THE CLASSIFICATION OF HOMOMORPHISM HOMOGENEOUS ORIENTED GRAPHS

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ABSTRACT. The modern theory of homogeneous structures begins with the work of Roland Fraïssé. The theory developed in the last seventy years is placed in the border area between combinatorics, model theory, algebra, and analysis. We turn our attention to its combinatorial pillar, namely, the work on the classification of structures for given homogeneity types, and focus onto the homomorphism homogeneous ones, introduced in 2006 by Cameron and Nešetřil. An oriented graph is called homomorphism homogeneous if every homomorphism between finite induced subgraphs extends to an endomorphism. In this paper we present a complete classification of the countable homomorphism homogeneous oriented graphs.

1. INTRODUCTION

The classical notion of homogeneity was introduced in the early fifties and thoroughly studied during the last seventy years. Recall that a relational structure is called *homogeneous* if every isomorphism between finite substructures extends to an automorphism. At the beginning of this century, the notion of homomorphism homogeneity was coined by Cameron and Nešetřil in their seminal paper on this topic [4]. A relational structure is called *homomorphism homogeneous* (shortly HH) if every homomorphism between two of its finite substructures extends to an endomorphism of the structure in question. In the mentioned paper, this phenomenon was studied for simple graphs and posets, with a number of inspiring and challenging questions posed. This initiated the research on the classification of countable HH relational structures with exactly one binary relation. First results were obtained relatively quickly by Cameron and Lockett [3], and Mašulović [13]

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who completely classified HH posets with strict and reflexive order relations, respectively. On the other hand, the classification of countable HH simple graphs is still wide open, and the subject of ongoing research [1,2]. Interestingly, already when allowing loops in undirected graphs the problem of classifying even finite HH structures becomes in a sense untractable. Namely, it was shown by Rusinov and Schweizer in [19] that the problem to decide whether a finite graph with loops allowed is HH is coNP-complete. Luckily it seems that HH digraphs with antisymmetric arc relation are more amenable to classification efforts: finite HH tournaments with loops allowed were classified in [11]. Finite HH uniform oriented graphs (i.e., oriented graphs with all / with no loops) were classified in [15].

In this paper we set out to classify the countable HH oriented graphs without loops (from now on, instead of "oriented graphs without loops", we will write "oriented graphs"). At this point it is worth mentioning that the parallel problem, namely the classification of homogeneous oriented graphs was carried out by Cherlin in [6]. It fills a whole volume of the Memoirs of the AMS, and already this fact was quite intimidating at the outset of the project. Fortunately, classifications of HH structures and of homogeneous structures are rather different in nature (but not completely unrelated). To our great delight it turned out that the classification of HH oriented graphs may be stratified by the countable homogeneous tournaments classified by Lachlan and Woodrow [12,21]. The key insight is that every countable HH oriented graph has a unique (up to isomorphism) homogeneous, HH core (as a consequence of a more general result from [17]), and that HH oriented graphs that are cores must be tournaments (and thus homogeneous). This allowed us to finish the classification of countable HH oriented graphs (Theorem 2.18 and Theorem 4.4).

Closely related to the notion of homomorphism homogeneity is the notion of polymorphism homogeneity. We say that a relational structure is *polymorphism homogeneous* (shortly PH) if each of its finite Cartesian powers is homomorphism homogeneous. The notion of polymorphism homogeneity has its origins in universal algebra where it has been used when studying polymorphism clones of countable homogeneous structures ([18]), and phenomena in universal algebraic geometry ([20]). First steps towards the classification theory of PH structures were done in [7,8,16]. Concerning the scope of this paper, we can mention that countable PH simple graphs, PH posets with strict and reflexive order relation, and countable PH tournaments (with loops allowed) are completely classified. This paper adds the countable oriented graphs to this list (Theorem 3.5 and Theorem 5.3).

Before starting with the exposition of our results, let us fix some notions and notation. Generally, we are using the usual graph theoretic terminology (see, e.g., [5]). Recall that an *oriented graph* is a digraph such that in between any two distinct vertices there is at most one arc. Formally, an oriented graph may be modelled as a pair $\Gamma = (V, E)$, where

- *V* is a non-empty set of vertices,
- $E \subseteq V^2$ is an asymmetric relation, i.e., $\forall x, y \in V : (x, y) \in E \Rightarrow (y, x) \notin E$.

The elements of *E* are called *arcs* of Γ . Generally, for every oriented graph Γ by V(Γ) and E(Γ) we denote the vertex set and the arc set of Γ , respectively.

For two subsets B_1 and B_2 of V(Γ) we write $B_1 \rightarrow B_2$ if $(x, y) \in E(\Gamma)$, for all $x \in B_1$ and $y \in B_2$. Instead of $\{b_1\} \rightarrow B_2, B_1 \rightarrow \{b_2\}$, and $\{b_1\} \rightarrow \{b_2\}$ we write $b_1 \rightarrow B_2, B_1 \rightarrow b_2$, and $b_1 \rightarrow b_2$, respectively.

For two vertices $x, y \in V(\Gamma)$ we write $x \sim y$ if either $x \rightarrow y$ or $y \rightarrow x$. Correspondingly we write $x \nsim y$ if it does not hold that $x \sim y$.

The notation $x \sim^* y$ means that there is semi-walk from x to y in Γ . In other words, there is a finite sequence z_0, \ldots, z_k (where $k \ge 0$) such that $x = z_0, y = z_k$, and for all $0 \le i < k$ we have $z_i \sim z_{i+1}$. Clearly, \sim^* is an equivalence relation on the vertex set of Γ . Its equivalence classes are the *weakly connected components* of Γ . If Γ has only one weakly connected component, then we say that it is *weakly connected*.

Throughout the paper, we are going to classify oriented graphs by whether or not they contain certain configurations. Here a configuration is nothing else but a finite (unlabelled) oriented graph, and a given oriented graph Γ contains a configuration Δ if some induced subgraph of Γ is isomorphic to Δ . By Age(Γ) we denote the class of all configurations contained in Γ .

We say that two oriented graphs Γ_1 and Γ_2 are *homomorphism equivalent* if there is a graph-homomorphism from Γ_1 to Γ_2 and vice versa.

Finally, throughout the paper ω denotes the set of non-negative integers. Moreover we use the convention that countable means finite or countably infinite.

Our results are presented in the four sections that follow. In Section 2 and 3 we classify weakly connected homomorphism homogeneous and polymorphism homogeneous oriented graphs, respectively. Sections 4 and 5 handle the case of disconnected oriented graphs.

2. Weakly connected homomorphism homogeneous oriented graphs

Our approach to the classification of weakly connected homomorphism homogeneous oriented graphs is based on the use of cores:

Definition 2.1. An oriented graph Γ is called a *core* if every endomorphism of Γ is a self-embedding. An oriented graph Δ is a *core of the oriented graph* Γ if it satisfies the following conditions:

- (1) Δ is a core,
- (2) Δ is an induced subgraph of Γ , and
- (3) there exists $h \in \text{End}(\Gamma)$ such that $h[V(\Gamma)] \subseteq V(\Delta)$.

The following observation is a special case of a model theoretic result about cores of relational structures:

Proposition 2.2 ([17, Corollary 6.7]). Every countable homomorphism homogeneous oriented graph has, up to isomorphism, a unique homomorphism homogeneous core. Moreover, this core is homogeneous.

Proof. First note that oriented graphs are special cases of relational structures with one binary relation.

[17, Corollary 6.7] says that the claim of the proposition holds for all countable weakly oligomorphic, homomorphism homogeneous relational structures.

It remains to observe that countable homomorphism homogeneous oriented graphs are weakly oligomorphic. However, this follows from the fact that oriented graphs, considered as relational structures, have a finite signature, together with [14, Proposition 2.3].

The strategy of classifying weakly connected homomorphism homogeneous oriented graphs is now clear, and can be divided in two steps:

- **Step 1:** Identify all weakly connected homogeneous homomorphism homogeneous oriented graphs that are cores.
- **Step 2:** For each of them classify weakly connected homomorphism homogeneous oriented graphs with a given core.

Concerning Step 1, there is a straightforward way of identifying cores among oriented graphs that are both homogeneous and homomorphism homogeneous, based on the classification of homogeneous tournaments by Lachlan and Woodrow in [12,21]:

Lemma 2.3. Homomorphism homogeneous oriented graphs that are cores must be tournaments.

Proof. Let Γ be a homomorphism homogeneous oriented graph that is a core. Suppose that Γ is not a tournament. Then there exist $v, w \in V(\Gamma)$ that are not connected by an arc. However, then the mapping $f: \{u, v\} \to V(\Gamma)$ that maps both, u and v to v is a local homomorphism of Γ (in general a *local homomorphism* of Γ is a homomorphism from a finite subgraph of Γ to Γ). As Γ is homomorphism homogeneous, f extends to an endomorphism of Γ that is not a self-embedding — a contradiction.

It is easy to see that every tournament is a core. Moreover, a simple back-andforth argument shows that a countable tournament is homomorphism homogeneous if and only if it is homogeneous. According to [12] all countable homogeneous tournaments are isomorphic to one of the tournaments from the following list:

- I_1 : the tournament that has just one vertex and no arc,
- C_3 : the oriented cycle of length 3,
- $(\mathbb{Q}, <)$: the rational numbers with the strict order,
 - *S*(2): the countable circular tournament. It is obtained by choosing a countable dense subset *S* of the unit circle in such a way that no two points of *S* are antipodal. For any two points $x, y \in S$ an arc is drawn from x to y whenever the angle traversed starting from x and going counter-clock wise to y is less then π ,
 - T^{∞} : the countable universal homogeneous tournament the Fraïssé limit of the class of all finite tournaments.

Remark. In the description of the classification of homogeneous tournaments above the term "Fraïssé limit" appears. This is because it was shown by Fraïssé (see [9]) that countable homogeneous structures are uniquely determined (up to

isomorphism) by their age. Therefore the age of a countable homogeneous structure is usually called a *Fraïssé class* and the unique countable homogeneous structure whose age is a given Fraïssé class C is called the *Fraïssé limit* of C (for a modern statement of this result, see, e.g., [10]).

Corollary 2.4. The only possible cores of homomorphism homogeneous oriented graphs are

 $I_1, C_3, (\mathbb{Q}, <), S(2), and T^{\infty}$.

Now we are ready to proceed with Step 2 of our strategy. Once the possible cores are identified, we move to the classification of weakly connected homomorphism homogeneous oriented graphs with a given core. It is an easy observation that I_1 and $(\mathbb{Q}, <)$ are the cores of acyclic weakly connected homomorphism homogeneous oriented graphs, and this defines our classification strategy — we conduct our further considerations in two directions, analyzing separately acyclic oriented graphs, and those that contain at least one cycle.

Acyclic homomorphism homogeneous oriented graphs. The crucial observation that enables the classification in this case is the following result:

Proposition 2.5. If Γ is an acyclic homomorphism homogeneous oriented graph, then its arc relation is transitive.

The proof of this proposition is based on a general property of homomorphism homogeneous oriented graphs:

Lemma 2.6. Let Γ be a homomorphism homogeneous oriented graph. Then Γ does not contain the configuration

$$\longrightarrow \longrightarrow \bigcirc \cdot$$

Proof. Suppose that Γ is a homomorphism homogeneous oriented graph that contains

as an induced subgraph. Consider the local homomorphism $f: \{x, z\} \to V(\Gamma)$ of Γ given by $f := \begin{pmatrix} x & z \\ x & x \end{pmatrix}$. Since Γ is homomorphism homogeneous, f can be extended to a local homomorphism $\hat{f}: \{x, y, z\} \to V(\Gamma)$. Observe that from $x \to y \to z$ it follows $\hat{f}(x) \to \hat{f}(y) \to \hat{f}(z)$, i.e. $x \to \hat{f}(y) \to x$, so either $\hat{f}(y) = x$ or both $x \to \hat{f}(x)$ and $\hat{f}(x) \to x$. In both situations we arrive at a contradiction with the asymmetry of $E(\Gamma)$. Hence, Γ has no induced subgraphs of the given shape. \Box

Proof of Proposition 2.5. Suppose that Γ is a homomorphism homogeneous acyclic oriented graph, and let $x, y, z \in V(\Gamma)$ be such that $x \to y \to z$. By Lemma 2.6, there is an oriented edge between x and z. Since Γ is acyclic, it follows that $x \to z$. Hence, $E(\Gamma)$ is transitive.

It is clear that transitive oriented graphs can be viewed as strict posets, so we obtain:

Corollary 2.7. Every acyclic homomorphism homogeneous oriented graph is a strict poset.

The homomorphism homogeneous strict posets were completely classified by Cameron and Lockett [3], and this enables us to give the classification in this case:

Proposition 2.8 ([3, Proposition 15]). Weakly connected acyclic homomorphism homogeneous oriented graphs are

(1) I_1 ,

- (2) (Q, <),
- (3) trees with no minimal elements such that no finite subset of vertices has a maximal lower bound,
- (4) *dual trees with no maximal elements such that no finite subset of vertices has a minimal upper bound,*
- (5) *posets such that:*
 - every finite subset of vertices is bounded from above and from below
 - no finite subset of vertices has a maximal lower bound or a minimal upper bound
 - no X₄-set has a midpoint,
- (6) extensions of the countable universal homogeneous strict poset.

Remark. A poset (P, <) is called a *tree* if for every $x \in P$ we have that the set of elements in *P* below *x* forms a chain. Moreover, (P, <) is called a *dual tree* if (P, >) is a tree. Finally, an X_4 -set in (P, <) is a 4-element subset of *P* that induces a subposet of the shape



We say that an X_4 -set has a *midpoint* if there is a fifth element that together with the X_4 -set induces a subposet of the shape:



Recall also that the countable universal homogeneous strict poset is the Fraïssé limit of the class of all finite posets. Finally, a poset $(A, <_2)$ is called an extension of a poset $(A, <_1)$ if $<_1$ is a subset of $<_2$.

Homomorphism homogeneous oriented graphs that contain cycles. Again, there is an easy but important observation that directs our strategy for the classification.

Lemma 2.9. Every induced oriented cycle in a homomorphism homogeneous oriented graph is isomorphic to C_3 .

Proof. Let Γ be a homomorphism homogeneous oriented graph. From Lemma 2.6 it follows that Γ contains no induced oriented path of length 2, so Γ cannot contain an induced oriented cycle of length greater than 3. Hence, every induced oriented cycle in Γ has to be isomorphic to C_3 .

Since homomorphism homogenous tournaments are just the homogeneous tournaments that were listed in front of Corollary 2.4, we turn our attention to homomorphism homogeneous oriented graphs that are not tournaments, but contain cycles.

Definition 2.10. Let Γ be a countable oriented graph, and let *S* be a countable set. Let $f: S \to V(\Gamma)$ be surjective. Then the oriented graph $\Gamma[f]$ is given by

$$V(\Gamma[f]) = S$$
, and $E(\Gamma[f]) = \{(s,t) \mid (f(s), f(t)) \in E(\Gamma)\}.$

Remark. In the definition above, if Γ is a tournament, then ker f is equal to the non-edge relation \neq of $\Gamma[f]$.

Proposition 2.11. Let Γ be a homomorphism homogeneous oriented graph that contains C_3 . Then Γ is isomorphic to $C_3[f]$, S(2)[f], or $T^{\infty}[f]$, for some f.

The proof of this proposition is based on the existence of certain configurations in Γ , as well as on the properties of the non-edge relation \nsim .

Lemma 2.12 (3-vertex configurations). Let Γ be a homomorphism homogeneous oriented graph that is not a tournament. Then Age(Γ) contains at least one of the following configurations:



Proof. Take $x, y \in V(\Gamma)$ with $x \neq y$. Since Γ is weakly connected, there exists a non-oriented path between x and y. Let $x \sim z_1 \sim \cdots \sim z_k \sim y$ be such a path of the shortest length. If k = 1, then we get $x \sim z_1 \sim y$. On the other hand, if $k \ge 2$, then we get $x \sim z_1 \sim z_2$, and $x \neq z_2$, since the observed path is the shortest one. In both cases, we find $u, v, w \in V(\Gamma)$ such that $u \sim v \sim w$, and $u \neq w$, so the possible induced subgraphs are $u \rightarrow v \leftarrow w$, $u \leftarrow v \rightarrow w$, $u \leftarrow v \leftarrow w$, and $u \rightarrow v \rightarrow w$. The last two can be disqualified by Lemma 2.6.

Lemma 2.13 (4-vertex configurations). Let Γ be a homomorphism homogeneous oriented graph that contains C_3 , and that is not a tournament. Then Γ contains the following configuration:



Proof. From Lemma 2.12 we have that Γ contains L_1 or L_2 . Suppose that it contains L_1 and consider the following two induced subgraphs of Γ :



as well as the local homomorphism $f: \{u, v\} \to V(\Gamma)$ of Γ given by $f := \begin{pmatrix} u & v \\ z & y \end{pmatrix}$. Since Γ is homomorphism homogeneous, f can be extended to a local homomorphism $\hat{f}: \{u, v, w\} \to V(\Gamma)$, with $z = \hat{f}(u) \to \hat{f}(w) \to \hat{f}(v) = y$. Note that $\hat{f}(w) \notin \{x, y, z\}$. This implies that depending on the relation between vertices x and $\hat{f}(w)$ the subgraph induced by $\{x, y, z, \hat{f}(w)\}$ is either of the following:

$$z \xrightarrow{f(w)} y \xrightarrow{f(w)} z \xrightarrow{f(w)} y \xrightarrow{f(w)} y \xrightarrow{f(w)} y \xrightarrow{f(w)} y \xrightarrow{f(w)} y$$

Note that the first two cases may not occur since both graphs contain an induced path of length 2 and are thus ruled out by Lemma 2.6. The third graph is isomorphic to A.

The case that Γ contains L_2 is handled analogously.

As an immediate consequence of Lemma 2.13 we obtain:

Corollary 2.14. Let Γ be a homomorphism homogeneous oriented graph that contains C_3 , and that is not a tournament. Then Γ contains configurations L_1 and L_2 .

Proof. Observe that both, L_1 and L_2 are induced subgraphs of A.

Proposition 2.15. Let Γ be a homomorphism homogeneous oriented graph that contains C_3 . Then the non-edge relation \nleftrightarrow is an equivalence relation.

In order to show that this claim holds we need to make one more auxiliary observation:

Lemma 2.16. *S*(2) *contains the following configuration:*



Proof. Recall the definition of S(2) on page 4 and observe the following picture:



Proof of Proposition 2.15. Note that the claim trivially holds if Γ is a tournament, so we continue under the assumption that Γ is not a tournament.

It is an easy observation that $\not\sim$ is an equivalence relation if and only if Γ does not contain configuration



Next we show that Γ contains configuration **B** if and only if it contains one of the following configurations:



So suppose that Γ contains configuration **B**. Let us fix in Γ the following two subgraphs:



The existence of the former in Γ is due to Corollary 2.14. Consider the local homomorphism $f: \{u, v\} \to V(\Gamma)$ of Γ given by $f := \begin{pmatrix} u & v \\ a & c \end{pmatrix}$. Since Γ is homomorphism homogeneous, f can be extended to a local homomorphism $\hat{f}: \{u, v, w\} \to V(\Gamma)$. Observe that from $u \to w \leftarrow v$ it follows $a = \hat{f}(u) \to \hat{f}(w) \leftarrow \hat{f}(v) = c$, so we conclude that $\hat{f}(w) \notin \{a, b, c\}$, implying that Γ has one of the following induced subgraphs:

The last one cannot appear by Lemma 2.6, since $c \to \hat{f}(w) \to b$, but $c \neq b$. For the proof of the other direction suppose that Γ contains one of the following induced subgraphs



Then $\{a, b, d\}$ induces configuration **B**.

It is now clear that the task of showing that $\not\sim$ is an equivalence relation reduces to the check of the (non-)containment of configurations C_1 and C_2 in Γ .

Suppose that there are $a, b, c, d \in V(\Gamma)$ such that

$$\begin{array}{c}c & \longrightarrow & 0 \\ a & 0 & 0 \\ b \end{array}$$

is an induced subgraph in Γ . Lemma 2.13 gives us the existence of $x, y, u, v \in V(\Gamma)$ such that



is an induced subgraph in Γ . Consider the local homomorphism $f: \{x, y, v\} \to V(\Gamma)$ of Γ given by $f := \begin{pmatrix} x & y & v \\ a & d & c \end{pmatrix}$. Since Γ is homomorphism homogeneous, f can be extended to a local homomorphism $\hat{f}: \{x, y, v, u\} \to V(\Gamma)$. Then $b \to d = \hat{f}(y) \to \hat{f}(u)$. From Lemma 2.6 it follows that $b \sim \hat{f}(u)$. But, if $b \to \hat{f}(u)$, then since $\hat{f}(u) \to \hat{f}(v) = c$ we get $b \sim c$. Similarly, if $\hat{f}(u) \to b$, then, since $a = \hat{f}(x) \to \hat{f}(u)$, we get $a \sim b$. In both cases we arrive at a contradiction.

Finally, suppose that there are $a, b, c, d \in V(\Gamma)$ such that

(1)
$$c \rightarrow d \\ a \rightarrow b \\ b \end{pmatrix}$$

is an induced subgraph in Γ . We proceed by studying the following two cases: **Case 1.** The core of Γ is C_3 . Then Γ has no induced subgraph of the shape



but the vertex set $\{a, c, d\}$ induces one — a contradiction. **Case 2.** The core of Γ is S(2) or T^{∞} . Then Γ contains the following induced subgraph



since both S(2) and T^{∞} contain **K** (this is clear for T^{∞} and follows from Lemma 2.16 for S(2)). Consider the local homomorphism $f: \{v, y, x\} \to V(\Gamma)$ of Γ given by $f:=\begin{pmatrix}v&y&x\\a&d&c\end{pmatrix}$ (see (1)). Since Γ is homomorphism homogeneous, f can be extended to a local homomorphism $\hat{f}: \{v, y, x, u\} \to V(\Gamma)$. Then $b \to d = \hat{f}(y) \to \hat{f}(u)$. Again, from Lemma 2.6 it follows that $b \sim \hat{f}(u)$. But, if $b \to \hat{f}(u)$, then, since $\hat{f}(u) \to \hat{f}(x) = c$, we get $b \sim c$. Similarly, if $\hat{f}(u) \to b$, then since $a = \hat{f}(v) \to \hat{f}(u)$ we get $a \sim b$. In both cases we arrive at a contradiction.

Thus Γ contains neither C_1 nor C_2 . Hence, $\not\sim$ is an equivalence relation. \Box

Proposition 2.17. Let Γ be a homomorphism homogeneous oriented graph that contains C_3 . Let A and B be distinct equivalence classes of \nsim . Then either $A \to B$ or $B \to A$ in Γ .

Proof. Suppose the opposite. Then $A \times B \notin E(\Gamma)$ and $B \times A \notin E(\Gamma)$, so there exist $a_1, a_2 \in A$, $b_1, b_2 \in B$ such that $(a_1, b_1), (b_2, a_2) \notin E(\Gamma)$. It follows that $(b_1, a_1), (a_2, b_2) \in E(\Gamma)$, and $a_1 \neq a_2$ or $b_1 \neq b_2$. Without loss of generality, suppose $a_1 \neq a_2$. If $b_1 = b_2$, then $a_2 \rightarrow b_2 = b_1 \rightarrow a_1$ — a contradiction with Lemma 2.6. Hence $b_1 \neq b_2$.

Since a_1 and b_2 are from different equivalence classes of \nsim , it follows that $a_1 \sim b_2$. If $a_1 \rightarrow b_2$, then $b_1 \rightarrow a_1 \rightarrow b_2$. If $b_2 \rightarrow a_1$, then $a_2 \rightarrow b_2 \rightarrow a_1$. In both cases we again arrive at a contradiction with Lemma 2.6.

Proof of Proposition 2.11. We show the claim for the case when the core of Γ is T^{∞} . The other two cases go analogously.

So, let Γ be a homomorphism homogeneous oriented graph that contains C_3 , and such that its core is T^{∞} . Let $(v_i)_{i < \omega}$ be a transversal of $V(\Gamma)/\not$. Let $f: V(\Gamma) \rightarrow V(\Gamma)$ be the function that assigns to each $x \in V(\Gamma)$ the unique v_i such that $[x]_{\not} = [v_i]_{\not}$. Then f is an endomorphism of Γ by Proposition 2.17, and Im finduces a tournament Δ in Γ . Moreover, since f is idempotent, it is a retraction. **Observation 1.** Age(Δ) = Age(T^{∞}). To see that this holds, recall that T^{∞} is the core of Γ , so T^{∞} and Γ are homomorphism equivalent. On the other hand, $f: \Gamma \rightarrow \Delta$, and $\Delta \leq \Gamma$, so Γ and Δ are homomorphism equivalent, implying that Δ and T^{∞} are homomorphism equivalent, too. Since both Δ and T^{∞} are tournaments, it follows that Δ imbeds into T^{∞} and T^{∞} imbeds into Δ . Hence, Age(Δ) = Age(T^{∞}). **Observation 2.** Δ is homogeneous. For the proof of this fact, let $g: A \rightarrow B$ be a local isomorphism of Δ , and let $v_j \in V(\Delta) \setminus A$. Then g is a local homomorphism of Γ . Since Γ is homomorphism homogeneous, there exists $\hat{g} \in \text{End}(\Gamma)$ that extends g. Let $k < \omega$ be such that $v_k = f(\hat{g}(v_j))$. Then $g': A \cup \{v_j\} \rightarrow B \cup \{v_k\}$ defined by

$$g' \colon x \mapsto \begin{cases} g(x), & \text{if } x \in A, \\ v_k, & \text{if } x = v_j \end{cases}$$

is a one-point-extension of g to v_j . Hence, Δ is weakly homogeneous, and thus homogeneous. From the previous two observations and from Fraïssé's Theorem we conclude that $\Delta \cong T^{\infty}$.

Observe now that $\Gamma = \Delta[f]$. Since $\Delta \cong T^{\infty}$, this finishes the proof.

Remark. In this proof we used the term "weakly homogeneous". An oriented graph Γ is called *weakly homogeneous* if for every local isomorphism f of Γ with domain A and for every finite superset $\hat{A} \supseteq A$ in $V(\Gamma)$ there exists a local isomorphism \hat{f} of Γ with domain \hat{A} that extends f. An easy back-and-forth argument shows that a countable oriented graph is weakly homogeneous if and only if it is homogeneous.

Combining Propositions 2.8 and 2.11 we finally obtain

Theorem 2.18. Let Γ be a countable homomorphism homogeneous weakly connected oriented graph. Then Γ is one of the following oriented graphs:

(1) I_1 , (2) (\mathbb{Q} , <),

- (3) a tree with no minimal elements such that no finite subset of vertices has a maximal lower bound,
- (4) a dual tree with no maximal elements such that no finite subset of vertices has a minimal upper bound,
- (5) a poset such that:
 - every finite subset of vertices is bounded from above and from below
 - no finite subset of vertices has a maximal lower bound or a minimal upper bound
 - *no* X₄-*set has a midpoint,*
- (6) an extension of the countable universal homogeneous strict poset,
- (7) an oriented graph isomorphic to $C_3[f]$, S(2)[f], or $T^{\infty}[f]$, for some f.
- 3. WEAKLY CONNECTED POLYMORPHISM HOMOGENEOUS ORIENTED GRAPHS

We turn now our attention to polymorphism homogeneous oriented graphs, an important subclass of the class of homomorphism homogeneous oriented graphs. As in the previous section, we will restrict our study to weakly connected oriented graphs, and the disconnected case will be treated separately. Our starting point will be the case of countable polymorphism homogeneous tournaments, that were fully classified in [8, Theorem 3.10]. An immediate consequence of this theorem is:

Corollary 3.1. Let *T* be a countable tournament. Then, *T* is polymorphism homogeneous if and only if it is isomorphic to either I_1 , C_3 , or $(\mathbb{Q}, <)$.

Moving on to non-tournaments we distinguish once more between the cyclic and acyclic among them motivated by the fact that a countable homomorphism homogeneous oriented graph is transitive if and only if it is acyclic. This allows us again to see countable acyclic non-tournaments as countable strict partially ordered sets. The latter have already been classified with respect to polymorphism homogeneity in [16].

Proposition 3.2 ([16, Theorem 6.29]). *The only countable polymorphism homogeneous acyclic non-tournaments are extensions of the countable universal homogeneous strict poset.*

At last, it remains to tackle the class of cyclic non-tournaments (recall that by Lemma 2.9, every cyclic polymorphism homogeneous oriented graph contains C_3). In the following we show that this class contains no polymorphism homogeneous oriented graph:

Lemma 3.3. If an oriented graph Γ contains configuration **K** (cf. Lemma 2.16), then Γ is not polymorphism homogeneous.

Proof. Let $a, b, c, d \in V(\Gamma)$ be such that they induce in Γ the following subgraph:



Observe that then $(b, d) \rightarrow (c, a) \rightarrow (d, b)$, and that $(b, d) \not\sim (d, b)$. By Lemma 2.6 we obtain that Γ^2 is not homomorphism homogeneous. Consequently, Γ is not polymorphism homogeneous.

Proposition 3.4. If Γ is a weakly connected polymorphism homogeneous oriented graph that contains an oriented cycle, then $\Gamma \cong C_3$.

Proof. Proposition 2.11 provides us with our only potential candidates for Γ .

First, let us show that if $\Gamma = C_3[f]$, for some f, then $\Gamma \cong C_3$.

Assuming the opposite, there would exist $x_1, x_2, y, z \in V(\Gamma)$ such that $x_1 \neq x_2$, but $x_1 \rightarrow y \rightarrow z \rightarrow x_1$ and $x_2 \rightarrow y \rightarrow z \rightarrow x_2$. Consider a local map of Γ^2 defined on $\{x_1, x_2\}^2$:



Notice how its domain is an independent set of vertices in Γ^2 . This makes g a local homomorphism of Γ^2 . As Γ is polymorphism homogeneous, then Γ^2 is homomorphism homogeneous. Therefore, there exists $\bar{g} \in \text{End}(\Gamma^2)$ which extends g. Let $(c_1, c_2) := \bar{g}(y, y)$. Since $\{(x_1, x_1), (x_2, x_2), (x_1, x_2), (x_2, x_1)\} \rightarrow (y, y)$, then $\bar{g}(\{(x_1, x_1), (x_2, x_2), (x_1, x_2), (x_2, x_1)\}) \rightarrow \bar{g}(y, y) = (c_1, c_2)$. However, this leads to the conclusion $\{x_1, x_2, z, y\} \rightarrow c_1$, but then, having in mind Proposition 2.17, c_1 could not belong to any of the tree equivalence classes of $\not{\sim}$ — a contradiction.

It remains to consider the cases $\Gamma \cong S(2)[f]$ and $\Gamma \cong T^{\infty}[f]$, for some f. Recall that since both, S(2) and T^{∞} , contain configuration **K** (cf. Lemma 2.16), so does Γ . Thus by Lemma 3.3 Γ is not polymorphism homogeneous.

The results of this section are summed up in the following theorem:

Theorem 3.5. Let Γ be a finite or countably infinite polymorphism homogeneous weakly connected oriented graph. Then Γ is one of the following oriented graphs:

- (1) I_1 ,
- (2) *C*₃,
- (3) (Q, <),
- (4) an extension of the countable universal homogeneous strict poset.
 - 4. DISCONNECTED HOMOMORPHISM HOMOGENEOUS ORIENTED GRAPHS

We continue our study of homomorphism homogeneous oriented graphs by considering the disconnected case. Our first observation is:

Proposition 4.1. If Γ is a homomorphism homogeneous disconnected oriented graph, then all of its weakly connected components are tournaments.

Proof. Assume the opposite, that there exists a non-edge within some weakly connected component. Thus there exist $x, y \in V(\Gamma)$ such that $x \sim^* y$, but $x \not\sim y$.

Because Γ is disconnected, there exists a $z \in V(\Gamma)$ that is not in the same weakly connected component like *x* and *y*.

Consider the map $f = \begin{pmatrix} x & y \\ z & y \end{pmatrix}$. Clearly it is a local homomorphism of Γ . Due to Γ 's homomorphism homogeneity, there exists $\hat{f} \in \text{End}(\Gamma)$ which extends f. Therefore, $z = f(x) = \hat{f}(x) \sim^* \hat{f}(y) = f(y) = y$, which is a contradiction.

Next, let us examine weakly connected components of homomorphism homogeneous oriented graphs in more detail.

Lemma 4.2. Weakly connected components of a homomorphism homogeneous disconnected oriented graph Γ are also homomorphism homogeneous.

Proof. Let *C* be a weakly connected component of Γ , and let *f* be a non-trivial local homomorphism of *C*. Then *f* is also a local homomorphism of Γ . Thus, there exists $\hat{f} \in \text{End}(\Gamma)$ which extends *f*. Clearly, $\hat{f}[C] \subseteq C$. Thus $\hat{f} \upharpoonright_C$ is an endomorphism of *C* that extends *f*.

Proposition 4.3. If Γ is a homomorphism homogeneous disconnected oriented graph, then all of its weakly connected components have the same age.

Proof. Let C_1 and C_2 be two different weakly connected components of Γ . Take any $x \in C_1$ and $y \in C_2$. Consider the following local homomorphism $f = \begin{pmatrix} x \\ y \end{pmatrix}$. By the homomorphism homogeneity of Γ , there exists $\hat{f} \in \text{End}(\Gamma)$ which extends f. Clearly, $\hat{f}[C_1] \subseteq C_2$. Since C_1 is a tournament, $\hat{f} \upharpoonright_{C_1}$ is an embedding. In other words, C_1 imbeds into C_2 . Analogously it can be shown that C_2 imbeds into C_1 . Consequently, $\text{Age}(C_1) = \text{Age}(C_2)$.

Remark. In the following, if an oriented graph Γ has exactly k weakly connected components each of which induces a subgraph isomorphic to a given oriented graph Δ , then we denote Γ also by $k \cdot \Delta$.

Theorem 4.4. Let Γ be a countable disconnected oriented graph. Then Γ is homomorphism homogeneous if and only if all of its weakly connected components are isomorphic to the same homogeneous tournament. In particular, it is isomorphic to one of the following oriented graphs:

(1) $k \cdot I_1$, (2) $k \cdot C_3$, (3) $k \cdot (\mathbb{Q}, <)$, (4) $k \cdot S(2)$, (5) $k \cdot T^{\infty}$,

for some $1 < k \leq \omega$.

Proof. Assume, at first, that Γ is homomorphism homogeneous. Proposition 4.1 together with Lemma 4.2 imply that all weakly connected components of Γ are homomorphism homogeneous tournaments. Recall that a countable tournament is homomorphism homogeneous if and only if it is homogeneous. Now, combining Proposition 4.3 with Fraïssé's Theorem, we come to the conclusion that all weakly connected components of Γ are isomorphic to one and the same homogeneous tournament.

Consider now the opposite direction, assuming that all of Γ 's weakly connected components are isomorphic to the same homogeneous tournament. Take any local homomorphism f of Γ . Let C_1, \ldots, C_k be all connected components of Γ such that dom $(f) \cap C_i \neq \emptyset$ for each $i \in \{1, 2, \ldots, k\}$. Each time denote dom $(f) \cap C_i$ by A_i . In particular,

$$\operatorname{dom}(f) = A_1 \,\dot{\cup}\, A_2 \,\dot{\cup}\, \ldots \,\dot{\cup}\, A_k.$$

Define $B_i := f[A_i]$, for all $i \in \{1, 2, ..., k\}$. Further define $f_i := f \upharpoonright_{A_i} : A_i \to B_i$, for all $i \in \{1, 2, ..., k\}$. Notice how each B_i is fully contained within some weakly connected component D_i of Γ . Now for each $i \ C_i$ and D_i are isomorphic homogeneous tournaments, so we may use Fraïssé's Theorem in order to conclude that there exists an isomorphism $\hat{f_i} : C_i \to D_i$ which extends f_i .

Finally, we define the following extension of f on Γ :

$$\hat{f}(x) := \begin{cases} \hat{f}_i(x), & \text{if } x \in C_i, i \in \{1, 2, \dots, k\} \\ x, & \text{otherwise.} \end{cases}$$

Clearly, \hat{f} is an endomorphism. Thus Γ is homomorphism homogeneous.

5. DISCONNECTED POLYMORPHISM HOMOGENEOUS ORIENTED GRAPHS

Now that we know all homomorphism homogeneous disconnected oriented graphs, let us see which of them are polymorphism homogeneous.

Proposition 5.1. All weakly connected components of a disconnected polymorphism homogeneous oriented graph are mutually isomorphic homogeneous polymorphism homogeneous tournaments.

Proof. Let Γ be a disconnected polymorphism homogeneous oriented graph with weakly connected components T_0, T_1, \ldots, T_m . By Theorem 4.4 all these components are mutually isomorphic homogeneous tournaments. For each $i \in \{1, \ldots, m\}$, let $\varphi_i : T_i \to T_0$ be an isomorphism. Fix a positive natural number n and consider a local homomorphism f of T^n . Then f is also a local homomorphism of Γ^n . Since Γ^n is homomorphism homogeneous, there exists $\overline{f} \in \text{End}(\Gamma^n)$ that extends f. Let

$$\varphi \colon \mathcal{V}(\Gamma) \to \mathcal{V}(\Gamma) \qquad x \mapsto \begin{cases} \varphi_i(x) & \text{if } x \in T_i, \, i \in \{1, \dots, m\}, \\ x & \text{if } x \in T_0. \end{cases}$$

Obviously, $\varphi \in \text{End}(\Gamma)$. Consequently,

$$\varphi^n \colon \mathcal{V}(\Gamma^n) \to \mathcal{V}(\Gamma^n) \qquad (x_1, \dots, x_n) \mapsto (\varphi(x_1), \dots, \varphi(x_n))$$

is an endomorphism of Γ^n . Finally, we define $\hat{f} := \varphi^n \circ \bar{f}$. Note that \hat{f} also extends f, but its image is completely contained in T_0^n . Thus $\hat{f} \upharpoonright_{T_0^n}$ is an endomorphism of T_0^n that extends f. Hence, T_0 is polymorphism homogeneous.

Lemma 5.2. For all $1 < k \leq \omega$ we have that $k \cdot (\mathbb{Q}, <)$ is not polymorphism homogeneous.

Proof. Fix a $k \ge 2$, and assume the opposite, that $k \cdot (\mathbb{Q}, <)$ is polymorphism homogeneous. Thus there exist two different weakly connected components C_1 and C_2 , and vertices $a, b, c \in C_1$ and $d \in C_2$ such that $a \to b \to c$. Observe that then $(a, b) \to (c, c) \leftarrow (b, a)$, and consider the local homomorphism of $(k \cdot (\mathbb{Q}, <))^2$:



Since $k \cdot (\mathbb{Q}, <)$ is polymorphism homogeneous, it follows that $(k \cdot (\mathbb{Q}, <))^2$ is homomorphism homogeneous, so f can be extended to $\overline{f} \in \text{End}((k \cdot (\mathbb{Q}, <))^2)$. But then $(a, b) = \overline{f}(a, b) \rightarrow \overline{f}(c, c) \leftarrow \overline{f}(b, a) = (a, d)$, but this cannot be satisfied, since b and d belong to distinct weakly connected components, and so we arrive at a contradiction.

This implies that $(k \cdot (\mathbb{Q}, <))^2$ is not homomorphism homogeneous nor is $k \cdot (\mathbb{Q}, <)$ polymorphism homogeneous.

Theorem 5.3. The only countable polymorphism homogeneous disconnected oriented graphs, up to isomorphism, are:

(1) $k \cdot I_1$ and (2) $k \cdot C_3$,

for any $1 < k \leq \omega$.

Proof. By Proposition 5.1 and [8, Theorem 3.10] the only candidates for weakly connected components are I_1 , C_3 , and $(\mathbb{Q}, <)$. Lemma 5.2 rules out $(\mathbb{Q}, <)$. As for I_1 and C_3 we note that for any n > 0

$$(k \cdot C_3)^n \cong (k^n \cdot 3^{n-1}) \cdot C_3$$

$$(k \cdot I_1)^n \cong k^n \cdot I_1$$
 are homomorphism homogeneous.

Thus $k \cdot C_3$ and $k \cdot I_1$ are indeed polymorphism homogeneous, for any $1 < k \le \omega$. \Box

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