U_{|R|} \otimes (\tau^0)_{R'W_2'W_3'E'\hat{E}M\hat{M}}\|_1 \leq \varepsilon_{\text{nmExt}}. \quad (41)$$

Suppose Υ denotes the CPTP map from registers $RR'W_3E'\hat{E}M\hat{M}$ to $M'\hat{M}$ (i.e. Υ maps state τ to η) in [Figure 13](#). We present two claims (proved in the next subsections) which allow us to conclude the proof. The first one stipulates that if the key is not recovered, the Bell basis measurement rejects with high probability:

Claim 4.5 (Key *not* recovered). $\Upsilon(U_{|R|} \otimes (\tau^0)_{R'W_2'W_3E'\hat{E}M\hat{M}}) \approx_{2 \cdot 2^{|W_3|-|E|}} \sigma_{\hat{M}} \otimes \perp_{M'}$.

In the second, if the key is recovered, then the 2-design functions as an authentication code. In this manner, we either recover the original message, or reject:

Claim 4.6 (Key recovered). $\Upsilon(U_{|R|} \otimes (\tau^1)_{W_2'W_3E'\hat{E}M\hat{M}}) \approx_{\frac{2}{(4^{|E|+|M|-1})} + \frac{2}{2^{|E|}}} p\sigma_{\hat{M}M} + (1-p)\sigma_{\hat{M}} \otimes \perp_{M'}$. Furthermore, p depends only on adversary $\mathcal{A} = (U, V, T, |\psi\rangle_{W_1W_2W_3})$.

We are now in a position to conclude the proof. First, leveraging [Equation \(41\)](#)

$$\begin{aligned} \eta_{M'\hat{M}} &= \Upsilon((\tau)_{RR'W_3E'\hat{E}M\hat{M}}) = p_{\text{same}}\Upsilon((\tau^1)_{RR'W_3E'\hat{E}M\hat{M}}) + (1-p_{\text{same}})\Upsilon((\tau^0)_{RR'W_3E'\hat{E}M\hat{M}}) \\ &\approx_{\varepsilon_{\text{nmExt}}} p_{\text{same}}\Upsilon(U_{|R|} \otimes (\tau^1)_{W_2'W_3E'\hat{E}M\hat{M}}) + (1-p_{\text{same}})\Upsilon(U_{|R|} \otimes (\tau^0)_{R'W_2'W_3E'\hat{E}M\hat{M}}) \end{aligned} \quad (42)$$

Next, by applying [Claim 4.6](#), proceeded by [Claim 4.5](#):

$$\approx_{\frac{2}{(4^{|E|+|M|-1})} + \frac{2}{2^{|E|}}} p_{\text{same}} \cdot (p \cdot \sigma_{M\hat{M}} + (1-p)(\perp_{M'} \otimes \sigma_{\hat{M}})) + (1-p_{\text{same}})\Upsilon(U_{|R|} \otimes (\tau^0)_{R'W_2'W_3E'\hat{E}M\hat{M}}) \quad (43)$$

$$\approx_{2 \cdot \frac{2^{|W_3|}}{2^{|E|}}} p_{\text{same}} \cdot p \cdot \sigma_{M\hat{M}} + p_{\text{same}}(1-p)(\perp_{M'} \otimes \sigma_{\hat{M}}) + (1-p_{\text{same}})\perp_{M'} \otimes \sigma_{\hat{M}} \quad (44)$$

The total error is thus $\leq \varepsilon_{\text{nmExt}} + 6 \cdot 2^{|W_3|-|E|}$. The observation that p_{same} and p depend only on the adversary $\mathcal{A} = (U, V, T, |\psi\rangle_{W_1W_2W_3})$ completes the proof. \square

4.2.2 Proof of [Claim 4.5](#)

Proof. Let ν^0, χ^0, μ^0 be the intermediate states and η^0 be the final state when we run the CPTP map Υ on $U_{|R|} \otimes \tau_{R'W_2'W_3E'\hat{E}M\hat{M}}^0$ (see [Figure 13](#)). Since, $\tau_{R'W_2'W_3E'\hat{E}M\hat{M}}^0 = \tau_{R'W_2'W_3E'\hat{E}}^0 \otimes \sigma_{M\hat{M}}$, using [Fact 2.5](#) (Cliffords are 1-Designs) it follows that in the state ν^0 the two registers E and \hat{E} are decoupled:

$$\nu_{R'W_2'W_3E'\hat{E}M\hat{M}}^0 = \tau_{R'W_2'W_3E'}^0 \otimes \sigma_{\hat{M}} \otimes U_{\hat{E}M} \quad (45)$$

We fix $R' = r'$ and argue that we output $\perp_{M'}$ with high probability for every such fixing. Let $\tau^{0,r'} \stackrel{\text{def}}{=} \tau^0|(R' = r')$ and similarly define $\nu^{0,r'}, \chi^{0,r'}, \mu^{0,r'}$ and $\eta^{0,r'}$. Note $\nu_{W_2'W_3E'\hat{E}M\hat{M}}^{0,r'} = \tau_{W_2'W_3E'}^{0,r'} \otimes \sigma_{\hat{M}} \otimes U_{\hat{E}M}$. This implies that the Schmidt number of the state $\nu^{0,r'}$ across the bipartition $(W_2'E', W_3\hat{E}M\hat{M})$ is $\leq 2^{|W_3|}$.

The states $\chi^{0,r'}, \mu^{0,r'}$ can be prepared from $\nu^{0,r'}$ using just local operations on each side of the cut $(W_2'E', W_3\hat{E}M\hat{M})$. From [Proposition 2.1](#), their Schmidt numbers are at most $2^{|W_3|}$. Moreover, again by [Proposition 2.1](#), the reduced density matrix $\mu_{E'\hat{E}}^{0,r'}$ also has Schmidt number at most $2^{|W_3|}$. [Claim 3.5](#) then ensures that the EPR test on state $\mu^{0,r'}$ fails with probability at least $1 - 2^{|W_3|-|E|}$. We conclude $\eta_{M\hat{M}}^{0,r'} \approx_{2 \cdot 2^{|W_3|-|E|}} \sigma_{\hat{M}} \otimes \perp_{M'}$.

Since the above argument works for every fixing of $R' = r'$ and $\eta^0 = \mathbb{E}_{r'} \eta^{0,r'}$, we have the desired. \square

4.2.3 Proof of Claim 4.6

Proof. Let ν^1, χ^1, μ^1 be the intermediate states and η^1 be the final state when we run the CPTP map Υ on $U_{|R|} \otimes \tau_{W'_2 W_3 E' \hat{E} M \hat{M}}^1$ (see Figure 13).

Note we have $(\tau^1)_{W'_2 W_3 E' \hat{E} M \hat{M}} = (\tau^1)_{W'_2 W_3 E' \hat{E}} \otimes \sigma_{M \hat{M}}$. Consider the state μ^1 , i.e. the state obtained by the action of C_R on τ^1 followed by CPTP map T , followed by C_R^\dagger . Using, Lemma 2.2, we have

$$\mu_{\hat{M} M' \hat{E}' E' W'_3}^1 \approx_{2/(4^{|M|+|\hat{E}|-1})} T_1((\tau^1)_{\hat{M} M \hat{E} E' W_3}) + T_2((\tau^1)_{\hat{M} E' W_3} \otimes U_{M \hat{E}}),$$

where $T_1(\cdot) : \mathcal{L}(\mathcal{H}_{W_3}) \rightarrow \mathcal{L}(\mathcal{H}_{W'_3})$, $T_2(\cdot) : \mathcal{L}(\mathcal{H}_{W_3}) \rightarrow \mathcal{L}(\mathcal{H}_{W'_3})$ are CP maps such that $T_1 + T_2$ is trace preserving and they depend only on adversary CPTP map $T(\cdot)$. Note both

$$T_1((\tau^1)_{\hat{M} M \hat{E} E' W_3}) \quad ; \quad T_2((\tau^1)_{\hat{M} E' W_3} \otimes U_{M \hat{E}})$$

are sub-normalized density operators. Let $p_1 \stackrel{\text{def}}{=} \text{Tr}(T_1((\tau^1)_{\hat{M} M \hat{E} E' W_3}))$. Let

$$\mu^{1,0} \stackrel{\text{def}}{=} \frac{1}{p_1} T_1((\tau^1)_{\hat{M} M \hat{E} E' W_3}) \quad ; \quad \mu^{1,1} \stackrel{\text{def}}{=} \frac{1}{1-p_1} T_2((\tau^1)_{\hat{M} E' W_3} \otimes U_{M \hat{E}}).$$

Note $\mu^1 \approx_{2/(4^{|M|+|\hat{E}|-1})} p_1 \mu^{1,0} + (1-p_1) \mu^{1,1}$. Let the final states be $\eta^{1,0}, \eta^{1,1}$ when we run the EPR test followed by Rep on $\mu^{1,0}, \mu^{1,1}$ respectively. Since, $\mu^1 \approx_{2/(4^{|M|+|\hat{E}|-1})} p_1 \mu^{1,0} + (1-p_1) \mu^{1,1}$, we conclude,

$$\eta^1 \approx_{2/(4^{|M|+|\hat{E}|-1})} p_1 \eta^{1,0} + (1-p_1) \eta^{1,1}. \quad (46)$$

In the first case, since $(\mu^{1,0})_{W'_3 E' \hat{E}' M' \hat{M}} = (\mu^{1,0})_{W'_3 E' \hat{E}'} \otimes \sigma_{M \hat{M}}$, we can conclude that

$$\eta^{1,0} = p_2 \sigma_{\hat{M} M} + (1-p_2) \sigma_{\hat{M}} \otimes \perp_{M'}, \quad (47)$$

and furthermore p_2 depends only on CP map $T_1(\cdot)$ and state τ^1 , which further depends only on tampering adversary \mathcal{A} .

In the second case, since $(\mu^{1,1})_{W'_3 E' \hat{E} M' \hat{M}} = (\mu^{1,1})_{W'_3 E' \hat{E}'} \otimes U_{\hat{E}' M} \otimes \sigma_{\hat{M}}$, we can conclude

$$\eta_{\hat{M} M'}^{1,1} \approx_{2 \cdot 2^{-|E|}} \sigma_{\hat{M}} \otimes \perp_{M'}, \quad (48)$$

using Claim 3.5 as EPR test rejects with probability atleast $1 - 2^{-|E|}$.

Combining Equation (46), Equation (47) and Equation (48), we have the following:

$$\begin{aligned} & \Upsilon(U_{|R|} \otimes (\tau^1)_{W'_2 W_3 E' \hat{E} M \hat{M}}) \\ &= \eta_{\hat{M} M'}^1 \\ &\approx_{2/(4^{|M|+|\hat{E}|-1})} p_1 \eta^{1,0} + (1-p_1) \eta^{1,1} \\ &= p_1 p_2 \sigma_{\hat{M} M} + p_1 (1-p_2) \sigma_{\hat{M}} \otimes \perp_{M'} + (1-p_1) \eta^{1,1} \\ &\approx_{2 \cdot 2^{-|E|}} p_1 \cdot p_2 \sigma_{\hat{M} M} + p_1 (1-p_2) \sigma_{\hat{M}} \otimes \perp_{M'} + (1-p_1) \sigma_{\hat{M}} \otimes \perp_{M'} \\ &= p_1 \cdot p_2 \sigma_{\hat{M} M} + (1-p_1 \cdot p_2) \sigma_{\hat{M}} \otimes \perp_{M'}. \end{aligned}$$

Considering $p = p_1 \cdot p_2$, we have the desired. Through p_1, p_2 , p depends only on tampering adversary \mathcal{A} . \square

5 Tamper-Detecting Secret Sharing Schemes

In this section, we prove our main result on the construction of secret sharing schemes which detect local tampering, restated below. We refer the reader to [Section 2.5.4](#) for the definition of tamper-detecting secret sharing schemes.

Theorem 5.1 ([Theorem 1.4](#), restatement). *For every p, t s.t. $4 \leq t \leq p - 2$, there exists a $(p, t, 0, 0)$ secret sharing scheme for k bit messages which is $(\text{LO}^p, t + 2, 2^{-\max(p, k)^{\Omega(1)}})$ -tamper detecting.*

In other words, no $t - 1$ shares reveal any information about the message, any t honest shares uniquely determine the message, and one can detect LO tampering on any $(t + 2)$ shares.

We organize this section as follows. We begin in [Section 5.1](#) by introducing the relevant code components, overviewing the construction, as well as presenting the construction and analysis of the “triangle gadgets” discussed in [Section 1.3.3](#). Next, in [Section 5.2](#) we present our secret sharing scheme, and in the subsequent [Section 5.3](#) we present its analysis.

5.1 Code Components and the Triangle Gadget

In [Section 5.1.1](#), we describe the ingredients in our secret sharing scheme, in [Section 5.1.2](#) we introduce the “triangle gadget”, and in [Section 5.1.3](#) we prove that it inherits two basic properties: (weak) tamper detection, and 3-out-of-3 secret sharing.

5.1.1 Ingredients and Overview

We combine:

1. $(\text{Share}_{\text{LRSS}}, \text{Rec}_{\text{LRSS}})$: A classical $(p, t, 0, 0)$ -secret sharing scheme which is ε_{l_r} -leakage resilient to μ qubits of the 3-local leakage model $\mathcal{F}_{3, \mu}^{p, t}$, such as that of [Theorem 2.6](#).

For basic definitions of leakage-resilient secret sharing, see [Section 2.5.2](#).

2. $(\text{Enc}_{\text{TD}}^\lambda, \text{Dec}_{\text{TD}}^\lambda)$ and $\lambda \in \mathbb{N}$: The 3-split-state quantum tamper-detection code in the bounded storage model of [Theorem 4.2](#) with λ EPR pairs, comprised of:
 - (a) $\text{nmExt} : \{0, 1\}^q \times \{0, 1\}^{\delta \cdot q} \rightarrow \{0, 1\}^r$, a quantum secure two source non-malleable extractor, with error $\varepsilon_{\text{NM}} = 2^{-q^{\Omega_\delta(1)}}$ and output size $r = (1/2 - \delta)q$ from [Fact 2.7](#).
 - (b) The family of 2-design unitaries C_R from [Fact 2.6](#) on $\frac{1}{5} \cdot r$ qubits.

From [Theorem 4.2](#), $(\text{Enc}_{\text{TD}}^\lambda, \text{Dec}_{\text{TD}}^\lambda)$ has message length $\frac{r}{5} - \lambda$ and is $\varepsilon_{\text{TD}} \leq 2(\varepsilon_{\text{NM}} + 2^{a-\lambda})$ secure against $\text{LO}_{(a, *, *)}$. Definitions of these code components can be found in [Section 4](#) and [Section 2](#).

Our approach to secret sharing attempts to extend the compiler by [\[ADN⁺19\]](#) to a tripartite setting. First, the classical message m is shared into the secret sharing scheme, resulting in classical shares $(M_1, \dots, M_p) \leftarrow \text{Share}_{\text{LRSS}}(m)$. Then every triplet of parties $a < b < c$ jointly encodes their shares M_a, M_b, M_c into a code Enc_Δ supported on a tripartite register, the shares of which are redistributed among a, b, c .

In this subsection, we present our gadget code Enc_Δ and prove two of its relevant properties: a strong form of 3-out-of-3 secret sharing, as well as a weak form of tamper detection. In the next subsections, we show how to leverage these gadget properties together with that of the underlying leakage-resilient secret sharing scheme to achieve tamper detection, globally.

5.1.2 The Triangle Gadget

Algorithm 5: Enc_Δ : The “triangle gadget”

Input: Three parties $p_0 < p_1 < p_2 \in [p]$, three messages M_0, M_1, M_2 , and an integer parameter λ .

Output: A quantum state $\text{Enc}_\Delta^\lambda(M_0, M_1, M_2)$ on a tripartite register A, B, C .

- 1: Each party $i \in \{0, 1, 2\}$ encrypts their message M_i into the tamper-detection code of [Algorithm 3](#), into a tripartite register $(Q_i, Y_i \hat{E}_i, X_i)$. Explicitly, party i :
 1. Samples uniformly random classical registers X_i, Y_i and evaluates the key $R_i = \text{nmExt}(X_i, Y_i)$.
 2. Prepares $\lambda_i = (i + 1) \cdot \lambda$ EPR pairs, on a bipartite register $E_i \hat{E}_i$.
 3. Samples the Clifford unitary C_{R_i} using the sampling process **Samp** in [Fact 2.6](#). Applies C_{R_i} on registers M_i, E_i to generate Q_i .
 4. Hands register X_i to party $i - 1 \pmod{3}$, registers Y_i, \hat{E}_i to party $i + 1 \pmod{3}$, and keeps the authenticated register Q_i .
- 2: Output registers A, B, C for parties p_0, p_1, p_2 respectively, where

$$A = (Q_0, Y_2 \hat{E}_2, X_1), B = (Q_1, Y_0 \hat{E}_0, X_2), C = (Q_2, Y_1 \hat{E}_1, X_0) \quad (49)$$

Dec_Δ : To decode, we simply run the decoder Dec_{TD} for the tamper-detection code on the registers $(Q_i, Y_i \hat{E}_i, X_i)$ corresponding to $\text{Enc}_{\text{TD}}(M_i)$. If any of the three are \perp , output \perp . If otherwise, output the resulting messages (M'_0, M'_1, M'_2) . Note that if no tampering is present, then from the perfect correctness of the tamper-detection code $(\text{Enc}_\Delta, \text{Dec}_\Delta)$ is also perfectly correct.

5.1.3 Analysis of the Triangle Gadget

Before moving on to our global construction, we prove two important properties of the gadget $(\text{Enc}_\Delta, \text{Dec}_\Delta)$. The first is that not only is Enc_Δ 3-out-of-3 secret sharing, but in fact any two shares Enc_Δ are near maximally mixed. Note that 3-out-of-3 secret sharing only implies that the two-party reduced density matrices are independent of the message, but apriori they could still be entangled.

Claim 5.2 (Pairwise Independence). *For every share $W \in \{A, B, C\}$ and fixed strings m_0, m_1, m_2 , the two party marginal of $\text{Enc}_\Delta(m_0, m_1, m_2)$ without W is δ_{PI} -close to maximally mixed, where $\delta_{\text{PI}} \leq 6 \cdot (\varepsilon_{\text{NM}} + 2^{-\lambda})$.*

Proof. Recall that $\text{Enc}_{\text{TD}}^\varepsilon$ is pairwise independent with error $\leq 2(\varepsilon_{\text{NM}} + 2^{-e})$ from [Theorem 4.4](#). The result follows from a triangle inequality. \square

The second property concerns the resilience of Enc_Δ to tampering in LO^3 . We prove that two of the three inputs, namely m_1, m_2 , are individually tamper-detected:

Claim 5.3 (Share-wise Tamper Detection). *Fix $\Lambda \in \text{LO}^3$, three strings m_0, m_1, m_2 , and an integer λ . Let $(M'_0, M'_1, M'_2) \leftarrow \text{Dec}_\Delta \circ \Lambda \circ \text{Enc}_\Delta(m_0, m_1, m_2)$ denote the distribution over the recovered shares. Then for both $i \in \{1, 2\}$,*

$$\mathbb{P}[M'_i \notin \{m_i, \perp\}] \leq \varepsilon_{\text{Share}}, \quad (50)$$

where $\varepsilon_{\text{Share}} \leq \varepsilon_{\text{NM}} + 2^{4-\lambda}$.

By a union bound, we have $\mathbb{P}[(M'_1, M'_2) \notin \{(m_1, m_2), \perp\}] \leq 2 \cdot \varepsilon_{\text{Share}}$. However, here we already exhibit the selective bot problem. The event $M'_1 = \perp$ may correlate with M'_2 , breaking the tamper detection guarantee.

Proof. [of Claim 5.3] Fix m_0, m_1, m_2 , the tampering channel $\Lambda \in \text{LO}^3$, and $i \in \{1, 2\}$. In Algorithm 5, note that the i th share of Enc_Δ is comprised of registers $(Q_i, Y_{i-1} \hat{E}_{i-1}, X_{i+1})$, where

1. Q_i is a share of $\text{Enc}_{\text{TD}}^{\lambda_i}(m_i)$
2. The only other quantum register contains $|\hat{E}_{i-1}| \leq \lambda_{(i-1) \bmod 3}$ qubits.

We can now consider the marginal distribution over the i th message m_i after the tampering channel. The marginal distribution over the recovered share m'_i in the gadget Enc_Δ can be simulated using a channel Λ' directly on $\text{Enc}_{\text{TD}}(m_i)$:

$$\text{Tr}_{\neq i} \text{Dec}_\Delta \circ \Lambda \circ \text{Enc}_\Delta(m_1, m_2, m_3) = \text{Dec}_{\text{TD}} \circ \Lambda' \circ \text{Enc}_{\text{TD}}(m_i), \quad (51)$$

where Λ' consists of a local tampering channel on $(Q_i, Y_i \hat{E}_i, X_i)$ aided by pre-shared entanglement in a particular form: the adversary who receives share Q_i holds at most $a = \lambda_{i-1}$ qubits of pre-shared entanglement (but the other two may hold unbounded-size registers). This is precisely the model $\text{LO}_{(a,*,*)}^3$ of Definition 2.10, and we recall by Theorem 4.2 that Enc_{TD}^e detects tampering against $\text{LO}_{(a,*,*)}^3$ with error $2\varepsilon_{\text{NM}} + 2^{4+a-e}$. Since parties $i \in \{1, 2\}$ hold more EPR pairs in their tamper-detection codes than the corresponding adversaries pre-shared, i.e $\lambda_i - \lambda_{i-1} = \lambda > 0$ when $i \in \{1, 2\}$, we conclude their shares are tamper-detected:

$$i \in \{1, 2\} : \mathbb{P}[M'_i \notin \{m_i, \perp\}] \leq 2\varepsilon_{\text{NM}} + 2^{4-\lambda}. \quad (52)$$

□

5.2 Code Construction

We describe our encoding algorithm in Algorithm 6 below:

Algorithm 6: Enc: A Tamper-Detecting Ramp Secret Sharing Scheme

Input: A k bit message m .

- 1: Encode m into the LRSS, $(M_1, \dots, M_p) \leftarrow \text{Share}_{\text{LRSS}}(m)$.
- 2: For every triplet of parties $a < b < c \in [p]$,
 1. Encode the shares (M_a, M_b, M_c) into the “triangle gadget” described in Algorithm 5 supported on triplets of quantum registers

$$A_{(a,b,c)}, B_{(a,b,c)}, C_{(a,b,c)} \leftarrow \text{Enc}_\Delta(M_a, M_b, M_c) \quad (53)$$

2. Hand the $A_{(a,b,c)}$ register to party a , $B_{(a,b,c)}$ to b , and $C_{(a,b,c)}$ to c .
- 3: Let the resulting i th share be the collection of quantum registers

$$S_i = \{A_{(i,b,c)} : \forall b, c \text{ s.t. } i < b < c\} \cup \{B_{(a,i,c)} : \forall a, c \text{ s.t. } a < i < c\} \cup \{C_{(a,b,i)} : \forall a, b \text{ s.t. } a < b < i\} \quad (54)$$

We are now in a position to describe our decoding algorithm. Upon receiving the locally tampered shares of any authorized subset T of size equal to $t + 2$, our decoder partitions T into two un-authorized subsets, and decodes the gadgets only within each partition.

Algorithm 7: Dec: A Bipartite Decoding Algorithm.

Input: An (authorized) subset T of size $t + 2$, and a collection of tampered quantum registers

$$S'_i : i \in T.$$

Output: A k bit message M'

- 1: Partition T into a subset U of the three smallest indexed shares, and $T \setminus U$.
 - 2: For every triplet of parties $a < b < c$ contained entirely in $T \setminus U$ or entirely in U :
 1. Apply the triangle gadget decoding algorithm Dec_Δ on registers $A_{(a,b,c)}, B_{(a,b,c)}, C_{(a,b,c)}$.
 2. Output \perp if so does the decoder. Otherwise, let $M'_{(a,b,c)}, M'_{(b,a,c)}, M'_{(c,a,b)}$ be the recovered shares.
 - 3: If $M'_{(a,b,c)} \neq M'_{(a,d,e)}$ for any two triangles $(a, b, c), (a, d, e)$ on the same side of the partition, output \perp . Otherwise, let $M'_U, M'_{T \setminus U}$ be the recovered shares from either side of the partition.
 - 4: Remove the lowest index share of U to obtain $V \subset U$ and $T \setminus U$ to obtain $W \subset T \setminus U$. Note $|V \cup W| \geq t$.
 - 5: Run the decoder $M' \leftarrow \text{Rec}_{\text{LRSS}}(M'_V, M'_W)$ on the classical shares of $V \cup W$.
-

The main result of this section proves that the secret sharing scheme described in [Algorithm 6](#) and [Algorithm 7](#) above detects unentangled tampering, when handed at least $t + 2$ shares.

Lemma 5.4. *Assuming the LRSS is ε_{lr} -resilient to $\mu \geq 10 \cdot p^2 \cdot \lambda$ qubits of leakage, then (Enc, Dec) described above is a $(p, t, 0, 0)$ -secret sharing scheme which is $(\text{LO}^p, t + 2, \varepsilon)$ -tamper-detecting with error $\varepsilon = O(\varepsilon_{lr} + p^4 \cdot (\varepsilon_{\text{NM}}^{1/2} + 2^{-\lambda/2}))$.*

We dedicate the next subsection to its proof. By instantiating [Lemma 5.4](#) above,

Theorem 5.5. *There exists an efficient $(p, t, 0, 0)$ secret sharing scheme for k bit messages which is $(\text{LO}^p, t + 3, 2^{-\max(k,p)^{\Omega(1)}})$ -tamper-detecting.*

Proof. We use the construction of the $(p, t, 0, 0)$ LRSS from [Theorem 2.6](#) with message length k bits, shares of length $s = k + p^2 \cdot \lambda$ bits, and error $2^{-\Omega(\lambda)}$; together with the tamper-detection codes Enc^λ from [Theorem 4.2](#) with message size s , error $2^{-\lambda} + 2^{-s^{\Omega(1)}}$, and selecting $\lambda = \max(k, p)$, we conclude the corollary. \square

5.3 Analysis

Our proof is comprised of two key claims, which we state and use to prove [Lemma 5.4](#) and then analyze in the subsequent sections. We begin by arguing that our secret sharing scheme inherits a weak form of tamper detection from the “outer” split-state tamper-detection codes.

Claim 5.6 (Weak Tamper Detection). *For every tampering channel $\Lambda \in \text{LO}^p$, authorized subset $T \subset [p]$ of size $\geq t + 2$ and message m , the distribution over the recovered message $M' \leftarrow \text{Dec}_T \circ \Lambda \circ \text{Enc}(m)$ satisfies*

$$\mathbb{P}[M' \notin \{m, \perp\}] \leq \eta_{\text{TD}} = O(p^3 \cdot (\varepsilon_{\text{Share}} + \delta_{\text{PI}} \cdot p)) = O(p^4(\varepsilon_{\text{NM}} + 2^{-\lambda})) \quad (55)$$

In fact, all the shares of $V \cup W$ are recovered whp: $\mathbb{P}[\exists i \in V \cup W : M'_i \notin \{m_i, \perp\}] \leq \eta_{\text{TD}}$

In other words, (Enc, Dec) is a secret sharing scheme which is weak tamper-detecting against LO^p , so long as the decoder receives at least $t + 2$ shares. As previously discussed, the reason this is only *weak* tamper detecting is due to the selective bot problem - the probability with which we reject may a priori depend on m . We leverage the leakage resilience of the underlying secret sharing scheme to ensure this cannot occur:

Claim 5.7 (The Selective Bot Problem). *For every tampering channel $\Lambda \in \text{LO}^p$ and authorized subset $T \subset [p]$ of size $\geq t + 2$, the probability the decoding algorithm Dec in [Algorithm 7](#) rejects is near independent of the message. That is,*

$$\forall \Lambda \in \text{LO}^p, m_0, m_1 : |\mathbb{P}[\text{Dec}_T \circ \Lambda \circ \text{Enc}(m_0) = \perp] - \mathbb{P}[\text{Dec}_T \circ \Lambda \circ \text{Enc}(m_1) = \perp]| \leq \eta_r, \quad (56)$$

where $\eta_{lr} = O(\varepsilon_{lr} + p^2 \cdot \sqrt{\delta_{\text{PI}} + \varepsilon_{\text{Share}}}) = O(\varepsilon_{lr} + p^2(\varepsilon_{\text{NM}}^{1/2} + 2^{-\lambda/2}))$.

Put together, we now conclude the proof:

Proof. [of [Lemma 5.4](#)] Let the random variable M denote the uniform distribution over messages, and let m be any fixed value of M . From [Claim 5.6](#), for a fixed $\Lambda \in \text{LO}^p$ and $T \subset [p]$ the distribution over the recovered message M' is near a convex combination of M and rejection, with bias γ_m (dependent on m):

$$M'|_{M=m} \approx_{\eta_{\text{TD}}} \gamma_m \cdot m + (1 - \gamma_m) \cdot \perp. \quad (57)$$

However, by [Claim 5.7](#), the probability of \perp can barely depend on M :

$$1 - \gamma_m \approx_{\eta_{\text{TD}}} \mathbb{P}[M' = \perp | M = m] \approx_{\eta_{lr}} \mathbb{P}[M' = \perp] \equiv p_{\text{Adv}} \quad (58)$$

By the triangle inequality, $M'|_{M=m} \approx_{3 \cdot \eta_{\text{TD}} + 2\eta_{lr}} (1 - p_{\text{Adv}}) \cdot m + p_{\text{Adv}} \cdot \perp$, for some fixed p_{Adv} dependent on Λ, T . With $\eta_{\text{TD}}, \eta_{lr}$ as in the claims above we conclude the theorem. \square

5.3.1 Proof of Weak Tamper Detection ([Claim 5.6](#))

Proof. [of [Claim 5.6](#)] Consider any triangle $a < b < c \in [p]$ contained on the same side of the partition $(U, T \setminus U)$, and let (m_a, m_b, m_c) be any fixing of their classical shares. Note that the reduced density matrix on their quantum registers $(S_a, S_b, S_c) |_{((M_a, M_b, M_c) = (m_a, m_b, m_c))}$ contains the triangle $\text{Enc}_{\Delta}(m_a, m_b, m_c)$, as well as the registers of all the triangles (a', b', c') which intersect (a, b, c) . However, since each individual register of Enc_{Δ} is maximally mixed, and all pairs of registers of Enc_{Δ} are δ_{PI} approximately pairwise independent ([Claim 5.2](#)), we have

$$(S_a, S_b, S_c) |_{((M_a, M_b, M_c) = (m_a, m_b, m_c))} \approx_{3 \cdot \delta_{\text{PI}} \cdot p} \text{Enc}_{\Delta}(m_a, m_b, m_c) \otimes \frac{\mathbb{I}}{\text{Tr } \mathbb{I}} \quad (59)$$

The copies $\sigma = \frac{\mathbb{I}}{\text{Tr } \mathbb{I}}$ of the maximally mixed state are separable across the shares a, b, c , and independent of m , and act as ancillas. Therefore, any channel $\Lambda \in \text{LO}^3$ which tampers with the registers (S_a, S_b, S_c) using σ can be simulated in LO^3 without them:

$$\begin{aligned} \text{Dec}_{\Delta} \circ \Lambda \circ (S_a, S_b, S_c) |_{((M_a, M_b, M_c) = (m_a, m_b, m_c))} &\approx_{3 \cdot \delta_{\text{PI}} \cdot p} \text{Dec}_{\Delta} \circ \Lambda(\text{Enc}_{\Delta}(m_a, m_b, m_c) \otimes \sigma) \\ &= \text{Dec}_{\Delta} \circ \Lambda' \circ \text{Enc}_{\Delta}(m_a, m_b, m_c) \end{aligned}$$

By [Claim 5.3](#) and the triangle inequality, this implies that two of the three shares in each triangle (a, b, c) are weak tamper detected:

$$\mathbb{P}[(M'_a, M'_b) \notin \{(m_a, m_b), \perp\}] \leq 2\varepsilon_{\text{Share}} + 3 \cdot \delta_{\text{PI}} \cdot p \quad (60)$$

By a union bound over all the triangles, either all the shares in $V \subset U, W \subset (T \setminus U)$ are recovered, or we reject with probability

$$\mathbb{P}[\forall i \in V \cup W : M'_i \notin \{m_i, \perp\}] \leq t^3 \cdot (2\varepsilon_{\text{Share}} + 3 \cdot \delta_{\text{PI}} \cdot p). \quad (61)$$

Since $|V \cup W|$ contains t (honest) shares, $\mathbb{P}[M' \notin \{m, \perp\}] \leq t^3 \cdot (2\varepsilon_{\text{Share}} + 3 \cdot \delta_{\text{PI}} \cdot p)$. \square

5.3.2 Proof of Leakage-Resilience ([Claim 5.7](#))

The decoder Dec_T of [Algorithm 7](#) rejects a tampered message $\Lambda \circ \text{Enc}(m)$ if either subset U or $(T \setminus U)$ reject (up to step 3), or if the decoder for the LRSS rejects.

We begin by analyzing the first of these two possibilities. In [Claim 5.8](#), we use the leakage resilience of the underlying secret sharing scheme to argue that the event that either subset U or $(T \setminus U)$ rejects cannot depend on the message m . To do so, we show that the measurement of whether either U or $(T \setminus U)$ reject can be simulated by a leakage channel from U to $(T \setminus U)$ on the shares of the classical LRSS.

Claim 5.8. Fix a tampering channel Λ and subsets $T \subset [n]$, $|T| \geq t + 2$ and $U \subset T$ of the smallest 3 shares in T . The probability Dec_T Algorithm 7 has rejected before step 4, does not depend on the message: $\forall m_0, m_1$:

$$\mathbb{P}[\text{Dec}_T \text{ aborts before Step 4} \mid M = m_0] - \mathbb{P}[\text{Dec}_T \text{ aborts before Step 4} \mid M = m_1] \leq p^3 \cdot \delta_{\text{PI}} + \varepsilon_{lr}. \quad (62)$$

Proof. Fix a message m , and let us condition on the classical LRSS shares M_T of m in $T \subset [p]$ being a fixed value $M_T = m_T = \{m_i : i \in T\}$. Now, consider the reduced density matrix $(\text{Enc}(m)_T)_{|M_T=m_T}$ on the shares of $T \subset [p]$ conditioned on the classical shares. It is comprised of multiple shares of Enc_Δ , for every triangle $a < b < c$ which intersects T . However, similarly to the proof of Claim 5.6, the pairwise independence of Enc_Δ (Claim 5.2) ensures any less than three shares are near maximally mixed. It suffices to then consider the triangles contained entirely in T :

$$\left\| (\text{Enc}(m)_T)_{|M_T=m_T} - \left(\bigotimes_{a,b,c \in T} \text{Enc}_\Delta(m_a, m_b, m_c) \right) \otimes \frac{\mathbb{I}}{\text{Tr} \mathbb{I}} \right\|_1 \leq p^3 \cdot \delta_{\text{PI}} \quad (63)$$

Now, recall that $\text{Enc}_\Delta^\lambda$ can be implemented using local operations between the three parties and at most $O(\lambda)$ qubits of shared entanglement. This implies that the parties in U and $T \setminus U$ could prepare the reduced density matrix T , $(\text{Enc}(m)_T)_{|M_T=m_T}$, using only their own shares $m_U, m_{T \setminus U}$ and

1. Joint operations within the subsets $U, T \setminus U$.
2. At most $\mu = O(\lambda \cdot p^2)$ qubits of one-way communication from U to $T \setminus U$.

If we denote L as the μ qubit ‘‘leakage’’ register, then one can formalize the above by defining two channels $\mathcal{E}_{L,U}$ and $\mathcal{E}_{T \setminus U, L}$. $\mathcal{E}_{L,U}$ acts on the shares of U producing the leakage register L , and $\mathcal{E}_{T \setminus U, L}$ acts on $L, T \setminus U$ respectively:

$$(\text{Enc}(m)_T)_{|M_T=m_T} \approx_{p^3 \delta_{\text{PI}}} (\mathcal{E}_{T \setminus U, L} \otimes \mathbb{I}_U) \circ (\mathbb{I}_{T \setminus U} \otimes \mathcal{E}_{L,U}) \circ (m_{T \setminus U}, m_U) \quad (64)$$

Moreover, before step 4 in Algorithm 7 the decoding channel Dec_T factorizes as a tensor product of channels $\text{Dec}_U \otimes \text{Dec}_{T \setminus U}$, as does the tampering channel $\Lambda_U \otimes \Lambda_{T \setminus U}$. The output of the decoder (up to step 4), conditioned on the message m and its shares $M_T = m_T$ can thereby be simulated using

$$(\text{Dec}_T \circ \Lambda_T \circ \text{Enc}(m)_T)_{|M_T=m_T} \approx_{p^3 \delta_{\text{PI}}} (\mathcal{N}_{T \setminus U, L} \otimes \mathbb{I}_U) \circ (\mathbb{I}_{T \setminus U} \otimes \mathcal{N}_{L,U}) \circ (m_{T \setminus U}, m_U) \quad (65)$$

$$\text{where } \mathcal{N}_{U,L} = \text{Dec}_U \circ \Lambda_U \circ \mathcal{E}_{U,L} \text{ and } \mathcal{N}_{L,T \setminus U} = \text{Dec}_{T \setminus U} \circ \Lambda_{T \setminus U} \circ \mathcal{E}_{T \setminus U, L} \quad (66)$$

Let $\mathcal{N}'_{U,L}$ denote the channel which applies $\mathcal{N}_{U,L}$ on m_U and then checks whether the decoder rejected (a bit b), leaks b and L to $T \setminus U$, and traces out U . Note that $\mathcal{N}'_{U,L}$ is in the leakage family $\mathcal{F}_{3,\mu+1}^{p,t}$, since $|U| = 3$ leaks $\mu + 1$ qubits to a subset of size $|T \setminus U| < t$. We conclude that for any two messages m_0, m_1 , by monotonicity of trace distance under CPTP maps:

$$\left| \mathbb{P}[\text{Dec}_T \text{ aborts before Step 4} \mid M = m_0] - \mathbb{P}[\text{Dec}_T \text{ aborts before Step 4} \mid M = m_1] \right| \leq \quad (67)$$

$$\leq 2 \cdot p^3 \cdot \delta_{\text{PI}} + \|(\mathcal{N}'_{U,L} \otimes \mathbb{I}_{T \setminus U}) \circ \text{Share}(m_0)_T - (\mathcal{N}'_{U,L} \otimes \mathbb{I}_{T \setminus U}) \circ \text{Share}(m_1)_T\|_1 \leq \quad (68)$$

$$\leq 2 \cdot p^3 \cdot \delta_{\text{PI}} + \varepsilon_{lr}, \text{ where last we leverage the leakage resilience of the LRSS against } \mathcal{F}_{3,\mu}^{p,t}. \quad \square$$

Observe that in Algorithm 7 the LRSS can only reject if the subsets U or $(T \setminus U)$ did not already reject. However, from Claim 5.6, we know that if U or $(T \setminus U)$ didn't reject, then (roughly) with high probability we must have recovered honest shares from the subsets $V \subset U, W \subset (T \setminus U)$ - which the LRSS must accept. We can now conclude the proof of Claim 5.7:

Proof. [of Claim 5.7] The decoder Dec_T rejects a tampered message $\Lambda \circ \text{Enc}(m)$ with probability

$$\mathbb{P}[\text{Dec} \circ \Lambda \circ \text{Enc}(M) = \perp \mid M = m] = \mathbb{P}[\text{Dec}_T \text{ aborts before Step 4} \mid M = m] + \quad (69)$$

$$+ \mathbb{P}[\text{Dec}_T \text{ doesn't abort before Step 4 and } \text{Rec}_{\text{LRSS}}(M'_V, M'_W) = \perp \mid M = m]. \quad (70)$$

From [Claim 5.8](#), the probability either half of the partition U or $T \setminus U$ rejects is $\gamma = 2p^3 \cdot \delta_{\text{PI}} + \varepsilon_{lr}$ close to independent of the message:

$$\mathbb{P}[\text{Dec}_T \text{ aborts before Step 4} \mid M = m_0] \approx_{\gamma} \mathbb{P}[\text{Dec}_T \text{ aborts before Step 4} \mid M = m_1] \quad (71)$$

Fix a real parameter δ . We divide into two cases on $\mathbb{P}[\text{Dec}_T \text{ aborts before Step 4} \mid M = m_0]$:

1. $\mathbb{P}[\text{Dec}_T \text{ doesn't abort before Step 4} \mid M = m_0] \leq \delta$, then $\forall m$,

$$\mathbb{P}[\text{Dec}_T \text{ doesn't abort before Step 4 and } \text{Rec}_{\text{LRSS}}(M'_V, M'_W) = \perp \mid M = m] \leq \quad (72)$$

$$\leq \mathbb{P}[\text{Dec}_T \text{ doesn't abort before Step 4} \mid M = m] \leq \delta + \gamma. \quad (73)$$

By the triangle inequality,

$$\mathbb{P}[\text{Dec} \circ \Lambda \circ \text{Enc}(m_0) = \perp] \approx_{\delta+\gamma} \mathbb{P}[\text{Dec} \circ \Lambda \circ \text{Enc}(m_1) = \perp]. \quad (74)$$

2. $\mathbb{P}[\text{Dec}_T \text{ doesn't abort before Step 4} \mid M = m_0] \geq \delta$. From the weak tamper detection guarantee in [Claim 5.6](#), $\forall m$:

$$\mathbb{P}[\text{Dec}_T \text{ doesn't abort before Step 4, and } (M'_V, M'_W) \neq (m_V, m_W) \mid M = m] \leq \eta_{\text{TD}} \quad (75)$$

Since under message m_0 , Dec_T doesn't abort before Step 4 with non-negligible probability, then conditioned on that event, the shares of V, W are untampered with high probability:

$$\mathbb{P}[(M'_V, M'_W) = (m_V, m_W) \mid \text{Dec}_T \text{ doesn't abort before Step 4, } M = m_0] \geq 1 - \frac{\eta_{\text{TD}}}{\delta}. \quad (76)$$

A similar statement holds for m_1 since $\mathbb{P}[\text{Dec}_T \text{ doesn't abort before Step 4} \mid M = m_1] \geq \delta - \gamma$, and therefore

$$\mathbb{P}[(M'_V, M'_W) = (m_V, m_W) \mid \text{Dec}_T \text{ doesn't abort before Step 4, } M = m_1] \geq 1 - \frac{\eta_{\text{TD}}}{\delta - \gamma}. \quad (77)$$

Finally, since Rec_{LRSS} has perfect correctness, if we condition on $(M'_V, M'_W) = (m_V, m_W)$, Rec_{LRSS} recovers the message with probability 1. Therefore,

$$\forall m : \mathbb{P}[\text{Dec}_T \text{ doesn't abort before Step 4, and } \text{Rec}_{\text{LRSS}}(M'_V, M'_W) \neq \perp \mid M = m] \quad (78)$$

$$\approx \frac{\eta_{\text{TD}}}{\delta - \gamma} \mathbb{P}[\text{Dec}_T \text{ doesn't abort before Step 4} \mid M = m]. \quad (79)$$

We conclude $|\mathbb{P}[\text{Dec} \circ \Lambda \circ \text{Enc}(m_0) = \perp] - \mathbb{P}[\text{Dec} \circ \Lambda \circ \text{Enc}(m_1) = \perp]| \leq \eta_{lr}$, where by the triangle inequality and optimizing over δ :

$$\eta_{lr} \leq \gamma + \max\left(\delta, \frac{\eta_{\text{TD}}}{\delta - \gamma}\right) \leq 2\gamma + \sqrt{\eta_{\text{TD}}} \leq O(\varepsilon_{lr} + p^2 \cdot \sqrt{\delta_{\text{PI}} + \varepsilon_{\text{Share}}}), \quad (80)$$

where we used the values of γ and η_{TD} from [Claim 5.8](#) and [Claim 5.6](#). \square

6 A Single-Bit Non-Malleable Code against LOCC²

In this section, our main result is a 2-split-state non-malleable code for a single bit message, secure against LOCC². Please refer to [Definition 2.11](#) for a description of LOCC channels.

Theorem 6.1. *For every $n, e \in \mathbb{N}$, there exists a non-malleable code against LOCC_e² (with e qubits of shared entanglement) for single-bit messages of blocklength $2n$ qubits and error 2^{1+e-n} .*

Our construction is simple and based on the LOCC data hiding scheme [\[TDL00\]](#). However, whereas our code is non-malleable against LOCC, curiously it is not data-hiding. That is, using LOCC the adversaries could distinguish the message, but even in doing so they wouldn't be able to tamper with it.

Algorithm 8: Enc: An NMC against LOCC² encoding a single bit.

Input: A bit $m \in \{0, 1\}$, and an integer n .

- 1: If $m = 0$, then output n EPR pairs $|\Phi^{\otimes n}\rangle$ on registers AB .
 - 2: If $m = 1$,
 - (a) Sample two vectors $a, b \in \mathbb{F}_2^n$ uniformly at random, conditioned on either a or $b \neq 0^n$
 - (b) Apply and output $(X^a \otimes Z^b) |\Phi^{\otimes n}\rangle$.¹⁸
-

If we didn't desire perfect correctness, we could have chosen $\text{Enc}(1)$ to be the maximally mixed state. The decoding algorithm measures each maximally entangled pair in the Bell basis, resulting in $a, b \in \{0, 1\}^n$, and outputs $m = 0$ if $a, b = 0^n$ and 1 otherwise.

6.1 Analysis

Put together, [Claim 6.2](#) and [Claim 6.3](#) argue that no LOCC tampering channel can change an encoding of 1 to one of 0 with non-negligible bias. Following a result by [\[DKO13\]](#) (see [Theorem 2.3](#)) on NMCs for single-bit messages, this will imply the non-malleability of our construction.

Claim 6.2. *The encoding of $m = 1$ is statistically close to the maximally mixed state, $\text{Enc}(1) \approx_{4^{1-n}} 4^{-n} \cdot \mathbb{I} \otimes \mathbb{I}$.*

Proof. If $\text{Enc}(1)$ were a uniformly random Bell basis state, it would be equal to the maximally mixed state. Instead, we sample a, b from a distribution which is $2 \cdot 4^{-n}$ close to the uniform distribution over $\{0, 1\}^{2n}$, which implies the desired bound. □

Claim 6.3. *For every $\Lambda \in \text{LOCC}^2$, $\mathbb{P}[\text{Dec} \circ \Lambda \circ \text{Enc}(1) = 0] \leq 2^{1+e-n}$.*

Together with [Theorem 2.3](#), we obtain [Theorem 6.1](#).

Proof. If $\Lambda \in \text{LOCC}^2$, from [Claim 6.2](#) and [Proposition 2.1](#), $\Lambda \circ \text{Enc}(1)$ remains 4^{1-n} close to a separable state σ . If $\Lambda \in \text{LOCC}_e^2$, then $\Lambda \circ \text{Enc}(1)$ is 4^{1-n} close to a state of Schmidt number 2^e .

However, from [Claim 3.5](#), any state of Schmidt number R separable state has fidelity at most $R \cdot 2^{-n}$ with any maximally entangled state on n pairs of qubits. We conclude

$$\mathbb{P}[\text{Dec} \circ \Lambda \circ \text{Enc}(1) = 0] \approx_{4^{1-n}} \text{Tr}[\Phi^{\otimes n} \sigma] \leq 2^{e-n}. \quad (81)$$

□

¹⁸We remark that for $a \in \mathbb{F}_2^n$, we refer to the n -qubit Pauli operator $X^a = \otimes_{i \in [n]} X^{a_i}$ (respectively Z^a). See [Definition A.1](#).

7 Non-Malleable Codes against LOCC⁴

The main result of this section is a construction of non-malleable codes for classical messages in the 4-split-state model which is secure against arbitrary local operations and classical communication between the parties:

Theorem 7.1. *For every sufficiently large blocklength n , $\lambda = \Omega(\log n) \in \mathbb{N}$, there exists an efficient split-state non-malleable code against LOCC⁴ with rate $\Omega(\lambda^{-2})$, security and correctness error $2^{-\Omega(\min(\lambda, n))}$.*

Our code is based on a compiler which combines (bipartite) LOCC data-hiding schemes and non-malleable codes in the 2-split-state model. We refer the reader to [Section 2.6](#) for formal definitions of data hiding schemes, and to [Section 2.4](#) for basic definitions of non-malleable codes.

In [Section 7.1](#) we present our construction and parameter choices. In [Section 7.2](#) we present our proof of security.

7.1 Code Construction

We combine:

1. $(\text{Enc}_{\text{NM}}, \text{Dec}_{\text{NM}})$, a classical 2-split-state non-malleable code with error ε_{NM} and rate r_{NM} .
2. $(\text{Enc}_{\text{LOCC}}, \text{Dec}_{\text{LOCC}})$, a family of bipartite $(\varepsilon_{\text{LOCC}}, \delta_{\text{LOCC}})$ -LOCC data hiding schemes of rate r_{LOCC} .

Algorithm 9: Enc: A non-malleable code against LOCC⁴ channels.

Input: A k bit message $m \in \{0, 1\}^k$

- 1: Encode m into shares of the classical non-malleable code $(L, R) \leftarrow \text{Enc}_{\text{NM}}(m)$.
- 2: Encode L into two shares L_1, L_2 of the LOCC data hiding scheme, $\text{Enc}_{\text{LOCC}}(L)$.
- 3: Encode R into two shares R_1, R_2 of the LOCC data hiding scheme, $\text{Enc}_{\text{LOCC}}(R)$.
- 4: Output (L_1, L_2, R_1, R_2) as 4 Shares respectively.

$$\text{Enc}(m) = \text{Enc}_{\text{LOCC}}^{\otimes 2} \circ \text{Enc}_{\text{NM}}(m) \tag{82}$$

Lemma 7.2. *Let $(\text{Enc}_{\text{LOCC}}, \text{Dec}_{\text{LOCC}})$ be a 2-out-of-2 $(\varepsilon_{\text{LOCC}}, \delta_{\text{LOCC}})$ -LOCC data hiding scheme and let $(\text{Enc}_{\text{NM}}, \text{Dec}_{\text{NM}})$ be a classical non-malleable code against 2-split-state adversaries with error ε_{NM} . Then $\text{Enc} = \text{Enc}_{\text{LOCC}}^{\otimes 2} \circ \text{Enc}_{\text{NM}}$, $\text{Dec} = \text{Dec}_{\text{NM}} \circ \text{Dec}_{\text{LOCC}}^{\otimes 2}$ is a $(2 \cdot \varepsilon_{\text{LOCC}} + \varepsilon_{\text{NM}}, 2 \cdot \delta_{\text{LOCC}})$ 4-split-state non-malleable code against LOCC channels.*

Moreover, the rate of Enc is the product of the rates of Enc_{LOCC} and Enc_{NM} .

7.1.1 Parameters

We present two instantiations of [Lemma 7.2](#), the first to obtain an efficient code, and the second using existential constructions of the component codes to optimize rate. First, leveraging the constant rate, inverse-exponential error 2 split-state non-malleable code by [\[Li23b\]](#), together with the LOCC data hiding scheme in [Theorem 2.9](#) by [\[EW02\]](#), we obtain:

Theorem 7.3. *For every sufficiently large blocklength n , $\lambda = \Omega(\log n) \in \mathbb{N}$, there exists an efficient split-state non-malleable code against LOCC⁴ with rate $\Omega(\lambda^{-2})$, security and correctness error $2^{-\Omega(\min(\lambda, n))}$.*

Using the (existential) capacity achieving 2-split-state non-malleable codes [Theorem 2.5](#) by [\[CG13a\]](#), together with the rate $1/2 - o(1)$ LOCC data hiding schemes in [Theorem 2.8](#) [\[HLSW03\]](#),

Theorem 7.4. *For every sufficiently large blocklength n , there exists a split-state non-malleable code against LOCC⁴ with rate $1/4 - o(1)$, security and correctness error $2^{-\Omega(n/\log n)}$.*

7.2 Analysis

Note that for any k bit message $m \in \{0, 1\}^k$, the encoding of m is given by a separable state

$$\text{Enc}(m) = \mathbb{E}_{(L,R) \leftarrow \text{Enc}_{\text{NM}}(m)} \text{Enc}_{\text{LOCC}}(L) \otimes \text{Enc}_{\text{LOCC}}(R) \quad (83)$$

where $\text{Enc}_{\text{NM}}(m)$ indicates the joint distribution over left and right shares L, R of the non-malleable code.

Now, suppose we fix $L = l, R = r$ and consider a fixed LOCC “tampering” channel Λ . The classical communication transcript $c \in \{0, 1\}^*$ of the protocol is some classical random variable, which is drawn from some conditional distribution $\mathbb{P}_{c|L=l, R=r}$. We begin by claiming that if the distribution over the transcript was correlated with l, r , then that would break the data-hiding guarantee:

Claim 7.5. *The distribution over the transcript is independent of l, r : $\|\mathbb{P}_{c|L=l, R=r} - \mathbb{P}_c\|_1 \leq 2 \cdot \varepsilon_{\text{LOCC}}$.*

Moreover, since the tampering is performed using a separable channel, the post-tampered density matrices on the left and right splits remain separable and can only depend on the transcript:

Claim 7.6. *For every value of $c \in \{0, 1\}^*$ and shares l, r , there exists a pair of density matrices $\sigma^{c,l}, \sigma^{c,r}$ such that the post-tampered state is given by the mixture*

$$\Lambda(\text{Enc}(m)) = \mathbb{E}_{L,R \leftarrow \text{Enc}_{\text{NM}}(m)} \mathbb{E}_{c \leftarrow \mathbb{P}_{c|L,R}} \left(\sigma^{c,L} \otimes \sigma^{c,R} \right) \quad (84)$$

Put together, we conclude our “non-malleable reduction” from 4-split-state LOCC tampering to 2-split tampering:

Proof. [of Lemma 7.2] From Claim 7.6, we have that after running the decoding channel Dec_{LOCC} , the distribution over tampered shares l', r' forms a Markov Chain, i.e. $\forall l', r', l, r, c$:

$$\mathbb{P}[(L', R') = (l', r') | L = l, R = r, C = c] = \mathbb{P}[L' = l' | L = l, C = c] \cdot \mathbb{P}[R' = r' | R = r, C = c] \quad (85)$$

Moreover, from Claim 7.5, we have the distribution over the transcript is independent of l, r : $\|\mathbb{P}_{c|L=l, R=r} - \mathbb{P}_c\|_1 \leq 2 \cdot \varepsilon_{\text{LOCC}}$. Therefore, the distribution over l', r' conditioned on l, r is approximately the same as that of split state tampering with the transcript acting as shared randomness:

$$\sum_{l', r'} \left| \mathbb{P}[(L', R') = (l', r') | L = l, R = r] - \sum_c \mathbb{P}[L' = l' | L = l, C = c] \cdot \mathbb{P}[R' = r' | R = r, C = c] \cdot \mathbb{P}[C = c] \right| \leq \quad (86)$$

$$\leq \sum_{l', r', c} \mathbb{P}[L' = l' | L = l, C = c] \cdot \mathbb{P}[R' = r' | R = r, C = c] \cdot \left| \mathbb{P}[C = c | L = l, R = r] - \mathbb{P}[C = c] \right| \leq \quad (87)$$

$$\leq \sum_c \left| \mathbb{P}[C = c | L = l, R = r] - \mathbb{P}[C = c] \right| \leq 2 \cdot \varepsilon_{\text{LOCC}} \quad (88)$$

Since the non-malleable code is secure against split state tampering, from Definition 2.12 we have that the distribution over the recovered message M' is approximately a convex combination of the original m with probability p , and an uncorrelated message distribution \mathcal{D} :

$$\text{Dec} \circ \Lambda \circ \text{Enc}(m) \approx_{2\varepsilon_{\text{LOCC}}} \mathbb{E}_{c \leftarrow \mathbb{P}_c} \text{Dec}_{\text{NM}} \circ (\Lambda_1^c \otimes \Lambda_2^c) \circ \text{Enc}(m) \approx_{\varepsilon_{\text{NM}}} p \cdot m + (1 - p) \cdot \mathcal{D} \quad (89)$$

with error $2\varepsilon_{\text{LOCC}} + \varepsilon_{\text{NM}}$. \square

7.2.1 Proofs of Deferred Claims

Proof. [of Claim 7.5] Suppose for the purposes of contradiction, there exists (x, y) such that $\|\mathbb{P}_{c|L=x, R=y} - \mathbb{P}_c\|_1 > 2 \cdot \varepsilon_{\text{LOCC}}$. Since $\mathbb{P}_c = \mathbb{E}_{L, R \leftarrow \text{Enc}_{\text{NM}}(m)} \mathbb{P}_{c|L, R}$, by an averaging argument, there exists x', y' such that

$$\|\mathbb{P}_{c|L=x, R=y} - \mathbb{P}_{c|L=x', R=y'}\|_1 > 2 \cdot \varepsilon_{\text{LOCC}} \quad (90)$$

by the triangle inequality, one of $\|\mathbb{P}_{c|x, y} - \mathbb{P}_{c|x', y}\|_1$ or $\|\mathbb{P}_{c|x', y} - \mathbb{P}_{c|x', y'}\|_1$ is $\geq \varepsilon_{\text{LOCC}}$, WLOG the first of the two. Now, we argue that this violates the indistinguishability of the data hiding scheme. Crucially, since $R = y$ is fixed, one of the two parties holding the encoding of L could generate the encoding of R by themselves, and simulate the LOCC protocol which discriminates (x, y) from (x', y) . Thereby,

$$\varepsilon_{\text{LOCC}} < \|\mathbb{P}_{c|x, y} - \mathbb{P}_{c|x', y}\|_1 \leq \|(\text{Enc}_{\text{LOCC}}(x) - \text{Enc}_{\text{LOCC}}(x'))\|_{\text{LOCC}}, \quad (91)$$

which by definition violates the data hiding property. \square

Proof. [of Claim 7.6] Recall the density matrix describing the encoding of a message m is

$$\text{Enc}(m) = \mathbb{E}_{L, R \leftarrow \text{Enc}_{\text{NM}}(m)} \text{Enc}_{\text{LOCC}}(L) \otimes \text{Enc}_{\text{LOCC}}(R) \quad (92)$$

Fix L, R and a communication transcript $c \in \{0, 1\}^*$. Since the tampering is performed using only LOCC operations, the post-tampered states must still be separable across the left-right partition. Thereby, WLOG, the post-tampered states are described by CP maps $\mathcal{E}^{L, c}, \mathcal{E}^{R, c}$,

$$\Lambda \circ \text{Enc}(m) = \sum_c \mathbb{E}_{L, R \leftarrow \text{Enc}_{\text{NM}}(m)} \left(\mathcal{E}^{L, c} \circ \text{Enc}_{\text{LOCC}}(L) \right) \otimes \left(\mathcal{E}^{R, c} \circ \text{Enc}_{\text{LOCC}}(R) \right). \quad (93)$$

If we denote the real numbers $q_{L, c} = \text{Tr } \mathcal{E}^{L, c} \circ \text{Enc}_{\text{LOCC}}(L)$, $q_{R, c} = \text{Tr } \mathcal{E}^{R, c} \circ \text{Enc}_{\text{LOCC}}(R)$, and corresponding density matrices

$$\sigma^{L, c} = \mathcal{E}^{L, c} \circ \text{Enc}_{\text{LOCC}}(L) / q_{L, c} \text{ and } \sigma^{R, c} = \mathcal{E}^{R, c} \circ \text{Enc}_{\text{LOCC}}(R) / q_{R, c}, \quad (94)$$

Then the probability the communication transcript is c is $\mathbb{P}_{c|L, R} = q_{L, c} \cdot q_{R, c}$, and the resulting post-tampered state is precisely as in the statement of Claim 7.6. \square

8 Non-Malleable Secret Sharing Schemes against LOCC

The main result of this section is a construction of a threshold secret sharing scheme for classical messages which is both data hiding and non-malleable against local operations and classical communication between the parties. For basic definitions of data hiding and non-malleable secret sharing, we refer the reader to [Section 2.6](#) and [Section 2.5.3](#) respectively.

Theorem 8.1 ([Theorem 1.3](#), restatement). *For every number of parties p , threshold $t \geq 3$ and message length $k = \Omega(\log p)$, there exists a t -out-of- p threshold secret sharing scheme which is non-malleable against LOCC with reconstruction, privacy and non-malleability error $2^{-(kp)^{\Omega(1)}}$, as well as inverse-polynomial rate $p^{-O(1)}$.*

Our construction is inspired by a compiler from [\[ADN⁺19\]](#), who constructed leakage-resilient and non-malleable secret sharing schemes from generic secret sharing schemes. In this section, we show how to combine bipartite LOCC data hiding schemes with non-malleable secret sharing schemes within their compiler, to construct secret sharing schemes which are non-malleable against LOCC.

We organize the rest of this section as follows. In [Section 8.1](#) we present the ingredients and overview our code construction. In [Section 8.2.1](#) we prove its data hiding property, and in [Section 8.2.2](#) we present the proof of non-malleability.

8.1 Code Construction

8.1.1 Ingredients

We refer the reader to [Section 2.6](#) and [Section 2.5.3](#) for basic definitions of the following ingredients:

1. An $(\varepsilon_{\text{LOCC}}, \delta_{\text{LOCC}})$ -2-out-of-2 data hiding scheme $(\text{Enc}_{\text{LOCC}}, \text{Dec}_{\text{LOCC}})$ with separable state encodings, such as that of [Theorem 2.9](#) from [\[EW02\]](#).
2. A $(p, t, \varepsilon_{\text{priv}}, \varepsilon_c, \varepsilon_{\text{NM}})$ -non-malleable secret sharing scheme $(\text{Enc}_{\text{SS}}, \text{Dec}_{\text{SS}})$ which is non-malleable against $\mathcal{F}_{p,t}^{\text{joint}}$, such as that of [Theorem 2.7](#) from [\[GK18a\]](#).

As previously discussed in the introduction, we require non-malleable secret sharing schemes which are secure against a (relatively mild) joint tampering model. However, we appeal to constructions secure against $\mathcal{F}_{p,t}^{\text{joint}}$ ([Definition 2.21](#)) for simplicity, as it is the most well studied joint tampering model.

8.1.2 Overview

We formally present the encoding channel Enc in [Algorithm 10](#) and decoding channel [Algorithm 11](#). An outline follows: First, a message $m \in \{0, 1\}^k$ is encoded into classical shares (M_1, \dots, M_p) , using a classical non-malleable secret sharing scheme. Then, each party $i \in [p]$ creates $(p - 1)$ copies of their share M_i and encodes each of them into the LOCC data hiding scheme $\text{Enc}_{\text{LOCC}}(M_i)$. This results in $p - 1$ bipartite registers $(X_{i,j}, Y_{i,j})_{j \in [p] \setminus \{i\}}$ for each i , and the register $Y_{i,j}$ is handed to the j th party (while i keeps all of the $X_{i,j}$).

Algorithm 10: Enc:

Input: A k bit message $m \in \{0, 1\}^k$, and an integer λ .

Output: A code-state defined on p registers S_1, \dots, S_p

- 1: Encode m into a secret sharing scheme $(M_1, \dots, M_p) \leftarrow \text{Enc}_{\text{SS}}(m)$.
- 2: For each $i \in [p]$, produce $p - 1$ copies $\text{Enc}_{\text{LOCC}}(M_i)$ of a 2-out-of-2 data hiding scheme encoding of the i th share. For $i \in [p], j \in [p] \setminus \{i\}$, let $X_{i,j}, Y_{i,j}$ be the corresponding bipartite register.
- 3: For each $i \in [p]$, define the i th share S_i to be the concatenation of quantum registers

$$S_i = \left\{ X_{i,j} \text{ and } Y_{j,i} : j \in [p] \setminus \{i\} \right\} \quad (95)$$

Algorithm 11: Dec:

Input: A subset $T \subset [p]$, $|T| \geq t$ and a density matrix ρ_T supported on S_T .

Output: A classical k bit message m' .

- 1: Partition $T = T_1 \cup T_2 \cup \dots \cup T_{\lfloor \frac{|T|}{2} \rfloor}$ into pairs or triples of vertices.
- 2: For each subset T_j , decode the data hiding schemes within T_j by applying Dec_{LOCC} on all registers $X_{u,v}, Y_{u,v}$ for pairs $u, v \in T_j$ until every share in T_j has been decoded. Let M'_u for $u \in T$ be the recovered classical shares.
- 3: Decode shares $\{M'_u\}_{u \in T}$ using Dec_{SS} .

Due to the redundancy in the encoding, if any two parties u, v had access to quantum communication, then they could jointly learn their shares M_u, M_v . From the threshold secrecy of the secret sharing scheme, if only $T \subset [p], |T| \leq t - 1$ parties has access to quantum communication, then naturally they would learn a negligible amount of information about the message. We begin by arguing that **Enc** above actually has an even stronger data hiding property: it is a data hiding scheme with a threshold access structure ([Definition 2.27](#)). In [Lemma 8.2](#), we prove that even if the parties in $([p] \setminus T)$ could aid T by means of local operations and unbounded classical communication, T would still not be able to distinguish the message.

Lemma 8.2. *Enc defined in [Algorithm 10](#) defines a $(p, t, \varepsilon'_c, \varepsilon'_{\text{priv}})$ -secret sharing scheme which is LOCC data hiding, with reconstruction error $\varepsilon'_c = p \cdot \delta_{\text{LOCC}} + \varepsilon_c$ and privacy error $\varepsilon'_{\text{priv}} = O(p^2 \cdot \varepsilon_{\text{LOCC}} + \varepsilon_{\text{priv}})$. Moreover, if the share size of Enc_{LOCC} is s_{LOCC} , then the share size of the resulting construction is $2 \cdot s_{\text{LOCC}} \cdot p$.*

Note that this “data hiding” or leakage resilience (against classical communication) follows directly from the privacy property of the underlying classical secret sharing scheme, and doesn’t require any non-malleability.

Lemma 8.3. *(Enc, Dec) defined in [Algorithm 10](#), [Algorithm 11](#) are non-malleable with respect to LOCC_p with error $\varepsilon'_{\text{NM}} = O(\varepsilon_{\text{NM}} + \varepsilon_{\text{LOCC}} \cdot p^2)$.*

8.1.3 Parameters

By instantiating our compiler with

1. The efficient, separable-state $(2^{-\Omega(\lambda)}, 2^{-\Omega(\lambda)})$ bipartite data hiding scheme [Theorem 2.9 \[EW02\]](#) message length b and blocklength $2b \cdot \lambda^2$.
2. The $(p, t, 0, 0, 2^{-b^{\Omega(1)}})$ -NMSS against $\mathcal{F}_{p,t}^{\text{joint}}$ of share size b and rate $p^{-O(1)}$ guaranteed by [Theorem 2.7 \[GK18a\]](#).

If we fix a message length k , then $b = k \cdot p^{O(1)}$, and we can further fix $\lambda = b^{\Theta(1)}$. We obtain

Theorem 8.4. *For every number of parties p , threshold $t \geq 3$ and message length $k = \Omega(\log p)$, there exists a t -out-of- p threshold secret sharing scheme which is non-malleable against LOCC^p with reconstruction, privacy and non-malleability error $2^{-(kp)^{\Omega(1)}}$, as well as inverse-polynomial rate $p^{-O(1)}$.*

8.2 Analysis

8.2.1 Correctness and Privacy

In the below, we refer to LOCC^p as the set of measurements performed by LOCC on p parties.

Claim 8.5 (Correctness). *Suppose $\text{Dec}_{\text{LOCC}}, \text{Dec}_{\text{SS}}$ have reconstruction error $\delta_{\text{LOCC}}, \varepsilon_c$ respectively. Then the reconstruction error for Dec is $\leq p \cdot \delta_{\text{LOCC}} + \varepsilon_c$.*

Proof. For a fixed access structure and a valid subset of parties $T \subset [p]$, the parties in T first attempt to recover their classical shares, and then use these shares to decode the classical secret sharing scheme. The reconstruction succeeds if the p Dec_{LOCC} protocols and the Dec_{SS} both succeed, which occurs with probability $\geq 1 - p \cdot \delta_{\text{LOCC}} - \varepsilon_c$. \square

Fix a subset $T \subset [p]$, and let LOCC_T be the tampering channels corresponding to arbitrary quantum communication within T while the parties in $[p] \setminus T$ are restricted to LOCC operations. Suppose now we fix two messages $m \neq \tilde{m}$. We prove privacy against measurements in LOCC_T , $|T| \leq t - 1$ by considering the following hybrids:

Hyb₀: The message m is encoded into the code, $\text{Enc}(m)$.

Hyb₁: The message m is encoded into the code, however, all registers $(X_{i,j}, Y_{i,j})$ when either of $i, j \notin T$, are replaced by encodings of a fixed string $\text{Enc}_{\text{LOCC}}(x)$.

Hyb₂: The message \tilde{m} is encoded into the code, however, all registers $(X_{i,j}, Y_{i,j})$ when either of $i, j \notin T$, are replaced by encodings of a fixed string $\text{Enc}_{\text{LOCC}}(x)$.

Hyb₃: The message \tilde{m} is encoded into the code, $\text{Enc}(\tilde{m})$.

Let ρ_{Hyb_i} be the density matrix on the p shares during Hybrid i . Our first claim argues that using LOCC between $T, [p] \setminus T$ cannot help to distinguish messages:

Claim 8.6. $\|\rho_{\text{Hyb}_0} - \rho_{\text{Hyb}_1}\|_{\text{LOCC}_T}, \|\rho_{\text{Hyb}_2} - \rho_{\text{Hyb}_3}\|_{\text{LOCC}_T} \leq p^2 \cdot \varepsilon_{\text{LOCC}}$.

Together with the statistical privacy of the secret sharing scheme, we prove statistical privacy against LOCC_T :

Claim 8.7 (Privacy). *For any two messages $m \neq \tilde{m}$ and every $T \subset [p], |T| \leq t - 1$, the privacy error*

$$\|\text{Enc}(m) - \text{Enc}(\tilde{m})\|_{\text{LOCC}_T} \leq 2 \cdot p^2 \cdot \varepsilon_{\text{LOCC}} + \varepsilon_{\text{priv}}$$

Proof. [of Claim 8.7] Fix $|T| \leq t - 1$. Since $\text{Enc}_{\text{LOCC}}(x)$ is separable, both $\rho_{\text{Hyb}_0}, \rho_{\text{Hyb}_1}$ can be written as tensor products $\rho_T^m \otimes \sigma, \rho_T^{\tilde{m}} \otimes \sigma$ for some fixed density matrix σ , and where ρ_T^m only depends of the shares in $\text{Enc}_{\text{SS}}(m)_T$ (similarly for \tilde{m}). Thus, by the statistical privacy of the secret sharing scheme,

$$\|\rho_{\text{Hyb}_1} - \rho_{\text{Hyb}_2}\|_{\text{LOCC}_T} = \|(\rho_T^m - \rho_T^{\tilde{m}}) \otimes \sigma\|_{\text{LOCC}_T} \leq \|\text{Enc}_{\text{SS}}(m)_T - \text{Enc}_{\text{SS}}(\tilde{m})_T\|_1 \leq \varepsilon_{\text{priv}}. \quad (96)$$

From Claim 8.6 and the triangle inequality,

$$\begin{aligned} \|\text{Enc}(m_a) - \text{Enc}(m_b)\|_{\text{LOCC}_T} &\leq \|\rho_{\text{Hyb}_0} - \rho_{\text{Hyb}_1}\|_{\text{LOCC}_T} + \|\rho_{\text{Hyb}_1} - \rho_{\text{Hyb}_2}\|_{\text{LOCC}_T} + \|\rho_{\text{Hyb}_2} - \rho_{\text{Hyb}_3}\|_{\text{LOCC}_T} \leq (97) \\ &\leq 2p^2 \cdot \varepsilon_{\text{LOCC}} + \varepsilon_{\text{priv}}. \quad \square \end{aligned}$$

Proof. [of Claim 8.6] It suffices to prove the hybrid $\text{Hyb}_0, \text{Hyb}_1$ are indistinguishable for all messages m , as the case $\text{Hyb}_2, \text{Hyb}_3$ is analogous. To do so, we introduce another sequence of hybrids, which replaces each data hiding scheme $(X_{u,v}, Y_{u,v})$ where either $u, v \notin T$ one by one.

Suppose we define an ordering $1, \dots, p$ to the vertices in $[p]$, and an ordering $1, \dots, p - |T|$ to the vertices of $[p] \setminus T$. For every $a \in \{1, \dots, p\}, b \in \{1, \dots, p - |T|\}$, we define:

$\text{Hyb}_{a,b}$: The message m is encoded into the code, however, if i is in the first $(a - 1)$ vertices of $[p]$ then we replace the registers $(X_{i,j}, Y_{i,j})$ with the fixed state $\text{Enc}_{\text{LOCC}}(x)$ for each $j \in [p] \setminus T$. Additionally, for $j \in \{1, \dots, b\}$ the registers $(X_{a,j}, Y_{a,j})$ are replaced with $\text{Enc}_{\text{LOCC}}(x)$.

We denote $\text{Hyb}_{a+1,0}$ by the hybrid which has already replaced all the data hiding schemes encoding s_i (where $i \leq a$) held by parties outside T by encodings of x . Note that $\text{Hyb}_{a,p-|T|} = \text{Hyb}_{a+1,0}$. By a triangle inequality,

$$\|\rho_{\text{Hyb}_0} - \rho_{\text{Hyb}_1}\|_{\text{LOCC}_T} \leq p^2 \cdot \max_{a \in \{1, \dots, p\}} \max_{b \in \{0, 1, \dots, p - |T| - 1\}} \|\rho_{\text{Hyb}_{a,b}} - \rho_{\text{Hyb}_{a,b+1}}\|_{\text{LOCC}_T} \quad (98)$$

However, $\rho_{\text{Hyb}_{a,b}}, \rho_{\text{Hyb}_{a,b+1}}$ differ only in the bipartite register $(X_{a,u_{b+1}}, Y_{a,u_{b+1}})$, where a is the a th vertex of $[p]$ and u_{b+1} is the $(b + 1)$ st neighbor of a outside T . Thus, there is some separable state σ such that $\rho_{\text{Hyb}_{a,b}} = \text{Enc}_{\text{LOCC}}(M_a) \otimes \sigma, \rho_{\text{Hyb}_{a,b+1}} = \text{Enc}_{\text{LOCC}}(x) \otimes \sigma$. By an averaging argument,

$$\begin{aligned} \|\rho_{\text{Hyb}_{a,b}} - \rho_{\text{Hyb}_{a,b+1}}\|_{\text{LOCC}_T} &= \|(\text{Enc}_{\text{LOCC}}(M_a) - \text{Enc}_{\text{LOCC}}(x)) \otimes \sigma\|_{\text{LOCC}_T} \leq \quad (99) \\ &\leq \max_{\otimes_u^p \psi_u} \|(\text{Enc}_{\text{LOCC}}(M_a) - \text{Enc}_{\text{LOCC}}(x)) \otimes_u^p \psi_u\|_{\text{LOCC}_T} = \|\text{Enc}_{\text{LOCC}}(M_a) - \text{Enc}_{\text{LOCC}}(x)\|_{\text{LOCC}} \leq \varepsilon_{\text{LOCC}}, \quad (100) \end{aligned}$$

where finally we used that vertices a, u_{b+1} can distinguish between $\text{Enc}_{\text{LOCC}}(M_a), \text{Enc}_{\text{LOCC}}(x)$ with a bipartite LOCC protocol with the same bias as any LOCC_T protocol for $\text{Enc}_{\text{LOCC}}(M_a) \otimes \sigma, \text{Enc}_{\text{LOCC}}(x) \otimes_u^p \psi_u$, by preparing $\otimes_u^p \psi_u$ using LOCC and simulating the LOCC_T protocol. \square

8.2.2 Non-Malleability

Our proof proceeds via a non-malleable reduction, from individual tampering on the quantum secret sharing scheme LOCC^p to joint tampering $\mathcal{F}_{p,t}^{\text{joint}}$ on a classical non-malleable secret sharing scheme. To do so, we reason that the distribution over the recovered shares $M'_T = (M'_u, u \in T)$, during the decoding algorithm in Algorithm 11 and conditioned on the shares before tampering $M_T = (M_u, u \in T)$, is near a convex combination over joint tampering functions $\mathcal{F}_{p,t}^{\text{joint}}$ on S_T . We prove:

Claim 8.8 (Non-Malleability). *If Enc_{LOCC} has privacy error $\varepsilon_{\text{LOCC}}$ and $(\text{Enc}_{\text{SS}}, \text{Dec}_{\text{SS}})$ is non-malleable against $\mathcal{F}_{p,t}^{\text{joint}}$ with error ε_{NM} , then (Enc, Dec) is non-malleable against LOCC^p with error $\leq 3 \cdot p^2 \cdot \varepsilon_{\text{LOCC}} + \varepsilon_{\text{NM}}$.*

To prove Claim 8.8, we begin similarly to the proof of privacy and establish a sequence of hybrids. Our goal will be to start with the encoding of a fixed message $\text{Enc}(m)$, and then to (gradually) replace all the data hiding schemes which are between parties not in the same partition in \mathcal{P} , by encodings of a fixed string $\text{Enc}_{\text{LOCC}}(x)$. Let $N_{\text{pairs}} = \sum_{i \neq j \in \mathcal{P}} |P_i| |P_j| + |T|(p - |T|)$ denote the number of bipartite data hiding schemes between different partitions of $[p]$. We define

Hyb_0 : The message m is encoded into the code, $\text{Enc}(m)$.

For every integer $0 \leq a \leq N_{\text{pairs}}$,

Hyb_a : The message m is encoded into the code, however, the first a data hiding schemes $(X_{i,j}, Y_{i,j})$ where i, j are in different sides of the $(T, [p] \setminus T)$ cut or in different partitions of T in \mathcal{P} , are replaced by encodings of a fixed string $\text{Enc}_{\text{LOCC}}(x)$.

$\text{Hyb}_{\text{Final}}$: The message m is encoded into the code, however, all registers $(X_{i,j}, Y_{i,j})$ where either of i, j are in different partitions of T in \mathcal{P} , are replaced by encodings of a fixed string $\text{Enc}_{\text{LOCC}}(x)$.

Recall that the random variable M'_T is the collection of recovered classical shares of the subset T (after tampering). If we denote $\mathbb{P}_{M'_T|M_T=m_T, m}$ the distribution over S'_T conditioned on the shares S_T before tampering (and the message), then clearly it is captured by the 0th Hybrid $\mathbb{P}_{M'_T|M_T=m_T, m} = \mathbb{P}_{M'_T|M_T=m_T, m}^{\text{Hyb}_0}$. Since Hybrids Hyb_0 and $\text{Hyb}_{\text{Final}}$ differ only in separable state data hiding schemes, then following the proof of privacy in [Claim 8.6](#), we reason that their distributions over recovered shares are near indistinguishable. As otherwise, their distinguishability could be turned into a bipartite LOCC protocol which distinguishes a data-hidden share from the fixed string $\text{Enc}_{\text{LOCC}}(x)$:

Claim 8.9. For all m, m_T , $\|\mathbb{P}_{M'_T|M_T=m_T, m}^{\text{Hyb}_0} - \mathbb{P}_{M'_T|M_T=m_T, m}^{\text{Hyb}_{\text{Final}}}\|_1 \leq p^2 \cdot \varepsilon_{\text{LOCC}}$

Now, note that once one conditions on the shares m_T , the density matrix in the final Hybrid $\text{Hyb}_{\text{Final}}$ can be expressed as a product state across the partitions of \mathcal{P} (which depend on the shares s_T), and a global separable state:

$$\rho^{\text{Hyb}_{\text{Final}}}|_{M_T=m_T} = \gamma_{m_T} \otimes \sigma = \otimes_{j \in \mathcal{P}} \gamma_{m_{P_j}} \otimes \sigma. \quad (101)$$

γ_{m_T} is supported on the bipartite registers $(X_{u,v}, Y_{u,v})$ where u, v both are in the same partition of \mathcal{P} , and thereby factorizes into a product state across different partitions. $\sigma = \mathbb{E}_{z \leftarrow Z} \psi^z$, where the ψ^z are product states, represents all the copies of the fixed state $\text{Enc}_{\text{LOCC}}(x)$.

Suppose we fix $m_T, Z = z$, and the LOCC^p tampering channel Λ . If the random variable $C \in \{0, 1\}^*$ denotes the classical transcript of the protocol, then let $\mathcal{D}_{|M_T=m_T, Z=z}^\Lambda$ be the conditional distribution over the transcript. We begin by claiming that, conditioned on M_T, Z and the transcript C , the post-tampered density matrix remains a tensor product state across the partition \mathcal{P} :

Claim 8.10. For all $M_T = m_T, Z = z, C = c$, there exist density matrices $\sigma_{m_P}^{c,z}$ for each partition $P \in \mathcal{P}$ and $\sigma_u^{c,z}$ for $u \in [p] \setminus T$ such that the post-tampered state

$$\Lambda(\rho^{\text{Hyb}_{\text{Final}}}|_{M_T=m_T, Z=z}) = \mathbb{E}_{c \leftarrow \mathcal{D}_{|M_T=m_T, Z=z}} \left(\bigotimes_{P \in \mathcal{P}} \sigma_{m_P}^{c,z} \bigotimes_{u \in [p] \setminus T} \sigma_u^{c,z} \right) \quad (102)$$

Moreover, we claim the distribution over the transcript $\mathcal{D}_{|M_T=m_T, Z=z}$ is almost independent of the shares, as otherwise one could break the bipartite security of the data hiding scheme:

Claim 8.11. For all $\Lambda \in \text{LOCC}^p$ and for every fixed value $Z = z, M_T = m_T$: $\|\mathcal{D}_{|M_T=m_T, Z=z}^\Lambda - \mathcal{D}_{Z=z}^\Lambda\|_1 \leq (p+1) \cdot \varepsilon_{\text{LOCC}}$.

We can now put everything together and prove the main claim of this subsection:

Proof. [of [Claim 8.8](#)] From [Claim 8.11](#) and [Claim 8.10](#)

$$\left\| \Lambda(\rho^{\text{Hyb}_{\text{Final}}}|_{M_T=m_T, Z=z}) - \mathbb{E}_z \mathbb{E}_{c \leftarrow \mathcal{D}_z^\Lambda} \left(\bigotimes_{P \in \mathcal{P}} \sigma_{m_P}^{c,z} \otimes \sigma_{[p] \setminus T}^{c,z} \right) \right\|_1 \leq (p+1) \cdot \varepsilon_{\text{LOCC}} \quad (103)$$

By monotonicity of trace distance, the fact our decoding algorithm only acts only within the same partition of \mathcal{P} ,

$$\left\| \mathbb{P}_{M'_T|M_T=m_T, m}^{\text{Hyb}_{\text{Final}}} - \mathbb{E}_z \mathbb{E}_{c \leftarrow \mathcal{D}_z^\Lambda} \prod_{j \in \mathcal{P}} \mathbb{P}_{M'_P|M_P=m_P, c, z}^{\text{Hyb}_{\text{Final}}} \right\|_1 \leq 2 \cdot p^2 \cdot \varepsilon_{\text{LOCC}} \quad (104)$$

Thereby, the distribution over the tampered shares in $\text{Hyb}_{\text{Final}}$ is near that of a convex combination of functions in $\mathcal{F}_{p,t}^{\text{joint}}$. However, via [Claim 8.9](#), the distribution over the tampered shares in Hyb_0 must

also be near a convex combination $\mathcal{F}_{p,t}^{joint}$. By the non-malleability of the classical secret sharing scheme and the triangle inequality, we conclude the distribution over recovered messages M' in $\text{Hyb}_{\text{Final}}$ must be $\leq \varepsilon_{\text{NM}} + 3 \cdot p^2 \cdot \varepsilon_{\text{LOCC}}$ close in statistical distance to a convex combination of the original message or an uncorrelated message. \square

8.2.3 Deferred Proofs

Proof. [of Claim 8.9] Consider the hybrids $\text{Hyb}_a : a \in [N_{\text{pairs}}]$, which replace all the data hiding schemes outside T by encodings of the fixed message x , one by one. By the triangle inequality, there exists a^* such that

$$\|\mathbb{P}_{M'_T | M_T=m_T, m}^{\text{Hyb}_0} - \mathbb{P}_{M'_T | M_T=m_T, m}^{\text{Hyb}_{\text{Final}}}\|_1 \leq p^2 \cdot \|\mathbb{P}_{M'_T | M_T=m_T, m}^{\text{Hyb}_{a^*}} - \mathbb{P}_{M'_T | M_T=m_T, m}^{\text{Hyb}_{a^*+1}}\|_1 \quad (105)$$

However, Hyb_{a^*} and Hyb_{a^*+1} differ only in a single bipartite register $(X_{u,v}, Y_{u,v})$, and all the remaining registers in Hyb_{a^*} and Hyb_{a^*+1} are fixed to a separable density matrix σ_{a^*} . Thereby, since u, v are in different partitions of T , parties u, v are able to distinguish between encodings of the share and the fixed string $\text{Enc}_{\text{LOCC}}(m_u), \text{Enc}_{\text{LOCC}}(x)$ using a bipartite LOCC protocol which simply simulates the decoding algorithm in Algorithm 11 to obtain the post-tampered shares M'_T . If the conditional distribution on M'_T is distinguishable, then so is $\text{Enc}_{\text{LOCC}}(M_u), \text{Enc}_{\text{LOCC}}(x)$:

$$\|\mathbb{P}_{M'_T | M_T=m_T, m}^{\text{Hyb}_{a^*}} - \mathbb{P}_{M'_T | M_T=m_T, m}^{\text{Hyb}_{a^*+1}}\|_1 \leq \|(\text{Enc}_{\text{LOCC}}(m_u) - \text{Enc}_{\text{LOCC}}(x))_{u,v} \otimes \sigma_{a^*}\|_{\text{LOCC}_{\mathcal{P}}} = \quad (106)$$

$$= \|\text{Enc}_{\text{LOCC}}(m_u) - \text{Enc}_{\text{LOCC}}(x)\|_{\text{LOCC}} \leq \varepsilon_{\text{LOCC}}, \quad (107)$$

where in the above we referred to $\text{LOCC}_{\mathcal{P}}$ as the class of channels implementable via quantum communication within partitions of \mathcal{P} and LOCC between partitions and out of T . \square

Proof. [of Claim 8.10] Recall that LOCC^P tampering channels act as collections of CP maps Λ_u^c on each $u \in [p]$, which depend only on Λ and the *transcript* $c \in \{0, 1\}^*$ of the classical communication in the protocol. The post-tampered state can thereby be expressed as a mixture over CP maps on γ_{m_T} :

$$\Lambda(\rho^{\text{Hyb}_{\text{Final}}} |_{S_T=s_t, Z=z}) = \sum_c \left(\bigotimes_{u \in [p]} \Lambda_u^c \right) \circ (\rho^{\text{Hyb}_{\text{Final}}} |_{M_T=m_T, Z=z}) = \quad (108)$$

$$= \sum_c \left(\bigotimes_{u \in [p]} \Lambda_u^c \right) \circ (\gamma_{m_T} \otimes \psi_z) = \sum_c \left(\bigotimes_{P \in \mathcal{P}} \tilde{\sigma}_{m_P}^{c,z} \bigotimes_{u \in [p] \setminus T} \tilde{\sigma}_u^{c,z} \right). \quad (109)$$

Where each $\tilde{\sigma}$ is a PSD matrix (or an unnormalized density matrix), and note we used the fact that γ_{m_T} factorizes across \mathcal{P} . Moreover, on average over c , Λ^c must be trace preserving: $\sum_c \text{Tr} \left(\bigotimes_{u \in [p]} \Lambda_u^c \right) (A) = \text{Tr} A$. By normalizing each $\tilde{\sigma}$ by its trace, we can treat the transcript c as a random variable with conditional distribution $\mathcal{D}_{|M_T=m_T, Z=z}(c) = \prod_{P \in \mathcal{P}} \text{Tr} \tilde{\sigma}_{m_P}^{c,z} \prod_{u \in [p] \setminus T} \text{Tr} \tilde{\sigma}_u^{c,z}$, such that:

$$\Lambda(\rho^{\text{Hyb}_{\text{Final}}} |_{M_T=m_T, Z=z}) = \mathbb{E}_{c \leftarrow \mathcal{D}_{|M_T=m_T, Z=z}} \left(\bigotimes_{P \in \mathcal{P}} \sigma_{m_P}^{c,z} \bigotimes_{u \in [p] \setminus T} \sigma_u^{c,z} \right) \quad (110)$$

\square

Proof. [of Claim 8.11] Following the now standard hybrid argument and triangle inequality, suppose two conditional distributions over transcripts $\mathcal{D}_{|M_T=a, Z=z}, \mathcal{D}_{|M_T=b, Z=z}$ for two different sets of shares $M_T = a, b$ were distinguishable. Then, there is a pair of parties $u, v \in T$ within the same subset in the partition of T and a fixed separable state σ , which is distinguishable with comparable bias using LOCC^P :

$$\|\mathcal{D}_{|M_T=a, Z=z} - \mathcal{D}_{|M_T=b, Z=z}\|_1 \leq (p+1) \cdot \|(\text{Enc}_{\text{LOCC}}(a_u) - \text{Enc}_{\text{LOCC}}(b_u))_{u,v} \otimes \sigma\|_{\text{LOCC}^P} = \quad (111)$$

$$= (p+1) \cdot \|\text{Enc}_{\text{LOCC}}(a_u) - \text{Enc}_{\text{LOCC}}(b_u)\|_{\text{LOCC}} \leq (p+1) \cdot \varepsilon_{\text{LOCC}}, \quad (112)$$

where we note the LOCC^p protocol distinguishing $(\text{Enc}_{\text{LOCC}}(a_u) - \text{Enc}_{\text{LOCC}}(b_u))_{u,v} \otimes \sigma$ can be simulated by a bipartite protocol between u, v . \square

9 Connections to Quantum Encryption

A remarkable property of quantum authentication schemes [BCG⁺01] is that they encrypt the quantum message¹⁹. Since quantum non-malleable and tamper-detection codes are relaxations of quantum authentication codes, it may not seem too surprising that they inherit similar properties. In this section, we show they satisfy a related notion of encryption:

Definition 9.1. *We refer to a quantum code Enc on t registers as single share encrypting with error δ if the reduced density matrix on each register is independent of the message: $\forall \psi_0, \psi_1$ and $i \in [t]$,*

$$\| \text{Tr}_{-i} \text{Enc}(\psi_0) - \text{Tr}_{-i} \text{Enc}(\psi_1) \|_1 \leq \delta \quad (113)$$

We prove that each share of a split-state quantum tamper-detection code must be encrypted:

Theorem 9.1 (Theorem 1.5, restatement). *Any t -split quantum tamper-detection code against LO^t with error ε must be single share encrypting with error $\Delta \leq 4 \cdot \varepsilon^{1/2}$.*

As previously discussed, classical non-malleable codes with 3 or more shares do *not* satisfy single share encryption. However, due to limitations in our proof approach in Theorem 9.1, it still remains open whether quantum non-malleable codes do. Nevertheless, by combining Theorem 9.1 with our reduction from Section 3, we show that one can always easily turn a quantum non-malleable code into one which encrypts its shares, with arbitrarily small changes to its rate and error:

Corollary 9.2. *Let (Enc, Dec) be a t -split quantum non-malleable code against LO^t with error ε and message length k . For $0 < \lambda < k$, consider the quantum code $(\text{Enc}_\lambda, \text{Dec}_\lambda)$ of message length $k - \lambda$ defined by encoding ψ together with λ random bits into Enc . Then, $(\text{Enc}_\lambda, \text{Dec}_\lambda)$ is a quantum non-malleable code with error ε , and is single share encrypting with error $4 \cdot \sqrt{\varepsilon + 2^{1-\lambda}}$.*

In a nutshell, Enc_λ simply “pads” the message state using λ random bits, which are ignored during decoding. To conclude this section, our last connection to encryption is an analog to a well known classical result by [ADKO15] for 2-split-state non-malleable codes. We prove that quantum non-malleable codes against entangled 2 split-state adversaries must always be 2-out-of-2 secret sharing:

Lemma 9.3. *Fix $\varepsilon \leq 1/16$. Any ε -secure quantum non-malleable code against entangled 2-split-state adversaries (Enc, Dec) is single share encrypting with privacy error $\delta \leq 32 \cdot \varepsilon$.*

We emphasize that this is not implied by Corollary 9.2, which requires padding the input. Our proof approach is similar to that of [ADKO15]. Informally, we reason that if two code-states $|a\rangle, |b\rangle$ are distinguishable by acting only on a subset of the qubits of the code-state, then we will be able to swap between encodings of $|a\rangle, |b\rangle$ with some non-trivial bias. We dedicate Section 9.2 to its proof.

9.1 t -Split Tamper-Detection Codes Encrypt their Shares

To prove Theorem 9.1, we follow a proof approach is analogous to that of [BCG⁺01]. At a high level, we leverage an equivalence between distinguishing two states $|a\rangle, |b\rangle$ (even with some tiny bias) and mapping $\frac{1}{\sqrt{2}}(|a\rangle + |b\rangle)$ to $\frac{1}{\sqrt{2}}(|a\rangle - |b\rangle)$. We make fundamental use of a lemma by [GJMZ22] (and shown similarly in [AAS20]):

Lemma 9.4 ([GJMZ22], Lemma 6.6.ii). *Suppose a distinguisher D implemented using a binary projective measurement $(\Pi, \mathbb{I} - \Pi)$ distinguishes $|x\rangle, |y\rangle$ with bias Δ , i.e. $|\langle x | \Pi | x \rangle - \langle y | \Pi | y \rangle| \geq \Delta$. Then, the unitary (reflection) $U = \mathbb{I} - 2\Pi$ maps the between the following superpositions with fidelity bias:*

$$\left| \left(\frac{|x\rangle - |y\rangle}{\sqrt{2}} \right) U \left(\frac{|x\rangle + |y\rangle}{\sqrt{2}} \right) \right| \geq \Delta \quad (114)$$

¹⁹That is, any adversary which is oblivious to the internal randomness shared by the encoder & decoder, can learn nothing about the encoded state. This naturally is in contrast to classical authentication, where oftentimes a short authentication tag suffices, leaving the message “in the clear”.

As a consequence, we reason that if one of the t adversaries is able to distinguish information about the message using just *their share*, then they can also map between messages with some non-negligible advantage.

Proof. [of [Theorem 9.1](#)] For the purpose of contradiction, suppose two messages $|a'\rangle, |b'\rangle$ are distinguishable on one of their shares, with statistical distance $\geq 4\Delta$. Then, by the triangle inequality and an averaging argument, there exists a pair of orthogonal states $|a\rangle, |b\rangle$ with statistical distance $\geq 2\Delta$ on that share. We thus restrict our attention to the distinguishability of orthogonal states.

If $\|\text{Tr}_{-i} \text{Enc}(|a\rangle) - \text{Tr}_{-i} \text{Enc}(|b\rangle)\|_1 \geq 2\Delta$ for some $i \in [t]$, then let $(\Pi_D, \mathbb{I} - \Pi_D)$ be the measurement which optimally distinguishes the two reduced density matrices on the i th share with bias $\geq \Delta$. By [Lemma 9.4](#), that implies the i th adversary can map between $\text{Enc}(\psi_+), \text{Enc}(\psi_-)$ with fidelity $\geq \Delta$ using $U_i = \mathbb{I} - 2\Pi$, where $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|a\rangle \pm |b\rangle)$. That is,

$$F(\text{Enc}(\psi_-), (\mathbb{I}_{-i} \otimes U_i) \circ \text{Enc}(\psi_+)) \geq \Delta \quad (115)$$

By monotonicity of fidelity, we conclude that after applying the decoding channel, i is able to swap the messages with non-trivial bias:

$$F(\psi_-, \text{Dec} \circ (\mathbb{I}_{-i} \otimes U_i) \circ \text{Enc}(\psi_+)) = F(\text{Dec} \circ \text{Enc}(\psi_-), \text{Dec} \circ (\mathbb{I}_{-i} \otimes U_i) \circ \text{Enc}(\psi_+)) \geq \Delta. \quad (116)$$

By the the tamper detection definition, there must exist $p \in [0, 1]$ such that

$$\text{Dec} \circ (\mathbb{I}_{-i} \otimes U_i) \circ \text{Enc}(\psi_+) \approx_\varepsilon p \cdot \psi_+ + (1 - p) \cdot \perp \quad (117)$$

$$\Rightarrow F^2(\psi_-, \text{Dec} \circ (\mathbb{I}_{-i} \otimes U_i) \circ \text{Enc}(\psi_+)) = \langle \psi_- | \text{Dec} \circ (\mathbb{I}_{-i} \otimes U_i) \circ \text{Enc}(\psi_+) | \psi_- \rangle \leq \varepsilon \quad (118)$$

which violates the tamper detection definition unless $\Delta \leq \varepsilon^{1/2}$. \square

Leveraging the secrecy of tamper-detection codes, we now prove our result on non-malleable secret sharing.

Proof. [of [Corollary 9.2](#)] Let (Enc, Dec) be a t -split-state non-malleable code with error ε , and let $(\text{Enc}_\lambda, \text{Dec}_\lambda)$ be the $(t+1)$ -split-state tamper-detection code guaranteed by [Theorem 3.1](#) with $\lambda < k$ EPR pairs and error $\varepsilon + 2^{1-\lambda}$. By [Theorem 9.1](#), $(\text{Enc}_\lambda, \text{Dec}_\lambda)$ is single share encrypting with error $4 \cdot \sqrt{\varepsilon + 2^{1-\lambda}}$. However, note that the reduced density matrix of $\text{Enc}_\lambda(\psi)$ on any single share $i \in [t]$, for any $k - \lambda$ qubit message state ψ , is the same as that of the non-malleable encoding of ψ together with λ random bits: $\text{Tr}_{-i} \text{Enc}(\psi \otimes \mathbb{I}/2^\lambda) = \text{Tr}_{-i} \text{Enc}_\lambda(\psi)$. Since $\|\text{Tr}_{-i} \text{Enc}_\lambda(\psi) - \rho_i\|_1 \leq 4 \cdot \sqrt{\varepsilon + 2^{1-\lambda}}$ for some fixed state ρ_i by [Theorem 9.1](#), we conclude that Enc must be single share encrypting with error $4 \cdot \sqrt{\varepsilon + 2^{1-\lambda}}$ if ψ is padded with λ random bits. \square

9.2 2-Split Non-Malleable Codes are 2-out-of-2 Secret Sharing

In this subsection we prove [Lemma 9.3](#), that 2-split-state non-malleable codes against entangled adversaries are secret sharing. For the purpose of contradiction, suppose two states $|\psi'_0\rangle, |\psi'_1\rangle$ are distinguishable on their left halves, with statistical distance $\geq \delta$. Then, by the triangle inequality and an averaging argument, there exists a pair of orthogonal states $|\psi_0\rangle, |\psi_1\rangle$ with statistical distance $\geq \delta/2$ on their left shares, and thereby we restrict our attention to the distinguishability of orthogonal states.

Our proof proceeds by constructing channels which leverage this distinguishability to break the non-malleability guarantee. To do so, first, in [Claim 9.5](#) we show that it suffices to construct a CPTP map Λ_{LR} ²⁰, which converts the encoding $\text{Enc}(\psi_b) = \rho_{LR}^b$ of ψ_b into that of ψ_{-b} (the opposite bit) with some fidelity bias to show a contradiction:

Claim 9.5. *Let $\rho_{out}^b = \text{Dec} \circ \Lambda \circ \text{Enc}(\psi_b)$ denote the state output by the decoder. If (Enc, Dec) is an ε non-malleable code against Λ , then the fidelity of ρ_{out}^{-b} with ψ_b can never be much larger than that of ρ_{out}^b with ψ_b :*

$$F^2(\rho_{out}^{-b}, \psi_b) - F^2(\rho_{out}^b, \psi_b) = \langle \psi_b | \rho_{out}^{-b} - \rho_{out}^b | \psi_b \rangle \leq 2 \cdot \varepsilon \quad (119)$$

²⁰Which can be implemented by local operations of the left and right adversaries with access to pre-shared copies of $\rho^b = \text{Enc}(\psi_b)$ for $b \in \{0, 1\}$

Now, assume there exists a pair of orthogonal states $|\psi_0\rangle, |\psi_1\rangle$ whose split-state shares are distinguishable, i.e.

$$\|\mathrm{Tr}_R \mathrm{Enc}(\psi_0) - \mathrm{Tr}_R \mathrm{Enc}(\psi_1)\|_1 = \delta_L \quad ; \quad \|\mathrm{Tr}_L \mathrm{Enc}(\psi_0) - \mathrm{Tr}_L \mathrm{Enc}(\psi_1)\|_1 = \delta_R. \quad (120)$$

and WLOG $\delta_L = \max\{\delta_L, \delta_R\}$. Recall the distinguishability implies the existence of a pair of POVMs $(M_L, \mathbb{I} - M_L)$ (and $(M_R, \mathbb{I} - M_R)$) supported only on L (and R), which distinguish the encodings of the pair with certain bias:

$$\mathbb{P}[\text{outcome } M_L = 0 | b = 0] = \mathrm{Tr} \left[\left(M_L \otimes \mathbb{I}_R \right) \mathrm{Enc}(\psi_0) \right] = \frac{1}{2} + \frac{\delta_L}{4}, \quad (121)$$

and analogously for $b = 1$ and R . In [Algorithm 12](#) and [Algorithm 13](#), we use the measurements M_L, M_R to construct two channels Λ_1, Λ_2 with large fidelity bias (in terms of on δ_L, δ_R). Together with [Claim 9.5](#), we obtain upper bounds on δ_L, δ_R .

Algorithm 12: Λ_1 : Right's output is entangled

Input: The bipartite state $\rho_{LR}^b = \mathrm{Enc}(\psi_b)$ for some $b \in \{0, 1\}$

Output: Some bipartite state $\Lambda_1(\rho_{LR}^b)$.

- 1: The split state adversaries share valid encodings ρ_{LR}^0, ρ_{LR}^1 of both $|\psi_0\rangle, |\psi_1\rangle$ in advance.
 - 2: Upon receiving ρ_{LR}^b , the left adversary measures $(M_L, \mathbb{I} - M_L)$, obtaining a bit $m \in \{0, 1\}$.
 - 3: The left outputs their pre-shared copy of the encoding of $-b$, and the right their pre-shared copy of the encoding of a random bit $r \in \{0, 1\}$.
-

Informally, Λ_1 attempts to distinguish b , and is able to output the encoding of $-b$ with some non-trivial bias:

Claim 9.6. $\forall b \in \{0, 1\}$, the state $\rho_{out}^b = \mathrm{Dec} \circ \Lambda_1 \circ \mathrm{Enc}(\psi_b)$ output by the receiver can be written as

$$\rho_{out}^b = \left(1 - \frac{\delta_L}{2}\right) \cdot \sigma + \frac{\delta_L}{4} \cdot \psi_{-b} + \frac{\delta_L}{4} \cdot \mathrm{Dec}(\rho_L^{-b} \otimes \rho_R^b) \quad (122)$$

where we used WLOG $\delta_L = \max\{\delta_L, \delta_R\}$, and σ is some fixed density matrix independent of b .

Unfortunately, this intuition is just slightly not enough to prove our result, as it is technically possible that the unentangled state $\mathrm{Dec}(\rho_L^{-b} \otimes \rho_R^b) \approx \psi_b$, ‘‘spoiling’’ our advantage in breaking the non-malleability. However, this can only happen if, coincidentally, an unentangled copy of ρ_R^b somehow determines the message. Leveraging this intuition, we formulate a set of channels which ensure that if the code-states are near separable, then the channels certainly break the non-malleability guarantee:

Algorithm 13: $\Lambda_2^{r,l}$: Right's output is fixed (and unentangled).

Input: The bipartite state $\rho_{LR}^b = \mathrm{Enc}(\psi_b)$ for some $b \in \{0, 1\}$, and two bits $r, l \in \{0, 1\}$

Output: A product state $\rho_L^m \otimes \rho_R^r$ for some $m \in \{0, 1\}$ correlated with b and some fixed $r \in \{0, 1\}$

- 1: The right adversary outputs the R half of a fresh encoding of ρ_R^r .
 - 2: The left adversary measures $(M_L, \mathbb{I} - M_L)$ obtaining outcome $m \in \{0, 1\}$, and outputs $\rho_L^{m \oplus l}$.
-

Claim 9.7. $\forall b, r, l \in \{0, 1\}$, the state $\rho^{b,r,l} = \mathrm{Dec} \circ \Lambda_2^{r,l} \circ \mathrm{Enc}(\psi_b)$ output by the receiver satisfies

$$\rho^{b,r,l} = \left(1 - \frac{\delta_L}{2}\right) \cdot \sigma_{r,l} + \frac{\delta_L}{2} \cdot \mathrm{Dec}(\rho_L^{b \oplus l} \otimes \rho_R^r), \quad (123)$$

where $\sigma_{r,l}$ doesn't depend on b .

We now put these claims together to prove the lemma:

Proof. [of Lemma 9.3] Applying Claim 9.5 to the channel $\Lambda_2^{r,l=0}$ we obtain (from Claim 9.7) that for all $b, r \in \{0, 1\}$:

$$2\varepsilon \geq \frac{\delta_L}{2} \cdot \langle \psi_b | \text{Dec} \left((\rho_L^{-b} - \rho_L^b) \otimes \rho_R^r \right) | \psi_b \rangle \quad (124)$$

And analogously, with $l = 1$, for all $b, r \in \{0, 1\}$:

$$\frac{\delta_L}{2} \cdot \langle \psi_b | \text{Dec} \left((\rho_L^{-b} - \rho_L^b) \otimes \rho_R^r \right) | \psi_b \rangle \geq -2\varepsilon. \quad (125)$$

By combining the above (with $r = b$) with Holder's inequality, we observe

$$| \langle \psi_b | \text{Dec} \left((\rho_L^{-b} - \rho_L^b) \otimes \rho_R^b \right) | \psi_b \rangle | \leq \min \left\{ \frac{4\varepsilon}{\delta_L}, \delta_L \right\} \leq 2\sqrt{\varepsilon} \quad (126)$$

Analogously, we obtain for the RHS:

$$\forall b : | \langle \psi_b | \text{Dec} \left(\rho_L^b \otimes (\rho_R^b - \rho_R^{-b}) \right) | \psi_b \rangle | \leq \min \left\{ \frac{4\varepsilon}{\delta_R}, \delta_R \right\} \leq 2\sqrt{\varepsilon} \quad (127)$$

Put together with Claim 9.6, and again applying Claim 9.5:

$$2\varepsilon \geq \frac{\delta_L}{4} + \frac{\delta_L}{4} \cdot \langle \psi_b | \text{Dec}(\rho_L^b \otimes \rho_R^{-b} - \rho_L^{-b} \otimes \rho_R^b) | \psi_b \rangle \geq \frac{\delta_L}{4} - \delta_L \cdot \varepsilon^{1/2} \quad (128)$$

Which implies $\delta_L = \max\{\delta_L, \delta_R\} \leq 16\varepsilon$ assuming $\varepsilon \leq 1/16$. □

9.2.1 Proofs of deferred claims

Proof. [of Claim 9.5] Since (Enc, Dec) is ε -non-malleable, there exists input-independent $p = p_{\text{same}}, \sigma$ such that

$$\max_b \|\rho_{out}^b - (p\psi_b + (1-p)\sigma)\|_1 \leq \varepsilon \quad (129)$$

$$\Rightarrow \langle \psi_b | \rho_{out}^{-b} - \rho_{out}^b | \psi_b \rangle \leq 2 \cdot \varepsilon + p \cdot \langle \psi_b | \psi_{-b} - \psi_b | \psi_b \rangle = 2\varepsilon - p \cdot (1 - F^2(\psi_b, \psi_{-b})) \leq 2\varepsilon \quad (130)$$

□

Proof. [of Claim 9.6] Let $\mathbb{P}[m_L|b]$ denote the distribution over left measurement outcomes m_L , where $\mathbb{P}[m_L = 0|b = 0] = \mathbb{P}[m_L = 1|b = 1] = \mathbb{P}[m_L = b|b] = \frac{1}{2} + \frac{\delta_L}{4}$. One can then re-express the state ρ_{out}^b output by the receiver as

$$\rho_{out}^b = \frac{1}{2} \cdot \mathbb{P}[b = m_L|b] \cdot \left(\text{Dec}(\rho^{-b}) + \text{Dec}(\rho_L^{-b} \otimes \rho_R^b) \right) + \frac{1}{2} \cdot \mathbb{P}[b \neq m_L|b] \cdot \left(\text{Dec}(\rho^b) + \text{Dec}(\rho_L^b \otimes \rho_R^{-b}) \right) \quad (131)$$

$$= \left(\frac{1}{4} - \frac{\delta_L}{8} \right) \cdot \text{Dec} \left(\rho^{-b} + \rho^b + \rho_L^{-b} \otimes \rho_R^b + \rho_L^b \otimes \rho_R^{-b} \right) + \frac{\delta_L}{4} \cdot \text{Dec}(\rho^{-b} + \rho_L^{-b} \otimes \rho_R^b) \quad (132)$$

Note that the first term in the line above is independent of b , and thereby we denote their sum as the fixed density matrix $(1 - \frac{\delta_L}{2}) \cdot \sigma$ of trace $1 - \frac{\delta_L}{2}$. Also, $\text{Dec}(\rho^{-b}) = \psi_{-b}$. Thus,

$$\rho_{out}^b = \left(1 - \frac{\delta_L}{2}\right) \cdot \sigma + \frac{\delta_L}{4} \cdot \psi_{-b} + \frac{\delta_L}{4} \cdot \text{Dec}(\rho_L^{-b} \otimes \rho_R^b) \quad (133)$$

which gives the claim. □

Proof. [of Claim 9.7] The decoded state is given by

$$\rho^{b,r,l} = \left(\frac{1}{2} + \frac{\delta_L}{4}\right) \text{Dec}(\rho_L^{b\oplus l} \otimes \rho_R^r) + \left(\frac{1}{2} - \frac{\delta_L}{4}\right) \text{Dec}(\rho_L^{b\oplus l\oplus 1} \otimes \rho_R^r) = \quad (134)$$

$$= \frac{\delta_L}{2} \cdot \text{Dec}(\rho_L^{b\oplus l} \otimes \rho_R^r) + \left(\frac{1}{2} - \frac{\delta_L}{4}\right) \text{Dec}(\rho_L^{\bar{b}} \otimes \rho_R^r + \rho_L^b \otimes \rho_R^r) = \quad (135)$$

$$= \left(1 - \frac{\delta_L}{2}\right) \cdot \sigma_r + \frac{\delta_L}{2} \cdot \text{Dec}(\rho_L^{b\oplus l} \otimes \rho_R^r) \quad (136)$$

for some fixed density matrix σ_r which doesn't depend on b . □

10 On the Capacity of Separable Non-Malleable Codes

In this section, we prove an upper bound for the rate of split-state non-malleable codes where the code-states are separable states. Recall that by separable, we mean the code-states are unentangled across one or more of the shares (see [Definition 2.6](#)). Although imposing separability may seem like a strong constraint on the family of codes we consider, we emphasize that prior to this work the only known constructions of non-malleable codes were separable [[BGJR23](#), [BBJ23](#)].

Our main result in this section is that quantum non-malleable codes which have separable state encodings, inherit their capacity from that of classical non-malleable codes. Our proof revisits and simplifies a result of Cheraghchi and Guruswami [[CG13a](#)], who proved a similar bound for the classical t -split-state model.

Theorem 10.1 ([Theorem 1.6](#), restatement). *Let (Enc, Dec) be a quantum non-malleable code against LO^t with error ε_{NM} and blocklength $n \in \mathbb{N}$, and assume that $\text{Enc}(\psi)$ is a separable state (across each share) for every message ψ . If the rate of Enc exceeds $1 - \frac{1}{t} + \delta$ for some $\delta \geq 4 \cdot \log n/n$, then the error must exceed $\varepsilon_{\text{NM}} = \Omega(\delta^2)$.*

At a high level, [[CG13a](#)]’s proof leverages information-theoretic techniques to argue that if the rate of the code is too high (exceeding $1 - \alpha$), then any large enough share (of size $\geq \alpha \cdot n$) must correlate with the message. They show how an adversary holding said share, could then leverage the correlations with the message to tamper with their share and break non-malleability.

In contrast, we appeal directly to the distinguishability of the message. [[CG13a](#)]’s arguments are used to show that if the rate of the code is too high, then the code is not single-share encrypting. That is, an adversary holding said share could guess the message by themselves. However, by appealing to our results in [Section 9](#), we know that quantum codes in the split-state model must encrypt their shares, leading to a contradiction.

10.1 Preliminaries

We dedicate this subsection to a background on quantum entropies, as well as the necessary statements from prior sections on breaking non-malleability and tamper detection. Refer to [[Wil11](#)] for a review.

10.1.1 Quantum Entropies

We use the following standard notions of quantum entropy, conditional quantum entropy, mutual information, and conditional quantum mutual information.

Definition 10.1 (von Neumann Entropy). *The von Neumann entropy of a quantum state ρ_A is defined as,*

$$S(A)_\rho \stackrel{\text{def}}{=} -\text{Tr}(\rho_A \log \rho_A).$$

Definition 10.2 (Conditional Quantum Entropy). *The conditional quantum entropy of a state ρ_{AB} is defined as,*

$$S(A|B)_\rho \stackrel{\text{def}}{=} S(AB)_\rho - S(B)_\rho, \quad \text{and} \quad -|A| \leq S(A|B)_\rho \leq |A|,$$

where we recall $|A| = \log \dim(\mathcal{H}_A)$.

Definition 10.3 (Mutual Information). *Let ρ_{AB} be a quantum state. We define the mutual information as follows.*

$$I(A : B)_\rho \stackrel{\text{def}}{=} S(A)_\rho + S(B)_\rho - S(AB)_\rho = S(A)_\rho - S(A|B)_\rho.$$

Furthermore, $0 \leq I(A : B)_\rho \leq 2 \min\{|A|, |B|\}$.

Definition 10.4 (Conditional Mutual Information). *Let ρ_{ABC} be a quantum state. We define the conditional mutual information as follows.*

$$I(A : B|C)_\rho \stackrel{\text{def}}{=} I(A : BC)_\rho - I(A : C)_\rho = S(A|C)_\rho - S(A|BC)_\rho.$$

Furthermore, $0 \leq I(A : B|C)_\rho \leq 2 \min\{|A|, |B|\}$.

The following three facts correspond to important properties of the conditional quantum entropy. In particular, concavity, non-decrease under local operations, and non-negativity when one of the registers is classical:

Fact 10.1 (Concavity of the Conditional Quantum Entropy). *Let ρ_{AB} be a state such that*

$$\rho_{AB} = \sum_i p_i \sigma_{AB}^i, \quad \forall i \quad p_i \geq 0, \quad \sum_i p_i = 1.$$

Then,

$$\mathsf{S}(A|B)_\rho \geq \sum_i p_i \mathsf{S}(A|B)_{\sigma^i}.$$

Fact 10.2 (Non-negativity of the Conditional Quantum Entropy of Separable States). *Let ρ_{XB} be a separable mixed state, i.e., $\rho_{XB} = \sum_i p_i \sigma_X^i \otimes \sigma_B^i$. Then, $\mathsf{S}(X|B)_\rho \geq 0$.*

Fact 10.3 (Data-Processing). *Let ρ_{AB} be a state and $\mathcal{E} : \mathsf{L}(\mathcal{H}_B) \rightarrow \mathsf{L}(\mathcal{H}_C)$ be a CPTP map acting on register B . Let $\sigma_{AC} = (\mathbb{I} \otimes \mathcal{E})(\rho_{AB})$. Then, $\mathsf{S}(A|B)_\rho \leq \mathsf{S}(A|C)_\sigma$.*

Finally, we make use of continuity of the von Neumann entropy:

Fact 10.4 (The Fannes Inequality). *Let ρ_A, σ_A be two states such that $\frac{1}{2} \|\rho_A - \sigma_A\|_1 \leq \delta$. Then,*

$$|\mathsf{S}(\sigma_A) - \mathsf{S}(\rho_A)| \leq \delta \cdot |A| + \frac{1}{e \ln 2}. \quad (137)$$

10.1.2 Tamper-resilient Codes Encrypt their Shares

At a high level, our proof proceeds by arguing that if the rate of the non-malleable code is too high, then one of the split-state adversaries (holding only one of the shares) would locally be able to distinguish the message. In some cases (see [Section 9](#))²¹, this can be shown to break the non-malleability guarantee. In this subsection we state two useful claims on breaking non-malleability by violating single-share encryption.

Aggarwal et. al [[ADKO15](#)] proved that non-malleable codes in the split-state model are 2-out-of-2 secret sharing. That is, the marginal distribution on either right or left shares is nearly independent of the message.

Lemma 10.2 ([\[ADKO15\]](#)). *Let (Enc, Dec) be a non-malleable code in the 2-split state model with error $\varepsilon < 1/2$. For any pair of messages m_0, m_1 , let $(X_i, Y_i) \leftarrow \text{Enc}(m_i)$ and let p_{X_i} denote the marginal distribution over the right share X_i . Then,*

$$\frac{1}{2} \cdot \|p_{X_0} - p_{X_1}\|_1 \leq 2 \cdot \varepsilon \quad (138)$$

In [Section 9](#), we showed that a similar behavior was present in tamper-detection codes, regardless of the number t of splits, following a connection between quantum authentication codes and quantum encryption scheme. In [Corollary 9.2](#) (restated below), we extended a limited version of this connection to non-malleable codes.

Claim 10.3 ([Corollary 9.2](#), restatement). *Let (Enc, Dec) be a non-malleable code in the t -split state model with error $\varepsilon < 1/2$, and let $(\text{Enc}_\lambda, \text{Dec}_\lambda)$ denote the non-malleable code in the t -split state model which pads the message with λ random bits and encodes it into Enc . Then, for any pair of messages m_0, m_1 and share $j \in [t]$:*

$$\frac{1}{2} \|\text{Tr}_{-j} \text{Enc}_\lambda(m_0) - \text{Tr}_{-j} \text{Enc}_\lambda(m_1)\|_1 \leq 2(\varepsilon^{1/2} + 2^{1-\lambda/2}). \quad (139)$$

²¹But not all - recall the 3 split-state counter-example for classical codes.

10.2 Proof of Theorem 10.1

Let (Enc, Dec) be a quantum non-malleable code with error ε_{NM} and rate $r = \frac{k}{n}$. Consider the cq state $\rho_{\hat{X}Q}$ defined by encoding half of a maximally correlated mixed state $\sigma_{\hat{X}X}$ into the code:

$$\sigma_{\hat{X}X} = 2^{-k} \sum_x |x\rangle\langle x|_{\hat{X}} \otimes |x\rangle\langle x|_X \rightarrow \rho_{\hat{X}Q} = 2^{-k} \sum_x |x\rangle\langle x|_{\hat{X}} \otimes \text{Enc}_{X \rightarrow Q}(|x\rangle\langle x|) \quad (140)$$

We begin by observing that Q has full information about \hat{X} .

Claim 10.4. *The conditional entropy $S(\hat{X}|Q)_\rho = S(\hat{X}Q)_\rho - S(Q)_\rho = 0$.*

Proof. Note that $S(\hat{X}|X) = 0$. By the data-processing inequality in [Fact 10.3](#), $S(\hat{X}|X)_\sigma \leq S(\hat{X}|Q)_\rho$. Moreover, if the non-malleable code is perfectly correct, we must have

$$\sigma_{\hat{X}X} = [\mathbb{I}_{\hat{X}} \otimes (\text{Dec} \circ \text{Enc})_X](\sigma_{\hat{X}X}). \quad (141)$$

Once again by the data-processing inequality, $S(\hat{X}|Q) \leq S(\hat{X}|X)$. \square

Based on these properties we prove a separable state analog of a statement by [\[CG13a\]](#), that there must exist at least one message whose left share has significantly less entropy than that of the encoding of a random message.

Claim 10.5. *Let the first share have size $|Q_1| = \alpha \cdot n$, and assume that $\rho_{\hat{X}, Q_1, Q_2 \dots Q_t}$ is separable across the $\hat{X}Q_1 : Q_2 \dots Q_t$ cut. Then,*

$$I(\hat{X} : Q_1) \geq k - (1 - \alpha) \cdot n. \quad (142)$$

Proof. [of [Claim 10.5](#)] From the chain rule of the mutual information, $I(Q_1 : \hat{X})_\rho = I(Q : \hat{X})_\rho - I(Q_2 \dots Q_t : \hat{X}|Q_1)_\rho$. From [Claim 10.4](#),

$$I(Q : \hat{X})_\rho = S(\hat{X}) - S(\hat{X}|Q) = S(\hat{X}) = k \quad (143)$$

If we assume ρ is separable across the $Q_1, Q \setminus Q_1$ cut, then the conditional entropy is non-negative $S(Q_2 \dots Q_t|Q_1, \hat{X}) \geq 0$ from [Fact 10.2](#). Then, by subadditivity,

$$I(Q_2 \dots Q_t : \hat{X}|Q_1)_\rho = S(Q_2 \dots Q_t|Q_1) - S(Q_2 \dots Q_t|Q_1, \hat{X}) \leq \quad (144)$$

$$\leq S(Q_2 \dots Q_t|Q_1) \leq \sum_{i \neq 1} |Q_i| = (1 - \alpha) \cdot n \quad (145)$$

\square

We can now conclude the proof of the theorem.

Proof. [of [Theorem 10.1](#)] By an averaging argument, at least one of the t shares (WLOG the first one) has size $|Q_1| = \alpha \cdot n \geq \frac{1}{t} \cdot n$. We begin by leveraging [Corollary 9.2](#). By our compiler, any quantum non-malleable code (Enc, Dec) of rate $r > \delta$ and error ε can be converted into another quantum non-malleable code $(\text{Enc}', \text{Dec}')$ of rate $r' = r - \delta$, which is single-share encrypting. That is, if $\rho_{Q_1}^x = \text{Tr}_{Q_2 \dots Q_t} \text{Enc}'_Q(|x\rangle\langle x|)$ denotes the reduced density matrix on Q_1 for a fixed message x , then

$$\forall x, y : \|\rho_{Q_1}^x - \rho_{Q_1}^y\|_1 \leq 4(\varepsilon^{1/2} + 2^{1-\delta \cdot n/2}). \quad (146)$$

We now prove that since $(\text{Enc}', \text{Dec}')$ is single-share encrypting, we must have an upper bound on its rate r' . Indeed, assume for the purposes of contradiction that the rate exceeds $r' \geq (1 - t^{-1}) + \delta$. From [Claim 10.5](#), we must then have $I(\hat{X} : Q_1) \geq \delta \cdot n$. From the concavity of conditional entropy [Fact 10.1](#), there exists an $x \in \{0, 1\}^k$ such that

$$S(Q_1|\hat{X} = x)_\rho \leq S(Q_1|\hat{X})_\rho = S(Q_1)_\rho - I(Q_1 : \hat{X})_\rho \leq S(Q_1)_\rho - \delta \cdot n. \quad (147)$$

However, by the the Fannes inequality [Fact 10.4](#), the entropy difference gives us a lower bound on the TV distance between the marginals:

$$\mathcal{S}(Q_1)_\rho - \mathcal{S}(Q_1)_{\text{Enc}_Q(|x\rangle\langle x|)} \leq \|\rho_{Q_1} - \rho_{Q_1}^x\|_1 \cdot \alpha \cdot n + \frac{2}{3}. \quad (148)$$

and by an averaging argument, this tells us there must exist two strings x, y whose marginals are statistically close:

$$\frac{1}{2} \|\rho_{Q_1}^x - \rho_{Q_1}^y\|_1 \geq \frac{\delta}{6\alpha} \quad (\text{Assuming } \delta \geq 1/n) \quad (149)$$

However, if we assume $\delta \geq 2 \log n/n$, this will contradict the assumption that Enc' is single-share encrypting, unless the error ε satisfies

$$\frac{\delta}{6\alpha} \leq 4(\varepsilon^{1/2} + 2^{1-\delta \cdot n/2}) \leq 4(\varepsilon^{1/2} + \frac{2}{n}) \Rightarrow \varepsilon = \Omega(\delta^2). \quad (150)$$

In this manner, we conclude that any quantum NMC of rate $r \geq 1 - t^{-1} + \gamma$ for some $\gamma \geq 4 \log n/n$ must have error $\Omega(\gamma^2)$. \square

A Background on Pauli and Clifford Operators

Here we review Pauli operators and the associated Pauli and Clifford groups.

Definition A.1 (Pauli Operators). *The single-qubit Pauli operators are given by*

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

An n -qubit Pauli operator is given by the n -fold tensor product of single-qubit Pauli operators. We denote the set of all $|A|$ -qubit Pauli operators on \mathcal{H}_A by $\mathcal{P}(\mathcal{H}_A)$, where $|\mathcal{P}(\mathcal{H}_A)| = 4^{|A|}$. Any linear operator $L \in \mathbb{L}(\mathcal{H}_A)$ can be written as a linear combination of $|A|$ -qubit Pauli operators with complex coefficients as $L = \sum_{P \in \mathcal{P}(\mathcal{H}_A)} \alpha_P P$. This is called the Pauli decomposition of a linear operator.

We remark that for $a \in \mathbb{F}_2^n$, we refer to the n -qubit Pauli operator $X^a = \otimes_{i \in [n]} X^{a_i}$ (respectively Z^a).

Definition A.2 (Pauli Group). *The single-qubit Pauli group is given by*

$$\{+P, -P, iP, -iP : P \in \{I, X, Y, Z\}\}.$$

The Pauli group on $|A|$ -qubits is the group generated by the operators described above applied to each of $|A|$ -qubits in the tensor product. We denote the $|A|$ -qubit Pauli group on \mathcal{H}_A by $\tilde{\mathcal{P}}(\mathcal{H}_A)$.

Definition A.3 (Clifford Group). *The Clifford group $\mathcal{C}(\mathcal{H}_A)$ is defined as the group of unitaries that normalize the Pauli group $\tilde{\mathcal{P}}(\mathcal{H}_A)$, i.e.,*

$$\mathcal{C}(\mathcal{H}_A) = \{V \in \mathcal{U}(\mathcal{H}_A) : V\tilde{\mathcal{P}}(\mathcal{H}_A)V^\dagger = \tilde{\mathcal{P}}(\mathcal{H}_A)\}.$$

The Clifford unitaries are the elements of the Clifford group.

We will also need to work with subgroups of the Clifford group with certain special properties. The following fact describes these properties and guarantees the existence of such subgroups.

Fact A.1 (Restatement of [Fact 2.6 \[CLLW16\]](#)). *There exists a subgroup $\mathcal{SC}(\mathcal{H}_A)$ of the Clifford group $\mathcal{C}(\mathcal{H}_A)$ such that given any non-identity Pauli operators $P, Q \in \mathcal{P}(\mathcal{H}_A)$ we have that*

$$|\{C \in \mathcal{SC}(\mathcal{H}_A) | C^\dagger P C = Q\}| = \frac{|\mathcal{SC}(\mathcal{H}_A)|}{|\mathcal{P}(\mathcal{H}_A)| - 1} \quad \text{and} \quad |\mathcal{SC}(\mathcal{H}_A)| = 2^{5|A|} - 2^{3|A|}.$$

Informally, applying a random Clifford operator from $\mathcal{SC}(\mathcal{H}_A)$ (by conjugation) maps P to a Pauli operator chosen uniformly at random over all non-identity Pauli operators. Furthermore, we have that $\mathcal{P}(\mathcal{H}_A) \subset \mathcal{SC}(\mathcal{H}_A)$.

Additionally, there exists a procedure **Samp** which given as input a uniformly random string $R \leftarrow \{0, 1\}^{5|A|}$ outputs in time $\text{poly}(|A|)$ a Clifford operator $C_R \in \mathcal{SC}(\mathcal{H}_A)$ such that

$$C_R \approx_{2^{-2|A|}} U_{\mathcal{SC}(\mathcal{H}_A)}, \tag{151}$$

where $U_{\mathcal{SC}(\mathcal{H}_A)}$ denotes the uniform distribution over $\mathcal{SC}(\mathcal{H}_A)$.

Twirling and related facts. The analysis of our construction will require the use of several facts related to Pauli and Clifford twirling. We collect them below, beginning with the usual version of the Pauli twirl.

Fact A.2 (Pauli 1-Design). *Let ρ_{AB} be a state. Then,*

$$\frac{1}{|\mathcal{P}(\mathcal{H}_A)|} \sum_{Q \in \mathcal{P}(\mathcal{H}_A)} (Q \otimes \mathbb{I}) \rho_{AB} (Q^\dagger \otimes \mathbb{I}) = U_A \otimes \rho_B.$$

Fact A.3 (Pauli Twirl [DCEL09]). Let $\rho \in \mathcal{D}(\mathcal{H}_A)$ be a state and $P, P' \in \mathcal{P}(\mathcal{H}_A)$ be Pauli operators such that $P \neq P'$. Then,

$$\sum_{Q \in \mathcal{P}(\mathcal{H}_A)} Q^\dagger P Q \rho Q^\dagger P' Q = 0.$$

Fact A.4 (1-Design). Let ρ_{AB} be a state. Let $\mathcal{SC}(\mathcal{H}_A)$ be the subgroup of Clifford group as defined in Fact A.1. Then,

$$\frac{1}{|\mathcal{SC}(\mathcal{H}_A)|} \sum_{C \in \mathcal{SC}(\mathcal{H}_A)} (C \otimes \mathbb{I}) \rho_{AB} (C^\dagger \otimes \mathbb{I}) = U_A \otimes \rho_B.$$

Fact A.5 (Clifford Subgroup Twirl, Lemma 1 in [BBJ23]). Let state ρ_{AB} be a state. Let $P, Q \in \mathcal{P}(\mathcal{H}_A)$ be any two Pauli operators. Let $\mathcal{SC}(\mathcal{H}_A)$ be the sub-group of Clifford group as defined in Fact A.1.

1. If $P \neq Q$, then

$$\frac{1}{|\mathcal{SC}(\mathcal{H}_A)|} \sum_{C \in \mathcal{SC}(\mathcal{H}_A)} (\mathbb{I} \otimes C^\dagger P C) \rho_{BA} (\mathbb{I} \otimes C^\dagger Q^\dagger C) = 0. \quad (152)$$

2. If $P = Q \neq \mathbb{I}_A$, then

$$\frac{1}{|\mathcal{SC}(\mathcal{H}_A)|} \sum_{C \in \mathcal{SC}(\mathcal{H}_A)} (\mathbb{I} \otimes C^\dagger P C) \rho_{BA} (\mathbb{I} \otimes C^\dagger Q^\dagger C) = \frac{|\mathcal{P}(\mathcal{H}_A)|(\rho_B \otimes U_A) - \rho_{BA}}{|\mathcal{P}(\mathcal{H}_A)| - 1}. \quad (153)$$

Lemma A.1 (Restatement of Lemma 2.2). Consider Figure 14. Let $\psi_R = U_R$ be a state independent of $\psi_{A\hat{A}E}$ and $|R| = 5|A|$. Let $\Lambda : \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_E) \rightarrow \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_E)$ be any CPTP map. Let $\mathcal{SC}(\mathcal{H}_A)$ be the sub-group of Clifford group as defined in Fact A.1. Let

$$\rho_{\hat{A}AE} = \frac{1}{|\mathcal{SC}(\mathcal{H}_A)|} \sum_{C \in \mathcal{SC}(\mathcal{H}_A)} (C^\dagger \Lambda(C(\psi_{\hat{A}AE})C^\dagger)C).$$

Then,

$$\rho_{\hat{A}AE} \approx \frac{2}{2^{2|A|-1}} \Phi_1(\psi_{\hat{A}AE}) + (\Phi_2(\psi_{\hat{A}E}) \otimes U_A),$$

where $\Phi_1, \Phi_2 : \mathcal{L}(\mathcal{H}_E) \rightarrow \mathcal{L}(\mathcal{H}_E)$ are CP (completely positive) maps acting only on register E, depending only on Λ , and $\Phi_1(\cdot) + \Phi_2(\cdot)$ is a CPTP map.

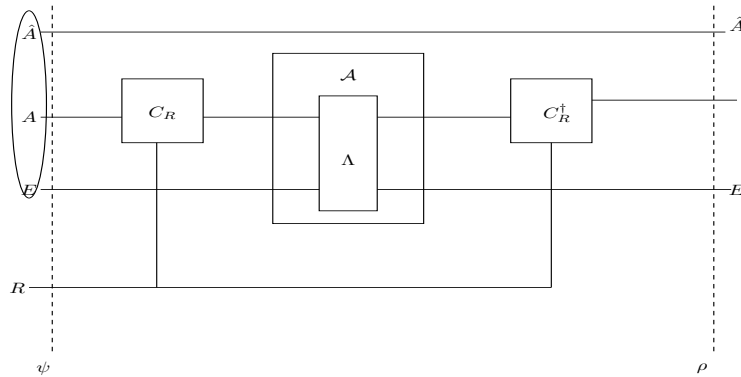


Figure 14: Clifford Twirling with Side Information

Proof. Let $\Lambda : \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_E) \rightarrow \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_E)$ be the CPTP map. Let $\{M_i\}_i$ be the set of Kraus operators corresponding to Λ , and its Pauli basis decomposition $M_i = \sum_j \alpha_{ij} P^{ij}$ for $P^{ij} \in \mathcal{P}(\mathcal{H}_{AE})$. We denote P_A^{ij} to use Pauli operator corresponding to register A of P^{ij} . Let $M_i^Q \stackrel{\text{def}}{=} \sum_{j: P_A^{ij}=Q} \alpha_{ij} P_E^{ij}$ for every $Q \in \mathcal{P}(\mathcal{H}_A)$. Note that since Λ is CPTP, we have $\sum_i M_i^\dagger M_i = \mathbb{I}_{AE}$.

We begin by showing that if we restrict the M_i to their operation on the register E , they still form a CPTP map. For this we need to show $\sum_{i,Q} (M_i^Q)^\dagger M_i^Q = \mathbb{I}_E$. Consider,

$$\begin{aligned}
2^{|A|} \mathbb{I}_E &= \text{Tr}_A(\mathbb{I}_{AE}) \\
&= \text{Tr}_A \left(\sum_i M_i^\dagger M_i \right) \\
&= \text{Tr}_A \left(\sum_i \left(\sum_{j'} \alpha_{ij'}^* P^{ij'} \right) \left(\sum_j \alpha_{ij} P^{ij} \right) \right) \\
&= \text{Tr}_A \left(\sum_{i,j,j'} \alpha_{ij} \alpha_{ij'}^* (P_A^{ij'} P_A^{ij}) \otimes (P_E^{ij'} P_E^{ij}) \right) \\
&= \left(\sum_{i,j,j': P_A^{ij}=P_A^{ij'}} \alpha_{ij} \alpha_{ij'}^* \text{Tr}(\mathbb{I}_A) \otimes P_E^{ij'} P_E^{ij} + \sum_{i,j,j': P_A^{ij} \neq P_A^{ij'}} \alpha_{ij} \alpha_{ij'}^* \text{Tr}(P_A^{ij'} P_A^{ij}) \otimes P_E^{ij'} P_E^{ij} \right) \\
&= 2^{|A|} \left(\sum_{i,j,j': P_A^{ij}=P_A^{ij'}} \alpha_{ij} \alpha_{ij'}^* P_E^{ij'} P_E^{ij} \right) \\
&= 2^{|A|} \sum_{i,Q} \left(\sum_{j,j': P_A^{ij}=P_A^{ij'}=Q} \alpha_{ij} \alpha_{ij'}^* P_E^{ij'} P_E^{ij} \right) \\
&= 2^{|A|} \sum_{i,Q} \left(\sum_{j': P_A^{ij'}=Q} \alpha_{ij'}^* P_E^{ij'} \right) \left(\sum_{j: P_A^{ij}=Q} \alpha_{ij} P_E^{ij} \right) \\
&= 2^{|A|} \sum_{i,Q} (M_i^Q)^\dagger (M_i^Q)
\end{aligned}$$

We can now turn our attention to $\rho_{\hat{A}AE}$. Consider,

$$\begin{aligned}
\rho_{\hat{A}AE} &= \frac{1}{|\mathcal{SC}(\mathcal{H}_A)|} \sum_{C \in \mathcal{SC}(\mathcal{H}_A)} C^\dagger \Lambda(C \psi_{\hat{A}AE} C^\dagger) C \\
&= \sum_i \left(\frac{1}{|\mathcal{SC}(\mathcal{H}_A)|} \sum_{C \in \mathcal{SC}(\mathcal{H}_A)} (C^\dagger M_i C)(\psi_{\hat{A}AE})(C^\dagger M_i^\dagger C) \right) \\
&= \sum_i \left(\frac{1}{|\mathcal{SC}(\mathcal{H}_A)|} \sum_{j,j'} \alpha_{ij} \alpha_{ij'}^* \sum_{C \in \mathcal{SC}(\mathcal{H}_A)} (C^\dagger P^{ij} C)(\psi_{\hat{A}AE})(C^\dagger P^{ij'} C) \right) \\
&= \sum_{i,j,j': P_A^{ij} = P_A^{ij'}} \left(\alpha_{ij} \alpha_{ij'}^* \frac{1}{|\mathcal{SC}(\mathcal{H}_A)|} P_E^{ij} \left(\sum_{C \in \mathcal{SC}(\mathcal{H}_A)} (C^\dagger P_A^{ij} C)(\psi_{\hat{A}AE})(C^\dagger P_A^{ij'} C) \right) P_E^{ij'} \right) \\
&+ \sum_{i,j,j': P_A^{ij} \neq P_A^{ij'}} \left(\alpha_{ij} \alpha_{ij'}^* \frac{1}{|\mathcal{SC}(\mathcal{H}_A)|} P_E^{ij} \left(\sum_{C \in \mathcal{SC}(\mathcal{H}_A)} (C^\dagger P_A^{ij} C)(\psi_{\hat{A}AE})(C^\dagger P_A^{ij'} C) \right) P_E^{ij'} \right) \\
&= \sum_{i,j,j': P_A^{ij} = P_A^{ij'}} \left(\alpha_{ij} \alpha_{ij'}^* \frac{1}{|\mathcal{SC}(\mathcal{H}_A)|} P_E^{ij} \left(\sum_{C \in \mathcal{SC}(\mathcal{H}_A)} (C^\dagger P_A^{ij} C)(\psi_{\hat{A}AE})(C^\dagger P_A^{ij'} C) \right) P_E^{ij'} \right) \quad (\text{Fact A.5.1}) \\
&= \sum_{i,j,j': P_A^{ij} = P_A^{ij'} = \mathbb{1}_A} \left(\alpha_{ij} \alpha_{ij'}^* \frac{1}{|\mathcal{SC}(\mathcal{H}_A)|} P_E^{ij} \left(\sum_{C \in \mathcal{SC}(\mathcal{H}_A)} (C^\dagger P_A^{ij} C)(\psi_{\hat{A}AE})(C^\dagger P_A^{ij'} C) \right) P_E^{ij'} \right) \\
&+ \sum_{i,j,j': P_A^{ij} = P_A^{ij'} \neq \mathbb{1}_A} \left(\alpha_{ij} \alpha_{ij'}^* \frac{1}{|\mathcal{SC}(\mathcal{H}_A)|} P_E^{ij} \left(\sum_{C \in \mathcal{SC}(\mathcal{H}_A)} (C^\dagger P_A^{ij} C)(\psi_{\hat{A}AE})(C^\dagger P_A^{ij'} C) \right) P_E^{ij'} \right) \\
&= \sum_{i,j,j': P_A^{ij} = P_A^{ij'} = \mathbb{1}_A} \left(\alpha_{ij} \alpha_{ij'}^* P_E^{ij} (\psi_{\hat{A}AE}) P_E^{ij'} \right) \\
&+ \sum_{i,j,j': P_A^{ij} = P_A^{ij'} \neq \mathbb{1}_A} \left(\alpha_{ij} \alpha_{ij'}^* P_E^{ij} \left(\frac{|\mathcal{P}(\mathcal{H}_A)| (\psi_{\hat{A}E} \otimes U_A) - \psi_{\hat{A}EA}}{|\mathcal{P}(\mathcal{H}_A)| - 1} \right) P_E^{ij'} \right) \quad (\text{Fact A.5.2}) \\
&\approx \frac{2}{|\mathcal{P}(\mathcal{H}_A)| - 1} \sum_i M_i^{\mathbb{1}_A} \psi_{A\hat{A}E} (M_i^{\mathbb{1}_A})^\dagger + \sum_i \sum_{Q \in \mathcal{P}(\mathcal{H}_A) \wedge Q \neq \mathbb{1}_A} M_i^Q (U_A \otimes \psi_{\hat{A}E}) (M_i^Q)^\dagger.
\end{aligned}$$

The approximation in the result follows from the fact that

$$\left\| \frac{|\mathcal{P}(\mathcal{H}_A)| (U_A \otimes \psi_{\hat{A}E}) - \psi_{A\hat{A}E}}{|\mathcal{P}(\mathcal{H}_A)| - 1} - (U_A \otimes \psi_{\hat{A}E}) \right\|_1 \leq \frac{2}{|\mathcal{P}(\mathcal{H}_A)| - 1}.$$

and the monotonicity of trace distance [Fact 2.1](#), since the operators $\{M_i^Q\}_{i,Q}$ define a CPTP set of Krauss operators. Finally, we let $\{M_i^{\mathbb{1}}\}_{i,\mathbb{1}}$ define the CP map Φ_1 and $\{M_i^Q\}_{i,Q \neq \mathbb{1}}$ define Φ_2 . \square

B Non-Malleable Codes against LO^2

In a recent work, [BGJR23] introduced the notion of a non-malleable code for quantum messages, and presented constructions in the split-state model with entangled adversaries LO_*^2 . Their constructions were based on quantum secure classical non-malleable codes and families of unitary 2-designs, and had inverse-polynomial rate. In this section, we revisit [BGJR23]’s construction of 2-split-state quantum non-malleable codes under the assumption that the adversaries are unentangled, that is, against LO channels. We show that in this model, simply an *augmented* classical non-malleable code (instead of quantum secure) suffices to achieve security with much better parameters. Our main result here is a 2-split non-malleable code for quantum messages with *constant* rate:

Theorem B.1. *For every $n \in \mathbb{N}$, there exists a quantum non-malleable code with efficient encoding and decoding against LO^2 , with message length k , block length $2n$, rate $k/(2n) = \Omega(1)$ and error $2^{-\Omega(k)}$.*

We leverage this 2-split construction to instantiate our tamper-detection reduction in [Theorem 3.1](#). To prove [Theorem B.1](#), we first show the average-case non-malleability against LO^2 which is stated in the following [Theorem B.2](#). We next make use of average-case to worst-case reduction as stated in [Theorem B.3](#) to conclude the desired main result of this section, i.e. [Theorem B.1](#).

Theorem B.2. *For every $n \in \mathbb{N}$, there exists an average-case quantum non-malleable code with efficient encoding and decoding against LO^2 , with message length k , block length $2n$, rate $k/(2n) = \Omega(1)$ and error $2^{-\Omega(k)}$.*

Theorem B.3 (Worst-case to average-case reduction, [BGJR23, Lemma 8]). *Let $t \geq 2$. Any average-case quantum non-malleable code against LO^t for messages of length b and error ε is also a worst-case quantum non-malleable code against LO^t for messages of length b with error $\varepsilon' \leq 2^b \cdot \varepsilon$.*

We next proceed to provide the construction of average-case quantum non-malleable code as stated in [Theorem B.2](#) and analyze the construction. Before that we refer the reader to [Appendix A](#) to background on Clifford operators, and dedicate next subsection to state basic fact on the transpose method.

B.0.1 The transpose method

The transpose method (see, e.g., [Ozo16]) is one of the most important tools for manipulating maximally entangled states. Roughly speaking, the transpose method corresponds to the statement that some local action on one half of the maximally entangled state is equivalent to performing the transpose of the same action on the other half of that state. We now state this formally.

Fact B.1 (Transpose method). *Let $\rho_{A\hat{A}}$ be the canonical purification of $\rho_A = U_A$. For any $M \in \mathcal{L}(\mathcal{H}_A)$ it holds that*

$$(M \otimes \mathbb{I}_{\hat{A}})\rho_{A\hat{A}}(M^\dagger \otimes \mathbb{I}_{\hat{A}}) = (\mathbb{I}_A \otimes M^T)\rho_{A\hat{A}}(\mathbb{I}_A \otimes (M^T)^\dagger).$$

B.1 Code Construction

Our goal is to provide a construction and prove the average-case non-malleability of quantum non-malleable code against LO^2 . However, to achieve constant rate, we need a really careful choice of components: we require a 2-split classical non-malleable code of constant rate and inverse exponential error, and a family of 2 designs with key-size linear in the number of qubits.

The design of 2-split-state NMCs with constant rate and inverse exponential error has been an outstanding open problem for many years, however, recently [Li23b] presented remarkable constructions of non-malleable extractors and codes with near-optimal parameters.

Theorem B.4 ([Li23b]). *For any $n \in \mathbb{N}$, there exists an augmented classical non-malleable code with efficient encoding and decoding against 2-split-state tampering, which has message length k , block length $2n$, rate $k/(2n) = \Omega(1)$ and error $2^{-\Omega(k)}$.*

B.1.1 Ingredients and Overview

We combine:

1. An augmented classical non-malleable code in the 2-split-state model, $(\text{Enc}^{\text{NM}}, \text{Dec}^{\text{NM}})$ from [Theorem B.4](#).
2. A family of unitary 2-designs, and in particular the subgroup \mathcal{SC} of the Clifford group guaranteed in [Fact A.1](#).

Consider the following quantum non-malleable code defined by encrypting the quantum message into a randomly chosen Clifford in the subgroup, and non-malleably protecting the randomness into Enc^{NM} :

$$\text{Enc}(\sigma_M) = \mathbb{E}_r \left(\text{Enc}^{\text{NM}}(r)_{YX} \otimes (C_r \sigma_M C_r^\dagger)_Z \right). \quad (154)$$

Registers (Y, XZ) form 2-split state codewords. After *unentangled* adversarial tampering (U, V) , the receiver first measures the classical registers obtaining X', Y' , decodes them obtaining a key $R' = \text{Dec}^{\text{NM}}(X', Y')$, and attempts to decrypt the quantum message using $C_{R'}$. We refer the reader to [Figure 15](#) for the quantum non-malleable code along with the tampering process. We remark that if the classical non-malleable code has perfect correctness, then the correctness of the quantum code is trivial. It remains to show security.

We prove that the construction above defines an *average-case* quantum non-malleable code against LO^2 , where the message $\sigma_M = U_M$ is a maximally mixed state.

Theorem B.5. *If $(\text{Enc}^{\text{NM}}, \text{Dec}^{\text{NM}})$ is an ε_{NM} -secure augmented 2-split-state classical non-malleable code, and \mathcal{SC} is the subgroup of the Clifford group guaranteed by [Fact A.1](#), then (Enc, Dec) above is an average-case $\leq \varepsilon_{\text{NM}} + 2^{2-|M|}$ -secure quantum non-malleable code against LO^2 (or simply unentangled adversaries).*

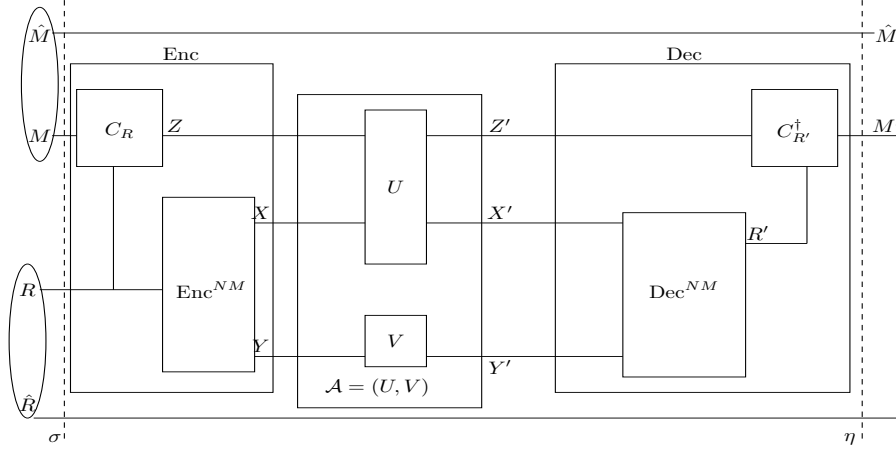


Figure 15: Quantum Non-Malleable Code against LO^2 .

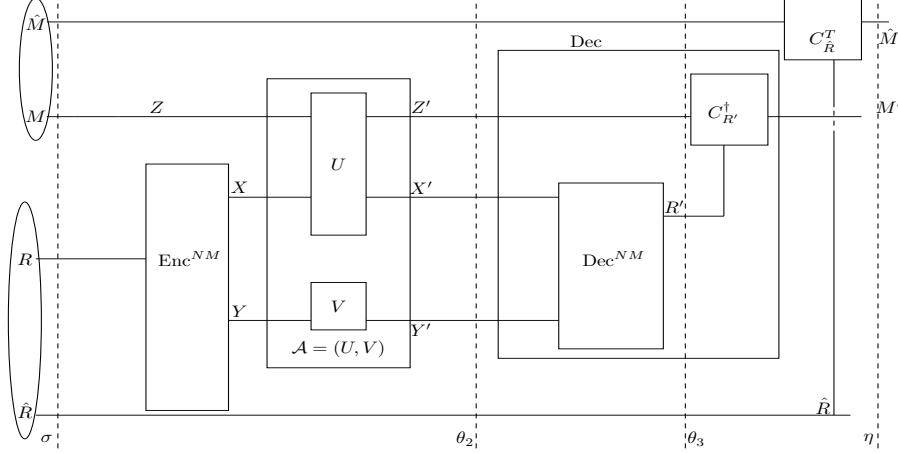


Figure 16: Quantum NMC along with the modified process against LO^2 .

B.1.2 Analysis

At a high level, the outline of our proof of security follows that of [BGJR23]. To prove this theorem, we begin with [Claim B.6](#), [Claim B.7](#) and the “transpose method” (see [Fact B.1](#)) to roughly show that the classical code remains secure in the presence of unentangled adversaries. We defer the proof of Claims to the next subsection, and use them to prove the theorem.

Our goal is to show that $\eta_{M'\hat{M}} \approx_{\varepsilon_{\text{NM}} + 2^{2-|M|}} p_{\mathcal{A}} \sigma_{M\hat{M}} + (1 - p_{\mathcal{A}}) \gamma_{M'}^{\mathcal{A}} \otimes U_{\hat{M}}$, where $p_{\mathcal{A}}, \gamma_{M'}^{\mathcal{A}}$ depend only on \mathcal{A} . We first note that the final states in both the [Figure 15](#) and [Figure 16](#) are same.

We now consider the split-state tampering experiment after applying the transpose method and delaying the application of the corresponding Clifford operator (see [Figure 16](#)). After the adversarial tampering, the receiver measures the classical registers, and the state collapses into a mixture where the quantum message only depends on the right share (x, x') . Let θ_2 be the resulting state:

Claim B.6.

$$\theta_2 = \sum_{x, y, x', y'} \mathbb{P}_2(x, y, x', y') \cdot |x', y'\rangle\langle x', y'| \otimes (\theta_2^{x, x'})_{Z'\hat{M}} \quad (155)$$

where $\mathbb{P}_2(x, y, x', y') = \mathbb{P}(x, y) \cdot \mathbb{P}(x'|x) \cdot \mathbb{P}(y'|y)$ is the joint distribution over split-state shares before and after tampering, and for each x, x' , $(\theta_2^{x, x'})_{Z'\hat{M}}$ is some bipartite density matrix (independent of y, y').

After the receiver attempts to decode the classical non-malleable code, they produce a mixture over original keys r and received keys r' , together with a quantum state dependent only on the right share x, x' :

Claim B.7.

$$\theta_3 = \sum_{r, r', x, x'} \mathbb{P}_3(r, r', x, x') \cdot |r\rangle\langle r| \otimes |r'\rangle\langle r'| \otimes (\theta_2^{x, x'})_{Z'\hat{M}} \quad (156)$$

Moreover, \mathbb{P}_3 is ε_{NM} close to a convex combination q , satisfying

$$q(r, r', x, x') = \mathbb{P}(r) \cdot [p_{\text{same}} \cdot \delta_{r=r'} \cdot q'(x, x') + (1 - p_{\text{same}}) \cdot q''(r', x, x')] \quad (157)$$

for some $p_{\text{same}} \in [0, 1]$, and distributions q', q'' which depend only on adversary \mathcal{A} .

Since $\mathbb{P}(r)$ is uniform to begin with, this implies

$$\theta_3 \approx_{\varepsilon_{\text{NM}}} p_{\text{same}} \cdot U_R U_{R'=R} \otimes (\theta_3^1)_{Z'\hat{M}} + (1 - p_{\text{same}}) \cdot U_R \otimes (\theta_3^2)_{R'Z'\hat{M}}, \quad (158)$$

where $(\theta_3^1)_{Z'\hat{M}} = \sum_{(x, x')} q'(x, x') (\theta_2^{x, x'})_{Z'\hat{M}}$ and $(\theta_3^2)_{R'Z'\hat{M}} = \sum_{(r', x, x')} q''(r', x, x') (|r'\rangle\langle r'| \otimes (\theta_2^{x, x'})_{Z'\hat{M}})$. Intuitively, this shows that either $R = R'$ and the quantum component is uncorrelated from R , or, R is

uniform and uncorrelated from R' and the quantum state. Together with the additional proof techniques from [BGJR23], we complete the proof as stated below.

Proof. [of Theorem B.5]

Let $D_{\hat{R}\hat{M}}$ on registers (\hat{R}, \hat{M}) be the application of the controlled Clifford gates as shown in Figure 16 (Similarly $D_{R'Z'}$ on registers (R', Z')). We consider $D_{\hat{R}\hat{M}}$ on registers (\hat{R}, \hat{M}) and $D_{R'Z'}$ on registers R', Z' , on the approximate state to θ_3 . First, if the key is received, we expect to recover the quantum message if there is no tampering on it. From Fact A.5 (the Clifford Twirl), Fact 2.6 (approximate sampling from \mathcal{SC}) and using techniques from [BGJR23], there exists $p_{\text{EPR}} \in [0, 1]$ and $\delta \leq 2 \cdot 2^{-|M|}$, such that the reduced density matrix on M', \hat{M} satisfies:

$$\text{Tr}_{R', \hat{R}} \left[\left(D_{\hat{R}\hat{M}} \otimes D_{R'Z'} \right) \circ \left(U_R U_{R'=R} \otimes (\theta_3^1)_{Z'\hat{M}} \right) \right] \approx_{\delta} p_{\text{EPR}} \sigma_{M\hat{M}} + (1 - p_{\text{EPR}}) U_M \otimes U_{\hat{M}}. \quad (159)$$

If the key is not received, we expect the controlled Clifford to decouple the quantum message. Note that,

$$\text{Tr}_{R', \hat{R}} \left[\left(D_{\hat{R}\hat{M}} \otimes D_{R'Z'} \right) \circ \left(U_R \otimes (\theta_3^2)_{R'Z'\hat{M}} \right) \right] \approx_{\delta'} \gamma_{M'}^A \otimes U_{\hat{M}} \quad (160)$$

by Fact 2.5 (Cliffords are 1-designs) and again Fact 2.6 (approximate sampling from \mathcal{SC}), for $\delta' \leq 2 \cdot 2^{-|M|}$ and some choice of $\gamma_{M'}$ that depends only on adversary. Put together, if $p_{\mathcal{A}} = p_{\text{EPR}} \cdot p_{\text{same}}$, we have from Equation (158) and data processing:

$$\eta_{M'\hat{M}} \approx_{\varepsilon_{\text{NM}} + \delta + \delta'} p_{\mathcal{A}} \sigma_{M\hat{M}} + (1 - p_{\mathcal{A}}) \gamma_{M'}^A \otimes U_{\hat{M}} \quad (161)$$

again for some (unentangled) choice of $\gamma_{M'}^A$. This is a convex combination of the original message σ_M and a fixed state, as intended. \square

B.1.3 Deferred Proofs

Claim B.7 is a consequence of the augmented classical non-malleable code in the 2-split-state model, $(\text{Enc}^{\text{NM}}, \text{Dec}^{\text{NM}})$. We now proceed to provide the proof of Claim B.6.

Proof. [of Claim B.6] The state before tampering is given by

$$\theta = \sum_{x,y} \mathbb{P}(x,y) \cdot |x,y\rangle\langle x,y| \otimes \sigma_{M\hat{M}} \quad (162)$$

Now, the state θ_2 after the tampering,

$$\theta_2 = \sum_{x,y,x',y'} \mathbb{P}(x,y) \cdot \mathbb{P}(y'|y) \cdot \mathbb{P}(x'|x) \cdot |x',y'\rangle\langle x',y'| \otimes (\theta_2^{x,x'})_{Z'\hat{M}} \quad (163)$$

where $\mathbb{P}(y'|y) = \text{Tr}[(|y'\rangle\langle y'|)V(|y\rangle\langle y|)]$, $\mathbb{P}(x'|x) = \text{Tr}[(|x'\rangle\langle x'| \otimes \mathbb{I})U(|x\rangle\langle x| \otimes \sigma)]$, and the post-measurement state (on registers $Z'\hat{M}$) is given by:

$$(\theta_2^{x,x'})_{Z'\hat{M}} = \frac{1}{\mathbb{P}(x'|x)} \cdot \text{Tr}_{(Z'\hat{M})^c} \left[(|x'\rangle\langle x'| \otimes \mathbb{I}) V \left(|x\rangle\langle x| \otimes \sigma_{M\hat{M}} \right) (|x'\rangle\langle x'| \otimes \mathbb{I}) \right]. \quad (164)$$

\square

B.2 [BGJR23]'s 2-Split-State NMCs are Augmented

In opening up [BGJR23]'s construction, we also make the observation that it is an augmented non-malleable code, under Definition 2.15. We dedicate Appendix B.2 to note the necessary changes in the proof of [BGJR23] to conclude the augmented property.

Claim B.8. *The 2-split-state quantum non-malleable code against LO_*^2 by [BGJR23] is a quantum augmented non-malleable code with inverse polynomial rate and error $\varepsilon \leq 2^{-n^c}$ for some tiny constant $c > 0$, where n is the length of the codeword.*

For simplicity, we simply state the modifications required to their proof. The key insight lies in the fact that they use an augmented quantum secure non-malleable randomness encoder (NMRE), based on a quantum secure non-malleable extractor. In this manner, the data-processing inequalities present in the proofs remain true in the presence of side entanglement (W_2 , in their proof) held by the adversary. The only major modification that we need is to the Lemma B.9 (which is Lemma 8 in [BGJR23] proof) which is now Lemma B.10 to note the augmented property.

Lemma B.9 (Lemma 8 in [BGJR23]). *Let $|\psi\rangle_{\hat{A}A}$ be the canonical purification of $\psi_A = U_A$, $\rho_{\hat{A}A}$ be any state, and $\text{SC}(\mathcal{H}_A)$ be the subgroup of Clifford group as defined in Fact A.1. Define $\Pi = |\psi\rangle\langle\psi|$. Then, we have that*

$$\begin{aligned} \frac{1}{|\text{SC}(\mathcal{H}_A)|} \sum_{C \in \text{SC}(\mathcal{H}_A)} (C^T \otimes C^\dagger) \rho_{\hat{A}A} ((C^T)^\dagger \otimes C) \\ = \text{tr}(\Pi\rho)\psi + (1 - \text{tr}(\Pi\rho)) \frac{|\text{P}(\mathcal{H}_A)|(U_{\hat{A}} \otimes U_A) - \psi_{A\hat{A}}}{|\text{P}(\mathcal{H}_A)| - 1} \\ \approx \frac{2}{4^{|\mathcal{A}|}} \text{tr}(\Pi\rho)\psi + (1 - \text{tr}(\Pi\rho))(U_{\hat{A}} \otimes U_A). \end{aligned} \quad (165)$$

We need the following modified version of the above lemma.

Lemma B.10. *Let $|\psi\rangle_{\hat{A}A}$ be the canonical purification of $\psi_A = U_A$, $\rho_{\hat{A}AE}$ be any state, and $\text{SC}(\mathcal{H}_A)$ be the subgroup of Clifford group as defined in Fact A.1. Define*

$$\Pi = |\psi\rangle\langle\psi| \quad ; \quad \gamma_E^0 = \text{tr}_{\hat{A}A} \left(\frac{\Pi\rho\Pi}{\text{tr}(\Pi\rho)} \right) \quad ; \quad \gamma_E^1 = \text{tr}_{\hat{A}A} \left(\frac{\bar{\Pi}\rho\bar{\Pi}}{1 - \text{tr}(\Pi\rho)} \right).$$

Then, we have that

$$\begin{aligned} \frac{1}{|\text{SC}(\mathcal{H}_A)|} \sum_{C \in \text{SC}(\mathcal{H}_A)} (C^T \otimes C^\dagger) \rho_{\hat{A}AE} ((C^T)^\dagger \otimes C) \\ = \text{tr}(\Pi\rho)(\psi \otimes \gamma_E^0) + (1 - \text{tr}(\Pi\rho)) \left(\left(\frac{|\text{P}(\mathcal{H}_A)|(U_{\hat{A}} \otimes U_A) - \psi_{A\hat{A}}}{|\text{P}(\mathcal{H}_A)| - 1} \right) \otimes \gamma_E^1 \right) \\ \approx \frac{2}{4^{|\mathcal{A}|}} \text{tr}(\Pi\rho)\psi + (1 - \text{tr}(\Pi\rho))(U_{\hat{A}} \otimes U_A \otimes \gamma_E^1). \end{aligned} \quad (166)$$

Proof. The proof proceeds along the lines of the proof of Lemma B.9. We provide the proof here for completeness.

Let $|\phi\rangle_{\hat{A}AE}$ be an eigenvector of $\rho_{\hat{A}AE}$. Consider the decomposition

$$|\phi\rangle_{\hat{A}AE} = \sum_{P \in \text{P}(\mathcal{H}_A)} \alpha_P (\mathbb{I} \otimes P) |\psi\rangle_{\hat{A}A} |\phi^P\rangle_E,$$

where $\sum_{P \in \text{P}(\mathcal{H}_A)} |\alpha_P|^2 = 1$. Define

$$\tau(P, Q) \stackrel{\text{def}}{=} (\mathbb{I} \otimes P) |\psi\rangle\langle\psi| (\mathbb{I} \otimes Q^\dagger).$$

Then, we have that

$$\begin{aligned}
& \frac{1}{|\mathcal{SC}(\mathcal{H}_A)|} \sum_{C \in \mathcal{SC}(\mathcal{H}_A)} (C^T \otimes C^\dagger) |\phi\rangle\langle\phi| ((C^T)^\dagger \otimes C) \\
&= \frac{1}{|\mathcal{SC}(\mathcal{H}_A)|} \sum_{C \in \mathcal{SC}(\mathcal{H}_A)} (C^T \otimes C^\dagger) \left(\sum_{P, Q \in \mathcal{P}(\mathcal{H}_A)} \alpha_P \alpha_Q^* \tau(P, Q) \otimes |\phi^P\rangle\langle\phi^Q| \right) ((C^T)^\dagger \otimes C) \\
&= \sum_{P, Q \in \mathcal{P}(\mathcal{H}_A)} \alpha_P \alpha_Q^* \left(\frac{1}{|\mathcal{SC}(\mathcal{H}_A)|} \sum_{C \in \mathcal{SC}(\mathcal{H}_A)} (C^T \otimes C^\dagger) \tau(P, Q) ((C^T)^\dagger \otimes C) \otimes |\phi^P\rangle\langle\phi^Q| \right) \\
&= \sum_{P, Q \in \mathcal{P}(\mathcal{H}_A) \wedge (P \neq Q)} \alpha_P \alpha_Q^* \left(\frac{1}{|\mathcal{SC}(\mathcal{H}_A)|} \sum_{C \in \mathcal{SC}(\mathcal{H}_A)} (C^T \otimes C^\dagger) \tau(P, Q) ((C^T)^\dagger \otimes C) \otimes |\phi^P\rangle\langle\phi^Q| \right) \\
&+ \sum_{P, Q \in \mathcal{P}(\mathcal{H}_A) \wedge (P=Q=\mathbb{I}_A)} \alpha_P \alpha_Q^* \left(\frac{1}{|\mathcal{SC}(\mathcal{H}_A)|} \sum_{C \in \mathcal{SC}(\mathcal{H}_A)} (C^T \otimes C^\dagger) \tau(P, Q) ((C^T)^\dagger \otimes C) \otimes |\phi^P\rangle\langle\phi^Q| \right) \\
&+ \sum_{P, Q \in \mathcal{P}(\mathcal{H}_A) \wedge (P=Q \neq \mathbb{I}_A)} \alpha_P \alpha_Q^* \left(\frac{1}{|\mathcal{SC}(\mathcal{H}_A)|} \sum_{C \in \mathcal{SC}(\mathcal{H}_A)} (C^T \otimes C^\dagger) \tau(P, Q) ((C^T)^\dagger \otimes C) \otimes |\phi^P\rangle\langle\phi^Q| \right) \\
&= |\alpha_{\mathbb{I}_A}|^2 |\psi\rangle\langle\psi| \otimes |\phi^{\mathbb{I}}\rangle\langle\phi^{\mathbb{I}}| \\
&+ \sum_{P \in \mathcal{P}(\mathcal{H}_A) \setminus \mathbb{I}_A} |\alpha_P|^2 \left(\frac{1}{|\mathcal{SC}(\mathcal{H}_A)|} \sum_{C \in \mathcal{SC}(\mathcal{H}_A)} (C^T \otimes C^\dagger) \tau(P, P) ((C^T)^\dagger \otimes C) \otimes |\phi^P\rangle\langle\phi^P| \right) \\
&= |\alpha_{\mathbb{I}_A}|^2 |\psi\rangle\langle\psi| \otimes |\phi^{\mathbb{I}}\rangle\langle\phi^{\mathbb{I}}| + (1 - |\alpha_{\mathbb{I}_A}|^2) \left(\frac{|\mathcal{P}(\mathcal{H}_A)|(U_A \otimes U_{\hat{A}}) - \psi_{A\hat{A}}}{|\mathcal{P}(\mathcal{H}_A)| - 1} \otimes \left(\frac{(\sum_{P \in \mathcal{P}(\mathcal{H}_A) \setminus \mathbb{I}_A} |\alpha_P|^2 |\phi^P\rangle\langle\phi^P|)}{(1 - |\alpha_{\mathbb{I}_A}|^2)} \right) \right).
\end{aligned}$$

The last equality follows from [Fact A.5](#) and the transpose method. Now, the first equality in [Equation \(166\)](#) from the lemma statement follows by observing that $\rho_{\hat{A}AE}$ is a convex combination of its eigenvectors, and the approximation in [Equation \(166\)](#) follows from [Fact 2.3](#) by observing that

$$\left\| \frac{|\mathcal{P}(\mathcal{H}_A)|(U_A \otimes U_{\hat{A}}) - \psi_{A\hat{A}}}{|\mathcal{P}(\mathcal{H}_A)| - 1} - (U_A \otimes U_{\hat{A}}) \right\|_1 \leq \frac{2}{|\mathcal{P}(\mathcal{H}_A)|}.$$

□

C Secret Sharing Schemes Resilient to Joint Quantum Leakage

In this section, we show that simple modifications to a recent construction of leakage resilient secret sharing schemes by [CKOS22] can be made secure against quantum leakage, even when the leakage adversaries are allowed to jointly leak a quantum state from an unauthorized subset (of size k) to another (of size $< t$). We refer to this leakage model as $\mathcal{F}_{k,\mu}^{n,t}$. The main result of this section

Theorem C.1. *For every $k < t \leq p < l, \mu \in \mathbb{N}$ there exists an $(p, t, 0, 0)$ threshold secret sharing scheme on messages of l bits and shares of size $l + \mu + o(l, \mu)$ bits, which is perfectly private and $p \cdot 2^{-\tilde{\Omega}(\sqrt[3]{\frac{l+\mu}{p}})}$ leakage resilient against the k local μ qubit leakage family $\mathcal{F}_{k,\mu}^{p,t}$.*

We organize the rest of this section as follows. In [Appendix C.1](#), we present the relevant background on quantum secure extractors, and recall the relevant secret sharing definitions. In [Appendix C.2](#), we present the code construction, and in [Appendix C.3](#) its proof of security. Finally, we instantiate our construction using specific secret sharing schemes and extractors in [Appendix C.3.1](#).

C.1 Preliminaries

C.1.1 Leakage Resilient Secret Sharing

We refer the reader to [Section 2.5.2](#) for a more comprehensive background on secret sharing.

Definition C.1 (Leakage-Resilient Secret Sharing). *Let $(\text{Share}, \text{Rec})$ be a secret sharing scheme with randomized sharing function $\text{Share} : \mathcal{M} \rightarrow \{\{0, 1\}^l\}^p$, and let \mathcal{F} be a family of leakage channels. Then Share is said to be $(\mathcal{F}, \varepsilon_{lr})$ leakage-resilient if, for every channel $\Lambda \in \mathcal{F}$,*

$$\forall m_0, m_1 \in \mathcal{M} : \Lambda(\text{Share}(m_0)) \approx_{\varepsilon_{lr}} \Lambda(\text{Share}(m_1)) \quad (167)$$

Definition C.2 (Quantum k local leakage model). *For any integer sizes p, t, k and leakage length (in qubits) μ , we define the (p, t, k, μ) -local leakage model to be the collection of channels specified by*

$$\mathcal{F}_{k,\mu}^{p,t} = \left\{ (T, K, \Lambda) : T, K \subset [p], |T| < t, |K| \leq k, \text{ and } \Lambda : \{0, 1\}^{l \cdot |K|} \rightarrow \mathcal{L}(\mathcal{H}_\mu) \right\} \quad (168)$$

Where $\log \dim(\mathcal{H}_\mu) = \mu$. A leakage query $(T, K, \Lambda) \in \mathcal{F}_{k,\mu}^{p,t}$ on a secret m is the density matrix:

$$(\mathbb{I}_T \otimes \Lambda_K)(\text{Share}(m)_{T \cup K}) \quad (169)$$

In their constructions [CKOS22] leverage secret sharing schemes augmented to satisfy a ‘‘local uniformity’’ property, where individual shares given out by the Share function are statistically close to the uniform distribution over the share space. To extend their construction to k local tampering, we require k wise independence:

Definition C.3. *A randomized sharing function $\text{Share} : \{0, 1\}^l \rightarrow \{\{0, 1\}^l\}^p$ is ε_u -approximately k wise independent if for every message $m \in \{0, 1\}^l$ and subset $S \subset [p]$ of size k :*

$$\text{Share}(m)_S \approx_{\varepsilon_u} U_l^{\otimes k} \quad (170)$$

We note that (p, t) Shamir Secret Sharing [Sha79] is exactly $(t - 1)$ wise independent.

C.1.2 Quantum Min Entropy and Randomness Extractors

Definition C.4 (Quantum conditional min-entropy). *Let X, Y be registers with state space \mathcal{X}, \mathcal{Y} and joint state ρ . We define the conditional min-entropy of X given Y as*

$$H_\infty(X|Y)_\rho = -\log \min_{\sigma \in \mathcal{Y}} \{ \min_{\lambda \in \mathbb{R}} \lambda \cdot \mathbb{I} \otimes \sigma \geq \rho \}. \quad (171)$$

When ρ is a cq-state, $H_\infty(X|Y)_\rho$ has an operational meaning in terms of the optimal guessing probability for X given Y . We remark that product states $\rho = \tau_X \otimes \sigma_Y$ have conditional min entropy $H_\infty(X|Y)_\rho = H_\infty(X)_\rho = -\log \lambda_{\max}(\tau)$ equal to the log of the largest eigenvalue of τ_X . When ρ is separable, it satisfies a Chain rule:

Lemma C.2 (Separable chain rule for quantum min-entropy [DD07], Lemma 7). *Let A, B, C be registers with some joint, separable state $\rho = \sum_i \tau_i^{AB} \otimes \sigma_i^C$. Then,*

$$H_\infty(A|B, C)_\rho \geq H_\infty(A|B)_\rho - \log |C| \quad (172)$$

Definition C.5 (Quantum-proof seeded extractor [DPVR12]). *A function $\text{Ext} : \{0, 1\}^\eta \times \{0, 1\}^d \rightarrow \{0, 1\}^l$ is said to be a $(\eta, \tau, d, l, \varepsilon_{\text{Ext}})$ -strong quantum-proof seeded extractor if for any cq-state $\rho \in \mathcal{H}^{\otimes n} \otimes \mathcal{Y}$ of the registers X, Y with $H_\infty(X|Y)_\rho \geq \tau$, we have*

$$\text{Ext}(X, S), Y, S \approx_\varepsilon U_l, Y, S \text{ where } S \leftarrow \{0, 1\}^d \quad (173)$$

Moreover, if $\text{Ext}(\cdot, s)$ is a linear function for all $s \in \{0, 1\}^d$, then Ext is called a linear seeded extractor.

Lemma C.3 ([Tre01, DPVR12]). *There is an explicit $(\eta, \tau, d, l, \varepsilon)$ -strong quantum-proof linear seeded extractor with $d = O(\log^3(\eta/\varepsilon))$ and $l = \tau - O(d)$.*

We require the extractor to support efficient *pre-image sampling*. Given a seed s and some $y \in \{0, 1\}^l$, the inverting function IExt needs to sample an element uniformly from the set $\text{Ext}(\cdot, s)^{-1}(y) = \{w : \text{Ext}(w; s) = y\}$. [CKOS22] showed that linear extractors always admit such sampling:

Lemma C.4 ([CKOS22]). *For every efficient linear extractor Ext , there exists an efficient randomized function $\text{IExt} : \{0, 1\}^l \times \{0, 1\}^d \rightarrow \{0, 1\}^\eta \cup \perp$ (termed inverter) such that*

1. $U_\eta, U_d, \text{Ext}(U_\eta; U_d) \equiv \text{IExt}(\text{Ext}(U_\eta; U_d), U_d), U_d, \text{Ext}(U_\eta; U_d)$
2. For each $(s, y) \in \{0, 1\}^d \times \{0, 1\}^l$:
 - (a) $\mathbb{P}[\text{IExt}(y, s) = \perp] = 1$, if and only if there exists no $w \in \{0, 1\}^\eta$ such that $\text{Ext}(w; s) = y$.
 - (b) $\mathbb{P}[\text{Ext}(\text{IExt}(y, s), s) = y] = 1$, if there exists some $w \in \{0, 1\}^\eta$ such that $\text{Ext}(w; s) = y$

C.2 Code Construction

Our code construction uses essentially the same ingredients as [CKOS22], with small modifications to the locality of their privacy parameters and to their compiler.

1. (MShare, MRec), an $(p, t, \varepsilon_{\text{priv}}, 0)$ threshold secret sharing scheme which is (k, ε_u) -locally uniform over the message space $\{0, 1\}^l$ and with share size l' .
2. (SdShare, SdRec), an $(p, k + 1, \varepsilon'_{\text{priv}}, 0)$ threshold secret sharing scheme over the message space $\{0, 1\}^d$ and share size d' .
3. Ext, a quantum-proof $(\eta, \tau \leq \eta - \mu, d, l', \varepsilon_{\text{ext}})$ -strong linear extractor. Let IExt be the inverter function corresponding to Ext given by Lemma C.4.

Share To share a message m , we begin by encoding it into $(M_1, \dots, M_p) \leftarrow \text{MShare}(m)$. Then, for each party $i \in [p]$, we sample a random seed $R_i \in \{0, 1\}^d$, and then use IExt to get the source $W_i \leftarrow \text{IExt}(M_i, R_i)$. If any of the $W_i = \perp$ rejects, output each of the share to be (\perp, M_i) . Else, concatenate the randomness $R = (R_1, \dots, R_p)$ and share it using SdShare(R) to get (S_1, \dots, S_p) , and finally set the i -th share to be (W_i, S_i) .

Rec Assuming the encoding doesn't reject, an authorized party $T \subset [p]$ of size $\geq t > k$ begins by recovering the randomness $R = (R_1, \dots, R_p)$ using SdRec on any $k + 1$ (honest) shares of T . Then, they recover the message shares $M_T = \{M_i : i \in T\}$ by running the extractor on (W_i, R_i) for each $i \in T$. Finally, using MRec on M_T it decodes the message m .

Theorem C.5. (Share, Rec) defines a $(p, t, \varepsilon_{\text{priv}}, 0)$ secret sharing scheme, which is $\leq 2(\varepsilon_{\text{priv}} + \varepsilon'_{\text{priv}}) + 2p \cdot (\varepsilon_{\text{Ext}} + \varepsilon_u)$ leakage resilient against $\mathcal{F}_{k,\mu}^{p,t}$.

C.3 Analysis

The correctness and privacy of the scheme are inherited from that of MShare, MRec. We analyze its rate in the next subsection [Appendix C.3.1](#), and dedicate the rest of this subsection to a proof of security.

Fix a leakage channel (T, K, Λ) and a message m . We assume $|T| \leq t - 1$ and $|K| \leq k$ are both unauthorized subsets. We proceed in a sequence of hybrids, where within the encoding map Share we replace the shares of K , $W_i \leftarrow \text{IExt}(M_i, R_i)$, by a uniformly random source $W_i \leftarrow U$:

$\text{Share}^0(m)$: To share a message m , we simply encode it into $\text{Share}(m)$.

$\text{Share}^1(m)$: To share a message m , we begin by encoding it into $(M_1, \dots, M_p) \leftarrow \text{MShare}(m)$. For each $i \in [n]$, sample a random seed $R_i \in \{0, 1\}^d$ and use IExt to get the source $W_i \leftarrow \text{IExt}(M_i, R_i)$. Then, concatenate the randomness $R = (R_1, \dots, R_p)$ and share it using SdShare(R) to get (S_1, \dots, S_p) , and finally set the i -th share to be (W_i, S_i) .

$\text{Share}^2(m)$: To share a message m , we begin by encoding it into $(M_1, \dots, M_p) \leftarrow \text{MShare}(m)$. For each $i \in K$, sample a random seed $R_i \in \{0, 1\}^d$ and source W_i . For each $i \in [p] \setminus K$, sample a random seed $R_i \in \{0, 1\}^d$ and use IExt to get the source $W_i \leftarrow \text{IExt}(M_i, R_i)$. Then, concatenate the randomness $R = (R_1, \dots, R_p)$ and share it using SdShare(R) to get (S_1, \dots, S_p) , and finally set the i -th share to be (W_i, S_i) .

Note that Share^0 differs from Share^1 in that it conditions on the extractor pre-image sampling succeeding. [\[CKOS22\]](#) begin by proving that it succeeds with high probability, and thus $\text{Share}(m) \approx \text{Share}^1(m)$.

Claim C.6 ([\[CKOS22\]](#)). For any message m , $\text{Share}(m) = ((\perp, M_1), \dots, (\perp, M_p))$ with probability $\leq p(\varepsilon_{\text{Ext}} + \varepsilon_u)$.

Moreover, note that $\text{Share}^2(m)$ is completely independent of the shares $M_i : i \in K$, and thus the reduced density matrix $\text{Share}_{T \cup K}^2$ only depends on the shares of $M_i : i \in T$ - where $|T| \leq t - 1$. By the privacy of MShare,

Claim C.7 ([\[CKOS22\]](#)). For any pair of messages m, m' , $\text{Share}_{T \cup K}^2(m) \approx_{\varepsilon_{\text{priv}}} \text{Share}_{T \cup K}^2(m')$.

It remains to argue that $\text{Share}^1(m)$ and $\text{Share}^2(m)$ are indistinguishable, given the shares in the unauthorized subset T and the leakage L .

Claim C.8. For any message m ,

$$(\mathbb{I}_T \otimes \Lambda_K)(\text{Share}_{T \cup K}^1(m)) \approx_{\delta} (\mathbb{I}_T \otimes \Lambda_K)(\text{Share}_{T \cup K}^2(m)) \quad (174)$$

where $\delta \leq 2(\varepsilon'_{\text{priv}} + \varepsilon_u + k \cdot \varepsilon_{\text{Ext}})$.

By the triangle inequality and the claims above, we conclude that for all m, m' :

$$(\mathbb{I}_T \otimes \Lambda_K)(\text{Share}_{T \cup K}(m)) \approx_{\delta'} (\mathbb{I}_T \otimes \Lambda_K)(\text{Share}_{T \cup K}(m')) \quad (175)$$

For some $\delta' \leq \delta + 2\varepsilon_{\text{priv}} + 2 \cdot p(\varepsilon_{\text{Ext}} + \varepsilon_u) \leq 2(\varepsilon_{\text{priv}} + \varepsilon'_{\text{priv}}) + 2p \cdot (\varepsilon_{\text{Ext}} + \varepsilon_u)$, which is exactly [Theorem C.5](#). Now we prove [Claim C.8](#):

Proof. [of [Claim C.8](#)] Fix a message m . Consider the quantum-classical mixed state comprised of the classical shares $M_K = \{M_i : i \in K\}$ of MShare, the seed R and its shares $S_K = \{S_i : i \in K\}$, and the quantum leakage register L . Our goal will be to show that this cq density matrix is nearly independent of the ‘‘source’’

register in the shares of K . That is, if we denote as $W_i, i \in K$ as uniformly random sources on η bits, then it suffices to show that for some δ ,

$$\Lambda(\{S_i, \text{Ext}(M_i, R)\}_{i \in K}, R, S_K, M_K) \approx_\delta \Lambda(\{S_i, W_i\}_{i \in K}, R, S_K, M_K). \quad (176)$$

This is since there is a CPTP map \mathcal{N}^m (dependent on the message) which given R, S_K, M_K and the leakage L produces

$$\mathcal{N}^m(\Lambda(\{S_i, \text{Ext}(M_i, R)\}_{i \in K}, R, S_K, M_K)) = (\mathbb{I}_T \otimes \Lambda_K)(\text{Share}_{T \cup K}^1(m)) \quad \text{and} \quad (177)$$

$$\mathcal{N}^m(\Lambda(\{S_i, W_i\}_{i \in K}, R, S_K, M_K)) = (\mathbb{I}_T \otimes \Lambda_K)(\text{Share}_{T \cup K}^2(m)). \quad (178)$$

Note that \mathcal{N}^m simply samples shares M_T, S_T consistent with m, R and M_K, S_K .²² Thereby by monotonicity of trace distance we obtain the desired $(\mathbb{I}_T \otimes \Lambda_K)(\text{Share}_{T \cup K}^1(m)) \approx_\delta (\mathbb{I}_T \otimes \Lambda_K)(\text{Share}_{T \cup K}^2(m))$.

It remains to show Equation (176). We begin by replacing the shares in K by fixed shares independent of $R, \hat{S}_K \leftarrow \text{SdShare}(0^d)_K$ using the privacy of SdShare , and leveraging the ε_u approximate k -wise independence of the shares M_K to replace them by the uniform distribution:

$$\Lambda(\{S_i, \text{Ext}(M_i, R)\}_{i \in K}, R, S_K, M_K) \approx_{\varepsilon'_{\text{priv}}} \Lambda(\{\hat{S}_i, \text{Ext}(M_i, R)\}_{i \in K}, R, \hat{S}_K, M_K), \quad \text{and} \quad (179)$$

$$\Lambda(\{\hat{S}_i, \text{Ext}(M_i, R)\}_{i \in K}, R, \hat{S}_K, M_K) \approx_{\varepsilon_u} \Lambda(\{\hat{S}_i, \text{Ext}(U_i, R_i)\}_{i \in K}, R, \hat{S}_K, (U_i)^{\otimes k}) \quad (180)$$

Recall W_1, \dots, W_K are uniformly random η bit sources. By Definition C.5, since Ext is a strong linear seeded extractor, $\text{Ext}(W_i, R_i) \approx_{\varepsilon_{\text{Ext}}} U_i$, and moreover by Lemma C.4(b) from [CKOS22] we have $W_i = \text{Ext}(\text{Ext}(W_i, R_i), R_i)$. Thus,

$$\Lambda(\{\hat{S}_i, \text{Ext}(U_i, R_i)\}_{i \in K}, R, \hat{S}_K, (U_i)^{\otimes k}) \approx_{k \cdot \varepsilon_{\text{Ext}}} \Lambda(\{\hat{S}_i, W_i\}_{i \in K}, R, \hat{S}_K, \{\text{Ext}(W_i, R_i)\}_{i \in K}) \quad (181)$$

We now replace each $\text{Ext}(W_j, R_j)$ by U_j in a sequence of hybrids, evoking *quantum-proof* extractor security. This is the main modification to the proof of [CKOS22]: Fix $0 \leq j \leq k$, and define the collection of classical registers Z_j :

$$Z_j = R_{[n] \setminus \{j\}}, \hat{S}_K, (U_i)^{\otimes (j-1)}, \{\text{Ext}(W_i, R_i)\}_{j < i \leq k} \quad (182)$$

Note that Z_j is independent of W_j . If L denotes the μ qubit leakage register, then from the chain rule for the min entropy of separable states, Lemma C.2 [DD07], $H_\infty(W_j | Z_j, L) \geq \eta - \mu \geq \tau$. By Definition C.5,

$$\Lambda(Z_j, W_j), Z_j, R_j, \text{Ext}(W_i, R_i) \approx_{\varepsilon_{\text{Ext}}} \Lambda(Z_j, W_j), Z_j, R_j, U_i \quad (183)$$

Which implies through the triangle inequality,

$$\Lambda(\{\hat{S}_i, W_i\}_{i \in K}, R, \hat{S}_K, \{\text{Ext}(W_i, R_i)\}_{i \in K}) \approx_{k \cdot \varepsilon_{\text{Ext}}} \Lambda(\{\hat{S}_i, W_i\}_{i \in K}, R, \hat{S}_K, U^{\otimes k}). \quad (184)$$

By once again appealing to k wise independence of M_K and the privacy of S_K , we conclude

$$\Lambda(\{S_i, \text{Ext}(M_i, R)\}_{i \in K}, R, S_K, M_K) \approx_{2(\varepsilon'_{\text{priv}} + \varepsilon_u + k \cdot \varepsilon_{\text{Ext}})} \Lambda(\{S_i, W_i\}_{i \in K}, R, S_K, M_K) \quad (185)$$

That is, $\delta \leq 2(\varepsilon'_{\text{priv}} + \varepsilon_u + k \cdot \varepsilon_{\text{Ext}})$. \square

C.3.1 Parameters

We combine

1. (MShare, MRec) is a $(p, t, \varepsilon_{\text{priv}} = 0, \varepsilon_c = 0)$ -Shamir secret sharing scheme for l bit messages and l bit shares, which is perfectly $t - 1$ wise independent.

²²This is also known as “consistent resampling” [CKOS22]

2. We set $\varepsilon_{\text{Ext}} = 2^{-\lambda^{-1/3}}$, and let Ext be the $(\eta = l + \mu + O(d), \tau = l + O(d), d, l, \varepsilon_{\text{Ext}})$ quantum proof strong linear extractor guaranteed by [Lemma C.3](#), where $d = O(\log^3 \frac{\eta}{\varepsilon_{\text{Ext}}}) = O(\lambda + \log^3(l + \mu))$.
3. $(\text{SdShare}, \text{SdRec})$ is a $(p, k + 1, \varepsilon'_{\text{priv}} = 0, \varepsilon'_c = 0)$ -Shamir secret sharing scheme for $p \cdot d$ bit messages and $p \cdot d$ bit shares.

The resulting share size of Share to handle μ qubits of leakage is $p \cdot d + \eta = l + \mu + O(p \cdot \lambda + p \cdot \log^3(l + \mu))$, which is $l + \mu + o(l, \mu)$ whenever $\lambda = o(\frac{l + \mu}{p})$ and $p = O(\frac{l + \mu}{\log^3(l + \mu)})$.

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