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CHAPTER 1

# Introduction

The moduli spaces of smooth projective curves of genus $g \geq 2$, and their compactifications by the moduli space of stable projective curves of genus $g$ are, quite possibly, the most studied of all algebraic varieties.

The aim of this book is to generalize the moduli theory of curves to surfaces and to higher dimensional varieties. In the introduction we start to outline how this is done, and, more importantly, to explain why the answer for surfaces is much more complicated than for curves. On the positive side, once we get the moduli theory of surfaces right, the higher dimensional theory works the same.

Section 1 is a quick review of the history of moduli problems, culminating in an outline of the basic moduli theory of curves. Section 2 introduces canonical models, which are the basic objects of moduli theory in higher dimensions. Starting from stable curves, Section 3 leads up to the definition of stable varieties; their higher dimensional analogs. Then we show, by a series of examples, why flat families of stable varieties are *not* the correct higher dimensional analogs of flat families of stable curves. Finding the correct replacement has been one of the main difficulties of the whole theory.

Section 4 is a collection of examples showing how easy it is to end up with rather horrible-looking moduli problems. Section 5 illustrates the differences between fine and coarse moduli spaces.

## 1. Short history of moduli problems

Let $\mathbf{V}$ be a "reasonable" class of objects in algebraic geometry, for instance, $\mathbf{V}$ could be all subvarieties of $\mathbb{P}^n$, all coherent sheaves on $\mathbb{P}^n$, all smooth curves or all projective varieties. The aim of the theory of moduli is understand all "reasonable" families of objects in $\mathbf{V}$ and to construct an algebraic variety (or scheme, or algebraic space) whose points are in "natural" one-to-one correspondence with the objects in $\mathbf{V}$. If such a variety exists, we call it the *moduli space* of $\mathbf{V}$ and denote it by $M_{\mathbf{V}}$. The simplest, classical examples are given by the theory of linear systems and families of linear systems.

1 (Linear systems). Let $X$ be a normal projective variety over an algebraically closed field $k$ and $L$ a line bundle on $X$. The corresponding linear system is

$$LinSys(X, L) = \{\text{effective divisors } D \text{ such that } \mathcal{O}_X(D) \cong L\}.$$

The objects in $LinSys(X, L)$ are in natural one-to-one correspondence with the points of the projective space $\mathbb{P}\big(H^0(X, L)^\vee\big)$ which is classically denoted by $|L|$. Thus, for every effective divisors $D$ such that $\mathcal{O}_X(D) \cong L$ there is a unique point $[D] \in |L|$.

Moreover, this correspondence between divisors and points is given by a universal family of divisors over $|L|$. That is, there is an effective Cartier divisor

$\mathrm{Univ}_L \subset |L| \times X$ with projection $\pi : \mathrm{Univ}_L \to |L|$ such that

$$\pi^{-1}\big([D]\big) = D$$

for every effective divisor $D$ linearly equivalent to $L$,

The classical literature never differentiates between the linear system as a set and the linear system as a projective space. There are, indeed, few reasons to distinguish them as long as we work over a fixed base field $k$. If, however, we pass to a field extension $K \supset k$, the advantages of viewing $|L|$ as a $k$-variety appear. For any $K \supset k$, the set of effective divisors $D$ defined over $K$ such that $\mathcal{O}_X(D) \cong L$ corresponds to the $K$-points of $|L|$. Thus the scheme theoretic version automatically gives the right answer over every field.

2 (Jacobians of curves). Let $C$ be a smooth projective curve (or Riemann surface) of genus $g$. As discovered by Abel and Jacobi, there is a variety $\mathrm{Jac}^0(C)$ of dimension $g$ whose points are in natural one-to-one correspondence with degree 0 line bundles on $C$. As before, the correspondence is given by a universal line bundle $L_{univ} \to C \times \mathrm{Jac}^0(C)$. That is, for any point $p \in \mathrm{Jac}^0(C)$, the restriction of $L_{univ}$ to $C \times \{p\}$ is the degree 0 line bundle corresponding to $p$.

A somewhat subtle point is that, unlike in (1), the uiversal line bundle $L_{univ}$ is not unique (and need not exist if the base field is not algebraically closed). This has to do with the fact that while a divisor $D \subset X$ has no automorphisms fixing $X$, any line bundle $L \to C$ has automorphisms that fix $C$: we can multiply every fiber of $L$ by the same nonzero constant.

3 (Chow varieties). Historically the next to emerge was the theory of Chow varieties, though it is a rather difficult moduli problem. It was defined by [**Cay62**] for curves in $\mathbb{P}^3$. See [**HP47**] for a classical introduction and [**Kol96**, Secs.I.3–4] for a more recent treatment.

Let $k$ be an algebraically closed field and $X$ a normal, projective $k$-variety. Fix a natural number $m$. An *$m$-cycle* on $X$ is a finite, formal linear combination $\sum a_i Z_i$ where the $Z_i$ are irreducible, reduced subvarieties of dimension $m$ and $a_i \in \mathbb{Z}$. We usually assume tacitly that all the $Z_i$ are distinct and $a_i \neq 0$. An $m$-cycle is called *effective* if $a_i > 0$ for every $i$.

Let $Y \subset X$ be a closed subscheme of dimension $m$. Let $Y_i \subset Y$ be its $m$-dimensional irreducible components, $Z_i := \mathrm{red}\, Y_i$ and $y_i \in Y_i$ the generic point. Let $a_i$ be the length of $\mathcal{O}_{y_i, Y_i}$ over $\mathcal{O}_{y_i, Z_i}$. We define the *fundamental cycle* of $Y$ as $[Y] := \sum a_i Z_i$. Thus the fundamental cycle ignores lower dimensional associated primes and from the $m$-dimensional components it keeps only the underlying reduced variety and the length at the generic points.

It turns out that there is a $k$-variety $\mathrm{Chow}_m(X)$, called the *Chow variety* of $X$ whose points are in "natural" one-to-one correspondence with the set of effective $m$-cycles on $X$. (Since we did not fix the degree of the cycles, $\mathrm{Chow}_m(X)$ is not actually a variety but a countable disjoint union of connected, projective, reduced $k$-schemes.) The point of $\mathrm{Chow}_m(X)$ corresponding to a cycle $Z = \sum a_i Z_i$ is also usually denoted by $[Z]$.

As for linear systems, it is best to describe the "natural correspondence" by a universal family. The situation is, however, more complicated than before.

There is a family (or rather an effective cycle) $\mathrm{Univ}_m(X)$ on $\mathrm{Chow}_m(X) \times X$ with projection $\pi : \mathrm{Univ}_m(X) \to \mathrm{Chow}_m(X)$ such that for every effective $m$-cycle $Z = \sum a_i Z_i$,

(1) the support of $\pi^{-1}\bigl([Z]\bigr)$ is $\sum Z_i$, and
(2) the fundamental cycle of $u^{-1}\bigl([Z]\bigr)$ equals $Z$ if $a_i = 1$ for every $i$.

If the characteristic of $k$ is 0, then the only problem in (2) is a clash between the traditional cycle-theoretic definition of the Chow variety and the scheme-theoretic definition of the fiber. It is easy to define a cycle-theoretic notion of fiber that restores equality in (2) for every $Z$; see [**Kol96**, I.3]. In positive characteristic the situation is more problematic; a possible solution is described in [**Kol96**, I.4].

4 (Hilbert schemes). The example of a "perfect" moduli problem is the theory of Hilbert schemes, introduced in [**Gro62**]. See [**Mum66**], [**Kol96**, I.1–2] or [**Ser06**, Sec.4.3] for detailed treatments.

Let $k$ be an algebraically closed field and $X$ a projective $k$-variety. Set

$$Hilb(X) = \{\text{closed subschemes of } X\}.$$

Then there is a $k$-scheme $\operatorname{Hilb}(X)$, called the *Hilbert scheme* of $X$ whose points are in a "natural" one-to-one correspondence with closed subschemes of $X$. Again, the point of $\operatorname{Hilb}(X)$ corresponding to a subscheme $Y \subset X$ is frequently denoted by $[Y]$. Moreover, there is a universal family $\operatorname{Univ}(X) \subset \operatorname{Hilb}(X) \times X$ such that

(1) the first projection $\pi : \operatorname{Univ}(X) \to \operatorname{Hilb}(X)$ is flat, and
(2) $\pi^{-1}\bigl([Y]\bigr) = Y$ for every closed subscheme $Y \subset X$.

The beauty of the Hilbert scheme is that it describes not just subschemes but all flat families of subschemes as well.

If $T$ is any scheme and $g : T \to \operatorname{Hilb}(X)$ is any morphism, then, by pulling back, we obtain a flat family of subschemes of $X$ parametrized by $T$

$$T \times_{g,\operatorname{Hilb}(X)} \operatorname{Univ}(X) \subset T \times X.$$

It turns out that every family is obtained this way:

(3) For every $T$ and for every closed subscheme $Z_T \subset T \times X$ that is flat and proper over $T$, there is a unique $g : T \to \operatorname{Hilb}(X)$ such that

$$Z_T = T \times_{g,\operatorname{Hilb}(X)} \operatorname{Univ}(X).$$

This takes us to the next, functorial approach to moduli problems.

5 (Hilbert functor and Hilbert scheme). Let $X \to S$ be a morphism of schemes. Define the *Hilbert functor* of $X/S$ as a functor that associates to a scheme $T \to S$ the set

$$Hilb_{X/S}(T) = \bigl\{\text{subschemes } Z \subset T \times_S X \text{ which are flat and proper over } T\bigr\}.$$

The basic existence theorem of Hilbert schemes then says that, if $X \to S$ is quasi projective, there is a scheme $\operatorname{Hilb}_{X/S}$ such that for any $S$ scheme $T$,

$$Hilb_{X/S}(T) = \operatorname{Mor}_S\bigl(T, \operatorname{Hilb}_{X/S}\bigr).$$

Moreover, there is a universal family $\pi : \operatorname{Univ}_{X/S} \to \operatorname{Hilb}_{X/S}$ such that the above isomorphism is given by pulling back the universal family.

We can summarize these results as follows

PRINCIPLE 6. $\pi : \operatorname{Univ}_{X/S} \to \operatorname{Hilb}_{X/S}$ *contains all the information about proper, flat families of subschemes of* $X/S$ *and does it in the most succinct way.*

This example leads us to a general definition:

Definition 7 (Fine moduli spaces). Let $\mathbf{V}$ be a "reasonable" class of projective varieties (or schemes, or sheaves, or ...). In practice "reasonable" may mean several restrictions, but for the definition we only need the following weak assumption:

(1) Let $K \supset k$ be a field extension. Then a $k$-variety $X_k$ is in $\mathbf{V}$ iff $X_K := X_k \times_{\operatorname{Spec} k} \operatorname{Spec} K$ is in $\mathbf{V}$.

Following (5), define the corresponding moduli functor as

$$Varieties_{\mathbf{V}}(T) := \left\{ \begin{array}{c} \text{Flat families } X \to T \text{ such that} \\ \text{every fiber is in } \mathbf{V}, \\ \text{modulo isomorphisms over } T. \end{array} \right\} \tag{7.2}$$

We say that a scheme $\mathrm{Moduli}_{\mathbf{V}}$, or, more precisely, a flat morphism

$$u : \mathrm{Univ}_{\mathbf{V}} \to \mathrm{Moduli}_{\mathbf{V}}$$

is a *fine moduli space* for the functor $Varieties_{\mathbf{V}}$ if the following holds:

(3) For every scheme $T$, pulling back gives an equality

$$Varieties_{\mathbf{V}}(T) = \operatorname{Mor}\bigl(T, \mathrm{Moduli}_{\mathbf{V}}\bigr).$$

Applying the definition to $T = \operatorname{Spec} K$, where $K$ is a field, we see that every fiber of $u : \mathrm{Univ}_{\mathbf{V}} \to \mathrm{Moduli}_{\mathbf{V}}$ is in $\mathbf{V}$ and the $K$-points of the fine moduli space $\mathrm{Moduli}_{\mathbf{V}}$ are in one-to-one correspondence with the $K$-isomorphism classes of objects in $\mathbf{V}$.

We consider the existence of a fine moduli space as the ideal possibility. Unfortunately, it is rarely achieved; see Section 5.

8 (Remarks on flatness). The definition (7) is very natural within Grothendieck's framework of algebraic geometry, but in fact it hides a very strong supposition:

**Assumption 8.1.** If $\mathbf{V}$ is a "reasonable" class then any flat family whose fibers are in $\mathbf{V}$ is a "reasonable" family.

In Grothendieck's foundations of algebraic geometry flatness is one of the cornerstones and there are many "reasonable" classes for which flat families are indeed the "reasonable" families. Nonetheless, (8.1) should not be viewed as self evident.

Even when the base of the family is a smooth curve, (8.1) needs arguing, but the assumption is especially surprising when applied to families over non-reduced schemes $T$. Consider, for instance, the case when $T$ is the spectrum of an Artinian $k$-algebra. Then $T$ has only one closed point $t \in T$. A flat family $p : X \to T$ has only one fiber $X_t$, and our only restriction is that $X_t$ be in our class $\mathbf{V}$. Thus (8.1) declares that we care only about $X_t$. Once $X_t$ is in $\mathbf{V}$, every flat deformation of $X_t$ over $T$ is automatically "reasonable."

In fact, a crucial conceptual point in the moduli theory of higher dimensional varieties is the realization that in (7.2), the flatness of the map $X \to T$ is not enough: allowing all flat families whose fibers are in a "reasonable" class leads to the wrong moduli problem. This difficulty arises even for families of surfaces over smooth curves.

Working out the correct concept has been one of the main stumbling blocks of the general theory.

Next we see what happens with the simplest case, for smooth curves of fixed genus.

9 (Moduli functor and moduli space of smooth curves). Following (7) we define the moduli functor of smooth curves of genus $g$ as

$$Curves_g(T) := \left\{ \begin{array}{c} \text{Smooth, proper families } S \to T, \\ \text{every fiber is a curve of genus } g, \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

It turns out that there is no fine moduli space for curves of genus $g$. In fact, every curve $C$ with nontrivial automorphisms causes problems; there can not be any point $[C]$ corresponding to it in a fine moduli space. Actually, problems arise already when $\mathbf{V}$ consist of a single curve! See Section 5 for such examples.

It has been, however, understood for a long time that there is some kind of an object, denoted by $M_g$, and called the *coarse moduli space* (or simply *moduli space*) of curves of genus $g$ that comes close to being a fine moduli space:

(1) For any algebraically closed field $k$, the $k$-points of $M_g$ are in a "natural" one-to-one correspondence with isomorphism classes of smooth curves of genus $g$ defined over $k$. Let us denote the correspondence by $C \mapsto [C] \in M_g$.
(2) For any family of smooth genus $g$ curves $h : S \to T$ there is a "moduli map" $m_{h,T} : T \to M_g$ such that for every geometric point $p \in T$, the image $m_{h,T}(p)$ is the point corresponding to the fiber $[h^{-1}(p)]$.

For elliptic curves we get $M_1 = \mathbb{A}^1$ and the moduli map is given by the $j$-invariant, as was known to Euler and Lagrange. They also knew that there is no universal family over $M_1$. The theory of Abelian integrals due to Abel, Jacobi and Riemann does essentially the same for all curves, though in this case a clear moduli theoretic interpretation seems to have been done only later. For smooth plane curves, and more generally for smooth hypersurfaces in any dimension, the invariant theory of Hilbert produces coarse moduli spaces. Still, a precise definition and proof of existence of $M_g$ appeared only in [**Tei44**] in the analytic case and in [**Mum65**] in the algebraic case.

10 (Coarse moduli spaces). As in (7), let $\mathbf{V}$ be a "reasonable" class. When there is no fine moduli space, we still can ask for a scheme that best approximates its properties.

We look for schemes $M$ for which there is a natural transformation of functors

$$T_M : Varieties_g(*) \longrightarrow \operatorname{Mor}(*, M).$$

Such schemes certainly exist, for instance, if we work over a field $k$ then $M = \operatorname{Spec} k$. All schemes $M$ for which $T_M$ exists form an inverse system which is closed under fiber products. Thus, as long as we are not unlucky, there is a universal (or largest) scheme with this property. Though it is not usually done, it should be called the *categorical moduli space*.

This object can be rather useless in general. For instance, fix $n, d$ and let $\mathbf{H}_{n,d}$ be the class of all hypersurfaces of degree $d$ in $\mathbb{P}^{n+1}_k$ up to isomorphisms. One can see (cf. (56)) that a categorical moduli space exists and it is $\operatorname{Spec} k$.

To get something more like a fine moduli space, we require that it give a one-to-one parametrization at least set theoretically. Thus we say that a scheme $\mathrm{Moduli}_{\mathbf{V}}$ is a *coarse moduli space* for $\mathbf{V}$ if the following hold.

(1) There is a natural transformation of functors

$$\mathrm{ModMap} : Varieties_{\mathbf{V}}(*) \longrightarrow \operatorname{Mor}(*, \mathrm{Moduli}_{\mathbf{V}}),$$

(2) $\mathrm{Moduli}_{\mathbf{V}}$ is universal satisfying (1), and
(3) for any algebraically closed field $K \supset k$,

$$\mathrm{ModMap} : \mathit{Varieties}_{\mathbf{V}}(\operatorname{Spec} K) \xrightarrow{\cong} \mathrm{Mor}(\operatorname{Spec} K, \mathrm{Moduli}_{\mathbf{V}}) = \mathrm{Moduli}_{\mathbf{V}}(K)$$

is an isomorphism (of sets).

11 (Moduli functors versus moduli spaces). While much of the early work on moduli, especially since [**Mum65**], put the emphasis on the construction of fine or coarse moduli spaces, recently the emphasis shifted towards the study of the families of varieties, that is towards moduli functors and moduli stacks. The main task is to understand what kind of objects form "nice" families. Once a good concept of "nice familes" is established, the existence of a coarse moduli space should be nearly automatic. The coarse moduli space is not the fundamental object any longer, rather it is only a convenient way to keep track of certain information that is only latent in the moduli functor or moduli stack.

12 (Compactifying $M_g$). While the basic theory of algebraic geometry is local, that is, it concerns affine varieties, most really interesting and important objects in algebraic geometry and its applications are global, that is, projective or at least proper.

The moduli functor of smooth curves discussed so far has a definitely local flavor. Most naturally occurring smooth families of curves $S \to T$ live over an affine scheme $T$. It is not easy to write down *any* family of smooth curves over a projective base. The best solution is to allow not just smooth curves but also singular curves in our families.

The moduli spaces $M_g$ are not compact and for many reasons it is useful to find geometrically meaningful compactifications of $M_g$. Concentrating on 1-parameter families, the main question is the following:

(12.1) *Let $B$ be a smooth curve, $B^0 \subset B$ an open subset and $\pi^0 : S^0 \to B^0$ a smooth family of genus $g$ curves. Find a "natural" extension*

$$\begin{array}{ccc} S^0 & \subset & S \\ \pi^0 \downarrow & & \downarrow \pi \\ B^0 & \subset & B, \end{array}$$

*where $\pi : S \to B$ is a flat family of (possibly singular) curves.*

We would like the extension to be unique and behave well with respect to pulling back families over curves and for families over higher dimensional bases.

The answer, proposed in [**DM69**] has been so successful that it is hard to imagine a time when it was not the "obvious" solution. Let us first review the definition of [**DM69**]. In Section 4 we see, by examples, why this concept has not been so obvious.

Definition 13 (Stable curve). A *stable curve* over an algebraically closed field $k$ is a proper, connected $k$-curve $C$ such that the following hold:

(Local property) The only singularities of $C$ are ordinary nodes.

(Global property) The canonical (or dualizing) sheaf $\omega_C$ is ample (33).

A *stable curve* over a scheme $T$ is a flat, proper morphism $\pi : S \to T$ such that every geometric fiber of $\pi$ is a stable curve. (The arithmetic genus of the fibers is a locally constant function on $T$, but we usually also tacitly assume that it is constant.)

The moduli functor of stable curves of genus $g$ is

$$\overline{Curves}_g(T) := \left\{ \begin{array}{c} \text{Stable curves of genus } g \text{ over } T, \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

THEOREM 14. [**DM69**] *For every $g \geq 2$, the moduli functor of stable curves of genus $g$ has a coarse moduli space $\bar{M}_g$. Moreover, $\bar{M}_g$ is projective, normal, has only quotient singularities and contains $M_g$ as an open dense subset.*

$\bar{M}_g$ has a rich and intriguing intrinsic geometry which is related to major questions in many branches of mathematics and theoretical physics.

15 (Moduli for varieties of general type).

The aim of this book is to use, as guideline, the moduli of stable curves, and develop a moduli theory for varieties of general type. (For the non-general type case, see (23).)

In some sense, this is a hopeless task since higher dimensional varieties are much more complicated than curves. For instance, even for smooth surfaces with ample canonical class, the moduli spaces can have arbitrarily complicated singularities and scheme structures [**Vak06**]. Thus we approach the question in four stages:

(1) Develop the correct higher dimensional analog of smooth, projective curves of genus $\geq 2$.
(2) Following the example of stable curves, define the notion of "stable" varieties in higher dimensions.
(3) Show that the functor of "stable" varieties with suitably fixed numerical invariants gives a well behaved moduli functor/stack and has a projective coarse moduli space.
(4) Show that, in many important cases, these moduli spaces are interesting and useful objects.

Let us now see in some detail how these goals are accomplished.

16 (Higher dimensional analogs of smooth curves of genus $\geq 2$). It has been understood since the beginnings of the theory of surfaces that, for surfaces of Kodaira dimension $\geq 0$, the correct moduli theory should be birational, not biregular. That is, the points of the moduli space should correspond not to *isomorphism* classes of surfaces but to *birational* equivalence classes of surfaces. There are two ways to deal with this problem.

First, one can work with smooth families but consider two families equivalent of there is a rational map between them that induces a birational equivalence on every fiber. This seems rather complicated technically.

The second, much more useful method relies on the observation that every birational equivalence class of surfaces of Kodaira dimension $\geq 0$ contains a unique *minimal model*, that is, a smooth projective surface $S^m$ whose canonical class is nef. Therefore, one can work with families of minimal models, modulo isomorphisms. With the works of [**Mum65, Art74, Gie77**] it became clear that, for surfaces of general type, it is even better to work with the *canonical model*, which is a mildly singular projective surface $S^c$ whose canonical class is ample. The resulting class of singularities has been since established in all dimensions; they are called *canonical singularities* (35). See Section **??** for details.

**Principle 16.1.** In moduli theory, the main objects of study are projective varieties with ample canonical class and with canonical singularities.

The correct definition of the higher dimensional analogs of stable curves was much less clear. An approach through geometric invariant theory was investigated [**Mum77**], but never fully developed. In essence, the GIT approach starts with a particular method of construction of moduli spaces and then tries to see for which class of varieties does it work.

A different framework was proposed in [**KSB88**]; see also [**Ale96**]. Instead of building on geometric invariant theory, it focuses on 1-parameter families and uses Mori's program as its basic tool. Before we give the definition, it is very helpful to go through a key point of the proof of (14), establishing that $\bar{M}_g$ is separated and proper. Keeping in mind the valuative criterion of separatedness and properness (21.1–2.), we expect that the study of 1-dimensional families is the key step. This is done in the next theorem.

THEOREM 17 (Stable reduction for curves). *Let $B$ be a smooth curve, $B^0 \subset B$ an open subset and $\pi^0 : S^0 \to B^0$ a flat family of genus $g$ stable curves. Then there is a finite surjection $p : A \to B$ such that there is a unique extension*

$$\begin{array}{ccccc} S^0 \times_B A & =: & T^0 & \subset & T \\ \downarrow & & \pi_A^0 \downarrow & & \downarrow \pi_A \\ B^0 \times_B A & =: & A^0 & \subset & A, \end{array}$$

*where $\pi_A : T \to A$ is a flat family of genus $g$ stable curves.*

18 (Outline of proof of (17)). Let us present the process in a way that generalizes to higher dimensions.

*Main case 18.1.* The generic fiber of $\pi^0 : S^0 \to B^0$ is smooth.

Step 1.1. Take any (possibly singular) projective surface $S_1 \supset S^0$ such that $\pi^0$ extends to a morphism $\pi_1 : S_1 \to B$.

Step 1.2. Resolve the singularities of $S_1$ to obtain a smooth surface $\pi_2 : S_2 \to B$ such that the reduced fibers of $\pi_2$ have only nodes as singularities.

Step 1.3. Run the relative minimal model program. That is, repeatedly contract all smooth rational curves $C \subset S_2$ that are contained in a fiber of $\pi_2$ and have negative intersection with the canonical class. The end result is $\pi_3 : S_3 \to B$ where $K_{S_3}$ has nonnegative degree on all curves contained in any fiber of $\pi_3$.

Step 1.4. Take the relative canonical model. That is, contract all smooth rational curves $C \subset S_3$ that are contained in a fiber of $\pi_3$ and have zero intersection with the canonical class. The end result is $\pi_4 : S_4 \to B$ where $K_{S_4}$ has positive degree on all curves contained in any fiber of $\pi_3$. Thus $K_{S_4}$ is relatively ample. Note that $S_4$ is, in general, not smooth, but has very mild (so called Du Val) singularities.

Step 1.5. Prove that $\pi_4 : S_4 \to B$ is the unique surface containing $S^0$ that has Du Val singularities and relatively ample canonical class.

Step 1.6. In general, the fibers of $\pi_4$ are not reduced and the construction of $S_4$ does not commute with base change $p : A \to B$. However, if the fibers of $\pi_2$ are reduced, then the fibers of $\pi_4$ are stable curves and the construction of $S_4$ does commute with base change. (Assuming only that the fibers of $\pi_4$ be reduced would not be enough.)

Step 1.7. Show that if $p : A \to B$ is sufficiently ramified and $T^0 := S^0 \times_B A$ then the analogously constructed $T := T_4 \to A$ satisfies the conclusion of (17). (Just to be concrete, in characteristic 0, the following ramification condition is sufficient:

For every $a \in A$, the ramification index of $p$ at $a$ is divisible by the multiplicity of every irreducible component of $\pi_2^{-1}\big(p(a)\big)$.)

*Singular case 18.2.* The generic fiber of $\pi^0 : S^0 \to B^0$ is not smooth.

Step 2.0. The generic fiber of $\pi^0 : S^0 \to B^0$ has nodes, and, correspondingly, $S^0$ has simple normal crossing singularities along a curve $C^0 \subset S^0$. Let $\bar{S}^0 \to S^0$ be the normalization, $D^0 \subset \bar{S}^0$ the preimage of the double curve and $\tau^0$ the involution of the degree 2 cover $D^0 \to C^0$.

Steps 2.1–7. Run the analog of Steps 1.1–7 for $\bar{S}^0 \to B^0$, with the difference of using

$$\text{(canonical class)} + \text{(birational transform of } D^0\text{)}$$

everywhere instead of the canonical class. The end result is $\pi_T : \bar{T} \to A$ with $D_T \subset \bar{T}$ the curve corresponding to $D^0$.

Step 2.8. Show that the involution $\tau^0$ extends to an involution $\tau_T$ on $D_T$. Construct a new, non-normal surface $\sigma : \bar{T} \to T$ such that $\sigma$ is an isomorphism outside $D_T$ and we identify every point $p \in D_T$ with its image $\tau_T(p)$.

19 (Higher dimensional analogs of stable curves of genus $\geq 2$). Now we can state the main theses of [**KSB88**] about higher dimensional moduli problems:

**Principle 19.1.** In higher dimensions, we should follow the proof of the Stable reduction theorem (17) as outlined in (18). The resulting fibers give the right class of stable varieties.

**Principle 19.2.** As in (13), a connected $k$-scheme $X$ is stable iff it satisfies the following two conditions:

(Local property) A restriction on the singularities of $X$ (so-caled "semi log canonical" singularities).

(Global property) The canonical (or dualizing) sheaf $\omega_X$ is ample.

The definition of semi log canonical is not important for now (44), the key point is that the only global restriction is the ampleness of $\omega_X$.

In general, Step 1.1 of (18) is still easy and Step 1.2 uses Hironaka's resolution of singularities. Steps 1.3–5 use Mori's program, also called the minimal model program. When [**KSB88**] was written, the relevant results were only known for families of surfaces, but [**BCHM06**] takes care of the higher dimensional cases as well, except that Step.1.4 is not yet fully known.

Steps 1.6–7 need very little change. As a starting point one could use the Semi stable reduction theorem [**KKMSD73**], but, as we see in Section **??**, one can get by without it.

The singular case, Steps 2.0–8, have not been worked out earlier. Steps 2.0–7 are conjectured to work as before, however the relevant results of the minimal model program have been fully established only for families of surfaces and 3-folds.

Step 2.8 turned out to be unexpectedly subtle. It is closely related to some basic questions concerning semi-log-canonical schemes. Much of Chapter ?? is devoted to its solution.

An alternative way to approach the singular case would be to develop the minimal model program for varieties with normal crossing singularities and apply it directly, without normalizing in Step 2.0. However, as we see in Section **??**, the minimal model program fails already for surfaces with normal crossing singularities.

20 (Moduli functor of stable varieties). In the moduli theory of curves, we go directly from the definition of stable curves over fields to the notion of stable curves over an arbitrary base (13). By contrast, for surfaces and in higher dimensions, a major difficulty remains. As we already mentioned in (8), not every flat family of stable surfaces can be allowed in a "reasonable" moduli theory. Examples illustrating this are given in Section 3. We must restrict to families $S \to T$ where the Hilbert polynomial of the fibers

$$\chi\big(S_t, \mathcal{O}_{S_t}(mK_{S_t})\big)$$

is independent of $t \in T$. The problem is that, for stable varieties, the canonical class $K$ need not be Cartier, and the sheaves $\mathcal{O}_{S_t}(mK_{S_t})$ do not form a flat family over $T$. It is actually quite difficult to define the right concept. Our final solution of this problem is in Chapter **??**?.

21 (Good properties of moduli problems). Let $\mathbf{V}$ be a "reasonable" class of varieties and $Varieties_{\mathbf{V}}$ the corresponding moduli functor. It is hard to pin down exactly what "reasonable" should mean, but it seems nearly impossible to do anything without the following assumption:

*Local closedness 21.0.* The functor $Varieties_{\mathbf{V}}$ is locally closed if for any flat morphism $X \to S$ there is a locally closed subscheme $S_{\mathbf{V}} \subset S$ such that for any $g : T \to S$, the pull-back $X \times_S T \to T$ is in $Varieties_{\mathbf{V}}(T)$ iff $\operatorname{im} g \subset S_{\mathbf{V}}$.

In most cases, $S_{\mathbf{V}} \subset S$ is even open. For instance, being reduced, normal or smooth are all open conditions. On the other hand, being a hyperelliptic curve is not an open condition but it is a locally closed condition. (This is subtler than it sounds. It is easy to see that in any flat family of projective curves, there is a unique *reduced* subscheme that parametrizes hyperelliptic curves. In order to endow it with a scheme structure, first we need to define what a "family of hyperelliptic curves" is over a nonreduced scheme. The geometric fibers need to be hyperelliptic, but this is not enough. The best is to define a family of hyperelliptic curves as a double cover of a $\mathbb{P}^1$-bundle.)

Local closedness also implies that membership in $Varieties_{\mathbf{V}}(T)$ can be tested on 0-dimensional subschemes of $T$, that is, on spectra of Artin rings. This is the reason why formal deformation theory is such a powerful tool [**Art76, Ill71, Ser06**].

Assume for the moment that there is a coarse moduli space $\mathrm{Moduli}_{\mathbf{V}}$. Our next aim is to understand how to recognize properties of $\mathrm{Moduli}_{\mathbf{V}}$ in terms of the functor $Varieties_{\mathbf{V}}$.

Let $X$ be a scheme of finite type over a field $k$. By the valuative criterion of separatedness, $X$ is separated iff the following holds.

Let $B$ be a smooth curve over $k$ and $B^0 \subset B$ an open subset. Then a morphism $\tau^0 : B^0 \to X$ has at most one extension to $\tau : B \to X$.

If $X = \mathrm{Moduli}_{\mathbf{V}}$ is a fine moduli space, then giving a morphism $U \to X$ is equivalent to specifying a proper, flat family $V_U \to U$ whose fibers are in $\mathbf{V}$. Thus the valuative criterion of separatedness translates to functors as follows:

*Separatedness 21.1.* The functor $Varieties_{\mathbf{V}}$ is separated iff for every smooth curve $B$ and every open subset $B^0 \subset B$, a proper, flat family $\pi^0 : V^0 \to B^0$ whose fibers are in **V** has at most one extension to

$$\begin{array}{ccc} V^0 & \subset & V \\ \pi^0 \downarrow & & \downarrow \pi \\ B^0 & \subset & B, \end{array}$$

where $\pi : V \to B$ is also a proper, flat family whose fibers are in **V**.

We obtain a similar translation of the valuative criterion of properness, but here we have to pay attention to the difference between coarse and fine moduli spaces.

*Valuative criterion of properness 21.2.* The functor $Varieties_{\mathbf{V}}$ satisfies the valuative criterion of properness iff the following holds:

Let $B$ be a smooth curve, $B^0 \subset B$ an open subset and $\pi^0 : V^0 \to B^0$ a proper, flat family whose fibers are in **V**. Then there is a finite surjection $p : A \to B$ such that there is an extension

$$\begin{array}{ccccc} V^0 \times_B A & =: & W^0 & \subset & W \\ \downarrow & & \downarrow & & \downarrow \pi_A \\ B^0 \times_B A & =: & A^0 & \subset & A, \end{array}$$

where $\pi_A : W \to A$ is also a proper, flat family whose fibers are in **V**. (For functors with a fine moduli space, we could take $A = B$, but for functors with a coarse moduli space, a finite base change may be needed.)

It is very convenient to roll these two concepts together. The resulting condition is then exactly the general version of the Stable reduction theorem (17).

The valuative criterion of properness implies properness for schemes of finite type, but not in general. The next condition is the functor version of finite type. It ensures that we do not have too many objects to parametrize.

*Boundedness 21.3.* The class of schemes **V** is called bounded if there is a flat morphism of schemes of finite type $u : U \to T$ such that for every algebraically closed field $K$, every $K$-scheme in **V** occurs as a fiber of $U_K \to T_K$. (Some authors also assume that every fiber of $u : U \to T$ is in **V**.)

*How important are these conditions? 21.4.*

As we already noted, the assumption in this book is that local closedness (21.0) is indispensable. When separatedness (21.1) fails, it usually either fails very badly or it can be restored by a judicious change of the definition; see Section 4 for such examples. (Note, however, that most moduli functors of sheaves behave differently. They are not separated but the notions of semi-stabilty and GIT quotients provide a good method to deal with this. See [**Mum65, HL97, Dol03**] for details.)

Properness (21.2) is considered a challenge: If a moduli functor does not satisfy the valuative criterion of properness, find out how to enlarge it to make it satisfy.

Finally, boundedness (21.3) seems to come automatically, though it can be very hard to prove that it holds. I do not know any natural moduli functor of projective varieties satisfying (21.1–2) with a coarse moduli space whose connected components are not of finite type. (In the proper but non-projective seeting this can, however, happen. The Hilbert scheme of curves on the Hironaka 3-fold described in [**Har77**, App.B.3.4.1] has a connected component with infinitely many irreducible components, each proper. I do not know any natural moduli functor with a coarse moduli space that has an irreducible component that is not of finite type.)

22 (From the moduli functor to the moduli space). Starting with [**Mum65**] and [**Mat64**], much effort was devoted to going from the moduli functor $Varieties_{\mathbf{V}}$ to the moduli space $\mathrm{Moduli}_{\mathbf{V}}$. In the quasi projective setting, this was solved in [**Vie95**], but the proofs are quite hard.

The construction of the moduli space as an algebraic space turns out to be much easier, and the general quotient theorems of [**Kol97, KM97**] take care of it completely.

Once we have a moduli space which is a proper algebraic space, it is not that hard to prove that it is projective [**Kol90**]. These results are treated in Chapter **??**?.

23 (Moduli for varieties of non-general type).

In contrast with varieties of general type, the moduli theory for varieties of non-general type is very complicated.

A general problem, illustrated by Abelian, elliptic and K3 surfaces is that a typical deformation of such an algebraic surface over $\mathbb{C}$ is a non-algebraic complex analytic surface. Thus any algebraic theory captures only a small part of the full analytic deformation theory.

The moduli question for analytic surfaces has been studied, especially for complex tori and K3 surfaces. In both cases it seems that one needs to add some extra structure (for instance, fixing a basis in some topological homology group) in order to get a sensible moduli space. (As an example of what could happen, note that the 3-dimensional space of Kummer surfaces is dense in the 20-dimensional space of all K3 surfaces, cf. [**PŠŠ71**].)

Even if one restricts to the algebraic case, compactifying the moduli space seems rather hopeless. Detailed studies of Abelian varieties and K3 surfaces show that there are many different compactifications depending on additional artificial choices, see [**KKMSD73, AMRT75**].

It is only with the works of [**Ale02**] that a geometrically meaningful compactification of the moduli of principally polarized Abelian varieties became available. This relies on the observation that a pair $(A, \Theta)$ consisting of a principally polarized Abelian variety $A$ and its theta divisor $\Theta$ behaves as if it were a variety of general type.

24 (Open problems). While we provide a solution to the basic general questions of the moduli theory of varieties of general types, there are many unsolved aspects. Some of the main ones are the following.

*Problem 24.1* (Positive characteristic). Most of our results work only in characteristic 0. This is partly caused by the need for resolution of singularities and minimal model theory. There are, however, many other difficulties that are unsettled in positive characteristic. Even the correct definition of stable families is problematic (46).

*Problem 24.2* (Boundedness). We show that in our moduli spaces, every irreducible component is projective. However, except for surfaces, we do not rule out the possibility of a connected component with infinitely many irreducible components. A solution of this question would follow from a series of interesting conjectures on various numerical invariants satisfying the ascending chain condition, see Section **??**?.

*Problem 24.3* (Effective results). Given a class of varieties of general type, we do not have good general methods to decide which stable varieties occur on the corresponding components of the moduli space. Even bounding basic numerical invariants, for instance the number of irreducible components, seems very hard. The methods in Section **??**? provide an answer in principle, but it does not seem feasible to work it out in practice, save in some very simple cases. A few results are discussed in Section **??**?, but it would be very useful to get much more information.

*Problem 24.4* (Moduli of pairs) It is very useful to study not just the moduli of curves, but also the moduli of curves with marked points. The corresponding higher dimensional question is to study the moduli of varieties with marked divisors. This case is especially fruitful in applications. By choosing the marked divisor carefully, one can get moduli spaces in many, seemingly unrelated, cases. The general case is, however, more complicated than the unmarked version and even some of the basic definitions are unsettled.

*Problem 24.5* (Fine moduli spaces). As we see, stable varieties have finite automorphism groups, and we get a fine moduli space iff the identity is the only automorphism; see Section 5. Hence the question: Is there a sensible way to kill automorphisms by additional structures. For curves over $\mathbb{C}$ this is achieved by introducing a "level $m$ structure" for some $m \geq 3$, that is, by fixing an isomorphism $H^1(C, \mathbb{Z}/m) \cong (\mathbb{Z}/m)^{2g}$. For smooth surfaces, similar topological invariants do not seem to be sufficient, but a completely different approach may work.

*Problem 24.6* (Applications). Many basic questions about smooth curves can be solved by investigating an analogous problem on stable curves, whose geometry is frequently much simpler. There are, so far, few such results in higher dimensions. Some of these are discussed in Section **??**?. One problem is that it is not easy to write down stable degenerations, the other is that the stable varieties themselves are still rather complicated.

## 2. From smooth curves to canonical models

In the theory of curves, the basic objects are smooth projective curves. We study any other curve by relating it to smooth projective curves. This is why the moduli functor/space of smooth curves is so important.

In higher dimensions, we define the moduli functor of smooth varieties as

$$Smooth(S) := \left\{ \begin{array}{c} \text{Smooth, proper families } X \to S, \\ \text{modulo isomorphisms over } S. \end{array} \right\}$$

This, however, gives a rather badly behaved and mostly useless moduli functor already for surfaces. First of all, it is very non-separated.

25 (Non-separatedness in the moduli of smooth surfaces of general type). We construct two smooth families of projective surfaces $f_i : X^i \to B$ over a pointed smooth curve $b \in B$ such that

(1) all the fibers are smooth, projective surfaces of general type,
(2) $X^1 \to B$ and $X^2 \to B$ are isomorphic over $B \setminus \{b\}$,
(3) the fibers $X^1_b$ and $X^2_b$ are *not* isomorphic.

As the construction shows, this type of behavior happens every time we look at deformations of a surface with at least 3 points blown-up.

Let $f : X \to B$ be a smooth family of projective surfaces over a smooth (affine) pointed curve $b \in B$. Let $C_1, C_2, C_3 \subset X$ be three sections of $f$, all passing through a point $x_b \in X_b$ that intersect pairwise transversally at $x_b$ and are disjoint elsewhere.

Set $X^1 := B_{C_1} B_{C_2} B_{C_3} X$, where we first blow-up $C_3 \subset X$, then the birational transform of $C_2$ in $B_{C_3} X$ and finally the birational transform of $C_1$ in $B_{C_2} B_{C_3} X$. Similarly, set $X^2 := B_{C_1} B_{C_3} B_{C_2} X$. Since the $C_i$ are sections, all these blow-ups are smooth families of projective surfaces over $B$.

Over $B \setminus \{b\}$ the curves $C_i$ are disjoint, thus $X^1$ and $X^2$ are both isomorphic to $B_{C_1+C_2+C_3} X$, the blow-up of $C_1 + C_2 + C_3 \subset X$.

We claim that, by contrast, the fibers of $X_b^1$ and $X_b^2$ are not isomorphic to each other for a general choice of $C_1$.

To see this, choose local coordinates $t$ at $b \in B$ and $(x, y, t)$ at $x_b \in X$. The curves $C_i$ are defined by equations

$$C_i = \bigl(x - a_i t - (\text{higher terms}) = y - b_i t - (\text{higher terms}) = 0\bigr).$$

The blow-up $B_{C_i} X$ is given by

$$B_{C_i} X = \bigl(u_i(x - a_i t - (\text{higher terms})) = v_i(y - b_i t - (\text{higher terms}))\bigr) \subset X \times \mathbb{P}^1_{u_i v_i}.$$

Thus we see that the birational transform of $C_j$ intersects the central fiber

$$\bigl(B_{C_i} X\bigr)_b = B_{x_b}\bigl(X_b\bigr) = (ux = vy) \subset X_b \times \mathbb{P}^1_{uv}$$

at the point

$$\frac{u}{v} = \frac{a_j - a_i}{b_j - b_i} \in \{x_b\} \times \mathbb{P}^1_{uv}.$$

The fibers $\bigl(B_{C_2} B_{C_3} X\bigr)_b$ and $\bigl(B_{C_3} B_{C_2} X\bigr)_b$ are isomorphic to each other since they are obtained from $B_{x_b}\bigl(X_b\bigr)$ by blowing up the same point

$$\frac{u}{v} = \frac{a_2 - a_3}{b_2 - b_3} \quad \text{resp.} \quad \frac{u}{v} = \frac{a_3 - a_2}{b_3 - b_2}.$$

When we next blow up the birational transform of $C_1$ on $\bigl(B_{C_2} B_{C_3} X\bigr)_b$ (resp. on $\bigl(B_{C_3} B_{C_2} X\bigr)_b$) this gives the blow-up of the point

$$\frac{a_1 - a_3}{b_1 - b_3} \quad \text{resp.} \quad \frac{a_1 - a_2}{b_1 - b_2}, \tag{25.4}$$

and these are different, unless $C_1 + C_2 + C_3$ is locally planar at $x_b$.

So far we have seen that the identity $X_b = X_b$ does not extend to an isomorphism between the fibers $X_b^1$ and $X_b^2$.

If $X_b$ is of general type, then $\operatorname{Aut} X_b$ is finite, hence, to ensure that $X_b^1$ and $X_b^2$ are not isomorphic, we need to avoid finitely many other possible coincidences in (25.4).

The main reason, however, why we do not study the moduli functor of smooth varieties up to isomorphism is that, in dimension two, smooth projective surfaces do not form the *smallest* basic class. Given any smooth projective surface $S$, one can blow up any set of points $Z \subset S$ to get another smooth projective surface $B_Z S$ which is very similar to $S$. Therefore, the basic object should be not a single smooth projective surface but a whole *birational equivalence class* of smooth projective surfaces. Thus it would be better to work with smooth, proper families $X \to S$

modulo birational equivalence over $S$. That is, with the moduli functor

$$GenType_{bir}(S) := \left\{ \begin{array}{c} \text{Smooth, proper families } X \to S, \\ \text{every fiber is of general type,} \\ \text{modulo birational equivalences over } S. \end{array} \right\} \tag{25.5}$$

In essence this is what we end up doing, but it is very cumbersome do deal with birational equivalence over a base scheme. Nonetheless, working with birational equivalence classes leads to a separated moduli functor.

PROPOSITION 26. *Let $f_i : X^i \to B$ be two smooth families of projective varieties over a smooth curve $B$. Assume that the generic fibers $X^1_{k(B)}$ and $X^2_{k(B)}$ are birational and the pluricanonical system $\left|mK_{X^1_{k(B)}}\right|$ is nonempty for some $m > 0$. Then, for every $b \in B$, the fibers $X^1_b$ and $X^2_b$ are birational.*

Proof. Pick a birational map $\phi : X^1_{k(B)} \dashrightarrow X^2_{k(B)}$ and let $\Gamma \subset X^1 \times_B X^2$ be the closure of the graph of $\phi$. Let $Y \to \Gamma$ be the normalization with projections $p_i : Y \to X^i$. Note that both of the $p_i$ are open embeddings on $Y \setminus \big(\operatorname{Ex} p_1 \cup \operatorname{Ex} p_2\big)$. Thus if we prove that neither $p_1\big(\operatorname{Ex} p_1 \cup \operatorname{Ex} p_2\big)$ nor $p_2\big(\operatorname{Ex} p_1 \cup \operatorname{Ex} p_2\big)$ contains a fiber of $f_1$ or $f_2$, then $p_2 \circ p_1^{-1} : X^1 \dashrightarrow X^2$ restricts to a birational map $X^1_b \dashrightarrow X^2_b$ for every $b \in B$.

We use the canonical class to compare $\operatorname{Ex} p_1$ and $\operatorname{Ex} p_2$. Since the $X^i$ are smooth,

$$K_Y \sim p_i^* K_{X^i} + E_i, \quad \text{where } E_i \geq 0 \text{ and } \operatorname{Supp} E_i = \operatorname{Ex} p_i. \tag{26.1}$$

Assume for simplicity that $B$ is affine and let $\operatorname{Bs}\left|mK_{X^i}\right|$ denote the set-theoretic base locus. By assumption, $\left|mK_{X^i}\right|$ is not empty and since $B$ is affine, $\operatorname{Bs}\left|mK_{X^i}\right|$ does not contain any of the fibers of $f_i$.

Every section of $\mathcal{O}(mK_Y)$ pulls back from $X^i$, thus

$$\operatorname{Bs}\left|mK_Y\right| = p_i^{-1}\Big(\operatorname{Bs}\left|mK_{X^i}\right|\Big) + \operatorname{Supp} E_i.$$

Comparing these for $i = 1, 2$, we conclude that

$$p_1^{-1}\Big(\operatorname{Bs}\left|mK_{X^1}\right|\Big) + \operatorname{Supp} E_1 = p_2^{-1}\Big(\operatorname{Bs}\left|mK_{X^2}\right|\Big) + \operatorname{Supp} E_2.$$

Therefore,

$$p_1\big(\operatorname{Supp} E_2\big) \subset p_1\big(\operatorname{Supp} E_1\big) + \operatorname{Bs}\left|mK_{X^1}\right|.$$

Since $E_1$ is $p_1$-exceptional, $p_1\big(E_1\big)$ has codimension $\geq 2$ in $X^1$, hence it does not contain any of the fibers of $f_1$. We saw that $\operatorname{Bs}\left|mK_{X^1}\right|$ does not contain any of the fibers either. Thus $p_1\big(\operatorname{Ex} p_1 \cup \operatorname{Ex} p_2\big)$ does not contain any of the fibers and similarly for $p_2\big(\operatorname{Ex} p_1 \cup \operatorname{Ex} p_2\big)$. □

REMARK 27. A result of [**MM64**] says that, more generally, (26) holds as long as the fibers $X^i_b$ are not birationally ruled, that is, not birational to a variety of the form $Z \times \mathbb{P}^1$. The proof of [**MM64**], relies on the study of exceptional divisors over a smooth variety; see [**KSC04**, Sec.4.5] for an overview. Exceptional divisors over a singular variety are much less understood. By contrast, the above proof focusses on the role of the canonical class. It is worthwhile to go back and check that the proof works if the $X^i$ are normal, as long as (26.1) holds.

It is precisely the property (26.1) and its closely related variants that lead us to the correct class of singular varieties for moduli purposes.

Since it is much harder to work with a whole equivalence class, it would be desirable to find a paticularly nice surface in every birational equivalence class. This was achieved by the theory of minimal models of algebraic surfaces. By a result of Enriques (cf. [**BPVdV84**, III.4.5]), every birational equivalence class of surfaces $\mathbf{S}$ contains a unique smooth projective surface whose canonical class is nef (that is, has nonnegative degree on every effective curve), except when $\mathbf{S}$ contains a ruled surface $C \times \mathbb{P}^1$ for some curve $C$. This unique surface is called the *minimal model* of $\mathbf{S}$.

It would seem at first sight that (26) implies that the moduli functor of minimal models is separated. There is, however, a quite subtle problem.

28 (Non-separatedness in the moduli of minimal models). We construct two smooth families of projective surfaces $f_i : X^i \to B$ over a pointed smooth curve $b \in B$ such that

(1) all the fibers are smooth, projective minimal models,
(2) $X^1 \to B$ and $X^2 \to B$ are isomorphic over $B \setminus \{b\}$,
(3) the fibers $X^1_b$ and $X^2_b$ are isomorphic, but
(4) $X^1 \to B$ and $X^2 \to B$ are *not* isomorphic.

While it is not clear from our construction, similar problems happen for any smooth family of surfaces where the general fiber has ample canonical class and a special fiber has nef (but not ample) canonical class, see [**Art74, Bri68, Rei80**].

Let $X_0 := \big(f(x_1, \dots, x_4) = 0\big) \subset \mathbb{P}^3$ be a surface of degree $n$ that has an ordinary double point (38) at $p = (0{:}0{:}0{:}1)$ as its sole singularity and contains the pair of lines $(x_1x_2 = x_3 = 0)$. Let $g$ be homogeneous of degree $n-1$ such that $x_4^{n-1}$ appears in it with nonzero coefficient. Consider the family of surfaces

$$X := \big(f(x_1, \dots, x_4) + tx_3 g(x_1, \dots, x_4) = 0\big) \subset \mathbb{P}^3_{\mathbf{x}} \times \mathbb{A}^1_t.$$

Note that $X_t$ is smooth for general $t \neq 0$ and $X$ contains the pair of smooth surfaces $(x_1x_2 = x_3 = 0)$.

For $i = 1, 2$, let $X^i := B_{(x_i,x_3)}X$ denote the blow-up of $(x_i = x_3 = 0)$ with induced morphisms $\pi_i : X^i \to X$ and $f_i : X^i \to \mathbb{A}^1$. There is a natural birational map $\phi := \pi_2^{-1} \circ \pi_1 : X^1 \dashrightarrow X^2$. Let $B_pX$ denote the blow-up of $p = \big((0{:}0{:}0{:}1), 0\big)$ with exceptional divisor $E \subset B_pX$.

We claim that the following hold.

(5) The $f_i : X^i \to \mathbb{A}^1$ are projective families of surfaces which are smooth over a neighborhood of $(t = 0)$.
(6) For $n \geq 5$, the fibers $X^i_t$ have ample canonical class for $t \neq 0$ and nef canonical class for $t = 0$.
(7) $X^1 \times_X X^2$ is isomorphic to $B_pX$, hence it is smooth and irreducible.
(8) The map $\phi$ is an isomorphism over $\mathbb{A}^1 \setminus \{0\}$ but it is not an isomorphism over 0.
(9) The fiber of $X^1 \times_X X^2$ over $(t = 0)$ has two irreducible components. One of these components is the graph of an isomorphism $X^1_0 \cong X^2_0$. The other component is $E \cong \mathbb{P}^1 \times \mathbb{P}^1$.
(10) Thus $\phi : X^1 \dashrightarrow X^2$ is an isomorphism over $\mathbb{A}^1 \setminus \{0\}$, the $X^i \to \mathbb{A}^1$ have isomorphic fibers over $0 \in \mathbb{A}^1$, but $\phi$ is not an isomorphism over $\mathbb{A}^1$.

(It is not hard to see that, for general choice of $f$ and $g$, the $X_t$ have no birational self-maps, thus the only possible isomorphism between $X^1$ and $X^2$ would be the identity on $X$. Thus, by (6), in this case, $X^1$ and $X^2$ are not isomorphic to each other.)

Note that $(x_i = x_3 = 0)$ defines a Weil divisor in $X$ which is Cartier outside the point $p$. Thus all 3 blow-ups are isomorphisms over $X \setminus \{p\}$. This means that all the above claims are local near $p$.

In (39) we show how to choose better local coordinates near $p$ that make all the claims (5–10) transparent.

All such problems go away when the canonical class is ample.

PROPOSITION 29. *Let $f_i : X^i \to B$ be two smooth families of projective varieties over a smooth curve $B$. Assume that the canonical classes $K_{X^i}$ are $f_i$-ample. Let $\phi : X^1_{k(B)} \cong X^2_{k(B)}$ be an isomorphism of the generic fibers.*

*Then $\phi$ extends to an isomorphism $\Phi : X^1 \cong X^2$.*

Proof. Let $\Gamma \subset X^1 \times_B X^2$ be the closure of the graph of $\phi$. Let $Y \to \Gamma$ be the normalization, with projections $p_i : Y \to X^i$ and $f : Y \to B$. As in (26), we use the canonical class to compare the $X^i$. Since the $X^i$ are smooth,

$$K_Y \sim p_i^* K_{X^i} + E_i \quad \text{where } E_i \text{ is effective and } p_i\text{-exceptional.} \tag{29.1}$$

Since $(p_i)_* \mathcal{O}_Y(mE_i) = \mathcal{O}_{X^i}$ for every $m \geq 0$, we get that

$$\begin{array}{rcll} (f_i)_* \mathcal{O}_{X^i}\big(mK_{X^i}\big) & = & (f_i)_*(p_i)_* \mathcal{O}_{X^i}\big(mp_i^* K_{X^i}\big) & = \\ & = & (f_i)_*(p_i)_* \mathcal{O}_{X^i}\big(mp_i^* K_{X^i} + mE_i\big) & = \\ & = & (f_i)_*(p_i)_* \mathcal{O}_Y\big(mK_Y\big) & = \quad f_* \mathcal{O}_Y\big(mK_Y\big). \end{array}$$

Since the $K_{X^i}$ are $f_i$-ample, $X^i = \operatorname{Proj}_B \sum_{m\geq 0} (f_i)_* \mathcal{O}_{X^i}\big(mK_{X^i}\big)$. Putting these together, we get the isomorphism

$$\begin{array}{rcll} \Phi : X^1 & \cong & \operatorname{Proj}_B \sum_{m\geq 0} (f_1)_* \mathcal{O}_{X^1}\big(mK_{X^1}\big) & \cong \\ & \cong & \operatorname{Proj}_B \sum_{m\geq 0} f_* \mathcal{O}_Y\big(mK_Y\big) & \cong \\ & \cong & \operatorname{Proj}_B \sum_{m\geq 0} (f_2)_* \mathcal{O}_{X^2}\big(mK_{X^2}\big) & \cong \quad X^2. \qquad \square \end{array}$$

REMARK 30. As in (27), it is again worthwhile to investigate the precise assumptions behind the proof. The smoothness of the $X^i$ is used only through the pull-back formula (29.1), which is weaker than (26.1).

If (29.1) holds, then, even if the $K_{X^i}$ are not $f_i$-ample, we obtain an isomorphism

$$\operatorname{Proj}_B \sum_{m\geq 0} (f_1)_* \mathcal{O}_{X^1}\big(mK_{X^1}\big) \cong \operatorname{Proj}_B \sum_{m\geq 0} (f_2)_* \mathcal{O}_{X^2}\big(mK_{X^2}\big). \tag{30.1}$$

Thus it is of interest to study objects as in (30.1) in general.

Let us start with the absolute case, when $X$ is a smooth projective variety over a field $k$. Its *canonical ring* is the graded ring

$$R(X, K_X) := \sum_{m\geq 0} H^0\big(X, \mathcal{O}_X(mK_X)\big).$$

In some cases the canonical ring tells us very little about $X$. For instance, if $X$ is rational or Fano then $R(X, K_X)$ is the base field $k$ and if $X$ is Calabi-Yau then $R(X, K_X)$ is isomorphic to the polynomial ring $k[t]$. One should thus focus on the cases when the canonical ring is large. The following theorem and the resulting definition is due to [**Iit71**]. See [**Laz04**, Sec.2.1.C] for a detailed treatment.

Theorem–Definition 31. *For a smooth projective variety $X$ of dimension $n$, the following are equivalent.*

(1) $h^0\big(X, \mathcal{O}_X(mK_X)\big) \geq \epsilon \cdot m^n$ *for some* $\epsilon > 0$ *and* $m \gg 1$.
(2) $\operatorname{Proj} R(X, K_X)$ *has dimension* $n$.
(3) *The natural map* $X \dashrightarrow \operatorname{Proj} R(X, K_X)$ *is birational.*

*If these hold, then we say that $X$ is of general type.*

This enables us to find a distinguished variety in any birational equivalence class.

Definition 32 (Canonical models). Let $X$ be a smooth projective variety of general type over a field $k$ such that its canonical ring $R(X, K_X)$ is finitely generated. We define its *canonical model* as

$$X^{can} := \operatorname{Proj}_k R(X, K_X).$$

If $Y$ is a smooth projective variety birational to $X$ then $Y^{can}$ is isomorphic to $X^{can}$. Thus $X^{can}$ is also the canonical model of the whole birational equivalence class containing $X$. (Taking Proj of a non-finitely generated ring may result in a quite complicated scheme. It does not seem profitable to contemplate what would happen in our case.)

Now we know [**BCHM06, Siu08**] that the canonical ring $R(X, K_X)$ is always finitely generated, thus $X^{can}$ is a projective variety. On the other hand, $X^{can}$ can be singular. Originally this was viewed as a major obstacle but now it seems only as a minor technical problem.

Definition 33 (Canonical class and canonical sheaf). Let $X$ be a smooth variety over a field $k$. As in [**Sha94**, III.6.3] or [**Har77**, p.180], the *canonical sheaf* of $X$ is $\omega_X := \wedge^{\dim X}\Omega_{X/k}$. Any divisor $D$ such that $\mathcal{O}_X(D) \cong \omega_X$ is called a *canonical divisor*. Their linear equivalence class is called the *canonical class*, denoted by $K_X$. (Note that both books assume that $X$ is nonsingular. However, they tacitly assume that $k$ is algebraically closed, hence nonsingularity implies smoothness. The definition, however, works over any field $k$ as long as $X$ is smooth over $k$.)

Let $X$ be a normal variety over a perfect field $k$. Let $j : X^{sm} \hookrightarrow X$ be the inclusion of the locus of smooth points. Then $X \setminus X^{sm}$ has codimension $\geq 2$, therefore, restriction from $X$ to $X^{sm}$ is a bijection on Weil divisors and on linear equivalence classes of Weil divisors. Thus there is a unique linear equivalence class $K_X$ of Weil divisors on $X$ such that $K_X|_{X^{sm}} = K_{X^{sm}}$. It is called the *canonical class* of $X$. In general, $K_X$ does not contain any Cartier divisors.

The push-forward $\omega_X := j_*\omega_{X^{sm}}$ is a rank 1 coherent sheaf on $X$, called the *canonical sheaf* of $X$. The canonical sheaf $\omega_X$ agrees with the *dualizing sheaf* $\omega_X^\circ$ as defined in [**Har77**, p.241]. (Note that [**Har77**] defines the dualizing sheaf only if $X$ is proper. In general, take a normal compactification $\bar{X} \supset X$ and use $\omega_{\bar{X}}^\circ|_X$ instead. For more details, see [**KM98**, Sec.5.5], [**Har66**] or [**Con00**].)

With this definition in place, we can give the following abstract characterization of canonical models.

Theorem 34. *A normal projective variety $Y$ is a canonical model iff*

(1) $m_0K_Y$ *is Cartier and ample for some* $m_0 > 0$, *and*

(2) *there is a resolution* $f : X \to Y$ *and an effective,* $f$*-exceptional divisor* $E$ *such that*

$$m_0 K_X \sim f^*(m_0 K_Y) + E.$$

Proof. For now we prove only the "if" part since this is what we need for the examples. For the coverse, see [**Rei80**] or (**??**?).

Note that for any $r > 0$, $f_* \mathcal{O}_X(rE) = \mathcal{O}_Y$ since $E$ is effective and $f$-exceptional. Thus, by the projection formula,

$$\begin{aligned} H^0\big(X, \mathcal{O}_X(rm_0K_X)\big) &= H^0\big(Y, f_*\mathcal{O}_X(rm_0K_X)\big) \\ &= H^0\big(Y, \mathcal{O}_Y(rm_0K_Y) \otimes f_*\mathcal{O}_X(rE)\big) \\ &= H^0\big(Y, \mathcal{O}_Y(rm_0K_Y)\big). \end{aligned}$$

Therefore

$$\begin{aligned} \operatorname{Proj} \textstyle\sum_m H^0\big(X, \mathcal{O}_X(mK_X)\big) &= \operatorname{Proj} \textstyle\sum_r H^0\big(X, \mathcal{O}_X(rm_0K_X)\big) \\ &= \operatorname{Proj} \textstyle\sum_r H^0\big(Y, \mathcal{O}_Y(rm_0K_Y)\big) = Y. \quad \square \end{aligned}$$

This makes it possible to give a local definition of the singularities that occur on canonical models.

Definition 35. We say that a normal variety $Y$ has *canonical singularities* if

(1) $m_0 K_Y$ is Cartier for some $m_0 > 0$, and
(2) there is a resolution $f : X \to Y$ and an effective, $f$-exceptional divisor $E$ such that $m_0 K_X \sim f^*(m_0 K_Y) + E$.

It is easy to see that this is independent of the resolution $f : X \to Y$ (**??**). (It is possible to define canonical singularities without assuming the existence of a resolution, but it is quite inconvenient, see, for instance [**Luo87**].)

Equivalently, $Y$ has canonical singularities iff every point $y \in Y$ has an étale neighborhood which is an open subset on some canonical model.

As an example, consider the cone $C_d(\mathbb{P}^n)$ over the Veronese embedding $\mathbb{P}^n \hookrightarrow \mathbb{P}\big(H^0(\mathbb{P}^n, \mathcal{O}(d))\big)$. It is easy to compute that $C_d(\mathbb{P}^n)$ has a canonical singularity iff $d \le n+1$ and its canonical class is Cartier iff $d | n+1$. (See (**??**) for the case of general cones.)

Definition 36 (Moduli of canonical models). The moduli functor of canonical models is

$$CanMod(S) := \left\{ \begin{array}{c} \text{Flat, proper families } X \to S, \\ \text{every fiber is a canonical model,} \\ \text{modulo isomorphisms over } S. \end{array} \right\} \tag{36.1}$$

This is an improved version of the birational moduli functor $GenType_{bir}(*)$ (25.5).

By a theorem of [**Siu98**], in a smooth, proper family of varieties of general type the canonical rings form a flat family and so do the canonical models. Thus there is a natural transformation

$$T_{\mathrm{CanMod}} : GenType_{bir}(*) \to CanMod(*).$$

By definition, if $X_i \to S$ are two smooth, proper families of varieties of general type then

$$T_{\mathrm{CanMod}}(X_1/S) = T_{\mathrm{CanMod}}(X_2/S) \quad \text{iff} \quad X_1 \text{ and } X_2 \text{ are birational,}$$

thus $T_{\mathrm{CanMod}}$ is injective. It is, however, not surjective, but we have the following partial surjectivity statement.

Let $Y \to S$ be a flat family of canonical models. Then there is a dense open subset $S^0 \subset S$ and a smooth, proper family of varieties of general type $Y^0 \to S^0$ such that

$$T_{\mathrm{CanMod}}(Y^0/S^0) = [X^0/S^0].$$

Some of the obstruction to surjectivity are obvious but some are quite subtle (**??**?).

REMARK 37. In retrospect, it seems only by luck that the definition (36.1) gives the correct functor. See (46) and the examples after it.

**Auxiliary results on double points.**

38 (Ordinary double points of surfaces). Let $S := \big(h(x_1, x_2, x_3) = 0\big) \subset \mathbb{C}^3$ be a surface with an ordinary double point at the origin. That is,

$$h = h_2(x_1, x_2, x_3) + (\text{higher order terms}),$$

where $h_2$ is a rank 3 quadric. It is easy to see, for instance using the Weierstrass preparation theorem, that one can choose complex analytic coordinates $y_i$ such that, in a neighborhood of the origin, $S = \big(y_1^2 + y_2^2 + y_3^2 = 0\big)$.

Over an arbitrary (possibly not even algebraically closed) field, one can choose formal coordinates $y_i$ such that $S = \big(h_2(y_1, y_2, y_3) = 0\big)$. In general, however, this can not be achieved with an algebraic change of coordinates.

Assume that $S$ contains the pair of lines $(x_1x_2 = x_3 = 0)$. Then $h$ can be written as

$$f(x_1, x_2, x_3)x_1x_2 - g(x_1, x_2, x_3)x_3.$$

If the quadratic part has rank 3 then $f(0, 0, 0) \neq 0$ and we can write $g = x_1g_1 + x_2g_2 + x_3g_3$ for some polynomials $g_i$. We can rewrite $h$ as

$$f\big(x_1 - f^{-1}g_1x_3\big)\big(x_2 - f^{-1}g_2x_3\big) - \big(g_3 + f^{-1}g_1g_2\big)x_3^2.$$

If the quadratic part has rank 3 then $g_3 + f^{-1}g_1g_2$ is nonzero at $(0, 0, 0)$ and we can set

$$y_1 := x_1 - f^{-1}g_1x_3,\ y_2 := \big(x_2 - f^{-1}g_2x_3\big)\big(g_3 + f^{-1}g_1g_2\big)^{-1} \quad \text{and} \quad y_3 := x_3$$

to bring the equation to the normal form $S = (y_1y_2 - y_3^2 = 0)$. The pair of lines is still $(y_1y_2 = y_3 = 0)$.

Now we consider 3 ways of resolving the singularity of $X$. First, one can blow up the origin $0 \in \mathbb{A}^3$. We get

$$B_0\mathbb{A}^3 \subset \mathbb{A}^3_{\mathbf{y}} \times \mathbb{P}^2_{\mathbf{s}}$$

defined by the equations $\{y_is_j = y_js_i : 1 \leq i, j \leq 3\}$. Besides these equations, $B_0S$ is defined by the vanishing of

$$y_1y_2 - y_3^2, s_1s_2 - s_3^2, y_1s_2 - y_3s_3, s_1y_2 - y_3s_3.$$

One can also blow up $(y_1, y_3)$. We get

$$B_{(y_1,y_3)}\mathbb{A}^3 \subset \mathbb{A}^3_{\mathbf{y}} \times \mathbb{P}^1_{u_1u_3}$$

defined by the equation $y_1u_3 = y_3u_1$. Besides this equation, $B_{(y_1,y_3)}S$ is defined by $y_1y_2 - y_3^2 = u_1y_2 - u_3y_3 = 0$.

These two blow-ups are actualy isomorphic, as shown by the embedding

$$\mathbb{A}^3_{\mathbf{y}} \times \mathbb{P}^1_{u_1u_3} \hookrightarrow \mathbb{A}^3_{\mathbf{y}} \times \mathbb{P}^2_{\mathbf{s}} \quad : \quad \big((y_1, y_2, y_3), (u_1{:}u_3)\big) \mapsto \big((y_1, y_2, y_3), (u_1^2{:}u_3^2{:}u_1u_3)\big)$$

restricted to $B_{(y_1,y_3)}S$.

The same things happen if we blow up $(y_2, y_3)$.

39 (Ordinary double points of 3-folds). Let $X := \big(h(x_1, \dots, x_4) = 0\big) \subset \mathbb{C}^4$ be a hypersurface with an ordinary double point at the origin. That is,

$$h = h_2(x_1, \dots, x_4) + (\text{higher order terms}),$$

where $h_2$ is a rank 4 quadric. As in (38), over an arbitrary (possibly not even algebraically closed) field, one can choose formal coordinates $y_i$ such that $X = \big(h_2(y_1, \dots, y_4) = 0\big)$. In general, however, this can not be achieved with an algebraic change of coordinates.

Assume that $X$ contains the pair of planes $(x_1x_2 = x_3 = 0)$. Then $h$ can be written as

$$f(x_1, \dots, x_4)x_1x_2 - g(x_1, \dots, x_4)x_3.$$

The quadratic part has rank 4 iff $f(0, \dots, 0) \neq 0$ and $x_4$ appears in $g$ with nonzero coefficient. In this case we can set

$$y_i := x_i \quad \text{for } i = 1, 2, 3, \text{ and} \quad y_4 := f^{-1}g$$

to bring the equation to the normal form $X = (y_1y_2 - y_3y_4 = 0)$. The original pair of planes is still $(y_1y_2 = y_3 = 0)$.

Now we consider 3 ways of resolving the singularity of $X$. First, one can blow up the origin $0 \in \mathbb{A}^4$. We get

$$B_0\mathbb{A}^4 \subset \mathbb{A}^4_{\mathbf{y}} \times \mathbb{P}^3_{\mathbf{s}}$$

defined by the equations $\{y_is_j = y_js_i : 1 \leq i, j \leq 4\}$. Besides these equations, $p : B_0X \to X$ is defined by the vanishing of

$$y_1y_2 - y_3y_4, s_1s_2 - s_3s_4, y_is_{3-i} - y_js_{7-j} : i \in \{1, 2\}, j \in \{3, 4\}.$$

The exceptional set is the smooth quadric $(s_1s_2 = s_3s_4) \subset \mathbb{P}^3$ lying over the origin $0 \in \mathbb{A}^4$.

One can also blow up $(y_1, y_3)$. We get

$$B_{(y_1,y_3)}\mathbb{A}^4 \subset \mathbb{A}^4_{\mathbf{y}} \times \mathbb{P}^1_{u_1u_3}$$

defined by the equation $y_1u_3 = y_3u_1$. Besides this equation, $B_{(y_1,y_3)}X$ is defined by $y_1y_2 - y_3y_4 = u_1y_2 - u_3y_4 = 0$. The exceptional set is the smooth rational curve $E \cong \mathbb{P}^1_{u_1u_3}$ lying over the origin $0 \in \mathbb{A}^4$.

Note furthermore that the birational transform $P^*_{24}$ of the plane $P_{24} := (y_2 = y_4 = 0)$ is the blown-up plane $B_0P_{24}$, but the birational transform $P^*_{14}$ of the plane $P_{14} := (y_1 = y_4 = 0)$ is the plane $(y_1 = u_1 = 0)$. The latter intersects $E$ at the point $(u_1 = 0) \in E$, thus $\big(P^*_{14} \cdot E\big) = 1$. Since $P^*_{14} + P^*_{24}$ is the pull-back of the Cartier divisor $(y_4 = 0)$, it has 0 intersection number with $E$. Thus $\big(P^*_{24} \cdot E\big) = -1$.

We claim that the rational map $p : \mathbb{A}^4_{\mathbf{y}} \times \mathbb{P}^3_{\mathbf{s}} \dashrightarrow \mathbb{A}^4_{\mathbf{y}} \times \mathbb{P}^1_{\mathbf{u}}$ given by

$$p_1 : (y_1, \dots, y_4, s_1{:}\cdots{:}s_4) \mapsto (y_1, \dots, y_4, s_1{:}s_3)$$

gives a morphism $p_1 : B_0X \to B_{(y_1,y_3)}X$.

To see this note that the quadric $Q := (s_1s_2 - s_3s_4 = 0)$ is isomorhic to $\mathbb{P}^1_{\mathbf{u}} \times \mathbb{P}^1_{\mathbf{v}}$, with the isomorphism given as

$$j : \big((u_0{:}u_1), (v_0{:}v_1)\big) \mapsto \big(u_0v_0{:}u_0v_1{:}u_1v_0{:}u_1v_1\big).$$

Thus the map $(s_1{:}\cdots{:}s_4) \mapsto (s_1{:}s_3)$ is the inverse of $j$ followed by the 1st coordinate projection. Thus $p_1$ restricts to a morphism on $\mathbb{A}^4_{\mathbf{y}} \times Q$ and $B_0X \subset \mathbb{A}^4_{\mathbf{y}} \times Q$.

Similarly, we obtain $p_2 : B_0X \to B_{(y_2,y_3)}X$. Putting these together, we get an isomorphism

$$p_1 \times p_2 : B_0X \cong B_{(y_1,y_3)}X \times_X B_{(y_2,y_3)}X.$$

(The above considerations show that this is an isomorphism of reduced schemes, and this is all we need. However, by explicit computation, the right hand side is reduced, so we have a scheme theoretic isomorphism.) In particular, this shows that the two maps $p_i : B_{(y_i,y_3)}X \to X$ are not isomorphic to each other.

Finally, set $S := (y_3 = y_4) \subset X$. By the computations of (38), the $p_i$ restrict to isomorphisms $p_i : B_0S \cong B_{(y_i,y_3)}S$. Thus $p^{-1}S = B_0S \cup E$ and $B_0S$ is the graph of the isomorphism $p_2 \circ p_1^{-1} : B_{(y_1,y_3)}S \cong B_{(y_2,y_3)}S$.

## 3. From stable curves to stable varieties

Let $C$ be a stable curve with normalized irreducible components $C_i$. We frequently view $C$ as an object assembled from the pieces $C_i$. Note that the restriction of $\omega_C$ to $C_i$ is not $\omega_{C_i}$, rather $\omega_{C_i}(P_i)$, where $P_i \subset C_i$ are the preimages of the nodes of $C$.

Similarly, if $X$ is a scheme with simple normal crossing singularities and irreducible components $X_i$, then the restriction of $\omega_X$ to $X_i$ is not $\omega_{X_i}$, rather $\omega_{X_i}(D_i)$ where $D_i \subset X_i$ is the preimage of $\operatorname{Sing} X$ on $X_i$.

This suggests that we should develop a theory of "canonical models" where the role of the canonical class is played by a divisor of the form $K_X + D$ where $D$ is a simple normal crossing divisor.

Definition 40 (Canonical models of pairs). Let $(X, D)$ be a pair consisting of a smooth projective variety $X$ and a simple normal crossing divisor $D \subset X$. (That is, $D = \sum D_i$ where the $D_i$ are distinct smooth divisors and all intersections are transversal (**??**?).) We define the *canonical ring* of the pair $(X, D)$ as

$$R(X, K_X + D) := \sum_{m \geq 0} H^0\bigl(X, \mathcal{O}_X(mK_X + mD)\bigr).$$

It is conjectured (but known only for $\dim X \leq 4$) that the canonical ring of a pair $(X, D)$ is finitely generated. If this holds then $X^{can} := \operatorname{Proj}_k R(X, K_X + D)$ is a normal projective variety. Let $D^{can} \subset X^{can}$ denote the image of $D$ under the natural birational map $X \dashrightarrow X^{can}$.

The pair $\bigl(X^{can}, D^{can}\bigr)$ is called the *canonical model* of $(X, D)$.

The proof of the "if" part of the following characterization goes exactly as in (34).

Theorem 41. *A pair $(Y, B)$, consiting of a proper normal variety $Y$ and an effective, reduced Weil divisor $B$, is a log canonical model iff*

1. *$m_0(K_Y + B)$ is Cartier and ample for some $m_0 > 0$, and*
2. *there is a resolution $f : X \to Y$, an effective, reduced simple normal crossing divisor $D \subset X$ such that $f(D) = B$ and an effective, $f$-exceptional divisor $E$ such that*

$$m_0(K_X + D) \sim f^*\bigl(m_0(K_Y + B)\bigr) + E.$$

Remark 42. Even if $B = 0$, the notion of log canonical model differs from the notion of canonical model (34). To see this, let $F_i \subset X$ be the $f$-exceptional divisors. If $B = 0$, in (41.2) we can still take $D = \sum F_i$. Thus (41.2) can be rewritten as

$$m_0 K_X \sim f^*\big(m_0 K_Y\big) + E - m_0 \sum F_i.$$

This looks like (34.2), but $E - m_0 \sum F_i$ need not be effective; it can contain divisors with coefficients $\geq -m_0$.

This is the source of some terminological problems. Originally $R(X, K_X + D)$ was called the "log canonical ring" and $\operatorname{Proj}_k R(X, K_X + D)$ the "log canonical model." Since the canonical ring is just the $D = 0$ special case of the "log canonical ring," it seems more convenient to drop the prefix "log." However, log canonical singularities are quite different from canonical singularities, so the "log" cannot be omitted there.

As in (35), this can be reformulated as a definition (For now we assume that every irreducible component of $B$ appears in $B$ with coefficient 1. Later (**??**?) we also consider cases when the coefficients are rational or real.)

DEFINITION 43. Let $(Y, B)$ be a pair consisting of a normal variety $Y$ and a reduced Weil divisor $B$. Then $(Y, B)$ is *log canonical*, or has *log canonical singularities* iff the condition (41.2) is satisfied.

We are now ready to define the higher dimensional analogs of stable curves.

DEFINITION 44 (Stable varieties or semi log canonical models). Let $k$ be a field and $Y$ a reduced, proper scheme over $k$. Let $Y_i \to Y$ be the irreducible components of the normalization of $Y$ and $D_i \subset Y_i$ the reduced preimage of the non-normal locus of $Y$. Then $Y$ is a *semi log canonical model* or a *stable variety* iff

(1) at codimension 1 points, $Y$ is either smooth or has a node,
(2) each $(Y_i, D_i)$ is log canonical, and
(3) $\omega_Y$, the canonical or dualizing sheaf of $X$ (33), is ample.

(Implicit in the definition is that the $D_i$ are divisors and that $\omega_Y$ being ample makes sense. The latter is a quite subtle condition that will be properly treated only Chapter 3. For now we will only deal with examples where this is clear.

We can now state the two cornerstones of the moduli theory of varieties of general type.

PRINCIPLE 45. *Stable varieties are the correct higher dimensional analogs of stable curves (13).*

PRINCIPLE 46. *Flat families of stable varieties $X \to T$ are the correct higher dimensional analogs of flat families of stable curves (13) if the canonical sheaves $\omega_{X_t}$ are locally free, but* **not** *in general.*

The correct analog will only be defined in Section **??**? for 1-parameter families and in Section **??**? in general.

I hope that the explanations given so far make (45) quite believable. It is more interesting to see examples that support the second assertion of (46). The simple fact is that basic numerical invariants, like the self intersection of the canonical class or even the Kodaira dimension fail to be locally constant in flat families of stable varieties, even when the singularities are quite mild. The rest of the section is devoted to such examples.

### Jump of $K^2$ and of the Kodaira dimension

We give examples of flat families of projective surfaces $\{S_t : t \in \mathbb{C}\}$ such that $S_t$ has log canonical singularities for every $t$ (that is, the pair $(S_t, 0)$ has log canonical singularities for every $t$) but the self intersection of the canonical class $K_{S_t}^2$ varies with $t$. We also give examples where $K_{S_t}$ is ample for $t = 0$ but not even big for $t \neq 0$. In the examples the $S_t$ are smooth for $t \neq 0$ and $S_0$ has only quotient singularities. Even among log canonical singularities, the quotient singularities are the mildest.

Example 47 (Degree 4 surfaces in $\mathbb{P}^5$). It is easy to see that there are 2 families of nondegenerate degree 4 smooth surfaces in $\mathbb{P}^5$.

One family consists of Veronese surfaces $\mathbb{P}^2 \subset \mathbb{P}^5$ embedded by $\mathcal{O}(2)$. The general member of the other family is $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^5$ embedded by $\mathcal{O}(2, 1)$, special members are embeddings of the ruled surface $\mathbb{F}_2$. The two families are distinct since

$$K_{\mathbb{P}^2}^2 = 9 \quad \text{and} \quad K_{\mathbb{P}^1 \times \mathbb{P}^1}^2 = 8.$$

For both of these surface, a smooth hyperplane section gives a degree 4 rational normal curve in $\mathbb{P}^4$.

There are also some nondegenerate degree 4 singular surfaces in $\mathbb{P}^5$. The most interesting is the cone over the degree 4 rational normal curve in $\mathbb{P}^4$; denote it by $T_0 \subset \mathbb{P}^5$. The minimal resolution of $T_0$ is the ruled surface $p : \mathbb{F}_4 \to T_0$. Let $E, F \subset \mathbb{F}_4$ be the exceptional curve and the fiber of the ruling. Then $K_{\mathbb{F}_4} = -2E - 6F$ and $p^*(2K_{T_0}) = -3E - 12F$. Thus

$$2\bigl(K_{\mathbb{F}_4} + E\bigr) = p^*(2K_{T_0}) + E$$

shows that $T_0$ has log canonical singularities. We also get that $K_{T_0}^2 = 9$.

For us the intersting feature is that one can write $T_0$ as a limit of smooth surfaces in two distinct way, corresponding to the two ways of writing the degree 4 rational normal curve in $\mathbb{P}^4$ as a hyperplane section of a surface. (See (**??**) for a concrete description of these deformations.)

From the first family, we get $T_0$ as the special fiber of a flat family whose general fiber is $\mathbb{P}^2$. This family is denoted by $\{T_t : t \in \mathbb{C}\}$. From the second family, we get $T_0$ as the special fiber of a flat family whose general fiber is $\mathbb{P}^1 \times \mathbb{P}^1$. This family is denoted by $\{T'_t : t \in \mathbb{C}\}$. (In general, one needs to worry about the possibility of getting embedded points at the vertex. However, by (**??**), in both cases the special fiber is indeed $T_0$.)

Note that $K^2$ is constant in the family $\{T_t : t \in \mathbb{C}\}$ but jumps at $t = 0$ in the family $\{T'_t : t \in \mathbb{C}\}$.

These are, however, families of rational surfaces with negative canonical class, and we are interested in stable varieties.

Next we take a suitable cyclic cover of the two families to get similar examples with ample canonical class.

Example 48 (Jump of Kodaira dimension I).

We give examples of two flat families of projective surfaces $S_t$ and $S'_t$ such that

(1) $S_0 \cong S'_0$ has log canonical singularities and ample canonical class,
(2) $S_t$ is a smooth surface with ample canonical class for $t \neq 0$, and
(3) $S'_t$ is smooth and elliptic with $K_{S'_t}^2 = 0$ for $t \neq 0$.

With $T_0$ as in (47), let $\pi_0 : S_0 \to T_0$ be a double cover, ramified along a smooth quartic hypersurface section. Note that $K_{T_0} \sim_{\mathbb{Q}} -\frac{3}{2}H$ where $H$ is the hyperplane class. Thus, by the Hurwitz formula,

$$K_{S_0} \sim_{\mathbb{Q}} \pi_0^*\big(K_{T_0} + 2H\big) \sim_{\mathbb{Q}} \tfrac{1}{2}\pi_0^* H.$$

So $S_0$ has ample canonical class and $K_{S_0}^2 = 2$. Since $\pi_0$ is étale over the vertex of $T_0$, $S_0$ has 2 singular points, locally (in the analytic or étale topology) isomorphic to the singularity on $T_0$. Thus $S_0$ is a stable surface.

Both of the smoothings in (47) lift to smoothings of $S_0$.

From $T_t$ we get a smoothing $S_t$ where $\pi_t : S_t \to \mathbb{P}^2$ is a double cover, ramified along a smooth octic. Thus $S_t$ is smooth, $K_{S_t} \sim_{\mathbb{Q}} \pi_t^* \mathcal{O}_{\mathbb{P}^2}(1)$ is ample and $K_{S_t}^2 = 2$.

From $T_t'$ we get a smoothing $S_t'$ where $\pi_t' : S_t' \to \mathbb{P}^1 \times \mathbb{P}^1$ is a double cover, ramified along a smooth curve of bidegree $(8, 4)$. One of the families of lines on $\mathbb{P}^1 \times \mathbb{P}^1$ pulls back to an elliptic pencil on $S_t'$ and $K_{S_t'}^2 = 0$. Thus $S_t'$ is not of general type for $t \neq 0$.

Example 49 (Jump of Kodaira dimension II). A similar pair of examples is obtained by working with triple covers ramified along a cubic hypersurface section. The family over $T_t$ has ample canonical class and $K^2 = 3$. As before, the family over $T_t'$ is elliptic and so $K^2 = 0$.

Example 50 (Jump of Kodaira dimension III).
Here are other examples of flat families of projective surfaces $S_t$ such that

(1) $S_0$ has quotient singularities and ample canonical class, and
(2) $S_t$ is a smooth, rational surface for $t \neq 0$.

First we construct $S_0$.

*Claim* 50.3. Let $L_n \subset \mathbb{P}^2$ be the union of $n$ general lines. Let $P \subset \mathbb{P}^2$ be the $\binom{n}{2}$ intersection points and $p : B_P\mathbb{P}^2$ the blow up. Let $E_n$ denote the sum of all exceptional curves and $L_n' \subset B_P\mathbb{P}^2$ the birational transform of $L_n$.

Then (the Stein factorization of) the map given by $|p^*\mathcal{O}_{\mathbb{P}^2}(n-1)(-E_n)|$ gives a morphism $q : B_P\mathbb{P}^2 \to X_n$. For $n \geq 7$, $X_n$ has ample canonical class and log canonical singularities.

Proof. Note that $L_n' \subset B_P\mathbb{P}^2$ is a union of $n$ disjoint smooth rational curves, each with self intersection $2-n$ and $p^*\mathcal{O}_{\mathbb{P}^2}(n-1)(-E_n)$ has zero intersection number with $L_n'$.

The sum of any $(n-1)$ lines in $L_n$ pulls back to a divisor in $|p^*\mathcal{O}_{\mathbb{P}^2}(n-1)|$ which contains $E_n$, giving a divisor in $|p^*\mathcal{O}_{\mathbb{P}^2}(n-1)(-E_n)|$. The intersection of all these divisors is empty, thus the linear system $|p^*\mathcal{O}_{\mathbb{P}^2}(n-1)(-E_n)|$ is base point free. Thus (the Stein factorization of) the map given by $|p^*\mathcal{O}_{\mathbb{P}^2}(n-1)(-E_n)|$ gives a morphism $q : B_P\mathbb{P}^2 \to X_n$ such that $p^*\mathcal{O}_{\mathbb{P}^2}(n-1)(-E_n) \cong q^* M_n$ for some ample line bundle $M_n$ on $X_n$. The lines in $L_n'$ are mapped to points by $q$ and they have self-intersection $2 - n$. Using the theorem of formal functions, we see that $R^i q_* \mathcal{O}_{B_P\mathbb{P}^2} = 0$ for $i > 0$.

For $n \geq 3$ the sum of all these sections of $p^*\mathcal{O}_{\mathbb{P}^2}(n-1)(-E_n)$ is supported on $E_n + L_n' = p^{-1}(L_n)$. Thus if $C \subset B_P\mathbb{P}^2$ is any irreducible curve different from the lines in $L_n'$ then $\deg_C p^*\mathcal{O}_{\mathbb{P}^2}(n-1)(-E_n) > 0$. Thus $q$ is birational and for $n \geq 4$, $X_n$ has $n$ singular points, corresponding to the lines in $L_n'$.

Since $K_{\mathbb{P}^2} \sim_{\mathbb{Q}} -\frac{3}{n} L_n$, we can write

$$K_{B_P\mathbb{P}^2} \sim_{\mathbb{Q}} -\tfrac{3}{n} L_n' + \big(1 - \tfrac{6}{n}\big) E_n.$$

Thus

$$K_{X_n} = q_* K_{B_P\mathbb{P}^2} \sim_{\mathbb{Q}} \bigl(1 - \tfrac{6}{n}\bigr) q_* E_n$$

is effective for $n \geq 6$. We can further write

$$\begin{array}{rcl} q^* K_{X_n} & \sim_{\mathbb{Q}} & \bigl(1-\frac{6}{n}\bigr)\bigl[E_n + \frac{n-1}{n-2} L_n'\bigr] \\ & \sim_{\mathbb{Q}} & \frac{n-6}{n-2} \cdot \bigl[p^* \mathcal{O}_{\mathbb{P}^2}(n-1)(-E_n)\bigr]. \end{array}$$

This shows that $K_{X_n}$ is ample for $n \geq 7$. Furthermore,

$$K_{B_P\mathbb{P}^2} + L_n' \sim_{\mathbb{Q}} q^* K_{X_n} + \tfrac{4}{n-2} L_n',$$

hence $X_n$ is log canonical. □

Next we construct a deformation of $X_n$ by moving the set of intersection points $P \subset \mathbb{P}^2$ into general position.

*Claim* 50.4. Let $R \subset \mathbb{P}^2$ be $\binom{n}{2}$ points in general position and $r : B_R\mathbb{P}^2 \to \mathbb{P}^2$ the blow up with exceptional curve $E_R$. Then $r^*\mathcal{O}_{\mathbb{P}^2}(n-1)(-E_R)$ is ample for $n \geq 6$.

Proof. It is enough to find one point set $R \subset \mathbb{P}^2$ such that $r^*\mathcal{O}_{\mathbb{P}^2}(n-1)(-E_R)$ is ample. Pick general degree $n-2$ curves $C_1, C_2$ such that every member of the pencil $|C_1, C_2|$ is irreducible and choose $R \subsetneq C_1 \cap C_2$. This is possible if $(n-2)^2 > \binom{n}{2}$, which holds for $n \geq 6$.

We can then write

$$r^*\mathcal{O}_{\mathbb{P}^2}(n-1)(-E_R) \sim_{\mathbb{Q}} \tfrac{n-1}{2(n-2)}\bigl(C_1' + C_2'\bigr) + \tfrac{1}{n-2} E_R,$$

where $C_i'$ denotes the birational transform of $C_i$. Note that the $C_i'$ are members of a pencil with base points all of whose members are irreducible. Since

$$\begin{array}{rcl} \bigl(r^*\mathcal{O}_{\mathbb{P}^2}(n-1)(-E_R)\cdot E_k\bigr) & = & 1 \quad \text{and} \\ \bigl(r^*\mathcal{O}_{\mathbb{P}^2}(n-1)(-E_R)\cdot C_i'\bigr) & = & (n-1)(n-2) - \binom{n}{2}, \end{array}$$

this shows, by the Nakai-Mosihezon criterion of ampleness that $r^*\mathcal{O}_{\mathbb{P}^2}(n-1)(-E_R)$ is ample. □

Fix $n \geq 7$ and let $Q_t \subset \mathbb{P}^2$ be a flat family of $\binom{n}{2}$ points such that $Q_0 = Q$ as above and $p_t^*\mathcal{O}_{\mathbb{P}^2}(n-1)(-E_{Q_t})$ is ample for $t \neq 0$ where $p_t : B_{Q_t}\mathbb{P}^2 \to \mathbb{P}^2$ denotes the blow up.

We claim that, for $m \gg 1$, $\bigl(p_t^*\mathcal{O}_{\mathbb{P}^2}(n-1)(-E_{Q_t})\bigr)^{\otimes m}$ is generated by global sections for every $t$ and has no higher cohomologies. (In fact this holds for all $m \geq 1$, but we do not need it.) For $t \neq 0$ this follows from Serre's vanishing since then $p_t^*\mathcal{O}_{\mathbb{P}^2}(n-1)(-E_{Q_t})$ is ample. For $t = 0$ we use that, by the construction of $X_n$, $p_t^*\mathcal{O}_{\mathbb{P}^2}(n-1)(-E_{Q_t}) \cong q^* M_n$ where $M_n$ is ample on $X_n$. As we saw, $R^i q_* \mathcal{O}_{B_P\mathbb{P}^2} = 0$ for $i > 0$. Thus, for $m \gg 1$,

$$H^i\bigl(B_P\mathbb{P}^2, \bigl(p_0^*\mathcal{O}_{\mathbb{P}^2}(n-1)(-E_{Q_0})\bigr)^{\otimes m}\bigr) = H^i\bigl(X_n, M_n^{\otimes m}\bigr) = 0.$$

These imply that

$$h^0\bigl(B_{Q_t}\mathbb{P}^2, \bigl(p_t^*\mathcal{O}_{\mathbb{P}^2}(n-1)(-E_{Q_t})\bigr)^{\otimes m}\bigr)$$

is independent of $t$ for $m \gg 1$. Therefore $\{S_t : t \in \mathbb{C}\}$ is a flat family, cf. [**Har77**, III.9.9].

REMARK 51. For $n = 6$ we get a surface $S_0 = X_6$ whose canonical class is numerically trivial. One can realize $X_6$ as a limit of smooth Enriques surfaces as follows.

Let $T \to \mathbb{P}^2$ be the double cover ramified along $L_6$. It is a K3 surface with 15 double points. If we resolve the double points, the preimages of the 6 lines in $L_6$ become disjoint rational curves with self intersection $-2$. If we contract them, we get a K3 surface $Y_6$. The involution on $T/\mathbb{P}^2$ gives an involution $\tau : Y_6 \to Y_6$ which fixes only the 6 double points and such that $X_6 = Y_6/\tau$.

By the deformation theory of K3 surfaces (cf. [**BPVdV84**, Chap.VIII]) we can smooth $Y_6$ while keeping the involution. This realizes $X_6$ as a limit of smooth Enriques surfaces.

REMARK 52. While (50) is just one set of examples, they are quite typical. In fact, if $S_0$ is any projective rational surface with quotient singularities, then there is a flat family of surfaces $\{S_t\}$ such that $S_t$ is a smooth rational surface for $t \neq 0$.

To see this, take a minimal resolution $S_0' \to S_0$. Let $H_0'$ be the pullback of an ample Cartier divisor from $S_0$. Since $S_0'$ is a smooth rational surface, it is obtained from a minimal smooth rational surface by blowing up points. A general deformation of a minimal smooth rational surface is $\mathbb{P}^1 \times \mathbb{P}^1$ or $B_0\mathbb{P}^2$ (or it is $\mathbb{P}^2$). Thus we see that if $S_0$ is singular then a general deformation $S_t'$ of $S_0'$ is obtained by blowing up points in $\mathbb{P}^2$ in general position. One can see, (cf. [**dF05**, 2.4]) that every smooth rational curve on $S_t'$ with negative self-itersection is a $(-1)$-curve. Thus no exceptional curve of $S_0' \to S_0$ lifts to $S_t'$ and so, as before, we get a flat deformation $\{S_t\}$ such that $S_t \cong S_t'$ for $t \neq 0$.

EXAMPLE 53 (More rational surfaces with ample canonical class). [**Kol08**, Sec.5] Given natural numbers $a_1, a_2, a_3, a_4$, consider the surface

$$S = S(a_1, a_2, a_3, a_4) := (x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_4 + x_4^{a_4}x_1 = 0) \subset \mathbb{P}(w_1, w_2, w_3, w_4),$$

where $w_i' = a_{i+1}a_{i+2}a_{i+3} - a_{i+2}a_{i+3} + a_{i+3} - 1$ (with indices modulo 4), and $w_i = w_i'/\gcd(w_1', w_2', w_3', w_4')$.

It is easy to see that $S$ has only quotient singularities (at the 4 coordinate vertices). It is proved in [**Kol08**, Thm.39] that $S$ is rational if $\gcd(w_1', w_2', w_3', w_4') = 1$. (By [**Kol08**, 38], this happens with probability $\geq 0.75$.)

$\mathbb{P}(w_1, w_2, w_3, w_4)$ has isolated singularities iff the $\{w_i\}$ are pairwise relatively prime. (It is easy to see that for $1 \leq a_i \leq N$, this happens for at least $c \cdot N^{4-\epsilon}$ of the 4-tuples.) In this case the canonical class of $S$ is

$$K_S = \mathcal{O}_{\mathbb{P}}\bigl(\textstyle\prod a_i - 1 - \sum w_i\bigr)|_S.$$

From this it is easy to see that if $a_1, a_2, a_3, a_4 \geq 4$ then $K_S$ is ample and $K_S^2$ converges to 1 as $a_1, a_2, a_3, a_4 \to \infty$.

## 4. Examples of bad moduli problems

The aim of this section is to present some examples where some quite reasonable looking moduli problems turn out to have rather bad properties.

### Moduli of hypersurfaces.

The Chow and Hilbert functors describe families of hypersurfaces in a fixed projective space $\mathbb{P}^n$. For many purposes it is more natural to consider the moduli functor of hypersurfaces modulo isomorphisms. We consider what kind of "moduli spaces" one can obtain in various cases.

Definition 54 (Hypersurfaces modulo linear isomorphisms).
Over an algebraically closed field $k$, we consider hypersurfaces $X \subset \mathbb{P}^n_k$ where $X_1, X_2 \subset \mathbb{P}^n_k$ are considered isomorphic if there is an automorphism $\phi \in \operatorname{Aut}(\mathbb{P}^n_k)$ such that $\phi(X_1) = X_2$. (One could also consider hypersurfaces modulo isomorphisms which do not necessarily extend to an isomorphism of the ambient projective space. It is easy to see that smooth hypersurfaces can have such nonlinear isomorphisms only for $(d,n) \in \{(3,2),(4,3)\}$. A smooth cubic curve in $\mathbb{P}^2$ has an infinite automorphism group, but only finitely many extend to an automorphism of $\mathbb{P}^2$. Similarly, a smooth quartic surface in $\mathbb{P}^3$ can have an infinite automorphism group, but only finitely many extend to an automorphism of $\mathbb{P}^3$.)

Over an arbitrary base scheme $S$, we consider pairs $(X \subset P)$ where $P/S$ is a $\mathbb{P}^n$-bundle for some $n$ and $X \subset P$ is a closed subscheme, flat over $S$ such that every fiber is a hypersurface. There are two natural invariants, the dimension of $P$ and the degree of $X$. Thus for any given $n, d$ we get a functor

$$HypSur_{n,d}(S) := \left\{ \begin{array}{c} \text{Flat families } X \subset P \\ \text{such that } \dim_S P = n,\ \deg X = d, \\ \text{modulo isomorphisms over } S. \end{array} \right\}$$

One can also consider various subfunctors, for instance $HypSur^{red}_{n,d}$, $HypSur^{norm}_{n,d}$, $HypSur^{can}_{n,d}$, $HypSur^{lc}_{n,d}$, or $HypSur^{sm}_{n,d}$) where we allow only reduced (resp. normal, canonical, log canonical or smooth) hypersurfaces.

Our aim is to investigate what the "coarse moduli spaces" of these functors look like. Our conclusion will be that in many cases there can not be any scheme or algebraic space which is a coarse moduli space; any "coarse moduli space" would have to have very strange topology.

Let $HS_{n,d}$ be any subfunctor of $HypSur_{n,d}$. We can try to construct its coarse moduli space $\mathrm{HS}_{n,d}$ step by step as follows. First, by definition, the set of $k$-points of $\mathrm{HS}_{n,d}$ is $HS_{n,d}(\operatorname{Spec} k)$. We can also get a good idea about the Zariski topology of $\mathrm{HS}_{n,d}$ using various families of hypersurfaces.

For instance, we can study the Zariski closure $\bar{U}$ of a subset $U \subset HS_{n,d}(\operatorname{Spec} k)$ using the following observation:

- Assume that there is a flat family of hypersurfaces $\pi : X \to S$ and a Zariski open subset $S^0 \subset S$ such that $[X_s] \in U$ for every $s \in S^0(k)$. Then $[X_s] \in \bar{U}$ for every $s \in S(k)$.

Next we write down flat families of hypersurfaces $\pi : X \to \mathbb{A}^1$ in $HS_{n,d}$ such that for $t \neq 0$ the fibers $X_t$ are isomorphic to each other but $X_0$ is not isomorphic to them. Our family corresponds to a morphism $\tau : \mathbb{A}^1 \to \mathrm{HS}_{n,d}$ such that $\tau\big(\mathbb{A}^1 \setminus \{0\}\big) = [X_1]$ but $\tau\big(\{0\}\big) = [X_0]$. This implies that the point $[X_1]$ is not closed and its closure contains $[X_0]$.

This is not very surprising in a scheme, but note that $X_1$ itself is defined over our base field $k$, so $[X_1]$ is a $k$-point. On a $k$-scheme, $k$-points are closed. Thus we can conclude that if there is any family as above, the moduli space $\mathrm{HS}_{n,d}$ can not be a $k$-scheme or algebraic space.

The simplest way to get such families is by the following construction.

EXAMPLE 55 (Deformation to cones). Let $f(x_0,\dots,x_n)$ be a homogeneous polynomial of degree $d$ and $X := (f=0)$ the corresponding hypersurface. For some $0 \le i < n$ consider the family of hypersurfaces

$$\mathbf{X} := (f(x_0,\dots,x_i,tx_{i+1},\dots tx_n)=0) \subset \mathbb{P}^n \times \mathbb{A}^1_t \tag{55.1}$$

with projection $\pi : \mathbf{X} \to \mathbb{A}^1_t$. If $t \neq 0$ then the substitution

$$x_j \mapsto x_j \quad \text{for } j \le i, \text{ and} \quad x_j \mapsto t^{-1}x_j \quad \text{for } j > i$$

shows that the fiber $X_t$ is isomorphic to $X$. If $t=0$ then we get the cone over $X \cap (x_{i+1} = \dots = x_n = 0)$:

$$X_0 = (f(x_0,\dots,x_i,0,\dots,0)=0) \subset \mathbb{P}^n.$$

Already these simple deformations show that various moduli spaces of hypersurfaces have very few closed points.

COROLLARY 56. *The sole closed point of* $\mathrm{HypSur}_{d,n}$ *is* $[(x_0^d=0)]$.

Proof. Take any $X=(f=0)\subset \mathbb{P}^n$. After a general change of coordinates, we can assume that $x_0^d$ appears in $f$ with nonzero coefficient. For $i=0$ consider the family (55.1).

Then $X_0 = (x_0^d=0)$, hence $[X]$ can not be closed point unless $X \cong X_0$. It is quite easy to see that if $X \to S$ is a flat family of hypersurfaces whose generic fiber is a $d$-fold plane, then every fiber is a $d$-fold plane. This shows that $[(x_0^d=0)]$ is a closed point. □

COROLLARY 57. *The only closed points of* $\mathrm{HypSur}^{red}_{d,n}$ *are* $[(f(x_0,x_1)=0)]$ *where* $f$ *has no multiple roots.*

Proof. If $X$ is a reduced hypersurface of degree $d$, there is a line that intersects it in $d$ distinct points. We can assume that this is the line $(x_2=\dots=x_n=0)$. For $i=1$ consider the family (55.1).

Then $X_0=(f(x_0,x_1,0,\dots,0)=0)$ where $f(x_0,x_1)$ has $d$ distinct roots. Since $X_0$ is reduced, we see that none of the other hypersurfaces correspond to closed points.

It is not obvious that the points corresponding to $(f(x_0,x_1,0,\dots,0)=0)$ are closed, but this can be easily established by studying the moduli of $d$ points in $\mathbb{P}^1$; see [**Mum65**, Chap.3] or [**Dol03**, Sec.10.2]. □

A similar argument establishes the normal case:

COROLLARY 58. *The only closed points of* $\mathrm{HypSur}^{norm}_{d,n}$ *are* $[(f(x_0,x_1,x_2)=0)]$ *where* $(f(x_0,x_1,x_2)=0)\subset\mathbb{P}^2$ *is a smooth curve.* □

In the above examples the trouble comes from cones. Cones can be normal, but they are very singular by other measures; they have a singular point whose multiplicty equals the degree of the variety. So one could hope that high multiplicity points cause the problems. This is true to some extent as the next theorems and examples show. For proofs see [**Mum65**, Sec.4.2], [**Dol03**, Sec.10.1], (**??**) and (**??**).

THEOREM 59. *Each of the following functors has a coarse moduli space which is a quasi projective variety.*

(1) *The functor of smooth hypersurfaces* $HypSur^{sm}_{d,n}$.

(2) *For* $d \geq n+1$, *the functor* $HypSur^{can}_{d,n}$ *of hypersurfaces with canonical singularities.*
(3) *For* $d > n+1$, *the functor* $HypSur^{lc}_{d,n}$ *of hypersurfaces with log canonical singularities.*
(4) *For* $d > n+1$, *the functor* $HypSur^{low-mult}_{d,n}$ *of those hypersurfaces that have only points of multiplicity* $< \frac{d}{n+1}$.

EXAMPLE 60. Consider the family of degree $2d$ hypersurfaces

$$\bigl((x_0^d + t^{2d}x_1^d)x_1^d + x_2^{2d} + \cdots + x_n^{2d} = 0\bigr) \subset \mathbb{P}^n \times \mathbb{A}^1_t.$$

For $t \neq 0$ the substitution

$$(x_0{:}x_1{:}x_2{:}\cdots x_n) \mapsto (tx_0{:}t^{-1}x_1{:}x_2{:}\cdots x_n).$$

transforms the equation of $X_t$ to

$$X := \bigl((x_0^d + x_1^d)x_1^d + x_2^{2d} + \cdots + x_n^{2d} = 0\bigr) \subset \mathbb{P}^n.$$

$X$ has a single singular point which is at $(1{:}0{:}\cdots{:}0)$ and has multiplicity $d$.

For $t = 0$ we obtain the hypersurface

$$X_0 := \bigl(x_0^d x_1^d + x_2^{2d} + \cdots + x_n^{2d} = 0\bigr).$$

$X_0$ has 2 singular points of multiplicity $d$, hence it is not isomorphic to $X$.

Thus we conclude that $[X]$ is not a closed point of the “moduli space” of those hypersurfaces of degree $2d$ that have only points of multiplicity $\leq d$.

EXAMPLE 61. Consider the family of degree $d$ smooth hypersurfaces

$$\bigl((x_0^{d-1} + t^{(d-1)^2}x_1^{d-1})x_1 + x_2^d + \cdots + x_n^d = 0\bigr) \subset \mathbb{P}^n \times \mathbb{A}^1_t.$$

For $t \neq 0$ the substitution

$$(x_0{:}x_1{:}x_2{:}\cdots x_n) \mapsto (tx_0{:}t^{1-d}x_1{:}x_2{:}\cdots x_n).$$

transforms the equation of $X$ to

$$X := \bigl((x_0^{d-1} + x_1^{d-1})x_1 + x_2^d + \cdots + x_n^d = 0\bigr) \subset \mathbb{P}^n,$$

which is a smooth hypersurface. For $t = 0$ we obtain

$$X_0 := \bigl(x_0^{d-1}x_1 + x_2^d + \cdots + x_n^d = 0\bigr),$$

which has a unique singular point at $(0{:}1{:}0{:}\cdots{:}0)$ of multiplicity $d-1$.

This is especially interesting when $d \leq n$ since in this case $X_0$ has canonical singularities (35, **??**).

Thus we see that for $d \leq n$, the functor $HypSur^{can}_{d,n}$ parametrizing hypersurfaces with canonical singularities does not have a coarse moduli scheme. By contrast, by (59), for $d > n$ the coarse moduli scheme $\mathrm{HypSur}^{can}_{d,n}$ exists and is quasi projective.

Let us end our study of hypersurfaces with a different type of example. This shows that the moduli problem for hypersurfaces usually includes smooth limits that are not hypersurfaces. These pose no problem for the general theory, but they show that it is not always easy to see what schemes one needs to include in a moduli space.

EXAMPLE 62 (Smooth limits of hypersurfaces). [**Mor75**]

Fix integers $a, b > 1$ and $n \geq 2$. We construct a family of smooth $n$-folds $X_t$ such that $X_t$ is a smooth hypersurface of degree $ab$ for $t \neq 0$ and $X_0$ is not isomorphic to a smooth hypersurface.

It is not known if similar examples exist for $n \geq 3$ and $\deg X$ a prime number.

Fix $\mathbb{P}(1^{n+1}, a)$ with coordinates $x_0, \dots, x_n, z$. Let $f_a, g_{ab}$ be general homogeneous forms of degree $a$ (resp. $ab$) in $x_0, \dots, x_n$. Consider the family of complete intersections

$$X_t := \bigl(tz - f_a(x_0, \dots, x_n) = z^b - g_{ab}(x_0, \dots, x_n) = 0\bigr) \subset \mathbb{P}(1^{n+1}, a).$$

For $t \neq 0$ we can eliminate $z$ to obtain a degree $ab$ smooth hypersurface

$$X_t \cong \bigl(f_a^b(x_0, \dots, x_n) = g_{ab}(x_0, \dots, x_n)\bigr) \subset \mathbb{P}^{n+1}.$$

For $t = 0$ we see that $\mathcal{O}_{X_0}(1)$ is not very ample but realizes $X_0$ as a $b$-fold cyclic cover

$$X_0 \to \bigl(f_a(x_0, \dots, x_n) = 0\bigr) \subset \mathbb{P}^{n+1}$$

of a degree $a$ smooth hypersurface. In particular, $X_0$ is not isomorphic to a smooth hypersurface.

**Moduli of genus 2 curves.**

Here we consider what happens if we try to replace the moduli functor of genus 2 stable curves with some other variant that uses only irreducible curves.

DEFINITION 63. Let $\mathcal{M}_2^{irr}$ be the moduli functor of flat families of irreducible curves of arithmetic genus 2 which are either

(1) smooth,
(2) nodal,
(3) rational with 2 cusps or
(4) rational with a triple point whose complete local ring is $\mathbb{C}[[x, y, z]]/(xy, yz, zx)$.

The aim of this subsection is to prove the following:

PROPOSITION 64. (1) *The coarse moduli space $M_2^{irr}$ exists and equals the GIT quotient $S^6\mathbb{P}^1 /\!/ \operatorname{Aut}(\mathbb{P}^1)$ (cf.* [**Mum65**, Chap.3] *or* [**Dol03**, Sec.10.2]*).*
(2) *$\mathcal{M}_2^{irr}$ is a very bad moduli functor.*

Proof. A smooth curve of genus 2 can be uniquely written as a double cover $\tau : C \to \mathbb{P}^1$, ramified at 6 distinct points $p_1, \dots, p_6 \in \mathbb{P}^1$, up to automorphisms of $\mathbb{P}^1$. Thus, $M_2$ is isomorphic to the space of 6 distinct points in $\mathbb{P}^1$, modulo the action of $\operatorname{Aut}(\mathbb{P}^1)$. If some of the 6 points coincide, we get singular curves as double covers.

It is easy to see the following (cf. [**Mum65**, Chap.3], [**Dol03**, Sec.10.2]).

(3) A point set is semi-stable iff it does not contain any point with multiplicity $\geq 4$. Equivalently, if the corresponding genus 2 cover has only nodes and cusps.
(4) The properly semi stable point sets are of the form $3p_1 + p_2 + p_3 + p_4$ where the $p_2, p_3, p_4$ are different from $p_1$ but may coincide with each other. Equivalently, the corresponding genus 2 cover has cusp(s).

(5) Point sets of the form $2p_1 + 2p_2 + 2p_3$ where the $p_1, p_2, p_3$ are different from each other give the only semistable case when the double cover is reducible. It has two smooth rational components meeting each other at 3 points.

In the properly semi stable case, generically the double cover is an elliptic curve with a cusp over $p_1$. As a special case we can have $3p_1+3p_2$, giving as double cover a rational curve with 2 cusps. Note that the curves of this type have a 1-dimensional moduli (the crosss ratio of the points $p_1, p_2, p_3, p_4$ or the $j$-ivarint of the elliptic curve), but they all correspond to the same point in $S^6\mathbb{P}^1/\!/\operatorname{Aut}(\mathbb{P}^1)$. (See (68) for an explicit construction.) Our definition (63) aims to remedy this non-uniqueness by always taking the most degenerate case; a rational curve with 2 cusps (63.3).

In case (5), write the reducible double cover as $C = C_1 + C_2$. The only obvious candidate to get an irreducible curve is to contract one of the two components $C_i$. We get an irreducible rational curve; denote it by $C'_j$ where $j = 3-i$. Note that $C'_j$ has one singular point which is analytically isomorphic to the 3 coordinate axes in $\mathbb{A}^3$. The resulting singular rational curves $C'_j$ are isomorphic to each other. These are listed in (63.4).

Let $p : X \to S$ be any flat family of irreducible, reduced curves of arithmetic genus 2. The trace map (cf. [**BPVdV84**, III.12.2]) shows that $R^1p_*\omega_{X/S} \cong \mathcal{O}_S$. Thus, by cohomology and base change (cf. [**Har77**, III.12.11]), $p_*\omega_{X/S}$ is locally free of rank 2. Set $P := \mathbb{P}_S\bigl(p_*\omega_{X/S}\bigr)$. Then $P$ is a $\mathbb{P}^1$-bundle over $S$ and we have a rational map $\pi : X \to P$. If $X_s$ has only nodes and cusps, then $\omega_{X_s}$ is locally free and generated by global sections, thus $\pi$ is a morphism along $X_s$.

If $X_s$ is as in (63.4), then $\omega_{X_s}$ is not locally free and $\pi$ is not defined at the singular point. $\pi|_{X_s}$ is birational and the 3 local brances of $X_s$ at the singular point correspond to 3 points on $\mathbb{P}\bigl(H^0(X_s, \omega_{X_s})\bigr)$.

The branch divisor of $\pi$ is a degree 6 multisection of $P \to S$ all of whose fibers are stable point sets. Thus we have a natural transformation

$$\mathcal{M}_2^{irr}(*) \to \operatorname{Mor}\bigl(*, S^6\mathbb{P}^1/\!/\operatorname{Aut}(\mathbb{P}^1)\bigr).$$

We have already seen that we get a bijection

$$\mathcal{M}_2^{irr}(\mathbb{C}) \cong \bigl(S^6\mathbb{P}^1/\!/\operatorname{Aut}(\mathbb{P}^1)\bigr)(\mathbb{C}).$$

Since $S^6\mathbb{P}^1/\!/\operatorname{Aut}(\mathbb{P}^1)$ is normal, we conclude that it is the coarse moduli space. This completes the proof of (64.1).

The assertion (64.2) is more a personal opinion. There are 3 main things "wrong" with the functor $\mathcal{M}_2^{irr}(*)$. Let us consider them one at a time.

64.6 (Stable reduction questions).

At the set-theoretic level, we have our moduli space $M_2^{irr} = S^6\mathbb{P}^1/\!/\operatorname{Aut}(\mathbb{P}^1)$, but what about at the level of familes?

The first indications are good. Let $\pi_B : S_B \to B$ be a stable family of genus 2 curves. Assume that no fiber has 2 smooth rational components. Let $b_i \in B$ be the points corresponding to fibers with 2 components of arithmetic genus 1. Let $p : A \to B$ be a double cover ramified at the points $b_i$. Consider the pull-back family $\pi_A : S_A \to A$. Set $a_i = p^{-1}(b_i)$ and let $s_i \in \pi_A^{-1}(a_i)$ be the separating node. Since we took a ramified double cover, each $s_i \in S_A$ is a double point. Thus if we blow up every $s_i$, the exceptional curve appears in the fiber with multiplicity 1. We can now contract the birational transforms of the elliptic curves to get a family where all these reducibe fibers are replaced by a rational curve with 2 cusps. We have proved the following analog of (17):

*Lemma 64.6.1.* Let $\pi : S \to B$ be a stable family of genus 2 curves such that no fiber has 2 smooth rational components. Then, after a suitable double cover $A \to B$, the pull-back $S \times_B A$ is birational to another family where each reducible fiber is replaced by a rational curve with 2 cusps. $\square$

This solved our problem for 1-parameter families, but, as it turns out, we have problems over higher dimensional bases. In particular, there is no universal family over any base scheme $Y$ that finitely dominates $S^6\mathbb{P}^1 /\!/ \operatorname{Aut}(\mathbb{P}^1)$, not even locally in any neighborhood of the properly semi stable point.

*Proposition 64.6.2.* Let $\pi : X \to Y$ be a proper, flat family of curves of arithmetic genus 2. Assume that $X_0$ is a rational curve with 2 cusps for some $0 \in Y$ and that $\dim_0 Y \geq 3$. Then there is a curve $0 \in C \subset Y$ such that $X_y$ has a cusp for every $y \in C$.

Proof. This follows from the deformation theory of the cusp. (See [**Art76**] or [**AGZV85**] for good introductions.) We need that every flat deformation of a cusp is induced by pull-back from the 2-parameter family

$$\begin{array}{ccc} (y^2 = x^3 + ux + v) & \subset & \mathbb{A}^2_{xy} \times \mathbb{A}^2_{uv} \\ p \downarrow & & \downarrow \\ \mathbb{A}^2_{uv} & = & \mathbb{A}^2_{uv}. \end{array}$$

Thus our family $\pi$ gives an analytic morphism $\tau : Y \to \mathbb{A}^2_{uv}$ (defined in some neighborhood of $0 \in Y$) and $C = \tau^{-1}(0,0)$ is the required curve. $\square$

64.7 (Failure of local closedness).

Following (64.6.2), consider the universal deformation of the rational curve with 2 cusps. This is given as

$$\begin{array}{ccc} \big(z^2 = (x^3 + uxy^2 + vy^3)(y^3 + syx^2 + tx^3)\big) & \subset & \mathbb{P}^2(1,1,3) \times \mathbb{A}^4_{uvst} \\ p \downarrow & & \downarrow \\ \mathbb{A}^4_{uvst} & = & \mathbb{A}^4_{uvst}. \end{array}$$

Let us work in a neighbourhood of $(0,0,0,0) \in \mathbb{A}^4$ where the 2 factors $x^3+uxy^2+vy^3$ and $y^3 + syx^2 + tx^3$ have no common roots. There are 3 types of fibers of $p$.

i) $p^{-1}(0,0,0,0)$ is a rational curve with 2 cusps.
ii) $p^{-1}(a,b,0,0)$ and $p^{-1}(0,0,a,b)$ are irreducible with exactly 1 cusp if $(a,b) \neq (0,0)$.
iii) $p^{-1}(a,b,c,d)$ is irreducible with at worst nodes otherwise.

Thus the curves that we allow in our moduli functor $\mathcal{M}_2^{irr}$ do not form a locally closed family. Even worse, the subfamily

$$\begin{array}{ccc} \big(z^2 = (x^3 + uxy^2 + vy^3)y^3\big) & \subset & \mathbb{P}^2(1,1,3) \times \operatorname{Spec} k[[u,v]] \\ p \downarrow & & \downarrow \\ \operatorname{Spec} k[[u,v]] & = & \operatorname{Spec} k[[u,v]]. \end{array}$$

is not allowed in our moduli functor $\mathcal{M}_2^{irr}$, but the family

$$\begin{array}{ccc} \big(z^2 = (x^3 + uxy^2 + vy^3)(y^3 + u^n yx^2 + v^n x^3)\big) & \subset & \mathbb{P}^2(1,1,3) \times \operatorname{Spec} k[[u,v]] \\ p \downarrow & & \downarrow \\ \operatorname{Spec} k[[u,v]] & = & \operatorname{Spec} k[[u,v]]. \end{array}$$

is allowed. Over $\operatorname{Spec} k[u,v]/(u^n,v^n)$ the two families are isomorphic. Since deformation theory is essentially a study of families over Artin rings, this means that the usual methods can not be applied to understand the functor $\mathcal{M}_2^{irr}$.

64.8 (Unusual non-separatedness).

A quite different type of problem arises at the curve corresponding to $2p_1 + 2p_2 + 2p_3$.

Write the double cover as $C = C_1 + C_2$. As before, if we contract one of the two components $C_i$, we get an irreducible rational curve $C'_j$ where $j = 3 - i$ as in (63.4).

Since the curves $C'_1$ and $C'_2$ are isomorphic, from the set-theoretic point of view this is a good solution. However, as in (28), something strange happens with families. Let $p : S \to \mathbb{A}^1$ be a family of stable curves whose central fiber $S_0 := p^{-1}(0)$ is isomorphic to $C = C_1 + C_2$. We have two ways to construct a family with an irreducible central fiber: contract either of the two irreducible components $C_i$. Thus we get two families

$$S \xrightarrow{\pi_i} S_i \xrightarrow{p_i} \mathbb{A}^1 \quad \text{with } p_i^{-1}(0) \cong C'_{3-i}.$$

Over $\mathbb{A}^1 \setminus \{0\}$ the two families are naturally isomorphic to $S \to \mathbb{A}^1$, hence to each other, yet this isomorphism does not extend to an isomorphism of $S_1$ and $S_2$. Indeed, the closure of the graph of the resulting birational map is given by the image $(\pi_1, \pi_2) : S \to S_1 \times_{\mathbb{A}^1} S_2$. Thus the corresponding moduli functor is not separated.

We claimed above that, by contrast, the coarse moduli space is $\bar{M}_2$, hence separated. A closer study reveals the source of this discrepancy: we have been thinking of schemes instead of algebraic spaces. The occurrence of such problems in moduli theory was first observed by [**Art74**]. The aim of the next paragraph is to show how such examples arise.

64.9 (Bug-eyed covers). [**Art74, Kol92**]

A non-separated scheme always has "extra" points. The typical example is when we take two copies of a scheme $X \times \{i\}$ for $i = 0, 1$, an open dense subscheme $U \subsetneq X$ and glue $U \times \{0\}$ to $U \times \{1\}$ to get $X \amalg_U X$. The non-separatedness arises from having 2 points in $X \amalg_U X$ for each point in $X \setminus U$.

By contrast, an algebraic space can be non-separated by having no extra points, only extra tangent directions. The simplest example is the following.

On $\mathbb{A}^1_t$ consider two equivalence relations. The first is $R_1 \rightrightarrows \mathbb{A}^1$ given by

$$(t_1 = t_2) \cup (t_1 = -t_2) \subset \mathbb{A}^1_{t_1} \times \mathbb{A}^1_{t_2}.$$

Then $\mathbb{A}^1_t / R_1 \cong \mathbb{A}^1_u$ where $u = t^2$.

The second is the étale equivalence relation $R_2 \rightrightarrows \mathbb{A}^1$ given by

$$\mathbb{A}^1 \xrightarrow{(1,1)} \mathbb{A}^1 \times \mathbb{A}^1 \quad \text{and} \quad \mathbb{A}^1 \setminus \{0\} \xrightarrow{(1,-1)} \mathbb{A}^1 \times \mathbb{A}^1.$$

(Note that we take the disconnected union of the two components, instead of their union as 2 lines in $\mathbb{A}^1 \times \mathbb{A}^1$ intersecting at the origin.)

One can also obtain $\mathbb{A}^1_t / R_2$ by taking the quotient of the nonseparated scheme $\mathbb{A}^1 \amalg_{\mathbb{A}^1 \setminus \{0\}} \mathbb{A}^1$ by the (fixed point free) involution that interchanges $(t, 0)$ and $(-t, 1)$.

The morphism $\mathbb{A}^1_t \to \mathbb{A}^1_t / R_2$ is étale, thus $\mathbb{A}^1_t / R_2 \neq \mathbb{A}^1_t / R_1$. Nonetheless, there is natural morphism

$$\mathbb{A}^1_t / R_2 \to \mathbb{A}^1_t / R_1$$

which is one-to-one and onto on closed points. The difference between the 2 spaces is seen by the tangent vectors at the origin. The tangent space of $\mathbb{A}^1_t/R_2$ at the origin is spanned by $\partial/\partial t$ while the tangent space of $\mathbb{A}^1_t/R_1$ at the origin is spanned by

$$\frac{\partial}{\partial u} = \frac{1}{2t}\cdot\frac{\partial}{\partial t}.$$

**Other compactifications of $M_g$.**

While $M_g$ has many compactifications besides $\bar{M}_g$, it is only recently that a systematic search begun for other geometrically meaningful examples. The papers [**Sch91, HH08, Smy08**] contain many examples.

Our attempt to replace the moduli functor of stable curves of genus 2 with another one that parametrizes only irreducible curves was not successful, but the problems seemed to have arisen from the symmetry that forced us to make artificial choices.

We try to avoid such choices for other values of the genus using the following observation.

Let $\pi : S \to B$ be a flat family of curves with smooth general fiber and reduced special fibers. If $C_b := \pi^{-1}(b)$ is a singular fiber and $C_{bi}$ are the irreducible components of its normalization then

$$\begin{array}{rcl}\sum_i h^1\big(C_{bi}, \mathcal{O}_{C_{bi}}\big) & \leq & h^1\big(C_b, \mathcal{O}_{C_b}\big) = 1 - \chi\big(C_b, \mathcal{O}_{C_b}\big) = 1 - \chi\big(C_{gen}, \mathcal{O}_{C_{gen}}\big) \\ & = & h^1\big(C_{gen}, \mathcal{O}_{C_{gen}}\big),\end{array}$$

where $C_{gen}$ is the general smooth fiber. In particular, there can be at most 1 irreducible component with geometric genus $> \frac{1}{2}g(C_{gen})$.

From this it is easy prove the following:

Let $B$ be a smooth curve and $S^0 \to B^0$ a smooth family of genus $g$ curves over an open subset of $B$. Then there is at most one normal surface $S \to B$ extending $S^0$ such that every fiber of $S \to B$ is irreducible and of geometric genus $> \frac{1}{2}g(C_{gen})$.

Moreover, if $S_{stab} \to B$ is a stable family extending $S^0$ end every fiber of $S_{stab} \to B$ contains an irreducible curve of geometric genus $> \frac{1}{2}g(C_{gen})$, then we obtain $S$ from $S_{stab}$ by contracting all connected components of curves of geometric genus $< \frac{1}{2}g(C_{gen})$ that are contained in the fibers.

In fact, this way we obtain a partial compactification $M_g \subset M'_g$ such that

(1) $M'_g$ parametrizes smoothable irreducible curves of arithmetic genus $g$ and geometric genus $> \frac{1}{2}g$.
(2) Let $M_g \subset M''_g \subset \bar{M}_g$ be the largest open subset parametrizing curves that contain an irreducible component of geometric genus $> \frac{1}{2}g$. Then there is a nartural morphism $M''_g \to M'_g$.

So far so good, but, as we see next, we can not extend $M'_g$ to a compactification in a geometrically meaningful way. This happens for every $g \geq 3$; the following example with $g = 13$ is given by simple equations.

This illustrates a general pattern: one can easily propose partial compactifications that work well for some families but lead to contradictions for some others.

Example 65. Consider the surface $F := \big(x^8 + y^8 + z^8 = u^2\big) \subset \mathbb{P}^3(1,1,1,4)$ and on it the curve $C := F \cap (xyz = 0)$. $C$ has 3 irreducible components $C_x = (x = 0), C_y = (y = 0), C_z = (z = 0)$ which are smooth curves of genus 3. $C$ itself has arithmetic genus 13.

We work with a 3-parameter family of deformations

$$T := \bigl(xyz - ux^3 - vy^3 - wz^3 = 0\bigr) \subset F \times \mathbb{A}^3_{uvw}. \tag{65.1}$$

For general $uvw \neq 0$ the fiber of the projection $\pi : T \to \mathbb{A}^3$ is a smooth curve of genus 13. If one of the $u, v, w$ is zero, then generically we get a curve with 2 nodes hence with geometric genus 11.

If two of the coordinates are zero, say $v = w = 0$, then we have a family

$$T_x := \bigl(x(yz - ux^2) = 0\bigr) \subset F \times \mathbb{A}^1_u.$$

For $u \neq 0$, the fiber $C_{u,0,0}$ has 2 irreducible components. One is $C_x = (x = 0)$, the other is $(yz - tx^2 = 0)$ which is a smooth genus 7 curve.

Thus the proposed rule says that we should contract $C_x \subset C_{u,0,0}$.

Similarly, by working over the $v$ and the $w$-axes, the rule tells us to contract $C_y \subset C_{0,v,0}$ for $v \neq 0$ and $C_z \subset C_{0,0,w}$ for $w \neq 0$.

It is easy to see that over $\mathbb{A}^3 \setminus \{(0,0,0)\}$ these contractions can be performed (at least among algebraic spaces). Thus we obtain

$$\begin{array}{ccc} T \setminus \{\pi^{-1}(0,0,0)\} & \stackrel{p_0}{\to} & T_0^* \\ \pi \downarrow & & \pi_0^* \downarrow \\ \mathbb{A}^3 \setminus \{(0,0,0)\} & = & \mathbb{A}^3 \setminus \{(0,0,0)\} \end{array} \tag{65.2}$$

where $\pi_0^*$ is flat with irreducible fibers.

*Claim 65.3.* There is no proper family of curves $\pi^* : T^* \to \mathbb{A}^3$ that extends $\pi_0^*$. (We do not require $\pi^*$ to be flat.)

Proof. Assume to the contrary that $\pi^* : T^* \to \mathbb{A}^3$ exists and let

$$\Gamma \subset T \times_{\mathbb{A}^3} T^*$$

be the closure of the graph of $p_0$. Since $p_0$ is a morphism on $T \setminus \{\pi^{-1}(0,0,0)\}$, we see that the first projection $\pi_1 : \Gamma \to T$ is an isomorphism away from $\pi^{-1}(0,0,0)$. Since $T \times_{\mathbb{A}^3} T^* \to \mathbb{A}^3$ has 2-dimensional fibers, we conclude that $\dim \pi_1^{-1}\bigl(\pi^{-1}(0,0,0)\bigr) \leq 2$. $T$ is, however, a smooth 4-fold, hence the exceptional set of any birational map to $T$ has pure dimension 3. Thus $\Gamma \cong T$ and so $p_0$ extend to a morphism $p : T \to T^*$.

Now we see that the rule lands us in a contradiction over the origin $(0,0,0)$. Here all 3 components $C_x, C_y, C_z \subset C_{0,0,0} = C$ should be contracted. This is impossible to do since this would give that the central fiber of $T^* \to \mathbb{A}^3$ is a point.

**Mild failures of local closedness.**

Here are 2 examples of moduli functors that are not locally closed (21.0) yet this does not cause any problems.

EXAMPLE 66. Let $S \subset \mathbb{P}^3$ be a smooth surface of degree 4 with infinite automorphism group (**??**). We claim that $Isotriv_S(*)$, defined in (73), is not locally closed.

Let $\mathbf{S} \to W$ be the universal family of smooth degree 4 surfaces in $\mathbb{P}^3$. The isomorphisms classes of the pairs $\bigl(S, \mathcal{O}_S(1)\bigr)$ correspond to the $\mathrm{Aut}(\mathbb{P}^3)$-orbits in $W$. We see below that the fibers isomorphic to $S$ form countably many $\mathrm{Aut}(\mathbb{P}^3)$-orbits. Thus $Isotriv_S(*)$ is not locally closed.

For any $g \in \mathrm{Aut}\, S$, $g^*\mathcal{O}_S(1)$ gives another embedding of $S$ into $\mathbb{P}^3$. Two such embedding are projectively equivalent iff $g^*\mathcal{O}_S(1) \cong \mathcal{O}_S(1)$, that is, when $g \in \mathrm{Aut}\bigl(S, \mathcal{O}_S(1)\bigr)$. The latter can be viewed as the group of automorphisms of $\mathbb{P}^3$

that map $S$ to itself. Thus $\operatorname{Aut}\bigl(S,\mathcal{O}_S(1)\bigr)$ is a closed subscheme of the algebraic variety $\operatorname{Aut}(\mathbb{P}^3)\cong PGL_4$. Since $\operatorname{Aut} S$ is discrete, this implies that $\operatorname{Aut}\bigl(S,\mathcal{O}_S(1)\bigr)$ is finite. Hence the fibers of $\mathbf{S}\to W$ that are isomorphic to $S$ lie over countably many $\operatorname{Aut}(\mathbb{P}^3)$-orbits, corresponding to $\operatorname{Aut} S/\operatorname{Aut}\bigl(S,\mathcal{O}_S(1)\bigr)$.

Example 67. We construct a smooth, proper family of surfaces $X\to C$ over a smooth curve such that

(1) every fiber has nef canonical class,
(2) the generic fiber has ample canonical class,
(3) $X\to C$ is locally projective but
(4) $X\to C$ is not projective.

Start with a general hypersurface $Y\subset\mathbb{P}^4$ of degree $d\geq 5$ that contains a 2-plane $L$. It is easy to see that $Y$ has $(d-1)^2$ ordinary double points as its singularities and a general hyperplane containing $L$ intersects $Y$ in $L+S$ where $S$ is also smooth. It is harder to prove that the class group of $Y$ is generated by $L$ and the hyperplane class $H$ [**?**].

Each ordinary double point can be resolved either by blowing up $L$ or by blowing up $S$ (39). Either of these results in a projective variety, but now we mix these up.

Partition the set of ordinary double points into two nonempty subsets $D_1, D_2$. Let $Y_1 := B_L(Y\setminus D_2)$ and $Y_2 := B_S(Y\setminus D_1)$. Both of these contain $Y\setminus(D_1+D_2)$ as an open subset. By gluing them together, we get a proper variety $Y^*$. We claim that $Y^*$ is not projective.

Indeed, let $E_i\subset Y^*$ be an exceptional curve mapping to a node in $D_i$. Let $L^*\subset Y^*$ (resp. $H^*\subset Y^*$) denote the birational transforms of $L$ (resp. $H$). Then, by (39), $L^*\cdot E_1=-1$, $L^*\cdot E_2=1$ and $H^*\cdot E_i=0$. Thus no linear combination $aL^*+bH^*$ has positive degree on both $E_1$ and $E_2$. Since $\operatorname{Pic} Y^*$ is generated by $L^*$ and $H^*$, this implies that there is no ample divisor on $Y^*$. Moreover, this also shows that if $X^*\to Y^*$ is a proper birational morphism that is an isomorphism near $E_1+E_2$ and $X\subset X^*$ is an open set that contains $E_1+E_2$, then $X$ is not quasi projective.

It is now easy to construct a family of surfaces as required. Let $H_1,H_2\subset\mathbb{P}^4$ be general hyperplanes and $Y' := B_{H_1\cap H_2\cap Y}Y$ the blow up. The pencil $|H_1,H_2|$ defines a morphism $f' : Y'\to\mathbb{P}^1$. Since the $H_i$ are general, we may assume that there are finite sets $B_0,B_1,B_2\subset\mathbb{P}^1$ such that the following holds

(4) for $b\notin\cup B_i$, the fiber $Y'_b$ is smooth,
(5) for $b\in B_1$ (resp. $b\in B_2$), the fiber $Y'_b$ has a single node which is at one of the points of $D_1$ (resp. $D_2$).

Set $X^* : B_{H_1\cap H_2\cap Y}Y^*$ and $f^* : X^*\to Y^*\to\mathbb{P}^1$. Finally let $C:=\mathbb{P}^1\setminus B_0$ and $X:=(f^*)^{-1}(C)\subset X^*$ with $f:=f^*|_X$.

By the computations of (39), $f : X\to C$ is smooth. By construction, $f$ is projective over $C\setminus B_i$ for $i=1,2$ but $X$ itself is not quasi projective.

**Other non-separated examples.**

As we noted in (27), by a result of [**MM64**], nonseparated examples in smooth families tend to involve birationally ruled fibers. Next we consider some examples of nonseparatedness where the varieties are not even uniruled. The bad behavior is due to the singularities and not to the global structure.

EXAMPLE 68 (Double covers of $\mathbb{P}^1$). Let $f(x,y)$ and $g(x,y)$ be two cubic forms without multiple roots, neither divisible by $x$ or $y$. Consider 2 families of curves

$$\begin{array}{l} S_1 := \big(f(x_1,y_1)g(t^2x_1,y_1) = z_1^2\big) \subset \mathbb{P}(1,1,3)\times\mathbb{A}^1 \quad \text{and} \\ S_2 := \big(f(x_2,t^2y_2)g(x_2,y_2) = z_2^2\big) \subset \mathbb{P}(1,1,3)\times\mathbb{A}^1. \end{array}$$

Note that $\omega_{S_i/\mathbb{A}^1}$ is relatively ample and the general fiber of $\pi_1 : S_i \to \mathbb{A}^1$ is a smooth curve of genus 2.

The central fibers are $\big(f(x_1,y_1)g(0,y_1) = z_1^2\big)$ resp. $\big(f(x_2,0)g(x_2,y_2) = z_2^2\big)$. By assumption $g(0,y_1) = a_1y_1^3$ and $f(x_2,0) = a_2x_2^3$ where the $a_i \neq 0$. Setting $z_1 = a_1^{1/2}w_1y_1$ and $z_2 = a_2^{1/2}w_2x_2$ gives the normalizations. Hence the central fibers are elliptic curves with a cusp. Their normalization is isomorphic to $\big(f(x_1,y_1)y_1 = w_1^2\big)$ resp. $\big(x_2g(x_2,y_2) = w_2^2\big)$, and these are, in general, not isomorphic to each other.

This also shows that along the central fibers, the only singularities are at $(1{:}0{:}0;0)$ and at $(0{:}1{:}0;0)$. Up to canceling units, the local equations are $g(t^2,y_1) = z_1^2$ resp. $f(x_2,t^2) = z_2^2$. (These are simple elliptic with minimal resolution a single smooth elliptic curve of self intersection $-1$.) Hence the $S_i$ are normal surfaces, each having 1 simple elliptic singular point.

Finally, the substitution

$$(x_1 : y_1 : z_1; t) = (x_2 : t^2y_2 : t^3z_2; t)$$

transforms $f(x_1,y_1)g(t^2x_1,y_1) - z_1^2$ into

$$f(x_2,t^2y_2)g(t^2x_2,t^2y_2) - t^6z_2^2 = t^6\big(f(x_2,t^2y_2)g(x_2,y_2) - z_2^2\big),$$

thus the two families are isomorphic over $\mathbb{A}^1 \setminus \{0\}$

EXAMPLE 69 (Limits of double covers of $\mathbb{P}^3$). Let $a_i(x,y)$ and $b_i(u,v)$ be homogeneous forms of degree $n$. Consider 2 families of threefolds

$$\begin{array}{l} X_1 := \big(a_1(x,y)+t^{2n}b_1(u,v)\big)\big(a_2(x,y)+b_2(u,v)\big) = w^2 \subset \mathbb{P}(1^4,n)\times\mathbb{A}^1, \quad \text{and} \\ X_2 := \big(a_1(x,y)+b_1(u,v)\big)\big(t^{2n}a_2(x,y)+b_2(u,v)\big) = w^2 \subset \mathbb{P}(1^4,n)\times\mathbb{A}^1. \end{array}$$

*Claim.*

(1) For general $a_i, b_i$, the central fibers of the $X_i \to \mathbb{A}^1$ are normal. Their singularities are canonical iff $n \le 3$, and log-canonical iff $n \le 4$.
(2) The central fibers are of general type if $n \ge 7$, have Kodaira dimension 1 if $n = 5, 6$ and are rationally connected if $n \le 4$.
(3) The general fibers of $X_i \to \mathbb{A}^1$ have only $cA_1$-singularities and their canonical class is trivial if $n = 4$ and ample if $n \ge 5$.
(4) The two families are isomorphic over $\mathbb{A}^1 \setminus \{0\}$ but not isomorphic over $\mathbb{A}^1$.

Proof. For general $a_i, b_i$, the surface $S := (a_2(x,y) + b_2(u,v) = 0) \subset \mathbb{P}^3$ is smooth and $T := (a_1(x,y) = 0)$ has only transverse intersection with it away from the line $L := (x = y = 0)$. The central fiber $X_{10}$ of $X_1 \to \mathbb{A}^1$ is the double cover $\pi : X_{10} \to \mathbb{P}^3$ ramified along $S \cup T$. At a general point of $L$ the function $b_2(u,v)$ is nonzero and the local equation of the double cover can be made into $p^2 = a_1(x,y)$. At special points $b_2$ can have simple zeros. Here the equation is $p^2 = s \cdot a_1(x,y)$.

Let $g : P' := B_L\mathbb{P}^3 \to \mathbb{P}^3$ denote the blow up with exceptional divisor $E$. Let $S' \subset P'$ denote the birational transform of $S$ and $T' \subset P'$ the birational transform of $T$. Note that $T'$ is the union of $n$ disjoint planes from the linear system $M = |g^*\mathcal{O}_{\mathbb{P}^3}(1)(-E)|$ and $S' + T' + E$ is a snc divisor if the $a_i, b_i$ are general. The fiber product $P' \times_{\mathbb{P}^3} X_{10}$ can be realized as a double cover $X_{10}^* \to P'$ ramified along

$S' + T' + nE$. This is not normal along $E$. Its normalization $\pi' : X' \to X_{10}^* \to P'$ is again a double cover that ramifies along $S' + T' + E$ if $n$ is odd and along $S' + T'$ if $n$ is even. Since $S' + T' + E$ is a snc divisor, $X'_{10}$ has only canonical singularities (35). Let $g_X : X'_{10} \to X_{10}$ denote the induced morphism.

The canonical classes of $X_{10}$ and of $X'_{10}$ are computed by the Hurwitz formulas

$$K_{X_{10}} \sim \pi^*\mathcal{O}_{\mathbb{P}^3}(n-4) \quad \text{and} \quad K_{X'_{10}} \sim {\pi'}^*\bigl(g^*\mathcal{O}_{\mathbb{P}^3}(n-4)(-\lfloor \tfrac{n-2}{2} \rfloor E)\bigr).$$

Thus we obtain that

$$K_{X'_{10}} \sim g_X^* K_{X_{10}} - \lfloor \tfrac{n-2}{2} \rfloor {\pi'}^* E.$$

This shows that $X_{10}$ has canonical singularities if $n \le 3$ and log canonical singularities if $n = 4$, proving (2). (Note that for $n = 5$ the formula gives $K_{X'_{10}} \sim g_X^* K_{X_{10}} - {\pi'}^* E$, but $\pi'$ ramifies along $E$ so ${\pi'}^* E$ is a divisor with multiplicity 2.)

Furthermore, if $n \ge 7$ then $n - 5 \ge \lfloor \frac{n-2}{2} \rfloor$, thus

$$g^*\mathcal{O}_{\mathbb{P}^3}(n-4)(-\lfloor \tfrac{n-2}{2} \rfloor E) \supset g^*\mathcal{O}_{\mathbb{P}^3}(n-4)(-(n-5)E) = g^*\mathcal{O}_{\mathbb{P}^3}(1)\bigl((n-5)M\bigr),$$

which shows that $X'_{10}$ is of general type.

If $n = 5, 6$ then $X'_{10}$ has Kodaira dimension 1 and ${\pi'}^* M$ is a pencil of K3 surfaces. For a general plane $M$ in this pencil, we get a double cover ramified along the quintic curve $M \cap S$ plus the line $L$ when $n = 5$. The ramification is along the sextic curve $M \cap S$ when $n = 6$.

The computations for the central fiber of $X_2 \to \mathbb{A}^1$ are the same.

The general fibers of $X_i \to \mathbb{A}^1$ are double covers of $\mathbb{P}^3$ ramified along two smooth surfaces which intersect transversally. This gives the singularities $(p^2 = qr)$. The Hurwitz formula computes the canonical class.

Finally, the substitution

$$(x : y : u : v : w; t) \mapsto (t^2x : t^2y : u : v : t^n w; t)$$

transforms $\bigl(a_1(x,y) + t^{2n}b_1(u,v)\bigr)\bigl(a_2(x,y) + b_2(u,v)\bigr) - w^2$ into

$$\begin{aligned} &\bigl(a_1(t^2x, t^2y) + t^{2n}b_1(u,v)\bigr)\bigl(a_2(t^2x,t^2y) + b_2(u,v)\bigr) - t^{2n}w^2 \\ &= t^{2n}\Bigl(\bigl(a_1(x,y) + b_1(u,v)\bigr)\bigl(t^{2n}a_2(x,y) + b_2(u,v)\bigr) - w^2\Bigr). \end{aligned}$$

## 5. Coarse and fine moduli spaces

As in (7), let $\mathbf{V}$ be a "reasonable" class of projective varieties (or schemes, or ...) and $Varieties_{\mathbf{V}}$ the corresponding functor. The aim of this section is to study the difference between coarse and fine moduli spaces, mosty through a few examples. We are guided by the following:

PRINCIPLE 70. *Let* $\mathbf{V}$ *be a "reasonable" class as above and assume that it has a coarse moduli space* $\mathrm{Moduli}_{\mathbf{V}}$. *Then* $\mathrm{Moduli}_{\mathbf{V}}$ *is a fine moduli space iff* $\mathrm{Aut}(V)$ *is trivial for every* $V \in \mathbf{V}$.

From the point of view of algebraic stacks, a precise version is given in [**LMB00**, 8.1.1]. Our construction of the moduli spaces in Section **??** also shows that this principle is true for various moduli spaces of polarized varieties.

The rest of the section is devoted to some simple examples illustrating (70). The direction $\Rightarrow$ is rather easy to see if $\mathrm{Aut}(V)$ is finite for every $V \in \mathbf{V}$, see (73.2). However, (70) fails in some cases, as shown by (73.3). The direction $\Leftarrow$ is subtler. It again holds for polarized varieties but a precise version needs careful attention to descent theory and the difference between schemes and algebraic spaces.

71 (Moduli of varieties without automorphisms). As above, let $\mathbf{V}$ be a "reasonable" class of varieties with a coarse moduli space $\mathrm{Moduli}_{\mathbf{V}}$. Let us make the following

*Assumption 71.1.* $\mathrm{Aut}(V) = \{1\}$ is an open condition in flat families with fibers in $\mathbf{V}$.

If this holds then there is an open subscheme $\mathrm{Moduli}^0_{\mathbf{V}} \subset \mathrm{Moduli}_{\mathbf{V}}$ which is a coarse moduli space for varieties in $\mathbf{V}$ without automorphisms. By (70), $\mathrm{Moduli}^0_{\mathbf{V}}$ is a fine moduli space. In many important cases $\mathrm{Moduli}^0_{\mathbf{V}}$ is dense in $\mathrm{Moduli}_{\mathbf{V}}$, thus one can understand much about the coarse moduli space $\mathrm{Moduli}_{\mathbf{V}}$ by studying the fine moduli space $\mathrm{Moduli}^0_{\mathbf{V}}$.

We see in (**??**) that (71.1) holds if $\mathbf{V}$ satisfies the valuative criterion of separatedness (21.1). The following example, however, shows that (71.1) does not hold for all smooth projective surfaces.

*Example 71.2.* Let $S$ be a smooth projective surface such that $G := \mathrm{Aut}(S) = \langle \tau \rangle \cong \mathbb{Z}/p$ has prime order $\geq 3$ and there is a $\tau$-fixed point $s \in S$ such that the $G$ action on $\mathbb{P}(T_s S)$ is faithful.

For instance, if $f(x, y, z)$ is a general homogeneous form of degree $pd$ then we can take $S$ to be the degree $p$ cyclic cover $\big(u^p = f(x, y, z)\big) \subset \mathbb{P}^3(1, 1, 1, d)$ and $s$ to be any branch point.

Take now a smooth (affine) curve $s \in C \subset S$ such that the stabilizer of $T_s C \subset T_s S$ is trivial. For $0 \leq i < m$ let $C_i \subset S \times C$ be the image of $(\tau^i, 1) : C \to S \times C$. By shrinking $C$ we may assume that the $C_i$ intersect only at $(s, s)$, and there pairwise transversally.

Let $X_0 \to S \times C$ denote the blow up of $C_0$. The birational transforms $C'_i$ are disjoint for $0 < i < m$. We can now blow up the $C'_i$ for $0 < i < m$ simultaneously to obtain

$$\pi : X \to S \times C \to C.$$

If $c \neq s$ then the fiber $X_c$ is obtained from $S$ by blowing up the $G$-orbit of the point $c \in C \subset S$. Thus the $G$-action on $S$ lifts to a $G$-action on $X_c$.

For $c = s$ we get a fiber $X_s$ which is obtained from $S$ in two steps.

First we blow up $s$ to get $B_s S$ with exceptional curve $E \subset B_s S$. The $G$-action on $S$ lifts to a $G$-action on $B_s S$. Second, we blow up the $(m-1)$ intersection points $E \cap C'_i$ for $0 < i < m$ but we do not blow up the point $E \cap C'_0$. There is no $G$-orbit with $m-1$ elements, thus the $G$-action on $B_s S$ does not lift to $X_s$ and $\mathrm{Aut}(X_s) = \{1\}$.

*Example 71.3.* A similar jump of the automorphism group also happens for Enriques surfaces. By the works of [**BP83, Dol84, Kon86**], the automorphism group of a general Enriques surface is infinite, but there are special Enriques surfaces with finite automorphism group.

Next we see what goes wrong in the presence of automorphisms. We start with a concrete example.

EXAMPLE 72 (Moduli theory of the curve $(z^2 = x^{2n} - 1)$, I.).

A seemingly trivial, but actualy quite subtle and revealing, example is the moduli theory of the hyperelliptic curve $C$, given by a projective equation as

$$C = \big(z^2 = x^{2n} - y^{2n}\big) \subset \mathbb{P}^2(1, 1, n).$$

Let $k$ be an algebraically closed field. Following the pattern of (9), as a first approximation, our moduli functor should be

$$Curves_C(T) := \left\{ \begin{array}{c} \text{Smooth families } S \to T \text{ such that} \\ \text{every fiber is isomorphic to } C, \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

This is the right definition if $T$ is reduced, but not otherwise, so for now we restrict ourselves to reduced base schemes. See (**??**) for the general case.

Since the $k$-points of the coarse moduli space are in one-to-one correspondance with the $k$-isomorphism classes of objects, a coarse moduli space for $Curves_C$ has a unique $k$-point.

The only choice for the universal family is now

$$u : C \to \operatorname{Spec} k.$$

Any $k$-scheme $T$ has a unique morphism $g : T \to \operatorname{Spec} k$ and by pull-back we obtain the trivial family

$$g^* u : C \times T \to T.$$

It is easy to see, however, that for many schemes $T$, there are other families in $Curves_C(T)$. Take, for instance, $T = \mathbb{A}^* := \mathbb{A}^1 \setminus \{0\}$ and consider the surface

$$S_1^* := \left(z^2 = x^{2n} - ty^{2n}\right) \subset \mathbb{P}^2(1,1,n)_{xyz} \times \mathbb{A}^*_t.$$

$S_1^*$ is smooth and the fibers of the projection $\pi_1 : S_1^* \to \mathbb{A}^*$ are smooth hyperelliptic curves of genus $n-1$. The substitution $y' := \sqrt[2n]{t} \cdot y$ shows that each fiber is isomorphic to the curve $C := \left(z^2 = x^{2n} - y^{2n}\right) \subset \mathbb{P}^2(1,1,n)$. We claim, however, that, for $n \geq 3$, the family $\pi_1 : S_1^* \to \mathbb{A}^*$ is different from the trivial family $\pi_2 : S_2^* := \left(C \times \mathbb{A}^*\right) \to \mathbb{A}^*$. We can write the latter as

$$S_2^* := \left(z^2 = x^{2n} - y^{2n}\right) \subset \mathbb{P}^2(1,1,n)_{xyz} \times \mathbb{A}^*_t.$$

To see the difference note that a hyperelliptic curve (of genus $\geq 2$) has a unique degree 2 map to $\mathbb{P}^1$. In our two families the corresponding maps are the coordinate projection

$$\mathbb{P}^2(1,1,n)_{xyz} \times \mathbb{A}^*_t \to \mathbb{P}^1_{xy} \times \mathbb{A}^*_t$$

restricted to $S_1^*$ (resp. $S_2^*$).

The branch curve of $S_1^* \to \mathbb{P}^1_{xy} \times \mathbb{A}^*_t$ is the irreducible curve

$$B_1^* := \left(x^{2n} - ty^{2n} = 0\right) \subset \mathbb{P}^2_{xy} \times \mathbb{A}^*_t,$$

whereas the branch curve of $S_2^* \to \mathbb{P}^1_{xy} \times \mathbb{A}^*_t$ is the reducible curve

$$B_2^* := \left(x^{2n} - y^{2n} = 0\right) \subset \mathbb{P}^2_{xy} \times \mathbb{A}^*_t.$$

Thus the two families are not isomorphic.

We also see that the two families become isomorphic after a finite and surjective base change. Consider the substitution $t = u^{2n}$. By pulling back $S_1^*$, we get the family

$$T_1^* := \left(z^2 = x^{2n} - u^{2n} y^{2n}\right) \subset \mathbb{P}^2(1,1,n)_{xyz} \times \mathbb{A}^*_u.$$

By setting $y_1 := uy$, $T_1^*$ becomes isomorphic to the trivial family

$$T_2^* := \left(z^2 = x^{2n} - y_1^{2n}\right) \subset \mathbb{P}^2(1,1,n)_{xy_1z} \times \mathbb{A}^*_u,$$

which is also obtained by pulling back the trivial family $S_2^*$ to $\mathbb{A}^*_u$.

We can put these considerations in a somewhat more general setting as follows.

73 (Isotrivial families). Let $X$ be a smooth projective variety over $\mathbb{C}$ and assume for simplicity that $\mathrm{Aut}(X)$ is a discrete group. We are interested in the functor, which to a reduced scheme $T$ associates the set

$$Isotriv_X(T) := \left\{ \begin{array}{c} \text{Smooth families } \mathbf{X} \to T \text{ such that} \\ \text{every fiber is isomorphic to } X, \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

More precisely, we should distinguish between the algebraic and the complex analytic versions $Isotriv_X^{alg}(*)$ and $Isotriv_X^{an}(*)$. It turns out that allowing $T$ to be a complex analytic space is a minor difference, but allowing $\mathbf{X}$ to be complex analytic creates a substantial change. Let us start complex analytically.

*Lemma 73.1.* Assume that $\mathrm{Aut}(X)$ is a discrete group. Then families in $Isotriv_X^{an}(T)$ are in one-to-one correspondance with the $\mathrm{Aut}(X)$-conjugacy classes of group homomorphisms $\mathrm{Hom}\big(\pi_1(T,t), \mathrm{Aut}(X)\big)$.

Proof. Since $\mathrm{Aut}(X)$ is a discrete group, over any contractible subset of $T$ the family has a unique trivialization. Thus, if we fix a point $t \in T$ and an isomorphism $\mathbf{X}_t \cong X$ then the various families are classified by the monodromy representation

$$\rho : \pi_1(T,t) \to \mathrm{Aut}(X).$$

If we do not fix an isomorphism $\mathbf{X}_t \cong X$, then we have to work with conjugacy classes of such homomorphisms. $\square$

It is not hard to go from an analytic classification to an algebraic one.

*Lemma 73.2.* Notation and assumptions as above.

(1) Two such algebraic families $\mathbf{X}_i \to T$ are algebraically isomorphic iff they are analytically isomorphic.
(2) $\mathbf{X} \to T$ is projective iff the image of $\rho$ is finite.
(3) $\mathbf{X} \to T$ is an algebraic space iff $\mathbf{X} \to T$ is projective.

Proof. Assume that $\mathbf{X}_i \to T$ are algebraic and consider the scheme parametrizing relative isomorphisms $\mathrm{Isom}_T(\mathbf{X}_1, \mathbf{X}_2)$ (cf. [**Kol96**, Sec.I.1]). By our assumptions $\mathrm{Isom}_T(\mathbf{X}_1, \mathbf{X}_2) \to T$ is étale, thus it has an algebraic section iff it has an analytic section. This proves (1).

Assume that $\mathbf{X} \to T$, corresponding to $\rho : \pi_1(T,t) \to \mathrm{Aut}(X)$, is projective and let $L$ be a relatively ample divisor on $\mathbf{X}$. Then $c_1\big(L|_X\big) \in H^2(X,\mathbb{Z})$ is invariant under $\mathrm{im}\,\rho$. For some $d > 0$, the Néron-Severi group $\mathrm{NS}(X)$ is generated by effective divisors of degree $\le d$ (with respect to $c_1\big(L|_X\big)$). There are only finitely many such divisor classes, hence a finite index subgroup of the image of $\rho$ acts trivially on $\mathrm{NS}(X)$. For any projective variety $X$, the subgroup $\mathrm{Aut}^\tau(X)$ of $\mathrm{Aut}(X)$ that acts trivially on $\mathrm{NS}(X)$ is an algebraic group (cf. [**Kol96**, I.1.10.2]). Since $\mathrm{Aut}(X)$ is assumed discrete, $\mathrm{Aut}^\tau(X)$ is finite. Thus $\mathrm{im}\,\rho$ is finite, proving one direction of (2).

Conversely, assume that $G := \mathrm{im}\,\rho$ is finite and let $T' \to T$ be the étale cover corresponding to $G$. On the trivial family $X \times T'$ consider the action of $G$ where we act on $T'$ by deck transformations and on $X$ by $\rho$. The quotient $\mathbf{X} := \big(X \times T'\big)/G$ exists and is projective (cf. **??**).

The proof of (3) is left to the reader; we will not use it. $\square$

*Corollary 73.3.* Let $X$ be a smooth projective variety over $\mathbb{C}$ such that $\mathrm{Aut}(X)$ is a discrete group. Then $X \to \operatorname{Spec}\mathbb{C}$ is a fine moduli space for $Isotriv_X^{an}(*)$ iff $\mathrm{Aut}(X) = \{1\}$.

Proof. If $\mathrm{Aut}(X) \neq \{1\}$ then there is a nontrivial homomorphism $\mathbb{Z} \to \mathrm{Aut}(X)$. This gives a locally trivial but globally nontrivial complex analytic family over $\mathbb{C}^*$ (or over any elliptic curve) that can not be the pull-back of $X \to \operatorname{Spec}\mathbb{C}$. Conversely, if $\mathrm{Aut}(X) = \{1\}$ then $Isotriv_X^{an}(T)$ consists of the trivial family for any $T$.

*Corollary 73.4.* Let $X$ be a smooth projective variety over $\mathbb{C}$ such that $\mathrm{Aut}(X)$ is discrete and torsion free. Then for any $T$, the trivial family $X \times T$ gives the only algebraic family in $Isotriv_X^{alg}(T)$. In particular, $X \to \operatorname{Spec}\mathbb{C}$ is a fine moduli space for $Isotriv_X^{alg}(*)$.

Proof. By our assumption, the only homomorphism $\rho : \pi_1(T, t) \to \mathrm{Aut}(X)$ with finite image is the trivial one. It corresponds to the trivial family $X \times T \to T$. $\square$

There are K3 surfaces with discrete and torsion free automorphism group. The next construction gives another example which is birational to an Abelian surface.

*Example 73.5.* Let $0 \in E$ be an elliptic curve such that $\mathrm{End}(0 \in E) \cong \mathbb{Z}$, (that is without complex multiplication). Then the automorphism group of its square is

$$\mathrm{Aut}\big((0,0) \in E \times E\big) \cong GL(2, \mathbb{Z})$$

and the isomorphism is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \big[(x, y) \mapsto (ax + by, cx + dy)\big].$$

Take 3 points $P_1 = (0, 0)$, $P_2 = (x_2, 0)$ and $P_3 = (0, x_3)$ where $x_3 \in E$ is 3-torsion and $x_2 \in E$ is non-torsion. It is easy to see that $\{0\} \times E$ (resp. $E \times \{0\}$) is the only elliptic curve in $E \times E$ that contains 2 of the points and their difference is torsion (resp. non-torision). Thus we conclude that

$$\mathrm{Aut}\big(E \times E, P_1 + P_2 + P_3\big) = \left\{ \begin{pmatrix} 1 & 3m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z} \right\}.$$

Let now $X$ be the surface obtained from $E \times E$ by blowing up the 3 points $P_i$. Since the only rational curves on $X$ are the 3 exceptional curves, we conclude that

$$\mathrm{Aut}(X) = \mathrm{Aut}\big(E \times E, P_1 + P_2 + P_3\big) \cong \mathbb{Z}.$$

EXAMPLE 74 (Moduli theory of the curve $(z^2 = x^{2n} - 1)$, II.).

Another reincarnation of the phenomenon observed in (72) occurs if we notice that $C$ is already defined over $\mathbb{Q}$ and we try to construct the moduli space as $\operatorname{Spec}\mathbb{Q}$.

Over an algebraically closed field, $C$ is isomorphic to any of the curves

$$C_{ab} = \big(z^2 = ax^{2n} - by^{2n}\big) \subset \mathbb{P}^2(1, 1, n) \quad \text{for } a, b \neq 0.$$

Over other fields, however, the curves $C_{ab}$ need not be isomorphic. For instance, over $\mathbb{R}$, we can obtain $\big(z^2 = x^{2n} + y^{2n}\big)$ whose set of real points consists of 2 circles, $\big(z^2 = x^{2n} - y^{2n}\big)$ whose set of real points consists of 1 circle and $\big(z^2 = -x^{2n} - y^{2n}\big)$ whose set of real points is empty.

The situation is even worse over $\mathbb{Q}$. For instance, as $p$ runs through all prime numbers, the curves $C_{1p} = \big(z^2 = x^{2n} - py^{2n}\big)$ are pairwise non-isomorphic for $n \geq 4$.

A simple way to see this is to note that the ramification locus of the projection $C_{1p} \to \mathbb{P}^1_{xy}$ is an isomorphism invariant of $C_{1p}$. In our case, the ramification locus is the scheme $\operatorname{Spec}_{\mathbb{Q}} \mathbb{Q}\bigl(\sqrt[2n]{p}\bigr)$, and these fields are different from each other for different values of $p$. For instance, the only ramified primes in $\mathbb{Q}\bigl(\sqrt[2n]{p}\bigr)/\mathbb{Q}$ are $p$ and possibly some divisors of $2n$. Thus as $p$ runs through the set of primes not dividing $2n$, we get pairwise non-isomorphic fields and hence non-isomorphic curves $C_{1p}$.

75 (Field of moduli). Let $X \subset \mathbb{P}^n$ be a projective variety over $\mathbb{C}$. Any set of defining equations involves only finitely many elements of $\mathbb{C}$, thus $X$ can be defined over a finitely generated subextension of $\mathbb{C}$. It is a natural question to ask: Is there a smallest subfield $K \subset \mathbb{C}$ such that $X$ can be defined by equations over $K$.

There are three variants for this question.

(1) Fix coordinates on $\mathbb{P}^n$ and view $X$ as a specific subvariety. In this case a smallest subfield exists; see [**Wei46**, Sec.I.7] or [**KSC04**, Sec.3.4]. This is a special case of the existence of Hilbert schemes (5).
(2) No embedding of $X$ is fixed. We see in (78) that this may lead to rather complicated behaviour.
(3) As an intermediate choice, fix an embedding $X \hookrightarrow \mathbb{P}^n$ but do not fix the coordinates on $\mathbb{P}^n$. Equivalently, we work with a pair $(X, L)$ where $L$ is a very ample line bundle on $X$. This is the question that we consider next. Note that, if the canonical line bundle on $X$ is ample or anti-ample, we can harmlessly identify $X$ with the pair $\bigl(X, \mathcal{O}_X(mK_X)\bigr)$ if $mK_X$ is very ample. (There are two further natural variants of this approach. We may decide not to distinguish between the pairs $(X, L)$ and $(X, L^m)$ for $m > 0$ or we may identify $(X, L)$ and $(X, L')$ if $L$ is numerically equivalent to $L'$. Both of these lead to minor technical differences only.)

How is this connected with moduli theory?

Let $\mathbf{V}$ be a class of varieties with a coarse moduli space $\mathrm{Moduli}_{\mathbf{V}}$. Assume that $X \in \mathbf{V}$ can be defined by equations over a field $K$; that is, there is a $K$-scheme $X_K \to \operatorname{Spec} K$ whose geometric fiber is isomorphic to $X$. By the definition of a coarse moduli space, this corresponds to a morphism $\operatorname{Spec} K \to \mathrm{Moduli}_{\mathbf{V}}$. In particular, we get an injection of the residue field of $\mathrm{Moduli}_{\mathbf{V}}$ at $[X]$ into $K$. Conversely, if $\mathrm{Moduli}_{\mathbf{V}}$ is a fine moduli space, then $X$ can be defined over the residue field. Thus we have proved the following:

*Lemma 75.4.* If $\mathrm{Moduli}_{\mathbf{V}}$ is a fine moduli space then the residue field of $\mathrm{Moduli}_{\mathbf{V}}$ at $[X]$ is the smallest field $K$ such that $X$ can be defined by equations over $K$. $\square$

An consequence is that, for fine moduli spaces, the residue field of $\mathrm{Moduli}_{\mathbf{V}}$ at $[X]$ depends only on $X$ and not on the choice of $\mathbf{V}$.

In general, let us define the *field of moduli* of $X$ as the (function field of) the coarse moduli space of the functor $Isotriv_X(*)$, where, generalizing the concept in (73) from $\mathbb{C}$ to arbitrary fields, for any reduced scheme $T$ we set

$$Isotriv_X(T) := \left\{ \begin{array}{c} \text{Smooth families } \mathbf{X} \to T \text{ such that} \\ \text{every geometric fiber is isomorphic to } X, \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

As we see in (78), $Isotriv_X(*)$ need not have a coarse moduli space. We thus introduce the following variant. For a pair $(X, L)$, where $L$ is a very ample line bundle on $X$, set

$$Isotriv_{(X,L)}(T) := \left\{ \begin{array}{c} \text{Smooth families } \mathbf{X} \to T \text{ plus a} \\ \text{relatively ample line bundle } \mathbf{L} \text{ such that} \\ \text{every geometric fiber is isomorphic to } (X, L), \\ \text{modulo isomorphisms over } T. \end{array} \right\}$$

By (**??**), $Isotriv_{(X,L)}(*)$ always has a coarse moduli space.

In order to avoid some problems with infinite Galois groups (78), the following lemma is stated for number fields only.

*Lemma 75.5.* Let $X$ be a smooth projective variety defined over a number field $L$. For a field $K$ the following are equivalent.

(1) The field of moduli of $X$ is contained in $K$.
(2) There is a $K$-scheme $T$ such that $Isotriv_X(T) \neq \emptyset$.
(3) For any $\sigma \in \mathrm{Gal}(\bar{K}/K)$, the variety $X^\sigma$ is isomorphic to $X$ over $\bar{K}$. (Here $X^\sigma$ is obtained by applying $\sigma$ to a set of defining equations of $X$.)

Proof. The interesting part is (3) $\Rightarrow$ (1). Choose a finite extension $K(\alpha)/K$ such that $L \subset K(\alpha)$, where $\alpha$ is a root of a polynomial $p(t) \in K[t]$ of degree $d$. Let

$$f_i(x_0, \dots, x_m) \in K(\alpha)[x_0, \dots, x_m] \ : \ i = 1, \dots, r$$

be defining equations of $X$ (in some projective embedding) over $K(\alpha)$. Since $K(\alpha) = K + \alpha K + \cdots + \alpha^{d-1}K$, we can also think of the $f_i$ as

$$f_i(\alpha, x_0, \dots, x_m) \in K[\alpha, x_0, \dots, x_m],$$

where $\deg_\alpha f_i < d$. Consider now the $K$-scheme

$$Y_K := \big(f_1(t, x_0, \dots, x_m) = \cdots = f_r(t, x_0, \dots, x_m) = p(t) = 0\big) \subset \mathbb{P}^m_K \times \mathbb{A}^1_t.$$

The second projection gives $\pi : Y_K \to \mathrm{Spec}_K K[t]/\big(p(t)\big)$. One of the geometric fibers of $\pi$ is $X_{\bar{L}}$, the others are its conjugates $X^\sigma_{\bar{L}}$. If (3) holds then $\pi : Y_K \to \mathrm{Spec}_K K(\alpha)$ is an isotrivial family over the $K$-scheme $\mathrm{Spec}_K K(\alpha)$, which shows (2). $\square$

In (77) we construct a hyperelliptic curve whose field of moduli is $\mathbb{Q}$ yet it can not be defined over $\mathbb{R}$.

76 (Field of moduli for hyperelliptic curves). Let $A$ be a smooth hyperelliptic curve of genus $\geq 2$. Over an algebraically closed field, $A$ has a unique degree 2 map to $\mathbb{P}^1$. Let $B \subset \mathbb{P}^1$ be the branch locus, that is, a collection of $2g+2$ points in $\mathbb{P}^1$. If the base field $k$ is not closed, then $A$ has a unique degree 2 map to a smooth genus 0 curve $Q$. (One can always think of $Q$ as a conic in $\mathbb{P}^2$.) Thus $A$ is defined over a field $k$ iff the pair $(B \subset \mathbb{P}^1)$ can be defined over $k$.

The latter problem is especially transparent if $A$ is defined over $\mathbb{C}$ and we want to know if it is defined over $\mathbb{R}$ or if its field of moduli is contained in $\mathbb{R}$.

Up to isomorphism, there are 2 real forms of $\mathbb{P}^1$. One is $\mathbb{P}^1$, corresponding to the anti-holomorphic involution $(x{:}y) \mapsto (\bar{x}{:}\bar{y})$, which, in suitable coordinates, can also be writen as $\sigma_1 : (x{:}y) \mapsto (\bar{y}{:}\bar{x})$. The other is the "empty" conic, corresponding to the anti-holomorphic involution $\sigma_2 : (x{:}y) \mapsto (-\bar{y}{:}\bar{x})$. Thus (75.5) gives the following.

*Lemma 76.1.* Let $A \to \mathbb{P}^1$ be a smooth hyperelliptic curve of genus $\geq 2$ over $\mathbb{C}$ and $B \subset \mathbb{CP}^1$ the branch locus. Then

(1) $A$ can be defined over $\mathbb{R}$ iff there is a $g \in \operatorname{Aut}(\mathbb{CP}^1)$ such that $gB$ is invariant under $\sigma_1$ or $\sigma_2$.
(2) The field of moduli of $A$ is contained in $\mathbb{R}$ iff there is $h \in \operatorname{Aut}(\mathbb{CP}^1)$ such that $hB$ equals $B^{\sigma_1}$ or $B^{\sigma_2}$.

Note that if $(gB)^\sigma = gB$ then $B^\sigma = \big(g^\sigma\big)^{-1} gB$ shows that (1) $\Rightarrow$ (2). Conversely, if $B^\sigma = hB$ and we can write $h = \big(g^\sigma\big)^{-1} g$ then $(gB)^\sigma = gB$.

EXAMPLE 77. Here is an example of a hyperelliptic curve $C$ whose field of moduli is $\mathbb{Q}$ but $C$ can not be defined over $\mathbb{R}$.

Pick $\alpha = a + ib$ where $a, b$ are rational. Consider the hyperelliptic curve

$$C(\alpha) := \Big(z^2 - \big(x^8 - y^8\big)\big(x^2 - \alpha y^2\big)\big(\bar{\alpha} x^2 + y^2\big) = 0\Big) \subset \mathbb{P}^3(1,1,6).$$

Its complex conjugate is

$$C(\bar{\alpha}) := \Big(z^2 - \big(x^8 - y^8\big)\big(x^2 - \bar{\alpha} y^2\big)\big(\alpha x^2 + y^2\big) = 0\Big) \subset \mathbb{P}^3(1,1,6).$$

Note that $C(\alpha)$ and $C(\bar{\alpha})$ are isomorphic, as shown by the substitution

$$(x, y, z) \mapsto (iy, x, z).$$

In particular, over the $\mathbb{Q}$-scheme $\operatorname{Spec}_{\mathbb{Q}} \mathbb{Q}[t]/(t^2+1)$ we have a curve

$$C(a, b) := \Big(z^2 - \big(x^8 - y^8\big)\big(x^2 - (a+tb)y^2\big)\big((a-tb)x^2 + y^2\big) = 0\Big) \subset \mathbb{P}^3(1,1,6).$$

whose geometric fibers are isomorphic to $C(\alpha)$. Thus the field of moduli of $C(\alpha)$ is $\mathbb{Q}$.

We claim that, for sufficiently general $a, b$, the curve $C(\alpha)$ can not be defined over $\mathbb{Q}$, not even over $\mathbb{R}$. By (76) we need to show that there is no anti-holomorphic involution that maps the branch locus to itself. In the affine chart $y \neq 0$, the ramification points of $C(\alpha) \to \mathbb{P}^1$ are:

(1) the 8th roots of unity corresponding to $x^8 - y^8$, and
(2) the 4 points $\pm\beta, \pm i/\bar{\beta}$ where $\beta^2 = \alpha$.

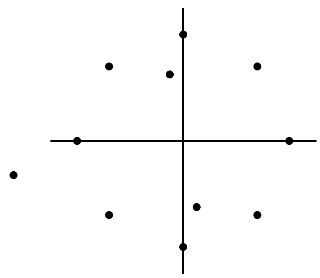
The anti-holomorphic automorphisms of the Riemann sphere map circles to circles. Out of our 12 points, the 8 roots of unity lie on the circle $|z| = 1$, but no other 8 can lie on a circle. Thus any anti-holomorphic automorphism that maps our configuration to itself, must fix the unit circle $|z| = 1$ and map the 8th roots of unity to each other.

The only such anti-holomorphic involutions are

(3) Reflection on the line $\mathbb{R} \cdot \epsilon$ where $\epsilon$ is a 16th root of unity, and
(4) $z \mapsto 1/\bar{z}$ or $z \mapsto -1/\bar{z}$.

Thus, as long as $\beta \notin \mathbb{R} \cdot \eta$ for a 16th root of unity, we conclude that $C(\alpha)$ is not isomorphic (over $\mathbb{C}$) to a real curve.

The configuration depicted below shows 12 points $p_1, \ldots, p_{12}$ on $\mathbb{C}$ that are invariant under $z \mapsto i/\bar{z}$ but not invariant under any anti-holomorphic involution.



Example 78. We give an example of a smooth projective surface $S$ such that if $S$ is defined over a field extension $K/\mathbb{C}$ then $\operatorname{trdeg} K \geq 2$ but the intersection of all such fields of definition is $\mathbb{C}$.

Let $X$ be a smooth projective variety such that

(1) $\operatorname{Aut}(X)$ is an infinite discrete group whose general orbit is Zariski dense in $X$ and
(2) $\operatorname{Aut}(X)$ is generated by 2 finite subgroups $G_1, G_2$.

By (73.5), one such example is $B_0(E \times E)$, the blow up of the square of an elliptic curve at a point. There are also K3 surfaces with infinite automorphism group that is generated by 2 involutions (**??**).

Let $\Delta \subset X \times X$ be the diagonal and, using one of the projections, consider the family of smooth varieties

$$f : Y := B_\Delta X \times X \to X.$$

Note that $Y \to X$ is the universal family of the varieties of the form $B_x X$ for $x \in X$. This shows that $f : Y \to X$ can not be obtained by pull-back from any family over a lower dimensional base.

In particular, if $x \in X$ is general, then $\operatorname{Aut}(B_x X) = \mathbb{Z}/2$ if $X = B_0(E \times E)$ and $\operatorname{Aut}(B_x X) = 1$ if $X$ is a K3 surface. The action of $\operatorname{Aut}(X)$ lifts to the diagonal action on $Y$.

Let $G \subset \operatorname{Aut}(X)$ be a finite subgroup. There is an open subset $U_G \subset X$ such that $G$ operates on $U_G$ without fixed points. Thus $f/G : Y/G \to X/G$ is a family of smooth varieties over $U_G/G$ and $Y|_{U_G} \cong Y/G \times_{X/G} U_G$.

Let $K = \mathbb{C}(X)$ denote the function field of $X$. The variety we are interested in is $Y_K$, the generic fiber of $Y \to X$. The above considerations show that $Y_K$ can be defined over $\mathbb{C}(X/G) = K^G$ for every finite subgroup $G \subset \operatorname{Aut}(X)$.

Note that $K = \mathbb{C}(X)$ is a function field of transcendence degree $\dim X$ over $\mathbb{C}$ and so are the subfields $K^G$. On the other hand, the intersection $K^{G_1} \cap K^{G_2}$ is $\mathbb{C}$. Indeed, any function in $K^{G_1} \cap K^{G_2}$ is constant on every $G_1$-orbit and also on every $G_2$-orbit. By assumption (2), it is also constant along every $\operatorname{Aut}(X)$-orbit, hence constant by assumption (1).

This phenomenon is also connected with the behaviour of ample line bundles on $\pi_i : Y \to Y/G_i$. Although both of the $Y/G_i$ are projective, there are no ample line bundles $L_i$ on $Y/G_i$ such that $\pi_1^* L_1 \cong \pi_2^* L_2$.

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