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# Finite-Time $H_\infty$ Control for Time-Delay Markovian Jump Systems with Partially Unknown Transition Rate via General Controllers

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**Abstract:** This paper deals with the problems of finite-time boundedness (FTB) and  $H_\infty$  FTB for time-delay Markovian jump systems with a partially unknown transition rate. First of all, sufficient conditions are provided, ensuring the FTB and  $H_\infty$  FTB of systems given by linear matrix inequalities (LMIs). A new type of partially delay-dependent controller (PDDC) is designed so that the resulting closed-loop systems are finite-time bounded and satisfy a given  $H_\infty$  disturbance attenuation level. The PDDC contains both non-time-delay and time-delay states, though not happening at the same time, which is related to the probability distribution of the Bernoulli variable. Furthermore, the PDDC is extended to two other cases; one does not contain the Bernoulli variable, and the other experiences a disordering phenomenon. Finally, three numerical examples are used to show the effectiveness of the proposed approaches.

**Keywords:** finite-time boundedness;  $H_\infty$  control; Markovian jump system; time-delay



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## 1. Introduction

In actual industrial processes, the transient performance of systems is sometimes particularly important. For example, aircraft control systems require that the states not exceed a given limit [1]; the temperature of a chemical reaction needs to be strictly controlled within a certain range [2]; the angular location of a robot arm should be limited to a particular scope [3]. In recent years, an increasing number of academics have focused on the finite-time stability (FTS) problem. Different from the traditional Lyapunov stability [4–7], FTS discusses the transient performance of systems in the finite-time interval. In fact, the stable systems in the Lyapunov sense may have very bad transient performances, such as severe oscillation. The definition of FTS (or short-time stability [8]) was first proposed by Kamenkov in [9]. According to FTS, a system state is limited to a certain critical value within a certain time region, if the initial state is norm bounded. The authors of [10] extended FTS to the concept of FTB and took external disturbances into account. The studies of FTS and FTB have been further developed with the evolution of LMI theory [11–19]. For example, in [11], sufficient conditions for FTB of closed-loop systems were given in the form of LMIs by designing a dynamic feedback controller. Meanwhile, finite-time  $H_\infty$  control/filtering problems [20–24] have received much attention in order to reduce influences on a system caused by external disturbances.

On the other hand, abrupt changes are often encountered in the industrial process due to a component fault, invalidation, an associated change between subsystems, a sudden environmental disturbance [25], and so on. The occurrence of these situations causes the structure and parameters of a system to switch between various subsystems, such as

networked control systems or power electrics. The Markov jump system [26] is used to deal with this kind of practical system via the transition probabilities of the jump process. In recent decades, many researchers have performed studies on these types of systems, such as [27–32]. Moreover, some research results have been utilized in many engineering fields, such as power systems [33], manufacturing systems [34], communication systems [35], etc. Although there have been lots of research achievements about Markov jump systems, most assume that the transition probabilities are all known; however, it is difficult to ascertain precise transition probabilities in real life due to instrument and measurement limitations. Therefore, further research on a Markov jump system with a partially unknown transition rate is really vital and necessary. Readers may refer to [36–38].

Time delay, as an inevitable phenomenon, widely exists in communication [39], the chemical industry [40], transportation [41], and other systems. The existence of a time delay may make the performance of a system deteriorate, destroy the balance and stability of systems, and even produce a chaos phenomenon. This leads to the development of and changes to the systems such that they depend not only on the present state but also on the previous state [42–46]. By studying the FTS of various time-delay systems [47–50], it is found that most of the results mainly consider the controller with or without a time delay. However, in practice, data transmission events with or without a delay occur randomly, which inspires us to consider designing a controller with both a time delay and no time delay, or non-simultaneous occurrences according to probability.

In this paper, we handle the FTB and  $H_\infty$  FTB problems of time-delay Markov jump systems with a partially unknown transition rate via some general PDDCs. The following are the main contributions: (1) With LMIs, we give sufficient conditions for FTB for the defined system. (2) A new kind of PDDC is designed to make the resulting system  $H_\infty$  FTB. The PDDC contains both non-time-delay and time-delay states; however, these are not happening at the same time. In comparison to conventional state feedback controllers [47–49,51], the probability distributions play an important role in the PDDC. (3) Different from the existing results of [46], the PDDC is extended to two new cases: one does not contain the Bernoulli variable, and the other experiences a disordering phenomenon.

The rest of this paper is arranged as follows: In Section 2, the preparation and problem statement are presented. Section 3 discusses the main results for the FTB and  $H_\infty$  FTB of the system defined by LMIs via the PDDC's design. Three examples are given to show the effectiveness of the obtained results in Section 4. Some conclusions are given in Section 5.

Notation:  $\lambda_{\max}(Q)$  ( $\lambda_{\min}(Q)$ ) means the maximum (minimum) eigenvalue of a real symmetric matrix  $Q$ ;  $E[\cdot]$  refers to the mathematical expectation operator; the superscript  $T$  is the transposition of the matrix. In the matrices,  $\text{diag}\{\cdot\cdot\cdot\}$  stands for the block-diagonal matrix, the symbol  $*$  is the symmetric term of a matrix, and  $(P)^* = P + P^T$ . The  $\sigma$ -algebras of the sample space subsets are represented by  $\mathcal{F}$ .  $\text{Pr}$  denotes the mathematic probability.

## 2. Problem Statement and Preliminaries

Consider a linear time-delay Itô stochastic Markovian switching system

$$\begin{cases} dx(t) = [S(\sigma_t)x(t) + S_\tau(\sigma_t)x(t - \tau) + L(\sigma_t)u(t) + G(\sigma_t)v(t)]dt \\ \quad + [U(\sigma_t)x(t) + U_\tau(\sigma_t)x(t - \tau) + J(\sigma_t)u(t) + F(\sigma_t)v(t)]d\omega(t), \\ z(t) = H(\sigma_t)x(t) + H_\tau(\sigma_t)x(t - \tau) + D(\sigma_t)v(t), \quad \forall t \in [0, \tilde{T}], \\ x(t) = \psi(t), \quad \sigma_t = \sigma_0, \quad \forall t \in [-\tau, 0], \end{cases} \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the system state,  $u(t) \in \mathbb{R}^m$  is the control input, and  $z(t) \in \mathbb{R}^q$  is the control output.  $S(\sigma_t)$ ,  $S_\tau(\sigma_t)$ ,  $L(\sigma_t)$ ,  $G(\sigma_t)$ ,  $U(\sigma_t)$ ,  $U_\tau(\sigma_t)$ ,  $J(\sigma_t)$ ,  $F(\sigma_t)$ ,  $H(\sigma_t)$ ,  $H_\tau(\sigma_t)$ , and  $D(\sigma_t)$  are constant matrices, for simplicity. When  $\sigma_t = i$ , they are denoted as  $S_i$ ,  $S_{\tau i}$ ,  $L_i$ ,  $G_i$ ,  $U_i$ ,  $U_{\tau i}$ ,  $J_i$ ,  $F_i$ ,  $H_i$ ,  $H_{\tau i}$ , and  $D_i$ . The time delay is  $\tau \geq 0$ . The continuous vector-valued function  $\psi(t)$  is defined on  $[-\tau, 0]$ ;  $\omega(t)$  is the standard one-dimensional Wiener process

defined on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  satisfying  $E[d\omega(t)] = 0, E[d^2\omega(t)] = dt$ ; and  $v(t)$  is the external disturbance satisfying

$$\int_0^t v^T(s)v(s)ds < d^2, d > 0. \tag{2}$$

The transition rate of the Markovian process  $\{\sigma_t, t \geq 0\}$  is given by

$$Pr(\sigma_{t+\Delta t} = j | \sigma_t = i) = \begin{cases} \pi_{ij}\Delta t + o(\Delta t), i \neq j, \\ 1 + \pi_{ii}\Delta t + o(\Delta t), i = j, \end{cases} \tag{3}$$

where  $\{\sigma_t, t \geq 0\}$  takes the values in  $\mathbb{S} = \{1, 2, \dots, N\}$ ,  $o(\Delta t)$  is the order of  $\Delta t$  that satisfies  $\Delta t > 0$ , and  $\lim_{\Delta t \rightarrow 0} \frac{o(\Delta t)}{\Delta t} = 0$ .  $\pi_{ij} \geq 0 (i \neq j, i, j \in \mathbb{S})$  is the transition rate of  $\sigma(t)$  from the mode  $i$  at the time  $t$  to the mode  $j$  at the time  $t + \Delta t$ , such that  $\pi_{ii} = -\sum_{j \neq i} \pi_{ij}$ . All of the transition rates  $\pi_{ij}, i, j \in \mathbb{S}$ , can be collected into the following transition rate matrix

$$\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1N} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2N} \\ \vdots & \cdots & \ddots & \vdots \\ \pi_{N1} & \pi_{N2} & \cdots & \pi_{NN} \end{bmatrix}.$$

Assume that the transition rate is partially unknown, for example, there is a  $2 \times 2$  transition rate matrix

$$\Pi_1 = \begin{bmatrix} \pi_{11} & \pi_{12} \\ ? & ? \end{bmatrix}$$

where “?” is an unknown element and  $\pi_{ij}$  is known. For all  $\pi_{ij} \in \mathbb{S}$ , define  $\mathbb{S} = L_k^i + L_{uk}^i$ , where

$$L_k^i = \{j : \text{if } \pi_{ij} \text{ is known}\}, L_{uk}^i = \{j : \text{if } \pi_{ij} \text{ is unknown}\}.$$

If  $L_k^i$  is non-empty, it is described as follows

$$L_k^i = \{k_1^i, k_2^i, \dots, k_m^i\}, \quad 0 \leq m \leq N,$$

where  $k_m^i \in \mathbb{S}$  denotes the  $m$ th known element in the matrix  $\Pi$ 's  $i$ th row.

**Definition 1 (FTB).** For the given scalars  $c_2 > c_1 > 0, \tilde{T} > 0$  and the matrix  $R_i > 0 (i \in \mathbb{S})$ , system (1) with  $u(t) = 0$  is FTB with respect to  $(c_1, c_2, \tilde{T}, R_i, d)$ , if

$$E[x^T(t_1)R_i x(t_1)] \leq c_1 \Rightarrow E[x^T(t_2)R_i x(t_2)] < c_2, \tag{4}$$

and (2) holds, where  $t_1 \in [-\tau, 0], t_2 \in [0, \tilde{T}]$ .

**Remark 1.** FTB can be simplified to FTS with respect to  $(c_1, c_2, \tilde{T}, R_i)$  when  $v(t) = 0$ . The FTB/FTS can be used to solve some practical problems, such as the chemical reaction process, electronic circuit systems, and medicine. For example, the body's normal systolic blood pressure is 90–140 mmHg. If the body's systolic blood pressure is greater than 140 mmHg, then one suffers from high blood pressure disease. One must take blood pressure medicine.

**Definition 2** ( $H_\infty$  FTB). For the given scalar  $\gamma > 0$ , system (1) with  $u(t) = 0$  is  $H_\infty$  FTB with respect to  $(c_1, c_2, \tilde{T}, R_i, d, \gamma)$ . If system (1) is FTB and under zero initial condition, for any non-zero disturbance  $v(t)$ , the control output  $z(t)$  satisfies

$$E\left[\int_0^{\tilde{T}} z^T(t)z(t)dt\right] < \gamma^2 E\left[\int_0^{\tilde{T}} v^T(t)v(t)dt\right]. \tag{5}$$

When the control problem is considered, the following definition is needed.

**Definition 3** ( $H_\infty$  FTB stabilization). System (1) is finite-time  $H_\infty$  stabilizable if there exists a controller  $u(t)$  such that the resulting closed-loop system is  $H_\infty$  FTB.

**Lemma 1** (Gronwall–Bellman inequality [52,53]). Let  $g(t)$  be a nonnegative continuous function. If there are positive constants  $r, q$  such that

$$g(t) \leq r + q \int_0^t g(s)ds, \quad 0 \leq t \leq \tilde{T}, \tag{6}$$

then

$$g(t) \leq \text{rexp}(qt), \quad 0 \leq t \leq \tilde{T}. \tag{7}$$

**Remark 2.** Lemma 1 can be reformulated with sharp inequalities. The proof is given in Appendix A.

**Lemma 2** (Schur’s complement lemma [54]). For the real matrix  $H$ , the real symmetric matrix  $S$ , and the positive-definite matrix  $U$ , the below inequalities are equivalent:

$$S + HU^{-1}H^T < 0$$

and

$$\begin{bmatrix} S & H \\ H^T & -U \end{bmatrix} < 0.$$

### 3. Main Results

Firstly, we discuss the FTB problem for system (1) (when  $u(t) = 0$ ) in this section.

**Theorem 1.** System (1) (when  $u(t) = 0$ ) is FTB with respect to  $(c_1, c_2, \tilde{T}, R_i, d)$ , if for a real scalar  $\eta \geq 0$ , there exist the scalars  $\lambda_{i1} > 0, \lambda_{i2} > 0$ , symmetric matrices  $P_i > 0, Q_i > 0$ , and  $O_i > 0$  satisfying

$$\begin{bmatrix} \Psi_{i1} & P_i S_{\tau i} & P_i G_i & U_i^T \\ * & -O_i & 0 & U_i^T \\ * & * & -Q_i & F_i^T \\ * & * & * & -P_i^{-1} \end{bmatrix} < 0, \tag{8}$$

$$O_i < R_i, \tag{9}$$

$$\lambda_{i1} I < \bar{P}_i < \lambda_{i2} I, \tag{10}$$

$$c_1(\lambda_{i2} + \tau) + \lambda_{\max}(Q_i)d^2 < c_2 \exp(-\eta\tilde{T})\lambda_{i1}, \tag{11}$$

where

$$\Psi_{i1} = \sum_{j \in L_{uk}^i} \pi_{ij} [(P_i S_i)^* + O_i + P_j] + \zeta_k^i + (1 + \pi_k^i) [(P_i S_i)^* + O_i],$$

$$\zeta_k^i = \sum_{j \in L_k^i} \pi_{ij} P_j, \quad \pi_k^i = \sum_{j \in L_k^i} \pi_{ij}, \quad \bar{P}_i = R_i^{-\frac{1}{2}} P_i R_i^{-\frac{1}{2}}.$$

**Proof.** For system (1), we choose a stochastic Lyapunov functional as

$$V(x_t, \sigma_t) = x^T(t) P(\sigma_t) x(t) + \int_{t-\tau}^t x^T(s) O(\sigma_t)(s) x(s) ds. \tag{12}$$

For each  $\sigma_t = i \in \mathbb{S}$ , let  $\mathcal{L}$  be the differential generating operator of system (1). According to the Itô formula, it follows that

$$\begin{aligned} \mathcal{L}V(x_t, \sigma_t = i) &= x^T(t) [(P_i S_i)^* + \sum_{j=1}^N \pi_{ij} P_j + O_i] x(t) + [\Xi]^T P_i [\Xi] - x^T(t - \tau) O_i x(t - \tau) \\ &\quad + 2x^T(t) P_i S_{\tau i} x(t - \tau) + 2x^T(t) P_i G_i v(t) \\ &= x^T(t) [(P_i S_i)^* + O_i + \sum_{j \in L_{uk}^i} \pi_{ij} P_j + \zeta_k^i + \sum_{j \in L_k^i} \pi_{ij} ((P_i S_i)^* + O_i)] x(t) \\ &\quad + 2x^T(t) P_i [S_{\tau i} x(t - \tau) + G_i v(t)] + [\Xi]^T P_i [\Xi] - x^T(t - \tau) O_i x(t - \tau) \\ &= x^T(t) [(1 + \pi_k^i) ((P_i S_i)^* + O_i) + \zeta_k^i + \sum_{j \in L_{uk}^i} \pi_{ij} ((P_i S_i)^* + O_i + P_j)] x(t) \\ &\quad + 2x^T(t) P_i [S_{\tau i} x(t - \tau) + G_i v(t)] + [\Xi]^T P_i [\Xi] - x^T(t - \tau) O_i x(t - \tau) \end{aligned} \tag{13}$$

where  $\Xi = F_i v(t) + U_i x(t) + U_{\tau i} x(t - \tau)$ .

From (8) and (13), it is easy to obtain

$$\mathcal{L}V(x_t, \sigma_t = i) < \eta V_1(x_t, \sigma_t = i) + v^T(t) Q_i v(t), \forall t \in [0, \tilde{T}],$$

where  $V_1(x_t, \sigma_t = i) = x^T(t) P_i x(t)$ , so

$$\mathcal{L}V(x_t, \sigma_t = i) < \eta V(x_t, \sigma_t = i) + \lambda_{\max}(Q_i) v^T(t) v(t). \tag{14}$$

Integrating both sides of (14) from 0 to  $t$  ( $t \in [0, \tilde{T}]$ ) yields

$$V(x_t, \sigma_t = i) - V(x_0, \sigma_0) < \eta \int_0^t V(x_s, \sigma_s) ds + \lambda_{\max}(Q_i) \int_0^t v^T(s) v(s) ds. \tag{15}$$

Taking the mathematical expectation on both sides of (15), the following is concluded

$$E[V(x_t, \sigma_t = i)] - E[V(x_0, \sigma_0)] < \eta E[\int_0^t V(x_s, \sigma_s) ds] + \lambda_{\max}(Q_i) E[\int_0^t v^T(s) v(s) ds],$$

i.e.,

$$E[V(x_t, \sigma_t = i)] < E[V(x_0, \sigma_0)] + \eta \int_0^t E[V(x_s, \sigma_s)] ds + \lambda_{\max}(Q_i) E[\int_0^t v^T(s) v(s) ds]. \tag{16}$$

Applying Lemma 1 or the Gronwall–Bellman-type inequality for the three functions [55] to (16) yields

$$E[V(x_t, \sigma_t = i)] < E[V(x_0, \sigma_0)] \exp(\eta t) + \lambda_{\max}(Q_i) E[\int_0^t v^T(s) v(s) ds] \exp(\eta t). \tag{17}$$

Set  $\check{\lambda}_i = \min_{i \in \mathbb{S}} \lambda_{\min}(\bar{P}_i)$  and  $\hat{\lambda}_i = \max_{i \in \mathbb{S}} \lambda_{\max}(\bar{P}_i)$ . Together with (10), we have

$$\begin{aligned} E[V(x_t, \sigma_t = i)] &= E\left[\int_{t-\tau}^t x^T(s) O_i x(s) ds\right] + E[V_1(x_t, \sigma_t = i)] \\ &\geq E[V_1(x_t, \sigma_t = i)] \geq \check{\lambda}_i E[x^T(t) R_i x(t)] \geq \lambda_{i1} E[x^T(t) R_i x(t)], \end{aligned} \tag{18}$$

$$\begin{aligned} E[V(x_0, \sigma_0 = i)] \exp(\eta t) &= E[x^T(0) R_i^{\frac{1}{2}} \bar{P}_i R_i^{\frac{1}{2}} x(0)] \exp(\eta t) + E\left[\int_{-\tau}^0 x^T(s) O_i x(s) ds\right] \exp(\eta t) \\ &\leq c_1 (\hat{\lambda}_i + \tau) \exp(\eta \tilde{T}) \leq c_1 (\lambda_{i2} + \tau) \exp(\eta \tilde{T}), \end{aligned} \tag{19}$$

$$\lambda_{\max}(Q_i) E\left[\int_0^t v^T(s) v(s) ds\right] \exp(\eta t) < \lambda_{\max}(Q_i) d^2 \exp(\eta \tilde{T}). \tag{20}$$

From conditions (17) to (20), it is derived

$$E[x^T(t) R_i x(t)] \leq \exp(\eta \tilde{T}) \left[ \frac{(\tau + \lambda_{i2}) c_1 + \lambda_{\max}(Q_i) d^2}{\lambda_{i1}} \right]. \tag{21}$$

For all  $t \in [0, \tilde{T}]$ ,  $E[x^T(t) R_i x(t)] < c_2$  holds, which is obtained by

$$[c_1 (\lambda_{i2} + \tau) + \lambda_{\max}(Q_i) d^2] \exp(\eta \tilde{T}) \lambda_{i1}^{-1} < c_2,$$

which is (11). The proof is complete.  $\square$

**Remark 3.** If  $F_i = G_i = 0$ , then Theorem 1 is reduced to Theorem 1 in [29].

In the following, we propose three novel types of partially delay-dependent controllers. One of the controllers is

$$u(t) = (1 - \delta(t)) K_\tau(\sigma_t) x(t - \tau) + \delta(t) K(\sigma_t) x(t), \tag{22}$$

where  $K_\tau(\sigma_t)$  and  $K(\sigma_t)$  represent the control gains, and  $\delta(t)$  is the Bernoulli variable defined as

$$\delta(t) = \begin{cases} 1, & \text{if } x(t) \text{ is available,} \\ 0, & \text{if } x(t - \tau) \text{ is available,} \end{cases}$$

and satisfies

$$Pr\{\delta(t) = 1\} = \delta, \quad Pr\{\delta(t) = 0\} = 1 - \delta.$$

Furthermore,

$$E[(\delta(t) - \delta)^2] = \delta(1 - \delta) = \beta^2, \quad E[\delta(t) - \delta] = 0.$$

Substituting (22) in (1), we have

$$\begin{cases} dx(t) = [\hat{S}(\sigma_t) x(t) + \hat{S}_\tau(\sigma_t) x(t - \tau) + G(\sigma_t) v(t) + (\delta(t) - \delta) W(\sigma_t)] dt \\ \quad + [\hat{U}(\sigma_t) x(t) + \hat{U}_\tau(\sigma_t) x(t - \tau) + F(\sigma_t) v(t) + (\delta(t) - \delta) Z(\sigma_t)] d\omega(t), \\ z(t) = H(\sigma_t) x(t) + H_\tau(\sigma_t) x(t - \tau) + D(\sigma_t) v(t), \quad \forall t \in [0, \tilde{T}], \\ x(t) = \psi(t), \quad \sigma_t = \sigma_0, \quad \forall t \in [-\tau, 0], \end{cases} \tag{23}$$

where

$$\begin{aligned} \hat{S}(\sigma_t) &= S(\sigma_t) + \delta L(\sigma_t)K(\sigma_t), \hat{S}_\tau(\sigma_t) = S_\tau(\sigma_t) + (1 - \delta)L(\sigma_t)K_\tau(\sigma_t), \\ W(\sigma_t) &= -L(\sigma_t)K_\tau(\sigma_t)x(t - \tau) + T(\sigma_t)K(\sigma_t)x(t), \hat{U}(\sigma_t) = U(\sigma_t) + \delta J(\sigma_t)K(\sigma_t), \\ \hat{U}_\tau(\sigma_t) &= U_\tau(\sigma_t) + (1 - \delta)J(\sigma_t)K_\tau(\sigma_t), Z(\sigma_t) = -J(\sigma_t)K_\tau(\sigma_t)x(t - \tau) + J(\sigma_t)K(\sigma_t)x(t). \end{aligned}$$

The following theorem gives the sufficient condition of  $H_\infty$  FTB for the closed-loop system (23) via controller (22).

**Theorem 2.** System (23) is  $H_\infty$  FTB with respect to  $(c_1, c_2, \tilde{T}, R_i, d, \gamma)$ , if for a real scalar  $\eta \geq 0$ , there exist the scalars  $\gamma > 0, \lambda_{i1} > 0, \lambda_{i2} > 0$ , matrices  $X_i > 0, \bar{O}_i > 0$ , and  $Y_i, Y_{\tau i}$  satisfying

$$\begin{bmatrix} \tilde{\Psi}_{i1} & \tilde{\Psi}_{i2} & G_i X_i & H_i^T & \tilde{\Psi}_{i3} & \tilde{\Psi}_{i4} & X_i \\ * & \tilde{\Psi}_{i5} & 0 & H_{\tau i}^T & \tilde{\Psi}_{i6} & \tilde{\Psi}_{i7} & 0 \\ * & * & -\gamma^2 I & D_i & F_i^T & 0 & 0 \\ * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & -X_i & 0 & 0 \\ * & * & * & * & * & -X_i & 0 \\ * & * & * & * & * & * & -\bar{O}_i \end{bmatrix} < 0, \tag{24}$$

$$R_i^{-1} < \bar{O}_i, \tag{25}$$

$$\begin{bmatrix} -\lambda_{i1} & R_i^{-\frac{1}{2}} \\ * & -X_i \end{bmatrix} < 0, \tag{26}$$

$$-2R_i^{-\frac{1}{2}} + X_i + \lambda_{i1}I < 0, \tag{27}$$

$$c_1(\lambda_{i2} + \tau) + \gamma^2 d^2 < c_2 e^{-\eta \tilde{T}} \lambda_{i1}, \tag{28}$$

where

$$\begin{aligned} \tilde{\Psi}_{i1} &= (1 + X_i \pi_k^i)(S_i X_i + \delta L_i Y_i)^* + X_i \zeta_k^i X_i^T \\ &\quad + X_i \sum_{j \in L_{uk}^i} \pi_{ij} [(S_i X_i + \delta L_i Y_i)^* + X_j^{-1} X_i^T] - \eta X_i, \\ \tilde{\Psi}_{i2} &= S_{\tau i} X_i + (1 - \delta)L_i Y_{\tau i}, \tilde{\Psi}_{i3} = X_i U_i^T + \delta Y_i^T J_i^T, \tilde{\Psi}_{i4} = \beta Y_i^T J_i^T, \\ \tilde{\Psi}_{i5} &= -2X_i + \bar{O}_i, \tilde{\Psi}_{i6} = X_i U_{\tau i}^T + (1 - \delta)Y_{\tau i}^T J_i^T, \tilde{\Psi}_{i7} = -\beta Y_{\tau i}^T J_i^T. \end{aligned}$$

Moreover, the gains of controller (22) are

$$K_i = Y_i X_i^{-1}, \quad K_{\tau i} = Y_{\tau i} X_i^{-1}.$$

**Proof.** Choosing the Lyapunov functional (12) for system (23), we obtain

$$\begin{aligned}
 & \mathcal{L}V(x_t, \sigma_t = i) \\
 &= x^T(t)[(P_i \hat{S}_i)^* + \sum_{j=1}^N \pi_{ij} P_j]x(t) + 2x^T(t)P_i G_i v(t) + 2x^T(t)P_i \hat{S}_{\tau i} x(t - \tau) \\
 & \quad + x^T(t)O_i x(t) - x^T(t - \tau)O_i x(t - \tau) + \beta^2 Z_i^T P_i Z_i + \tilde{\Xi}^T P_i \tilde{\Xi} \\
 &= x^T(t)[(1 + \pi_k^i)(P_i \hat{S}_i)^* + \zeta_k^i + \sum_{j \in L_{uk}^i} \pi_{ij}((P_i \hat{S}_i)^* + P_j) + O_i]x(t) \\
 & \quad + 2x^T(t)P_i G_i v(t) + \beta^2 Z_i^T P_i Z_i + \tilde{\Xi}^T P_i \tilde{\Xi} \\
 & \quad - x^T(t - \tau)O_i x(t - \tau) + 2x^T(t)P_i \hat{S}_{\tau i} x(t - \tau),
 \end{aligned} \tag{29}$$

where  $\tilde{\Xi} = F_i v(t) + \hat{U}_i x(t) + \hat{U}_{\tau i} x(t - \tau)$ .

Let  $\bar{O}_i = O_i^{-1}$  per (24) and the following inequality

$$-X_i O_i X_i \leq -X_i - X_i + \bar{O}_i. \tag{30}$$

The following result is obtained

$$\begin{bmatrix}
 \tilde{\Psi}_{i1} & \tilde{\Psi}_{i2} & G_i X_i & H_i^T & \tilde{\Psi}_{i3} & \tilde{\Psi}_{i4} & X_i \\
 * & \tilde{\Psi}_{i5} & 0 & H_{\tau i}^T & \tilde{\Psi}_{i6} & \tilde{\Psi}_{i7} & 0 \\
 * & * & -\gamma^2 I & D_i & F_i^T & 0 & 0 \\
 * & * & * & -I & 0 & 0 & 0 \\
 * & * & * & * & -X_i & 0 & 0 \\
 * & * & * & * & * & -X_i & 0 \\
 * & * & * & * & * & * & -\bar{O}_i
 \end{bmatrix} < 0, \tag{31}$$

where  $\tilde{\Psi}_{i5} = -X_i O_i X_i$ .

By pre- and post-multiplying both sides of (31), respectively, by  $\text{diag} \{X_i^{-1}, X_i^{-1}, I, I, I, I, I\}$  and  $\text{diag} \{X_i^{-1}, X_i^{-1}, I, I, I, I, I\}^T$ ; denoting  $X_i = P_i^{-1}, Y_i = K_i X_i, Y_{\tau i} = K_{\tau i} X_i$ ; and according to Lemma 2, one obtains

$$\begin{bmatrix}
 \Pi_{i1} & \Pi_{i2} & \Pi_{i3} \\
 * & -\gamma^2 I + F_i^T P_i F_i & D_i \\
 * & * & -I
 \end{bmatrix} < 0, \tag{32}$$

where

$$\begin{aligned}
 \Pi_{i1} &= \begin{bmatrix} \hat{\Omega}_{i1} & \hat{\Omega}_{i2} \\ * & \hat{\Omega}_{i3} \end{bmatrix}, \\
 \hat{\Omega}_{i1} &= (1 + \pi_k^i)(P_i S_i + \delta P_i L_i K_i)^* + \zeta_k^i \\
 & \quad + \sum_{j \in L_{uk}^i} \pi_{ij}((P_i S_i + \delta P_i L_i K_i)^* + P_j) + O_i - \eta P_i \\
 & \quad + (U_i + \delta J_i K_i)^T P_i (U_i + \delta J_i K_i) + \beta^2 (J_i K_i)^T P_i (J_i K_i), \\
 \hat{\Omega}_{i2} &= P_i S_{\tau i} + P_i (L_i K_{\tau i}) + \beta^2 (J_i K_i)^T P_i (J_i K_{\tau i}) \\
 & \quad + (U_i + \delta J_i K_i)^T P_i (U_{\tau i} + (1 - \delta) J_i K_{\tau i}), \\
 \hat{\Omega}_{i3} &= -O_i + \beta^2 (J_i K_{\tau i})^T P_i (J_i K_{\tau i}) \\
 & \quad + (U_{\tau i} + \delta J_i K_{\tau i})^T P_i (U_{\tau i} + (1 - \delta) J_i K_{\tau i}), \\
 \Pi_{i2} &= \begin{bmatrix} P_i G_i + (U_i + \delta J_i K_i)^T P_i F_i \\ (U_{\tau i} + (1 - \delta) J_i K_{\tau i})^T P_i F_i \end{bmatrix}, \quad \Pi_{i3} = [H_i \quad H_{\tau i}]^T.
 \end{aligned}$$



By pre- and post-multiplying (32) by  $\text{diag} [x^T(t) \quad x^T(t - \tau) \quad v^T(t) \quad z^T(t)]$  and its transpose, respectively, and comparing it with (29), it is seen that

$$\mathcal{L}V(x_t, \sigma_t = i) < \eta V_1(x_t, \sigma_t = i) + \gamma^2 v^T(t)v(t) - z^T(t)z(t).$$

Then, one has

$$\mathcal{L}V(x_t, \sigma_t = i) < \eta V(x_t, \sigma_t = i) + \gamma^2 v^T(t)v(t) - z^T(t)z(t). \tag{33}$$

Under zero initial condition, taking mathematical expectation, and integrating both sides of (33) from 0 to  $t$  ( $t \in [0, \tilde{T}]$ ), by applying Lemma 1, it is deduced that

$$E[V(x_t, \sigma_t = i)] < e^{\eta \tilde{T}} \{ \gamma^2 E[\int_0^{\tilde{T}} v^T(t)v(t)dt] - E[\int_0^{\tilde{T}} z^T(t)z(t)dt] \}. \tag{34}$$

It is also clear that (34) implies

$$E[\int_0^{\tilde{T}} z^T(t)z(t)dt] < \gamma^2 E[\int_0^{\tilde{T}} v^T(t)v(t)dt].$$

By (33), we obtain

$$\mathcal{L}V(x_t, \sigma_t = i) < \eta V(x_t, \sigma_t = i) + \gamma^2 v^T(t)v(t). \tag{35}$$

Because of  $R_i > 0$ , it is easy to see that (9) is the actual condition (26). For (10), it is equivalent to  $\bar{P}_i < \lambda_{i2}I$  and  $\bar{P}_i < \lambda_{i2}I$ , that is,

$$-\lambda_{i2}I + R_i^{-\frac{1}{2}} P_i R_i^{-\frac{1}{2}} < 0, \tag{36}$$

and

$$\lambda_{i1}I - R_i^{-\frac{1}{2}} P_i R_i^{-\frac{1}{2}} < 0. \tag{37}$$

According to Lemma 2, (26) is equivalent to (36), and (37) is acquired by (27) and (30). From Theorem 1, if  $Q_i = \gamma^2 I$ , it is concluded that (14) and (35) are equivalent. The rest is similar to the proof of (16)–(21), which is obtained by conditions (9), (10), and (28). This completes the proof.  $\square$

**Remark 4.** Compared with the literature [42–44], controller (22) combines two traditional controllers,  $u(t) = Kx(t)$  and  $u(t) = K_\tau(t)x(t - \tau)$ , and therefore is more general and has broader applications, such as networked control systems [45].

With the idea behind controller (22), another stabilizing controller without a Bernoulli variable is devised

$$u(t) = K_\tau(\sigma_t)x(t - \tau) + K(\sigma_t)x(t). \tag{38}$$

Using controller (38) in system (1), which includes the Bernoulli variable, one obtains

$$\begin{cases} dx(t) = [S(\sigma_t)x(t) + S_\tau(\sigma_t)x(t - \tau) + G(\sigma_t)v(t) + \delta(t)L(\sigma_t)u(t)]dt \\ \quad + [U(\sigma_t)x(t) + U_\tau(\sigma_t)x(t - \tau) + (1 - \delta(t))J(\sigma_t)u(t) + F(\sigma_t)v(t)]d\omega(t), \\ z(t) = H(\sigma_t)x(t) + H_\tau(\sigma_t)x(t - \tau) + D(\sigma_t)v(t), \quad \forall t \in [0, \tilde{T}], \\ x(t) = \psi(t), \quad \sigma_t = t_0, \quad \forall t \in [-\tau, 0], \end{cases}$$

which is rewritten as follows

$$\begin{cases} dx(t) = [\hat{S}(\sigma_t)x(t) + \bar{S}_\tau(\sigma_t)x(t - \tau) + G(\sigma_t)v(t) + (\delta(t) - \delta)\bar{W}(\sigma_t)]dt \\ \quad + [\bar{U}(\sigma_t)x(t) + \bar{U}_\tau(\sigma_t)x(t - \tau) + F(\sigma_t)v(t) + (\delta(t) - \delta)\bar{Z}(\sigma_t)]d\omega(t), \\ z(t) = H(\sigma_t)x(t) + H_\tau(\sigma_t)x(t - \tau) + D(\sigma_t)v(t), \forall t \in [0, \tilde{T}], \\ x(t) = \psi(t), \sigma_t = t_0, \forall t \in [-\tau, 0], \end{cases} \quad (39)$$

where

$$\begin{aligned} \bar{S}_\tau(\sigma_t) &= S_\tau(\sigma_t) + \delta L(\sigma_t)K_\tau(\sigma_t), \\ \bar{W}(\sigma_t) &= L(\sigma_t)K(\sigma_t)x(t) + L(\sigma_t)K_\tau(\sigma_t)x(t - \tau), \\ \bar{U}(\sigma_t) &= U(\sigma_t) + (1 - \alpha)J(\sigma_t)K(\sigma_t), \\ \bar{Z}(\sigma_t) &= J(\sigma_t)K(\sigma_t)x(t) + J(\sigma_t)K_\tau(\sigma_t)x(t - \tau). \end{aligned}$$

The following theorem is developed, which is a sufficient condition of  $H_\infty$  FTB for the closed-loop system (39).

**Theorem 3.** System (39) is  $H_\infty$  FTB with respect to  $(c_1, c_2, \tilde{T}, R_i, d, \gamma)$ , if for a real scalar  $\eta \geq 0$ , there exist the constants  $\gamma > 0, \lambda_{i1} > 0, \lambda_{i2} > 0$ , the symmetric matrix  $X_i > 0$ , the matrices  $\bar{O}_i > 0$  and  $Y_i, Y_{\tau i}$  satisfying (25)–(28), and

$$\begin{bmatrix} \hat{\Psi}_{i1} & \hat{\Psi}_{i2} & G_i X_i & H_i^T & \hat{\Psi}_{i3} & \hat{\Psi}_{i4} & X_i \\ * & \hat{\Psi}_{i5} & 0 & H_{\tau i}^T & \hat{\Psi}_{i6} & \hat{\Psi}_{i7} & 0 \\ * & * & -\gamma^2 I & D_i & F_i^T & 0 & 0 \\ * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & -X_i & 0 & 0 \\ * & * & * & * & * & -X_i & 0 \\ * & * & * & * & * & * & -\bar{O}_i \end{bmatrix} < 0, \quad (40)$$

where

$$\hat{\Psi}_{i2} = S_{\tau i} X_i + \delta L_i Y_{\tau i}, \hat{\Psi}_{i7} = \beta Y_{\tau i}^T J_i^T, \hat{\Psi}_{i3} = X_i U_i^T + (1 - \delta) Y_i^T J_i^T.$$

The gains of controller (38) are presented by

$$K_i = Y_i X_i^{-1}, \quad K_{\tau i} = Y_{\tau i} X_i^{-1}.$$

**Proof.** Choosing the Lyapunov functional (12) for system (39), then  $\mathcal{L}V(x_t, \sigma_t = i)$  satisfies

$$\begin{aligned} &\mathcal{L}V(x_t, \sigma_t = i) \\ &= x^T(t) [(P_i \hat{S}_i)^* + \sum_{j=1}^N \pi_{ij} P_j] x(t) + 2x^T(t) P_i \bar{S}_{\tau i} x(t - \tau) \\ &\quad + 2x^T(t) P_i G_i v(t) + \beta^2 \bar{Z}_i^T P_i \bar{Z}_i + x^T(t) O_i x(t) \\ &\quad + \hat{\Xi}^T P_i \hat{\Xi} - x^T(t - \tau) O_i x(t - \tau) \\ &= x^T(t) [(1 + \pi_k^i) (P_i \hat{S}_i)^* + \zeta_k^i + \sum_{j \in L_{uk}^i} \pi_{ij} ((P_i \hat{S}_i)^* + P_j) + O_i] x(t) \\ &\quad + 2x^T(t) P_i \bar{S}_{\tau i} x(t - \tau) + \beta^2 \bar{Z}_i^T P_i \bar{Z}_i + \hat{\Xi}^T P_i \hat{\Xi} \\ &\quad + 2x^T(t) P_i G_i v(t) - x^T(t - \tau) O_i x(t - \tau), \end{aligned} \quad (41)$$

where  $\hat{\Xi} = F_i v(t) + \bar{U}_i x(t) + \bar{U}_{\tau i} x(t - \tau)$ .

The next steps are the same as those for the proof of Theorem 2. Pre- and post-multiply (40) by  $\text{diag} \{X_i^{-1}, X_i^{-1}, I, I, \dots, I\}$  and its transpose, respectively. Then, by Schur's com-

plement and pre- and post-multiplying both sides by  $[x^T(t) \ x^T(t - \tau) \ v^T(t) \ z^T(t)]$  and its transpose, respectively, and, by comparing it with (41), one obtains

$$\mathcal{L}V(x_t, \sigma_t = i) < \eta V(x_t, \sigma_t = i) + \gamma^2 v^T(t)v(t) - z^T(t)z(t).$$

The following step is similar to Theorem 2 and is omitted here. The proof ends.  $\square$

**Remark 5.** If  $K_\tau(\sigma_i) = 0$ , then Theorem 3 is reduced to Theorem 3.3 in [51].

For system (1), another controller experiencing a disordering phenomenon is described as

$$u(t) = [(1 - \delta(t))K_\tau(\sigma_t) + \delta(t)K(\sigma_t)]x(t) + [(1 - \delta(t))K(\sigma_t) + \delta(t)K_\tau(\sigma_t)]x(t - \tau), \tag{42}$$

which implies

$$u(t) = \begin{cases} K_\tau(\sigma_t)x(t - \tau) + K(\sigma_t)x(t), & \text{if } \delta(t) = 1 \text{ or without disordering,} \\ K(\sigma_t)x(t - \tau) + K_\tau(\sigma_t)x(t), & \text{if } \delta(t) = 0 \text{ or with disordering.} \end{cases}$$

It is easy to see that (42) is the same as

$$u(t) = [(1 - \delta)K_\tau(\sigma_t) + \delta K(\sigma_t) + (\delta(t) - \delta)(K(\sigma_t) - K_\tau(\sigma_t))]x(t) + [(1 - \delta)K(\sigma_t) + \delta K_\tau(\sigma_t) + (\delta(t) - \delta)(K_\tau(\sigma_t) - K(\sigma_t))]x(t - \tau). \tag{43}$$

Controller (43) is applied to system (1), and let  $\delta_t = \delta(t) - \delta$ . Then, we have

$$\left\{ \begin{aligned} dx(t) &= [\tilde{S}(\sigma_t)x(t) + \tilde{S}_\tau(\sigma_t)x(t - \tau) + Gv(t) \\ &\quad + \delta_t L(\sigma_t)(K(\sigma_t) - K_\tau(\sigma_t))x(t) \\ &\quad + \delta_t L(\sigma_t)(K_\tau(\sigma_t) - K(\sigma_t))x(t - \tau)]dt \\ &\quad + [\tilde{U}(\sigma_t)x(t) + Fv(t) + \tilde{U}_\tau(\sigma_t)x(t - \tau) \\ &\quad + \delta_t J(\sigma_t)(K(\sigma_t) - K_\tau(\sigma_t))x(t) \\ &\quad + \delta_t J(\sigma_t)(K_\tau(\sigma_t) - K(\sigma_t))x(t - \tau)]d\omega, \\ z(t) &= H(\sigma_t)x(t) + H_\tau(\sigma_t)x(t - \tau) + D(\sigma_t)v(t), \forall t \in [0, \tilde{T}], \\ x(t) &= \psi(t), \sigma_t = t_0, \forall t \in [-\tau, 0], \end{aligned} \right. \tag{44}$$

where

$$\begin{aligned} \tilde{S}(\sigma_t) &= S(\sigma_t) + L(\sigma_t)[\delta K(\sigma_t) + (1 - \delta)K_\tau(\sigma_t)], \\ \tilde{S}_\tau(\sigma_t) &= S_\tau(\sigma_t) + L(\sigma_t)[\delta K_\tau(\sigma_t) + (1 - \delta)K(\sigma_t)], \\ \tilde{U}(\sigma_t) &= U(\sigma_t) + J(\sigma_t)[\delta K(\sigma_t) + (1 - \delta)K_\tau(\sigma_t)], \\ \tilde{U}_\tau(\sigma_t) &= U_\tau(\sigma_t) + J(\sigma_t)[\delta K_\tau(\sigma_t) + (1 - \delta)K(\sigma_t)]. \end{aligned}$$

Then, the following theorem is developed.

**Theorem 4.** For the given real scalar  $\eta \geq 0$ , system (44) is  $H_\infty$  FTB with respect to  $(c_1, c_2, \bar{T}, R_i, d, \gamma)$ , if there exist  $\gamma > 0, \lambda_{i1} > 0, \lambda_{i2} > 0$ , the matrices  $X_i > 0, \bar{O}_i > 0$  and  $Y_i, Y_{\tau i}$  satisfying (25)–(28), and

$$\begin{bmatrix} \check{\Psi}_{i1} & \check{\Psi}_{i2} & G_i X_i & H_i^T & \check{\Psi}_{i3} & \check{\Psi}_{i4} & X_i \\ * & \check{\Psi}_{i5} & 0 & H_{\tau i}^T & \check{\Psi}_{i6} & \check{\Psi}_{i7} & 0 \\ * & * & -\gamma^2 I & D_i & F_i^T & 0 & 0 \\ * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & -X_i & 0 & 0 \\ * & * & * & * & * & -X_i & 0 \\ * & * & * & * & * & * & -\bar{O}_i \end{bmatrix} < 0. \tag{45}$$

where

$$\begin{aligned} \check{\Psi}_{i1} &= (1 + X_i \pi_k^i)(S_i X_i + \delta L_i Y_i + (1 - \delta) L_i Y_{\tau i})^* + X_i \zeta_k^i X_i^T - \eta X_i \\ &\quad + X_i \sum_{j \in L_{ik}^i} \pi_{ij} [(S_i X_i + \delta L_i Y_i + (1 - \delta) L_i Y_{\tau i})^* + X_j^{-1} X_i^T], \\ \check{\Psi}_{i2} &= S_{\tau i} X_i + \delta L_i Y_{\tau i} + (1 - \delta) L_i Y_i, \quad \check{\Psi}_{i3} = X_i U_i^T + \delta Y_i^T J_i^T + (1 - \delta) Y_{\tau i}^T J_i^T, \\ \check{\Psi}_{i4} &= \beta (Y_i^T J_i^T - Y_{\tau i}^T J_i^T), \quad \check{\Psi}_{i7} = \beta (Y_{\tau i}^T J_i^T - Y_i^T J_i^T), \\ \check{\Psi}_{i6} &= X_i U_{\tau i}^T + \delta Y_{\tau i}^T J_i^T + (1 - \delta) Y_i^T J_i^T. \end{aligned}$$

Then, the gains of controller (42) are obtained by

$$K_i = Y_i X_i^{-1}, \quad K_{\tau i} = Y_{\tau i} X_i^{-1}.$$

**Proof.** Choosing the Lyapunov functional (12) for system (44), it is obtained that

$$\begin{aligned} &\mathcal{L}V(x_t, \sigma_t = i) \\ &= x^T(t) [(P_i \tilde{S}_i)^* + \sum_{j=1}^N \pi_{ij} P_j] x(t) + \beta^2 \check{Z}_i^T P_i \check{Z}_i \\ &\quad + 2x^T(t) P_i G_i v(t) + 2x^T(t) P_i \tilde{S}_{\tau i} x(t - \tau) \\ &\quad + x^T(t) O_i x(t) + \check{\Xi}^T P_i \check{\Xi} - x^T(t - \tau) O_i x(t - \tau) \\ &= x^T(t) [(1 + \pi_k^i)(P_i \tilde{S}_i)^* + \zeta_k^i + P_j + O_i] x(t) \\ &\quad + 2x^T(t) P_i \tilde{S}_{\tau i} x(t - \tau) + 2x^T(t) P_i G_i v(t) \\ &\quad + \beta^2 \check{Z}_i^T P_i \check{Z}_i + \check{\Xi}^T P_i \check{\Xi} - x^T(t - \tau) O_i x(t - \tau). \end{aligned} \tag{46}$$

where

$$\begin{aligned} \check{\Xi} &= \tilde{U}_i x(t) + \tilde{U}_{\tau i} x(t - \tau) + F_i v(t), \\ \check{Z}_i &= J_i (K_i - K_{\tau i}) x(t) + J_i (K_{\tau i} - K_i) x(t - \tau). \end{aligned}$$

Pre- and post-multiply (45), respectively, by  $\text{diag} \{X_i^{-1}, X_i^{-1}, I, I, \dots, I\}$  and its transpose. Then, from Lemma 2, by pre- and post-multiplying both sides by  $[x^T(t) \ x^T(t - \tau) \ v^T(t) \ z^T(t)]$  and its transpose, respectively, and comparing it with (46), one obtains

$$\mathcal{L}V(x_t, \sigma_t = i) < \eta V(x_t, \sigma_t = i) + \gamma^2 v^T(t) v(t) - z^T(t) z(t).$$

The next steps are same as Theorem 2 and are omitted here. The proof is complete.  $\square$

**Remark 6.** If  $\delta(t) = 1$  or controller (42) does not experience a disordering phenomenon, then Theorem 4 is reduced to Theorem 3.

### 4. Numerical Examples

In this part, three examples are given to illustrate the effectiveness of the proposed results.

**Example 1.** Consider system (1) with the following parameters:

Model1:

$$S_1 = \begin{bmatrix} -3.1 & 0.3 \\ 1 & 0.1 \end{bmatrix}, S_{\tau 1} = \begin{bmatrix} -1.7 & 1.1 \\ 0 & 0.2 \end{bmatrix}, U_1 = \begin{bmatrix} 0.61 & 0.13 \\ 0.17 & 0.15 \end{bmatrix}, U_{\tau 1} = \begin{bmatrix} -0.2 & 0.1 \\ 0 & -0.1 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 0.1 & -1.3 \\ 0.2 & -0.9 \end{bmatrix}, F_1 = \begin{bmatrix} -0.2 & -0.7 \\ 1.9 & 0 \end{bmatrix}, L_1 = \begin{bmatrix} 2.1 \\ 0.9 \end{bmatrix}, J_1 = \begin{bmatrix} 1.1 \\ 0.5 \end{bmatrix}, R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} -0.4 & 0.1 \\ 0.7 & 0.1 \end{bmatrix}, H_{\tau 1} = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & -0.3 \end{bmatrix}, D_1 = \begin{bmatrix} -0.2 & 0.5 \\ 0.3 & -0.1 \end{bmatrix}.$$

Mode2:

$$S_2 = \begin{bmatrix} -3.9 & 0.9 \\ 1.1 & 0 \end{bmatrix}, S_{\tau 2} = \begin{bmatrix} -0.7 & 1.2 \\ 0 & 0.3 \end{bmatrix}, U_2 = \begin{bmatrix} 0.5 & 0.3 \\ 0.1 & 0.3 \end{bmatrix}, U_{\tau 2} = \begin{bmatrix} -0.2 & 0.3 \\ 0.5 & -0.1 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 0.3 & -0.9 \\ 0.7 & -1.1 \end{bmatrix}, F_2 = \begin{bmatrix} -0.1 & -1 \\ 0.2 & 0.1 \end{bmatrix}, L_2 = \begin{bmatrix} 1.6 \\ 1.5 \end{bmatrix}, J_2 = \begin{bmatrix} 1.3 \\ 0.4 \end{bmatrix}, R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} -0.3 & 0.1 \\ 0.9 & 0.3 \end{bmatrix}, H_{\tau 2} = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & -0.3 \end{bmatrix}, D_2 = \begin{bmatrix} -0.2 & 0.6 \\ 0.2 & -0.1 \end{bmatrix}.$$

The partially unknown transition rate matrix is

$$\Pi = \begin{bmatrix} -0.5 & 0.5 \\ ? & ? \end{bmatrix}.$$

Moreover,  $\tilde{T} = 10, c_1 = 0.5, \tau = 1, \delta = 0.6, d = 1, x_0 = [0.1 \ -0.05]^T$ , and  $v(t) = \frac{1}{(1+t^2)}$ . From Theorem 2, the feasible solution can be found when  $\eta \in [0, 1.90]$ . The relationship curves of  $\eta$  with  $c_2$  and  $\gamma$  are shown in Figures 1 and 2, respectively. From Figure 1, it is seen that the minimum value of  $c_2$  is 32.3726 and the corresponding  $\gamma = 2.8132$  when  $\eta = 0.05$ .

When  $\eta = 0.05$ , the gains of controller (22) are

$$K_1 = [ -1.8478 \quad -1.1423 ], K_{\tau 1} = [ 0.1438 \quad -0.2441 ],$$

$$K_2 = [ -1.1312 \quad -1.4924 ], K_{\tau 2} = [ 0.0100 \quad -0.2335 ].$$

This indicates that under controller (22), when  $E[x^T(t_1)R_ix(t_1)] \leq 0.5, t_1 \in [-1, 0]$ , then  $E[x^T(t_2)R_ix(t_2)] < 32.3726, t_2 \in [0, 1]$ , and  $E[\int_0^1 z^T(t)z(t)dt] < 2.8132^2 E[\int_0^1 v^T(t)v(t)dt]$ .

According to the conditions mentioned above, Figure 3 shows the state response of system (23), where the small figures represent the curves of a possible Markovian mode evolution and the evolution of the Bernoulli variable  $\delta(t)$  with  $\delta = 0.6$ . The evolution of  $E[x^T(t)Rx(t)]$  is shown in Figure 4, which implies that the closed-loop system (23) is  $H_\infty$  FTB.

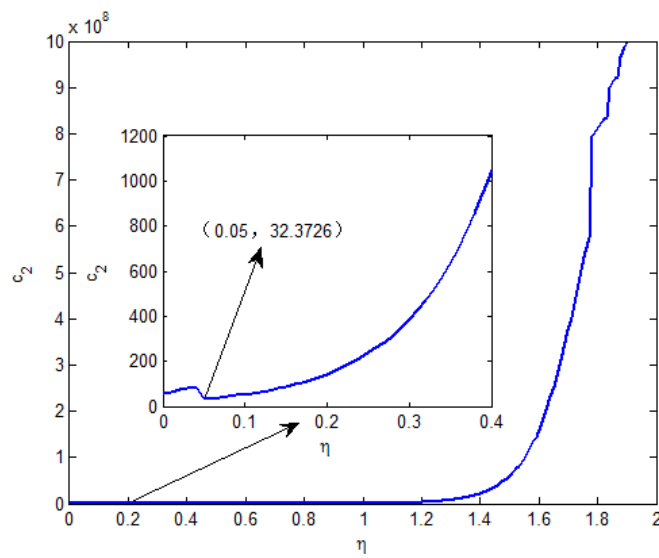


Figure 1. When  $\eta \in [0, 1.90]$ , the curve of  $c_2$ .

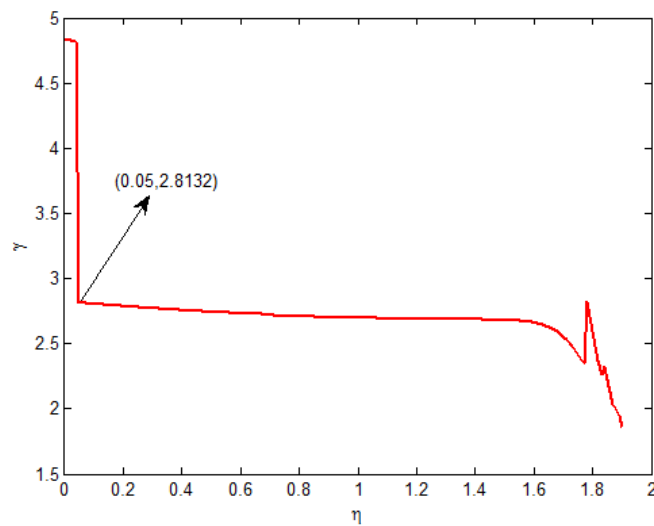


Figure 2. When  $\eta \in [0, 1.90]$ , the curve of  $\gamma$ .

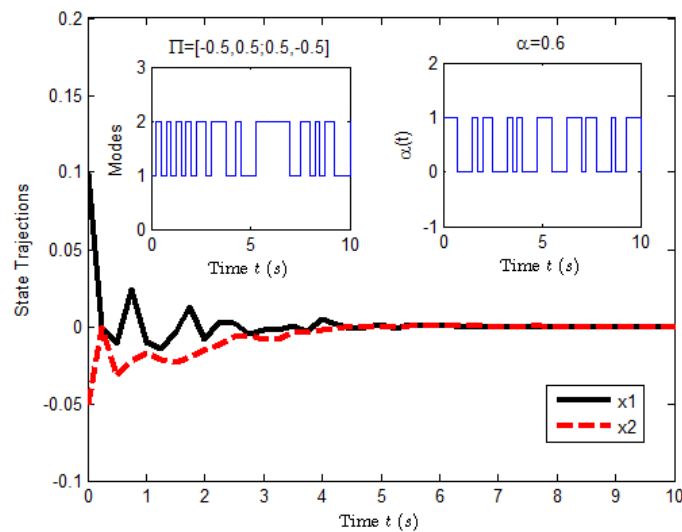


Figure 3. The state response of system (23).

In order to show the advantages of Theorem 2 and the influence of the probability  $\delta$ , Figure 5 depicts the relationship between  $c_2$  and  $\delta$ . It is seen that  $c_2$  takes the minimum value when  $\delta = 0.78$ . This means controller (22) has less conservatism.

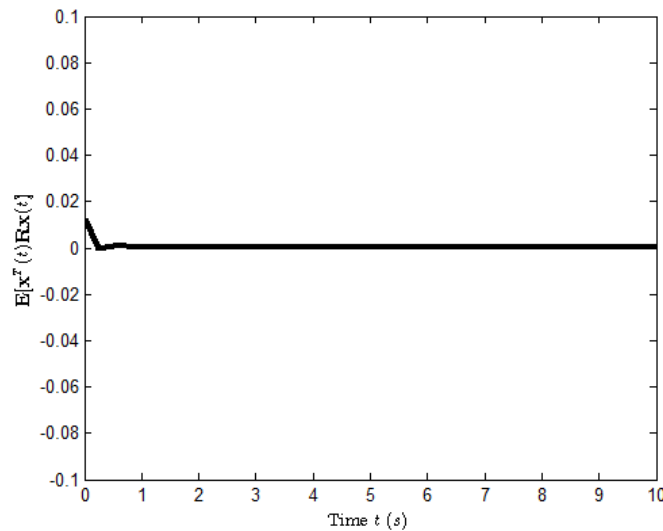


Figure 4. The evolution of  $E[x^T(t)Rx(t)]$  for system (23).

**Example 2.** Consider system (39) with the parameters of Example 1. By Theorem 3, we obtain the feasible solution when  $\eta \in [0, 1.90]$ . The minimum value of  $c_2$  is 67.8162 and the corresponding  $\gamma = 5.0148$  when  $\eta = 0$ . The gains of controller (38) are

$$K_1 = [ 2.1007 \quad -1.3952 ], K_{\tau 1} = [ 0.3184 \quad -0.1772 ],$$

$$K_2 = [ -0.2674 \quad -1.8616 ], K_{\tau 2} = [ 0.1328 \quad -0.4017 ].$$

Figures 6 and 7 show the state response of system (39) and the evolution of  $E[x^T(t)Rx(t)]$ , respectively. From these figures, it is seen that the closed-loop system (39) is  $H_\infty$  FTB by the designed controller (38). This implies that Theorem 3 is valid.

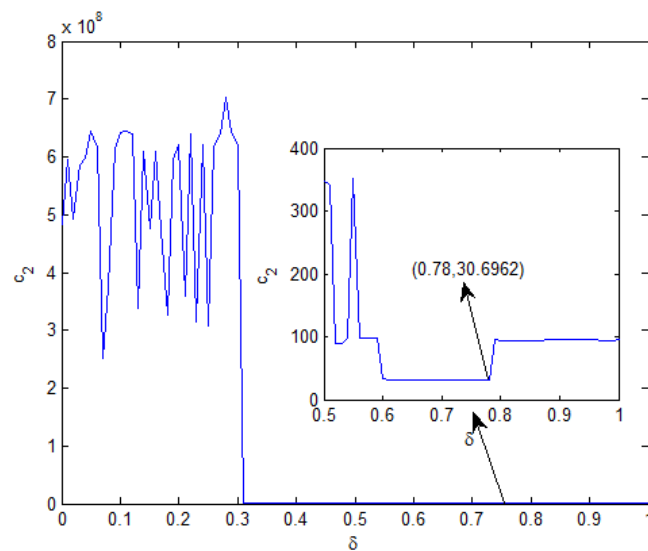


Figure 5. The relationship between  $\delta$  and  $c_2$ .

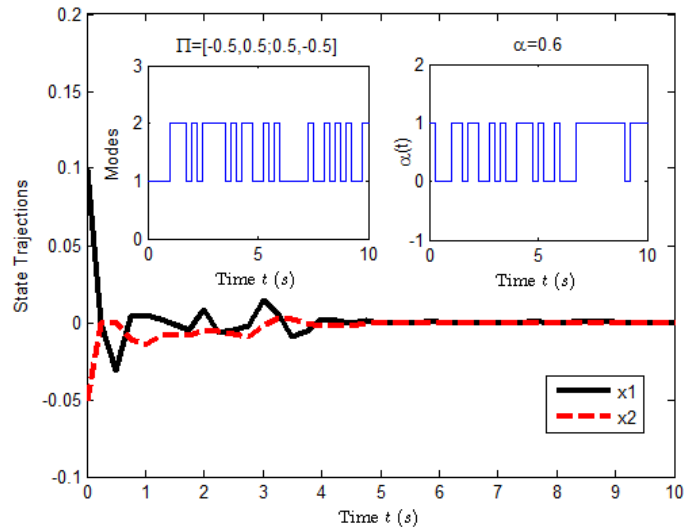


Figure 6. The state response of system (39).

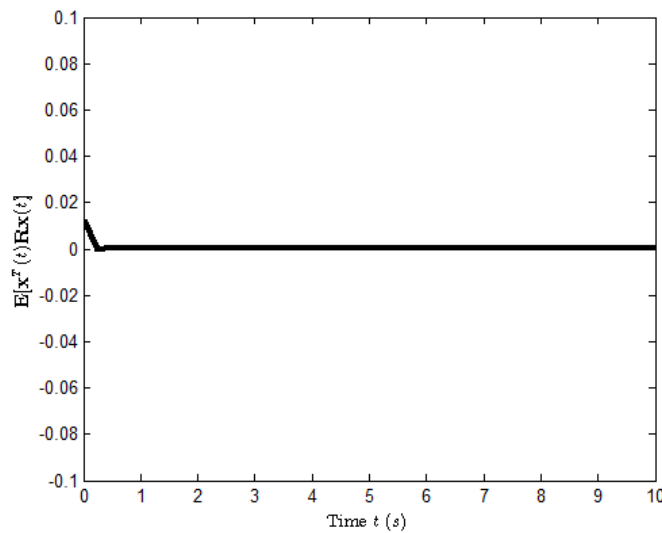


Figure 7. The evolution of  $E[x^T(t)Rx(t)]$  for system (39).

**Example 3.** Consider system (44) with the system parameters of Example 1. By Theorem 4, the feasible region is  $\eta \in [0, 1.89]$ . When  $\eta = 0.03$ , the minimum value of  $c_2$  is 310.9813, and the corresponding  $\gamma = 9.7316$ . The gains of controller (42) are

$$K_1 = [ -0.5970 \quad -0.3385 ], K_{\tau 1} = [ 0.8791 \quad -0.2409 ],$$

$$K_2 = [ -0.6966 \quad -0.3475 ], K_{\tau 2} = [ 0.6689 \quad -0.2812 ].$$

Similar to Example 2, the state response of system (44) is shown in Figure 8, and the evolution of  $E[x^T(t)Rx(t)]$  is drawn in Figure 9. It is concluded from these plots that the closed-loop system (44) is  $H_\infty$  FTB, by the designed controller (42). Therefore, Theorem 4 is valid.



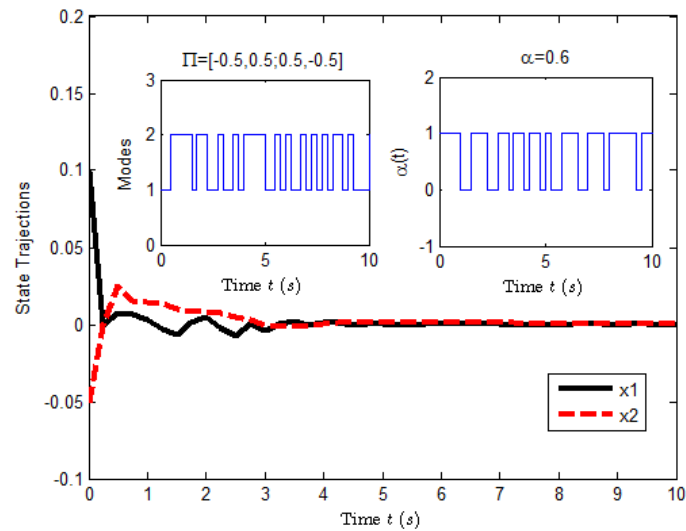


Figure 8. The state response of system (44).

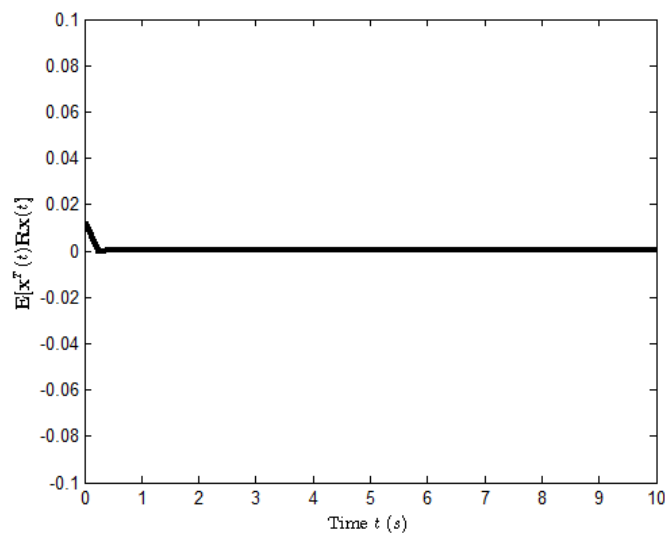


Figure 9. The evolution of  $E[x^T(t)Rx(t)]$  for system (44).

## 5. Conclusions

In this paper, the FTB and  $H_\infty$  FTB problems of time-delay Markovian jump systems with a partially unknown transition rate have been studied. A sufficient condition of FTB for the given system is obtained by the LMIs technique and the Lyapunov functional method. A new controller that is partially time delay-dependent is designed. This controller has the advantages of strong generality and less conservative property. Based on PDDCs, two new kinds of controllers are derived; one does not contain the Bernoulli variable, and the other describes controllers experiencing a disordering phenomenon. Combined with LMIs, some sufficient conditions of  $H_\infty$  FTB for closed-loop systems are given via the designed controllers. Three numerical examples illustrate that the proposed methods are effective. The results in this paper can be extended to the  $H_\infty$  filtering problem for Markovian jump systems with time-varying delays. In the future, the FTB and  $H_\infty$  FTB problems of fractional systems will be considered by means of the theories of fractional calculus and negative probabilities [56].

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## Appendix A

In this section, Lemma 1 is reformulated with sharp inequalities and also proved.

Let  $g(t)$  be a nonnegative continuous function. If there are positive constants  $r, q$  such that

$$g(t) < r + q \int_0^t g(s) ds, \quad 0 \leq t \leq \tilde{T}, \quad (\text{A1})$$

then

$$g(t) < r \exp(qt), \quad 0 \leq t \leq \tilde{T}. \quad (\text{A2})$$

**Proof.** Let

$$U(t) = r + q \int_0^t g(s) ds, \quad 0 \leq t \leq \tilde{T}. \quad (\text{A3})$$

Then, the derivative of  $U(t)$  is  $\dot{U}(t) = qg(t)$ . From (A1), we have  $\dot{U}(t) < qU(t)$ , i.e.,  $\dot{U}(t) - qU(t) < 0$ . Then, it is deduced that

$$\dot{U}(t)\exp(-qt) - qU(t)\exp(-qt) < 0,$$

which implies that the derivative of  $U(t)\exp(-qt)$  satisfies  $(U(t)\exp(-qt))' < 0$ . From the monotonicity,  $U(t)\exp(-qt) < U(0) = r$ , which guarantees that  $U(t) < r \exp(qt)$ . Together with (A1) and (A3), (A2) is obtained.  $\square$

## References

- Langton, R. *Stability and Control of Aircraft Systems: Introduction to Classical Feedback Control*; John Wiley and Sons, Ltd.: Hoboken, NJ, USA, 2006. [\[CrossRef\]](#)
- Liu, G.; Ma, L.; Liu, J. *Handbook of Chemistry and Chemical Material Property Data*; Chemical Industry Press: Beijing, China, 2002.
- Zivanovic, M.; Vukobratovic, M. *Multi-Arm Cooperating Robots: Dynamics and Control*, 1st ed.; Springer: Berlin/Heidelberg, Germany, 2006.
- Fridman, E.; Shaked, U. Parameter dependent stability and stabilization of uncertain time-delay systems. *IEEE Trans. Autom. Control.* **2003**, *48*, 861–866. [\[CrossRef\]](#)
- Yakoubi, K.; Chitour, Y. Linear systems subject to input saturation and time delay: Global asymptotic stabilization. *IEEE Trans. Autom. Control.* **2007**, *52*, 874–879. [\[CrossRef\]](#)
- Hai, L.; Antsaklis, P. Stability and stabilizability of switched linear systems: A survey of recent results. *IEEE Trans. Autom. Control* **2009**, *54*, 308–322. [\[CrossRef\]](#)
- Mancilla-Aguilar, J.; Haimovich, H.; Garcia, R. Global stability results for switched systems based on weak Lyapunov functions. *IEEE Trans. Autom. Control* **2017**, *62*, 2764–2777. [\[CrossRef\]](#)
- Dorato, P. Short time stability in linear time-varying systems. *Proc. IRE Int. Conv. Rec.* **1961**, *4*, 83–87.
- Kamenkov, G. On stability of motion over a finite interval of time. *J. Appl. Math. Mech.* **1953**, *17*, 529–540.
- Amato, F.; Ariola, M.; Dorato, P. Finite-time control of linear systems subject to parametric uncertainties and disturbances. *Automatica* **2001**, *37*, 1459–1463. [\[CrossRef\]](#)
- Amato, F.; Ariola, M.; Cosentino, C. Finite-time stabilization via dynamic output feedback. *Automatica* **2005**, *42*, 337–342. [\[CrossRef\]](#)
- Amato, F.; Ariola, M.; Cosentino, C.; De Tommasi, G. Input-output finite time stabilization of linear systems. *Automatica* **2010**, *46*, 1558–1562. [\[CrossRef\]](#)

13. Amato, F.; Ambrosino, R.; Ariola, M.; De Tommasi, G.; Pironti, A. On the finite-time boundedness of linear systems. *Automatica* **2019**, *107*, 454–466. [[CrossRef](#)]
14. Liu, X.; Liu, Q.; Li, Y. Finite-time guaranteed cost control for uncertain mean-field stochastic systems. *J. Frankl. Inst.* **2020**, *357*, 2813–2829. [[CrossRef](#)]
15. Tartaglione, G.; Ariola, M.; Cosention, C.; De Tommasi, G.; Pironti, A.; Amato, F. Annular finite-time stability analysis and synthesis of stochastic linear time-varying systems. *Int. J. Control.* **2019**, *94*, 2252–2263. [[CrossRef](#)]
16. Tartaglione, G.; Ariola, M.; Amato, F. Conditions for annular finite-time stability of Itô stochastic linear time-varying systems with Markov switching. *IET Control. Theory Appl.* **2019**, *14*, 626–633. [[CrossRef](#)]
17. Bai, Y.; Sun, H.; Wu, A. Finite-time stability and stabilization of Markovian jump linear systems subject to incomplete transition descriptions. *Int. J. Control. Autom. Syst.* **2021**, *19*, 2999–3012. [[CrossRef](#)]
18. Gu, Y.; Shen, M.; Ren, Y.; Liu, H.  $H_\infty$  finite-time control of unknown uncertain systems with actuator failure. *Appl. Math. Comput.* **2020**, *383*, 125375. [[CrossRef](#)]
19. Liu, X.; Wang, J.; Li, Y. Finite-time stability analysis for discrete-time stochastic nonlinear systems with time-varying delay. *J. Shandong Univ. Sci. Technol. (Nat. Sci.)* **2021**, *40*, 110–120. [[CrossRef](#)]
20. Shen, H.; Li, F.; Yan, H.; Zhang, C.; Yan, H. Finite-time event-triggered  $H_\infty$  control for T-S fuzzy Markov systems. *IEEE Trans. Fuzzy Syst.* **2018**, *26*, 3122–3135. [[CrossRef](#)]
21. Liu, H.; Shen, Y. Finite-time bounded stabilisation for linear systems with finite-time  $H_2$ -gain constraint. *IET Control. Theory Appl.* **2020**, *14*, 1266–1275. [[CrossRef](#)]
22. Li, M.; Sun, L.; Yang, R. Finite-time  $H_\infty$  control for a class of discrete-time nonlinear singular systems. *J. Frankl. Inst.* **2018**, *355*, 5384–5393. [[CrossRef](#)]
23. Xiang, Z.; Qiao, C.; Mahmoud, M. Finite-time analysis and  $H_\infty$  control for switched stochastic systems. *J. Frankl. Inst.* **2011**, *349*, 915–927. [[CrossRef](#)]
24. Zhuang, J.; Liu, X.; Li, Y. Event-triggered annular finite-time  $H_\infty$  filtering for stochastic network systems. *J. Frankl. Inst.* **2022**, *359*, 11208–11228. [[CrossRef](#)]
25. De la Sen, M.; Ibeas, A.; Nistal, R. On the entropy of events under eventually global inflated or deflated probability constraints. Application to the supervision of epidemic models under vaccination controls. *Entropy* **2020**, *22*, 284. [[CrossRef](#)]
26. Li, Y.; Zhang, W.; Liu, X.  $H_-$  index for discrete-time stochastic systems with Markovian jump and multiplicative noise. *Automatica* **2018**, *90*, 286–293. [[CrossRef](#)] [[PubMed](#)]
27. Liu, X.; Zhang, W.; Li, Y.  $H_-$  index for continuous-time stochastic systems with Markov jump and multiplicative noise. *Automatica* **2019**, *105*, 167–178. [[CrossRef](#)]
28. Zhuang, J.; Liu, X.; Li, Y.  $H_-$  index for Itô stochastic systems with Poisson jump. *J. Frankl. Inst.* **2021**, *358*, 9929–9950. [[CrossRef](#)]
29. Wang, G.; Liu, L.; Zhang, Q.; Yang, C. Finite-time stability and stabilization of stochastic delayed jump systems via general controllers. *J. Frankl. Inst.* **2016**, *354*, 938–966. [[CrossRef](#)]
30. Yan, Z.; Ju, H.; Zhang, W. Finite-time guaranteed cost control for Itô stochastic Markovian jump systems with incomplete transition rates. *Int. J. Robust Nonlinear Control.* **2016**, *27*, 66–83. [[CrossRef](#)]
31. Yan, G.; Zhang, W.; Zhang, G. Finite-time stability and stabilization of Itô stochastic systems with Markovian switching: Mode-dependent parameter approach. *IEEE Trans. Autom. Control* **2015**, *60*, 2428–2433. [[CrossRef](#)]
32. Liu, X.; Li, W.; Yao, C.; Li, Y. Finite-time guaranteed cost control for Markovian jump systems with time-varying delays. *Mathematics* **2022**, *10*, 2028. [[CrossRef](#)]
33. Liu, Z.; Miao, S.; Fan, Z.; Han, J. Markovian switching model and non-linear DC modulation control of AC/DC power system. *IET Gener. Transm. Distrib.* **2017**, *11*, 2654–2663. [[CrossRef](#)]
34. Liu, X.; Zhuang, J.; Li, Y.  $H_\infty$  filtering for Markovian jump linear systems with uncertain transition probabilities. *Int. J. Control. Autom. Syst.* **2021**, *19*, 2500–2510. [[CrossRef](#)]
35. Zhang, L.; Guo, G. Control of a group of systems whose communication channels are assigned by a semi-Markov process. *Int. J. Syst. Sci.* **2019**, *50*, 2306–2315. [[CrossRef](#)]
36. Qi, W.; Gao, X. Finite-time  $H_\infty$  control for stochastic time-delayed Markovian switching systems with partly known transition rates and nonlinearity. *Int. J. Syst. Sci.* **2015**, *47*, 500–508. [[CrossRef](#)]
37. Cheng, J.; Zhu, H.; Zhong, S. Finite-time  $H_\infty$  filtering for discrete-time Markovian jump systems with partly unknown transition probabilities. *J. Frankl. Inst.* **2013**, *91*, 1020–1028. [[CrossRef](#)]
38. Wu, Z.; Yang, L.; Jiang, B. Finite-time  $H_\infty$  control of stochastic singular systems with partly known transition rates via an optimization algorithm. *Int. J. Control. Autom. Syst.* **2019**, *17*, 1462–1472. [[CrossRef](#)]
39. Jiang, W.; Liu, K.; Charalambous, T. Multi-agent consensus with heterogeneous time-varying input and communication delays in digraphs. *Automatica* **2022**, *135*, 109950. [[CrossRef](#)]
40. Pourdehi, S.; Kavimaghaee, P. Stability analysis and design of model predictive reset control for nonlinear time-delay systems with application to a two-stage chemical reactor system. *J. Process Control* **2018**, *71*, 103–115. [[CrossRef](#)]
41. Truong, D. Using causal machine learning for predicting the risk of flight delays in air transportation. *J. Air Transp. Manag.* **2021**, *91*, 101993. [[CrossRef](#)]

42. Li, X.; Li, P. Stability of time-delay systems with impulsive control involving stabilizing delays. *Automatica* **2020**, *124*, 109336. [[CrossRef](#)]
43. Li, L.; Zhang, H.; Wang, Y. Stabilization and optimal control of discrete-time systems with multiplicative noise and multiple input delays. *Syst. Control. Lett.* **2021**, *147*, 104833. [[CrossRef](#)]
44. Zong, G.; Wang, R.; Zheng, W. Finite-time  $H_\infty$  control for discrete-time switched nonlinear systems with time delay. *Int. J. Robust Nonlinear Control.* **2015**, *25*, 914–936. [[CrossRef](#)]
45. Qiu, L.; He, L.; Dai, L.; Fang, C.; Chen, Z.; Pan, J.; Zhang, B.; Xu, Y.; Chen, C.R. Networked control strategy of dual linear switched reluctance motors based time delay tracking system. *ISA Trans.* **2021**, *129*, 605–615. [[CrossRef](#)]
46. Liu, X.; Liu, W.; Li, Y. Finite-time  $H_\infty$  control of stochastic time-delay Markovian jump systems. *J. Shandong Univ. Sci. Technol. (Natural Sci.)* **2022**, *41*, 75–84. [[CrossRef](#)] [[PubMed](#)]
47. Yan, Z.; Song, Y.; Park, J. Finite-time stability and stabilization for stochastic Markov jump systems with mode-dependent time delays. *ISA Trans.* **2017**, *68*, 141–149. [[CrossRef](#)]
48. Liu, L.; Zhang, X.; Zhao, X.; Yang, B. Stochastic finite-time stabilization for discrete-time positive Markov jump time-delay systems. *J. Frankl. Inst.* **2022**, *359*, 84–103. [[CrossRef](#)]
49. Wang, G.; Li, Z.; Zhang, Q.; Yang, C. Robust finite-time stability and stabilization of uncertain Markovian jump systems with time-varying delay. *Appl. Math. Comput.* **2017**, *293*, 377–393. [[CrossRef](#)]
50. Liu, X.; Li, W.; Wang, J.; Li, Y. Robust finite-time stability for uncertain discrete-time stochastic nonlinear systems with time-varying delay. *Entropy* **2022**, *24*, 828. [[CrossRef](#)]
51. Tian, G. Finite-time  $H_\infty$  control for stochastic Markovian jump systems with time-varying delay and generally uncertain transition rates. *Int. J. Syst. Sci.* **2021**, *52*, 1–14. [[CrossRef](#)]
52. Oksendal, B. *Stochastic Differential Equations: An Introduction with Applications*, 6th ed.; Springer: New York, NY, USA, 2006. [[CrossRef](#)]
53. Bellman, R.; Cooke, K. *Differential-Difference Equations*; Academic Press: New York, NY, USA, 1963. [[CrossRef](#)]
54. Ouellette, D. Schur Complements and Statistics. *Linear Algebra Appl.* **1981**, *36*, 187–295. [[CrossRef](#)]
55. Vrabel, R. On local asymptotic stabilization of the nonlinear systems with time-varying perturbations by state-feedback control. *Int. J. Gen. Syst.* **2018**, *48*, 80–89. [[CrossRef](#)]
56. Tenreiro Machado, J. Fractional derivatives and negative probabilities. *Commun. Nonlinear Sci. Numer. Simul.* **2019**, *79*, 104913. [[CrossRef](#)]

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