

Trust, But Verify: A Survey of Randomized Smoothing Techniques

Anupriya Kumari*
ECE dept.
IIT Roorkee
anupriya_k@ece.iitr.ac.in

Devansh Bhardwaj*
ECE dept.
IIT Roorkee
d_bhardwaj@ece.iitr.ac.in

Sukrit Jindal*
DSAI dept.
IIT Roorkee
sukrit_j@dfs.iitr.ac.in

Sarthak Gupta
Mathematics dept.
IIT Roorkee
sarthak_g@ma.iitr.ac.in

Abstract—Machine learning models have demonstrated remarkable success across diverse domains but remain vulnerable to adversarial attacks. Empirical defence mechanisms often fall short, as new attacks constantly emerge, rendering existing defences obsolete. A paradigm shift from empirical defences to certification-based defences has been observed in response. Randomized smoothing has emerged as a promising technique among notable advancements. This study reviews the theoretical foundations, empirical effectiveness, and applications of randomized smoothing in verifying machine learning classifiers. We provide an in-depth exploration of the fundamental concepts underlying randomized smoothing, highlighting its theoretical guarantees in certifying robustness against adversarial perturbations. Additionally, we discuss the challenges of existing methodologies and offer insightful perspectives on potential solutions. This paper is novel in its attempt to systemise the existing knowledge in the context of randomized smoothing.

Index Terms—Deep learning, adversarial attacks, randomized smoothing, certification-based defense

I. INTRODUCTION

Machine Learning (ML) has significantly advanced, notably since the development of deep neural networks (DNNs) [79]. However, a significant challenge is the susceptibility of state-of-the-art networks to adversarial examples [65], [68] limiting their application in safety-critical areas [77]. Adversarial attacks include techniques like DeepFool [95], AutoAttack [96], and patch-based attacks [97]. Existing defenses against adversarial attacks have proven ineffective against stronger methods, creating a continual cat-and-mouse game between attackers and defenders where attackers continually devise stronger attacks [81], and defenders attempt to provide robustness against these attacks citeb111. Empirical and heuristic defenses include denoising autoencoders [98], using training-based defenses like adversarial training [99], and defensive distillation [100]. Despite efforts to fortify ML models, it has been suggested that creating DNNs inherently robust to adversarial examples may be inevitable [101], [102]. Some approaches include those explicitly based on the architecture of the DNN or otherwise [84], [106]. Further, it has been shown that enhancing robustness might come at the cost of a drop in accuracy [73], [78]. This ongoing struggle has led researchers to explore certified robustness as an alternative.

A. Adversarial Examples

Adversarial examples are subtle input perturbations causing the classifier to give incorrect outputs. Consider a classifier from \mathbb{R}^d to classes C , mathematically defined as, $f : \mathbb{R}^d \rightarrow [C]$, that assigns a label to an input $x \in X$, we define an adversarial example x_0 as one where $D(x, x_0) < \epsilon$ for a small $\epsilon > 0$, and $f(x) \neq f(x_0)$. Here, $D(\cdot, \cdot)$ represents a distance metric, often expressed using an l_p norm, with l_1 , l_2 and l_∞ norms being the most commonly used in the literature.

B. Empirical and Certified Robustness

Given a classifier f , We say f is adversarially robust if, when provided with an input x with the actual label y , it can consistently classify all inputs within a certain p -norm ball ($B_{p,r}(x)$) of radius r centered at x correctly. In simpler terms, f is robust if it correctly classifies nearby inputs. Further, we define empirical and certified defenses. Empirical defenses against adversarial examples refer to strategies implemented to safeguard machine learning models from adversarial examples. These empirical strategies are developed and validated based on experimental findings rather than theoretical guarantees. In contrast, certified defenses against adversarial examples refer to strategies that provide mathematically provable assurances that a machine learning model can withstand adversarial attacks within certain constraints. Empirical robustness is often determined by attempting to find inputs (x_0) within the ball $B_{p,r}(x)$ that are misclassified by f , providing an upper estimate of the classifier’s actual robustness. On the other hand, certified robustness serves as a provable lower bound on the classifier’s robustness, ensuring a certain level of correctness within the specified region.

C. Randomized Smoothing

Randomized smoothing [2], [72] has emerged as a powerful certification-based defense against adversarial attacks. The fundamental idea behind randomized smoothing is to create a smoothed classifier by applying a convolution with Gaussian noise to the base classifier. Given a base classifier f , We create a smoothed version of this classifier, denoted as g , which predicts the most likely class c that the base classifier

*Equal Contribution

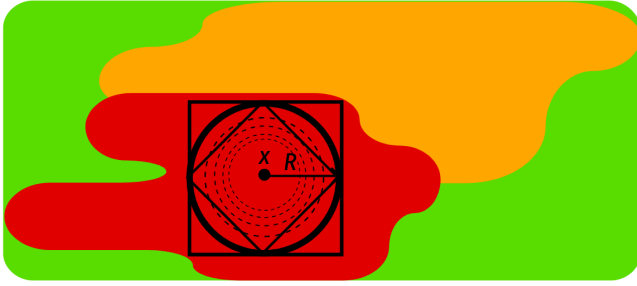


Fig. 1. This figure showcases the decision boundaries for l_1 , l_2 , and l_∞ norms, visually representing their sensitivity to adversarial perturbations. The diamond, circle, and square-shaped boundaries signify l_1 , l_2 , and l_∞ norms, respectively, showcasing the technique’s impact on different normative spaces for a given certified radius.

f will predict for a noised version of the input $x + \epsilon$, where $\epsilon \sim \mathcal{N}(0, \sigma^2 I)$. Thus, g can be defined as:

$$g(x + \epsilon) = \arg \max_{c \in [C]} P(f(x + \epsilon) = c), \text{ where } \epsilon \sim \mathcal{N}(0, \sigma^2 I) \quad (1)$$

Lecuyer et al. [72] and Li et al. [2] showed that the smoothed classifier g will consistently classify within a certified radius around the input x under l_2 norm considerations. However, the guarantees provided by these papers were loose due to a noticeable gap between the theoretical lower bounds and the empirical accuracy, suggesting the existence of a tighter radius around the input x than those predicted.

Cohen et al. [1] was the first paper to prove tight robustness guarantees for a randomized smoothing classifier against adversarial attacks constrained under the l_2 norm. Given the top two classes, c_A and c_B , that the base classifier is most likely to predict for the sample, the robust radius r is given by:

$$r = \frac{\sigma}{2} (\Phi^{-1}(p_A) - \Phi^{-1}(p_B)) \quad (2)$$

where $p_A = P(f(x + \epsilon) = c_A)$ and $p_B = P(f(x + \epsilon) = c_B)$, with the condition that $p_A > 0.5$ and Φ represents the CDF of the standard Gaussian distribution. This demonstrates that if g correctly predicts the input x , we can provide provable l_2 robustness of our model in a probabilistic sense. However, Cohen et al. [1] highlighted that it is practically not possible to calculate the exact values of the top two class probabilities, i.e., p_A and p_B . Hence, they introduce a modified version of the above statement that utilizes the Clopper-Pearson bounds for p_A and p_B . For p_A , the Clopper-Pearson lower bound, denoted as \underline{p}_A , and for p_B , the Clopper-Pearson upper bound, denoted as \overline{p}_B , can be calculated using Monte-Carlo sampling. To theoretically prove that using this combination of Clopper-Pearson and Monte-Carlo sampling will give us a lower bound on the actual certifiable radius r , they use Neyman-Pearson lemma (NP lemma). Furthermore, to reduce time complexity, they approximate \overline{p}_B with $1 - \underline{p}_A$, meaning that if the base classifier correctly classifies the input x , then any example x_0 within the l_2 distance of x , $D(x, x_0) < \sigma \Phi^{-1}(\underline{p}_A)$ will be correctly classified by the smoothed classifier g .

Following Cohen et al.’s seminal work [1], the concept of randomized smoothing has spurred a proliferation of studies. This paper endeavours to provide a comprehensive summary of these works, elucidating significant challenges, notable advancements and extracting key insights. The organization of the survey is structured into three main sections. Section II critically examines the major issues associated with randomized smoothing, categorizing these challenges into three broad domains. Subsequently, a detailed discussion ensues on these challenges, followed by an exploration of future research directions.

Moving to Section III, we delve into significant advancements that focus on enhancing radius certifications beyond those outlined in Cohen et al. [1]. This section also addresses the challenges presented in Section II, providing discussions for the same. Section IV broadens the perspective by discussing applications of randomized smoothing, which extend beyond the original focus of Cohen et al. [1], which was primarily limited to image classification tasks.

Rethinking Scalability in Randomized Smoothing: An Analytical Perspective

Due to its theoretical guarantees, randomized smoothing is commonly hailed as a crucial certification-based defense mechanism. However, through our analysis, we raise a fundamental question:

Despite all the theoretical assurances, is randomized smoothing truly scalable?

Throughout our examination, we highlight the limitations of existing works that predominantly emphasize theoretical guarantees while overlooking the critical aspect of computational complexity, especially during inference, and dimensionality considerations. This concern holds significant weight, particularly for machine learning applications requiring minimal inference time, where randomized smoothing may fall short as a practical solution. Hence, in addition to serving as a survey of randomized smoothing techniques, we scrutinize each work from the perspective of scalability. At the end of each section, we offer insights and discussions on how the various works discussed within that section may influence the scalability of randomized smoothing.

II. CHALLENGES

Randomized smoothing is a powerful architecture-independent technique for certified robustness, and it has only gained popularity over the years; however, as with any other method, it also has its limitations. In the following subsections we define and discuss three significant challenges in RS that commonly encountered in literature. These challenges affect the scalability and applicability of RS in many domains, which has currently offered promising empirical results in certified robustness.

A. Curse of Dimensionality

In their work, Cohen et al. [1] left the case of general l_p norms as an open problem, suspecting that smoothing with

other noise distributions might provide robustness guarantees for these cases, and only discussed randomized smoothing in the context of l_2 norms. Additionally, they state that, despite the certified radius being independent of the input dimension d , randomized smoothing can be easily scaled to higher dimension images as in the case of higher dimension images, we can increase the variance of the noise, which will eventually lead to increase in the robust radius. This approach has its issues, as will be discussed in the next section. Later, two independent works, Blum, Avrim, et al. [7] and Kumar, Aounon, et al. [11], showed that as the dimension of the input increases, it becomes increasingly difficult to defend against the general l_p norm cases for $p > 2$ as the certified radius decreases.

Blum, Avrim, et al. [7] showed that, given any noise distribution \mathcal{D} and the noise vector $\epsilon \sim \mathcal{D}$, 99% pixels of the noise vector, must satisfy $\mathbb{E}\epsilon_i^2 = \Omega(d^{1-\frac{2}{p}} r^2 \frac{1-\delta}{\delta^2})$, where r is the robust radius and δ is the difference between the probability scores of the top two classes predicted by the smoothed classifier (highest score-class c_A and runner-up class c_B). They further proved that the problem of finding the certified radius in the l_∞ case can be approximated as finding the l_2 radius times $1/\sqrt{d}$ of that radius.

Subsequently, Kumar, Aounon, et al. [11] show, in the specific case of a generalized Gaussian distribution, tighter bounds than those attained by Blum, Avrim, et al. [7]. Their work states that the certified radius for an l_p norm decreases as $O\left(d^{\frac{1}{2}-\frac{1}{p}}\right)$ with data dimension d for $p > 2$. The family of distributions discussed in this paper includes the Laplacian, Gaussian, and uniform distributions, commonly employed in literature for randomized smoothing techniques. Wu, Yihan, et al. [75] then extend the discussion for the case of spherical symmetric distributions (which includes the Gaussian distribution) and present upper bounds for $p > 2$. It also discusses upper bounds for $p = 2$ for many popular smoothing distributions, which have yet to be discussed. Yang, Greg, et al. [4] go into more detail about l_1 , l_2 , and l_∞ threat models to derive from the class of i.i.d. distributions, the most optimal smoothing distribution using the concept of Wulff Crystals. Overall, their results in this regard can be summarised as follows: the uniform distribution is the most optimal choice for l_1 attacks, the Gaussian distribution, as mentioned by Cohen et al. [1], is the most optimal for l_2 attacks, and that for l_∞ , the Gaussian distribution performs best using appropriate approximations. There is a tight $O\left(\min\left(1, d^{\frac{1}{2}-\frac{1}{p}}\right)\right)$ dimensionality bound for l_p norms without using any additional information other than the Neyman-Pearson technique.

Ettegui, Raphael, et al. [66] provide a different perspective and argue about the information-theoretic limitations faced by RS and how they are not intrinsic but a byproduct of the current certification methods. They further discuss that these certificates need to include more information about the classifier and ignore the local curvature of the decision boundary, which leads to sub-optimal robustness guarantees as the dimension of the problem increases. It is possible to

overcome this by generalizing the Neyman-Pearson Lemma and collecting more information about the classifier. The paper shows that it is possible to approximate the optimal certificate with arbitrary precision by probing the decision boundary with several noise distributions. More specifically, the paper achieves this without sampling in high dimensions by combining uniform and Gaussian distribution and leveraging the isotropic properties of the latter. Further, this process retains natural accuracy as it is executed at certification time. The paper then gives theoretical insight into how to mitigate the computational cost of a classifier-specific certification.

Pfrommer, Samuel et al. [41] present notable work in the sense that they implicitly propose a solution to the curse of dimensionality, suggested by Blum, Avrim, et al. [7]. It combines the limitations of randomized smoothing in light of the curse of dimensionality with the manifold hypothesis [74] and shows that using various dimensionality reduction techniques improves robust guarantees in theory, as opposed to heuristic approaches in previous works by Mustafa, Aamir, et al. [76] and Kurakin, Alexey et al. [77].

So far, only Li, Linyi, et al. [3] have provided proofs and a possible solution to the issue of the curse of dimensionality using a technique called double sampling randomized smoothing (DSRS). They utilize a primary distribution in the form of a general Gaussian distribution and additional information in the form of a sampled probability from a secondary truncated distribution that is not necessarily different from the primary distribution. For l_2 norm, they provide a tighter radius and prove that their method can certify $\Theta\left(\sqrt{d}\right)$ robust radius bounds. As mentioned by Cohen et al. [1], for the l_∞ case, the certified radius is equivalent to $1/\sqrt{d}$ times the l_2 robust radius, which on using [3] would remove the dependency on the dimensions of the input.

We note that there has been an in-depth theoretical exploration of the issue termed the curse of dimensionality. The base conclusion of all papers remains that there is a tight bound on certified radii that decreases with increasing dimensionality, which arises from statistical, probabilistic, and information-theoretic perspectives based on the existing methodology for RS. While we certainly encourage more research into that aspect, we notice the dire lack of solutions for the same. Apart from DSRS, no major breakthrough has been achieved to solve this issue. As a result, we encourage works to explore various solutions to this issue.

B. Robustness-Accuracy Trade-off

In adversarial robustness literature, not limited to RS, there is a focus on increasing the robustness bounds of a classifier. While this seems to be an optimal goal, there has been literature suggesting that increasing robustness comes with a tradeoff in accuracy [73], [78]. It has been studied for adversarial robustness in much detail, including theoretical considerations. In [78], the authors study the accuracy costs of improving robustness in a model. This was further explored in [73] and [80]. Both empirically and theoretically, it is seen that robustness and accuracy exist in a trade-off, contrary to what

one would expect intuitively. Randomized smoothing is also vulnerable to this trade-off, and there is much to explore in this regard, leaving this an important open aspect of research for randomized smoothing techniques. Naturally, this can be seen in RS as well. Randomized smoothing involves convoluting, or smoothing, the base classifier with Gaussian noise. While this provides certified robustness, it can be inferred that there is some loss in accuracy due to noisy inputs. Termed as the robustness-accuracy tradeoff, we explore this issue concerning RS in this subsection.

Specific to RS, the theoretical and numerical aspects of Gaussian smoothing are explored in the work by Mohapatra, Jeet, et al. [51], along with which they provide theoretical results showing the limitations of RS in terms of classification accuracy. Mainly, they characterize and identify the conditions under which Gaussian smoothing leads to a decrease in classification accuracy, provide theoretical lower bounds for the magnitude of this effect, numerically inspect the behaviour of the certified radius, and use tools from information theory to analyse the effects of Gaussian smoothing during training augmentation, concluding that it leads to information loss and finally validate their results empirically.

A theoretical analysis of robustness-accuracy in terms of the benign risk associated with applying randomized smoothing on a classifier trained using noise augmentation for the base classifier is given by Pal, Ambar et al. [33]. This arises from the observation of improved performance when using a noise-augmented base classifier instead of one without any such augmentation. They derive an upper bound for any data distribution used for smoothing, which suggests randomized smoothing harms the classifier’s accuracy. This practice has pervaded most literature with ample empirical support. The paper then suggests certain distributions where smoothing may be beneficial based on the separation regime. Also, it suggests optimisation for randomized smoothing, using different noise parameters to train the base classifier. They also discuss the theoretical intuition behind this, viz., the empirical observation that real data lies on a lower dimension than the actual data dimension.

A compositional architecture termed ACES, which aims to tackle this robustness-accuracy trade-off, is introduced in [49]. It decides on a per-sample basis whether to use the robust-but-less-accurate smoothed model or the accurate-but-less-robust base classifier for prediction using a learnable selection algorithm that can again be made robust by further smoothing. This approach improves the trade-off between robustness and accuracy in current models and can be used as an orthogonal optimisation to other methods. This is done in the light of optimising randomized smoothing specific robustness and accuracy using compositional architectures, as introduced and proposed earlier by Mueller, Mark Niklas et al. [93].

A notable work by Mohapatra, Ko, et al. [32] articulates and proves two major limitations regarding RS: (I) the decision boundary of the smoothed classifier will shrink, resulting in a discrepancy in class-wise accuracy, and (II) applying noise augmentation in the training process does not resolve the

shrinking issue because of inconsistent learning objectives. The paper proceeds to review adversarial robustness certification, exposes the significant hidden cost of RS, which includes biased predictions using evidence from both real-life and synthetic datasets, provides a comprehensive theory exposing the root of the biased prediction-shrinking phenomenon and then discusses the effects of data augmentation on the shrinking phenomenon and its implications. In essence, they arrive at two observations - the need to limit the use of high values of smoothing factor σ and a data geometry-dependent augmentation scheme to counteract the shrinking effect caused by smoothing properly. This challenges what we already know about randomized smoothing, which, as introduced by Cohen et al., uses data-independent noise parameters. Namely, the Gaussian noise, $\epsilon \sim \mathcal{N}(0, \sigma^2 I)$ which is convoluted with the base classifier is data-independent and σ can be thought of as a hyperparameter representing a trade-off between accuracy and robustness. Too low of a value will give smaller certified radii, which are given, after appropriate approximations, as the certified radius $r = \sigma \Phi^{-1}(p_A)$, which is directly proportional to σ . Using too large of a value for σ will reduce the accuracy in calculating p_A due to highly noisy training inputs.

Overall, we notice ample theoretical and empirical evidence supporting the existence, and its pervasivity in most RS techniques. However, there is very little literature aimed at actually tackling this problem. We strongly suggest more works to explore solutions to this challenge and improve existing methods like ACES using newer techniques and methodology, not just for RS but also for adversarial robustness in general.

C. High Inference Cost

Randomized smoothing utilizes Monte Carlo sampling, which requires multiple passes of the noisy samples through the model, leading to high inference costs, at the same time it has been shown that certified radius increases with the no. of passes or no. of samples (n) and for significant radius size typically 10^5 or more samples are required [1]. This implies that during inference we need to do a forward pass to have a significant certified radius and essentially increases the computational cost by a large margin.

A controlled way to trade off the average certified radius (ACR) with the number of samples required was introduced by Chen, Ruoxin, et al. [23]. They changed their sampling scheme from an input-agnostic scheme to an input-specific scheme. To achieve this input-specific sampling, first a relatively loose two-sided Clopper-Pearson interval is calculated for a given input using a smaller sample size. Then, a mapping of sample size to the relative decline in average radius is computed for the input. This mapping shows a trade-off between the number of samples and the average certified radius.

Another way of reducing the computation cost could be by lowering the inference time of the smoothed model by techniques such as pruning, quantization of the base model, or knowledge transfer. But this essentially means we have changed the actual model and the theoretical guarantees may not hold. Ugare, Shubham, et al. [10] study this scenario and

formulate an incremental certification technique for randomized smoothing called Incremental Randomized Smoothing (IRS). This improves performance on probabilistic techniques like randomized smoothing compared to existing deterministic techniques for incremental certification. In particular, incremental certification deals with certifying robustness bounds for the smoothed classifier of a given modification of a base classifier, reusing information from the smoothed classifier of the base classifier to reduce computational costs. It does so based on the disparity between the classifiers and certain approximation techniques. On a related note, Vaishnavi, Pratik, et al. [120] propose a similar knowledge transfer scheme, Certified Robustness Transfer (CRT), for randomized smoothing for training l_2 certifiably robust image classifiers with comparable levels of robustness using a pre-trained classifier that is certifiably robust as a teacher. Both techniques demonstrate significantly reduced costs, indicating the empirical effectiveness of their methods.

It is pretty clear from the number of works in this section that this aspect of randomized smoothing is virtually untouched and has a lot of scope for improvements and future directions.

Challenges in the Scalability of RS

In this section, we explored the challenges which affect RS. While the robustness-accuracy tradeoff, as noted, is a challenge which affects the entire field of adversarial robustness and even ML as a whole, the other two, viz., the curse of dimensionality and the issue of high inference costs, expressly limit the scalability of RS. Further, on a broad level, we note these challenges are acknowledged and understood in their impact, whether implicitly, as is true for computational costs, or explicitly, as is seen for the curse of dimensionality. However, contrary to expectations, little research is devoted to developing solutions that truly address these issues and improve the scalability of RS.

III. IMPROVING RANDOMIZED SMOOTHING

Randomized smoothing offers a tight lower bound on the certified radius for any classifier. As with any certified robustness technique, these bounds are lower than the upper bounds arising from empirical verification techniques, which are based on heuristics and empirical evidence instead of theoretical guarantees. Thus, there has been much research into improving the performance of randomized smoothing and improving its certified radii to an optimal limit. Further, there is a need to address the challenges faced in RS and develop various techniques. In this section, we explore specific optimisations that improve the robustness certificates given by RS and solve various issues encountered in RS.

A. Randomized Smoothing for Training

The idea of incorporating randomized smoothing objectives in adversarial training [87] roots from the intuition that doing so will result in boosted performance and improved robustness. Further, addressing the challenge of training a good base

classifier that is accurate and robust when smoothed [34] led to significantly better results as discussed in this section. Salman et al. [87] consider a smoothed classifier which is a generalization of equation (2), to soft classifiers, namely, functions $f : \mathbb{R}^d \rightarrow P(y)$ where $P(y)$ is the set of probability distributions over y . They then train a hard classifier which essentially takes the argmax of the soft classifier, using adversarial examples generated by describing an adversarial attack against the smoothed soft classifier [108], [109]. This results in a boosted empirical robustness and substantially improved certifiable robustness using the certification method of Cohen et al. [1]. Feng et al. [34] improved upon this by deriving a new regularized risk in which the regularization can 'adaptively encourage' the training of the base classifier. Specifically, they proposed ADRE (ADaptive Radius Enhancing regularizer), which penalizes examples misclassified by the smoothed classifier while encouraging the certified radius of the correctly classified examples. It is computationally efficient and can be implemented in parallel with other empirical defense methods, under standard (non-adversarial) and adversarial training schemes. They also designed a new certification algorithm, T-CERTIFY, that can leverage the regularization effect to provide tighter robustness and lower bounds that hold high probability. Alfara, Motasem, et al. [13] challenged these attack-dependent and time costly adversarial training techniques [1], [87] by introducing an algorithm called MACER, which learns robust models without using adversarial training. More specifically, they break the standard training loss into two subsequent losses: a standard cross-entropy loss and a hinge-based loss function applied to the robust radius. A modified version of the original randomized smoothing obtains the robust radius. Since it is a non-differentiable method, to make it differentiable, they take expectation over the probability scores each time a noisy sample is passed through the model and then take the maximum over all the classes. It is observed that MACER also outperforms Feng, Huijie, et al. [34] by a significant margin in terms of Certified top-1 accuracy on CIFAR-10 at various l_2 radii. On Imagenet, however, the latter gives better accuracy.

Thus, the challenge of training a base classifier that is accurate and robust stands resolved [34] along with some noteworthy contributions [13] that ultimately prove that adversarial training is not a must for robust training and larger average certified radii can be achieved in less training time than state-of-the-art adversarial training algorithms.

B. Improving Performance of Pre-Trained Classifiers

Improving the performance of pre-trained classifiers by turning them from non-robust into provably robust ones is desired to maintain their confidentiality, integrity, or availability. For example, it might be beneficial to public vision API providers and users. In their work towards achieving this, Salman, Hadi, et al. [18] apply randomized smoothing to pre-trained classifiers by attaching a denoiser before the pre-trained model. Intuitively, they remove the Gaussian noise added in the randomized smoothing, unlike traditional methods that focus on increasing the quality of images. They train the

denoiser with a modified loss function, which, instead of using the usual MSE, uses cross-entropy loss to train the combined denoiser and pre-trained classifier architecture, keeping the pre-trained classifiers’ weights frozen. This process also increases the classification accuracy. The methodology can be extended for other l_p cases by simply changing the noise used during training to a noise sampled from the distribution corresponding to the specific l_p case. Denoised smoothing can be applied to both white and black box settings, making it the state-of-the-art approach to defending pre-trained classifiers. Sheikholeslami, Fatemeh, et al. [54] challenge the low natural accuracy of this state-of-the-art approach and contribute further to the methodology by Salman, Hadi, et al. [18] by augmenting the joint system with a “rejector” and exploiting adaptive sample rejection, i.e., intentionally abstain from providing a prediction. The reject class is used, through Monte Carlo (MC) sampling, to reject noisy samples whose prediction is inconsistent with the prediction of the clean sample. This approach to preventing misclassified examples in the non-rejected classes has substantially improved the natural accuracy and the certification radius for computationally inexpensive denoisers. They achieve considerably better accuracy with their rejection methodology on CIFAR10 with both DnCNN-based denoising and MemNet-based denoising [110], [111]. Lee, Kyungmin, and Seyoon Oh. [46] also improve on previous works [18], [54] by introducing an improved score-based architecture for denoisers instead of classification loss, however, faced challenges concerning the novelty of their work and the possibility that the proposed method might exacerbate the weakness of randomized smoothing (i.e., slow prediction), especially in high-dimensions. Hence, while they produced higher certified accuracy of ResNet-110 on CIFAR-10 at various l_2 radii compared to previous works [1], [18], they failed to achieve similar improvements in certified accuracy of ResNet-50 on ImageNet, indicating that their method seems to be effective for low-resolution images only. While the work so far in this domain seems somewhat promising, there remains scope for future research to further improve the certified accuracy of high-dimensional datasets with increasing σ (σ as defined in section I, subsection C).

C. Leveraging Local Information

The original randomized smoothing technique makes use of Monte-Carlo sampling to derive certified radii. This information is fully summarised for an input x by $p_A(x)$, the probability of the top class, and does not require any other information beyond the standard deviation σ of the Gaussian. Levine et al., [14] propose the use of additional gradient information, viz. $p'_A(x)$, in a technique known as second-order smoothing (SoS) to provide more robust bounds. Their approach is based on the Lipschitz property of randomized smoothing classifiers [112], which globally bounds the Lipschitz constant of the gradient of a smoothed network. Thus, instead of using the Lipschitz bounds directly on the classifier, it utilizes the Lipschitz bounds on the gradient to obtain what the paper terms as second-order smoothing certificates, which

can be used to compute a lower bound on the distance of a point to its closest adversarial example and improve robustness certificates. It also introduces Gaussian dipole smoothing, which is an optimisation based on SoS and Gaussian randomized smoothing to provide similar radius bounds. The results in this paper are significant as they show that all randomized smoothing-based classifiers suffer from this curvature constraint, restricting their application and prompting further research into certificates obtained using higher-order derivative information. This is what Mohapatra, Ko, et al. [24] do by extending the usage of gradient information to provide a general framework for evaluating more optimal certified radii using higher-order derivatives around the input. They do so by reformulating radius calculation as an optimisation problem. It also evaluates the certified radii obtained using Gaussian-smoothed classifiers, which utilize zeroth and first-order local information for general l_p attacks. It also introduces a theoretical result for Gaussian-smoothed classifiers, which states that it is possible to achieve arbitrarily tight bounds using sufficient local information, thereby addressing the curse of dimensionality, which is a major challenge in RS and is discussed later on. Anderson et al. [20] utilize a locally optimal robust classifier based on Weierstrass transformations using decision boundary information. Their model not only improves on local information methods mentioned here but also addresses the issue of the robustness-accuracy trade-off for smoothing classifiers as discussed previously. Their study is limited to binary classifiers but remains unexplored for other cases, and this is a potential area for future works to explore.

Thus, one can see that there is an improvement in certified bounds using local information, such as the gradient and Hessian, and there is a possibility to explore and apply this approach orthogonally to other optimisations for randomized smoothing. A drawback of this approach is that the computational overheads associated with calculating local information add heavily to the already expensive smoothing convolution despite the gains being limited, as they also are tightly bound.

D. Ensemble-based methods

Randomized smoothing bounds can be improved further using ensemble-based methods. These methods work by aggregating the results of multiple classifiers, somewhat similar to how RS involves smoothing the base classifier with multiple noisy inputs and using those results to create the base classifier.

Horváth, Müller, et al. [57] theoretically motivate why ensembles are a particularly suitable choice as base models for RS and then empirically verify this choice, obtaining state-of-the-art results in multiple settings. The insight around which their work revolves is that reduced variance of ensembles over the perturbations introduced in RS leads to significantly more consistent classifiers. Additionally, they introduce a data-dependent adaptive sampling scheme for RS that enables a significant decrease in sample complexity of RS for predetermined radii, thus reducing its computational overhead. The modeling approach is mathematically related to the work by Tumer & Ghosh [91], which focuses on analyzing data and

model choices to reduce classifier correlation for all ensemble methods, not just randomized smoothing.

RS requires augmenting data with large amounts of noise, which leads to a drop in accuracy. To overcome this, there is a training-free, modified smoothing approach, Smooth-Reduce, that leverages patching of images and aggregation to improve classifier certificates [27]. This approach is different from a simple ensemble technique, which is computationally expensive, and rather involves patch sampling. For this, the authors study two aggregation schemes and show that both approaches provide better certificates in certified accuracy compared to concurrent approaches. They also provide theoretical guarantees for such certificates and empirically show significant improvements over other methods that require expensive retraining and even extend this approach to videos.

The Smoothed WEighted ENsembling (SWEEN) scheme improves the robustness of RS classifiers using ensemble techniques that generally can help achieve optimal certified robustness [45]. The authors make use of a γ -robustness index as a measure of optimality. Theoretical analysis further proves that the optimal SWEEN model can be obtained from training under mild assumptions. Additionally, SWEEN does not limit how individual candidate classifiers are trained. The model adopts a data-dependent weighted average of neural networks to serve as the base model for smoothing. To find the optimal SWEEN model, one must minimize the surrogate loss of g_{ens} over the training set and obtain appropriate weights. The major drawback of ensembling is the high execution cost during inference, for which the paper also develops an adaptive prediction algorithm to reduce the prediction and certification cost of the models.

From here, we see that ensemble-based methods for randomized smoothing have empirical and theoretical guarantees. Further, despite the prohibitive costs of ensemble training, adaptive algorithms for their training have been shown to reduce the computational overhead. Overall, ensemble methods provide a satisfactory approach to improve robustness guarantees with reduced costs as compared to other similarly expensive approaches. As with other approaches, we encourage the orthogonal application of these approaches and also a study on how to reduce ensemble training overheads further, possibly by advantageously using the noise augmentation already employed for randomized smoothing.

E. Input Specific Information

As discussed previously, RS uses a data-independent hyperparameter, σ , for the Gaussian noise $\epsilon \sim \mathcal{N}(0, \sigma^2 I)$ to be convoluted with the base classifier. Even though this forms a basis of the robustness-accuracy tradeoff, it is also important to note that the issue of an optimal choice of σ is important to improve certified robustness using TS. To deal with this, Alfara et al. [15] propose the use of additional information, as is suggested in the curse of dimensionality section, which can be extracted from the data itself, in a scheme known as Data Dependent Randomized Smoothing (DDRS). DDRS aims to improve the choice of the hyperparameter σ to achieve a

larger certified radius, a parameter traditionally fixed in the original Randomized Smoothing (RS) method. Specifically, DDRS addresses this by solving an optimisation problem, $\sigma_x^* = \arg \max_x R(x, \mu)$.

This optimisation is performed using gradient ascent, and it defines the smoothed classifier G by introducing sampling noise $\mathcal{N}(0, \sigma_x^{*2} I)$ for each input x . This approach has empirically demonstrated improved certified isotropic regions for individual data points. [44] also develop a related sample-wise choice of σ which they term as Insta-RS along with adaptive training optimisations for the same. A major difference is that Insta-RS uses a multi-search, gradient-free paradigm, while DDRS assumes concavity and opts for gradient ascent.

The concept of anisotropic smoothing aims to improve further on the results given by DDRS, which only certifies isotropic boundaries ([17], [115]). Anisotropic smoothing aims to certify, as the name suggests, different certified radii along each axis, essentially acting as a hyperelliptic curve instead of a hypersphere when considering the l_2 norm. Where normal randomized smoothing uses a constant variance noise of the form $\mathcal{N}(0, \sigma^2 I)$ and DDRS uses a data-dependent variance of the form $\mathcal{N}(0, \sigma_x^2 I)$ to certify robust bounds, anisotropic smoothing uses $\mathcal{N}(\mu, \Sigma)$ where Σ is a diagonal matrix of variance generated using data-dependent techniques. Both papers follow a similar approach, differing only in the fact that in [115], $\mu = 0$ is fixed, and they term their algorithm as ANCER. Their methodology is such that the certified bounds along each direction exceed not only the bounds presented in Cohen et al. [1], but also encapsulate the bounds given by isotropic data-dependent smoothing as in [15].

In an unrelated but still data-dependent approach, Wang et al. [60] propose a smoothing algorithm that optimises the noise level choice (measured as the variance of the isotropic Gaussian noise) by dividing the input space into multiple regions and optimizing the noise level for each region, pre-training and fine-tuning the model using testing data. This approach allows one to create a sample-wise robust classifier that addresses the accuracy-robustness trade-off and improves the certified radii. Their approach maintains robustness in each region of the input space as per Cohen’s results without affecting other regions.

Súkeník et al. [22] offers a concise overview of issues associated with a fixed global σ value including (I) Using lower confidence bounds for class probability estimation, leading to smaller certified radii, (II) Balancing the trade-off between robustness and accuracy, as previously discussed and (III) The effect of Randomized Smoothing (RS) on the classifier’s decision boundary, where bounded or convex regions contract with increasing σ while unbounded and anti-convex regions expand. The paper then tackles these issues by introducing Input-Specific Randomized Smoothing (ISRS) which employs the Neyman-Pearson lemma in a general case. However, they also state the effect of the curse of dimensionality and propose a framework to evaluate the efficacy of data-dependent RS methods in high dimensionality. It is a fundamental work

in that it identifies the shortcomings of previous works like DDRS, Insta-RS, ANCER, and [60] which do have significant empirical results, but weak theoretical foundations.

Riemannian data-dependent randomized smoothing (RD-DRS) [38] further optimises the approach in anisotropic smoothing by allowing it to be robust to rotations. It does so using Riemannian optimisation to obtain \mathcal{C} , a covariance matrix that is not restricted to a scalar matrix (as in normal randomized smoothing and DDRS) nor a diagonal matrix (as in anisotropic randomized smoothing). Here, the smoothed classifier is constructed using Gaussian noise of the form $\mathcal{N}(0, \mathcal{C})$. RANCER [116] also employs a non-axis-aligned approach. These methods empirically improve on the certified bounds introduced by even anisotropic smoothing techniques. As can be seen, there is clearly a pattern on increasing the complexity of the covariance matrix for the Gaussian noise to derive better results. As motivated by [22], however, we also note the weak theoretical justifications and encourage future work to analyse strongly the theoretical foundations to potentially justify and back the empirical results of approaches like DDRS, Insta-RS, ANCER, RDDRS, and RANCER.

Geometrically Informed Certified Robustness [5] is an approach which exploits the underlying geometric properties of a network using information around a point, such as the certified radii of nearby points, and exploits the transitivity of certifications. It is an optimisation that can be combined orthogonally with previous approaches, which only certify the region around a point without using any local information. We strongly suggest future empirical work to explore GICR in conjunction with other optimisations.

Randomized smoothing has primarily been applied to single output cases. [61] propose a localised smoothing algorithm for multi-output settings instead of using independent RS schemes. Their approach combines anisotropic smoothing and also segments each input into various parts, where different anisotropic noise distributions are used for smoothing. In practice, they utilise soft locality, i.e. when the entire input influences the output but specific subsets have higher importance, to optimise the certified bounds and aggregate the result for the overall classifier. Their approach is a novel approach which extends and optimises RS for the multi-output scenario, and can be explored further, potentially with non-axis aligned distributions as well, similar to how RANCER extends ANCER.

One of the issues mentioned in [22] is that of fairness where class boundary disparities affect classwise prediction accuracy and robustness. This also relates heavily to the discussion in robustness-accuracy tradeoff, where a paper highlighted the need to limit data geometry-dependent augmentation schemes for similar reasons [32]. Addressing this, Kumar et al. [21] use the average prediction score of a class and the margin by which the average prediction score of one class exceeds that of another. Following this, using a modified version of the Neyman-Pearson lemma, which forms the basis of RS and IDRS, a procedure is designed to compute the certified radius where the confidence is guaranteed to stay above a certain

threshold, thus achieving a significantly better-certified radius, as is empirically shown. They raise a valid point regarding certifying confidence (soft classifiers) instead of labels (hard classifiers), as has been the primary focus of most RS based techniques so far. We urge further exploration and extension of this notion of understanding classifier confidence and base classifier properties and leveraging them for RS.

Addressing the high cost challenge in RS, Quasi-Convexity-based Randomized Smoothing (QCRS) [59] presents an optimisation which exploits the quasiconcavity condition for the radius-variance curve in randomized smoothing, which is a weaker form of concavity. While DDR assumed concavity for the certified radii, empirically, it is seen that this is not the case despite it showing significant certified radii improvements. On the other hand, quasiconcavity is shown to hold for most data points. Based on this, QCRS is presented as a computationally efficient approach with less overheads compared to other input-specific inference-based optimisation methods. Further, it is an entirely orthogonal approach to many other optimisation techniques in randomized smoothing, and as with other orthogonal approaches, we encourage empirical testing with other methods.

Noting the dearth of inference cost-reducing methods like [59], particularly for input-specific techniques which add a significant overhead to RS, which itself is computationally expensive, we also encourage works to work towards reducing these costs, since they prohibit scalable applications of RS. Further, even though many techniques use information beyond the Neyman-Pearson lemma, such as classifier-specific information [21], data-dependent information and local geometric information [5], there is not much research to say that this works towards addressing the curse of dimensionality in the input-specific case. Further, these input-specific schemes also suffer from robustness-accuracy issues as with base RS and there should be more research into addressing this issue.

F. Other Methods

Prior works on randomized smoothing against l_1 adversarial attacks use additive smoothing noise and provide probabilistic robustness guarantees, which provide a certain level of confidence but lack absolute certainty. Levine et al. [90] propose a l_1 certification method based on data quantization. They term this method SSN, Smoothing with Splitting Noise. They also propose a de-randomized version of SSN, DSSN, which yields deterministic certificates instead of probabilistic certificates in normal randomized smoothing. Whether this non-additive smoothing technique can be extended to other l_p norms remains unexplored. Another work [43], presents a technique to obtain larger l_1 certificates when using randomized smoothing. Current l_1 methods augment the data with noise beyond the domain constraints in most problems like image classification which is trivial to detect. They show that leveraging the domain constraint that input points have to lie in the image domain $[0, 1]^d$ improves l_1 certificates significantly without the need to adapt the classifiers or change smoothing distributions. However, there remains a gap to what seems possible in

empirical l_1 robustness giving way for future research. Both works essentially improve l_1 certificates with the differences lying in their core methodology.

So far, randomized smoothing has only been used for l_p robustness for $p \geq 1$ in the majority of research, and techniques have been developed to provably guarantee the robustness of a classifier to adversarial perturbations of bounded l_1 and l_2 magnitudes by using randomized smoothing. This has been extended considerably in works that focus on offering adversarial robustness guarantees for cases where the adversary is l_0 bounded [89], [83]. Lee, Guang-He, et al. [89] consider a distribution where random noise, $\sigma \sim \{0, 1\}^d$, is multiplied by the image. This multiplication is done pixel-wise; hence, they are masking the pixels out. Moreover, they theoretically prove the superiority of this discrete distribution against Gaussian noise to provide a better-certified radius for l_0 attacks. They empirically show that their certificates are consistently better than the ones derived from the Gaussian distribution. Further, their model outperforms other models by a large margin across different radii measures. Challenging these results, Levine, Alexander, et al. [83] introduce a novel method that addresses l_0 attacks within the framework of randomized smoothing, deviating from the conventional use of additive Gaussian noise. They propose a unique l_0 ablation scheme, wherein pixels of the image are randomly ablated. They also introduce an encoding of the ablated classifier, representing the absence of information, which is carefully designed to ensure unbiased representation in the classifier’s training process. This modification is used in sparse adversarial attacks, for l_0 attacks where additive noise is not useful, and for data with a sparse structure. They point out that in the scheme by Lee et al. [89] each pixel is retained with a fixed probability κ and is otherwise assigned to a random value from the remaining possible pixel values in S , leading to considerably lower median certified robustness compared to Levine, Alexander, et al. [83], on the datasets used in both works. These works provide robustness guarantees in various resolutions and perspectives, from a point-wise certificate to a regional certificate, and open up many avenues for future extensions at different levels.

Another concept that is focused on improving adversarial robustness is that of colored noise injection-based smoothing (CNI smoothing) [113]. It comes from extending the idea of adding white Gaussian noise to the network weights and activations during adversarial training (PNI) [114] to the injection of colored noise for defense against common white-box and black-box attacks and is exploited by Nemcovsky, Yaniv, et al. [42]. They present an inference-only method for improving empirical adversarial robustness by taking into account the spatial information in the area of the data points. As an adversarial defense technique, CNI smoothing is based on noise injection to the model’s weights or parameters using multivariate Gaussian noise. When used in constructing an RS-based approach where both the inputs and parameters have Gaussian noise added to them, this technique yields improved adversarial robustness compared to CNI, improving model

performance. However, at the same time, it does not provide a proper ablation approach which can be pursued by future studies. Another point of concern is that the performance of this method declines as the number of attack iterations increases. Overcoming this comes at a cost of increased computational complexity which paves a path for future work in this domain towards lowering costs.

Coming back to the challenge of designing an efficient classifier with a sufficient certified radius, Delattre, Blaise, et al. [58] show that the variance introduced by the Monte-Carlo sampling in the randomized smoothing procedure estimate closely interacts with two other important properties of the classifier, i.e. its Lipschitz constant and margin. They utilize the Lipschitz constant for the base classifier to further improve the bounds on the Lipschitz constant for the smoothed classifier [86]. They also present a novel technique named Lipschitz-Variance-Margin Randomized Smoothing (LVM-RS) based on optimizing the variance-margin trade-off in the base classifier to improve the certified robust radius while maintaining reliability. This is based on the relation between the variance of the smoothed classifier, the Lipschitz constant, and the margin of the base classifier. This new certification method encourages the use of pre-trained models in addition to randomized smoothing, leading to improvement in the current certification radius. This also brings into the picture the scope of integrating LVM-RS with margin maximization strategies.

Other works ([6], [56]) propose a solution to improve the certified bounds obtained using generalized Gaussian distributions and formulating non-Gaussian distributions. While this does not address the curse of dimensionality directly, it optimises the bounds obtained despite this issue. The former [6] optimises the distribution for each l_p threat model, whereas the latter [56] generalizes and provides a universal robustness certification framework by the name of Universally Approximated Certified Robustness which can also be used to derive optimal certified radii against l_p perturbation generated by a continuous probability distribution.

The necessity of certified robustness for top-k predictions, instead of just the top prediction as has been the case for RS methods till now, has been pointed out in the work by Jia et al. [62] along with the need to derive tight robustness in the l_2 norm for the same using Gaussian noise. They address the technical challenges faced in implementing the same, which can majorly be summarised as (I) difficulty in computing the probability p_l that a label l is verifiably among the top-k labels predicted by the smoothed classifier, (II) difficulty in computing the k-largest probabilities p_i for the top-k labels and (III) finding an analytical solution to the equation for the certified radius of each label. They overcome these challenges using simultaneous confidence intervals of the label probabilities and an algorithm to solve the equation to obtain a lower bound of the certified radius. As noted in the work itself, there needs to be future work to derive similar bounds for l_1 and l_0 attacks and the optimal noise distributions for the case of top-k predictions.

Dvijotham, Krishnamurthy Dj, et al. [92] introduce a way

to provide robustness guarantees for arbitrary smoothing distributions using f-divergences. They formulate the task of verification in a way so that it becomes solvable using a simple optimisation problem. They were mainly focused on binary classifiers. Their approach reframes the problem in the space of probability measures, employing f-divergences to define tractable constraint sets. Kou, Yiwen et al. [31] discuss an extension of this: they utilize distance-based likelihood metrics, Wasserstein distance and total-variation distance to solve the optimisation problem introduced in the work by Dvijotham, Krishnamurthy Dj, et al. [92]. They also show a comparison of different uniform probability measures over different norm-based attacks and show a comparison of their performances. They also show that the curse of dimensionality exists even for distance-based metrics.

Anderson, Brendon G. et al. [36] discuss randomized smoothing with the consideration of generic smoothing distributions, which can be input-dependent. They theoretically discuss that the main task of randomized smoothing is to maximize the average certified radius, where, in actuality, this problem is an infinite dimensional optimisation problem, which can be lower bound by cone-based linear programming if the confidence intervals of the classes are used. They provide a semi-infinite approach to solving this problem and mainly focus on the theoretical side with a simple yet unpractical experiment as a proof of concept.

Improving Randomized Smoothing for Scalability

Throughout our examination in this section, we observe that most new optimizations in randomized smoothing do not directly address the curse of dimensionality or the issue of high inference costs. Instead, they tend to introduce new techniques, each with its own computational overhead and dimensionality challenges, and then propose solutions for that additional cost. One such example is that of input-dependent techniques, which add input-specific components over an already costly randomized smoothing process, as pointed out in that specific subsection. While this certainly helps improve the applicability of a technique individually, it does not enhance the overall scalability of randomized smoothing, hence this foundational issue persists.

IV. DIVERSE APPLICATIONS OF RANDOMIZED SMOOTHING

In this section, we discuss applications of randomized smoothing in various fields. While introducing these applications, domain-specific optimisations to improve RS for that particular application have also been discussed. While we have tried to group some related application cases and present them in a consolidated manner, certain use cases stand out with no proper classification possible. For those papers and applications, we simply present a summary of the work without elaborating on the domain and required background.

A. RS in other types of Adversarial Attacks

Patch attacks involve strategically placing visually imperceptible modifications on images leading to misclassifications and compromised system security. Levine, Alexander, et al. [23] deal with (De-) Randomized Smoothing for certifiable defense against Patch Attacks, which utilizes the fact that patch attacks are more constrained than general sparse attacks. This paper achieves much better results than Chiang et al. (2020), who defend against patch attacks using interval-bound propagation. They propose a structured ablation scheme (block smoothing and band smoothing). Experimentally, their method produces higher certified accuracy than l_0 randomized ablation. This method stands out as it even scales to larger datasets like Imagenet, unlike previous works.

Malware-based attacks pose a serious cybersecurity threat, making robust malware detection imperative to safeguard against data breaches, system compromise, and potential economic and operational disruptions. Zhang, Jiawei, et al. [67] introduce DRSM (De-randomized smoothed MalConv), a certified defense created by redesigning the de-randomized smoothing technique for the domain of malware detection through executables [94]. It is found that DRSM faces a marginal accuracy drop at the cost of robustness. DRSM proves to be empirically robust to some extent against a diverse set of attacks.

Adversarial label-flipping attacks manipulate the labeling of data to deceive machine learning models, leading to incorrect predictions and compromising the system's integrity. Rosenfeld, Elan, et al. [52] propose a strategy to build a linear least squares classifier based on pre-trained deep features incorporated with the Chernoff inequality that are certifiably robust against a strong variant of label flipping where the adversary targets each test example independently. They do so without additional runtime cost over standard classification. Notably, although they employ a randomized approach, the final algorithm does not use random sampling but relies on a convex optimisation problem to compute the certified robustness.

A major vulnerability of RS against adversarial attacks is where the attacker can backdoor even the randomness in a model. This applies to many randomization-based methods, particularly RS, seeing its widespread usage for security purposes. Dahiya, Pranav, et al. [103] demonstrate that there exist PRNG-attacks in which an attacker can alter the randomness of the model ever so slightly such that it cannot be detected by existing standards of randomness [104], but affect the performance of randomized smoothing by overestimating certification bounds. This prompts research into developing more extensive randomness tests, particularly in the backdrop of adversarial robustness, specifically for randomness in RS, but even for machine learning models in general.

B. Watermark Neural Networks

Watermarks are a strategy to preserve a creator's intellectual rights over their property. This has led to the development of watermarking strategies using neural networks that aim

to prevent adversaries from removing watermarks. However, these models fail when stronger or fine-tuned attacks are used. Thus, as in the general field of adversarial ML, there was a shift to certified robustness, so was this change observed in watermark networks. Randomized smoothing is one of the techniques that can be used for this purpose [28]. Their certified bounds, based on l_2 adversary attacks, guarantee watermark persistence and provide state-of-the-art robustness against adversarial attacks.

Ren, Jiayang, et al. [48] extend the use of randomized smoothing for watermark neural networks [28] for better application against l_p attacks for any $p > 1$, which, as mentioned earlier, suffer from the curse of dimensionality. Their approach leverages Mollifier theory and the dimension-independent Lipschitz constant to develop a smoothing technique that gives dimension-independent bounds. Notably, they also developed an efficient version of their proposed classifier, Certified Watermarks, using approximations, which performs effectively and efficiently. Thus, they have tried to adapt and overcome both the curse of dimensionality and high inference costs. We encourage those interested in this domain to test some of the above-mentioned optimisations to improve the application of RS for watermark neural networks.

C. Differential Privacy and Federated Learning

Randomized smoothing and certified robustness can also be viewed from the lens of differential privacy and federated learning in the context of AI security. For federated learning, Chen, Cheng, et al. [55] present a randomized smoothing approach in existing frameworks to enable certified robustness. Its performance is comparable to existing certifiably robust models trained by centralized training. However, there is much to explore regarding how randomized smoothing can be implemented for more complex federated learning frameworks.

In the context of differential privacy, Wang, Wenxiao, et al. [53] propose DPLs, which improves differentially private stochastic gradient descent. It does so by smoothing the learning function to become less irregular and more noise-tolerant. The paper, focusing on DP-SGD, also opens the door for randomized smoothing to be applied to other differential privacy techniques like the private aggregation of teacher ensembles (PATE) by realizing that the fundamental idea behind PATE is noisy data labeling and the parameter noise during training, which is essentially the label noise itself, can be smoothed using RS.

D. Graph Neural Networks

Graph Neural Networks (GNNs) are a powerful class of neural networks that rely on graph-based data representation through nodes and edges. Like other technologies, they are not immune to adversarial attacks, such as structural perturbations that affect graph and node classification. Since randomized smoothing is an architecture-independent approach to introduce certified robustness, it certainly can be used for GNNs. A few papers have been published for certifying robustness in GNNs, but much remains to be explored. Wang, Binghui, et al.

[47] develop a robustness guarantee for randomized smoothing in GNNs. They note that their bounds apply for both node and graph classification and any classifier with binary data without making any assumptions about its architecture. Bojchevski et al. [121] extend RS for discrete and sparse data, highlighting its relevance as a new approach to RS by applying their method to GNNs, which often deal with such data.

A message-interception randomized smoothing approach to improve robustness in GNNs [63] is specific to the commonly used architecture of message-passing neural networks [82]. This is based on a gray box, since some knowledge of the classifier is known, certification method where adversarial messages are intercepted so as not to affect any target nodes containing sensitive data. They use the concept of randomized ablation to mask nodal features and deleted edges. Randomized ablation is a less explored technique which is based on randomized smoothing and offers sufficient protection against various attacks, including l_0 attacks. We encourage more work to identify and use randomized ablation in other applications alongside theoretical studies of the same.

E. Reinforcement Learning

Reinforcement learning (RL) has found usage in many domains like robotics, trading, and autonomous driving. It is concerned with how an intelligent agent should act in a dynamic environment to maximize the cumulative reward. The sensitivity of its applications makes it essential to make RL robust to adversarial attacks, and RS is successful in doing so. This also represents a new approach to RS for dynamic cases, as are commonly encountered in RL, instead of just static problems. Kumar, Aounon, et al. [85] discuss the application of RL policy smoothing to provide robustness by adding Gaussian noise to the input data. Their approach offers robust certificates against adversaries, which aim to minimize the reward. Their guarantees ensure rewards stay above a certain threshold, even if intermediary steps may be non-optimal. They also utilise and prove an adaptive version of the Neyman-Pearson lemma, which forms the base of RS, to protect against adaptive adversarial attacks. CROP [86] is a generalised framework based on randomized smoothing to certify robustness in RL policies, which extends to [85] by allowing certified robustness for cumulative reward and action. They provide empirical results for CROP. We encourage the use of the adaptive RS, as introduced in these papers, for other applications and propose further theoretical and empirical research into RS for RL.

F. Miscellaneous

Tree-based models are used in domains such as finance and medicine, where robustness is paramount. Constructing robust tree-based models helps reduce overfitting, improve generalization, and enhance the model's resilience to variations in the data. Horváth, Miklós, et al. [69] propose a novel (De-)Randomized Smoothing approach for the same, with deterministic l_p norm guarantees. The critical insight of their work is that they leverage the nature of decision stump ensembles to

obtain deterministic certificates instead of probabilistic ones. A minor limitation in their work is that DRS can not handle arbitrary ensembles of decision trees.

Another interesting usage of RS is in the convergence of stochastic optimisation procedures. Duchi et al. [30] and Yousefian et al. [37] present two techniques to discuss this: accelerated gradient techniques and convex analysis. In [117], the authors propose randomized Differential Dynamic Programming (DDP) for RL and optimal control theory (OC), which extends the applicability of the classic Differential Dynamic Programming in OC for non-smooth dynamic deterministic systems. Minin et al. [50] discuss the problem of optimisation of near-convex functions, which can be represented as a sum of strongly convex and bounded functions. The paper presents an algorithm for minimizing such functions using the RS technique. This technique theoretically justifies the global optimisation properties of SPSA(Simultaneous Perturbation Stochastic Approximation)-like algorithms.

In the work by Ding, Liang, et al. [19], randomized smoothing is presented as a form of regularisation that prevents overfitting by adding noise and forcing the model to learn generalized features. Thus, using various noise distributions, one can treat it as a convolution-based smoothing kernel. They obtain estimators with fast convergence, irrespective of the dimension, using techniques like early stopping and weight decay. They present future research areas for classification and other models beyond standard gradient descent.

The application of randomized smoothing further extends to automatic speech recognition task [16]. Unlike image classification, automatic speech recognition is a sequential task; the main challenge lies in the fact that as the input size increases, the no. of outputs becomes difficult and hence it becomes difficult to provide the most likely class. Their work does not change the theoretical model of randomized smoothing, but it modifies the voting criteria to ROVER, a voting technique typically utilized in ASR. The drawback of this approach is the increased time complexity.

Fischer, Marc, et al. [8] modify the existing randomized smoothing technique and apply it in image segmentation. The downside in their approach is that it is highly dependent on pixel values, and a slight change in the pixel values can significantly affect the Clopper-Pearson lower bound, p_A which they tackle by considering a threshold τ , which would only provide a class to a pixel if the highest class probability exceeds τ ; abstaining otherwise..

Applications of randomized smoothing in natural language processing are not as intensively explored as other domains. Zhao, Haiteng, et al. [35] introduce a novel framework called Causal Intervention by Semantic Smoothing (CISS) to enhance the robustness of NLP models against adversarial attacks. Other works [118], [119] introduce RSMI, a novel two-stage framework that combines randomized smoothing (RS) with masked inference (MI) to improve the adversarial robustness of text classification models.

Gaussian noise augmentation, a novel certification method that relies on the exact mechanism as randomized smoothing,

has been proposed by Chiang, Ping-yeh, et al. [71], with the difference being that it introduces median-based smoothing instead of mean-based smoothing, i.e. using percentiles instead of expectation for classification. At the same time, it also converts an object detection problem to a regression problem for a black-box evaluation of certified robustness, extending RS from classification to regression.

In their work, Xu, Xiao, et al. [40] introduce a modified attention module, extracting attention maps for noisy samples and then merging these maps of robust features with clean image features to obtain an attention map focusing more on robust features to improve certified bounds obtained by RS.

Transformation-specific smoothing, TSS, is a unified framework for certifying ML robustness against general adversarial semantic transformations presented by Li, Linyi, et al. [64]. Their approach is to divide common transformations into two categories depending on the properties of each transformation. Transformation-specific RS strategies obtain strong robustness certification for resolvable transformations, while for differentially resolvable transformations, which cover interpolation errors, a novel approach based on stratified sampling is used. Combined with consistency-enhanced training, TSS provides rigorous certification of robustness.

Muravev, Nikita, et al. [39] discuss the usage of randomized smoothing for multiplicative parameters, instead of additive ones. They mainly focus on the case of gamma correction in images. However, in their certification experiments on large datasets, they failed to approximate actual robustness while on smaller datasets, a significant drop in accuracy is seen when they switch from the idealized setting to the realistic one. Both of these arise from the computational complexity of RS, which highlights the need to explore solutions to these issues.

A practical limitation in RS-based methods that affects their performance in certain kinds of applications has been presented by Voráček, Václav et al. [105]. They analyze randomized smoothing for numerical errors that may arise due to floating-point arithmetic and show how the current approach is numerically unsound and may overestimate or underestimate certified radii. Further, they propose a modified version of randomized smoothing which improves numerical stability however, it comes with a slight computational overhead.

Applications highlight the need for scalable RS

This section underscores the significance of randomized smoothing by showcasing its widespread adaptation and utilization across numerous applications. Despite grappling with theoretical challenges, such as the curse of dimensionality and high inference costs, directly impacting these applications, addressing these challenges could substantially enhance the scalability and practicality of randomized smoothing. We also observe that many empirical verifications for these applications limit their tests to smaller datasets or less expensive models, which is a direct consequence of the scalability issues in RS. Hence, this section well reinforces our assertion that randomized smoothing needs to be made more scalable.

Limitations and Conclusion: While this paper has thoroughly investigated randomized smoothing techniques, including their robustness, applications, limitations, and potential solutions while emphasising on its scalability, it is important to recognize the boundaries of this research. By concentrating solely on randomized smoothing, our perspective has become somewhat narrow, emphasizing it as the dominant state-of-the-art approach for tackling adversarial attacks in terms of certified robustness. Thus, one limitation of this exclusive focus is the possibility of missing alternative methods that could perform just as effectively or even surpass randomized smoothing in specific applications.

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