The structure of bull-free graphs II — elementary trigraphs

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Abstract

The *bull* is a graph consisting of a triangle and two pendant edges. A graph is called *bull-free* if no induced subgraph of it is a bull. This is the second paper in a series of three. The goal of the series is to give a complete description of all bull-free graphs. We call a bull-free graph *elementary* if it does not contain an induced three-edge-path P such that some vertex $c \notin V(P)$ is complete to $V(P)$, and some vertex $a \notin V(P)$ is anticomplete to $V(P)$. In this paper we prove that every elementary graph either belongs to one a few basic classes, or admits a certain decomposition.

1 Introduction

All graphs in this paper are finite and simple, unless stated otherwise. The *bull* is a graph with vertex set $\{x_1, x_2, x_3, y, z\}$ and edge set

$$\{x_1x_2, x_2x_3, x_1x_3, x_1y, x_2z\}.$$

Let G be a graph. We say that G is *bull-free* if no induced subgraph of G is isomorphic to the bull. The complement of G is the graph $\overline{G}$, on the same vertex set as G, and such that two vertices are adjacent in G if and only if they are non-adjacent in $\overline{G}$. A *clique* in G is a set of vertices, all pairwise adjacent. A *stable set* in G is a clique in the complement of G. A clique of size three is called a *triangle* and a stable set of size three is a *triad*. For a subset A of $V(G)$ and a vertex $b \in V(G) \setminus A$, we say that b is *complete* to A if b is adjacent to every vertex of A, and that b is *anticomplete* to A if b is not adjacent to any vertex of A. For two disjoint subsets A and B of

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$V(G)$, A is *complete* to B if every vertex of A is complete to B, and A is *anticomplete* to B every vertex of A is anticomplete to B. For a subset X of $V(G)$, we denote by $G|X$ the subgraph induced by G on X, and by $G \setminus X$ the subgraph induced by G on $V(G) \setminus X$.

Let us call a bull-free graph G *elementary* if it does not contain an induced three-edge-path P such that some vertex $c \notin V(P)$ is complete to $V(P)$ and some vertex $a \notin V(P)$ is anticomplete to $V(P)$. In this paper we prove that every elementary graph either belongs to a one of a few basic classes, or admits a decomposition.

This paper is organized as follows. In the next section we define an object called a "trigraph", which is a generalization of a graph, but is more convenient for stating the main result of this series of papers. Most of the definitions of Section 2 appeared in [1], but we include them here for the reader's convenience. In Section 3 we state the main theorem of this paper, 3.4, giving all the necessary definitions. We also define the class of "unfriendly trigraphs", which is the subject of most of the theorems in this paper. In Section 4, we study unfriendly trigraphs, that contain a "prism" (an induced subtrigraph consisting of two disjoint cliques and a matching between them, for a precise definition see Section 4). We prove that every such trigraph satisfies one of the outcomes of 3.4. Section 5 contains a few useful lemmas about unfriendly trigraphs. In Section 6, we study the behavior of an unfriendly trigraph relative to an induced triangle-free subtrigraph (again, see Section 6 for the definitions). We prove that one of the outcomes of 3.4 holds for every unfriendly trigraph that contains an induced three-edge path. We finish Section 6 with a proof of 3.4, using a result from [1].

2 Trigraphs

In order to prove our main result, we consider objects, slightly more general than bull-free graphs, that we call "bull-free trigraphs". A *trigraph* G consists of a finite set $V(G)$, called the *vertex set* of G, and a map $\theta : V(G)^2 \to \{-1, 0, 1\}$, called the *adjacency function*, satisfying:

- for all $v \in V(G)$, $\theta_G(v, v) = 0$
- for all distinct $u, v \in V(G)$, $\theta_G(u, v) = \theta_G(v, u)$
- for all distinct $u, v, w \in V(G)$, at most one of $\theta_G(u, v), \theta_G(u, w) = 0$.

Two distinct vertices of G are said to be *strongly adjacent* if $\theta(u, v) = 1$, *strongly antiadjacent* if $\theta(u, v) = -1$, and *semi-adjacent* if $\theta(u, v) = 0$. We say that u and v are *adjacent* if they are either strongly adjacent, or semi-adjacent; and *antiadjacent* of they are either strongly antiadjacent, or semi-adjacent. If u and v are adjacent (antiadjacent), we also say that u is *adjacent (antiadjacent) to* v, or that u is a *neighbor (antineighbor)* of v.

Similarly, if u and v are strongly adjacent (strongly antiadjacent), then u is a *strong neighbor (strong antineighbor)* of v. Let $\eta(G)$ be the set of all strongly adjacent pairs of G, $\nu(G)$ the set of all strongly antiadjacent pairs of G, and $\sigma(G)$ the set of all pairs $\{u, v\}$ of vertices of G, such that u and v are distinct and semi-adjacent. Thus, a trigraph G is a graph if $\sigma(G)$ empty.

Let G be a trigraph. The complement $\overline{G}$ of G is a trigraph with the same vertex set as G, and adjacency function $\overline{\theta} = -\theta$. Let $A \subset V(G)$ and $b \in V(G) \setminus A$. For $v \in V(G)$ let $N(v)$ denote the set of all vertices in $V(G)\setminus\{v\}$ that are adjacent to v, and let $S(v)$ denote the set of all vertices in $V(G)\setminus\{v\}$ that are strongly adjacent to v. We say that b is *strongly complete* to A if b is strongly adjacent to every vertex of A, b is *strongly anticomplete* to A if b is strongly antiadjacent to every vertex of A, b is *complete* to A if b is adjacent to every vertex of A, and b is *anticomplete* to A if b is antiadjacent to every vertex of A. For two disjoint subsets A, B of $V(G)$, B is *strongly complete (strongly anticomplete, complete, anticomplete)* to A if every vertex of B is strongly complete (strongly anticomplete, complete, anticomplete, respectively) to A. We say that b is *mixed* on A, if b is not strongly complete and not strongly anticomplete to A. A *clique* in G is a set of vertices all pairwise adjacent, and a *strong clique* is a set of vertices all pairwise strongly adjacent. A *stable set* is a set of vertices all pairwise antiadjacent, and a *strongly stable set* is a set of vertices all pairwise strongly antiadjacent. A (strong) clique of size three is a *(strong) triangle* and a (strong) stable set of size three is a *(strong) triad*. For $X \subset V(G)$, the trigraph *induced by G on X* (denoted by $G|X$) has vertex set X, and adjacency function that is the restriction of θ to X^2. Isomorphism between trigraphs is defined in the natural way, and for two trigraphs G and H we say that H is an *induced subtrigraph* of G (or G *contains H as an induced subtrigraph*) if H is isomorphic to $G|X$ for some $X \subseteq V(G)$. We denote by $G \setminus X$ the trigraph $G|(V(G) \setminus X)$.

A *bull* is a trigraph with vertex set $\{x_1, x_2, x_3, v_1, v_2\}$ such that $\{x_1, x_2, x_3\}$ is a triangle, v_1 is adjacent to x_1 and antiadjacent to x_2, x_3, v_2, and v_2 is adjacent to x_2 and antiadjacent to x_1, x_3. For a trigraph G, a subset X of $V(G)$ is said to be a *bull* if $G|X$ is a bull. We say that a trigraph is *bull-free* if no induced subtrigraph of it is a bull, or, equivalently, no subset of its vertex set is a bull.

Let G be a trigraph. An induced subtrigraph P of G with vertices $\{p_1, \dots, p_k\}$ is a *path* in G if either $k = 1$, or for $i, j \in \{1, \dots, k\}$, p_i is adjacent to p_j if $|i - j| = 1$ and p_i is antiadjacent to p_j if $|i - j| > 1$. Under these circumstances we say that P is a path *from p_1 to p_k*, its *interior* is the set $P^* = V(P) \setminus \{p_1, p_k\}$, and the *length* of P is $k - 1$. We also say that P is a $(k-1)$-edge-path. Sometimes we denote P by p_1-$\dots$-p_k. An induced subtrigraph H of G with vertices $h_1, \dots, h_k$ is a *hole* if $k \geq 4$, and for $i, j \in \{1, \dots, k\}$, h_i is adjacent to h_j if $|i - j| = 1$ or $|i - j| = k - 1$; and h_i is antiadjacent to h_j if $1 < |i - j| < k - 1$. The *length* of a hole is the number of vertices in it. Sometimes we denote H by h_1-...-h_k-h_1. An *antipath* (*antihole*) is an induced subtrigraph of G whose complement is a path (hole) in $\overline{G}$.

Let G be a trigraph, and let $X \subseteq V(G)$. Let G_c be the graph with vertex set X, and such that two vertices of X are adjacent in G_c if and only if they are adjacent in G, and let G_a be be the graph with vertex set X, and such that two vertices of X are adjacent in G_a if and only if they are strongly adjacent in G. We say that X (and $G|X$) is *connected* if the graph G_c is connected, and that X (and $G|X$) is *anticonnected* if $\overline{G_a}$ is connected. A *connected component* of X is a maximal connected subset of X, and an *anticonnected component* of X is a maximal anticonnected subset of X. For a trigraph G, if X is a component of $V(G)$, then $G|X$ is a component of G.

We finish this section by two easy observations from [1].

2.1 *If G be a bull-free trigraph, then so is $\overline{G}$.*

2.2 *Let G be a trigraph, let $X \subseteq V(G)$ and $v \in V(G) \setminus X$. Assume that $|X| > 1$ and v is mixed on X. Then there exist vertices $x_1, x_2 \in X$ such that v is adjacent to x_1 and antiadjacent to x_2. Moreover, if X is connected, then x_1 and x_2 can be chosen adjacent.*

3 The main theorem

In this section we state our main theorem. We start by describing a few special types of trigraphs.

Clique connectors. Let G be a trigraph. Let $K = \{k_1, \ldots, k_t\}$ be a strong clique in G, and let A, B, C, D be strongly stable sets, such that the sets K, A, B, C, D are pairwise disjoint and $A \cup B \cup C \cup D \cup K = V(G)$. Let $A_1, \ldots, A_t$ be disjoint subsets of A with $\bigcup_{i=1}^{t} A_i = A$, and let $B_1, \ldots, B_t, C_1, \ldots, C_t, D_1, \ldots, D_t$ be defined similarly. Let us now describe the adjacencies in G:

- For $i \in \{1, \ldots, t\}$
  A_i is strongly complete to $\{k_1, \ldots, k_{i-1}\}$,
  A_i is complete to $\{k_i\}$,
  A_i is strongly anticomplete to $\{k_{i+1}, \ldots, k_t\}$,
  B_i is strongly complete to $\{k_{t-i+2}, \ldots, k_t\}$,
  B_i is complete to $\{k_{t-i+1}\}$, and
  B_i is strongly anticomplete to $\{k_1, \ldots, k_{t-i}\}$.

Let A'_i be the set of vertices of A_i that are semi-adjacent to k_i, and let B'_{t-i+1} be the set of vertices of B_{t-i+1} that are semi-adjacent to k_i. (Thus $|A'_i| \leq 1$ and $|B'_{t-i+1}| \leq 1$.)

- For $i, j \in \{1, \ldots, t\}$, if $i + j \neq t$ and A_i is not strongly complete to B_j, then $|A| = |B| = |K| = 1$ and A is complete to B.

- A'_i is strongly complete to B_{t-i}, B'_{t-i} is strongly complete to A_i, and the adjacency between $A_i \setminus A'_i$ and $B_{t-i} \setminus B'_{t-i}$ is arbitrary.
- $A \cup K$ is strongly anticomplete to D, and $B \cup K$ is strongly anticomplete to C.
- For $i \in \{1, \ldots, t\}$, C_i is strongly complete to $\bigcup_{j<i} A_j$, and C_i is strongly anticomplete to $\bigcup_{j>i} A_j$.
- For $i \in \{1, \ldots, t\}$, C_i is strongly complete to A'_i, every vertex of C_i has a neighbor in A_i, and otherwise the adjacency between C_i and $A_i \setminus A'_i$ is arbitrary.
- For $i \in \{1, \ldots, t\}$, D_i is strongly complete to $\bigcup_{j<i} B_j$, and D_i is strongly anticomplete to $\bigcup_{j>i} B_j$.
- For $i \in \{1, \ldots, t\}$, D_i is strongly complete to B'_i, every vertex of D_i has a neighbor in B_i, and otherwise the adjacency between D_i and $B_i \setminus B'_i$ is arbitrary.
- For $i, j \in \{1, \ldots, t\}$, if $i + j > t$, then C_i is strongly complete to D_j, and otherwise the adjacency between C_i and D_j is arbitrary.

If $A_t \neq \emptyset$ and $B_t \neq \emptyset$, then G is a (K, A, B, C, D)-*clique connector.*

3.1 *Every clique connector is bull-free.*

Proof. Let G be a (K, A, B, C, D)-clique connector. Let $|K| = t$.

(1) Let $a \in A$ and $b \in B$, and suppose that k_i is adjacent to both a and b for some $i \in \{1, \ldots, t\}$. Then every vertex of K is strongly adjacent to at least one of a, b.

Since k_i is adjacent to a, if follows that $a \in \bigcup_{j \geq i} A_i$, and since b is adjacent to k_i, it follows that $b \in \bigcup_{j \geq t-i+1} B_j$. Therefore, a is strongly complete to $\{k_1, \ldots, k_{i-1}\}$, and b is strongly complete to $\{k_{i+1}, \ldots, k_t\}$. Since both a and b are adjacent to k_i, and at most one of a, b is semi-adjacent to k_i, (1) follows.

(2) There do not exist $k, k' \in K$ and $a, a' \in A$, such that the pairs $ak, a'k'$ are adjacent, and the pairs $ak', a'k$ are antiadjacent.

Suppose such a, a', k, k' exist, say $k = k_p$ and $k' = k_q$ for $p, q \in \{1, \ldots, t\}$. We may assume that $p > q$. Then, since a is adjacent to k_p, it follows that $a \in \bigcup_{j \geq p} A_j$, and therefore a is strongly adjacent to k_q, a contradiction. This proves (2).

(3) Let $a \in A$ and $b \in B$, and suppose that k_i is adjacent to both a

and b for some $i \in \{1, \dots, t\}$. The either a is strongly adjacent to b, or $|A| = |B| = |K| = 1$.

We may assume at least one of A, B, K has size at least two. Since a is adjacent to k_i, it follows that $a \in \bigcup_{j \geq i} A_j$, and since b is adjacent to k_i, it follows that $b \in \bigcup_{j \geq t-i+1} B_j$, and therefore a is strongly adjacent to b. This proves (3).

(4) Let $a \in A$ and $b \in B$, and suppose that k_i is antiadjacent to both a and b for some $i \in \{1, \dots, t\}$. Then a is strongly adjacent to b.

Suppose a is antiadjacent to b. Then $a \notin A'_i$ and $b \notin B'_{t-i+1}$. Let $p, q \in \{1, \dots, t\}$ such that $a \in A_p$ and $b \in B_q$. Since a is antiadjacent to k_i, it follows that $p < i$, and since b is antiadjacent to k_i, it follows that $q < t-i+1$. But then $p + q < t$, a contradiction. This proves (4).

(5) There do not exist $a, a' \in A$, $k \in K$ and $c \in C$, such that the pairs ak, ac are adjacent, and the pairs $a'c, a'k$ are antiadjacent.

Let $i, p, q, r \in \{1, \dots, t\}$ such that $k = k_i$, $a \in A_p$, $a' \in A_q$ and $c \in C_r$. Since a is adjacent to k_i and a' is antiadjacent to k_i, it follows that $p \geq i$ and $q \leq i$. Since c is adjacent to a and antiadjacent to a', it follows that $r \geq p$ and $r \leq q$. Consequently, $p = q = r = i$, and $a' \in A'_i$. But C_i is strongly complete to A'_i, a contradiction. This proves (5).

Suppose that there is a bull T in G. Let $T = \{b_1, b_2, b_3, b_4, b_5\}$, where the pairs $b_1b_2, b_2b_3, b_2b_4, b_3b_4, b_4b_5$ are adjacent, and all the remaining pairs are antiadjacent.

Since $A \cup D$ and $B \cup C$ are strongly stable sets, it follows that at least one of b_2, b_3, b_4 belongs to K.

Suppose first that $|K \cap \{b_2, b_3, b_4\}| = 1$. Assume first that $b_3 \in K$, say $b_3 = k_i$ for some $i \in \{1, \dots, t\}$. Then, since each of A, B is strongly stable, and K is strongly anticomplete to $C \cup D$, we may assume from the symmetry that $b_2 \in A$ and $b_4 \in B$. Let $s \in \{1, \dots, t\}$ such that $b_2 \in A_s$. Then $s \geq i$. Since b_1 is antiadjacent to b_3 and adjacent to b_2, it follows that $b_1 \in B \cup C$. Similarly, $b_5 \in A \cup D$. Suppose $b_5 \in A$. If $b_1 \in B$, then, since both b_1 and b_5 are antiadjacent to b_3, (4) implies that b_1 is strongly adjacent to b_5, a contradiction. So $b_1 \in C$. But then b_2 is adjacent to both b_3, b_1, and b_5 is antiadjacent to both b_3, b_5, contrary to (5). This proves that $b_5 \in D$, and, from the symmetry, $b_1 \in C$. Then $b_1 \in \bigcup_{j \geq s} C_j \subseteq \bigcup_{j \geq i} C_j$, and, similarly, $b_5 \in \bigcup_{j \geq t-i+1} D_j$, and so b_1 is strongly adjacent to b_5, a contradiction. This proves that $b_3 \notin K$. From the symmetry we may assume that $b_2 \in K$, say $b_2 = k_i$ for some $i \in \{1, \dots, t\}$. Let $\{x, y\} = \{b_3, b_4\}$. Then, since each of A, B is strongly stable, and K is strongly anticomplete to $C \cup D$, we may assume from the symmetry that $x \in A$ and $y \in B$. Since b_1 is adjacent to b_2, we may assume from the symmetry, that $b_1 \in K \cup A$. Since b_1 is antiadjacent to both b_3, b_4, (1) implies that $b_1 \notin K$. Therefore $b_1 \in A$, and so, by (3), b_1 is strongly adjacent to y, a contradiction. This proves that $|K \cap \{b_2, b_3, b_4\}| > 1$.

Next suppose that $|K \cap \{b_2, b_3, b_4\}| = 2$. Assume first that $b_3 \notin K$. Then $b_2, b_4 \in K$. Then we may assume from the symmetry that $b_3 \in A$. Since b_1 is antiadjacent to b_4, and b_5 to b_2, it follows that $b_1, b_5 \in A \cup B$. By (2), it follows that not both of b_1, b_5 are in A, and not both are in B. Thus we may assume that $b_1 \in A$, and $b_5 \in B$, but now both b_3, b_5 are adjacent to b_4, and yet b_3 is antiadjacent to b_5, contrary to (3). This proves that $b_3 \in K$. From the symmetry we may assume that $b_2 \in K$ and $b_4 \in A$. Then $b_1 \in A \cup B$. Since b_2 is adjacent to both b_1 and b_4, and since b_1 is antiadjacent to b_4, (3) implies that $b_1 \in A$. Since b_5 is adjacent to b_4, and antiadjacent to b_2, it follows that $b_5 \in B \cup C$. If $b_5 \in B$, then, since b_3 is antiadjacent to both b_1, b_5, (4) implies that b_1 is strongly adjacent to b_5, a contradiction. So $b_5 \in C$. But then b_4 is adjacent to both b_3, b_5, and b_1 is antiadjacent to both b_3, b_5, contrary to (5). This proves that $|K \cap \{b_2, b_3, b_4\}| > 2$, and therefore $b_2, b_3, b_4 \in K$.

Then $b_1, b_5 \in A \cup B$. By (2), not both b_1, b_5 are in A, and, from the symmetry not both are in B. So we may assume that $b_1 \in A$, and $b_5 \in B$. But now, since b_3 is antiadjacent to both b_1, b_5, (4) implies that b_1 is strongly adjacent to b_5, a contradiction. This proves 3.1. ▮

Melts. Let G be a trigraph, such that $V(G)$ is the disjoint union of four sets K, M, A, B, where A and B are strongly stable sets, and K and M are strong cliques. Assume that $|A| > 1$ and $|B| > 1$. Let $K = \{k_1, \dots, k_m\}$ and $M = \{m_1, \dots, m_n\}$. Let A be the union of pairwise disjoint subsets $A_{i,j}$ where $i \in \{0, \dots, m\}$ and $j \in \{0, \dots, n\}$, and let B be the disjoint union of subsets $B_{i,j}$ where $i \in \{0, \dots, m\}$ and $j \in \{0, \dots, n\}$. Let $A_{0,0} = B_{0,0} = \emptyset$. Assume also that

- K is strongly anticomplete to M
- for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ $A_{i,j}$ is
  strongly complete to $\{k_1, \dots, k_{i-1}\} \cup \{m_{n-j+2}, \dots, m_n\}$,
  complete to $\{k_i\} \cup \{m_{n-j+1}\}$,
  strongly anticomplete to $\{k_{i+1}, \dots, k_m\} \cup \{m_1, \dots, m_{n-j}\}$,
  and the set $B_{i,j}$ is
  strongly complete to $\{k_{m-i+2}, \dots, k_m\} \cup \{m_1, \dots, m_{j-1}\}$,
  complete to $\{k_{m-i+1}\} \cup \{m_j\}$,
  strongly anticomplete to $\{k_1, \dots, k_{m-i}\} \cup \{m_{j+1}, \dots, m_n\}$.
- for $i \in \{1, \dots, m\}$, $A_{i,0}$ is
  strongly complete to $\{k_1, \dots, k_{i-1}\}$,
  complete to $\{k_i\}$,
  strongly anticomplete to $\{k_{i+1}, \dots, k_m\} \cup M$

- for $j \in \{1, \dots, n\}$, $A_{0,j}$ is
strongly complete to $\{m_{n-j+2}, \dots, m_n\}$,
complete to $\{m_{n-j+1}\}$,
strongly anticomplete to $K \cup \{m_1, \dots, m_{n-j}\}$

- for $i \in \{1, \dots, m\}$, $B_{i,0}$ is
strongly complete to $\{k_{m-i+2}, \dots, k_m\}$,
complete to $\{k_{m-i+1}\}$,
strongly anticomplete to $\{k_1, \dots, k_{m-i}\} \cup M$

- for $j \in \{1, \dots, n\}$, $B_{0,j}$ is
strongly complete to $\{m_1, \dots, m_{j-1}\}$,
complete to $\{m_j\}$,
strongly anticomplete to $K \cup \{m_{j+1}, \dots, m_n\}$

- the sets $\bigcup_{0 \le j \le n} A_{m,j}$, $\bigcup_{0 \le j \le n} B_{m,j}$, $\bigcup_{0 \le i \le m} A_{i,n}$ and $\bigcup_{0 \le i \le m} B_{i,n}$ are all non-empty

- Let $i, i' \in \{0, \dots, m\}$ and $j, j' \in \{0, \dots, n\}$, and suppose that $i' > i$ and $j' > j$. Then at least one of the sets $A_{i,j}$ and $A_{i',j'}$ is empty, and at least one of the sets $B_{i,j}$ and $B_{i',j'}$ is empty

- For $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, $A_{i,j}$ is strongly complete to B, and $B_{i,j}$ is strongly complete to A

- For $i, i' \in \{1, \dots, m\}$ and $j, j' \in \{1, \dots, n\}$, $A_{i,0}$ is strongly complete to $B_{i',0}$, and $A_{0,j}$ is strongly complete to $B_{0,j'}$

- for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots n\}$, $A_{i,0}$ is the disjoint union of sets $A^k_{i,0}$ with $k \in \{0, \dots, n\}$, and $A_{0,j}$ is the disjoint union of sets $A^k_{0,j}$ with $k \in \{0, \dots, m\}$,

- for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots n\}$, $B_{i,0}$ is the disjoint union of sets $B^k_{i,0}$ with $k \in \{0, \dots, n\}$, and $B_{0,j}$ is the disjoint union of sets $B^k_{0,j}$ with $k \in \{0, \dots, m\}$.

- for $i \in \{1, \dots, m\}$, every vertex of $A^0_{i,0}$ is strongly anticomplete to $\bigcup_{1 \le j \le n} B_{0,j}$, and has a neighbor in $\bigcup_{1 \le j \le m} \bigcup_{1 \le k \le n} B_{j,k}$

- for $j \in \{1, \dots, n\}$, every vertex of $A^0_{0,j}$ is strongly anticomplete to $\bigcup_{1 \le i \le m} B_{i,0}$, and has a neighbor in $\bigcup_{1 \le i \le m} \bigcup_{1 \le k \le n} B_{i,k}$

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- for $j \in \{1, \dots, n\}$, every vertex of $B^0_{0,j}$ is strongly anticomplete to $\bigcup_{1 \le i \le m} A_{i,0}$, and has a neighbor in $\bigcup_{1 \le i \le m} \bigcup_{1 \le k \le n} A_{i,k}$

- for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$,
every vertex of $A^i_{0,j}$ has a neighbor in $B_{i,0}$,
every vertex of $B^j_{i,0}$ has a neighbor in $A_{0,j}$,
every vertex of $A^j_{i,0}$ has a neighbor in $B_{0,j}$,
every vertex of $B^i_{0,j}$ has a neighbor in $A_{i,0}$,
$A^i_{0,j}$ is strongly complete to $\bigcup_{1 \le s < i} B_{s,0}$
$A^i_{0,j}$ is strongly anticomplete to $\bigcup_{i < s \le m} B_{s,0}$
$A^j_{i,0}$ is strongly complete to $\bigcup_{1 \le s < j} B_{0,s}$
$A^j_{i,0}$ is strongly anticomplete to $\bigcup_{j < s \le n} B_{0,s}$
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$B^i_{0,j}$ is strongly complete to $\bigcup_{1 \le s < i} A_{s,0}$
$B^i_{0,j}$ is strongly anticomplete to $\bigcup_{i < s \le m} A_{s,0}$

- for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ let
$A'_{i,0}$ be the set of vertices of $A_{i,0}$ that are semi-adjacent to k_i
$A'_{0,j}$ be the set of vertices of $A_{0,j}$ that are semi-adjacent to m_{n-j+1},
$B'_{i,0}$ be the set of vertices of $B_{i,0}$ that are semi-adjacent to k_{m-i+1},
$B'_{0,j}$ be the set of vertices of $B_{0,j}$ that are semi-adjacent to m_j.
Then
$A'_{i,0}$ is strongly complete to $\bigcup_{1 \le s \le n} B^i_{0,s}$,
$A'_{0,j}$ is strongly complete to $\bigcup_{1 \le s \le m} B^j_{s,0}$,
$B'_{i,0}$ is strongly complete to $\bigcup_{1 \le s \le n} A^i_{0,s}$,
$B'_{0,j}$ is strongly complete to $\bigcup_{1 \le s \le m} A^j_{s,0}$.

- there exist $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ such that either $A_{i,j} \neq \emptyset$, or $B_{i,j} \neq \emptyset$.

- Let $i, s, s' \in \{1, \dots, m\}$ and $j, t, t' \in \{1, \dots, n\}$ such that $t' \ge j \ge n + 1 - t$ and $s \ge i \ge m + 1 - s'$. Then at least one of $A_{s,t}$ and $B_{s',t'}$ is empty.

Under these circumstances we say that G is a *melt*. We say that a melt is an *A-melt* if $B_{i,j} = \emptyset$ for every $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$. We say that a melt is a *B-melt* if $A_{i,j} = \emptyset$ for every $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$. We say that a melt is a *double melt* if there exist $i, i' \in \{1, \dots, m\}$ and $j, j' \in \{1, \dots, n\}$ such that $A_{i,j} \neq \emptyset$, and $B_{i',j'} \neq \emptyset$.

3.2 *Every melt is bull-free.*

Proof. Let G be a melt. We use the notation from the definition of a melt. Suppose there is a bull $C = \{c_1, c_2, c_3, c_4, c_5\}$ in G, where the pairs $c_1c_2, c_2c_3, c_3c_4, c_2c_4, c_4c_5$ are adjacent, and the pairs $c_1c_3, c_1c_4, c_1c_5, c_2c_5, c_3c_5$

are antiadjacent. Let $X = \bigcup_{1\leq j\leq n} = A_{0,j}$, $Y = \bigcup_{1\leq j\leq n} B_{0,j}$, $Z = A \setminus X$, $W = B \setminus Y$. We observe that the graph $G \setminus M$ is a (K, Z, W, Y, X)-clique connector. Therefore, 3.1 implies that $C \cap M \neq \emptyset$, and, similarly, $C \cap K \neq \emptyset$. Since $\{c_2, c_3, c_4\}$ is a clique and since K is strongly anticomplete to M, we may assume that $M \cap \{c_2, c_3, c_4\} = \emptyset$. Since $M \cap C \neq \emptyset$, and c_1 is antiadjacent to c_5, and M is a strong clique, we may assume that $c_1 \in M$ and $c_5 \notin M$. Then $c_2 \in A \cup B$, and from the symmetry we may assume that $c_2 \in A$. Let $i \in \{1, \ldots, m\}$ and $j, k \in \{1, \ldots, n\}$ be such that $c_1 = m_j$ and $c_2 \in A_{i,k}$. Since c_2 is adjacent to c_1, it follows that $j \geq n - k + 1$. Since A, B are both strongly stable sets, it follows that at least one of c_3, c_4 belongs to K, and therefore, since c_2 is adjacent to both c_3, c_4, we deduce that $i > 0$. Consequently, c_2 is strongly complete to B. Let

$$B' = \bigcup_{0\leq i\leq m} \bigcup_{j\leq s\leq n} B_{i,s}.$$

Then $G|(K \cup \{m_j\} \cup A \cup (B \setminus B'))$ is a $(K, Z, W \setminus B', (Y \cup \{m_j\}) \setminus B', X)$-clique connector, and so 3.1 implies that $C \cap B' \neq \emptyset$. Since c_1 is anticomplete to $\{c_3, c_4, c_5\}$, it follows that $C \cap B_{s,t} = \emptyset$ for every $s \in \{0, \ldots, m\}$ and $t \in \{j+1, \ldots, n\}$, and there exists $s \in \{0, \ldots, m\}$ and $b \in B_{s,j} \cap C$ such that b is semi-adjacent to c_1. Since c_2 is strongly complete to B, it follows that $b \in \{c_3, c_4\}$, and the vertex of $\{c_3, c_4\} \setminus \{b\}$ belongs to K, say it is k_p. Then both c_2 and b is adjacent to both k_p and m_j, contrary to the last condition in the definition of a melt. This proves 3.2. ∎

Let H be a graph. For a vertex $v \in V(H)$, the *degree* of v in H, denoted by $deg(v)$, is the number of edges of H incident with v. If H is the empty graph let $maxdeg(H) = 0$, and otherwise we define $maxdeg(H) = \max_{v \in V(H)} deg(v)$.

The class $\mathcal{T}_1$. Before giving a precise definition of the class $\mathcal{T}_1$, let us describe roughly what a trigraph in this class looks like. The idea is the following. Every trigraph in $\mathcal{T}_1$ consists of a triangle-free part X (in what follows $V(X)$ is the union of L, the sets $h(e)$, and the sets $h(e, v) \cap B$), and a collection of pairwise disjoint and pairwise anticomplete strong cliques Y_v (in what follows Y_v is the union of $h(v)$ and the sets $h(e, v) \setminus B$ for all edges e incident with v). Every vertex of X attaches in at most two cliques Y_v. Each Y_v, together with vertices of X at distance at most two from Y_v, induces a clique connector. If every vertex of X has neighbors in at most one Y_v, this describes the graph completely. Describing the adjacency rules for vertices of X that attach in two different cliques, Y_u and Y_v is more complicated (we need to explain how the clique connectors for Y_u and Y_v overlap). Without going into details, the structure there is locally a melt.

Let us now turn to the precise definition of $\mathcal{T}_1$. Let H be a loopless triangle-free graph with $maxdeg(H) \leq 2$ (H may be empty, and may have parallel edges). We say that a trigraph G admits an *H-structure* if there exist a subset L of $V(G)$ and a map

$$h : V(H) \cup E(H) \cup (E(H) \times V(H)) \to 2^{V(G)\setminus L}$$

such that

- every vertex of $V(G) \setminus L$ is in $h(x)$ for exactly one element x of $V(H) \cup E(H) \cup (E(H) \times V(H))$, and
- $h(v) \neq \emptyset$ for every $v \in V(H)$ of degree zero, and
- $h(e) \neq \emptyset$ for every $e \in E(H)$, and
- $h(e, v) \neq \emptyset$ if e is incident with v, and
- $h(e, v) = \emptyset$ if e is not incident with v, and
- for $u, v \in V(H)$, $h(u)$ is strongly anticomplete to $h(v)$, and
- $h(v)$ is a strong clique for every $v \in V(H)$, and
- every vertex of L has a neighbor in at most one of the sets $h(v)$ where $v \in V(H)$, and
- $G|(L \cup (\bigcup_{e \in E(H)} h(e)))$ has no triangle, and
- for every $e \in E(H)$, every vertex of L is either strongly complete or strongly anticomplete to $h(e)$, and
- $h(e)$ is either strongly complete or strongly anticomplete to $h(f)$ for every $e, f \in E(H)$; if e and f share an endpoint, then $h(e)$ is strongly complete to $h(f)$, and
- for every $e \in E(H)$ and $v \in V(H)$, $h(e)$ is strongly anticomplete to $h(v)$, and
- for $v \in V(H)$, let S_v be the vertices of L with a neighbor in $h(v)$, and let T_v be the vertices of $(L \cup (\bigcup_{e \in E(H)} h(e))) \setminus S_v$ with a neighbor in S_v. Then there is a partition of S_v into two sets A_v, B_v, and a partition of T_v into two sets C_v, D_v such that $G|(h(v) \cup S_v \cup T_v)$ is an $(h(v), A_v, B_v, C_v, D_v)$-clique connector, and
- for $v \in V(H)$, if there exist $a \in A_v$ and $b \in B_v$ antiadjacent with a common neighbor in $h(v)$, then v has degree zero in H.

Moreover, let e be an edge of H with ends u, v. Then

- if $f \in E(H) \setminus \{e\}$ is incident with v, then $h(e, v)$ is strongly complete to $h(f, v)$, and

- $G|(h(e) \cup h(e,v) \cup h(e,u))$ is an $h(e)$-melt, such that if (K, M, A, B) are as in the definition of a melt, then $K \subseteq h(e,v)$, $M \subseteq h(e,u)$, $A = h(e)$, $B \subseteq h(e,v) \cup h(e,u)$, every vertex of $h(e,v) \cap B$ has a neighbor in K, and every vertex of $h(e,u) \cap B$ has a neighbor in M (and, in particular, $h(e,v)$ is strongly anticomplete to $h(e,u)$); and

- $h(e,v)$ is strongly complete to $h(v)$, and $h(e,v)$ is strongly anticomplete to $h(w)$ for every $w \in V(H) \setminus \{v\}$, and

- $h(e,v)$ is strongly anticomplete to $h(f,w)$ for every $f \in E(H) \setminus \{e\}$, and $w \in V(H) \setminus \{v\}$, and

- $h(e,v)$ is strongly anticomplete to $h(f)$ for every $f \in E(H) \setminus \{e\}$.

Furthermore, either the following statements all hold, or they all hold with the roles of $A_u \cup A_v$ and $B_u \cup B_v$ switched:

- $h(e)$ is strongly complete to $B_u \cup B_v$, and

- $h(e,v)$ is strongly complete to A_v and strongly anticomplete to $L \setminus A_v$, and, and

- every vertex of $(L \cup (\bigcup_{f \in E(H)} h(f))) \setminus (A_u \cup A_v)$ with a neighbor in $A_u \cup A_v$ is strongly complete to $h(e)$.

Let us say that G belongs to $\mathcal{T}_1$ if either G is a double melt, or G admits an H structure for some loopless triangle-free graph H with maximum degree at most two.

In the definition of an H-structure, we did not specify the adjacencies between the sets $h(e)$ for disjoint edges e of H, except that

- $h(e)$ is either strongly complete or strongly anticomplete to $h(f)$ for every $e, f \in E(H)$.

In fact, the only constraints on these adjacencies come from the condition that

- $G|(L \cup (\bigcup_{e \in E(H)} h(e)))$ has no triangle.

To tighten the structure, one might want to add another ingredient, which is a triangle-free supergraph F of the line graph of H, that would "record" for which pairs of disjoint edges e, f of H, the sets $h(e)$ and $h(f)$ are strongly complete to each other. We did not do that here, since such a graph F can be easily reconstructed from the H-structure. The situation concerning the adjacencies between the vertices of L and the sets $h(e)$ is similar.

We observe the following:

3.3 *Every trigraph in $\mathcal{T}_1$ is bull-free.*

Proof. Let $G \in \mathcal{T}_1$. If G is a double melt, then 3.3 follows from 3.2, so we may assume not. Let H, h and L be as in the definition of $\mathcal{T}_1$. We use the notation of the definition of $\mathcal{T}_1$. Suppose there is a bull B in G. Let $B = \{v_1, v_2, v_3, v_4, v_5\}$, where the pairs $v_1v_2, v_2v_3, v_2v_4, v_3v_4, v_4v_5$ are adjacent, and all the remaining pairs are antiadjacent. Since $G|(L \cup (\bigcup_{e \in E(H)} h(e)))$ is triangle-free, it follows that at least one of v_2, v_3, v_4 belongs to $h(v) \cup h(e,v)$ for some $v \in V(H)$ and $e \in E(H)$. If $\{v_2, v_3, v_4\} \cap h(e,v) = \emptyset$ for every $e \in E(H)$ and $v \in V(H)$, then $B \subseteq h(v) \cup S_v \cup T_v$ for some $v \in V(H)$, contrary to the 3.1, since $G|(h(v) \cup S_v \cup T_v)$ is a clique connector. So we may assume that at least one of v_2, v_3, v_4 belongs to $h(e,v)$ for some $v \in V(H)$ and $e \in E(H)$. Let u be the other end of e, and if v has degree two in H, let f be the other edge incident with v. If v has degree one in H, let $X = Y = \emptyset$, and if v has degree two in H, let $X = h(f)$ and $Y = h(f,v)$. Let Z be the set of vertices of $L \cup ((\bigcup_{g \in E(H) \setminus \{e,f\}} h(g)) \setminus (S_v \cup T_v))$ that are strongly complete to $h(e)$. Then

$$B \subseteq h(v) \cup h(e,v) \cup h(e) \cup h(e,u) \cup S_v \cup T_v \cup X \cup Y \cup Z.$$

We observe that $h(v) \cup h(e,v) \cup h(e) \cup S_v \cup T_v \cup X \cup Y \cup Z$ is a clique connector, and so $B \cap h(e,u) \neq \emptyset$. Since each of v_2, v_3, v_4 has distance at most two from every vertex of B, it follows that $\{v_2, v_3, v_4\} \cap (h(v) \cup Y) = \emptyset$. Since $h(e,u)$ is strongly anticomplete to $h(e,v)$, it follows that $B \cap h(e,u) \subseteq \{v_1, v_5\}$, and we may assume from the symmetry that $v_1 \in B \cap h(e,u)$. Then $v_2 \notin h(e,v)$, and $\{v_3, v_4\} \cap h(e,v) \neq \emptyset$. Since v_2 is complete to $\{v_1, v_3, v_4\}$, it follows that $v_2 \in h(e)$. Now, since $\{v_2, v_3, v_4\}$ is a triangle, $v_2 \in h(e)$, $h(e)$ is strongly anticomplete to $h(v)$, there is no triangle in $h(e) \cup S_v$, and no vertex of S_v has both a neighbor in $h(e)$ and a neighbor in $h(e,v)$, it follows that $\{v_3, v_4\} \subseteq h(e,v)$. Since v_5 is adjacent to v_4 and antiadjacent to v_3, it follows that $v_5 \in h(e,v) \cup h(e)$. But now $B \subseteq h(e) \cup h(e,u) \cup h(e,v)$, contrary to 3.2. This proves 3.3. ∎

Next let us describe some decompositions (these definitions appear in [1], but we repeat them for completeness). Let G be a trigraph. A proper subset $X$of $V(G)$ is a *homogeneous set* in G if every vertex of $V(G) \setminus X$ is either strongly complete or strongly anticomplete to X. We say that G admits *a homogeneous set decomposition*, if there is a homogeneous set in G of size at least two.

For two disjoint subsets A and B of $V(G)$, the pair (A, B) is a *homogeneous pair* in G, if A is a homogeneous set in $G \setminus B$ and B is a homogeneous set in $G \setminus A$. We say that the pair (A, B) is *tame* if

- $|V(G)| - 2 > |A| + |B| > 2$, and
- A is not strongly complete and not strongly anticomplete to B.

G admits a *homogeneous pair decomposition* if there is a tame homogeneous pair in G.

Let $S \subseteq V(G)$. A *center* for S is a vertex of $V(G) \setminus S$ that is complete to S, and an *anticenter* for S is a vertex of $V(G) \setminus S$ that is anticomplete to S. A vertex of G is a *center (anticenter)* for an induced subgraph H of G if it is a center (anticenter) for $V(H)$.

We say that a trigraph G is *elementary* if there does not exist a path P of length three in G, such that some vertex c of $V(G) \setminus V(P)$ is a center for P, and some vertex a of $V(G) \setminus V(P)$ is an anticenter for P. The main result of this paper is the following:

3.4 *Let G be an elementary bull-free trigraph. Then either*

- *one of $G, \overline{G}$ belongs to $\mathcal{T}_1$, or*
- *G admits a homogeneous set decomposition, or*
- *G admits a homogeneous pair decomposition.*

Let us call a bull-free trigraph that does not admit a homogeneous set decomposition, or a homogeneous pair decomposition, and does not contain a path of length three with a center *unfriendly.* In view of the main result of [1], in this paper we deal mainly with unfriendly graphs (for a precise explanation, see the end of Section 6).

4 Prisms

Let $k \geq 3$ be an integer. A *k-prism* is a trigraph whose vertex set is the disjoint union of two cliques $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$; and such that for every $i, j \in \{1, \ldots, k\}$, a_i is adjacent to b_j if $i = j$ and a_i is antiadjacent to b_j if $i \neq j$. A *prism* is a 3-prism. For a trigraph G, an *n-prism in G* is an induced subtrigraph of G that is an n-prism.

We start by listing some properties of a prism in an unfriendly trigraph.

4.1 *Let G be an unfriendly trigraph, and let P be a k-prism in G. Let A and B be as in the definition of a k-prism. Then*

- *A and B are strong cliques,*
- *a_i is strongly antiadjacent to b_j for every $1 \leq i \neq j \leq k$,*
- *no vertex $x \in V(G) \setminus V(P)$ is complete to $\{a_i, b_i, a_j, b_j\}$ for any $1 \leq i < j \leq k$.*

Proof. Let i, j, m be three distinct integers in $\{1, \ldots, k\}$. Since $\{a_i, b_i, b_m, b_j, a_j\}$ is not a bull, it follows that a_i is strongly adjacent to a_j. Therefore, A, and from the symmetry B, is a strong clique. This proves the first assertion of 4.1

If a_i is adjacent to b_j, then a_i is a center for the path a_m-a_j-b_j-b_i, contrary to the fact that G is unfriendly. This proves the second assertion of 4.1.

Finally, if some vertex $x \in V(G) \setminus V(P)$ is complete to $\{a_i, b_i, a_j, b_j\}$, then since a_i-x-b_j-b_m is not path with center b_i, it follows that x is adjacent to b_m. But now a_i-a_j-b_j-b_m is a path of length three with center x, contrary to the fact that G is unfriendly. This completes the proof of 4.1. ∎

The main result of this section is the following:

4.2 *Let G be an unfriendly trigraph. Assume that for some integer $n \geq 3$, G contains an induced subtrigraph that is an n-prism. Then G is a prism.*

Proof. Let $A_1, \dots A_k, B_1, \dots, B_k$ be pairwise disjoint non-empty subsets of $V(G)$ such that for $i, j \in \{1, \dots, k\}$

- A_i is complete to A_j and B_i is complete to B_j
- if $i \neq j$, then A_i is anticomplete to B_j
- every vertex of A_i has a neighbor in B_i
- every vertex in B_i has a neighbor in A_i
- $k \geq 3$.

Let $W = \bigcup_{i=1}^{k} (A_i \cup B_i)$. In these circumstances we call $G|W$ a *hyperprism* in G. Since G contains an n-prism, there is a hyperprism in G. We may assume that W is maximal subject to $G|W$ being a hyperprism in G. Let $A = \bigcup_{i=1}^{k} A_i$ and $B = \bigcup_{i=1}^{k} B_i$.

(1) Let $i, j \in \{1, \dots, k\}$ such that $i \neq j$. Then A_i is strongly complete to A_j, and strongly anticomplete to B_j.

Let $m \in \{1, \dots, k\} \setminus \{i, j\}$. Let $a_i \in A_i$ and $a_j \in A_j$. Choose $b_i \in B_i$ adjacent to a_i and $b_j \in B_j$ adjacent to a_j. Choose $a_m \in A_m$ and $b_m \in B_m$ adjacent. Then $G|\{a_i, b_i, a_j, b_j, a_m, b_m\}$ is a 3-prism, and so by 4.1 a_i is strongly adjacent to a_j, and a_i is strongly antiadjacent to b_j. Now if follows from the symmetry that A_i is strongly complete to A_j. Similarly, since every vertex of B_j has a neighbor in A_j, it follows that A_i is strongly anticomplete to B_j. This proves (1).

(2) Let $v \in V(G) \setminus W$ and let $i \in \{1, \dots, k\}$. Suppose v has a neighbor $a_i \in A_i$ and a neighbor $b_i \in B_i$. Then a_i is strongly antiadjacent to b_i.

Assume a_i is adjacent to b_i. From the symmetry we may assume that $i = 1$. Suppose v has a neighbor $a_2 \in A_2$ and a neighbor $b_2 \in B_2$. Since G is unfriendly a_2-a_1-b_1-b_2 is not a three edge path with center v, and therefore a_2 is strongly adjacent to b_2. Let $a_3 \in A_3$ and $b_3 \in B_3$ be adjacent. Then

$G|\{a_1, a_2, a_3, b_1, b_2, b_3\}$ is a 3-prism and v is complete to $\{a_1, a_2, b_1, b_2\}$, contrary to 4.1. This proves, using symmetry, that for every $j \in \{2, \ldots, k\}$, v is strongly anticomplete to at least one of A_j, B_j. Suppose that for some $j, m \in \{2, \ldots, k\}$, v has a neighbor $a_j \in A_j$ and $b_m \in B_m$. Then $j \neq m$, and a_j-a_1-b_1-b_m is a path with center v, a contradiction. This proves that v is strongly anticomplete to at least one of $A \setminus A_1$ and $B \setminus B_1$. From the symmetry we may assume that v is strongly anticomplete to $B \setminus B_1$. If for some $j \in \{2, \ldots, k\}$, v has an antineighbor $a_j \in A_j$, then $\{a_j, a_1, v, b_1, b_m\}$ is a bull for every $b_m \in B_m$ with $m \in \{2, \ldots, k\} \setminus \{j\}$. This proves that v is strongly complete to $A \setminus A_1$. But now the sets $A_1 \cup \{v\}, \ldots, A_k, B_1, \ldots, B_k$ form a hyperprism in G, contrary to the maximality of W. This proves (2).

(3) Let $v \in V(G) \setminus W$ and let $i, j, m \in \{1, \ldots, k\}$ be all distinct. Suppose $b_i \in B_i$ is adjacent to v, and $b_j \in B_j, a_m \in A_m$ and $b_m \in B_m$ are antiadjacent to v. Then a_m is antiadjacent to b_m.

If a_m is adjacent to b_m, then $\{v, b_i, b_j, b_m, a_m\}$ is a bull, a contradiction. This proves (3).

(4) Let $v \in V(G) \setminus W$ and let $i \in \{1, \ldots, k\}$. Then v is strongly anticomplete to at least one of A_i, B_i.

Suppose not. We may assume that v has a neighbor in A_1 and a neighbor in B_1. For $j \in \{1, \ldots, k\}$, let A'_j be the set of neighbors of v in A_j, and $A''_j = A_j \setminus A'_j$. Let B'_j and B''_j be defined similarly. By (2), A'_j is strongly anticomplete to B'_j. Since every vertex in A_j has a neighbor in B_j, it follows that if A'_j is non-empty, then so is B''_j; and if B'_j is non-empty, then so is A''_j. In particular, A'_1, B'_1, A''_1 and B''_1 are all non-empty.

Suppose that some $a_2 \in A''_2$ is adjacent to some $b_2 \in B''_2$. By (3), and the symmetry, it follows that v is strongly complete to $A_3 \cup B_3$, and so $A''_3 = B''_3 = \emptyset$, a contradiction. This proves, using symmetry, that for every $j \in \{2, \ldots, k\}$, A''_j is strongly anticomplete to B''_j. Since every vertex of A_j has a neighbor in B_j, it follows that $A''_j \neq \emptyset$ if and only if $B'_j \neq \emptyset$, and, symmetrically, $B''_j \neq \emptyset$ if and only if $A'_j \neq \emptyset$.

If v is anticomplete to $B \setminus B_1$, then v is complete to $A \setminus A_1$, and the sets $A_1 \cup \{v\}, \ldots, A_k, B_1, \ldots, B_k$ form a hyperprism, contrary to the maximality of W. This proves that for some $2 \leq s \leq k$, $B'_s \neq \emptyset$, and, from the symmetry, for some $2 \leq t \leq k$, $A'_t \neq \emptyset$. It follows that $A''_s \neq \emptyset$ and $B''_t \neq \emptyset$. Now, by (3) (and from the symmetry if $s = t$), A''_1 is strongly anticomplete to B''_1.

Next we claim that for $j \in \{1, \ldots, k\}$, A'_j is strongly complete to A''_j, and B'_j to B''_j. Suppose there exist $a'_j \in A'_j$ and $a''_j \in A''_j$ antiadjacent. Choose $b \in B \setminus B_j$ adjacent to v (such a vertex b exists for j is different from at least one of $1, t$). Let $b_j \in B_j$ be adjacent to a''_j. Then $b_j \in B'_j$, and so, by (2), b_j is strongly antiadjacent to a'_j. Now $\{a'_j, v, b, b_j, a''_j\}$ is a bull, a contradiction.

This proves that A'_j is strongly complete to A''_j, and from the symmetry B'_j is strongly complete to B''_j.

Let $\mathcal{J} = \{j \in \{1, \dots, k\} : A'_j \neq \emptyset\}$. Then $B''_j \neq \emptyset$ for $j \in \mathcal{J}$. Moreover, for $j \in \{1, \dots, k\} \setminus \mathcal{J}$, $B''_j = \emptyset$. Then $|\mathcal{J}| \geq 2$. Let

$$\tilde{A}_0 = \bigcup_{j=1}^{k} A''_j \cup \{v\},$$

$$\tilde{B}_0 = \bigcup_{j=1}^{k} B'_j$$

and for $j \in \mathcal{J}$, let

$$\tilde{A}_j = A'_j \text{ and } \tilde{B}_j = B''_j.$$

Now, since $|\mathcal{J}| \geq 2$, the sets $\tilde{A}_0, \{\tilde{A}_j\}_{j\in\mathcal{J}}, \tilde{B}_0, \{\tilde{B}_j\}_{j\in\mathcal{J}}$ form a hyperprism, contrary to the maximality of W. This proves (4).

(5) Let $v \in V(G) \setminus W$. Then v is strongly anticomplete to at least one of A, B.

Suppose v has neighbors $a_1 \in A$ and $b_2 \in B$. From the symmetry we may assume that $a_1 \in A_1$. By (4), $b_2 \notin B_1$, and therefore we may assume that $b_2 \in B_2$. Now by (4), v is strongly anticomplete to $B_1 \cup A_2$.

Suppose v is strongly complete to $B \setminus B_1$. By (4), this implies that v is strongly anticomplete to $A \setminus A_1$. But now the sets $A_1, \dots, A_k, B_1 \cup \{v\}, \dots, B_k$ form a hyperprism, contrary to the maximality of W. This proves that v has an antineighbor in $b \in B \setminus B_1$. From the symmetry, renumbering $B_2, \dots, B_k$ if necessary, we may assume that $b \notin B_2$. Now since v has a neighbor in B_2, and since every vertex in A_1 has a neighbor in B_1, (3) implies that v is strongly complete to A_1. From the symmetry, it follows that for every $i \in \{1, \dots, k\}$, v is either strongly complete or strongly anticomplete to A_i, and the same for B_i. Consequently, v is strongly complete to $A_1 \cup B_2$, and strongly anticomplete to $B_1 \cup A_2$. Now by (3) and (4), for every $i \in \{3, \dots, k\}$, v is strongly complete to one of A_i, B_i, and strongly anticomplete to the other. From the symmetry between A and B, we may assume that v is strongly complete to A_i for at least two values of i.

Let $\mathcal{I} = \{i \in \{1, \dots, k\} : v \text{ is strongly complete to } A_i\}$. Then v is strongly complete to $\bigcup_{i\notin\mathcal{I}} B_i$, and strongly anticomplete to $(\bigcup_{i\in\mathcal{I}} B_i) \cup (\bigcup_{i\notin\mathcal{I}} A_i)$. Let

$$\tilde{A}_0 = \bigcup_{i\notin\mathcal{I}} A_i \cup \{v\},$$

$$\tilde{B}_0 = \bigcup_{i\notin\mathcal{I}} B_i$$

and for $i \in \mathcal{I}$, let

$$\tilde{A}_i = A_i \text{ and } \tilde{B}_i = B_i.$$

Now, since $|\mathcal{I}| \geq 2$, it follows that the sets $\tilde{A}_0, \{\tilde{A}_i\}_{i\in\mathcal{I}}, \tilde{B}_0, \{\tilde{B}_i\}_{i\in\mathcal{I}}$ form a hyperprism, contrary to the maximality of W. This proves (5).

(6) Let $v \in V(G) \setminus (A \cup B)$. Then one of the following holds for v:

1. *possibly with A and B switched, for some $i \in \{1, \dots, k\}$, v strongly complete to $A \setminus A_i$ and strongly anticomplete to B*
2. *v is strongly anticomplete to $A \cup B$.*

We may assume that v has a neighbor $a_1 \in A_1$, for otherwise (5.2) holds. Now (5) implies that v is strongly anticomplete to B. If there exist distinct $i, j \in \{2, \dots, k\}$ such that v has an antineighbor $a_i \in A_i$ and $a_j \in A_j$, then, choosing $b_i \in B_i$ to be a neighbor of a_i, we get a contradiction to (3). So we may assume that v is strongly complete to $A \setminus (A_1 \cup A_2)$. By the same argument with the roles of A_1, and, say, A_3, exchanged, we deduce that v is strongly complete to A_1, and (5.2) holds with $i = 2$. This proves (6).

Let A_0 be the set of vertices of $V(G) \setminus W$ that are strongly complete to A, and for $1 \leq i \leq k$, let A'_i be the set of vertices of $V(G) \setminus (W \cup A_0)$ that are strongly complete to $A \setminus A_i$. Define $B_0, B'_1, \dots, B'_k$ similarly. Let N be the set of vertices of $V(G) \setminus W$ that are strongly anticomplete to W. By (6), the sets $A_0, A'_1, \dots, A'_k, B_0, B'_1, \dots, B'_k, N$ are pairwise disjoint and have union $V(G) \setminus W$.

(7) $N = \emptyset$.

Suppose not, and choose $n \in N$. Since G is unfriendly, it follows that G is connected, and, from the symmetry, we may assume that n has a neighbor a in $A_0 \cup A'_1$. Let $a_2 \in A_2$, $a_3 \in A_3$, and choose $b_2 \in B_2$ adjacent to a_2. Then $\{n, a, a_3, a_2, b_2\}$ is a bull, a contradiction. This proves (7).

(8) Let $i, j \in \{1, \dots, k\}$. Then $A_0 \cup A'_i$ is strongly anticomplete to $B_0 \cup B'_j$.

From the maximality of W, $A_0 \cup A'_i$ is strongly anticomplete to $B_0 \cup B'_i$ for every $i \in \{1, \dots, k\}$. Suppose $a \in A'_i$ has a neighbor $b \in B'_j$ where $1 \leq i < j \leq k$. Let $b_j \in B_j$ be antiadjacent to b, and let $a_j \in A_j$ be a neighbor of b_j. Choose $a_m \in A \setminus (A_i \cup A_j)$. Now $\{b_j, a_j, a_m, a, b\}$ is a bull, a contradiction. This proves (8).

(9) Let $i, j \in \{1, \dots, k\}$ such that $i \neq j$. Then A'_i is strongly complete to $A'_j \cup A_0$.

Suppose $a'_i \in A'_i$ has an antineighbor $a'_j \in A'_j \cup A_0$. Let $a_i \in A_i$ be antiadjacent to a'_i and let $b_i \in B_i$ be a neighbor of a_i. Choose $m \in \{1, \dots, k\} \setminus \{i, j\}$ and $a_m \in A_m$. Now $\{a'_i, a_m, a'_j, a_i, b_i\}$ is a bull, a contradiction. This proves (9).

By (1), (8) and (9), $(A_1 \cup A'_1 \cup A_0, B_1 \cup B'_1 \cup A_0)$ is a homogeneous pair in G. Since G is unfriendly, it follows that this is not a tame homogeneous pair, and G does not admit a homogeneous set decomposition, and therefore $A'_1 = B'_1 = A_0 = B_0 = \emptyset$, and $|A_1| = |B_1| = 1$. Form the symmetry, we deduce that $A'_i = B'_i = \emptyset$, and $|A_i| = |B_i| = 1$ for every $i \in \{1, \dots, k\}$. If $k > 3$, then $(A \setminus (A_1 \cup A_2), B \setminus (B_1 \cup B_2)$ is a tame homogeneous pair in G, a contradiction. Thus $k = 3$ and G is a prism. This proves 4.2. ∎

5 Lemmas about unfriendly trigraphs

In this section we prove a few lemmas about unfriendly trigraphs.

5.1 *Let G be unfriendly graph, let $m > 2$ be an integer, and let $Y_1, \dots, Y_m$ be pairwise disjoint anticonnected sets, such that for $i, j \in \{1, \dots, m\}$, Y_i is complete to Y_j. Let $v \in V(G) \setminus (\bigcup_{i=1}^m Y_i)$, assume that $|Y_1| > 1$ and v has a neighbor and an antineighbor in $\bigcup_{i=2}^m Y_i$. Then v is either strongly complete, or strongly anticomplete to Y_1.*

Proof. Suppose not. Then v has a neighbor a and an antineighbor a' in Y_1, and by 2.2 we may assume that a and a' are distinct and antiadjacent. From the symmetry, we may assume that v has a neighbor $x \in Y_2$ and an antineighbor $h \in Y_3$. But now v-a-h-a' is a path, and x is a center for it, contrary to the fact that G is unfriendly. This proves 5.1. ∎

5.2 *Let G be an unfriendly trigraph such that there is no prism in G, and let a_1-a_2-a_3-a_4-a_1 be a hole of length four. Let K be the set of vertices that are complete to $\{a_1, a_2\}$ and anticomplete to $\{a_3, a_4\}$. Then K is a strong clique.*

Proof. Suppose some two vertices of K are not strongly adjacent, and let C be an anti-component of K with $|C| > 1$. Since G is unfriendly, it follows that C is not a homogeneous set in G, and so, by 2.2 applied in $\overline{G}$, there exist vertices c, c', v such that $c, c' \in C$, $v \notin C$, v is adjacent to c' and antiadjacent to c, and c' is antiadjacent to c. Since $\{a_4, a_1, c', a_2, c\}$ is not a bull, it follows that $v \neq a_1$, and from the symmetry $v \neq a_2$. Since a_4-c'-a_2-c is not a path with center a_1, it follows that $v \neq a_4$, and from the symmetry $v \neq a_3$.

Suppose first that v is anticomplete to $\{a_1, a_2\}$. Since $\{v, c', a_2, a_1, a_4\}$ is not a bull, it follows that v is strongly adjacent to a_4, and, similarly,

v is strongly adjacent to a_3. But now $G|\{a_1, a_2, c', a_3, a_3, v\}$ is a prism, a contradiction. So we may assume that v is strongly adjacent to a_1, and by 5.1, v is strongly adjacent to a_2. Since $\{c, a_2, c', v, a_4\}$ is not a bull, it follows that v is strongly antiadjacent to a_4, and similarly to a_3. But now $v \in C$, a contradiction. This proves 5.2. ∎

5.3 *Let G be an unfriendly trigraph such that there is no prism in G, let a_1-a_2-a_3-a_4-a_1 be a hole in G, and let c be a center and a an anticenter for $\{a_1, a_2, a_3, a_4\}$. Then c is strongly antiadjacent to a.*

Proof. Suppose c is adjacent to a.

(1) Let $i \in \{1, \ldots, 4\}$. Then a_i is strongly adjacent to a_{i+1} (here the addition is performed mod 4*), c is strongly adjacent to a_i, and a is strongly antiadjacent to a_i.*

Since a_i-a_{i+3}-a_{i+2}-a_{i+1} is not a path with a center c, it follows that a_i is strongly adjacent to a_{i+1}. Since $\{a_i, a_{i+1}, a_{i+2}, c, a\}$ is not a bull, it follows that a_i is strongly adjacent to c. Finally, since a-a_i-a_{i+1}-a_{i+2} is not a path with center c, we deduce that a is strongly antiadjacent to a_i. This proves (1).

Let A_1, A_2, A_3, A_4 be connected subsets of $V(G)$, where $a_i \in A_i$ for $i \in \{1, \ldots, 4\}$, such that

- for $i \in \{1, \ldots, 4\}$, A_i is strongly complete to A_{i+1} (with addition *mod* 4),
- for $i = 1, 2$, A_i is anticomplete to A_{i+2},
- c is strongly complete to $A_1 \cup A_2 \cup A_3 \cup A_4$
- a is strongly anticomplete to $A_1 \cup A_2 \cup A_3 \cup A_4$.

Let $W = A_1 \cup A_2 \cup A_3 \cup A_4$, and assume that A_1, A_2, A_3, A_4 are chosen with W maximal. Since G is unfriendly, it follows that $A_1 \cup A_3$ is not a homogeneous set in G, and so some vertex v of $V(G) \setminus (A_1 \cup A_3)$ is mixed on $A_1 \cup A_3$. Then $v \notin A_2 \cup A_3 \cup \{a, c\}$. We may assume that v has a neighbor $v_1 \in A_1$, and antineighbor $v_3 \in A_3$. Since $A_1 \cup A_3, A_2 \cup A_4$ and $\{c\}$ are three anticonnected sets complete to each other, 5.1 implies that v is either strongly complete or strongly anticomplete to $A_2 \cup A_4 \cup \{c\}$.

Suppose first that v is strongly anticomplete to $A_2 \cup A_4 \cup \{c\}$. Since $\{v, v_1, a_2, c, a\}$ is not a bull, it follows that v is adjacent to a. But now v-a-c-v_1-v is a hole of length four, and a_2, a_4 are two antiadjacent vertices, each complete to $\{v_1, c\}$ and anticomplete to $\{v, a\}$, contrary to 5.2. This proves that v is strongly complete to $A_2 \cup A_4 \cup \{c\}$. Since a-v-a_2-v_3 is not a path with center c, it follows that v is strongly antiadjacent to a. If v is anticomplete to A_3, then replacing A_1 by $A_1 \cup \{v\}$ contradicts the maximality of W, so v has a strong neighbor in A_3, and therefore $A_3 \neq \{v_3\}$. Since A_3 is connected, 2.2 implies that there exist vertices $x, y \in A_3$, such that v is adjacent to x and antiadjacent to y, and x is adjacent to y. But now y-x-v-v_1 is a path, and c is a center for it, contrary to the fact that G is unfriendly. This proves 5.3. ∎

5.4 *Let H be a trigraph such that no induced subtrigraph of H is a path of length three. Then either*

1. *H is not connected, or*
2. *H is not anticonnected, or*
3. *there exist two vertices $v_1, v_2 \in V(H)$ such that v_1 is semi-adjacent to v_2, and $V(H) \setminus \{v_1, v_2\}$ is strongly complete to v_1 and strongly anticomplete to v_2.*

Proof. Let $X, Y \subseteq V(H)$ such that $X \neq \emptyset$, $Y \neq \emptyset$, X is either complete, or anticomplete to Y, and there is at most one semi-adjacent pair xy with $x \in X$ and $y \in Y$. Assume that X, Y are chosen with $X \cup Y$ maximal. Passing to the complement if necessary, we may assume that X is anticomplete to Y. First we show that $X \cup Y = V(H)$. Suppose not. Let $v \in V(H) \setminus (X \cup Y)$. Let X', Y' be the set of neighbors of v in X, Y, respectively. By the maximality of $X \cup Y$, it follows that $X' \neq \emptyset$ and $Y' \neq \emptyset$. Since x-x'-v-y' is not a path, where $x \in X \setminus X'$, $x' \in X'$ and $y' \in Y'$, it follows that X' is strongly anticomplete to $X \setminus X'$. Similarly, Y' is strongly anticomplete to $Y \setminus Y'$. Now $X' \cup Y' \cup \{v\}$ is anticomplete to $(X \setminus X') \cup (Y \setminus Y')$, and the only semi-adjacent pairs xy with $x \in X' \cup Y' \cup \{v\}$ and $y \in (X \setminus X') \cup (Y \setminus Y')$ are those with $x \in X$ and $y \in Y$. It follows from the maximality of $X \cup Y$ that $(X \setminus X') \cup (Y \setminus Y') = \emptyset$. Now $\{v\}$ is complete to $X \cup Y$, and since v is semi-adjacent to at most one vertex of H, it follows that there is at most one semi-adjacent pair with a vertex in $X \cup Y$ and a vertex in $\{v\}$, contrary to the maximality of $X \cup Y$. This proves that $X \cup Y = V(H)$.

If X is strongly anticomplete to Y, then the theorem holds. So we may assume that some $x \in X$ and $y \in Y$ are semi-adjacent. Since x'-x-y-y' is not a path for $x' \in X \setminus \{x\}$ and $y' \in Y \setminus \{y\}$, we may assume, from the symmetry, that x is strongly anticomplete to $X \setminus \{x\}$. If $X \neq \{x\}$, then $Y \cup \{x\}$ is strongly anticomplete to $X \setminus \{x\}$, and the theorem holds, so we may assume that $X = \{x\}$. Let Y_1 be the set of neighbors of y in Y, and Y_2 the set of strong antineighbors of y in Y. Since y is semi-adjacent to x, it follows that y is strongly complete to Y_1. If some $y_1 \in Y_1$ is adjacent to some $y_2 \in Y_2$, then x-y-y_1-y_2 is a path, a contradiction. So Y_1 is strongly anticomplete to Y_2. But now, if $Y_2 = \emptyset$, then the last outcome of the theorem holds, and if $Y_2 \neq \emptyset$ then the first outcome of the theorem holds. This proves 5.4. ∎

5.5 *Let G be an unfriendly trigraph with no prism, and let $u, v \in V(G)$ be adjacent. Let A, B be subsets of $V(G)$ such that*

- *u is strongly complete to A and strongly anticomplete to B,*
- *v is strongly complete to B and strongly anticomplete to A,*
- *No vertex of $V(G) \setminus (A \cup B)$ is mixed on A, and*
- *if $x, y \in B$ are adjacent, then no vertex of $V(G) \setminus (A \cup B)$ is mixed on $\{x, y\}$.*

Then $A = K \cup S$, where K is a strong clique and S is a strongly stable set.

Proof. Let K, S be subsets of A, such that K is a strong clique and K is strongly complete to $A \setminus (K \cup S)$, and S is a strongly stable set and S is strongly anticomplete to $A \setminus (K \cup S)$. Assume that K and S are chosen with $K \cup S$ maximal. Let $Z = A \setminus (K \cup S)$. We may assume that Z is non-empty, for otherwise the theorem holds.

(1) There do not exist $k, s \in Z$, such that k is semi-adjacent to s, k is strongly complete to $Z \setminus \{k, s\}$ and s is strongly anticomplete to $Z \setminus \{k, s\}$.

If such k, s exist, then $K \cup \{k\}$ and $S \cup \{s\}$ contradict the maximality of $K \cup S$. This proves (1).

(2) Z is anticonnected.

Suppose not. If some anticomponent Z_0 of Z has size one, then $K \cup Z_0, S$ contradict the maximality of $K \cup S$, so we may assume that there exist two anticomponents, Z_1, Z_2 of Z, each with at least two vertices. Since Z_1 is not a homogeneous set in G, it follows that there exists a vertex $v_1 \in V(G) \setminus Z_1$ such that v_1 is mixed on Z_1. Then $v_1 \notin A$. By 2.2, there exist vertices $z_1, z_1' \in Z_1$ such that z_1 is antiadjacent to z_1', and v_1 is adjacent to z_1 and antiadjacent to z_1'. Let v_2, z_2, z_2' be defined similarly. Then $v_1, v_2 \in B$. Since $\{v, v_1, z_1, z_2, z_1'\}$ is not a bull, it follows that v_1 is strongly antiadjacent to z_2. Similarly, v_2 is strongly antiadjacent to z_1. Since $\{v_1, z_1, u, z_2, v_2\}$ is not a bull, it follows that v_1 is strongly adjacent to v_2. But now $G|\{u, z_1, z_2, v, v_1, v_2\}$ is a prism, a contradiction. This proves (2).

Since u is complete to Z and G is unfriendly, it follows that there is no path of length three in $G|Z$. Now it follows from 5.4, (1), and (2) that Z is not connected. If some component Z_0 of Z has size one, then $K, S \cup Z_0$ contradict the maximality of $K \cup S$, so every component of Z has at least two vertices and, in particular, that there exist two components, Z_1, Z_2 of Z, each with at least two vertices. Let $i \in \{1, 2\}$. Since Z_i is not a homogeneous set in G, it follows that there exists a vertex $v_i \in V(G) \setminus Z_i$ such that

v_i is mixed on Z_i. Then $v_i \notin A$, and therefore $v_i \in B$. By 2.2, there exist vertices $z_i, z'_i \in Z_i$ such that z_i is adjacent to z'_i, and v_i is adjacent to z_i and antiadjacent to z'_i. Since for $z \in (Z \cup S) \setminus Z_i$, $\{v_i, z_i, z'_i, u, z\}$ is not a bull, it follows that v_i is strongly complete to $(Z \cup S) \setminus Z_i$. Let B_i be the set of all vertices of $V(G) \setminus \{u\}$ that are mixed on Z_i. Then $B_i \subseteq B$, B_i is strongly complete to $(A \cup S) \setminus Z_i$, and $B_1 \cap B_2 = \emptyset$.

Let $\{i, j\} = \{1, 2\}$.

(3) If $b \in V(G) \setminus (A \cup B_i)$ has a neighbor in B_i, then v is strongly anticomplete to Z_i.

Suppose not. Since $b \notin B_i$, it follows that b is strongly complete to Z_i. Let $b_i \in B_i$ be adjacent to b. By 2.2, there exist vertices $z, z' \in Z_i$ such that z is adjacent to z', and b_i is adjacent to z and antiadjacent to z'. Since G is unfriendly, it follows that b is not a center for the path v-b_i-z-z', and therefore b is strongly antiadjacent to v. Consequently, $b \notin B$, and so b is strongly complete to A. Choose $z_j \in Z_j$. Now z_j-b_i-z-z' is a path, and b is a center for it, contrary to the fact that G is unfriendly. This proves (3).

(4) Let $b \in V(G) \setminus (A \cup B_1 \cup B_2)$, $b_i \in B_i$ and $b_j \in B_j$, and assume that b is adjacent to b_i and antiadjacent to b_j. Then $b \in B$, and b is strongly anticomplete to B_j and strongly complete to Z_j.

By (3), b_i is strongly antiadjacent to b_j. By 2.2, there exist $z, z' \in Z_j$ such that z is adjacent to z', and b_j is adjacent to z and antiadjacent to z'. Since b_i is strongly complete to Z_j, and since $\{b, b_i, z', z, b_j\}$ is not a bull, it follows that b has a neighbor in Z_j. Since b is adjacent to b_i, (3) implies that b is strongly anticomplete to Z_i, and therefore b has a neighbor and an antineighbor in A. Since b is not in A, it follows that $b \in B$. Now by (3), b is strongly anticomplete to B_j, and since $b \notin B_j$, b is strongly complete to Z_j. This proves (4).

Let C_i be the set of all vertices of $V(G) \setminus (A \cup B_1 \cup B_2)$ that have a neighbor in B_i and an antineighbor in B_j. By (4), $C_i \subseteq B$ and C_i is strongly anticomplete to B_j. Let X be the vertices of $B \setminus (B_1 \cup B_2)$ that are strongly anticomplete to $B_1 \cup B_2$, and let Y be the vertices of $B \setminus (B_1 \cup B_2)$ that are strongly complete to $B_1 \cup B_2$. By (4), $B = B_1 \cup B_2 \cup C_1 \cup C_2 \cup X \cup Y$. Let X_i be the vertices of X with a neighbor in C_i, and let $X_0 = X \setminus (X_1 \cup X_2)$. By (3), B_i is strongly anticomplete to B_j. Since v is complete to B, and G is unfriendly, it follows that there is no path of length three in $G|B$, and therefore C_i is strongly anticomplete to $C_j \cup X_j$, X_i is disjoint from X_j, and the sets X_i, X_j, X_0 are pairwise strongly anticomplete to each other.

(5) K is strongly anticomplete to $B_1 \cup B_2$.

Suppose some k in K has a neighbor $b_1 \in B_1$. By 2.2, there exist $z_1, z_1' \in Z_1$ such that b_1 is adjacent to z and antiadjacent to z', and z is adjacent to z'. Let $z \in Z_2$. Then z is adjacent to b_1, and z-b_1-z-z' is a path with center k. This proves (5).

(6) Both C_1 and C_2 are non-empty.

Suppose C_1 is empty. We claim that (Z_1, B_1) is a homogeneous pair. Since Z_1 is a component of Z, no vertex of $V(G) \setminus B_1$ is mixed on Z_1. Suppose some $w \in V(G) \setminus (Z_1 \cup B_1)$ is mixed on B_1. Then $w \notin B_2$. Since $C_1 = \emptyset$, it follows that w has a neighbor in B_2. Since w has an antineighbor in B_1, we deduce that $w \in C_2 \cup A$, and since w has a neighbor in B_1, it follows that $w \in A$. Since B_1 is strongly complete to $(Z \cup S) \setminus Z_1$, it follows that $w \in K$, contrary to (5). This proves (6).

Let S_i be the vertices of S that are strongly complete to K and are not strongly complete to $C_i \cup X_i$. To complete the proof, we show that $(Z_i \cup S_i, B_i \cup C_i \cup X_i)$ is a homogeneous pair in G, contradicting the fact that G is unfriendly.

(7) Let $a, b, c \in B$ and $w \in V(G) \setminus B$, such that a is adjacent to b, c is anticomplete to $\{a, b\}$, and w is adjacent to a and anticomplete to $\{b, v\}$. Then $w \in A$ and w is strongly adjacent to c.

Since w is mixed on $\{a, b\}$, it follows that $w \in A$. Since $\{w, a, b, v, c\}$ is not a bull, it follows that w is strongly adjacent to c. This proves (7).

(8) No vertex of $V(G) \setminus (Z_i \cup S_i \cup B_i \cup C_i \cup X_i)$ is mixed on $B_i \cup C_i \cup X_i$.

First we claim that K is strongly anticomplete to $B_i \cup C_i \cup X_i$. Choose $w \in K$. By (5), w is strongly anticomplete to $B_i \cup B_j$. Since w is strongly anticomplete to B_j, and B_j is strongly anticomplete to $B_i \cup C_i \cup X_i$, it follows from (7) there there do not exit vertices $a, b \in B_i \cup C_i \cup X_i$, such that a is adjacent to b, and w is mixed on $\{a, b\}$. Now, since every vertex of C_i has a neighbor in B_i, it follows that w is strongly anticomplete to C_i; and since very vertex of X_i has a neighbor in C_i, it follows that w is strongly anticomplete to X_i. This proves the claim.

Next suppose that $r \in (Z \cup S \cup Y) \setminus (Z_i \cup S_i)$ is not strongly complete to $B_i \cup C_i \cup X_i$. Then r is strongly complete to B_i, and, since every vertex of C_i has a neighbor in B_i, and every vertex of X_i has a neighbor in C_i, there exist $p, q \in B_i \cup C_i \cup X_i$, such that p is adjacent to q, r is adjacent to p and antiadjacent to q, and $q \in C_i \cup X_i$. Assume first that $r \in Z \setminus Z_i$. By the maximality of $S \cup K$, it follows that every component of Z has size at least two, and so, from the symmetry we may assume that $r \in Z_j$. By (3), r is strongly anticomplete to C_j; and since C_j is strongly anticomplete to $\{p, q\}$ we get a contradiction to (7). This proves that $Z \setminus Z_i$ is strongly complete to $B_i \cup C_i \cup X_i$, and therefore $r \in (S \cup Y) \setminus S_i$. Choose $z_j \in Z_j$. If $r \in Y$, then z_j-r-p-q is a path with center v, contrary to the fact that G is unfriendly, so $r \in S \setminus S_i$. Since r is antiadjacent to q and $r \notin S_i$, we deduce that there exists $k \in K$ antiadjacent to r. Now $\{r, p, q, z_j, k\}$ is a bull, a contradiction. This proves that $(Z \cup S \cup Y) \setminus (Z_i \cup S_i)$ is strongly complete to $B_i \cup C_i \cup X_i$. Since $B_j \cup C_j \cup X_j \cup X_0$ is strongly anticomplete to $B_i \cup C_i \cup X_i$, it follows that no vertex of $(A \cup B) \setminus (Z_i \cup S_i \cup B_i \cup C_i \cup X_i)$ is mixed on $B_i \cup C_i \cup X_i$.

Let $w \in V(G) \setminus (Z_i \cup S_i \cup B_i \cup C_i \cup X_i)$, and assume that w is mixed on $B_i \cup C_i \cup X_i$. Then $w \notin (A \cup B \cup \{u, v\})$. Applying (4) twice, we deduce that w is not mixed on B_i. Since every vertex of C_i has a neighbor in B_i, and every vertex of X_i has a neighbor in C_i, it follows that there exist two adjacent vertices $a, b \in B_i \cup C_i \cup X_i$ such that w is adjacent to a and antiadjacent to b. But then $w \in A \cup B$, a contradiction. This proves (8).

(9) No vertex of $V(G) \setminus (Z_i \cup S_i \cup B_i \cup C_i \cup X_i)$ is mixed on $Z_i \cup S_i$.

Since no vertex of $V(G) \setminus (A \cup B)$ is mixed on A, it is enough to show that no vertex of $(A \cup B) \setminus (Z_i \cup S_i \cup B_i \cup C_i \cup X_i)$ is mixed on $Z_i \cup S_i$. Since K is strongly complete to $Z_i \cup S_i$, and $(Z \cup S) \setminus (Z_i \cup S_i)$ is strongly anticomplete to $(Z_i \cup S_i)$, it follows that no vertex of $A \setminus (Z_i \cup S_i)$ is mixed on $Z_i \cup S_i$. By (8) and symmetry, and since $Z_i \cup S_i$ is strongly complete to B_j, we deduce that $Z_i \cup S_i$ is strongly complete to $B_j \cup C_j \cup X_j$. We claim that no vertex of X_0 is mixed on $Z_i \cup S_i$. If $S_i = \emptyset$, then no vertex of $B \setminus B_i$ is mixed on $Z_i \cup S_i$, and the claim follows. So we may assume that $S_i \neq \emptyset$. Suppose $b \in X_0$ has an antineighbor $s \in Z_i \cup S_i$. Since b is strongly anticomplete to $B_i \cup C_i \cup X_i$, (7) implies that there do not exist adjacent vertices $p, q \in B_i \cup C_i \cup X_i$, such that s is mixed on $\{p, q\}$. Since every vertex of C_i has a neighbor in B_i, and every vertex of X_i has a neighbor in C_i, it follows that either s is mixed on B_i, or s is strongly complete to $B_i \cup C_i \cup X_i$, or s is strongly anticomplete to $B_i \cup C_i \cup X_i$. Since every vertex of S_i is strongly complete to B_i and has an antineighbor in $B_i \cup C_i \cup X_i$, it follows that $s \notin S_i$. Therefore $s \in Z_i$, and hence b is strongly anticomplete to Z_i. Consequently, there do not exist adjacent vertices $p, q \in B_i \cup C_i \cup X_i$, and $z \in Z_i$ such that z is mixed on $\{p, q\}$. By (3), C_i is strongly anticomplete to Z_i. Let $c_i \in C_i$ and let $b_i \in B_i$ be a neighbor of c_i. Then b_i has a neighbor $z \in Z_i$. But now z is adjacent to b_i and antiadjacent to c_i, a contradiction. This proves that $Z_i \cup S_i$ is strongly complete to X_0, and the claim follows.

By (3), Y is strongly anticomplete to Z_i. Suppose some vertex $y \in Y$ has a neighbor $s \in S_i$. Let $b_j \in B_j$, and let $b \in C_i \cup X_i$ be an antineighbor of s. Since $s \notin Z_j$, it follows that b_j is strongly adjacent to s. Since Y is strongly complete to B_i, (8) implies that y is strongly adjacent to b. Now $\{u, s, b_j, y, b\}$ is a bull, a contradiction. So Y is strongly anticomplete to S_i, and therefore to $Z_i \cup S_i$. Therefore, no vertex of $B \setminus (B_i \cup C_i \cup X_i)$ is mixed on $Z_i \cup X_i$. This proves (9).

Now, it follows from (8) and (9) that $(Z_i \cup S_i, B_i \cup C_i \cup X_i)$ is a homogeneous pair in G, contrary to the fact that G is unfriendly. This proves 5.5. ▮

5.6 *Let G be an unfriendly bull-free trigraph with no prism. Then there do not exist six vertices $a, b, c, d, x, y \in V(G)$ such that*

- *the pairs ab, cd, xy are adjacent,*
- *$\{a, b\}$ is anticomplete to $\{c, d\}$, and*
- *$\{x, y\}$ is complete to $\{a, b, c, d\}$.*

Proof. Since b-a-y-c is not a path with center x, it follows that y is strongly adjacent to b, and from the symmetry, $\{x, y\}$ is strongly adjacent to $\{a, b, c, d\}$.

Let $k \geq 2$ be an integer, and let $Y_0, \dots, Y_k$ be pairwise disjoint anticonnected sets, such that

- Y_0 is strongly complete to $\bigcup_{i=1}^{k} Y_i$,
- for $i, j \in \{1, \dots, k\}$, Y_i is complete to Y_j, and
- $\{a, b, c, d\} \subseteq Y_0$.

We may assume that $Y_0, \dots, Y_k$ are chosen with $W = \bigcup_{i=0}^{k} Y_i$ maximal.

(1) Let $v \in V(G) \setminus W$ and assume that v has a neighbor in Y_0. Then v is strongly anticomplete to $W \setminus Y_0$.

We may assume that v has a neighbor in $W \setminus Y_0$. Suppose first that v is mixed on Y_0. By 5.1, it follows that v strongly complete to $W \setminus Y_0$, and therefore $Y_0 \cup \{v\}, Y_1, \dots, Y_k$ contradict the maximality of W. This proves that v is strongly complete to Y_0.

Next suppose that v has a neighbor in Y_1, and v is not complete to Y_1. Then $|Y_1| > 1$, and 5.1 implies that v is strongly complete to $W \setminus Y_1$. But then replacing Y_1 with $Y_1 \cup \{v\}$ contradicts the maximality of W. Using the symmetry, this proves that if v has a neighbor in Y_i with $1 \leq i \leq k$, then v is complete to Y_i.

Let I be the set of all $i \in \{1, \dots, k\}$, such that v is complete to Y_i, and let $J = \{1, \dots, k\} \setminus I$. Then v is strongly anticomplete to $\bigcup_{j \in J} Y_j$. From the symmetry we may assume that $I = \{1, \dots, t\}$ for some $t \in \{1, \dots, k\}$. Let

$Z_{t+1} = \{v\} \cup \bigcup_{j \in J} Y_j$. Then $Y_0, Y_1, \ldots, Y_t, Z_{t+1}$ contradict the maximality of W. This proves (1).

Since $W \setminus Y_0$ is strongly complete to Y_0, and since Y_0 is not a homogeneous set in G, it follows that some vertex of $V(G) \setminus Y_0$ has a neighbor in Y_0. Let Z_0 be the set of all vertices of $V(G) \setminus W$ with a neighbor in Y_0. Then $Z_0 \neq \emptyset$, and by (1), Z_0 is strongly anticomplete to $W \setminus Y_0$. Moreover, no vertex of $V(G) \setminus (Y_0 \cup Z_0)$ is mixed on Y_0.

Since Y_0 is strongly complete to $W \setminus Y_0$, and Z_0 is strongly anticomplete to $W \setminus Y_0$, and since $W \setminus Y_0$ is not a homogeneous set in G, it follows that some vertex $z_1 \in V(G) \setminus (W \cup Z_0)$ is mixed on $W \setminus Y_0$. Then z_1 is strongly anticomplete to Y_0. We may assume that z_1 has a neighbor $y_1 \in Y_1$ and antineighbor $y_2 \in Y_2$.

(2) z_1 is strongly complete to Z_0.

Suppose $z_0 \in Z_0$ is antiadjacent to z_1. Let $y_0 \in Y_0$ be a neighbor of z_0. Then $\{z_0, y_0, y_2, y_1, z_1\}$ is a bull, a contradiction. This proves (2).

(3) Let $s, t \in Z_0$ be adjacent, and let $v \in V(G) \setminus (Y_0 \cup Z_0)$. Then v is not mixed on $\{s, t\}$.

Suppose that v is adjacent to s and antiadjacent to t. Let $y_s \in Y_0$ be adjacent to s, and y_t to t, choosing $y_s = y_t$ if possible. Since v is mixed on Z_0, it follows that $v \notin (W \setminus Y_0)$. Since $v \notin Z_0$, it follows that v is strongly antiadjacent to y_s, y_t.

Assume first that $y_s = y_t$. Since $\{v, s, t, y_t, w\}$ is not a bull for any $w \in W \setminus Y_0$, it follows that v is strongly complete to $W \setminus Y_0$. But now $Y_0 \cup \{v\}, Y_1, \ldots, Y_k$ contradict the maximality of W. This proves that $y_s \neq y_t$, and therefore s is antiadjacent to y_t, and t to y_s. Since $\{y_s, s, z_1, t, y_t\}$ is not a bull, it follows that y_s is strongly adjacent to y_t. But now $G|\{s, t, z_1, y_s, y_t, y_1\}$ is a prism, a contradiction. This proves (3).

Now y_1, z_1 are adjacent, and Y_0, Z_0 are subsets of $V(G)$ such that

- y_1 is strongly complete to Y_0 and strongly anticomplete to Z_0,
- z_1 is strongly complete to Z_0 and strongly anticomplete to Y_0,
- No vertex of $V(G) \setminus (Y_0 \cup Z_0)$ is mixed on Y_0, and
- if $s, t \in Z_0$ are adjacent, then no vertex of $V(G) \setminus (Y_0 \cup Z_0)$ is mixed on $\{s, t\}$.

By 5.5, we deduce that $Y_0 = K \cup S$, where K is a strong clique and S is a strongly stable set. But then at least one of a, b is in K, and at least one of c, d is in K, contrary to the fact that $\{a, b\}$ is strongly anticomplete to $\{c, d\}$. This proves 5.6. ∎

Let G be a trigraph, let $N \subseteq V(G)$ with $|N| = k$. We say that N, or $G|N$, is a *matching of size* k in G if $N = \{a_1, \ldots, a_k, b_1, \ldots, b_k\}$ and for distinct $i, j \in \{1, \ldots, k\}$ the pairs $a_i b_i$ are adjacent, and the pairs $a_i b_j$ are antiadjacent.

5.7 *Let G be a bull-free trigraph, let v be a vertex of G and let N be the set of neighbors of v. Let $H = G|N$. Let $a_1, a_2, b_1, b_2 \in N$ such that $H|\{a_1, a_2, b_1, b_2\}$ is a matching of size two in G, where the pairs $a_1 b_1$ and $a_2 b_2$ are adjacent. For $i = 1, 2$ let C_i be the component of H containing $\{a_i, b_i\}$, and let D_i be the set of vertices of $V(G) \setminus (N \cup \{v\})$ that are mixed on C_i. Then*

1. *$C_1 \cap C_2 = \emptyset$,*

2. *D_i is strongly complete to $N \setminus C_i$, and consequently $D_1 \cap D_2 = \emptyset$,*

3. *Let $i \in \{1, 2\}$ and let $x \in V(G) \setminus (N \cup D_i)$ have a neighbor $d_i \in D_i$. Then x is strongly anticomplete to C_i,*

4. *D_1 is strongly anticomplete to D_2.*

Proof. First we prove the first assertion of 5.7. It is enough to show that there is no path from $\{a_1, b_1\}$ to $\{a_2, b_2\}$ in H. First we claim that $\{a_1, b_1\}$ is strongly anticomplete to $\{a_2, b_2\}$. For suppose not, from the symmetry we may assume that a_1 is adjacent to a_2. Then b_1-a_1-a_2-b_2 is a path, an v is a center for it, contrary to the fact that G is unfriendly. This proves that $\{a_1, b_1\}$ is strongly anticomplete to $\{a_2, b_2\}$.

Next suppose that there is a path P from $\{a_1, b_1\}$ to $\{a_2, b_2\}$ in H. Since v is a weak center for P, it follows that P has length less than three, and so some vertex $p \in N$ has a neighbor in $\{a_1, b_1\}$ and a neighbor in $\{a_2, b_2\}$. From the symmetry we may assume that p is adjacent to a_1 and to a_2. Since b_1-a_1-p-a_2 is not a path with center v, it follows that p is adjacent to b_1, and similarly to b_2. But now the vertices a_1, b_1, a_2, b_2, v, p contradict 5.6. This proves the first assertion of 5.7.

To prove the second assertion of 5.7, let $d \in D_i$ and suppose that d has an antineighbor $n \in N \setminus C_i$. By 2.2, there exist $c_i, c_i' \in C_i$ such that c_i is adjacent to c_i', and d is adjacent to c_i and antiadjacent to c_i'. But now $\{d, c_i, c_i', v, n\}$ is a bull, a contradiction. This proves the second assertion of 5.7.

To prove the third assertion, suppose that x has a neighbor in C_i. Since $x \notin D_i \cup C_i$, it follows that x is strongly complete to C_i. Since $x \notin N$, it follows that x is strongly antiadjacent to v. By 2.2, there exist $c_i, c_i' \in C_i$ such that c_i is adjacent to c_i', and d_i is adjacent to c_i and antiadjacent to

c_i'. Now v-c_i'-x-d_i is a path, and c_i is a center for it, a contradiction. This proves the third assertion of 5.7.

Finally, the last assertion of 5.7 follows from the second and the third assertion. ∎

5.8 *Let G be an unfriendly bull-free trigraph with no prism, let $v \in V(G)$ and let N be the set of neighbors of v in G. Then no induced subtrigraph of $G|N$ is a matching of size three.*

Proof. Suppose not, and let $\{a_1, a_2, a_3, b_1, b_2, b_3\} \subseteq N$ be as in the definition of a matching, and let $H = G|N$. For $i \in \{1, 2, 3\}$ let C_i be the component of H containing $\{a_i, b_i\}$. By 5.7 C_1, C_2, C_3 are all distinct components of H. For $i \in \{1, 2, 3\}$ let D_i be the set of vertices of $V(G) \setminus C_i$ that are mixed on C_i. Since G is unfriendly, it follows that C_i is not a homogeneous set, and $(C_i, \{v\})$ is not a homogeneous pair, and therefore $D_i \neq \emptyset$. Since C_i is a component of N, it follows that v is strongly anticomplete to D_i. By 5.7, D_i is strongly complete to $N \setminus C_i$, the sets D_1, D_2, D_3 are pairwise disjoint, and D_i is strongly anticomplete to D_j.

(1) Let $i \in \{1, 2, 3\}$. No vertex of $V(G) \setminus (N \cup D_i)$ is mixed on D_i.

From the symmetry, may assume $i = 1$. Suppose $x \in V(G) \setminus (N \cup D_1)$ is mixed on D_1. Then $x \neq v$, and by 5.7, $x \notin D_2 \cup D_3$. Let $d_1 \in D_1$ be adjacent to x. By 5.7, d_1 is strongly complete to $C_2 \cup C_3$. By 5.6, $\{x, d_1\}$ is not complete to a_2, b_2, a_3, b_3, and, since $x \notin D_2 \cup D_3$, we may assume, from the symmetry, that x is strongly anticomplete to C_2. Let $d_2 \in D_2$. By 2.2, there exist $c_2, c_2' \in C_2$ such that c_2 is adjacent to c_2', and d_2 is adjacent to c_2 and antiadjacent to c_2'. Since $\{x, d_1, c_2', c_2, d_2\}$ is not a bull, it follows that x is adjacent to d_2, and therefore x is strongly complete to D_2. By 5.7, x is strongly anticomplete to C_1. But now, applying the previous argument with the roles of D_1 and D_2 exchanged, we deduce that x is strongly complete to D_1, a contradiction. This proves (1).

Now, since v is semi-adjacent to at most one vertex of G, we may assume that v is strongly complete to C_1. But then, by (1), (C_1, D_1) is a homogeneous pair in G, contrary to the fact that G is unfriendly. This proves 5.8. ∎

5.9 *Let G be an unfriendly bull-free trigraph, let $\{a_1, a_2, b_1, b_2\}$ be a matching of size two in G (with the usual notation), and let $c \in V(G) \setminus \{a_1, a_2, b_1, b_2\}$ be complete to $\{a_1, a_2, b_1, b_2\}$. Then the following statements hold:*

1. *For $i = 1, 2$ let $d_i \in V(G) \setminus (N(c) \cup \{c\})$ be mixed on $\{a_i, b_i\}$, and let $y \in V(G) \setminus \{a_1, a_2, b_1, b_2, d_1, d_2, c\}$ be adjacent to both d_1 and d_2, Then y is strongly adjacent to c.*

2. *Let $x \in V(G)$ be a neighbor of c, such that there is no path in $G|N(c)$ from x to $\{a_1, a_2, b_1, b_2\}$. Then x is strongly adjacent to c. Let $c' \in V(G)$ be an antineighbor of c, such that c' has a neighbor in $\{a_1, b_1\}$ and in $\{a_2, b_2\}$. Then x is strongly adjacent to c'.*

Proof. Let X be the set of neighbors of c. Let $\{i, j\} = \{1, 2\}$. For $i = 1, 2$ let X_i be the component of X containing a_i, b_i. By 5.7, $X_1 \cap X_2 = \emptyset$. Let $X' = X \setminus (X_1 \cup X_2)$. By 5.8, X' is strongly stable. If c is not strongly complete to X_i, let $C_i = \{c\}$, and otherwise let $C_i = \emptyset$. Let Y_i be the set of vertices of $V(G) \setminus (X \cup \{c\})$ that are mixed on X_i. Let C be the set of vertices of $V(G) \setminus \{c\}$ that are strongly complete to $X_1 \cup X_2$. By 5.6 $C \cup \{c\}$ is a strongly stable set. By 5.7 Y_i is strongly complete to $X \setminus X_i$, and Y_1 is strongly anticomplete to Y_2. Let Z_i be the set of vertices of $V(G) \setminus (C \cup \{c\} \cup X \cup Y_1 \cup Y_2)$ with a neighbor in Y_i and an antineighbor in Y_j.

We claim that $Z_i \neq \emptyset$. Suppose not. Since $(X_i, C_i \cup Y_i)$ is not a homogeneous pair in G, it follows that some vertex $v \in V(G) \setminus (X_i \cup C_i \cup Y_i)$ is mixed on $C_i \cup Y_i$. By 5.7, $v \notin X$. So v has a neighbor in Y_i and v is strongly antiadjacent to c. Since $Z_i = \emptyset$, it follows that v is strongly complete to Y_j. By 5.7, it follows that v is strongly anticomplete to $X_1 \cup X_2$. Let $y \in Y_i \cup C_i$ be antiadjacent to v. By 2.2, there exist $x, x' \in X_i$ such that y is adjacent to x and antiadjacent to x', and x is adjacent to x'. Let $y_2 \in Y_2$. Now $\{v, y_2, x', x, y\}$ is a bull, a contradiction. This proves that $Z_i \neq \emptyset$.

By 5.7, Z_i is strongly anticomplete to X_i. Let W_i be the set of vertices of $V(G) \setminus (C \cup \{c\} \cup X \cup Y_1 \cup Y_2 \cup Z_1 \cup Z_2)$ with a neighbor in Z_i and an antineighbor in Y_j.

(1) Z_i is strongly complete to X_j and strongly anticomplete to Y_j.

Suppose some $z_i \in Z_i$ has an antineighbor in X_j. Since $Z_i \cap (C \cup X \cup Y_j) = \emptyset$, it follows that z_i is strongly anticomplete to X_j. Let $y_j \in Y_j$ be antiadjacent to z_i. By 2.2, there exist $x_j, x'_j \in X_j$ such that x_j is adjacent to x'_j, and y_j is adjacent to x_j and antiadjacent to x'_j. Let $y_i \in Y_i$ be adjacent to z_i. Then, by 5.7, $\{z_i, y_i, x'_j, x_j, y_j\}$ is a bull, a contradiction. This proves that Z_i is strongly complete to X_j. Now it follows from 5.7 that Z_i is strongly anticomplete to Y_j. This proves (1).

(2) W_i is strongly complete to X_j and anticomplete to Y_j.

Suppose not, and let $w_i \in W_i$ and $x_j \in X_j$ be antiadjacent. Let $z_i \in Z_i$ be adjacent to w_i, and let $y_i \in Y_i$ be adjacent to z_i. Then y_i is strongly antiadjacent to w_i. But now, by (1), $\{w_i, z_i, y_i, x_j, c\}$ is a bull, a contradiction. Now it follows from 5.7 that W_i is strongly anticomplete to Y_j. This proves (2).

Since $W_i \cap (C \cup \{c\} \cup Y_i) = \emptyset$, it follows that W_i is strongly anticomplete to X_i.

(3) $Z_i \cup W_i$ is strongly anticomplete to Z_j.

Suppose $z_j \in Z_j$ has a neighbor $w \in Z_i \cup W_i$. Let $y_j \in Y_j$ be adjacent to z_j. Let $x_i \in X_i$. Then x_i is antiadjacent to w, by (1) x_i is adjacent to z_j, and by (2) w is antiadjacent to y_j. But now $\{c, x_i, y_j, z_j, w\}$ is a bull, a contradiction. This proves (3).

(4) W_1 is strongly anticomplete to W_2.

Suppose $w_1 \in W_1$ is adjacent to $w_2 \in W_2$. Let $z_2 \in Z_2$ be adjacent to w_2. Let $x_1 \in X_1$. Then x_1 is antiadjacent to w_1. By (2), x_1 is adjacent to w_2 and to z_2. But now $\{w_1, w_2, z_2, x_1, c\}$ is a bull, a contradiction. This proves (4).

(5) C is strongly anticomplete to Y_i. Every vertex of $V(G) \setminus X$ that has both a neighbor in X_1 and a neighbor in X_2 belongs to $Y_1 \cup Y_2 \cup C \cup \{c\}$.

Let $v \in C$. By 5.7, C is strongly anticomplete to Y_i. Now let v be a vertex with both a neighbor in X_1 and a neighbor in X_2. If v is mixed on one of X_1, X_2, then $v \in Y_1 \cup Y_2 \cup \{c\}$; and if v is strongly complete to $X_1 \cup X_2$, then $v \in C \cup \{c\}$. This proves (5).

Let $M = X_1 \cup X_2 \cup Y_1 \cup Y_2 \cup Z_1 \cup Z_2 \cup W_1 \cup W_2$.

(6) Suppose $a \in V(G) \setminus M$ is strongly complete to $Y_1 \cup Y_2$, and is antiadjacent to $\{c\}$. Then c is strongly complete to $X_1 \cup X_2$, and a is strongly complete to $Y_1 \cup Z_1 \cup W_1 \cup Y_2 \cup Z_2 \cup W_2$.

By 5.7, a is strongly anticomplete to $X_1 \cup X_2$. Suppose that c is not strongly complete to X_i. By 2.2, there exist $x_i, x'_i \in X_i$, such that x_i is adjacent to x'_i, and c is adjacent to x_i and antiadjacent to x'_i. Let $y_j \in Y_j$. Now $\{a, y_j, x'_i, x_i, c_i\}$ is a bull, a contradiction. This proves that c is strongly complete to $X_1 \cup X_2$.

Suppose a has an antineighbor $z_i \in Z_i$. Let $y_i \in Y_i$ be adjacent to z_i, and let $x_j \in X_j$. Then $\{a, y_i, z_i, x_j, c\}$ is a bull, a contradiction. This proves that a is strongly complete to $Z_1 \cup Z_2$. Next suppose that a has an antineighbor $w_i \in W_i$. Let $z_i \in Z_i$ be adjacent to w_i, and let $x_j \in X_j$. Then $\{a, z_i, w_i, x_j, c\}$ is a bull, a contradiction. This proves that a is strongly complete to $W_1 \cup W_2$, and completes the proof of (6).

(7) Suppose $a \in V(G) \setminus (M \cup C)$ has a neighbor in $Y_i \cup Z_i \cup W_i$ and is antiadjacent to $\{c\}$. Then a is strongly complete to $Y_1 \cup Z_1 \cup W_1 \cup Y_2 \cup Z_2 \cup W_2$.

Suppose first that a is strongly anticomplete to $Y_i \cup Z_i$. Then it follows from 5.7 that $a \notin X'$, and therefore a is antiadjacent to c. Let $w_i \in Y_i \cup Z_i \cup W_i$ be a neighbor of a. Then $w_i \in W_i$. Let $z_i \in Z_i$ be adjacent to w_i and let $x_j \in X_j$. Since $\{c, x_j, z_i, w_i, a\}$ is not a bull, it follows that x_j is adjacent to a. Let $y_i \in Y_i$ be adjacent to z_i. Now y_i-z_i-w_i-a is a path, and x_j is a center for it, a contradiction. This proves that a has a neighbor in $Y_i \cup Z_i$. We claim that a is strongly complete to Y_j. If $a \in X'$, the claim follows from 5.7, and if $a \notin X'$, the claim follows from the fact that $a \notin Z_i \cup W_i$. Similarly, a is strongly complete to Y_i. Now (7) follows from (6).

(8) Suppose that there exists $a \in V(G) \setminus (M \cup C)$ with a neighbor in $Y_1 \cup Y_2 \cup Z_1 \cup Z_2 \cup W_1 \cup W_2$ and antiadjacent to c. Then every vertex of X' is strongly complete to one of $Y_1 \cup Z_1 \cup W_1$ and $Y_2 \cup Z_2 \cup Y_2$.

By (7), a is strongly complete to $Y_1 \cup Y_2 \cup Z_1 \cup Z_2 \cup W_1 \cup W_2$. Suppose $x' \in X'$ has an antineighbor $b_1 \in Y_1 \cup Z_1 \cup W_1$ an an antineighbor $b_2 \in Y_2 \cup Z_2 \cup W_2$. Then $b_1 \in Z_1 \cup W_1$, and $b_2 \in Z_2 \cup W_2$.

First we claim that x' is strongly antiadjacent to a. Suppose not. Let P be a path from b_1 to x' with interior in $Z_1 \cup Y_1$. Let $y_2 \in Y_2$. Then b_1-P-x'-y_2 is a path of length at least three, and a is a center for it, a contradiction. This proves that x' is strongly antiadjacent to a.

Since x' is strongly complete to Y_1, it follows that there exist $b, b' \in Y_1 \cup Z_1 \cup W_1$ such that b is adjacent to b', and x' is adjacent to b and antiadjacent to b'. But now $\{x', b, b', a, b_2\}$ is a bull, a contradiction. This proves (8).

(9) Suppose that there exist

- *$a \in V(G) \setminus (M \cup C)$ with a neighbor in $Y_1 \cup Y_2 \cup Z_1 \cup Z_2 \cup W_1 \cup W_2$ and antiadjacent to c, and*
- *$b \in V(G) \setminus (X_i \cup Y_i \cup Z_i \cup W_i \cup C \cup \{c\})$ with a neighbor in X_i.*

Then b is strongly complete to X.

Since $b \notin Y_i$, it follows that b is strongly complete to X_i. We may assume that b has an antineighbor $x' \in X \setminus X_i$. Since $b \notin C$, it follows that b is not strongly complete to X_j. Since $b \notin Y_j$, it follows that b is strongly anticomplete to X_j. Since $b \notin X_i$, it follows that b is strongly antiadjacent to c. By 5.7, b is strongly anticomplete to Y_i, and so by (7) b is strongly anticomplete to $Y_j \cup Z_j$. Let $z_j \in Z_j$ and $y_j \in Y_j$ be adjacent. Let $x_j \in X_j$ be adjacent to y_j. Let $x_i \in X_i$. Then $\{b, x_i, z_j, y_j, x_j\}$ is a bull, a contradiction. This proves (9).

(10) Suppose that there exists $a \in V(G) \setminus (M \cup C)$ with a neighbor in $Y_1 \cup Y_2 \cup Z_1 \cup Z_2 \cup W_1 \cup W_2$ and antiadjacent to c. Then

- *if $v \in C$ is antiadjacent to a, then v is strongly anticomplete to $Y_1 \cup Z_1 \cup W_1 \cup Y_2 \cup Z_2 \cup W_2$, and*
- *every vertex of C is strongly anticomplete to either $Y_1 \cup Z_1 \cup W_1$ or $Y_2 \cup Z_2 \cup W_2$. Moreover, if $v \in C$ has a neighbor in $Y_i \cup Z_i \cup W_i$, then v has a neighbor in Z_i.*

By (5), C is strongly anticomplete to $Y_1 \cup Y_2$. By (7), a is strongly complete to $Y_1 \cup Z_1 \cup W_1 \cup Y_2 \cup Z_2 \cup W_2$.

Suppose first that v is antiadjacent to a. If v has a neighbor $z_i \in Z_i$, then, choosing $y_i \in Y_i$ adjacent to z_i, and $y_j \in Y_j$, we observe that $\{v, z_i, y_i, a, y_j\}$ is a bull, a contradiction. This proves that v is strongly anticomplete to Z_i. Next assume that v has a neighbor $w_i \in W_i$. Let $z_i \in Z_i$ be adjacent to w_i, and let $y_i \in Y_i$ be adjacent to z_i. Then v-w_i-z_i-y_i is a path, and every $x_j \in X_j$ is a center for it, contrary to the fact that G is unfriendly. This proves the first assertion of (10).

Now suppose that $v \in C$ has a neighbor $u_i \in Z_i \cup W_i$. Then v is strongly adjacent to a. Let P_i be a path from $u_i \in Z_i \cup W_i$ adjacent to v to some vertex $y_i \in Y_i$, with interior in $Y_i \cup Z_i \cup W_i$, and such that u_i is the only neighbor of v in P_i.

If v is strongly anticomplete to Z_i, then $u_i \in W_i$, y_i-P_i-u_i-v is a path, and every vertex of X_2 is a center for it, a contradiction. This proves that if v has a neighbor in $Z_i \cup W_i$, then v has a neighbor in Z_i.

Finally, if v has both a neighbor in $Z_1 \cup W_1$ and a neighbor in $Z_2 \cup W_2$, then y_1-P_1-u_1-v-u_2-P_2-y_2 is a path of length at least three (in fact, at least four), and a is a center for it, contrary to the fact that G is unfriendly. This proves (10).

(11) Every vertex of $V(G) \setminus (M \cup C)$ with a neighbor in $Y_1 \cup Y_2 \cup Z_1 \cup Z_2 \cup W_1 \cup W_2$ is strongly adjacent to c.

Suppose there exists $a \in V(G) \setminus M$ with a neighbor in $Y_1 \cup Y_2 \cup Z_1 \cup Z_2 \cup W_1 \cup W_2$ and antiadjacent to c. By (7), a is strongly complete to $Y_1 \cup Y_2 \cup Z_1 \cup Z_2 \cup W_1 \cup W_2$. By (6), $C_1 \cup C_2 = \emptyset$. Let X'_i be the the set of vertices of X' that are not strongly complete to $Y_i \cup Z_i \cup W_i$. By (8), $X'_1 \cap X'_2 = \emptyset$. Let C'_i be the vertices of C with a neighbor in $Y_i \cup Z_i \cup W_i$.

Then $(X_i \cup X'_i, Y_i \cup Z_i \cup Y_i \cup C'_i)$ is not a homogeneous pair in G. Since $X_2 \cup (X' \setminus X'_1)$ is strongly complete to $Y_1 \cup Z_1 \cup W_1$, and by (7), it follows that no vertex of $V(G) \setminus (X_1 \cup X'_1 \cup Y_1 \cup Z_1 \cup W_1 \cup C'_1)$ is mixed on $Y_1 \cup Z_1 \cup W_1$.

Suppose that some vertex v of $V(G) \setminus (X_1 \cup X'_1 \cup Y_1 \cup Z_1 \cup W_1 \cup C'_1)$ is mixed on $Y_1 \cup Z_1 \cup W_1 \cup C'_1$. Assume first that v has a neighbor in $Y_1 \cup Z_1 \cup W_1$. Then $v \notin C$. Then v is strongly complete to $Y_1 \cup Z_1 \cup W_1$, and has an antineighbor $c' \in C'_1$. By (10), c' has a neighbor $z_1 \in Z_1$, and

c' is strongly anticomplete to $Y_2 \cup Z_2 \cup W_2$. Let $y_1 \in Y_1$ be adjacent to z_1. Since $\{c', z_1, y_1, v, u\}$ is not a bull for any $u \in Y_2 \cup Z_2 \cup W_2 \cup \{c\}$, it follows that v is strongly anticomplete to $Y_2 \cup Z_2 \cup W_2 \cup \{c\}$ (we remind the reader that $C \cap \{c\}$ is a strongly stable set). Then $v \notin X$, and, since $v \notin Y_1 \cup Z_1 \cup W_1$, it follows that $v \notin M$, contrary to (7). This proves that v is strongly anticomplete to $Y_1 \cup Z_1 \cup W_1$, and has a neighbor $c' \in C_1'$. By (10), c' has a neighbor $z_1 \in Z_1$, and, again by (10), c' is strongly anticomplete to $Y_2 \cup Z_2 \cup W_2$. Let $y_1 \in Y_1$ be adjacent to z_1. Then v-c'-z_1-y_1 is a path, and since vertices of X_2 are not centers for it, it follows that v is strongly anticomplete to X_2. Since $\{v, c', z_1, x_2, c\}$ is not a bull for any $x_2 \in X_2$, it follows that v is strongly adjacent to c, and therefore $v \in X$. Since v is strongly anticomplete to Y_1, it follows that $v \in X_1$, a contradiction. This proves that no vertex of $V(G) \setminus (X_1 \cup X_1' \cup Y_1 \cup Z_1 \cup W_1 \cup C_1')$ is mixed on $Y_1 \cup Z_1 \cup W_1 \cup C_1'$.

Therefore, some vertex $v \in V(G) \setminus (X_1 \cup X_1' \cup Y_1 \cup Z_1 \cup W_1 \cup C_1')$ is mixed on $X_1 \cup X_1'$. By (6) and (7), c is strongly complete to $X_1 \cup X_1'$, and so $v \neq c$. Suppose first that v has a neighbor in X_1. Since $v \notin Y_1$, it follows that v is strongly complete to X_1, and has an antineighbor $x_1' \in X_1'$. By (9), $v \in C$. Since $v \notin C_1'$, it follows that v is strongly anticomplete to $Y_1 \cup Z_1 \cup W_1$. Since $x_1' \in X_1'$, it follows that there exist $p, q \in Y_1 \cup Z_1 \cup W_1$ such that p is adjacent to q, and x_1' is adjacent to p and antiadjacent to q. But now $\{v, x_2, q, p, x_1'\}$ is a bull for every $x_2 \in X_2$, a contradiction. This proves that v is strongly anticomplete to X_1. Then $v \notin C$; and since $v \notin Y_1 \cup Z_1 \cup W_1$, it follows that $v \notin M$. We deduce from (9) that v is strongly anticomplete to $X_1 \cup X_2$. Since v is mixed on $X_1 \cup X_1'$, it follows that v has a neighbor $x_1' \in X_1'$. Let $z_2 \in Z_2$, $y_2 \in Y_2$ adjacent to z_2, and $x_2 \in X_2$ adjacent to y_2. Since $\{v, x_1', z_2, y_2, x_2\}$ is not a bull, it follows that v is strongly adjacent to one of y_2, z_2. By 5.8 applies to $\{v, x_1'\}, \{a_1, b_1\}, \{a_2, b_2\}$ and c, it follows that v is strongly anticomplete to c, and so, by (7), v is strongly complete to $Y_1 \cup Z_1 \cup W_1 \cup Y_2 \cup Z_2 \cup W_2$. Let $y_2 \in Y_2$. Since $x_1' \in X_1'$, it follows that there exist $p, q \in Y_1 \cup Z_1 \cup W_1$ such that p is adjacent to q, and x_1' is adjacent to p and antiadjacent to q. Now q-p-x_1'-y_2 is a path of length three, and v is a center for it, a contradiction. This proves (11).

We can now prove the first assertion of the theorem. For $i = 1, 2$ let $d_i \in V(G) \setminus (N(c) \cup \{c\})$ be mixed on $\{a_i, b_i\}$, and let $y \in V(G) \setminus \{a_1, a_2, b_1, b_2, d_1, d_2, c\}$ be adjacent to both d_1 and d_2. We may assume that d_i is adjacent to a_i and antiadjacent to b_i. Suppose y is antiadjacent to c. Since $d_i \in Y_i$, it follows that y has a neighbor in Y_1, and a neighbor in Y_2. By (5), $y \notin C$, and so, by (11), $y \in M$. Since y has a neighbor in Y_1, it follows that $y \notin Y_2 \cup Z_2 \cup W_2$, and since y has a neighbor in Y_2, it follows that $y \notin Y_1 \cup Z_1 \cup W_1$. Therefore $y \in X_1 \cup X_2$, and, in particular, y is adjacent, and therefore semi-adjacent to c. From the symmetry, we may assume that $y \in X_1$. Since d_1-y-b_1-c is not a path with center a_1, it follows that y is not complete to $\{a_1, b_1\}$. Let

$p, q \in X_1 \setminus \{y\}$ be adjacent. Since $\{a_2, c, p, q, y\}$ is not a bull, it follows that v is either strongly complete or strongly anticomplete to $\{p, q\}$. But this implies that y is strongly anticomplete to $\{a_1, b_1\}$, and there is no path in $G|X_1$ from y to $\{a_1, b_1\}$, contrary to the fact that X_1 is connected. This proves the first assertion of the theorem.

To prove the second assertion, let $x \in V(G)$ be a neighbor of c, such that there is no path in $G|N(c)$ from x to $\{a_1, a_2, b_1, b_2\}$. Then $x \in X'$. By 5.7, x is strongly complete to $Y_1 \cup Y_2$, and therefore, by the first assertion of the theorem, x is strongly adjacent to c. Let $c' \in V(G)$ be an antineighbor of c, such that c' has a neighbor in $\{a_1, b_1\}$ and in $\{a_2, b_2\}$. Suppose that c' is antiadjacent to x. Then 5.7 implies that c' is not mixed on $\{a_1, b_1\}$, and so c' is strongly complete to $\{a_1, b_1\}$. Similarly, c' is strongly complete to $\{a_2, b_2\}$. By 5.6, c' is strongly anticomplete to c, and therefore, $c' \notin X_1 \cup X_2$. Now, since c' is strongly anticomplete to x', 5.7 implies that c' is strongly complete to $X_1 \cup X_2$, and therefore $c' \in C$. Choose $d_i \in Y_i$, and let $a'_i, b'_i \in X_i$ be such that a'_i is adjacent to b'_i, and y_i is adjacent to a'_i and antiadjacent to b'_i. By (5), c' is strongly antiadjacent to d_i. By 5.7, x' is adjacent to d_1, d_2. But now, applying the first assertion of the theorem to $\{a'_1, b'_1, a'_2, b'_2, c', x\}$ we deduce that c' is strongly adjacent to x, a contradiction. This proves 5.9. ∎

6 Frames

In this section we study unfriendly trigraphs that contain a three edge path and do not contain a prism. Let G be such a trigraph. We choose a maximal subtrigraph H of G such that there is no triangle in H, and analyze how the vertices of $V(G) \setminus V(H)$ attach to H. It turns out that each component of $V(G) \setminus V(H)$ is a strong clique, no vertex of H has neighbors in more than two components of $V(G) \setminus V(H)$, and we can describe how each of the cliques "connects" to H, thus proving that $G \in \mathcal{T}_1$.

We start with a lemma.

6.1 *Let G be an unfriendly trigraph with no prism, and let h_1-h_2-h_3-h_4-h_5-h_1 be a hole of length five in G, say H. Then no vertex of $V(G) \setminus V(H)$ is adjacent to h_1, h_2, h_5.*

Proof. Suppose some $v \in V(G) \setminus V(H)$ is adjacent to h_1, h_2, h_5. Since $\{h_2, v, h_1, h_5, h_4\}$ and $\{h_2, h_1, v, h_5, h_4\}$ are not bulls, it follows that h_2 is strongly complete to $\{v, h_1\}$, and from the symmetry, h_5 is strongly complete to $\{v, h_1\}$. Since h_5-v-h_2-h_3 is not a path with center h_1, it follows that h_3 is strongly antiadjacent to h_1, and therefore h_3 is strongly anticomplete to $\{v, h_1\}$. From the symmetry h_4 is strongly anticomplete to $\{v, h_1\}$.

Let X the set of vertices of $V(G) \setminus \{h_2, h_3, h_4, h_5\}$ that are strongly complete to $\{h_2, h_5\}$ and strongly anticomplete to $\{h_3, h_4\}$ and let C be a component of X such that $v, h_1 \in C$. Since G is unfriendly, it follows that C is not a homogeneous set in G, and therefore some vertex $w \in V(G) \setminus C$ is mixed on C. Then $w \notin V(H)$. By 2.2, there exists $c, c' \in C$ such that c is adjacent to c', and w is adjacent to c and antiadjacent to c'.

Assume first that w is antiadjacent to h_5. Since $\{w, c, c', h_5, h_4\}$ is not a bull, it follows that w is strongly adjacent to h_4. If w is antiadjacent to h_2, then, form the symmetry, w is strongly adjacent to h_3, and $\{h_2, h_3, w, h_4, h_5\}$ is a bull, a contradiction; thus w is strongly adjacent to h_2. Since c-h_2-h_3-h_4 is not a path with center w, it follows that w is strongly antiadjacent to h_3. But now, $\{h_5, c, w, h_2, h_3\}$ is a bull, a contradiction. This proves that w is strongly adjacent to h_5, and so, from the symmetry, w is strongly adjacent to h_2. Since h_5-c-h_2-h_3 is not a path with center w, it follows that w is strongly antiadjacent to h_3, and from the symmetry, w is strongly antiadjacent to h_4. But then $w \in C$, a contradiction. This proves 6.1. ∎

A *frame* is a trigraph T such that

- T is connected, and
- there is no triangle in T, and
- T has an induced subtrigraph which is a path of length three.

A trigraph is called *framed* if some induced subtrigraph of it is a frame. We prove the following:

6.2 *Every unfriendly framed trigraph with no prism is in $\mathcal{T}_1$.*

Proof. Let G be an unfriendly framed trigraph, and let F be an induced subtrigraph of G that is a frame. We may assume that there is a triangle in G, for otherwise G admits an H-structure where H is the empty graph. Since G is unfriendly, it follows that G is connected. Assume that F is chosen with $|V(F)|$ maximum, subject to that with $|\eta(F)| + |\sigma(F)|$ maximum (we remind the reader that $\eta(F)$ is the number of strongly adjacent pairs of vertices in F, and $\sigma(F)$ is the number of semi-adjacent pairs).

(1) Every vertex of $V(G) \setminus V(F)$ has a neighbor in $V(F)$.

Suppose some vertex of $V(G) \setminus V(F)$ is strongly anticomplete to $V(F)$. Since G is connected, there exist vertices $u, v \in V(G) \setminus V(F)$ such that u has a neighbor in $V(F)$, and v is strongly anticomplete to $V(F)$. Let N be the set of neighbors of u in $V(F)$, and let $M = V(F) \setminus N$. By the maximality of $|V(F)|$, there are two adjacent vertices in N. Let C be a component of N with $|C| > 1$. Since G is unfriendly, F contains a path of length three and u is complete to C, it follows that $C \neq V(F)$. Since F is connected, some vertex $f \in V(F)$ has a neighbor in C, and since C is a component of N, it follows that f belongs to M. Let $c \in C$ be adjacent to f. Since C is connected, it follows that c has a neighbor, say c', in C. Since F is triangle-free, we deduce that f is strongly antiadjacent to c'. But now $\{v, u, c', c, f\}$ is a bull, a contradiction. This proves (1).

For a vertex $v \in V(G) \setminus V(F)$, let $N_F(v)$ be the set of neighbors of v in $V(F)$, and let $M(v) = V(F) \setminus N_F(v)$.

(2) Let H be a triangle free trigraph, no induced subtrigraph of which is a path of length three, and assume that H is connected. Then $V(H) = S_1 \cup S_2$, where S_1 and S_2 are disjoint strongly stable sets, complete to each other. Moreover, if both $|S_2| > 1$ and $|S_2| > 1$, then S_1 is strongly complete to S_2.

By 5.4, and since H is connected, one of the following holds:

- H is not anticonnected, or
- there exist two vertices $v_1, v_2 \in V(H)$ such that v_1 is semi-adjacent to v_2, and $V(H) \setminus \{v_1, v_2\}$ is strongly complete to v_1 and strongly anticomplete to v_2.

Assume first that H is not anticonnected. Since H is triangle free, H has exactly two anti-components, and each of them is a strongly stable set, and (2) holds.

Next assume that there exist two vertices $v_1, v_2 \in V(H)$ such that v_1 is semi-adjacent to v_2, and $V(H) \setminus \{v_1, v_2\}$ is strongly complete to v_1 and strongly anticomplete to v_2. Since H is triangle free, it follows that $V(H) \setminus \{v_1\}$ is strongly stable, and again (2) holds. This proves (2).

(3) Let $v \in V(G) \setminus V(F)$. Then there exist non-empty strongly stable sets $S_1(v)$ and $S_2(v)$ in F, such that $N_F(v) = S_1(v) \cup S_2(v)$, $S_1(v)$ is complete to $S_2(v)$, and if both $|S_1(v)| > 1$ and $|S_2(v)| > 1$, then $S_1(v)$ is strongly complete to $S_2(v)$.

Let $H = F|N_F(v)$. Since G is unfriendly, it follows that no induced subtrigraph of H is a path of length tree. If H is connected, (3) follows from (2), so we may assume not. It follows from the maximality of $|V(F)|$ that some two vertices of $N_F(v)$ are adjacent. Let C be component of $N_F(v)$ with $|C| > 1$. Since H is not connected, it follows that $N_F(v) \neq C$. Since F is connected, some vertex $m \in V(F) \setminus C$ has a neighbor in C, and since C is a component of $N_F(v)$, we deduce that $m \in M(v)$. Let $c \in C$ be a neighbor of m. Since C is connected and F is triangle free, there exists $c' \in C$ such that c' is adjacent to c and antiadjacent to m. Since $\{m, c, c', v, n\}$ is not a bull for any $n \in N_F(v) \setminus C$, it follows that m is strongly complete to $N_F(v) \setminus C$. Since F is triangle-free, it follows that the set $N_F(v) \setminus C$ is strongly stable.

By (2), $C = C_1 \cup C_2$, such that C_1 and C_2 are disjoint non-empty strongly stable sets, and C_1 is complete to C_2. Let $n \in N_F(v) \setminus C$. If both

$|C_1| > 1$ and $|C_2| > 1$, then $G|C$ contains a hole of length four, with center v and anticenter n, contrary to 5.3. So we may assume that $|C_1| = 1$, say $C_1 = \{c_1\}$. Let $F' = G|((V(F) \setminus \{c_1\}) \cup \{v\})$. By the choice of F, $|\eta(F')| + |\sigma(F)'| \leq |\eta(F)| + |\sigma(F)|$, and therefore some vertex $m_1 \in M(v)$ is adjacent to c_1. By the argument in the previous paragraph with m replaced by m_1, we deduce that m_1 is strongly complete to $N_F(v) \setminus C$. Now c_1-m_1-n-v-c_1 is a hole of length four, and, since F is triangle-free, it follows that every vertex of C_2 is complete to $\{c_1, v\}$ and anticomplete to $\{m_1, n\}$. By 5.2, it follows that C_2 is a strong clique, and therefore $|C_2| = 1$, say $C_2 = \{c_2\}$. Exchanging the roles of c_1 and c_2, we deduce that some vertex $m_2 \in M(v)$ is adjacent to c_2 and to n. Since F is triangle-free, it follows that $m_1 \neq m_2$, and since $\{m_1, c_1, v, c_2, m_2\}$ is not a bull, it follows that m_2 is strongly adjacent to m_1. But now $\{m_1, m_2, n\}$ is a triangle in F, a contradiction. This proves (3).

(4) Let $u, v \in V(G) \setminus V(F)$ be adjacent. Then there exist $s_1, s_2 \in N_F(u) \cap N_F(v)$ such that s_1 is adjacent to s_2.

Let $S_1(u), S_1(v), S_2(u), S_2(v)$ be as in (3). Since $S_1(u), S_1(v), S_2(u), S_2(v)$ are non-empty strongly stable sets, and since $S_1(u)$ is complete to $S_2(u)$, and $S_1(v)$ to $S_2(v)$, we may assume that $S_1(u) \cap S_2(v) = S_2(u) \cap S_1(v) = \emptyset$.

If both $S_1(u) \cap S_1(v)$ and $S_2(u) \cap S_2(v)$ are non-empty then (3) holds, so we may assume that $S_2(u) \cap S_2(v) = \emptyset$. From the maximality of $|V(F)|$, there exist $t_u \in S_2(u)$ and $t_v \in S_2(v)$.

Suppose $S_1(u) \cap S_1(v) \neq \emptyset$, and choose $s \in S_1(u) \cap S_1(v)$. Since F is triangle free and s is adjacent to both t_u and t_v, it follows that t_u is antiadjacent to t_v. But now t_u-u-v-t_v is a path, and s is a center for it, contrary to the fact that G is unfriendly. This proves that $S_1(u) \cap S_1(v) = \emptyset$.

If $|S_1(u)| > 1$ and $|S_2(u)| > 1$, then $G|(S_1(u) \cup S_2(u))$ contains a hole of length four, say H; and u is a center for H and v is an anticenter for H, contrary to 5.3, since u is adjacent to v. So we may assume that $S_1(u) = \{s_u\}$, say. Similarly, we may assume that $S_1(v) = \{s_v\}$.

Suppose s_u is strongly antiadjacent to s_v. Let $F' = (F \setminus \{s_u, s_v\}) + \{u, v\}$. Then F' is triangle-free, and therefore $|\eta(F')| + |\sigma(F')| \leq |\eta(F)| + |\sigma(F)|$. Consequently, we may assume from the symmetry, that s_u has a neighbor $m \in M(u)$. Then m is strongly anticomplete to $S_2(u)$. Since $\{m, s_u, t_u, u, v\}$ is not a bull, it follows that $m \in N_F(v)$; and since s_u is strongly antiadjacent to s_v, we deduce that $m \in S_2(v)$. Now u-s_u-m-v-u is a hole of length four, and, since F is triangle free, $S_2(u)$ is complete to $\{u, s_u\}$ and anticomplete to $\{m, v\}$. Therefore, 5.2 implies that $S_2(u)$ is a strong clique, and therefore $|S_2(u)| = 1$, namely $S_2(u) = \{t_u\}$. Since F is triangle free, it follows that t_u is strongly antiadjacent to m. Since $G|\{u, s_u, t_u, v, m, s_v\}$ is not a prism, it follows that s_v is strongly antiadjacent to t_u. Let $F'' = (F \setminus \{t_u, s_v\}) + \{u, v\}$. Then F'' is triangle-free, and therefore $|\eta(F'')| + |\sigma(F'')| \leq |\eta(F)| + |\sigma(F)|$.

Consequently, either t_u has a neighbor in $M(u)$, or s_v has a neighbor in $M(v)$. If s_v has a neighbor $x \in M(v)$, then $x \neq s_u, t_u$, and so $\{x, s_v, m, v, u\}$ is a bull, a contradiction. Thus t_u has a neighbor $y \in M(u)$. Since $\{y, t_u, s_u, u, v\}$ is not a bull, it follows that $y \in S_2(v)$. Then $y \neq m$, and since F is triangle free, we deduce that y is strongly antiadjacent to s_u. But then $\{m, s_u, u, t_u, y\}$ is a bull, a contradiction. This proves that s_u is adjacent to s_v.

Now u-s_u-s_v-v-u is a hole of length four, $S_2(u)$ is complete to $\{u, s_u\}$ and anticomplete to $\{v, s_v\}$, and $S_2(v)$ complete to $\{v, s_v\}$ and anticomplete to $\{u, s_u\}$. Thus, 5.2 implies that $|S_2(u)| = |S_2(v)| = 1$, and therefore $S_2(u) = \{t_u\}$, and $S_2(v) = \{t_v\}$. Now, reversing the roles of $S_1(u)$ and $S_2(u)$, and of $S_1(v)$ and $S_2(v)$, we deduce that t_u is adjacent to t_v. But then, since F is triangle free, it follows that $G|\{u, s_u, t_u, v, s_v, t_v\}$ is a prism, a contradiction. This proves (4).

(5) Let $u, v \in V(G) \setminus V(F)$ be antiadjacent. Then $N_F(u) \cap N_F(v)$ is a strongly stable set.

Let $S_1(u), S_2(u), S_1(v), S_2(v)$ be as in (3). Suppose $s_1, s_2 \in N_F(u) \cap N_F(v)$ are adjacent. We may assume that $s_1 \in S_1(u) \cap S_1(v)$, and $s_2 \in S_2(u) \cap S_2(v)$. Then $S_2(u) \cap S_1(v) = S_1(u) \cap S_2(v) = \emptyset$.

First we claim that $N_F(u) = N_F(v)$. Suppose $S_2(u) \setminus S_2(v) \neq \emptyset$, and let $t \in S_2(u) \setminus S_2(v)$. Then t-u-s_2-v is a path, and s_1 is a center for it, contrary to the fact that G is unfriendly. Therefore, $S_2(u) \setminus S_2(v) = \emptyset$, and, form the symmetry, this implies that $N_F(u) = N_F(v)$, and the claim follows. Let $S_1(u) = S_1(v) = S_1$, and $S_2(u) = S_2(v) = S_2$.

Let C_0 be the set of all vertices of $V(G) \setminus V(F)$ that are complete to $S_1 \cup S_2$ and strongly anticomplete to $V(F) \setminus (S_1 \cup S_2)$. Let C be an anticomponent of C_0 with $u, v \in C$. Since C is not a homogeneous set in G, it follows from 2.2 that there exist $c_1, c_2 \in C$ and $x \in V(G) \setminus C$, such that c_1 is antiadjacent to c_2, and x is adjacent to c_1 and antiadjacent to c_2.

Suppose first that $x \notin S_1 \cup S_2$. By 5.1, it follows that x is either strongly complete or strongly anticomplete to $S_1 \cup S_2$. If x is strongly complete to $S_1 \cup S_2$, then, $x \in V(G) \setminus V(F)$, and since x is antiadjacent to c_2, the claim above implies that $N_F(x) = N_F(c_2) = S_1 \cup S_2$, contrary to the fact that $x \notin C$. Therefore x is strongly anticomplete to $S_1 \cup S_2$. Since $x \notin S_1 \cup S_2$, and since x is adjacent to c_1, it follows that $x \in V(G) \setminus V(F)$. But now (4) implies that $N_F(x) \cap N_F(c_1) \neq \emptyset$, contrary to the fact that x is strongly anticomplete to $S_1 \cup S_2$. This proves that $x \in S_1 \cup S_2$, say $x \in S_1$. Since for any $s \in S_1 \setminus \{x\}$, x-c_1-s-c_2 is not a path with center s_2, it follows that $S_1 = \{x\}$. Since $(C, \{x\})$ is not a homogeneous pair in G, it follows that some vertex $y \in S_2$ is mixed on C, and therefore $S_2 = \{y\}$ and y is semi-adjacent to some vertex $c_3 \in C$. Since x is semi-adjacent to c_2, it follows that $c_2 \neq c_3$. Suppose that there exist $x', y' \in V(F) \setminus \{x, y\}$ such that x'

is adjacent to x, and y' to y. Since F us triangle free, it follows that x' is strongly antiadjacent to y, and y' to x. Since $\{x', x, u, y, y'\}$ is not a bull, we deduce that x' is adjacent to y'. But now x-y-y'-x'-x is a hole of length four, and $\{u, v\}$ is complete to $\{x, y\}$ and anticomplete to $\{x', y'\}$, contrary to 5.2. So we may assume from the symmetry that y is strongly anticomplete to $V(F) \setminus \{x, y\}$. Since F is connected and since there is a three-edge path in F, it follows that there exists a vertex $x' \in V(F) \setminus \{x, y\}$ adjacent to x. Since $\{x', x, c_3, y, c_2\}$ is not a bull, it follows that c_2 is strongly adjacent to c_3. Since C is anticonnected, there is an antipath Q from c_2 to c_3 with $V(Q) \subseteq C$. Since x is complete to C and G is unfriendly, it follows that Q has a unique internal vertex, say q. Then q is complete to $\{x, y\}$ and strongly antiadjacent to x'. But now $\{x', x, q, y, c_2\}$ is a bull, a contradiction. This proves (5).

(6) Let C be a component of $V(G) \setminus V(F)$. Then C is a strong clique.

Suppose C is not a strong clique. Then, since C is connected, there exist vertices $x, y, z \in C$, such that y is adjacent to both x and z; and x is antiadjacent to z. By (4), there exist $a, b, c, d \in V(F)$ such that a is adjacent to b, c is adjacent to d, $\{x, y\}$ is complete to $\{a, b\}$ and $\{y, z\}$ is complete to $\{c, d\}$. By (5), z is not complete to $\{a, b\}$, and x is not complete to $\{c, d\}$; and therefore $\{a, b\} \neq \{c, d\}$. Suppose b is complete to $\{z, d\}$. Since F is triangle-free, it follows that a is strongly antiadjacent to d. By (5), x is strongly antiadjacent to d, and z to a. But now $\{x, a\}$ is anticomplete to $\{z, d\}$, and $\{y, b\}$ is complete to $\{x, a, z, d\}$, contrary to 5.6. This proves that b is not complete to $\{z, d\}$, and, in particular, $b \neq c$. From the symmetry, this implies that a is not complete to $\{z, c\}$, and that $\{a, b\} \cap \{c, d\} = \emptyset$. Since $a, b, c, d, \in N_F(y)$, by (3) and the symmetry we may assume that a is adjacent to c and b to d. Since F is triangle-free, it follows that b is strongly antiadjacent to c. Since b is adjacent to d, it follows that b is antiadjacent to z, and, since a is adjacent to c, it follows that a is antiadjacent to z. But now z-c-a-b is a path, and y is a center for it, contrary to the fact that G is unfriendly. This proves (6).

Let C be a component of $V(G) \setminus V(F)$, and let $f \in V(F)$. We denote by $C(f)$ the set of vertices of C that are adjacent to f, and by $N_F(C)$ the set of vertices of F with a neighbor in C.

(7) Let C be a component of $V(G) \setminus V(F)$, and let $c \in C$. For $i = 1, 2$ let $S_i(c)$ be defined as in (3). Then, for $i = 1, 2$ there exists $s_i \in S_i(c)$ such that s_i is complete to C.

Choose $s_1 \in S_1(c)$ with $C(s_1)$ maximal. We may assume that $C(s_1) \neq C$, for otherwise (7) holds. Let $c' \in C \setminus C(s_1)$. By (4), c' has a neighbor $s'_1 \in S_1(c)$.

It follows from the maximality of $C(s_1)$, there exists $c_1 \in C(s_1)$ such that s_1' is strongly antiadjacent to c_1. But now s_1-c_1-c'-s_1' is a path with center c, a contradiction. This proves (7).

(8) Let C be a component of $V(G) \setminus V(F)$. Then $N_F(C) = S_1(C) \cup S_2(C)$ where each of $S_1(C), S_2(C)$ is a non-empty strongly stable set.

Let $c \in C$, and let $S_1(c), S_2(c)$ be as in (3). By (7), for $i = 1, 2$ there exists $s_i \in S_i(c)$ such that C is complete to s_i. Now, by (3), we may assume that for every $c' \in C$, $S_1(c')$ is complete to s_2, and $S_2(c')$ is complete to s_1. For $i = 1, 2$, let $S_i(C) = \bigcup_{c' \in C} S_i(c')$. Then $N_F(C) = S_1(C) \cup S_2(C)$. But $S_1(C)$ is complete to s_2, and $S_2(C)$ is complete to s_1, and therefore, since F is triangle free, it follows that each of $S_1(C)$ and $S_2(C)$ is strongly stable. This proves (8).

For a component C of $V(G) \setminus V(F)$ we call the sets $S_1(C), S_2(C)$ defined in (8) the *anchors* of C.

(9) Let C be a component of $V(G) \setminus V(F)$. Let $S_1(C), S_2(C)$ be the anchors of C, for $i = 1, 2$ let $T_i(C)$ be the set of vertices of $V(F) \setminus (S_1(C) \cup S_2(C))$ with a neighbor in $S_i(C)$; and for $s_i \in S_i(C)$, let $T_i(s_i)$ be the set of neighbors of s_i in $V(F) \setminus (S_1(C) \cup S_2(C))$. Then

- *for every $s, s' \in S_1(C)$ either s is strongly complete to $C(s')$, or s' is strongly complete to $C(s)$,*
- *Let $s_1 \in S_1(C)$ be antiadjacent to $s_2 \in S_2(C)$. Then every vertex of C is strongly adjacent to one of s_1, s_2. If some $c \in C$ is adjacent to both s_1 and s_2, then $C = \{c\}$, $N_F(C) = \{s_1, s_2\}$ and s_1 is semi-adjacent to s_2.*
- *for every $s, s' \in S_1(C)$, if some vertex of $C(s')$ is antiadjacent to s, then s is strongly complete to $T(s')$.*
- *$T_1(s_1)$ is disjoint from and strongly complete to $T_2(s_2)$ for every $s_1 \in S_1(c)$, $s_2 \in S_2(c)$ and $c \in C$.*
- *let $c \in C$, $s_1 \in S_1(C)$ and $s_2 \in S_2(C)$ such that c is adjacent to both s_1 and s_2. Then every vertex of C is strongly adjacent to at least one of s_1, s_2.*

Let $s, s' \in S_1(C)$, and suppose there exist $c \in C$ adjacent to s and antiadjacent to s', and $c' \in C$ adjacent to s' and antiadjacent to s. By (4), there is $s_2 \in S_2(C)$ adjacent to both c, c'. By (3), s_2 is adjacent to both s and s'. But now s-c-c'-s' is a path, and s_2 is a center for it, contrary to the fact that G is unfriendly. This proves the first assertion of (9).

Next assume that $s_1 \in S_1(C)$ is antiadjacent to $s_2 \in S_2(C)$. Suppose first that some $c \in C$ is adjacent to both s_1 and s_2. By (3), it follows that $S_1(c) = \{s_1\}$, $S_2(c) = \{s_2\}$, and s_1 is semi-adjacent to s_2. Suppose there exists $c' \in C \setminus \{c\}$. By (4), c' is complete to $\{s_1, s_2\}$. Suppose c' has a neighbor $f \in V(F) \setminus \{s_1, s_2\}$. By (3), we may assume that f is adjacent to s_1 and antiadjacent to s_2. But now f-s_1-c-s_2 is a path, and c' is a center for it, a contradiction. Therefore, $N_F(C) = \{s_1, s_2\}$. Since s_1 is semi-adjacent to s_2, it follows that C is strongly complete to $N_F(C)$, and C is a homogeneous set in G, contrary to the fact that G is unfriendly. Thus $C = \{c\}$, and the second assertion of (9) holds. So we may assume that $C(s_1) \cap C(s_2) = \emptyset$. Suppose there exists a vertex $c \in C$ anticomplete to $\{s_1, s_2\}$. For $i = 1, 2$, let $c_i \in C$ be adjacent to s_i. If c, c_1, c_2 are all distinct, then $\{s_1, c_1, c, c_2, s_2\}$ is a bull, a contradiction. Thus we may assume that $c = c_1$. By (7), there exists a vertex $s \in S_2(C)$ adjacent to both c_1 and c_2. Since c_1 is semi-adjacent to s_1, it follows that c_1 is strongly antiadjacent to s_2, and so $s \neq s_2$. By (3), s is adjacent to s_1. But now $\{s_1, s, c_1, c_2, s_2\}$ is a bull, a contradiction. This proves the second assertion of (9).

Next let $s, s' \in S_1(C)$, and assume that some vertex $c' \in C(s')$ is antiadjacent to s, and some vertex $t' \in T_1(s')$ is antiadjacent to s. Let $s_2 \in S_2(C)$ be complete to C (such a vertex s_2 exists by (7)). By the second assertion of (9), and since both s, s' have neighbors in C, it follows that s_2 is adjacent to both s, s'. But now, since F is triangle-free, $\{t', s', c', s_2, s\}$ is a bull, a contradiction. This proves the third assertion of (9).

Next, let $c \in C$, and for $i = 1, 2$, let $s_i \in S_i(c)$, and let $t_i \in T_i(s_i)$. By (3), s_1 is adjacent to s_2. Since F is triangle free, s_1 is strongly antiadjacent to t_2, and s_2 to t_1, and therefore $t_1 \neq t_2$. Now since $\{t_1, s_1, c, s_2, t_2\}$ is not a bull, it follows that t_1 is strongly adjacent to t_2, and the fourth assertions of (9) follows.

Finally, suppose that there exist $c, c' \in C$, $s_1 \in S_1(C)$ and $s_2 \in S_2(C)$ such that c is adjacent to both s_1 and s_2, and c' is antiadjacent to both s_1, s_2. Since c is semi-adjacent to at most one of s_1, s_2, it follows that c is strongly adjacent to at least one of s_1, s_2, and so $c \neq c'$. By the second assertion of (9), s_1 is adjacent to s_2. Since c' is semi-adjacent to at most one of s_1, s_2, we may assume that s_1 is strongly antiadjacent to c'. By (7), there exists $s \in S_1(C)$ complete to C. Then $s \neq s_1$. By the second assertion of (9), since s_2 has a neighbor in C, it follows that s is adjacent to s_2. But now s_1-s_2-s-c' is a path, and c is a center for it, contrary to the fact that G is unfriendly. This proves the fifth assertion of (9), and completes the proof of (9).

(10) Let C be a component of $V(G) \setminus (F)$, with anchors S_1, S_2. For $i = 1, 2$, let T_i be the set of vertices of $V(F) \setminus (S_1 \cup S_2)$ with a neighbor in S_i. Then $G|(C \cup S_1 \cup S_2 \cup T_1 \cup T_2)$ is a (C, S_1, S_2, T_1, T_2)-clique connector.

Let $|C| = t$. By (9), we can number the vertices of C as $\{c_1, \ldots, c_t\}$ such that for every $s \in S_1$, $N(s) \cap C = \{c_1, \ldots, c_i\}$ for some $i \in \{1, \ldots, t\}$, and s is strongly complete to $\{c_1, \ldots, c_{i-1}\}$, and for every $s \in S_2$, $N(s) \cap C = \{c_{t-i+1}, \ldots, c_t\}$ for some $i \in \{1, \ldots, t\}$, and s is strongly complete to $\{c_{t-i+2}, \ldots, c_t\}$. Let $i \in \{1, \ldots, t\}$. Let A_i be the set of vertices of S_1 that are strongly complete to $\{c_1, \ldots, c_{i-1}\}$, adjacent to c_i and strongly anticomplete to $\{c_{i+1}, \ldots, c_t\}$. Let A'_i be the set of vertices of A_i that are semi-adjacent to c_i. Let B_i be the set of vertices of S_2 that are strongly complete to $\{c_{t-i+2}, \ldots, c_t\}$, adjacent to c_{t-i+1} and strongly anticomplete to $\{c_1, \ldots, c_{t-i}\}$. Let B'_i be the set of vertices of B_i that are semi-adjacent to c_{t-i+1}. Then $S_1 = \bigcup_{i=1}^t A_i$, and $S_2 = \bigcup_{i=1}^t B_i$. Let $i \in \{1, \ldots, t\}$. Let C_i be the set of vertices of T_1 with a neighbor in A_i, and that are strongly anticomplete to $\bigcup_{j>i} A_j$, and let D_i be the set of vertices of T_2 with a neighbor in B_i, and that are strongly anticomplete to $\bigcup_{j>i} B_j$. Then $T_1 = \bigcup_{i=1}^t C_i$, and $T_2 = \bigcup_{i=1}^t D_i$. We show that the sets $C, A_1, \ldots, A_t, B_1, \ldots, B_t, C_1, \ldots, C_t, D_1 \ldots, D_t$ satisfy the axioms of a clique connector.

If $i + j \neq t$, then either some vertex of C is complete to $A_i \cup B_j$, or some vertex of C is anticomplete to $A_i \cup B_j$. Therefore, (9) implies, that if $i+j \neq t$, and A_i is not strongly complete to A_j, then $|C| = |S_1| = |S_2| = 1$, and S_1 is complete to S_2. Since for every i, c_i is anticomplete to $A'_i \cup B_{t-i}$, it follows from (9) that A'_i is strongly complete to B_{t-i}, and from the symmetry B'_{t-i} is strongly complete to A_i.

Next we show that S_1 is strongly anticomplete to T_2. Suppose $s_1 \in S_1$ has a neighbor $t \in T_2$. Let $s_2 \in S_2$ be a neighbor of t. Then, since F is triangle-free, it follows that s is strongly antiadjacent to t, and so $s_1 \in A_i \setminus A'_i$ and $s_2 \in B_{t-i} \setminus B'_{t-i}$ for some $i \in \{1, \ldots, t\}$. Now c_i-c_{i+1}-s_2-t-s_1-c_i is a hole of length five. By (7), there exists $s'_1 \in S_1$ complete to C. Then $s'_1 \neq s_1$, and s'_1 is adjacent to c_i, c_{i+1}, and, by (9), s_2, contrary to 6.1. This proves that S_1 is strongly anticomplete to T_2. Similarly, S_2 is strongly anticomplete to T_1.

By (9), for $i \in \{1, \ldots, t\}$, C_i is strongly complete to $\bigcup_{j<i} A_j$, and D_i is strongly complete to $\bigcup_{j<i} B_j$.

We claim that for $i \in \{1, \ldots, t\}$, C_i is strongly complete to A'_i. Suppose $c \in C_i$ is antiadjacent to $a' \in A'_i$. Since a' is semi-adjacent to c_i, it follows that a' is strongly antiadjacent to c. Since $c \in C_i$, there is a vertex $a \in A_i \setminus \{a'\}$ that is adjacent to c. But then a is adjacent to both c_i and c, and a' is antiadjacent to both c_i and c, contrary to (9). This proves that C_i is strongly complete to A'_i. Similarly, for $i \in \{1, \ldots, t\}$, D_i is strongly complete to B'_i.

Finally, let $i, j \in \{1, \ldots, t\}$, such that $i + j > t$. We claim that C_i is strongly complete to D_j. Suppose $c \in C_i$ is antiadjacent to $d \in D_j$. Let $a_i \in A_i$ be adjacent to c, and let $b_j \in B_j$ be adjacent to d. Since $j > t - i$, it follows that b_j is adjacent to c_i. But now $\{c, a_i, c_i, b_j, d\}$ is a bull, a contradiction.

Finally, by (7), $A_t \neq \emptyset$ and $B_t \neq \emptyset$. Thus, all the axioms of a clique connector are satisfied. This proves (10).

Now, if $N_F(C_1) \cap N_F(C_2) = \emptyset$ for every two components C_1, C_2 of $V(G) \setminus V(F)$, then taking H to be the graph whose vertices are the components of $V(G) \setminus V(F)$, and with $E(H) = \emptyset$, we observe, using (10), that G admits an H-structure and thus $G \in \mathcal{T}_1$. Consequently, we may assume that there exist components C_1, C_2 of $V(G) \setminus V(F)$ with $N_F(C_1) \cap N_F(C_2) \neq \emptyset$ For $i, j \in \{1, 2\}$ let $S_i(C_j)$ be the anchors of C_1, C_2.

(11) Renumbering the anchors if necessary, we may assume that $S_1(C_1) \cap S_2(C_2) = S_2(C_1) \cap S_1(C_2) = \emptyset$.

From the symmetry, it is enough to show that at most one of the sets $S_1(C_1) \cap S_1(C_2)$ and $S_1(C_1) \cap S_2(C_2)$ is non-empty. Suppose there exist $s_1 \in S_1(C_1) \cap S_1(C_2)$ and $s_2 \in S_1(C_1) \cap S_2(C_2)$. Since, by (8), $S_1(C_1)$ is a strongly stable set, it follows that s_1 is strongly antiadjacent to s_2. By (9), $C_2(s_1) \cap C_2(s_2) = \emptyset$. Let $c_1 \in C_2(s_1)$, $c_2 \in C_2(s_2)$. Also by (9), there exists $c \in C_1(s_1) \cap C_1(s_2)$. Now s_1-c-s_2-c_2-c_1-s_1 is a hole of length five. By (7), there exists $s_2' \in S_2(C_2)$ complete to C_2. But now by (9), s_1 is adjacent to s_2', contrary to 6.1. This proves (11).

In view of (11), we may henceforth assume that $S_1(C_1) \cap S_1(C_2) \neq \emptyset$, and $S_1(C_1) \cap S_2(C_2) = S_2(C_1) \cap S_1(C_2) = \emptyset$

(12) Let $s \in S_1(C_1) \cap S_1(C_2)$, and $s' \in S_1(C_1) \setminus S_1(C_2)$. Then s' is strongly complete to $C_1(s)$.

Suppose not, and let $c \in C_1(s)$ be antiadjacent to s'. Let $c_2 \in C_2(s)$. By (7), there exists $s_2 \in S_2(C_1)$ complete to C_1. By (9), s_2 is strongly adjacent to both s, s'. Since $\{c_2, s, c, s_2, s'\}$ is not a bull, it follows that s_2 is strongly adjacent to c_2. But now $s_1, s_2 \in N_F(c) \cap N_F(c_2)$, contrary to (5). This proves (12).

(13) No vertex of F has a neighbor in three different components of $V(G) \setminus V(F)$.

Let $f \in V(F)$, and let C_1, C_2, C_3 be three distinct components of $V(G) \setminus V(F)$, such that f has a neighbor in each of C_1, C_2, C_3. For $i \in \{1, 2, 3\}$, let $c_i \in C_i$ be adjacent to f. We may assume that $f \in S_1(C_i)$. By (7), there exists a vertex $x_i \in S_2(C_i)$, that is strongly complete to C_i. By (9), f is adjacent to each of x_1, x_2, x_3, and therefore, by (5), x_i is strongly antiadjacent to c_j for $1 \leq i \neq j \leq 3$. Since F is triangle-free, it follows that $\{c_1, c_2, c_3, x_1, x_2, x_3\}$ is a matching of size three in $G|(N_F(c)$, contrary to 5.8. This proves (13).

(14) Every vertex of $V(G) \setminus (C_1 \cup C_2 \cup N_F(C_1) \cup N_F(C_2))$ with a neighbor in $S_1(C_1) \cap S_1(C_2)$ is strongly complete to $S_1(C_1) \cap S_1(C_2)$.

Suppose $x \in V(G) \setminus (C_1 \cup C_2 \cup N_F(C_1) \cup N_F(C_2))$ has a neighbor $s_1 \in S_1(C_1) \cap S_1(C_2)$. For $i = 1, 2$ let $a_i \in C_i$ be complete to $S_1(C_1) \cap S_1(C_2)$ (such a vertex exists by (9)), and let $b_i \in S_2(C_i)$ be complete to C_i (such a vertex exists by (7)). By (9), for $i = 1, 2$, b_i is complete to $S_1(C_1) \cap S_1(C_2)$.

We claim that there is no path in $G|(N(s_1)$ from x to $\{a_1, b_1, a_2, b_2\}$. Suppose there is, and let p be a neighbor of x in the path. Since $\{s_1, x, p\}$ is a triangle, and $s_1 \in V(F)$, it follows that at least one of $p, x \in V(G) \setminus V(F)$. Since $x \notin C_1 \cup C_2 \cup N_F(C_1) \cup N_F(C_2)$, it follows that $p \notin C_1 \cup C_2$, and so there exist a component C_3 of $V(G) \setminus V(F)$, different from C_1, C_2, such that one of $p, x \in C_3$. But now s_1 has a neighbor in three different components of $V(G) \setminus V(F)$, contrary to (13). This proves the claim.

Now, since every vertex of $S_1(C_1) \cap S_1(C_2)$, has a neighbor in $\{a_1, b_1\}$ (namely a_1) and a neighbor in $\{a_2, b_2\}$ (namely a_2), the second assertion of 5.9 implies that x is strongly complete to $S_1(C_1) \cap S_1(C_2)$. This proves (14).

(15) There exists $s_2 \in S_2(C_1)$, complete to C_1 and with a neighbor in $S_1(C_2) \setminus S_1(C_1)$.

The first assertion of (9) implies that there exists $c_1 \in C_1$ complete to $S_1(C_1)$. Let S be the set of neighbors of c_1 in $S_2(C_1)$. We may assume that c_1 is chosen with S minimal, and subject to that with the minimum number of strong neighbors in $S_2(C_1)$.

First we claim that every vertex of S is strongly complete to $C_1 \setminus \{c_1\}$. Suppose some $s \in S$ has an antineighbor $c \in C_1 \setminus \{c_1\}$. Since c_1 is adjacent to s and complete to $S_1(C_1)$, the last assertion of (9) implies that c is strongly complete to $S_1(C_1)$.

We claim that c has a neighbor in $S_2(C_1) \setminus S$. Suppose not. It follows from the choice of c_1 that c is complete to S and semi-adjacent to s, and so the first assertion of (9) implies that c_1 is strongly complete to S, contrary to the choice of c_1. This proves the claim. Let $s_2 \in S_2(C_1) \setminus S$ be a neighbor of c. But now s is adjacent to c_1 and antiadjacent to c, and s_2 is adjacent to c and strongly antiadjacent to c_1, contrary to (9). This proves that S is strongly complete to $C_1 \setminus \{c_1\}$.

Let X be the set of vertices of $S_1(C_1)$ that are semi-adjacent to a vertex of $S \cup \{c_1\}$. Since c_1 is complete to $S_1(C_1)$, (9) implies that either $X = \emptyset$, or X consists of the unique vertex semi-adjacent to c_1, or $|S_1(C_1)| = |S_2(C_1)| = |C_1| = 1$, and X consists of the unique vertex of $S_1(C_1)$ that is semi-adjacent to the unique vertex of $S_2(C_1) = S$. In all cases, $|X| \leq 1$. Since G is unfriendly, it follows that $S \cup \{c_1\}$ is not a homogeneous set in G, and $(S \cup \{c_1\}, X)$ is not homogeneous pair in G. Therefore, some vertex

$v \in V(G) \setminus (S \cup X \cup \{c_1\})$ is mixed on $S \cup \{c_1\}$.

Suppose first that v is strongly antiadjacent to c_1. Then v has a neighbor $s \in S$. Let $s_1 \in S_1(C_1) \cap S_1(C_2)$. Since both s, s_1 are adjacent to c_1, (9) implies that s is adjacent to s_1. Let $c_2 \in C_2$ be adjacent to s_1. By (5), c_2 is antiadjacent to s. By (5), and since v is strongly antiadjacent to c_1, it follows that v is strongly antiadjacent to s_1. Since $\{c_2, s_1, c_1, s, v\}$ is not a bull, it follows that v is strongly adjacent to c_2. Consequently, $v \in C_2 \cup N_F(C_2)$. If $v \in S_2(C_2)$, then, by (9), v is strongly adjacent to s_1, a contradiction. If $v \in S_1(C_2)$, then , since v is strongly antiadjacent to c_1, it follows that $v \in S_1(C_2) \setminus S_1(C_1)$, and s is a vertex complete to C_1 and adjacent to v; and thus (15) holds. So we may assume that $v \in C_2$. Then $s \in S_2(C_2)$. By the maximality of F, v has a neighbor $s_2 \in S_1(C_2)$. By (9), s_2 is adjacent to s. If $s_2 \in S_1(C_1)$, then c_1, v are both adjacent to s, s_2, contrary to (5). Consequently, $s_2 \in S_1(C_2) \setminus S_1(C_1)$, s is adjacent to s_2 and s is complete to C_1; and therefore again (15) holds.

This proves that we may assume that v is adjacent to c_1. Since $v \notin X$, v is strongly adjacent to c_1, and has a strong antineighbor in S. Since v is adjacent to c_1, it follows that $v \in C_1 \cup N_F(C_1)$. Since S is strongly complete to $C_1 \setminus \{c_1\}$, it follows that $v \in N_F(C_1)$. Since v is adjacent to c_1 and $v \notin S$, it follows that $v \notin S_2(C_1)$. Consequently, $v \in S_1(C_1)$. But by (9), since c_1 is complete to $S \cup S_1(C_1)$, it follows that S is complete to $S_1(C_1)$, a contradiction. This proves (15).

(16) Let T_1 be the set of vertices of $V(G) \setminus (C_1 \cup C_2 \cup N_F(C_1) \cup N_F(C_2))$ that are strongly complete to $S_1(C_1) \cap S_1(C_2)$. Then $S_1(C_1) \cup S_1(C_2)$ is strongly anticomplete to $V(F) \setminus (N_F(C_1) \cup N_F(C_2) \cup T_1)$.

Suppose some vertex $s_1 \in S_1(C_1)$ has a neighbor $f_1 \in V(F) \setminus (N_F(C_1) \cup N_F(C_2) \cup T_1)$. By (14), $s_1 \notin S_1(C_2)$ and f_1 is strongly anticomplete to $S_1(C_1) \cap S_1(C_2)$.

By (15), there exist vertices $p_1 \in S_2(C_1)$, $q_1 \in S_1(C_2) \setminus S_1(C_1)$, $p_2 \in S_2(C_2)$, $q_2 \in S_1(C_1) \setminus S_1(C_2)$, such that for $i = 1, 2$ p_i is complete to C_i and adjacent to q_i. Let $c \in C_2$ be adjacent to q_1. By (9), p_2 is adjacent to q_1.

Let $c' \in C_1$ be adjacent to s_1. By (9), s_1 is adjacent to p_1. Since $\{f_1, s_1, c', p_1, q_1\}$ is not a bull and F is triangle-free, it follows that f_1 is adjacent to q_1. Now, since $\{f_1, q_1, c, p_2, q_2\}$ is not a bull and F is triangle-free, it follows that f_1 is adjacent to q_2.

Let $s \in S_1(C_1) \cap S_1(C_2)$. For $i = 1, 2$, let $c_i \in C_i$ be adjacent to s. Then $\{c_1, c_2, p_1, p_2\}$ is a matching of size two in G, s is complete to $\{c_1, c_2, p_1, p_2\}$, q_1 is adjacent to p_1 and antiadjacent to c_1, q_2 is adjacent to p_2 and antiadjacent to c_2, and f_1 is adjacent to q_1, q_2 and antiadjacent to s, contrary to the first assertion of 5.9. This proves (16).

(17) $S_2(C_1) \cup S_2(C_2)$ is strongly complete to $S_1(C_1) \cap S_1(C_2)$; and con-

sequently if $S_1(C_1) \cap S_1(C_2) \neq \emptyset$, then $S_2(C_1) \cup S_2(C_2)$ is a strongly stable set.

Suppose not. We may assume that there exist vertices $a \in S_1(C_1) \cap S_1(C_2)$ and $v \in S_2(C_1)$ that are antiadjacent. For $i = 1, 2$, let V_i be the set of neighbors of a in C_i. Since $S_1(C_1) \cap S_1(C_2) \neq \emptyset$ and, by (15), we deduce that $S_1(C_1) \setminus S_1(C_2) \neq \emptyset$ and $S_1(C_2) \setminus S_1(C_1) \neq \emptyset$. Now it follows from (9) that v is strongly anticomplete to $V_1 \cup V_2$.

Let $p_1 \in S_2(C_1)$ be a vertex complete to C_1, and let $q_1 \in S_1(C_2) \setminus S_1(C_1)$ be adjacent to p_1. Let $p_2 \in S_2(C_2)$ be a vertex complete to C_2, and let $q_2 \in S_1(C_1) \setminus S_1(C_2)$ be adjacent to p_2 (such p_1, q_1, p_2, q_2 exist by (15)). Then $v \neq p_1, p_2$. By (9), p_1 is strongly adjacent to both q_2 and a, and p_2 is strongly adjacent to both q_1 and a. For $i = 1, 2$, let $v_i \in V_i$. Since v is antiadjacent to a, 5.9, applied to the matching $\{p_1, p_2, v_1, v_2\}$ implies that v is antiadjacent to at least one of q_1, q_2. Suppose first that v is antiadjacent to q_1. Let $c_1 \in C_1$ be adjacent to v. Then $\{v, c_1, v_1, p_1, q_1\}$ is a bull, a contradiction. So v is strongly adjacent to q_1, and therefore v is antiadjacent to q_2. From the symmetry, it follows that $v \notin S_2(C_2)$. Since p_2 is adjacent to q_1, and since $\{p_2, q_1, v\}$ and $\{q_1, p_2, q_2\}$ are not triangles in $G|F$, it follows that q_1 is strongly antiadjacent to q_2, and p_2 is strongly antiadjacent to v. Let $c_2 \in C_2$ be adjacent to q_1. Now $\{q_2, p_2, c_2, q_1, v\}$ is a bull, a contradiction. This proves the first assertion of (17). The second assertion now follows, since F is triangle-free. This proves (17).

Let $Q_0 = R_0 = T_0 = U_0 = \emptyset$, and let $P_0 = S_0 = S_1(C_1) \cap S_1(C_2)$. For $i \geq 1$, let us define the sets $P_i, Q_i, R_i, S_i, T_i, U_i$ recursively as follows:

- Let Q_i be the set of vertices of $C_1 \setminus (\bigcup_{j<i} Q_j)$ with a neighbor in P_{i-1}.
- Let R_i be the set of vertices of $S_2(C_1) \setminus (\bigcup_{j<i} R_j)$ with a neighbor in Q_i.
- Let S_i be the set of vertices of $S_1(C_2) \setminus (\bigcup_{j<i} S_j)$ with a neighbor in R_i.
- Let T_i be the set of vertices of $C_2 \setminus (\bigcup_{j<i} T_j)$ with a neighbor in S_{i-1}.
- Let U_i be the set of vertices of $S_2(C_2) \setminus (\bigcup_{j<i} U_j)$ with a neighbor in T_i.
- Let P_i be the set of vertices of $S_1(C_1) \setminus (\bigcup_{j<i} P_j)$ with a neighbor in U_i.

We observe that the definition above is symmetric under exchanging C_1 and C_2. Let $P = \bigcup_{i \geq 0} P_i$, and let Q, R, S, T, U be defined similarly. Let $W = P \cup Q \cup R \cup S \cup T \cup U$. The maximality of $|V(F)|$ implies that Q_1, R_1, T_1, U_1 are all non-empty, and, by (15), S_1 and P_1 are non-empty.

(18) Let $i \geq 1$. If $c \in C_1$ has a neighbor in U_i, then $c \in \bigcup_{j \leq i+1} Q_j$. If $c \in C_2$ has a neighbor in R_i, then $c \in \bigcup_{j \leq i+1} T_j$.

From the symmetry, it is enough to prove the first assertion of (18). Let $u \in U_i$ be adjacent to $c \in C_1$. Let $s \in S_1(C_1)$ be adjacent to c. By (9), u is adjacent to s, and therefore $s \in \bigcup_{j \leq i} P_j$. But then, since c is adjacent to s, it follows that $c \in \bigcup_{j \leq i+1} Q_j$. This proves (18).

(19) No vertex of $V(G) \setminus W$ is mixed on $P \cup S$.

Suppose some $v \in V(G) \setminus W$ is mixed on $P \cup S$. Let i be minimum such that v is mixed on $\bigcup_{j \leq i}(P_j \cup S_j)$. By (14), $i > 0$.

We claim that v is strongly complete to $\bigcup_{j<i}(P_j \cup S_j)$ and has an antineighbor in $P_i \cup S_i$. If v is strongly anticomplete to $P_i \cup S_i$, then, since v is mixed on $\bigcup_{j \leq i}(P_j \cup S_j)$, the claim follows from the minimality of i, and so we may assume that v has a neighbor in $P_i \cup S_i$. Now it follows from (16) that v is strongly complete to $P_0 = S_0$, and again, by the minimality of i, it follows that v is strongly complete to $\bigcup_{j<i}(P_j \cup S_j)$. This proves the claim.

From the symmetry, we may assume that v has an antineighbor $p \in P_i$. By the claim in the first paragraph, it follows that v is strongly complete to $\bigcup_{j<i}(P_j \cup S_j)$. Since $p \in P_i$, there exist $u \in U_i$, $t \in T_i$, and $s \in S_{i-1}$ such that $\{u, t, s\}$ is a triangle, and p is adjacent to u. Then v is strongly adjacent to s. Since $p \notin P_0$, it follows that p is strongly antiadjacent to t. Since F is triangle-free, p is strongly antiadjacent to s. If v is adjacent to t, then $v \in N_F(C_2)$, which, since v is adjacent to s, implies that $v \in S_2(C_2)$, and so $v \in U \subseteq W$, a contradiction. So v is strongly antiadjacent to t. If v is adjacent to u, then $\{s, u, v\}$ is a triangle, and so $v \notin V(F)$, but $\{t, v\}$ is complete to $\{s, u\}$, contrary to (5). So v is strongly antiadjacent to u. But now $\{v, s, t, u, p\}$ is a bull, a contradiction. This proves (19).

(20) No vertex of $V(G) \setminus W$ is mixed on $Q_1 \cup R_1$.

Suppose $v \in V(G) \setminus W$ is mixed on $Q_1 \cup R_1$. The last assertion of (9) implies that $C_1 \setminus Q_1$ is strongly complete to $Q_1 \cup R_1$; by the definition of R_1, $S_2(C_1) \setminus R_1$ is strongly anticomplete to $Q_1 \cup R_1$; and by (12), $S_1(C_1) \setminus P_0$ is strongly complete to Q_1. Now, by (15), $|S_1(C_1)| \neq 1$, and so, by (9), since every vertex of R_1 has a neighbor in Q_1, it follows that $S_1(C_1) \setminus P_0$ is strongly complete to R_1. This proves that no vertex in $(C_1 \cup S_1(C_1) \cup S_2(C_2)) \setminus W$ is mixed on $S_1 \cup R_1$, and so $v \notin C_1 \cup S_1(C_1) \cup S_2(C_1)$. Therefore, v is strongly anticomplete to Q_1. Since v is mixed on $Q_1 \cup R_1$, it follows that v has a neighbor $r \in R_1$. Then there exist $q \in Q_1$ and $p \in P_0$ such that $\{r, q, p\}$ is a triangle. Let $c_2 \in C_2$ be adjacent to p. By (5), c_2 is strongly antiadjacent to r. Since F is triangle-free and by (5), v is strongly antiadjacent to p. Since

$\{v, r, q, p, c_2\}$ is not a bull, it follows that v is strongly adjacent to c_2, and therefore $v \in C_2 \cup S_1(C_2) \cup S_2(C_2)$. Since $v \notin S_2$, it follows that $v \notin S_1(C_2)$. Since v is antiadjacent to p, (17) implies that $v \notin S_2(C_2)$. Therefore $v \in C_2$. But now, by (18), $v \in T$, contrary to the fact that $v \notin W$. This proves (20).

(21) No vertex of $V(G) \setminus W$ is mixed on $Q \cup R$ and no vertex of $V(G) \setminus W$ is mixed on $T \cup U$.

Suppose some $v \in V(G) \setminus W$ is mixed on $Q \cup R$ or on $T \cup U$. Let i be minimum such that v is mixed on $\bigcup_{j \leq i}(Q_j \cup R_j)$ or on $\bigcup_{j \leq i}(T_j \cup U_j)$. From the symmetry, we may assume that v is mixed on $\bigcup_{j \leq i}(Q_j \cup R_j)$. By (20), $i > 1$

From the minimality of i, it follows that either v is strongly anticomplete to $\bigcup_{j<i}(Q_j \cup R_j)$ and has a neighbor in $Q_i \cup R_i$, or v is strongly complete to $\bigcup_{j<i}(Q_j \cup R_j)$ and has an antineighbor in $Q_i \cup R_i$.

Suppose v is strongly anticomplete to $\bigcup_{j<i}(Q_j \cup R_j)$ and has a neighbor in $Q_i \cup R_i$. Assume first that v has a neighbor in Q_i. Then, since v is strongly anticomplete to Q_1, it follows that $v \notin C_1$, and by (12), $v \notin S_1(C_1)$. So $v \in S_2(C_1)$, but then $v \in R_i$, a contradiction. So v is strongly anticomplete to Q_i, and therefore v has a neighbor $r_i \in R_i$. Then that there exist $q_i \in Q_i$ and $p_{i-1} \in P_{i-1}$ such that $\{r_i, q_i, p_{i-1}\}$ is a triangle. Since $i > 1$, there exists $u_{i-1} \in U_{i-1}$, adjacent to p_{i-1}. We claim that v is adjacent to u_{i-1}. Suppose not. Since F it triangle-free and by (5), it follows that u_{i-1} is strongly antiadjacent to r_i, and v is strongly antiadjacent to p_{i-1}. Since $\{u_{i-1}, p_{i-1}, q_i, r_i, v\}$ is not a bull, it follows that u_{i-1} is adjacent to q_i, and therefore $u_{i-1} \in S_2(C_1) \cap S_2(C_2)$. But v is adjacent to r_i and antiadjacent to u_{i-1}, contrary to (16). This proves the claim that v is adjacent to u_{i-1}. It follows from the definition of U_{i-1} that there exist $t_{i-1} \in T_{i-1}$ and $s_{i-2} \in S_{i-2}$ such that $\{u_{i-1}, t_{i-1}, s_{i-2}\}$ is a triangle. From the minimality of i and since v is adjacent to u_{i-1}, we deduce that v is adjacent to t_{i-1}. Consequently, $v \in C_2 \cup S_1(C_2) \cup S_2(C_2)$. Since v is adjacent to u_{i-1}, it follows that $v \notin S_2(C_2)$. Since v is adjacent to r_i, and $v \notin T$, (18) implies that $v \notin C_2$. Therefore, $v \in S_1(C_1)$, and so, since v is adjacent to r_i, it follows that $v \in S_i$, contrary to the fact that $v \notin W$. This proves that v is strongly complete to $\bigcup_{j<i}(Q_j \cup R_j)$ and has an antineighbor in $Q_i \cup R_i$.

In particular, v has a neighbor in C_1, and so $v \in C_1 \cup S_1(C_1) \cup S_2(C_1)$. Since v is strongly complete to R_1, it follows that $v \notin S_2(C_1)$. Suppose $v \in C_1$. Then v is strongly complete to Q, and so v has an antineighbor $r \in R_i$. Since $v \notin Q_i$, it follows that v is strongly anticomplete to P_{i-1}. But some vertex of Q_i is adjacent adjacent to r and has a neighbor in P_{i-1}, contrary to the last assertion of (9). This proves that $v \notin C_1$, and so $v \in S_1(C_1)$. Since $v \notin P_0$, it follows that v is strongly anticomplete to C_2. By (9), and since $|S_1(C_1)| > 1$, we deduce that if v is strongly complete to Q_i, then v is strongly complete to R_i, and hence v has an antineighbor $q_i \in Q_i$. Since

$q_i \in Q_i$, there exist $p \in P_{i-1}$ adjacent to q_i. Since $i > 1$, there exists $u \in U_{i-1}$ adjacent to p. Since v is strongly anticomplete to C_2, it follows from the minimality of i that v is strongly antiadjacent to u. Let $q_1 \in Q_1$. Since $i > 1$, both p and v are adjacent to q_1. Since u is antiadjacent to v, (17) implies that $u \notin S_2(C_1)$. But now $\{u, p, q_i, q_1, v\}$ is a bull, a contradiction. This proves (21).

(22) For every $i > 0$, P_i is strongly complete to $\bigcup_{j \leq i}(Q_j \cup R_j)$.

Suppose $p_i \in P_i$ is antiadjacent to $q \in Q_j$ with $j \leq i$. By (12), $j > 1$. Let $p_{j-1} \in P_{j-1}$ be adjacent to q. Since $j > 1$, there exists $u \in U_{j-1}$ adjacent to p_{j-1}. But now, since $p_i \in P_i$, it follows that p_i is strongly antiadjacent to u, and therefore $u \notin N_F(C_1)$, contrary to the third assertion of (9). Now, since, by (15), $|S_1(C_1)| > 1$, P_i is strongly complete to $\bigcup_{j \leq i} Q_j$, and every vertex of $\bigcup_{j \leq i} R_j$ has a neighbor in $\bigcup_{j \leq i} Q_j$, (9) implies that P_i is strongly complete to $\bigcup_{j \leq i} R_j$. This proves (22).

(23) For every $i > 0$, R_i is strongly complete to $C_1 \setminus (\bigcup_{j \leq i} Q_j)$.

Suppose $r \in R_i$ has an antineighbor $c \in C_1 \setminus (\bigcup_{j \leq i} Q_j)$. Choose $q \in Q_i$ and $p \in P_{i-1}$ such that $\{p, q, r\}$ is a triangle (this is possible by the definition of Q_i and R_i, and by the maximality of $|V(F)|$). Since $c \notin \bigcup_{j \leq i} Q_j$, it follows that c is antiadjacent to both p and r, contrary to (9). This proves (23).

(24) For $i > 0$, R_i is strongly complete to $\bigcup_{j<i} S_j$.

Suppose $r_i \in R_i$ has an antineighbor $s \in S_j$ with $j < i$. By (17), $j > 0$, and so there exists $r_j \in R_j$ adjacent to s_j. Let $q \in Q_j$ be adjacent to r_j. Then, since $r_i \notin R_j$, it follows that q is strongly antiadjacent to r_i, contrary to the third assertion of (9). This proves (24).

(25) $P \cup S$ is strongly complete to $(S_2(C_1) \cup S_2(C_2)) \setminus W$, and strongly anticomplete to $(C_1 \cup C_2 \cup S_1(C_1) \cup S_1(C_2)) \setminus W$.

By (17), $S_2(C_1) \cup S_2(C_2)$ is strongly complete to P_0, and so by (19) $P \cup S$ is strongly complete to $(S_2(C_1) \cup S_2(C_2)) \setminus W$. Since each of $S_1(C_1)$, $S_1(C_2)$ is a strongly stable set, it follows that $(S_1(C_1) \cup S_2(C_2)) \setminus P_0$ is strongly anticomplete to P_0. Now (19) implies that $(S_1(C_1) \cup S_2(C_2)) \setminus W$ is strongly anticomplete to $P \cup S$. Finally, it follows from the definition of Q and T, that $(C_1 \cup C_2) \setminus W$ is strongly anticomplete to $P \cup S$. This proves (25).

(26) $Q \cup R$ is strongly complete to $(C_1 \cup S_1(C_1)) \setminus W$ and strongly anticomplete to $(S_2(C_1) \cup S_1(C_2) \cup S_2(C_2) \cup C_2) \setminus W$.

Since $Q \subseteq C_1$ and C_1 is a strong clique, it follows from (21) that $Q \cup R$ is strongly complete to $C_1 \setminus W$. By (12), $S_1(C_1) \setminus P_0$ is strongly complete to Q_1, and so by (21), $Q \cup R$ is strongly complete to $S_1(C_1) \setminus W$.

In order to show that $Q \cup R$ is strongly anticomplete to $(S_2(C_1) \cup S_1(C_2) \cup S_2(C_2) \cup C_2) \setminus W$, it is enough, by (21), to prove that every vertex of $(S_2(C_1) \cup S_1(C_2) \cup S_2(C_2) \cup C_2) \setminus W$ has an antineighbor in $Q \cup R$.

Since $C_2 \cup (S_2(C_2) \setminus S_2(C_1))$ is strongly anticomplete to C_1 and $Q \subseteq C_1$, it follows that every vertex of $C_2 \cup (S_2(C_2) \setminus S_2(C_1))$ is strongly anticomplete to Q. Since $S_2(C_1)$ is a strongly stable set and $R \subseteq S_2(C_1)$ it follows that every vertex of $S_2(C_1) \setminus W$ is a strongly anticomplete to R. Finally, by the definition of S, $S_1(C_2) \setminus W$ is strongly anticomplete to R. This proves (26).

(27) P is strongly complete to R.

Suppose $p \in P$ is antiadjacent to $r \in R$. Let i, j be integers such that $p \in P_i$ and $r \in R_j$. By (22) $i < j$. By (17), $i > 0$, and so there exists $u \in U_i$ adjacent to p. By (3), there exist $t \in T_i$ and $s \in S_{i-1}$ such that $\{s, t, u\}$ is a triangle. By (24), since $i < j$, it follows that r is strongly adjacent to s. But now, since F is triangle-free, and since, by (17), both p and r are strongly antiadjacent to t, it follows that $\{r, s, t, u, p\}$ is a bull, a contradiction. This proves (27).

It follows from (27) and the symmetry that S is strongly complete to U.

(28) If $W \cap S_2(C_1) \cap S_2(C_2) \neq \emptyset$, then $P = S_1(C_1)$, $Q = C_1$, $R = S_2(C_1)$, $S = S_1(C_2)$, $T = C_2$ and $U = S_2(C_2)$.

From the symmetry, we may assume that there exist $w \in R \cap S_2(C_1) \cap S_2(C_2)$. By (17), w is strongly complete to $S_1(C_2)$, therefore $S_1(C_2) \setminus S_1(C_1) \subseteq S$, and so $S = S_1(C_2)$. It follows that $T = C_2$, and, consequently $U = S_2(C_2)$; in particular, $w \in U$. But now, for, the symmetry, $P = S_1(C_1)$, $Q = C_1$ and $R = S_2(C_1)$. This proves (28).

(29) If $W \cap S_2(C_1) \cap S_2(C_2) \neq \emptyset$, then $V(G) = C_1 \cup C_2 \cup S_1(C_1) \cup S_2(C_1) \cup S_1(C_2) \cup S_2(C_2)$.

Suppose not. Then there exists $v \in V(G) \setminus (C_1 \cup C_2 \cup S_1(C_1) \cup S_2(C_1) \cup S_1(C_2) \cup S_2(C_2))$ with a neighbor in $C_1 \cup C_2 \cup S_1(C_1) \cup S_2(C_1) \cup S_1(C_2) \cup S_2(C_2)$. By (28), $P = S_1(C_1), Q = C_1, R = S_2(C_1), S = S_1(C_2), T = C_2$ and $U = S_2(C_2)$. Since $v \in V(G) \setminus (C_1 \cup C_2 \cup S_1(C_1) \cup S_2(C_1) \cup S_1(C_2) \cup S_2(C_2))$, it follows that v is strongly anticomplete to $C_1 \cup C_2$, and so (21) implies that v is strongly anticomplete to $C_1 \cup C_2 \cup S_2(C_1) \cup S_2(C_2)$. So v has a neighbor in $S_1(C_1) \cup S_1(C_2)$, and therefore, by (20), v is strongly complete to $S_1(C_1) \cup S_1(C_2)$. Let $s_2 \in S_2(C_1) \cap S_2(C_2)$. For $i = 1, 2$ let $c_i \in C_i$ be adjacent to s_2, and let $s_1 \in S_1(C_1)$ be adjacent to c_1. Then, by (9), s_1 is adjacent to s_2, and so by (5), s_1 is strongly antiadjacent to c_2. But now $\{v, s_1, c_1, s_2, c_2\}$ is a bull, a contradiction. This proves (29).

(30) $P \cup S$ and $R \cup U$ are strongly stable sets.

Since P_0 is strongly complete to $R \cup U$ and F is triangle-free, it follows that $R \cup U$ is a strongly stable set. Since $P \subseteq S_1(C_1)$ and $S \subseteq S_1(C_2)$, it follows that each of P, S is a strongly stable set. So it is enough to prove that $P \setminus S$ is strongly anticomplete to $S \setminus P$. Suppose $p \in P$ is adjacent to $s \in S$. Let i, j be integers such that $p \in P_i$ and $s \in S_j$. Then $i, j > 0$, and so there exists $r \in R_j$ adjacent to s. By (27), p is adjacent to r. But now $\{p, r, s\}$ is a triangle in F, a contradiction. This proves (30).

Let $Z = P \cup S$ and $L = R \cup U$.

(31) If $S_2(C_1) \cap S_2(C_2) \cap W = \emptyset$, then $G|(Q \cup T \cup Z \cup L)$ is a Z-melt, and if $S_2(C_1) \cap S_2(C_2) \cap W \neq \emptyset$, than $G|(Q \cup T \cup Z \cup L)$ is a double melt.

First we observe that Q, T are strong cliques, and, by (30), Z, L are strongly stable sets. By (15), $|Z| > 1$ and $|L| > 1$. Let $|Q| = m$ and $|T| = n$. By (9), we can number the vertices of Q as $\{q_1, \ldots, q_m\}$ such that for every $p \in P$, $N(p) \cap Q = \{q_1, \ldots, q_i\}$ for some $i \in \{1, \ldots, m\}$, and p is strongly complete to $\{q_1, \ldots, q_{i-1}\}$; and for every $r \in R$, $N(r) \cap Q = \{q_{m-i+1}, \ldots, q_m\}$ for some $i \in \{1, \ldots, m\}$, and r is strongly complete to $\{q_{m-i+2}, \ldots, q_m\}$. Similarly, we can number the vertices of T as $\{t_1, \ldots, t_n\}$ such that for every $s \in S$, $N(s) \cap T = \{t_{n+1-j}, \ldots, t_n\}$ for some $j \in \{1, \ldots, n\}$, and s is strongly complete to $\{t_{n+2-j}, \ldots, t_n\}$, and for every $u \in U$, $N(u) \cap T = \{t_1, \ldots, t_j\}$ for some $j \in \{1, \ldots, n\}$, and u is strongly complete to $\{t_1, \ldots, t_{j-1}\}$.

Let $A_{0,0} = B_{0,0} = \emptyset$. For $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ let $A_{i,j}$ be the set of vertices of Z that are strongly complete to $\{q_1, \ldots, q_{i-1}\} \cup \{t_{n-j+2}, \ldots, t_n\}$, complete to $\{q_i\} \cup \{t_{n-j+1}\}$, and strongly anticomplete to $\{q_{i+1}, \ldots, q_m\} \cup \{t_1, \ldots, t_{n-j}\}$; and let $B_{i,j}$ be the set of vertices of L that are strongly complete to $\{q_{m-i+2}, \ldots, q_m\} \cup \{t_1, \ldots, t_{j-1}\}$, complete to $\{q_{m-i+1}\} \cup \{t_j\}$, and strongly anticomplete to $\{q_1, \ldots, q_{m-i}\} \cup \{t_{j+1}, \ldots, t_n\}$. For $i \in \{1, \ldots, m\}$, let $A_{i,0}$ be the set of vertices of Z that are strongly complete to $\{q_1, \ldots, q_{i-1}\}$, complete to $\{q_i\}$, and strongly anticomplete to $\{q_{i+1}, \ldots, q_m\} \cup T$. For $j \in \{1, \ldots, n\}$, $A_{0,j}$ be the set of vertices of Z that are strongly complete to $\{t_{n-j+2}, \ldots, t_n\}$, complete to $\{t_{n-j+1}\}$, and strongly anticomplete to $Q \cup \{t_1, \ldots, t_{n-j}\}$. For $i \in \{1, \ldots, m\}$, let $B_{i,0}$ be the set of vertices of L that are strongly complete to $\{q_{m-i+2}, \ldots, q_m\}$, complete to $\{q_{m-i+1}\}$, and strongly anticomplete to $\{q_1, \ldots, q_{m-i}\} \cup T$. Finally, for $j \in \{1, \ldots, n\}$, let $B_{0,j}$ be the set of vertices of L that are strongly complete to $\{t_1, \ldots, t_{j-1}\}$, complete to $\{t_j\}$, and strongly anticomplete to $Q \cup \{t_{j+1}, \dots, t_n\}$. Then $Z = \bigcup_{0\leq i\leq m} \bigcup_{0\leq j\leq n} A_{i,j}$ and $L = \bigcup_{0\leq i\leq m} \bigcup_{0\leq j\leq n} B_{i,j}$.

Since every vertex of $Q \cup T$ has a neighbor in both Z and L, (9) implies that the sets $\bigcup_{0\leq j\leq n} A_{m,j}$, $\bigcup_{0\leq j\leq n} B_{m,j}$, $\bigcup_{0\leq i\leq m} A_{i,n}$ and $\bigcup_{0\leq i\leq m} B_{i,n}$ are all non-empty.

Let $i, i' \in \{0, \dots, m\}$ and $j, j' \in \{0, \dots, n\}$, such that $i' > i$ and $j' > j$, and let $a \in A_{i,j}$ and $a' \in A_{i',j'}$. Since $A_{0,0} = \emptyset$, we may assume that $i > 0$. Then a' is complete $\{q_i, q_{i'}, t_{n-j'+1}\}$, and a is anticomplete to $\{q_{i'}, t_{n-j'+1}\}$ and adjacent to q_i, and so $\{a, q_i, q_{i'}, a', t_{n-j'+1}\}$ is a bull, a contradiction. This proves that one of $A_{i,j}$ and $A_{i',j'}$ is empty. Similarly, one of the sets $B_{i,j}$ and $B_{i',j'}$ is empty.

By (17), for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, $A_{i,j}$ is strongly complete to L, and $B_{i,j}$ is strongly complete to Z. By (27), for every $i, i' \in \{1, \dots, m\}$ and $j, j' \in \{1, \dots, n\}$, $A_{i,0}$ is strongly complete to $B_{i',0}$, and $A_{0,j}$ is strongly complete to $B_{0,j'}$.

Let $i \in \{1, \dots, m\}$ and $j \in \{1, \dots n\}$. Let $A^j_{i,0}$ be the set of vertices of $A_{i,0}$ with that have a neighbor in $B_{0,j}$ are strongly anticomplete to $\bigcup_{j<k\leq n} B_{0,k}$. Let $A^0_{i,0}$ be the set of vertices of $A_{i,0}$ that are strongly anticomplete to $\bigcup_{1\leq k\leq n} B_{0,k}$. Let $A^i_{0,j}$ be the set of vertices of $A_{0,j}$ that have a neighbor in $B_{i,0}$ and are strongly anticomplete to $\bigcup_{i<k\leq m} B_{k,0}$. Let $A^0_{0,j}$ be the set of vertices of $A_{0,j}$ that are strongly anticomplete to $\bigcup_{1\leq k\leq m} B_{k,0}$. Let $B^j_{i,0}$ be the set of vertices of $B_{i,0}$ that have a neighbor in $A_{0,j}$ and are strongly anticomplete to $\bigcup_{j<k\leq n} A_{0,k}$. Let $B^0_{i,0}$ be the set of vertices of $B_{i,0}$ that are strongly anticomplete to $\bigcup_{1\leq k\leq n} A_{0,k}$. Let $B^i_{0,j}$ be the set of vertices of $B_{0,j}$ with a neighbor in $A_{i,0}$ that are strongly anticomplete to $\bigcup_{i<k\leq m} A_{k,0}$. Finally, let $B^0_{0,j}$ be the set of vertices of $B_{0,j}$ that are strongly anticomplete to $\bigcup_{1\leq k\leq m} A_{k,0}$. Then

$$A_{i,0} = \bigcup_{0\leq k\leq n} A^k_{i,0},$$

$$A_{0,j} = \bigcup_{0\leq k\leq m} A^k_{0,j},$$

$$B_{i,0} = \bigcup_{0\leq k\leq n} B^k_{i,0},$$

and

$$B_{0,j} = \bigcup_{0\leq k\leq m} B^k_{0,j}.$$

We observe that for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, $A_{i,0} \subseteq P \setminus P_0$, $A_{0,j} \subseteq S \setminus S_0$, $B_{i,0} \subseteq R$ and $B_{0,j} \subseteq U$. Therefore every vertex of $A^0_{i,0}$ has a neighbor in $\bigcup_{1\leq p\leq m} \bigcup_{1\leq q\leq n} B_{p,q}$, every vertex of $B^0_{i,0}$ has a neighbor in $\bigcup_{1\leq p\leq m} \bigcup_{1\leq q\leq n} A_{p,q}$, every vertex of $A^0_{0,j}$ has a neighbor in $\bigcup_{1\leq p\leq m} \bigcup_{1\leq q\leq n} B_{p,q}$, and every vertex of $B^0_{0,j}$ has a neighbor in $\bigcup_{1\leq p\leq m} \bigcup_{1\leq q\leq n} A_{p,q}$.

By (9), $A^i_{0,j}$ is strongly complete to $\bigcup_{1\leq s<i} B_{s,0}$, $A^j_{i,0}$ is strongly complete to $\bigcup_{1\leq s<j} B_{0,s}$, $B^j_{i,0}$ is strongly complete to $\bigcup_{1\leq s<j} A_{0,s}$ and $B^i_{0,j}$ is strongly complete to $\bigcup_{1\leq s<i} A_{s,0}$. For $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$ let $A'_{i,0}$ be the set of vertices of $A_{i,0}$ that are semi-adjacent to q_i, let $A'_{0,j}$ be the set of vertices of $A_{0,j}$ that are semi-adjacent to t_{n-j+1}, let $B'_{i,0}$ be the set of vertices of $B_{i,0}$ that are semi-adjacent to q_{m-i+1}, and let $B'_{0,j}$ be the set of vertices of $B_{0,j}$ that are semi-adjacent to t_j. Then, by (9), $A'_{i,0}$ is strongly complete to $\bigcup_{1\leq s\leq n} B^i_{0,s}$, $A'_{0,j}$ is strongly complete to $\bigcup_{1\leq s\leq m} B^j_{s,0}$, $B'_{i,0}$ is strongly complete to $\bigcup_{1\leq s\leq n} A^i_{0,s}$, and $B'_{0,j}$ is strongly complete to $\bigcup_{1\leq s\leq m} A^j_{s,0}$. Since $P_0 \neq \emptyset$, it follows that there exist $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$ such that either $A_{i,j} \neq \emptyset$. Finally, let $i,s,s' \in \{1,\ldots,m\}$ and $j,t,t' \in \{1,\ldots,n\}$ such that $t' \geq j \geq n+1-t$ and $s \geq i \geq m+1-s'$, and let $a \in A_{s,t}$ and $b \in B_{s',t'}$. Then $\{a,b\}$ is complete to $\{q_i,t_j\}$, and a is adjacent to b, contrary to (5). This proves that at least one of $A_{s,t}$, $B_{s',t'}$ is empty.

Thus all the conditions of the definition of a melt are satisfied, and so $G|(Q \cup T \cup Z \cup L)$ is a melt. Moreover, if $S_2(C_1) \cap S_2(C_2) \cap W = \emptyset$, then R is strongly anticomplete to T and U is strongly anticomplete to Q, and so $G|(Q \cup T \cup Z \cup L)$ is a Z-melt. If $S_2(C_1) \cap S_2(C_2) \cap W \neq \emptyset$, then, by (28), $R \cap U \neq \emptyset$, and so $G|(Q \cup T \cup Z \cup L)$ is a double melt. This proves (31).

Now, if $S_2(C_1) \cap S_2(C_2) \cap W \neq \emptyset$, (29) and (31) imply that G is a double melt, and so $G \in \mathcal{T}_1$. So we may assume that $S_2(C_1) \cap S_2(C_2) \cap W = \emptyset$.

If $S_2(C_1) \cap S_2(C_2) = \emptyset$, let $Q' = T' = Z' = L' = \emptyset$. Assume $S_2(C_1) \cap S_2(C_2) \neq \emptyset$. Let $P'_0 = S'_0 = S_2(C_1) \cap S_2(C_2)$, let $Q'_0 = R'_0 = T'_0 = U'_0 = \emptyset$, and for $i \geq 1$, define $P'_i, Q'_i, R'_i, S'_i, T'_i, U'_i$ similarly to $P_i, Q_i, R_i, S_i, T_i, U_i$. Let $P' = \bigcup_{i\geq 1} P'_i$, and let Q', R', S', T', U' be defined similarly. Let $W' = P' \cup Q' \cup R' \cup S' \cup T' \cup U'$. Let $Z' = P' \cup S'$ and $L' = R' \cup U'$. By the remark following (31), we may assume that $W' \cap S_1(C_1) \cap S_1(C_2) = \emptyset$. Now, by (31), $G|(Q' \cup T' \cup Z' \cup L')$ is a Z'-melt.

(32) $W \cap W' = \emptyset$.

Suppose $W \cap W' \neq \emptyset$, let $i \geq 0$ be minimum such that $(P_i \cup Q_i \cup R_i \cup S_i \cup T_i \cup U_i) \cap W' \neq \emptyset$, and let $v \in (P_i \cup Q_i \cup R_i \cup S_i \cup T_i \cup U_i) \cap W'$. Since $P_0 \cap W' = \emptyset$, it follows that $i > 0$.

Assume first that $v \in Q_i$. Then there there exists $p_{i-1} \in P_{i-1}$ adjacent to v. Since $Q \subseteq C_1$ and $W' \cap C_1 \subseteq Q'$, we deduce that $v \in Q'$, and so $p_{i-1} \in R'$, contrary to the minimality of i. This proves that $Q_i \cap W' = \emptyset$, and, from the symmetry, that $T_i \cap W' = \emptyset$.

Next assume that $v \in R_i$. Then there there exists $q \in Q_i$ adjacent to v. Since $v \in R_i$, and since $W \cap S_2(C_1) \cap S_2(C_2) = \emptyset$, it follows that $v \in S_2(C_1) \setminus S_2(C_2)$, and so $v \in P'$. But now $q \in Q'$, contrary to the fact that $Q_i \cap W' = \emptyset$. This proves that $R_i \cap W' = \emptyset$, and, from the symmetry,

$U_i \cap W' = \emptyset$.

Consequently, $v \in P_i \cup S_i$, and form the symmetry we may assume that $v \in P_i$. Since $i > 0$, it follows that there exists $u \in U_i$, adjacent to v. Also since $i > 0$, we deduce that $v \in S_1(C_1) \setminus S_1(C_2)$, and so $v \in R'$. But then $u \in S'$, contrary to the fact that $U_i \cap W' = \emptyset$. This proves (32).

Let $Z(C_1, C_2) = Z$, $Q(C_1, C_2) = Q$, $T(C_1, C_2) = T$, $R(C_1, C_2) = R$ and $U(C_1, C_2) = U$. Let $Z'(C_1, C_2) = Z'$, $Q'(C_1, C_2) = Q'$, $T'(C_1, C_2) = T'$, $R'(C_1, C_2) = R'$ and $U'(C_1, C_2) = U'$. For every pair of distinct components C'_1, C'_2 of $V(G) \setminus V(F)$ with $N_F(C'_1) \cap N_F(C'_2) \neq \emptyset$, we define $Z(C'_1, C'_2)$, $Q(C'_1, C'_2)$, $T(C'_1, C'_2)$, $R(C'_1, C'_2)$ $U(C'_1, C'_2)$, $Z'(C'_1, C'_2)$, $Q'(C'_1, C'_2)$, $T'(C'_1, C'_2)$, $R'(C'_1, C'_2)$ and $U'(C'_1, C'_2)$ similarly.

Let C'_1, C'_2 be two distinct components of $V(G) \setminus V(F)$. For $i, j \in \{1, 2\}$ let $S_i(C'_j)$ be their anchors. We may assume that $S_1(C'_1) \cap S_2(C'_2) = S_2(C'_1) \cap S_1(C'_2) = \emptyset$. Let $i(C'_1, C'_2)$ be the number of non-empty sets among $S_1(C'_1) \cap S_1(C'_2)$ and $S_2(C'_1) \cap S_2(C'_2)$.

Let H be the graph whose vertices are the components of $V(G) \setminus V(F)$, and such that if $C'_1, C'_2 \in V(H)$, then there are $i(C'_1 C'_2)$ edges with ends C'_1, C'_2. Then H is a loopless graph.

(33) H is triangle-free and $maxdeg(H) \leq 2$.

Let C_1, C_2, C_3 be components of $V(G) \setminus V(F)$. Suppose $S_1(C_1) \cap S_1(C_2) \neq \emptyset$. We claim that for $i \in \{1, 2\}$ $S_1(C_1) \cap S_i(C_3) = S_1(C_2) \cap S_i(C_3) = \emptyset$. For suppose there is a vertex $x \in S_1(C_1) \cap S_1(C_3)$. Let c be a vertex of C_3 adjacent to x. Then, by (16), c is strongly complete to $S_1(C_1) \cap S_1(C_2)$, contrary to (13). This proves the claim. It follows from the claim that $maxdeg(H) \leq 2$.

Suppose there is a triangle in H. That means that there exist component C_1, C_2, C_3, and, in view of the claim in the previous paragraph, renumbering the anchors, we may assume that there exist $u \in S_1(C_1) \cap S_1(C_2)$, $v \in S_2(C_2) \cap S_2(C_3)$, and $w \in S_1(C_3) \cap S_2(C_1)$. But now, by (17), $\{u, v, w\}$ is a triangle in F, a contradiction. This proves (33).

We show that G admits an H-structure. Let us define a map

$$h : V(H) \cup E(H) \cup (E(H) \times V(H)) \rightarrow 2^{V(G)}.$$

Let C_1, C_2 be distinct components of $V(G) \setminus V(F)$. If there is a unique edge e with ends C_1, C_2 let $h(e) = Z(C_1, C_2)$, $h(e, C_1) = Q(C_1, C_2) \cup R(C_1, C_2)$ and $h(e, C_2) = T(C_1, C_2) \cup U(C_1, C_2)$. Let C_3 be a component of $V(G) \setminus V(F)$, distinct from C_1, C_2, and assume that f is an edge of H with ends C_1, C_3. We observe that by (13) and (16), if $S_1(C_1)$ and $S_2(C_1)$ are the anchors of C_1, then, up to symmetry, $Z(C_1, C_2) \cap N_F(C_1) \subseteq S_1(C_1)$, and $Z(C_1, C_3) \cap N_F(C_1) \subseteq S_2(C_1)$. If there are two edges e, e'

with ends C_1, C_2 let $h(e) = Z(C_1, C_2)$, $h(e, C_1) = Q(C_1, C_2) \cup R(C_1, C_2)$ and $h(e, C_2) = T(C_1, C_2) \cup U(C_1, C_2)$; and $h(e') = Z'(C_1, C_2)$, $h(e', C_1) = Q'(C_1, C_2) \cup R'(C_1, C_2)$ and $h(e', C_2) = T'(C_1, C_2) \cup U'(C_1, C_2)$. For every component C of $V(G) \setminus V(F)$, let $h(C) = C \setminus (\bigcup_{e \in E(H)} \bigcup_{C \sim e} h(e, C))$. Let $L = V(G) \setminus h(V(H) \cup E(H) \cup (E(H) \times V(H)))$.

It follows from the definition of h that

- every vertex of $V(G) \setminus L$ is in $h(x)$ for exactly one element x of $V(H) \cup E(H) \cup (E(H) \times V(H))$, and
- $h(v) \neq \emptyset$ for every $v \in V(H)$ of degree zero, and
- $h(e) \neq \emptyset$ for every $e \in E(H)$, and
- $h(e, v) \neq \emptyset$ if e is incident with v, and
- $h(e, v) = \emptyset$ if e is not incident with v, and
- for $u, v \in V(H)$, $h(u)$ is strongly anticomplete to $h(v)$.

Since $L \cup (\bigcup_{e \in E(H)} h(e)) \subseteq V(F)$, it follows that $G|(L \cup (\bigcup_{e \in E(H)} h(e)))$ has no triangle. Since $h(C) \subseteq C$ for every component C of $V(G) \setminus V(F)$, it follows that $h(v)$ is a strong clique for every $v \in V(H)$. Since $h(e) = Z(C_1, C_2)$ for every edge C_1C_2 of H, it follows that every vertex of L has a neighbor in at most one of the sets $h(v)$ where $v \in V(H)$. By (19), for every $e \in E(H)$, every vertex of L is either strongly complete or strongly anticomplete to $h(e)$, and for every $e, f \in E(H)$, $h(e)$ is either strongly complete or strongly anticomplete to $h(f)$. By (25) and (32), if $e, f \in E(H)$, and e and f share an end, then $h(e)$ is strongly complete to $h(f)$. By (25), for every $e \in E(H)$ and $v \in V(H)$, $h(e)$ is strongly anticomplete to $h(v)$.

Let $v \in V(H)$, let S_v be the vertices of L with a neighbor in $h(v)$, and let T_v be the vertices of $(L \cup (\bigcup_{e \in E(H)} h(e))) \setminus S_v$ with a neighbor in S_v. Then S_v contains every every vertex of F with a neighbor in $h(v)$, and T_v contains every vertex of $V(F) \setminus S_v$ with a neighbor in S_v. Now, by (10) applied to the graph $G|(V(F) \cup h(v))$, it follows that there is a partition of S_v into two sets A_v, B_v, and a partition of T_v into two sets C_v, D_v such that $G|(h(v) \cup S_v \cup T_v)$ is an $(h(v), A_v, B_v, C_v, D_v)$-clique connector. By (9) and (15), for $v \in V(H)$, if there exist $a \in A_v$ and $b \in B_v$ antiadjacent with a common neighbor in $h(v)$, then v has degree zero in H.

Let e be an edge of H with ends u, v. Then (26) and (32) imply that if $f \in E(H) \setminus \{e\}$ is incident with v then $h(e, v)$ is strongly complete to $h(f, v)$. By (31), $G|(h(e) \cup h(e, v) \cup h(e, f))$ is an $h(e)$-melt, such that if (K, M, A, B) are as in the definition of a melt, then $K \subseteq h(e, v)$, $M \subseteq h(e, u)$, $A = h(e)$, $B \subseteq h(e, v) \cup h(e, u)$, every vertex of $h(e, v) \cap B$ has a neighbor in K, and every vertex of $h(e, u) \cap B$ has a neighbor in M (and, in particular, $h(e, v)$ is strongly anticomplete to $h(e, u)$). It follows from (21) and (26) that $h(e, v)$ is strongly complete to $h(v)$, and $h(e, v)$ is strongly anticomplete to $h(w)$

for every $w \in V(H) \setminus \{v\}$; and $h(e,v)$ is strongly anticomplete to $h(f,w)$ for every $f \in E(H) \setminus \{e\}$ and $w \in V(H) \setminus \{v\}$; and $h(e,v)$ is strongly anticomplete to $h(f)$ for every $f \in E(H) \setminus \{e\}$.

We may assume that $A_v = S_1(v) \cap L, A_u = S_1(u) \cap L, B_v = S_2(v) \cap L, B_u = S_2(u) \cap L$, and $S_1(u) \cap S_2(v) = S_2(v) \cap S_1(u) = \emptyset$. Switching the roles of $A_u \cup A_v$ an $B_u \cup B_v$ if necessary, we may assume that $h(e) \subseteq S_1(v) \cup S_1(u)$.

- (25) implies that $h(e)$ is strongly complete to $B_u \cup B_v$,
- (26) implies that $h(e,v)$ is strongly complete to A_v, and strongly anticomplete to $L \setminus A_v$,
- By (16), (19) and (25), every vertex of $(L \cup (\bigcup_{e \in E(H)} h(e))) \setminus (A_u \cup A_v)$ with a neighbor in $A_u \cup A_v$ is strongly complete to $h(e)$.

Thus, in view of (33), all the conditions of the definition of an H-structure are satisfied, and so G admits an H-structure, and therefore $G \in \mathcal{T}_1$. This completes the proof of 6.2. ∎

We can now prove 3.4, which we restate.

6.3 *Let G be an elementary bull-free trigraph. Then either*

- *one of $G, \overline{G}$ belongs to $\mathcal{T}_1$, or*
- *G admits a homogeneous set decomposition, or*
- *G admits a homogeneous pair decomposition.*

Let us first remind the reader the main result of [1].

6.4 *Let G be a bull-free trigraph. Let P and Q be paths of length three, and assume that there is a center for P and an anticenter for Q in G. Then either*

- *G admits a homogeneous set decomposition, or*
- *G admits a homogeneous pair decomposition, or*
- *G or $\overline{G}$ belongs to $\mathcal{T}_0$.*

Proof of 6.3. We may assume that G does not admit a homogeneous set decomposition or a homogeneous pair decomposition. Assume first that there are paths P and Q, each of length three, in G, and that there is a center for P and an anticenter for Q in G. By 6.4, either

- G admits a homogeneous set decomposition, or
- G admits a homogeneous pair decomposition, or
- G or $\overline{G}$ belongs to $\mathcal{T}_0$.

So one of $G, \overline{G}$ belongs to $\mathcal{T}_0$. But then G is not elementary, a contradiction. Consequently, no such paths P, Q exist in G, and therefore we may assume that either G or $\overline{G}$ is unfriendly. Since one of the outcomes of 6.3 holds for G if and only if one of the outcomes of 6.3 holds for $\overline{G}$, we may assume that G is unfriendly. Since if G is a prism, then $\overline{G}$ has no triangle, and therefore admits and H-structure with H being the empty graph, 4.2 implies that no induced subtrigraph of G is a prism.

If G is framed, then by 6.2, $G \in \mathcal{T}_1$, so we may assume that G is not framed. It follows that no induced subtrigraph of G is a path of length three. So by 5.4, one of the following holds:

- G is not connected, or
- G is not anticonnected, or
- there exist two vertices $v_1, v_2 \in V(G)$ such that v_1 is semi-adjacent to v_2, and $V(G) \setminus \{v_1, v_2\}$ is strongly complete to v_1 and strongly anticomplete to v_2.

Since G does not admit a homogeneous set decomposition, if G is not connected or G is not anticonnected, then $|V(G)| = 2$ and $G \in \mathcal{T}_1$. Thus we may assume that there exist two vertices $v_1, v_2 \in V(G)$ such that v_1 is semi-adjacent to v_2, and $V(G) \setminus \{v_1, v_2\}$ is strongly complete to v_1 and strongly anticomplete to v_2. Since G does not admit a homogeneous set decomposition, it follows that $|V(G) \setminus \{v_1, v_2\}| = 1$. But then $G \in \mathcal{T}_1$. This proves 6.3. ∎

References

[1] M. Chudnovsky, The Structure of bull-free graphs I— three-edge-paths with centers and anticenters, *submitted for publication*