The structure of bull-free graphs III—global structure

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Abstract

The *bull* is a graph consisting of a triangle and two pendant edges. A graph is called *bull-free* if no induced subgraph of it is a bull. This is the last in a series of three papers. In this paper we use the results of [1, 2] and give an explicit description of the structure of all bull-free graphs.

1 Introduction

All graphs in this paper are finite and simple. The *bull* is a graph with vertex set $\{x_1, x_2, x_3, y, z\}$ and edge set

$$\{x_1x_2, x_2x_3, x_1x_3, x_1y, x_2z\}.$$

Let G be a graph. We say that G is *bull-free* if no induced subgraph of G is isomorphic to the bull. The complement of G is the graph $\overline{G}$, on the same vertex set as G, and such that two vertices are adjacent in G if and only if they are non-adjacent in $\overline{G}$. A *clique* in G is a set of vertices, all pairwise adjacent. A *stable set* in G is a clique in the complement of G. A clique of size three is called a *triangle* and a stable set of size three is a *triad*. For a subset A of $V(G)$ and a vertex $b \in V(G) \setminus A$, we say that b is *complete* to A if b is adjacent to every vertex of A, and that b is *anticomplete* to A if b is not adjacent to any vertex of A. For two disjoint subsets A and B of $V(G)$, A is *complete* to B if every vertex of A is complete to B, and A is *anticomplete* to B every vertex of A is anticomplete to B. For a subset X of $V(G)$, we denote by $G|X$ the subgraph induced by G on X, and by $G \setminus X$ the subgraph induced by G on $V(G) \setminus X$.

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An obvious example of a bull free graph is a graph with no triangle, or a graph with no triad; but there are others. Let us call a graph G an *ordered split graph* if there exists an integer n such that the vertex set of G is the union of a clique $\{k_1, \ldots, k_n\}$ and a stable set $\{s_1, \ldots, s_n\}$, and s_i is adjacent to k_j if and only if $i + j \leq n + 1$. It is easy to see that every ordered split graph is bull-free. A large ordered split graph contains a large clique and a large stable set, and therefore the three classes (triangle-free, triad-free and ordered split graphs) are significantly different. Another way to make a bull-free graph that has both a large clique and a large stable set is by using the operation of substitution (this is a well known operation, but, for completeness, we define it in Section 4). It turns out, however, that we can give and explicit description of the structure of all bull-free graphs that are not obtained from smaller bull-free graphs by substitution. To do so, we first define "bull-free trigraphs", which are objects generalizing bull-free graphs: while in a graph every two vertices are either adjacent or nonadjacent, in a trigraph every pair of vertices is either adjacent, or antiadjacent or semi-adjacent (this is done in Section 2). In Section 3, we describe three special classes of bull-free trigraphs, $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2$ (in fact, two of the classes were defined in [1] and [2]), and state a theorem that says that, up to taking complements, every bull-free trigraph either belongs to one of these three classes, or admits a decomposition. In Section 4, we turn the decomposition theorem of Section 3 into a "composition theorem", which is our main result, 4.2. Roughly, 4.2 says that every bull-free trigraph that is not obtained from smaller bull-free trigraphs by substitution is an "expansion" of a trigraph in $\mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2$ (we postpone the definition of an "expansion" to Section 4). The organization of the rest of this paper is described at the end of Section 4.

2 Trigraphs

In order to prove our main result, we consider objects, slightly more general than bull-free graphs, that we call "bull-free trigraphs". A *trigraph* G consists of a finite set $V(G)$, called the *vertex set* of G, and an a map $\theta : V(G)^2 \to \{-1, 0, 1\}$, called the *adjacency function*, satisfying:

- for all $v \in V(G)$, $\theta_G(v, v) = 0$
- for all distinct $u, v \in V(G)$, $\theta_G(u, v) = \theta_G(v, u)$
- for all distinct $u, v, w \in V(G)$, at most one of $\theta_G(u, v), \theta_G(u, w) = 0$.

Two distinct vertices of G are said to be *strongly adjacent* if $\theta(u, v) = 1$, *strongly antiadjacent* if $\theta(u, v) = -1$, and *semi-adjacent* if $\theta(u, v) = 0$. We say that u and v are *adjacent* if they are either strongly adjacent, or semi-adjacent; and *antiadjacent* of they are either strongly antiadjacent, or semi-adjacent. If u and v are adjacent, we also say that u is *adjacent to* v, or that u is a *neighbor* of v. If u and v are antiadjacent, we also say that u is *antiadjacent to* v, or that u is an *anti-neighbor* of v. Similarly, if u and v are strongly adjacent (strongly antiadjacent), then u is a *strong neighbor* (*strong anti-neighbor*) of v. Let $\eta(G)$ be the set of all strongly adjacent pairs of G, $\nu(G)$ the set of all strongly antiadjacent pairs of G, and $\sigma(G)$ the set of all pairs $\{u, v\}$ of vertices of G, such that u and v are distinct and semi-adjacent. Thus, a trigraph G is a graph if $\sigma(G)$ empty.

Let G be a trigraph. The complement $\overline{G}$ of G is a trigraph with the same vertex set as G, and adjacency function $\overline{\theta} = -\theta$. Let $A \subset V(G)$ and $b \in V(G) \setminus A$. For $v \in V(G)$ let $N(v)$ denote the set of all vertices in $V(G) \setminus \{v\}$ that are adjacent to v, and let $S(v)$ denote the set of all vertices in $V(G) \setminus \{v\}$ that are strongly adjacent to v. We say that b is *strongly complete* to A if b is strongly adjacent to every vertex of A, b is *strongly anticomplete* to A if b is strongly antiadjacent to every vertex of A, b is *complete* to A if b is adjacent to every vertex of A and b is *anticomplete* to A if b is antiadjacent to every vertex of A. For two disjoint subsets A, B of $V(G)$, B is *strongly complete (strongly anticomplete, complete, anticomplete)* to A if every vertex of B is strongly complete (strongly anticomplete, complete, anticomplete, respectively) to every vertex of A. We say that b is *mixed* on A if b is not strongly complete and not strongly anticomplete to A. A *clique* in G is a set of vertices all pairwise adjacent, and a *strong clique* is a set of vertices all pairwise strongly adjacent. A *stable set* is a set of vertices all pairwise antiadjacent, and a *strongly stable set* is a set of vertices all pairwise strongly antiadjacent. A (strong) clique of size three is a *(strong) triangle* and a (strong) stable set of size three is a *(strong) triad*. For $X \subset V(G)$ the trigraph *induced by G on X* (denoted by $G|X$) has vertex set X, and adjacency function that is the restriction of θ to X^2. Isomorphism between trigraphs is defined in the natural way, and for two trigraphs G and H we say that H is an *induced subtrigraph* of G (or G *contains H as an induced subtrigraph*) if H is isomorphic to $G|X$ for some $X \subseteq V(G)$. We denote by $G \setminus X$ the trigraph $G|(V(G) \setminus X)$.

A *bull* is a trigraph with vertex set $\{x_1, x_2, x_3, v_1, v_2\}$ such that $\{x_1, x_2, x_3\}$ is a triangle, v_1 is adjacent to x_1 and antiadjacent to x_2, x_3, v_2, and v_2 is adjacent to x_2 and antiadjacent to x_1, x_3. For a trigraph G, a subset X of $V(G)$ is said to be a *bull* if $G|X$ is a bull. We say that a trigraph is *bull-free* if no induced subtrigraph of it is a bull, or, equivalently, no subset of its vertex set is a bull.

Let G be a trigraph. An induced subtrigraph P of G with vertices $\{p_1, \ldots, p_k\}$ is a *path* in G if either $k = 1$, or for $i, j \in \{1, \ldots, k\}$, p_i is adjacent to p_j if $|i - j| = 1$ and p_i is antiadjacent to p_j if $|i - j| > 1$. Under these circumstances we say that P is a path *from p_1 to p_k*, its *interior* is the set $P^* = V(P) \setminus \{p_1, p_k\}$, and the *length* of P is $k - 1$. We also denote P by p_1-$\ldots$-p_k, and say that P is a $(k-1)$-*edge path*. An induced subtrigraph H of G with vertices $h_1, \ldots, h_k$ is a *hole* if $k \geq 4$, and for $i, j \in \{1, \ldots, k\}$, h_i

is adjacent to h_j if $|i-j|=1$ or $|i-j|=k-1$; and h_i is antiadjacent to h_j if $1<|i-j|<k-1$. The *length* of a hole is the number of vertices in it. Sometimes we denote H by h_1-…-h_k-h_1. An *antipath* (*antihole*) is a path (hole) in $\overline{G}$.

Let G be a trigraph, and let $X \subseteq V(G)$. Let G_c be the graph with vertex set X, and such that two vertices of X are adjacent in G_c if and only if they are adjacent in G, and let G_a be be the graph with vertex set X, and such that two vertices of X are adjacent in G_a if and only if they are strongly adjacent in G. We say that X (and $G|X$) is *connected* if the graph G_c is connected, and that X (and $G|X$) is *anticonnected* if $\overline{G_a}$ is connected. A *connected component* of X is a maximal connected subset of X, and an *anticonnected component* of X is a maximal anticonnected subset of X. For a trigraph G, if X is a component of $V(G)$, then $G|X$ is a component of G.

We finish this section by two easy observations (these appeared in [1, 2], but we repeat them for completeness, and omit the proofs).

2.1 *If G be a bull-free trigraph, then so is $\overline{G}$.*

2.2 *Let G be a trigraph, let $X \subseteq V(G)$ and $v \in V(G) \setminus X$. Assume that $|X|>1$ and v is mixed on X. Then there exist vertices $x_1, x_2 \in X$ such that v is adjacent to x_1 and antiadjacent to x_2. Moreover, if X is connected, x_1 and x_2 can be chosen adjacent.*

3 The decomposition theorem for trigraphs

In this section we state a decomposition theorem for bull-free trigraphs. We start by describing a special type of trigraphs.

1-thin trigraphs. Let G be a trigraph. Let $a, b \in V(G)$ be distinct vertices, and let $A=\{a_1,\dots,a_n\}$ and $B=\{b_1,\dots,b_m\}$ be disjoint subsets of $V(G)$ such that $A \cup B = V(G) \setminus \{a,b\}$. Let us now describe the adjacency in G.

- a is strongly complete to A and strongly anticomplete to B.
- b is strongly complete to B and strongly anticomplete to A.
- a is semi-adjacent to b.
- If $i,j \in \{1,\dots,n\}$, and $i<j$, and a_i is adjacent to a_j, then a_i is strongly complete to $\{a_{i+1},\dots,a_{j-1}\}$, and a_j is strongly complete to $\{a_1,\dots,a_{i-1}\}$.
- If $i,j \in \{1,\dots,m\}$, and $i<j$, and b_i is adjacent to b_j, then b_i is strongly complete to $\{b_{i+1},\dots,b_{j-1}\}$, and b_j is strongly complete to $\{b_1,\dots,b_{i-1}\}$.

- If $p \in \{1, \dots, n\}$ and $q \in \{1, \dots, m\}$, and a_p is adjacent to b_q, then a_p is strongly complete to $\{b_{q+1}, \dots, b_m\}$, and b_q is strongly complete to $\{a_{p+1}, \dots, a_n\}$.

Under these circumstances we say that G is 1-*thin.* We call the pair (a, b) the *base* of G.

3.1 *Every* 1*-thin trigraph is bull-free.*

Proof. Let G be a 1-thin trigraph, and let a, b, A, B be as in the definition of a 1-thin trigraph. Let $|A| = n$ and $|B| = m$. Suppose there is a bull C in G. Let $C = \{c_1, c_2, c_3, c_4, c_5\}$, where the pairs $c_1c_2, c_2c_3, c_2c_4, c_3c_4, c_4c_5$ are adjacent, and all the remaining pairs are antiadjacent.

(1) There do not exist $a, a' \in A$ and $b, b' \in B$ such that the pairs $ab, a'b'$ are adjacent, and the pairs $ab', a'b$ are antiadjacent.

Suppose such a, a', b, b' exist. We may assume $a = a_i$, $a' = a_j$, and $i < j$. But then, since b is adjacent to a_i, it follows that b is strongly adjacent to a_j, a contradiction. This proves (1).

(2) Let $i, j, k \in \{1, \dots, n\}$ such that a_i is adjacent to a_j, and a_k is anticomplete to $\{a_i, a_j\}$. Then $k > i$ and $k > j$.

We may assume that $i > j$. If $k < j$, then a_i is strongly adjacent to a_k, and if $j < k < i$, then a_j is strongly adjacent to a_k, in both cases a contradiction. This proves (2).

(3) Let $i, j, k \in \{1, \dots, n\}$ such that a_i is adjacent to a_j and to a_k, and a_j is antiadjacent to a_k. Then $i < j$ and $i < k$.

From the symmetry we may assume that $j < k$. If $i > k$, then, since a_i is adjacent to a_j, it follows that a_j is strongly adjacent to a_k, a contradiction. If $j < i < k$, then, since a_k is adjacent to a_i, it follows, again, that a_j is strongly adjacent to a_k. This proves (3).

(4) $a \notin C$.

Suppose $a \in C$. Assume first that $a = c_3$. Since b is strongly antiadjacent to every other neighbor of a and strongly adjacent to every other anti-neighbor of a, it follows that $b \notin C$. Since c_2, c_4 are both adjacent to c_3, it follows that $c_2, c_4 \in A$; and since c_1, c_5 are both antiadjacent to c_3, it follows that $c_1, c_5 \in B$. But the pairs c_2c_1, c_4c_5 are adjacent, and the pairs c_2c_5, c_4c_1 are antiadjacent, contrary to (1). This proves that $a \neq c_3$. Next suppose that $a = c_2$. Since b is strongly antiadjacent to every other neighbor of a, it follows that $b \notin \{c_3, c_4\}$. Thus $c_3, c_4 \in A$, and since c_5 is antiadjacent to c_2, it follows that $c_5 \in B$. Since c_1 is antiadjacent to c_5, it follows that $c_1 \in A$. Let $a_i = c_4, a_j = c_3, a_k = c_1$. By (2), $k > i$. But now c_1 is strongly adjacent to c_5, a contradiction. This proves that $a \neq c_2$, and, from the symmetry, $a \neq c_4$. Now, using symmetry, we may assume that $a = c_1$. Then $c_3, c_4, c_5 \in B \cup \{b\}$. It follows from the symmetry between a and b that $b \neq c_3, c_4$; consequently $c_3, c_4 \in B$. This implies that $b \neq c_5$, and so $c_5 \in B$. Since c_2 is antiadjacent to c_5, it follows that $b \neq c_2$, and so $c_2 \in A$. Let $b_i = c_3$, $b_j = c_4$ and $b_k = c_5$. By (3), $j < i$ and $j < k$. But now, since c_2 is adjacent to c_4, it follows that c_2 is strongly adjacent to c_5, a contradiction. This proves (4).

(5) Not both c_2 and c_4 are in A.

Suppose that that both $c_2, c_4 \in A$. Let $i, j \in \{1, \dots n\}$ such that $a_i = c_2$ and $a_j = c_4$. We may assume that $i < j$. Since c_1 is adjacent to c_2 and antiadjacent to c_4, it follows that $c_1 \in A$. Let $a_k = c_1$. Then, by (3), $i < k$ and $i < j$. It follows that if $c_3 \in B$, then c_3 is strongly adjacent to c_1, and therefore $c_3 \notin B$. By (4) and the symmetry, $c_3 \neq a, b$, and so $c_3 \in A$. Since c_1 is strongly anticomplete to $\{c_3, c_4\}$, (2) implies that that $k > j$. Since c_5 is adjacent to c_4 and antiadjacent to c_1, it follows that $c_1 \notin B$, and so, by (4) and the symmetry, we deduce that $c_5 \in A$. Let $c_5 = a_s$. Since c_5 is anticomplete to $\{c_1, c_2\}$, it follows from (2) that $s > k$. But now, since c_5 is adjacent to c_4, and since $i < j < k < s$, it follows from (3) that c_5 is strongly adjacent to c_2, a contradiction. This proves (5).

By (4), (5) and the symmetry, we may assume that $c_2, c_3 \in A$, and $c_4 \in B$. Let $a_i = c_2$, $a_j = c_3$ and $b_k = c_4$. Suppose that $c_1 \in A$, say $c_1 = a_s$. By (3), it follows that $i < s$. But then, since c_4 is adjacent to c_2, it follows that c_4 is strongly adjacent to c_1, a contradiction. This proves that $c_1 \in B$, say $c_1 = b_s$. Since c_1 is adjacent to c_2 and antiadjacent to c_3, it follows that $j < i$. Since c_3 is adjacent to c_4 and antiadjacent to c_1, it follows that $k > s$. Now, since c_1 is anticomplete to $\{c_4, c_5\}$, (2) implies that $c_5 \notin B$. By (4) and the symmetry, it follows that $c_5 \in A$, say $c_5 = a_t$. By (3), $t > i$. But c_1 is adjacent to c_2, and antiadjacent to c_5, a contradiction. This proves 3.1. ∎

2-thin trigraphs. Let G be a trigraph. Let $x_{AK}, x_{AM}, x_{BK}, x_{BM}$ be pairwise distinct vertices of G, and let A, B, K, M be pairwise disjoint subsets of $V(G)$, such that K, M are strong cliques, A, B are strongly stable sets and

$$A \cup B \cup K \cup M \cup \{x_{AK}, x_{AM}, x_{BK}, x_{BM}\} = V(G).$$

Let $t, s \geq 0$ be integers and let $K = \{k_1, \dots, k_t\}$ and $M = \{m_1, \dots, m_s\}$ (so if $t = 0$ then $K = \emptyset$, and if $s = 0$ then $M = \emptyset$). Let A be the disjoint union of sets $A_{i,j}$, and B the disjoint union of sets $B_{i,j}$, where $i \in \{0, \dots, t\}$ and $j \in \{0, \dots, s\}$.

Assume that :

- A is strongly complete to B
- K is strongly anticomplete to M
- A is strongly complete to $\{x_{AK}, x_{AM}\}$ and strongly anticomplete to $\{x_{BK}, x_{BM}\}$
- B is strongly complete to $\{x_{BK}, x_{BM}\}$ and strongly anticomplete to $\{x_{AK}, x_{AM}\}$
- K is strongly complete to $\{x_{AK}, x_{BK}\}$ and strongly anticomplete to $\{x_{AM}, x_{BM}\}$
- M is strongly complete to $\{x_{AM}, x_{BM}\}$ and strongly anticomplete to $\{x_{AK}, x_{BK}\}$
- x_{AK} is semi-adjacent to x_{BM}
- x_{AM} is semi-adjacent to x_{BK}
- the pairs $x_{AK}x_{BK}$ and $x_{AM}x_{BM}$ are strongly adjacent, and the pairs $x_{AK}x_{AM}$ and $x_{BK}x_{BM}$ are strongly antiadjacent.

Let $i \in \{0, \dots, t\}$ and $j \in \{0, \dots, s\}$. Then

- if $i' \in \{0, \dots, t\}$ and $j' \in \{0, \dots, s\}$ such that $i > i'$ and $j > j'$, then at least one of the sets $A_{i,j}$, $A_{i',j'}$ is empty, and at least one of the sets $B_{i,j}$, $B_{i',j'}$ is empty.
- $A_{i,j}$ is strongly complete to $\{k_1, \dots, k_{i-1}\} \cup \{m_{s-j+2}, \dots, m_s\}$,
  $A_{i,j}$ is complete to $\{k_i, m_{s-j+1}\}$,
  $A_{i,j}$ is strongly anticomplete to $\{k_{i+1}, \dots, k_t\} \cup \{m_1, \dots, m_{s-j}\}$,
- $B_{i,j}$ is strongly complete to $\{k_{t-i+2}, \dots, k_t\} \cup \{m_1, \dots, m_{j-1}\}$,
  $B_{i,j}$ is complete to $\{k_{t-i+1}, m_j\}$,
  $B_{i,j}$ is strongly anticomplete to $\{k_1, \dots, k_{t-i}\} \cup \{m_{j+1}, \dots, m_s\}$.

Then G is 2-*thin with base* $(x_{AK}, x_{BM}, x_{BK}, x_{AM})$. We call (A, B, K, M) the *partition of* G *with respect to the base* $(x_{AK}, x_{BM}, x_{BK}, x_{AM})$.

3.2 *Every* 2*-thin trigraph is bull-free.*

Proof. Let G be 2-thin. We observe that G is 1-thin with base (x_{AK}, x_{BM}), and the result follows from 3.1. This proves 3.2. ∎

We will need the following three classes of trigraphs in order to state our main theorem. The class $\mathcal{T}_0$ was defined in [1], and the class $\mathcal{T}_1$ was defined in [2].

The class $\mathcal{T}_2$. Let G be a bull-free trigraph and let (A, B) be a homogeneous pair in G. Let C be the set of vertices of G that are strongly complete to A and strongly anticomplete to B, and let D be the set of vertices of G that are strongly complete to B and strongly anticomplete to A. We say that (A, B) is *doubly dominating* if $V(G) = A \cup B \cup C \cup D$, and both C and D are non-empty. For a semi-adjacent pair a_0b_0, we say that a_0b_0 is *doubly dominating* if the pair $(\{a_0\}, \{b_0\})$ is doubly dominating.

Let G_1, G_2 be bull-free trigraphs, and for $i = 1, 2$ let (a_i, b_i) be a doubly dominating semi-adjacent pair in G_i, let A_i be the set of vertices of G_i that are strongly complete to a_i, and let B_i be the set of vertices of G_i that are strongly complete to b_i. We say that G is obtained from G_1 and G_2 by *composing along* (a_1, b_1, a_2, b_2) if $V(G) = A_1 \cup A_2 \cup B_1 \cup B_2$, for $i = 1, 2$ $G|(A_i \cup B_i) = G_i|(A_i \cup B_i)$, A_1 is strongly complete to A_2 and strongly anticomplete to B_2, and B_1 is strongly complete to B_2 and strongly anticomplete to A_2. We observe that if $(x, y) \neq (a_i, b_i)$ is a doubly dominating semi-adjacent pair in G_i, then (x, y) is a doubly dominating semi-adjacent pair in G; and these are all the doubly dominating semi-adjacent pairs in G.

Let H be either the the complete graph on two vertices, or the complete graph on three vertices, or the graph on three vertices with no edges. We say that a trigraph G is an *H-pattern* if the vertex set of G consist of two distinct copies a_v, b_v of every vertex v of H, and such that

- for every $v \in V(H)$, a_v is semi-adjacent to b_v, and
- if $u, v \in V(H)$ are adjacent, then a_u is strongly adjacent to a_v and strongly antiadjacent to b_v, and b_u is strongly adjacent to b_v and strongly antiadjacent to a_v, and
- if $u, v \in V(H)$ are non-adjacent, then a_u is strongly adjacent to b_v and strongly antiadjacent to a_v, and b_u is strongly adjacent to a_v and strongly antiadjacent to b_v.

Thus for every $v \in V(H)$, (a_v, b_v) is a doubly dominating semi-adjacent pair in G, and there are no other semi-adjacent pairs in G. We say that G is a *triangle pattern* if H is the complete graph on three vertices, an *edge pattern* if H is the complete graph on two vertices, and a *triad pattern* if H is the graph on three vertices with no edges. We remark that edge patterns are 2-thin graphs, however, it is convenient to have a special name for them.

Let $k \geq 1$ be an integer, and let $G'_1, \ldots, G'_k$ be trigraphs, such that for $i \in \{1, \ldots, k\}$, G'_k is either a triangle pattern, or a triad pattern, or a 2-thin trigraph (possibly an edge pattern). For $i \in \{2, \ldots, k\}$, let (c_i, d_i) be a doubly dominating semi-adjacent pair in G'_i. For $j \in \{1, \ldots, k-1\}$, let (x_j, y_j) be a doubly dominating semi-adjacent pair in G'_q for some $q \in \{1, \ldots, j\}$, and such that the pairs $\{c_2, d_2\}, \dots \{c_k, d_k\}, \{x_1, y_1\}, \dots, \{x_{k-1}, y_{k-1}\}$ are all distinct (and therefore pairwise disjoint).

Let $G_1 = G'_1$. Then (x_1, y_1) is a doubly dominating semi-adjacent pair in G_1. For $i \in \{1, \dots, k-1\}$, let G_{i+1} be the trigraph obtained by composing G_i and G'_{i+1} along $(x_i, y_i, c_{i+1}, d_{i+1})$. Let $G = G_k$. We call such a trigraph G a *skeleton*. Every skeleton is in $\mathcal{T}_2$.

We observe that a semi-adjacent pair $\{u, v\}$ is doubly dominating in G if and only if (u, v) is a doubly dominating semi-adjacent pair in some G'_i with $i \in \{1, \dots, k\}$, and $\{u, v\}$ is not one of

$$\{c_2, d_2\}, \dots, \{c_k, d_k\}, \{x_1, y_1\}, \dots, \{x_{k-1}, y_{k-1}\}.$$

Let G'_0 be a skeleton, and for $i \in \{1, \dots, n\}$ let (a_i, b_i) be a doubly dominating semi-adjacent pair in G'_0, such that the pairs $\{a_1, b_1\}, \dots, \{a_n, b_n\}$ are all distinct (and therefore pairwise disjoint). For $i = \{1, \dots, n\}$, let G'_i be a trigraph such that

- $V(G'_i) = A_i \cup B_i \cup \{a'_i, b'_i\}$, and
- the sets $A_i, B_i, \{a'_i, b'_i\}$ are all non-empty and pairwise disjoint, and
- a'_i is strongly complete to A_i and strongly anticomplete to B_i, and
- b'_i is strongly complete to B_i and strongly anticomplete to A_i, and
- a'_i is semi-adjacent to b'_i, and either
  - both A_i, B_i are strong cliques, and there do not exist $a \in A_i$ and $b \in B_i$, such that a is strongly anticomplete to $B_i \setminus \{b\}$, b is strongly anticomplete to $A_i \setminus \{a\}$, and a is semi-adjacent to b, or
  - both A_i, B_i are strongly stable sets, and there do not exist $a \in A_i$ and $b \in B_i$, such that a is strongly complete to $B_i \setminus \{b\}$, b is strongly complete to $A_i \setminus \{a\}$, and a is semi-adjacent to b, or
  - one of $G'_i, \overline{G'_i}$ is a 1-thin trigraph with base (a'_i, b'_i), and G'_i is not a 2-thin trigraph.

Let $G_0 = G'_0$, and for $i \in \{1, \dots, n\}$, let G_i be obtained by composing G_{i-1} and G'_i along (a_i, b_i, a'_i, b'_i). Let $G = G_n$. Then $G \in \mathcal{T}_2$.

The following two results were proved in [1] and [2] respectively:

3.3 *Every trigraph in $\mathcal{T}_0$ is bull-free.*

3.4 *Every trigraph in $\mathcal{T}_1$ is bull-free.*

Here we prove that:

3.5 *Every trigraph in $\mathcal{T}_2$ is bull-free.*

Proof. We start with the following observations:

(1) Let H_1, H_2 be bull-free trigraphs with $V(H_1) \cap V(H_2) = \emptyset$, and for $i = 1, 2$ let (a_i, b_i) be a doubly dominating semi-adjacent pair in H_i. Let H be the trigraph obtained by composing H_1 and H_2 along (a_1, b_1, a_2, b_2). Then H is bull-free.

For $i = 1, 2$ let A_i be the set of neighbors of a_i in $V(H_i) \setminus \{a_i, b_i\}$, and let B_i be the set of neighbors of b_i in $V(H_i) \setminus \{a_i, b_i\}$. Then $A_1 \cap B_1 = A_2 \cap B_2 = \emptyset$ and $V(H_i) = A_i \cup B_i \cup \{a_i, b_i\}$. Suppose there is a bull B in G. Let $B = \{v_1, v_2, v_3, v_4, v_5\}$, where the pairs $v_1v_2, v_2v_3, v_2v_4, v_3v_4, v_4v_5$ are adjacent, and all the remaining pairs are antiadjacent. From the symmetry, we may assume that $\{v_2, v_3, v_4\} \cap A_1 \neq \emptyset$. Then either $\{v_2, v_3, v_4\} \subseteq A_1 \cup B_1$, or $\{v_2, v_3, v_4\} \subseteq A_1 \cup A_2$. Suppose first that $\{v_2, v_3, v_4\} \subseteq A_1 \cup B_1$. Since each of v_2, v_3, v_4 has at most one neighbor in $\{v_1, v_5\}$, it follows that $|B \cap A_2| \leq 1$, and $|B \cap B_2| \leq 1$, and so $(B \setminus (A_2 \cup B_2)) \cup \{a_1, b_1\}$ contains a bull. But $(B \setminus (A_2 \cup B_2)) \cup \{a_1, b_1\} \subseteq V(H_1)$, contrary to the fact that H_1 is bull-free. This proves that $\{v_2, v_3, v_4\} \subseteq A_1 \cup A_2$. We may assume from the symmetry that $|\{v_2, v_3, v_4\} \cap A_1| > 1$. This implies that $A_2 \cap B \subseteq \{v_2, v_3, v_4\}$, and therefore $|A_2 \cap B| \leq 1$. In turn, this implies that $|B \cap B_2| \leq 1$, and so $(B \setminus (A_2 \cup B_2)) \cup \{a_1, b_1\}$ contains a bull, contrary to the fact that $(B \setminus (A_2 \cup B_2)) \cup \{a_1, b_1\} \subseteq V(H_1)$ and H_1 is bull-free. This proves (1).

(2) Let H be a trigraph with $V(H) = A \cup B \cup \{a, b\}$ such that a is strongly complete to A and strongly anticomplete to B, b is strongly complete to B and strongly anticomplete to A, and a is semi-adjacent to b; and either both A and B are strong cliques, or both A and B are strongly stable sets. Then H is bull-free.

Since either there is no triangle in H, or there is no triad in H, (2) follows.

Now we observe that if $G \in \mathcal{T}_2$, then G is obtained by repeatedly composing pairs of trigraphs with a dominating semi-adjacent pair. By 3.1, 3.2, (2), and since triangle patterns and triad patterns are bull-free, it follows that all trigraphs used to build G are bull-free, and thus, by (1), G is bull-free. This proves 3.5. ▮

We observe the following:

3.6 *$\overline{G} \in \mathcal{T}_2$ for every trigraph $G \in \mathcal{T}_2$.*

The proof of 3.6 is easy and we omit it. Next let us describe some decompositions. Let G be a trigraph. We say that G admits a 1-*join*, if $V(G)$ is the disjoint union of four non-empty sets A, B, C, D such that

- B is strongly complete to C, A is strongly anticomplete to $C \cup D$, and B is strongly anticomplete to D;
- $|A \cup B| > 2$ and $|C \cup D| > 2$, and
- A is not strongly complete and not strongly anticomplete to B, and
- C is not strongly complete and not strongly anticomplete to D.

A proper subset $X$of $V(G)$ is a *homogeneous set* in G if every vertex of $V(G) \setminus X$ is either strongly complete or strongly anticomplete to X. We say that G admits *a homogeneous set decomposition*, if there is a homogeneous set in G of size at least two.

For two disjoint subsets A and B of $V(G)$, the pair (A, B) is a *homogeneous pair* in G, if A is a homogeneous set in $G \setminus B$ and B is a homogeneous set in $G \setminus A$. We say that the pair (A, B) is *tame* if

- $|V(G)| - 2 > |A| + |B| > 2$, and
- A is not strongly complete and not strongly anticomplete to B

A trigraph G admits a *homogeneous pair decomposition* if there is a tame homogeneous pair in G.

We need three special kinds of homogeneous pairs. Let (A, B) be a homogeneous pair in G. Let C be the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, D the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to B and strongly anticomplete to A, E the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to $A \cup B$, and F the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly anticomplete to $A \cup B$. We say that (A, B) is *a homogeneous pair of type zero in G* (this was defined in [1], but we repeat the definition here) if

- $D = \emptyset$, and
- some member of C is antiadjacent to some member of E, and
- A is a strongly stable set, and
- $|C \cup E \cup F| > 2$, and
- $|B| = 2$, say $B = \{b_1, b_2\}$, and b_1 is strongly adjacent to b_2, and
- let $\{i, j\} = \{1, 2\}$. Let A_i be the set of vertices of A that are adjacent to b_i. Then $A_1 \cap A_2 = \emptyset$, $A_1 \cup A_2 = A$, $1 \leq |A_i| \leq 2$, and if $|A_i| = 2$, then one of the vertices of A_i is semi-adjacent to b_i, and
- if $|A_1| = |A_2| = 1$, then F is non-empty.

We say that (A, B) is a *homogeneous pair of type one in G* if

- at least one member of C is adjacent to at least one member of F, and
- at least one member of D is adjacent to at least one member of F, and
- $E = \emptyset$, and
- $|A| + |B| > 2$, and A is not strongly complete and not strongly anticomplete to B, and
- both A and B are strongly stable sets.

A trigraph T is a forest of there are no holes and no triangles in T. Thus, for every two vertices of T, there is at most one path between them. A forest T is a *tree* if T is connected. A *rooted forest* is a $(k+1)$-tuple $(T, r_1, \dots, r_k)$, where T is a forest with components $T_1, \dots, T_k$, and $r_i \in V(T_i)$ for $i \in \{1, \dots, k\}$. Let $u, v \in V(F)$ be distinct. We say that u is a *child* of v, if for some $i \in \{1, \dots, k\}$, both $u, v \in V(T_i)$, and u is adjacent to v, and if P is the unique path of T_i from r_i to u, then $v \in V(P)$. We say that u is a *descendant* of v if for some $i \in \{1, \dots, k\}$, both $u, v \in V(T_i)$, and if P is the unique path of T_i from r_i to u, then $v \in V(P)$.

Let $(T, r_1, \dots, r_k)$ be a rooted forest. We say that the trigraph T' is *the closure* of $(T, r_1, \dots, r_k)$, if $V(T') = V(T)$, $\sigma(T) = \sigma(T')$, and u is adjacent to v in T' if and only if one of u, v is a descendant of the other.

Finally, we say that (A, B) is a *homogeneous pair of type two in G* if

- at least one member of C is adjacent to at least one member of F, and
- $D \neq \emptyset$, and
- D strongly anticomplete to F, and
- $E = \emptyset$, and
- $|A| + |B| > 2$, and A is not strongly complete and not strongly anticomplete to B, and
- A is strongly stable, and
- there exists a rooted forest $(T, r_1, \dots, r_k)$ such that $G|B$ is the closure of $(T, r_1, \dots, r_k)$, and
- if $b, b' \in B$ are semi-adjacent, then, possibly with the roles of b and b' exchanged, b is a leaf of T and a child of b', and
- if $a \in A$ is adjacent to $b \in B$, then a is strongly adjacent to every descendant of b in T, and
- let $u, v \in B$ and assume that u is a child of v. Let $i \in \{1, \dots, k\}$ and let T_i be the component of T such that $u, v \in V(T_i)$. Let P be the unique path of T_i from v to r_i, and let X be the component of $T_i \setminus (V(P) \setminus \{v\})$

containing v (and therefore u). Let Y be the set of vertices of X that are semi-adjacent to v. Let $a \in A$ be adjacent to u and antiadjacent to v. Then a is strongly complete Y and to $B \setminus (V(X) \cup V(P))$, and a is strongly anticomplete to $V(P) \setminus \{v\}$.

Please note that every homogeneous pair of type zero, one, or two is tame in both G and $\overline{G}$, and therefore if there is a homogeneous pair of type zero, one or two in either G or $\overline{G}$, then G admits a homogeneous pair decomposition.

Let G be a trigraph and let $S \subseteq V(G)$. A *center* for S is a vertex of $V(G) \setminus S$ that is complete to S, and an *anticenter* for S is a vertex of $V(G) \setminus S$ that is anticomplete to S. A vertex of G is a *center (anticenter)* for an induced subgraph H of G if it is a center (anticenter) for $V(H)$.

We say that a trigraph G is *elementary* if there does not exist a path P of length three in G, such that some vertex c of G is a center for P, and some vertex a of G is an anticenter for P.

The following two theorems are the main results of [1] and [2], respectively:

3.7 *Let G be a bull-free trigraph that is not elementary. Then either*

- *one of $G, \overline{G}$ belongs to $\mathcal{T}_0$, or*
- *one of $G, \overline{G}$ contains a homogeneous pair of type zero,or*
- *G admits a homogeneous set decomposition.*

3.8 *Let G be an elementary bull-free trigraph. Then either*

- *one of $G, \overline{G}$ belongs to $\mathcal{T}_1$, or*
- *G admits a homogeneous set decomposition, or*
- *G admits a homogeneous pair decomposition.*

Our first goal in this paper is to strengthen 3.8 to obtain the following:

3.9 *Let G be an elementary bull-free trigraph. Then either*

- *one of $G, \overline{G}$ belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$, or*
- *one of $G, \overline{G}$ contains a homogeneous pair of type one or two, or*
- *G admits a homogeneous set decomposition.*

Then we use 3.7 and 3.9 to prove our main theorem, which we state in the next section.

4 The main theorem

Let G be a bull-free trigraph, and let $a, b \in V(G)$ be semi-adjacent. Let C be the set of vertices of $V(G) \setminus \{a, b\}$ that are strongly adjacent to a and strongly antiadjacent to b, D the set of vertices of $V(G) \setminus \{a, b\}$ that are strongly adjacent to b and strongly antiadjacent to a, E the set of vertices of $V(G) \setminus \{a, b\}$ that are strongly complete to $\{a, b\}$, and F the set of vertices of $V(G) \setminus \{a, b\}$ that are strongly anticomplete to $\{a, b\}$. Then $V(G) = \{a, b\} \cup C \cup D \cup E \cup F$. We say that ab is a semi-adjacent pair of *type zero* if

- $D = \emptyset$, and
- some member of C is antiadjacent to some member of E, and
- $|C \cup E \cup F| > 2$.

We say that ab is a semi-adjacent pair of *type one* if

- at least one member of C is adjacent to at least one member of F, and
- at least one member of D is adjacent to at least one member of F, and
- $E = \emptyset$.

Finally, we say that ab is a semi-adjacent pair of *type two* if

- at least one member of C is adjacent to at least one member of F, and
- $D \neq \emptyset$, and
- D strongly anticomplete to F, and
- $E = \emptyset$.

We say that ab is of *complement type zero, one* or *two* if ab is of type zero, one or two in $\overline{G}$, respectively. We remark that the type of a semi-adjacent pair is well defined with one exception—a pair ab may be of both type zero, and complement type zero. Also, not every semi-adjacent pair in a bull-free trigraph needs to be of one of the types above, but it turns out that these are the only types of homogeneous pairs that are needed to describe the structure of bull-free trigraphs.

We say that H is an *elementary expansion* of G if for every vertex v of G there exists a non-empty subset X_v of $V(H)$, all pairwise disjoint and with union $V(H)$, such that

- for $u, v \in V(G)$, if u is strongly adjacent to v, then X_u is strongly complete to X_v, and if u is strongly antiadjacent to v, then X_u is strongly anticomplete to X_v,
- if $v \in V(G)$ does not belong to any semi-adjacent pair of type 1 or 2 or of complement type 1 or 2, then $|X_v| = 1$

- if u is semi-adjacent to v, and neither of uv, vu is a semi-adjacent pair of type 1 or 2 or of complement type 1 or 2, then the unique vertex of X_u is semi-adjacent to the unique vertex of X_v

- if uv is a semi-adjacent pair of type 1 or 2 in G, then either $|X_v| = |X_u| = 1$ and the unique vertex of X_u is semi-adjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type 1 or 2, respectively, in H

- if uv is a semi-adjacent pair of complement type 1 or 2 in G, then either $|X_v| = |X_u| = 1$ and the unique vertex of X_u is semi-adjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type 1 or 2, respectively, in $\overline{H}$.

We say that H is a *non-elementary expansion* of G if for every vertex v of G there exists a non-empty subset X_v of $V(H)$, all pairwise disjoint and with union $V(H)$, such that

- for $u, v \in V(G)$, if u is strongly adjacent to v, then X_u is strongly complete to X_v, and if u is strongly antiadjacent to v, then X_u is strongly anticomplete to X_v,

- if $v \in V(G)$ does not belong to any semi-adjacent pair of type 0 or of complement type 0, then $|X_v| = 1$

- if u is semi-adjacent to v, and neither of uv, vu is a semi-adjacent pair of type 0 or of complement type 0, then the unique vertex of X_u is semi-adjacent to the unique vertex of X_v

- if uv is a semi-adjacent pair that is both of type 0 and of complement type zero, then either $|X_v| = |X_u| = 1$ and the unique vertex of X_u is semi-adjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type 0 either in H or in $\overline{H}$

- if uv is a semi-adjacent pair of type 0 in G and not in $\overline{G}$, then either $|X_v| = |X_u| = 1$ and the unique vertex of X_u is semi-adjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type 0 in H

- if uv is a semi-adjacent pair of type 0 in $\overline{G}$ and not in G, then either $|X_v| = |X_u| = 1$ and the unique vertex of X_u is semi-adjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type 0 in $\overline{H}$.

We leave it to the reader to verify that an elementary expansion of an elementary bull-free trigraph is another elementary bull-free trigraph, and that a non-elementary expansion of a bull-free trigraph is another bull-free trigraph.

Before we can state our main theorem, we need to define an operation. Let G_1, G_2 be bull-free trigraphs with disjoint vertex sets. We say that G is obtained from G_1, G_2 by *substitution* if

- there exist a vertex $v \in V(G_1)$ such that no vertex of $V(G_1) \setminus \{v\}$ is semi-adjacent to v, and
- $V(G) = (V(G_1) \cup V(G_2)) \setminus \{v\}$, and
- $G|(V(G_1) \setminus \{v\}) = G_1 \setminus \{v\}$, and
- $G|V(G_2) = G_2$, and
- for $x \in V(G_1)$ and $y \in V(G_2)$, x is strongly adjacent to y if x is strongly adjacent to v, and x is strongly antiadjacent to y otherwise.

It is easy to check that a trigraph obtained from two bull-free trigraphs by substitution is another bull-free trigraph.

We can now describe the structure of all bull-free trigraphs (and therefore of all bull-free graphs). First let us state a theorem that describes the structure of elementary bull-free trigraphs that are not obtained from smaller bull-free trigraphs by substitutions.

4.1 *Let G be an elementary bull-free trigraph that is not obtained from smaller bull-free trigraphs by substitution. Then one of G, $\overline{G}$ is an elementary expansion of a member of $\mathcal{T}_1 \cup \mathcal{T}_2$; and every elementary expansion of a trigraph H such that either H or $\overline{H}$ is member of $\mathcal{T}_1 \cup \mathcal{T}_2$ is elementary.*

Finally, we describe the structure of all bull-free trigraphs.

4.2 *Let G be a bull-free trigraph. Then either*

- *G is obtained by substitution from smaller bull-free trigraphs, or*
- *G is a non-elementary expansion of an elementary bull-free trigraph, or*
- *one of $G, \overline{G}$ belongs to $\mathcal{T}_0$, or*
- *one of G, $\overline{G}$ is an elementary expansion of a member of $\mathcal{T}_1 \cup \mathcal{T}_2$,*

and every trigraph obtained this way is bull-free.

In the remainder of this section we prove 4.1 and 4.2 assuming 3.7 and 3.9, and using a few lemmas from Section 7.

Proof of 4.1 assuming 3.9. The proof that an elementary expansion of a trigraph in $\mathcal{T}_1 \cup \mathcal{T}_2$ is elementary consists of routine case checking, and we leave it to the reader. Let G be an elementary bull-free trigraph that is not obtained from smaller bull-free trigraphs by substitution. The proof is by induction on $|V(G)|$. By 3.9 either

- one of $G, \overline{G}$ belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$, or
- one of $G, \overline{G}$ contains a homogeneous pair of type one or two, or
- G admits a homogeneous set decomposition.

We may assume that neither of $G, \overline{G}$ belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$, for then 4.1 holds. If G admits a homogeneous set decomposition, then G is obtained from smaller bull-free trigraphs by substitution, a contradiction. Consequently, there exists a homogeneous pair (A, B) in G, such that (A, B) is of type 1 or 2 in one of $G, \overline{G}$. Since the conclusion of 4.1 is invariant under taking complements, we may assume that (A, B) is a homogeneous pair of type 1 or 2 in G. Let C be the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, D the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to B and strongly anticomplete to A, E the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to $A \cup B$, and F the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly anticomplete to $A \cup B$. Let G' be the trigraph obtained from $G|(C \cup D \cup E \cup F)$ by adding two new vertices a and b, such that a is strongly complete to $C \cup E$ and strongly anticomplete to $D \cup F$, b is strongly complete to $D \cup E$ and strongly anticomplete to $C \cup F$, and a is semi-adjacent to b. We observe that for $i = 1, 2$, if (A, B) is a homogeneous pair of type i in G, then ab is a semi-adjacent pair of type i in G'. Since $|V(G')| < |V(G)|$, it follows from the inductive hypothesis, that either G' is obtained by substitution from smaller bull-free trigraphs, or one of G', $\overline{G'}$ is an elementary expansion of a member of $\mathcal{T}_1 \cup \mathcal{T}_2$. It is easy to check that if G' is obtained by substitution from smaller elementary trigraphs then so is G, and so we may assume that one of G', $\overline{G'}$ is an elementary expansion of a member of $\mathcal{T}_1 \cup \mathcal{T}_2$. We observe that if G' is an elementary expansion of a trigraph K, then $\overline{G'}$ is an elementary expansion of $\overline{K}$. Thus there exists a trigraph K such that one of $K, \overline{K}$ belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$, and for every vertex v of K there exists a non-empty subset X_v of $V(G')$, all pairwise disjoint and with union $V(G')$, such that

- for $u, v \in V(K)$, if u is strongly adjacent to v, then X_u is strongly complete to X_v, and if u is strongly antiadjacent to v, then X_u is strongly anticomplete to X_v,
- if $v \in V(K)$ does not belong to any semi-adjacent pair of type 1 or 2 or of complement type 1 or 2, then $|X_v| = 1$
- if u is semi-adjacent to v, and neither of uv, vu is a semi-adjacent pair of type 1 or 2 or of complement type 1 or 2, then the unique vertex of X_u is semi-adjacent to the unique vertex of X_v
- if uv is a semi-adjacent pair of type 1 or 2 in K, then either $|X_v| = |X_u| = 1$ and the unique vertex of X_u is semi-adjacent to the unique

vertex of X_v, or (X_u, X_v) is a homogeneous pair of type 1 or 2, respectively, in G'

- if uv is a semi-adjacent pair of complement type 1 or 2 in K, then either $|X_v| = |X_u| = 1$ and the unique vertex of X_u is semi-adjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type 1 or 2, respectively, in $\overline{G'}$.

Suppose first that $a, b \in X_v$ for some $v \in V(K)$. Then, since $|X_v| > 2$, there exist $u \in V(K)$ such that uv is a semi-adjacent pair of type 1 or 2, or of complement type 1 or 2, and, consequently, some vertex $V(K) \setminus \{u, v\}$ is strongly adjacent to v. But then some vertex of $V(G')$ is strongly adjacent to both a and b, contrary to the fact that ab is a semi-adjacent pair of type 1 or 2 in G'. Thus there exist distinct $u, v \in V(K)$ such that $a \in X_u$ and $b \in X_v$. Since a is semi-adjacent to b, it follows that u is semi-adjacent to v in K.

We claim that uv is a semi-adjacent pair of type 1 or 2 in K. Since ab is of type 1 or 2 in G', it follows that no vertex of G' is adjacent to both a and b, and, consequently, no vertex of K is adjacent to both u and v, which implies that uv is not of complement type 1 or 2. Since uv is the only semi-adjacent pair of K involving u or v, if $|X_u| > 1$ or $|X_v| > 1$, then it follows from the definition of an elementary expansion that uv is of type 1 or 2 in K, and the claim holds. So we may assume that $X_u = \{a\}$ and $X_v = \{b\}$. But now uv has the same type in K as ab is in G', and therefore uv is of type 1 or 2 in K. This proves the claim.

Now, if uv is of type one in K, then 7.3 implies that $((X_u \setminus \{a\}) \cup A, (X_v \setminus \{b\}) \cup B)$ is a homogeneous pair of type one in G; and if uv is of type two in K, then 7.4 implies that $((X_u \setminus \{a\}) \cup A, (X_v \setminus \{b\}) \cup B)$ is a homogeneous pair of type two in G. In both cases, replacing X_u by $(X_u \setminus \{a\}) \cup A$ and X_v by $(X_v \setminus \{b\}) \cup B$, we observe that G is an elementary expansion of K. This proves 4.1. ∎

Proof of 4.2 assuming 3.7. By 3.3, 3.4 and 3.5, it follows that every trigraph in $\mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2$ is bull-free. We leave the rest of the proof the "only if" part of 4.2 to the reader.

For the "if" part, let G be a bull-free trigraph. The proof is by induction on $|V(G)|$. We may assume that G is not obtained from smaller trigraphs by substitutions. If G is elementary, then, by 4.1, one of $G, \overline{G}$ is an elementary expansion of a member of $\mathcal{T}_1 \cup \mathcal{T}_2$, and 4.2 holds. So we may assume that G is not elementary. So, by 3.7 either

- one of $G, \overline{G}$ belongs to $\mathcal{T}_0$, or
- one of $G, \overline{G}$ contains a homogeneous pair of type zero, or
- G admits a homogeneous set decomposition.

We may assume that neither of $G, \overline{G}$ belongs to $\mathcal{T}_0$, for then 4.2 holds. If G admits a homogeneous set decomposition, then G is obtained from smaller bull-free trigraphs by substitution, a contradiction. Consequently, there exists a homogeneous pair (A, B) in G, such that (A, B) is of type zero in one of $G, \overline{G}$. Since the conclusion of 4.2 is invariant under taking complements, we may assume that (A, B) is a homogeneous pair of type zero in G. Let C be the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, D the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to B and strongly anticomplete to A, E the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to $A \cup B$, and F the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly anticomplete to $A \cup B$. Since (A, B) is of type zero in G, it follows that $D = \emptyset$, and some vertex of C is antiadjacent to some vertex of E. Let G' be the trigraph obtained from $G|(C \cup D \cup E \cup F)$ by adding two new vertices a and b such that a is strongly complete to $C \cup E$ and strongly anticomplete to $D \cup F$, b is strongly complete to $D \cup E$ and strongly anticomplete to $C \cup F$, and a is semi-adjacent to b. Then ab is a semi-adjacent pair of type zero in G'. Since $|V(G')| < |V(G)|$, by the inductive hypothesis, one of the outcomes of 4.2 holds for G'. Therefore, either

- G' is obtained by substitution from smaller bull-free trigraphs, or
- one of $G', \overline{G'}$ is an elementary expansion of a member of $\mathcal{T}_1 \cup \mathcal{T}_2$, or
- one of $G', \overline{G'}$ belongs to $\mathcal{T}_0$, or
- G' is a non-elementary expansion of an elementary bull-free trigraph.

If G' is obtained by substitution from smaller bull-free trigraphs, then so is G, so we may assume not. If one of $G', \overline{G'}$ is an elementary expansion of a member of $\mathcal{T}_1 \cup \mathcal{T}_2$, then G' is an elementary trigraph, and so setting $X_v = \{v\}$, for $v \in V(G') \setminus \{a, b\}$, and setting $X_a = A$ and $X_b = B$, we observe that G is a non-elementary expansion of G'. So we may assume that neither of $G', \overline{G'}$ is an elementary expansion of a member of $\mathcal{T}_1 \cup \mathcal{T}_2$. We observe that if H is a trigraph such that either H or $\overline{H}$ belongs to $\mathcal{T}_0$, then for every semi-adjacent pair xy of H, there is a vertex of $V(H) \setminus \{x, y\}$ that is strongly adjacent to x and strongly antiadjacent to y, and a vertex of $V(H) \setminus \{x, y\}$ that is strongly adjacent to y and strongly antiadjacent to x, and hence there is no semi-adjacent pair of type zero in H. Consequently neither of $G', \overline{G'}$ belongs to $\mathcal{T}_0$. This implies that G' is a non-elementary expansion of an elementary bull-free trigraph. This means that there is an elementary trigraph K such that for every vertex v of K there exists a non-empty subset X_v of $V(G')$, all pairwise disjoint and with union $V(G')$, such that

- for $u, v \in V(K)$, if u is strongly adjacent to v, then X_u is strongly complete to X_v, and if u is strongly antiadjacent to v, then X_u is strongly anticomplete to X_v,

- if $v \in V(K)$ does not belong to any semi-adjacent pair of type 0 or of complement type 0, then $|X_v| = 1$
- if u is semi-adjacent to v, and neither of uv, vu is a semi-adjacent pair of type 0 or of complement type 0, then the unique vertex of X_u is semi-adjacent to the unique vertex of X_v
- if uv is a semi-adjacent pair that is both of type 0 and of complement type zero, then either $|X_v| = |X_u| = 1$ and the unique vertex of X_u is semi-adjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type 0 either in G' or in $\overline{G'}$
- if uv is a semi-adjacent pair of type 0 in K and not in $\overline{K}$, then either $|X_v| = |X_u| = 1$ and the unique vertex of X_u is semi-adjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type 0 in G'
- if uv is a semi-adjacent pair of type 0 in $\overline{K}$ and not in K, then either $|X_v| = |X_u| = 1$ and the unique vertex of X_u is semi-adjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type 0 in $\overline{G'}$.

Since for every $v \in V(K)$, X_v is either a strongly stable set or a strong clique, it follows that there exist distinct $u, v \in V(K)$ such that $a \in X_u$ and $b \in X_v$. Suppose that either $|X_u| > 1$ or $|X_v| > 1$. Then (X_u, X_v) is a homogeneous pair of type zero in either G' or $\overline{G'}$, and so (from the definition of a homogeneous pair of type zero) some vertex of G' is strongly adjacent to b and strongly antiadjacent to a, contrary to the fact that $D = \emptyset$. This proves that $|X_u| = |X_v| = 1$, and so $X_u = \{a\}$ and $X_v = \{b\}$. Since ab is a semi-adjacent pair of type zero in G', it follows that that uv is a semi-adjacent pair of type zero in K. But now, replacing X_u by A and X_v by B, we observe that G is a non-elementary expansion of K. This proves 4.2. ∎

The remainder of this paper is organized as follows. In the next section we list some theorems and definitions from [1] and [2] that are useful to us here. Section 6 is devoted to studying bull-free trigraphs with doubly dominating homogeneous pairs. We describe all such trigraphs (up to tame homogeneous pairs that are not doubly dominating) completely in 6.1. In Section 7 we classify tame homogeneous pairs in an elementary bull-free trigraphs, proving that (up to taking complements) every elementary bull-free trigraph either belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$, or admits a homogeneous set decomposition, or a 1-join, or a homogeneous pair of type one, two or three (7.1). Section 8 shows that homogeneous pairs of type three are in fact unnecessary (8.1). Finally, in Section 9 we prove that (up to taking complements), if an elementary bull-free trigraphs admits a 1-join, then it belongs to $\mathcal{T}_1$, thus proving 3.9.

5 Theorems and definitions from [1] and [2].

In this section we list theorems and definitions from [1] and [2] that we need in the remainder of this paper.

Let H be a graph and let $v \in V(H)$. The *degree* of v in H, denoted by $deg(v)$ is the number of edges of H incident with v. If H is the empty graph, let $maxdeg(H) = 0$; and otherwise we define $maxdeg(H) = \max_{v \in V(H)} deg(v)$. We call a bull-free trigraph that does not admit a homogeneous set decomposition, or a homogeneous pair decomposition, and does not contain a path of length three with a center, *unfriendly*.

Let $k \geq 3$ be an integer. A *k-prism* in G is a trigraph whose vertex set is the disjoint union of two cliques $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$; and such that for every $i, j \in \{1, \ldots, k\}$, a_i is adjacent to b_j if $i = j$, and a_i is antiadjacent to b_j if $i \neq j$. A *prism* is a 3-prism.

The following results are proved in [2] (these are theorems 4.2, 5.3, 5.4 and 5.5 of [2], respectively).

5.1 *Let G be an unfriendly trigraph. Assume that for some integer $n \geq 3$, G contains an induced subtrigraph that is an n-prism. Then G is a prism.*

5.2 *Let G be an unfriendly trigraphs, let a_1-a_2-a_3-a_4-a_1 be a hole in G, and let c be a center and a an anticenter for $\{a_1, a_2, a_3, a_4\}$. Then c is strongly antiadjacent to a.*

5.3 *Let H be a trigraph such that no induced subtrigraph of H is a path of length three. Then either*

1. *H is not connected, or*
2. *H is not anticonnected, or*
3. *there exist two vertices $v_1, v_2 \in V(H)$ such that v_1 is semi-adjacent to v_2, and $V(H) \setminus \{v_1, v_2\}$ is strongly complete to v_1 and strongly anticomplete to v_2.*

5.4 *Let G be an unfriendly trigraph, and let $u, v \in V(G)$ be adjacent. Let A, B be subsets of $V(G)$ such that*

- *u is strongly complete to A and strongly anticomplete to B,*
- *v is strongly complete to B and strongly anticomplete to A,*
- *No vertex of $V(G) \setminus (A \cup B)$ is mixed on A, and*
- *if $x, y \in B$ are adjacent, then no vertex of $V(G) \setminus (A \cup B)$ is mixed on $\{x, y\}$.*

Then $A = K \cup S$, where K is a strong clique and S is a strongly stable set.

We also need the main result (3.2) of [1], the following:

5.5 *Let G be a bull-free trigraph. Let P and Q be paths of length three, and assume that there is a center for P and an anticenter for Q in G. Then either*

- *G admits a homogeneous set decomposition, or*
- *G admits a homogeneous pair decomposition, or*
- *G or $\overline{G}$ belongs to $\mathcal{T}_0$.*

6 Doubly dominating homogeneous pairs

Let G be a bull-free trigraph, and let (A, B) be a homogeneous pair in G. We remind the reader that (A, B) is *doubly dominating* if every vertex of $V(G) \setminus (A \cup B)$ is either strongly complete to A and strongly anticomplete to B, or strongly complete to B and strongly anticomplete to A, and there is at least one vertex of each kind. For a semi-adjacent pair a_0b_0, we say that a_0b_0 is *doubly dominating* if the pair $(\{a_0\}, \{b_0\})$ is doubly dominating. In this section we study elementary bull-free trigraphs that admit a doubly dominating homogeneous pair. Our goal is to prove the following:

6.1 *Let G be an elementary bull-free trigraph. Assume that there is a doubly dominating tame homogeneous pair in G, and that every tame homogeneous pair in G is doubly dominating. Then either G admits a homogeneous set decomposition, or $G \in \mathcal{T}_2$.*

We start with three lemmas.

6.2 *Let G be a bull-free trigraph, let $a_0, b_0 \in V(G)$ be two distinct vertices such that a_0 is semi-adjacent to b_0, $V(G) \setminus \{a_0, b_0\} = A \cup B$, where a_0 is strongly complete to A and strongly anticomplete to B, and b_0 is strongly complete to B and strongly anticomplete to A. Then*

- *there do not exist $a_1, a_2, a_3 \in A$ and $b \in B$, such that a_1 is adjacent to a_2, a_3 is anticomplete to $\{a_1, a_2\}$, b is adjacent to a_1, and b is anticomplete to $\{a_2, a_3\}$.*
- *there do not exist $a_1, a_2, a_3 \in A$ and $b \in B$, such that a_1 is antiadjacent to a_2, a_3 is complete to $\{a_1, a_2\}$, b is antiadjacent to a_1, and b is complete to $\{a_2, a_3\}$*

Proof. Since the second assertion of 6.2 follows from the first one applied in $\overline{G}$, it is enough to prove the first assertion. Let $a_1, a_2, a_3 \in A$ and $b \in B$, such that a_1 is adjacent to a_2, a_3 is anticomplete to $\{a_1, a_2\}$, b is adjacent to a_1, and b is anticomplete to $\{a_2, a_3\}$. Then $\{b, a_1, a_2, a_0, a_3\}$ is a bull, a contradiction. This proves 6.2. ∎

Let $k > 0$ be an integer. A *k-edge matching* in a trigraph G is a subset X of $V(G)$, such that $X = \{a_1, \ldots, a_k, a'_1, \ldots, a'_k\}$, each of the sets $\{a_1, \ldots, a_k\}$ and $\{a'_1, \ldots, a'_k\}$ is a stable set, and for $i, j \in \{1, \ldots k\}$, if $i = j$ then a_i is adjacent to a'_j, and if $i \neq j$, then a_i is antiadjacent to a'_j.

6.3 *Let G be an unfriendly bull-free trigraph, let $a_0, b_0 \in V(G)$ be two distinct vertices such that a_0 is semi-adjacent to b_0, $V(G) \setminus \{a_0, b_0\} = A \cup B$, where a_0 is strongly complete to A and strongly anticomplete to B, and b_0 is strongly complete to B and strongly anticomplete to A. Then there is no two edge matching in $G|A$ or in $\overline{G}|A$.*

Proof. Suppose there exist $a_1, a'_1, a_2, a'_2 \in A$ such that the pairs $a_1a'_1, a_2a'_2$ are adjacent, and the pairs $a_1a_2, a_1a'_2, a'_1a_2, a'_1a'_2$ are antiadjacent. By 5.4, $A = K \cup S$, where K is a strong clique, and S is a strongly stable set. Since a_1 is antiadjacent to a_2, it follows that not both a_1, a_2 belong to K, and so we may assume that $a_1 \in S$. Since a'_1 is adjacent to a_1, it follows that $a'_1 \in K$. But now, since a'_1 is anticomplete to $\{a_2, a'_2\}$, it follows that both a_2, a'_2 are in S, contrary to the fact that S is a strongly stable set. This proves that there is no two edge matching in $G|A$.

Next suppose that there is a two edge matching in $\overline{G}|A$. Then there exist $a_1, a_2, a_3, a_4 \in A$, such that a_1-a_2-a_3-a_4-a_1 is a hole, say H, in G. But a_0 is a center for H, and b_0 is an anticenter for H, and a_0 is adjacent to b_0, contrary to 5.2. This proves that there is no two edge matching in $\overline{G}|A$ and completes the proof of 6.3. ∎

6.4 *Let G be a bull-free trigraph, and let (A, B) be a homogeneous pair in G such that some vertex $d \in V(G) \setminus (A \cup B)$ is strongly complete to B and strongly anticomplete to A. Assume also that either*

- *some vertex $c \in V(G) \setminus (A \cup B)$ is strongly complete to $A \cup \{d\}$, and strongly anticomplete to B, and G has no prism, or*
- *A is a stable set.*

Let $B' \subseteq B$ be a clique. Let $|B'| = m$. Then the vertices of B' can be ordered $b_1, \ldots, b_m$, so that if $a \in A$ is adjacent to b_i, then a is strongly complete to $\{b_{i+1}, \ldots, b_m\}$.

Proof. The proof is by induction of $|B'|$. Choose $b \in B'$ with $N(b) \cap A$ maximal, and subject to that $S(b) \cap A$ maximal. Inductively, the vertices of $B' \setminus \{b\}$ can be ordered $b_1, \ldots, b_{m-1}$, so that if $a \in A$ is adjacent to b_i, then a is strongly complete to $\{b_{i+1}, \ldots, b_{m-1}\}$. Let $b_m = b$. We need to show that if $a \in A$ is adjacent to b_i with $i \in \{1, \ldots, m-1\}$, then a is strongly adjacent to b_m. Suppose not. Let $i \in \{1, \ldots, m-1\}$, and assume that $a \in A$ is adjacent to b_i and antiadjacent to b_m. Since $\{a, b_i, d, b_m, a'\}$ is not a bull for any vertex $a' \in (N(b) \cap A) \setminus \{a\}$, and, if A is not a stable set, then

$G|\{a, a', c, b_i, b_m, d\}$ is not a prism and for any vertex $a' \in (N(b) \cap A) \setminus \{a\}$, it follows that every vertex of $(N(b) \cap A) \setminus \{a\}$ is strongly adjacent to b_i. By the maximality of $N(b) \cap A$, it follows that b is adjacent, and therefore semi-adjacent, to a. Since a is semi-adjacent to at most one vertex of $V(G)$, it follows that b_i is strongly adjacent to a. But this contradicts the maximality of $S(b) \cap A$. Thus a is strongly adjacent to b_m. This proves 6.4 ▮

First we need to understand unfriendly trigraphs that have a doubly dominating semi-adjacent pair. We start with the following:

6.5 *Let G be an unfriendly bull-free trigraph, let $a_0, b_0 \in V(G)$ be two distinct vertices such that a_0 is semi-adjacent to b_0, $V(G) \setminus \{a_0, b_0\} = A \cup B$, where a_0 is strongly complete to A and strongly anticomplete to B, and b_0 is strongly complete to B and strongly anticomplete to A. Then either G is a prism, or the vertices of A can be numbered $\{a_1, \dots, a_n\}$ and the vertices of B can be numbered $\{b_1, \dots, b_m\}$ such that the following conditions are satisfied:*

1. *for $i, j \in \{1, \dots, n\}$, with $i < j$, if a_i is adjacent to a_j, then a_j is strongly complete to $\{a_1, \dots, a_{i-1}\}$, and a_i is strongly complete to $\{a_{i+1}, \dots, a_{j-1}\}$*

2. *for $i, j \in \{1, \dots, m\}$, with $i < j$, if b_i is adjacent to b_j, then b_j is strongly complete to $\{b_1, \dots, b_{i-1}\}$, and b_i is strongly complete to $\{b_{i+1}, \dots, b_{j-1}\}$*

3. *for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, if a_i is adjacent to b_j, and b_j has a neighbor in $\{b_{j+1}, \dots, b_m\}$, then a_i is strongly complete to $\{b_{j+1}, \dots, b_m\}$,*

4. *for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, if a_i is adjacent to b_j, and a_i has a neighbor in $\{a_{i+1}, \dots, a_n\}$, then b_j is strongly complete to $\{a_{i+1}, \dots, a_n\}$.*

Proof. By 5.1, we may assume that there is no prism in G. Let K, S, X be pairwise disjoint subsets of A such that $K \cup X \cup S = A$. We say that (K, S, X) is a *calm partition* of A if the vertices of K can be numbered $\{k_1, \dots, k_k\}$ and the vertices of S can be numbered $\{s_1, \dots, s_s\}$ such that the following conditions are satisfied:

1. K is a strong clique

2. S is a strongly stable set

3. K is strongly complete to X

4. S is strongly anticomplete to X

5. for $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, s\}$, if k_i is adjacent to s_j, then s_j is strongly complete to $\{k_1, \dots, k_{i-1}\}$ and k_i is strongly complete to $\{s_1, \dots, s_{j-1}\}$

6. for $i \in \{1, \dots, k\}$, if $b \in B$ is adjacent to k_i, then b is strongly complete to $\{k_{i+1}, \dots, k_k\} \cup X \cup S$

7. if $b \in B$ has a neighbor in X, then b is strongly complete to S.

We call the orders $\{k_1, \dots, k_k\}$ and $\{s_1, \dots, s_s\}$ *the orders associated with the partition*.

A calm partition of B is defined similarly. Let (K, S, X) be a calm partition of A chosen with X minimal, and let (L, T, Y) be a calm partition of B chosen with Y minimal. Let $|K| = k, |S| = s, |L| = l$ and $|T| = t$, and let $\{k_1, \dots, k_k\}$, $\{s_1, \dots, s_s\}$, $\{l_1, \dots, l_l\}$ and $\{t_1, \dots, t_t\}$ be the associated orders of K, S, L and T, respectively. We observe that if $X = Y = \emptyset$, then ordering the vertices of A as $\{k_1, \dots, k_k, s_1, \dots, s_s\}$ and the vertices of B as $\{l_1, \dots, l_l, t_1, \dots, t_t\}$, we obtain a numbering that satisfies the conditions of 6.5. Thus we may assume that $X \neq \emptyset$.

(1) There do not exist $u, v \in X$ such that u is semi-adjacent to v, u is strongly complete to $X \setminus \{u, v\}$ and v is strongly anticomplete to $X \setminus \{u, v\}$.

Suppose such u, v exists. If $X \neq \{u, v\}$, then the difference between u and v is clear. However, if $X = \{u, v\}$, then there is symmetry between u and v. Since G is unfriendly, it follows that if $X = \{u, v\}$ then some vertex $b_1 \in B$ is mixed on $\{u, v\}$. In this case we will assume that b_1 can be chosen adjacent to v and antiadjacent to u. Let $K' = K \cup \{u\}$, $S' = S \cup \{v\}$ and $X' = X \setminus \{u, v\}$. We claim that (K', S', X') is a calm partition of A. Order the vertices of K' as $k_1, \dots, k_k, u$ and of S' as $v, s_1, \dots, s_s$. Since K is strongly complete to X, and S is strongly anticomplete to X, it follows that K' is a strong clique, and S' is a strong stable set. Since u is strongly complete to $X \setminus \{u, v\}$, it follows that K' is strongly complete to X'. Since v is strongly anticomplete to $X \setminus \{u, v\}$, it follows that S' is strongly anticomplete to X'. Since (K, S, X) is a calm partition of A, and since u is strongly anticomplete to S, and v is strongly complete to K, it follows that the fifth condition in the definition of a clam partition is satisfied. Since (K, S, X) is a calm partition of A, in order to check that the sixth condition of the definition of a calm partition is satisfied by (K', S', X'), it is enough to show that if $b \in B$ is adjacent to u, then b is strongly complete to $S \cup (X \setminus \{u\})$. Since (K, S, X) is a calm partition, it follows that b is strongly complete to S. Since u is complete to $X \cup \{v\}$, 6.2.2 implies that b is either strongly complete or strongly anticomplete to $X \cup \{v\}$. So we may assume that b is strongly anticomplete to $X \cup \{v\}$. If $X' \neq \emptyset$, we get a contraction to 6.2.1, since b is antiadjacent to v, and v is anticomplete to $X' \cup \{u\}$. Thus we may assume that $X' = \emptyset$, and so there exists $b_1 \in B$ adjacent to v and antiadjacent to u. Then $b_1 \neq b$. Since $\{u, b, b_0, b_1, v\}$ is not a bull, it follows that b is antiadjacent to b_1. But now $\{b, u, a_0, v, b_1\}$ is a bull, a contradiction. This proves that (K', S', X') satisfies the sixth condition of the definition of a calm partition. Finally, to check the seventh condition, since (K, S, X) is a calm partition, it is enough to show that if $b \in B$ has a neighbor in X', then b is strongly complete to S'. Since (K, S, X) is a calm partition, b is strongly complete to S. Suppose b is antiadjacent to v. By the sixth condition, it follows that b is strongly antiadjacent to u. Let $x \in X'$ be adjacent to b. Now setting $a_1 = x, a_2 = u, a_3 = v$ we obtain a contradiction to 6.2.1. This proves that b is strongly complete to S', and therefore (K', S', X') is a calm partition of A, contrary to the minimality of X. This proves (1).

(2) X is anticonnected.

Suppose not. Let $Y_1, \ldots, Y_p$ be the anticomponents of X. By 6.3, we may assume that $|Y_2| = \ldots = |Y_p| = 1$. If $|Y_1| > 1$, let $X' = Y_1$ and $X_2 = X \setminus X'$. If $|Y_1| = 1$, let $X' = \emptyset$, and $X_2 = X$. Thus, in both cases, X_2 is a strong clique. By 6.4, we can number the vertices of X_2 as $\{x_1, \ldots, x_q\}$, so that for $i \in \{1, \ldots, q\}$, if b is adjacent to x_i, then b is strongly complete to $\{x_{i+1}, \ldots, x_q\}$.

Let $K' = K \cup X_2$. We claim that (K', S, X') is a calm partition of A. For $i \in \{k+1, \ldots, k+q\}$, let $k_i = x_{i-k}$. Order the vertices of K' as $k_1, \ldots, k_{k+q}$ and the vertices of S as $s_1, \ldots, s_s$. Since X_2 is a strong clique, and X_2 is strongly complete to $K \cup X'$, it follows that (K', S, X') satisfies conditions (1)-(4) of the definition of a calm partition.

To check the fifth condition, let $i \in \{1, \ldots, k+q\}$ and $j \in \{1, \ldots, s\}$ and assume that k_i is adjacent to s_j. We claim that s_j is strongly complete to $\{k_1, \ldots k_{i-1}\}$, and k_i is strongly complete to $\{s_1, \ldots, s_{j-1}\}$. Since S is strongly anticomplete to X, it follows that $i \leq k$. But now the claim follows from the fact that (K, S, X) is a calm partition. Thus (K', S, X') satisfies the fifth condition of the definition of a calm partition.

To check the sixth condition, let $b \in B$ and $i \in \{1, \ldots, k+1\}$, and assume that b is adjacent to k_i. We need to show that b is strongly complete to $\{k_{i+1}, \ldots, k_{k+q}\} \cup X' \cup S$. If $i < k$, then, since (K, S, X) is a calm partition, it follows that b is strongly complete to $\{k_{i+1}, \ldots, k_k\} \cup X \cup S = \{k_{i+1}, \ldots, k_{k+q}\} \cup X' \cup S$, thus we may assume that $i > k$. Then b is strongly complete to $\{k_{i+1}, \ldots k_{k+q}\}$, and, since (K, S, X) is a calm partition and $k_i \in X$, it follows that b is strongly complete to S. Thus it remains to show that b is strongly complete to X'. Suppose not. Since b is adjacent to k_i, and k_i is complete to X', and X' is anticonnected, 6.2.2 implies that b is strongly anticomplete to X' and $X' \neq \emptyset$. Thus $|X'| > 1$, and so it follows that X' is not a homogeneous set in G. Consequently, some vertex $v \in V(G) \setminus X'$ is mixed on X'. Since $K \cup X_2$ is strongly complete to X', and S is strongly anticomplete to X', it follows that $v \in B$. By 6.2.2, v is strongly anticomplete to $K \cup X_2$. Let $x' \in X'$ be adjacent to v. Then since $\{b, k_i, a_0, x', v\}$ is not a bull, it follows that v is strongly adjacent to b. But now $G|\{k_i, x', a_0, b, v, b_0\}$ is a prism, a contradiction. This proves that b is strongly complete to X', and so (K', S, X') satisfies the sixth condition of the definition of a calm partition. Since $X' \subseteq X$, the seventh condition of the definition of a calm partition is satisfied. Thus (K', S, X') is a calm partition of A, contrary to the minimality of X. This proves (2).

Since G is unfriendly, there is no three edge path in X, and so (1),(2) and 5.3 imply that X is not connected. Let $Z_1, \dots, Z_p$ be the components of X. By 6.3, we may assume that $|Z_2| = \dots = |Z_p| = 1$. If $|Z_1| > 1$, let $X' = Z_1$ and $X_2 = X \setminus X'$. If $|Z_1| = 1$, let $X' = \emptyset$, and $X_2 = X$. Thus, in both cases, X_2 is a strongly stable set. Let $S' = S \cup X_2$.

From the symmetry, it follows that if $Y \neq \emptyset$, then Y is not connected. Let $Y' = Y_2 = \emptyset$ if $Y = \emptyset$, and define Y', Y_2 similarly to X', X_2 if $Y \neq \emptyset$. Let $T' = T \cup Y_2$.

(3) $X' \neq \emptyset$ and some vertex of B is strongly complete to X' and has an antineighbor in X_2.

Suppose that either $X' = \emptyset$, or no vertex of B is strongly complete to X' and has an antineighbor in X_2. We claim that (K, S', X') is a calm partition of A. Let $q = |X_2|$. Order the vertices of X_2 arbitrarily as $s'_1, \dots, s'_q$. For $i \in \{q+1, \dots, q+s\}$, let $s'_i = s_{i-q}$. Then $s'_1, \dots, s'_{s+q}$ is an ordering of the vertices of S'. Order the vertices of K as $k_1, \dots, k_k$.

Since X_2 is a strongly stable set, and X_2 is strongly anticomplete to $S \cup X'$, it follows that (K, S', X') satisfies conditions (1)-(4) of the definition of a calm partition.

To check the fifth condition, let $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, q+s\}$ and assume that k_i is adjacent to s'_j. We claim that s'_j is strongly complete to $\{k_1, \dots, k_{i-1}\}$, and k_i is strongly complete to $\{s'_1, \dots, s'_{j-1}\}$. If $j > q$, the claim follows from the fact that K is strongly complete to X_2, and that (K, S, X) is a calm partition, so we may assume that $j < q$. But now the claim follows from the fact that K is strongly complete to X_2. Thus (K, S', X') satisfies the fifth condition of the definition of a calm partition.

The sixth condition of the definition of a calm partition is satisfied since $X' \cup S' = X \cup S$.

To check the seventh condition, let $b \in B$ be adjacent to $x' \in X'$ (and so $X' \neq \emptyset$). We need to prove that b is strongly complete to S'. Since (K, S, X) is a calm partition of A, it follows that b is strongly complete to S. Suppose b has an antineighbor $x \in X_2$. Now, since $X' \neq \emptyset$, it follows that $|X'| > 1$ and X' is connected. But then 2.2 and 6.2.1 imply that b is strongly complete to X', which is a contradiction, since b has an antineighbor in X_2. This proves that the seventh condition of the definition of a calm partition is satisfied, and so (K, S', X') is a calm partition, contrary to the minimality of X. This proves (3).

In view of (3), let $b \in B$ be a vertex strongly complete to X' and with an antineighbor $x_2 \in X_2$. Also from (3), $X' \neq \emptyset$, and therefore $|X'| > 1$. Since X' is not a homogeneous set in G, and K is strongly complete to X', and S' is strongly anticomplete to X', it follows that some vertex of B is mixed on X'. Let B' be the set of vertices of B that are mixed on X'.

(4) B' is strongly anticomplete to K and strongly complete to $S \cup X_2$.

By 6.2.1 and 2.2, since X' is connected, it follows that B' is strongly complete to $X_2 \cup S$. Since every vertex of B' has an antineighbor in X' and is strongly adjacent to x_2, and since K is strongly complete to X, 6.2.2 implies that K is strongly anticomplete to B'. This proves (4).

(5) Let $c \in B$ be complete to X'. Then B' is strongly anticomplete to c. In particular, B' is strongly anticomplete to b.

Suppose $b' \in B'$ is adjacent to c. By 2.2, there exist $x, x' \in X'$ such that x is adjacent to x', and b' is adjacent to x and antiadjacent to x'. Now x'-x-b'-b_0 is a path, and c is a center for it, contrary to the fact that G is unfriendly. This proves (5).

(6) $b \notin L$.

Suppose $b \in L$. Choose $b' \in B'$. Let $x' \in X'$ be antiadjacent to b'. By (5) b' is strongly antiadjacent to b, and therefore $b' \in T$. But x' is adjacent to b and antiadjacent to b', contrary the fact that (L, T, Y) is a calm partition of B. This proves (6).

(7) If $b \in Y$, then $B' \subseteq Y$, and if $b \in T$, then $B' \subseteq T$.

Suppose $b \in Y$. It follows from (5) that $B' \cap L = \emptyset$. Suppose $B' \cap T \neq \emptyset$, and choose $b' \in B' \cap T$. Let $x' \in X'$ be antiadjacent to b'. Then x' is adjacent to b and antiadjacent to b', contrary to the fact that (L, T, Y) is a calm partition of B. This proves that $B' \subseteq Y$.

Next suppose that $b \in T$. Suppose $B' \cap Y \neq \emptyset$, an let $b' \in B' \cap Y$. Then, by (4), x_2 is adjacent to b' and antiadjacent to b, contrary to the fact that (L, T, Y) is a calm partition of B. This proves (7).

(8) $b \notin Y$.

Suppose $b \in Y$. By (7), $B' \subseteq Y$. By (4), B' is strongly anticomplete to K and strongly complete to $S \cup X_2$. Since Y_2 is a strongly stable set, and Y_2 is strongly complete to L and strongly anticomplete to $Y' \cup T$, and since (X', B') is not a tame homogeneous pair in G, it follows that $B' \cap Y' \neq \emptyset$. Since Y' is strongly complete to L and strongly anticomplete to $T \cup Y_2$, and since (X', B') is not a tame homogeneous pair in G, it follows that $Y' \setminus B' \neq \emptyset$. Since Y' is connected, some vertex $y' \in Y' \setminus B'$ has a neighbor $b' \in Y' \cap B'$, and (5) implies y' is strongly anticomplete to X'. Let $x \in X'$ be adjacent to b'. Let A' be the set of vertices of A that are mixed on Y'. Then, since x is antiadjacent to y', it follows that $A' \cap X' \neq \emptyset$. Since $Y' \neq \emptyset$, the symmetry between A and B has been restored. So, from the symmetry and by (3), it follows that some vertex $a \in A$ is strongly complete to Y' and has an antineighbor $y_2 \in Y_2$. Since $A' \cap X' \neq \emptyset$, it follows from (7) that $A' \cup \{a\} \subseteq X$. Moreover, since y' is strongly anticomplete to X', it follows that $a \in X_2$. Also from the symmetry, some vertex $x' \in X' \setminus A'$ is strongly anticomplete to Y', and so $b \in Y_2$.

Now K is strongly complete to $X' \cup A'$ and $S \cup (X_2 \setminus A')$ is strongly anticomplete to $X' \cup A'$. By (4), A' is strongly anticomplete to L and strongly complete to $T \cup (Y_2 \setminus B')$. Since $B' \subseteq Y$, and since $A' \cap X' \neq \emptyset$, it follows that $X' \cup A'$ is strongly anticomplete to L and strongly complete to $T \cup (Y_2 \setminus B')$. Consequently, no vertex of $V(G) \setminus (Y' \cup X' \cup A' \cup B')$ is mixed on $X' \cup A'$. Similarly, no vertex of $V(G) \setminus (Y' \cup X' \cup A' \cup B')$ is mixed on $Y' \cup B'$. But now, since $a, b, a_0, b_0 \notin A' \cup B' \cup X' \cup Y'$, we deduce that $(A' \cup X', B' \cup Y')$ is a tame homogeneous pair in G, a contradiction. This proves (8).

By (7) and (8), $B' \cup \{b\} \subseteq T$. By (4), B' is strongly complete to $X_2 \cup S$ and strongly anticomplete to K. Since B' is strongly anticomplete to $Y \cup (T \setminus B')$, and since (X', B') is not a homogeneous pair, it follows that some vertex $l \in L$ is mixed on B'. Since x_2 is antiadjacent to b, and since (L, T, Y) is a calm partition of B, it follows that x_2 is strongly antiadjacent to l. Let $b_1 \in B'$ be adjacent to l, let $b_2 \in B'$ be antiadjacent to l. Since x_2 is adjacent to b_1 and antiadjacent to both l and b, 6.2.1 implies that l is adjacent to b. Let $x' \in X'$ be antiadjacent to b_2. Since (L, T, Y) is a calm partition of B, it follows that x' is strongly antiadjacent to l. But now x' is adjacent to b and antiadjacent to l, and b_2 is anticomplete to $\{b_2, l\}$, contrary to 6.2.1. This proves 6.5. ∎

Let us now list a few properties of the class $\mathcal{T}_2$.

6.6 *Let G be a bull-free trigraph, let $a_0, b_0 \in V(G)$ be two distinct vertices such that a_0 is semi-adjacent to b_0, and let $V(G) \setminus \{a_0, b_0\} = A \cup B$, where a_0 is strongly complete to A and strongly anticomplete to B, and b_0 is strongly complete to B and strongly anticomplete to A. If both A and B are non-empty strongly stable sets, then $G \in \mathcal{T}_2$.*

Proof. If there do not exist vertices $a \in A$ and $b \in B$ such that a is strongly complete to $B \setminus \{b\}$, b is strongly complete to $A \setminus \{a\}$, and a is semi-adjacent to b, then, using the skeleton G_0 which is an edge pattern, we observe that $G \in \mathcal{T}_2$.

So we may assume that such a, b exist. Let k be an integer such that there exist vertices $a_1, \ldots, a_k$ in A, and $b_1, \ldots, b_k$ in B, such that a_i is strongly complete to $B \setminus \{b_i\}$, b_i is strongly complete to $A \setminus \{a_i\}$, and a_i is semi-adjacent to b_i. We may assume that there do not exist vertices $a \in A \setminus \{a_1, \ldots, a_k\}$ and $b \in B \setminus \{b_1, \ldots, b_k\}$ such that a is strongly complete to $B \setminus \{b\}$, b is strongly complete to $A \setminus \{a\}$, and a is semi-adjacent to b. Assume first that

$$V(G) = \{b_0, a_1, \ldots, a_k\} \cup \{a_0, b_1, \ldots, b_k\}.$$

Then G is either an edge pattern, or a triad pattern, or if $k > 2$, G is obtained by composing $k-1$ triad patterns. In all cases G is a skeleton, and so $G \in \mathcal{T}_2$.

So we may assume that $V(G) \neq \{b_0, a_1, \ldots, a_k\} \cup \{a_0, b_1, \ldots, b_k\}$. Next assume that $A = \{a_1, \ldots, a_k\}$. Let G_0 be the trigraph obtained from $G|\{a_0, a_1, \ldots, a_{k-1}, b_0, b_1, \ldots, b_{k-1}\}$ by adding two new vertices x and y, such that x is strongly complete to $\{a_0, b_1, \ldots, b_{k-1}\}$ and strongly anticomplete to $\{b_0, a_1, \ldots, a_{k-1}\}$, y is strongly complete to $\{b_0, a_1, \ldots, a_{k-1}\}$ and strongly anticomplete to $\{a_0, b_1, \ldots, b_{k-1}\}$, and x is semi-adjacent to y. Then G_0 is a skeleton (in fact, G_0 is either an edge pattern, or, if $k > 0$, G_0 is obtained by composing k triad patterns). Let G' be the trigraph obtained from $G \setminus \{a_0, a_1, \ldots, a_{k-1}, b_0, \ldots, b_1, \ldots, b_{k-1}\}$ by adding two new vertices x', y' such that x' is strongly complete to $V(G') \cap A$ and strongly anticomplete to $V(G') \cap B$, y' is strongly complete to $V(G') \cap B$ and strongly anticomplete to $V(G') \cap A$, and x' is semi-adjacent to y'. Then G' is a 2-thin trigraph, and since G is obtained by composing G_0 and G' along (x, y, x', y'), it follows that G is a skeleton, and in particular $G \in \mathcal{T}_2$.

Thus we may assume that $A \neq \{a_1, \ldots, a_k\}$ and $B \neq \{b_1, \ldots, b_k\}$. Let G_0 be the trigraph obtained from $G|\{a_0, a_1, \ldots, a_k, b_0, b_1, \ldots, b_k\}$ by adding two new vertices x and y, such that x is strongly complete to $\{a_0, b_1, \ldots, b_k\}$ and strongly anticomplete to $\{b_0, a_1, \ldots, a_k\}$, y is strongly complete to $\{b_0, a_1, \ldots, a_k\}$ and strongly anticomplete to $\{a_0, b_1, \ldots, b_k\}$, and x is semi-adjacent to y. Then G_0 is a skeleton (in fact, G_0 is either an edge pattern, or, if $k > 1$, G_0 is obtained by composing $k-1$ triad patterns). Let G' be the trigraph obtained from $G \setminus \{a_0, a_1, \ldots, a_k, b_0, b_1, \ldots, b_k\}$ by adding two new vertices x', y' such that x' is strongly complete to $V(G') \cap A$ and strongly anticomplete to $V(G') \cap B$, y' is strongly complete to $V(G') \cap B$ and strongly anticomplete to $V(G') \cap A$, and x' is semi-adjacent to y'. Then both $V(G') \cap A$ and $V(G') \cap B$ are non-empty, and since G is obtained by composing G_0 and G' along (x, y, x', y'), it follows that $G \in \mathcal{T}_2$. This proves 6.6. ∎

6.7 *Every* 1*-thin trigraph belongs to* T_2*.*

Proof. Suppose that G is 1-thin with base (a_0, b_0). Since very 2-thin graph is a skeleton and therefore belongs to $\mathcal{T}_2$, we may assume that G is not 2-thin. Now using the skeleton G_0 which is an edge pattern, we observe that $G \in \mathcal{T}_2$. This proves 6.7. ∎

6.8 *Let* $G_1, G_2 \in \mathcal{T}_2$*, for* $i = 1, 2$ *let* (a^i, b^i) *be a doubly dominating semi-adjacent pair in* G_i*, and let* G *be obtained by composing* G_1 *and* G_2 *along* (a^1, b^1, a^2, b^2)*. Then* $G \in \mathcal{T}_2$*.*

Proof. Since $G_i \in \mathcal{T}_2$, for $i = 1, 2$ there exist a skeleton G_0^i, a list of doubly dominating semi-adjacent pairs $(a_1^i, b_1^i), \ldots, (a_{n^i}^i, b_{n^i}^i)$ in G_0^i, and a list of trigraphs ${G_1^i}', \ldots, {G_{n^i}^i}'$, such that G_i is obtained from $G_0^i, {G_1^i}', \ldots, {G_{n^i}^i}'$ as in the definition of $\mathcal{T}_2$. Moreover, for $i \in 1, 2$ there exist trigraphs $F_1^i, \ldots, F_{k^i}^i$ each of which is a triangle pattern, a triad pattern or a 2-thin trigraph, and lists of semi-adjacent pairs

$$(c_2^i, d_2^i), \ldots, (c_{k^i}^i, d_{k^i}^i)$$

and

$$(x_1^i, y_1^i), \ldots, (x_{k^i-1}^i, y_{k^i-1}^i),$$

and G_0^i is obtained as in the definition of a skeleton.

Since for $i = 1, 2$ the pair (a^i, b^i) is a doubly dominating semi-adjacent pair in G_i, it follows that for some $j^i \in \{1, \ldots, k^i\}$, $a^i, b^i \in V(F_{j^i}^i)$ and (a^i, b^i) is distinct from

$$(c_2^i, d_2^i), \ldots, (c_{k^i}^i, d_{k^i}^i),$$
$$(x_1^i, y_1^i), \ldots, (x_{k^i-1}^i, y_{k^i-1}^i),$$
$$(a_1^i, b_1^i), \ldots, (a_{n^i}^i, b_{n^i}^i).$$

It is not difficult to see that the composition operation is commutative and associative, and so we may assume that $j^1 = k^1$ and $j^2 = 1$. Set

$$F_p = F_p^1 \text{ for } p \in \{1, \ldots, k^1\},$$

$$F_{k^i+q} = F_q^2 \text{ for } q \in \{1, \ldots, k^2\},$$

$$(c_p, d_p) = (c_p^1, d_p^1) \text{ for } p \in \{2, \ldots, k^1\},$$

$$(c_{j^1+1}, d_{j^1+1}) = (a^2, b^2),$$

$$(c_{j^1+q}, d_{j^1+q}) = (c_q^2, d_q^2) \text{ for } q \in \{2, \ldots, k^2\},$$

$$(x_p, y_p) = (x_p^1, y_p^1) \text{ for } p \in \{1, \ldots, k^1 - 1\},$$

$$(x_{j^1}, y_{j^1}) = (a^1, b^1),$$

$$(x_{j^1+q}, y_{j^1+q}) = (x^2_q, y^2_q) \text{ for } q \in \{1, \dots, k^2 - 1\}.$$

Let $s = k^1 + k^2$. Let G_0 the trigraph obtained from $F_1, \dots, F_s$ using the semi-adjacent pairs

$$(c_2, d_2), \dots (c_s, d_s)$$

and

$$(x_1, y_1), \dots (x_{s-1}, y_{s-1})$$

as in the definition of a skeleton. Then G_0 is a skeleton.

For $i = 1, 2$, the pairs $(a^i_1, b^i_1), \dots, (a^i_{n^i}, b^i_{n^i})$ are doubly dominating semi-adjacent pairs in G_0. But now G is obtained from G_0 and ${G^1_1}', \dots, {G^1_{n^1}}', {G^2_1}', \dots, {G^2_{n^2}}'$ as in the defending of $\mathcal{T}_2$, and therefore $G \in \mathcal{T}_2$. This proves 6.8. ▮

Now we can describe unfriendly trigraphs with a doubly dominating semi-adjacent pair completely.

6.9 *Let G be an unfriendly bull-free trigraph, let $a_0, b_0 \in V(G)$ be two distinct vertices such that a_0 is semi-adjacent to b_0, and let $V(G) \setminus \{a_0, b_0\} = A \cup B$, where a_0 is strongly complete to A and strongly anticomplete to B, and b_0 is strongly complete to B and strongly anticomplete to A. Assume that both A and B are non-empty. Then either G is a prism, or each of A, B is strongly stable, or G is 1-thin with base (a_0, b_0), and in all cases $G \in \mathcal{T}_2$.*

Proof. By 6.6, 6.7 and 3.6, it follows that if G is a prism, or each of A, B is strongly stable, or G is 1-thin with base (a_0, b_0), then $G \in \mathcal{T}_2$.

Since if G is a prism, then $G \in \mathcal{T}_2$, by 5.1 we may assume that no induced subtrigraph of G is a prism. We may also assume that not both A and B are strongly stable sets. By 6.5, the vertices of A can be numbered $a_1, \dots, a_n$ and the vertices of B can be numbered $b_1, \dots, b_m$ such that the following conditions are satisfied:

1. for $i, j \in \{1, \dots, n\}$, with $i < j$, if a_i is adjacent to a_j, then a_j is strongly complete to $\{a_1, \dots, a_{i-1}\}$, and a_i is strongly complete to $\{a_{i+1}, \dots, a_{j-1}\}$
2. for $i, j \in \{1, \dots, m\}$, with $i < j$, if b_i is adjacent to b_j, then b_j is strongly complete to $\{b_1, \dots, b_{i-1}\}$, and b_i is strongly complete to $\{b_{i+1}, \dots, b_{j-1}\}$
3. for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, if a_i is adjacent to b_j, and b_j has a neighbor in $\{b_{j+1}, \dots, b_m\}$, then a_i is strongly complete to $\{b_{j+1}, \dots, b_m\}$,
4. for $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, if a_i is adjacent to b_j, and a_i has a neighbor in $\{a_{i+1}, \dots, a_n\}$, then b_j is strongly complete to $\{a_{i+1}, \dots, a_n\}$.

Let us all call a pair of orderings of A, B satisfying the four conditions above *laminar*; we say that the orderings are *laminar orderings.*

(1) *If A is not a strongly stable set, then a_1 is adjacent to a_2.*

Suppose not, and let $a_i, a_j \in A$ be adjacent. We may assume that $i < j$. Since (A, B) is a laminar pair, it follows that a_j is adjacent to a_1. But now, again since (A, B) is a laminar pair, it follows that a_1 is complete to $\{a_2, \ldots, a_j\}$, a contradiction. This proves (1).

(2) There do not exist $a, a' \in A$ and $b, b' \in B$ such that the pairs $ab, a'b'$ are adjacent, and the pairs $ab', a'b$ are antiadjacent.

Suppose such a, a', b, b' exist. Let $A' \subseteq A$ and $B' \subseteq B$. We say that the pair (A', B') is *matching-connected* if for every partition A_1, A_2 of A' there exist $a_1 \in A_1, a_2 \in A_2$ and $b_1, b_2 \in B'$ such that the pairs a_1b_1, a_2b_2 are adjacent, and the pairs a_1b_2, a_2b_1 are antiadjacent; and the same with the roles of A', B' switched.

Thus the pair $(\{a, a'\}, \{b, b'\})$ is matching connected. Choose $A' \subseteq A$ and $B' \subseteq B$ such that $a, a' \in A'$, $b, b' \in B$, the pair (A', B') is matching connected, and subject to that with $A' \cup B'$ maximal.

We claim that no vertex of $V(G) \setminus (A' \cup B')$ is mixed on A'. Suppose some $v \in V(G) \setminus (A' \cup B')$ is mixed on A'. Then $v \in A \cup B$. If v is complete to A', let A_1 be the set of strong neighbors of v in A', and let $A_2 = A' \setminus A_1$. If v is not complete to A', let A_1 be the set of neighbors of v in A_1, and let $A_2 = A' \setminus A_1$. Since (A', B') is matching-connected, there exist $i, j \in \{1, \ldots, n\}$ and $p, q \in \{1, \ldots, m\}$ such that $a_i \in A_1, a_j \in A_2$ and $b_p, b_q \in B'$ such that the pairs a_ib_p, a_jb_q are adjacent, and the pairs a_ib_q, a_jb_p are antiadjacent. Thus v is adjacent to a_i and antiadjacent to a_j. Since $G|(\{a_i, a_j, a_0, b_p, b_q, b_0\})$ is not a prism, and $\{b_p, a_i, a_0, a_j, b_q\}$ and $\{a_i, b_p, b_0, b_q, a_j\}$ are not bulls in G, it follows that a_i is strongly antiadjacent to a_j, and b_p is strongly antiadjacent to b_q. Suppose first that $v \in A \setminus A'$, By 6.2.1 applied to a_i, a_j, v and b_p, it follows that v is strongly adjacent to b_p. Let $t \in \{1, \ldots, n\}$ be such that $v = a_t$. Since a_i is adjacent to a_t and a_j is antiadjacent to both a_i, a_t, it follows that $j > i$ and $j > t$. Let $s = \min(i, t)$. Then b_p is adjacent to a_s and antiadjacent to a_j, and a_s has a neighbor in $\{a_{s+1}, \ldots, a_n\}$, a contradiction. This proves that $v \notin A$, and therefore $v \in B$. But now, since the pair (A', B') is matching connected, the pairs a_iv, a_jb_q are adjacent, and the pairs a_ib_q, va_j are antiadjacent, it follows that $(A', B' \cup \{v\})$ is a matching connected pair, contrary to the maximality of $A' \cup B'$. This proves that no vertex of $V(G) \setminus (A' \cup B')$ is mixed on A'. From the symmetry, no vertex of $V(G) \setminus (A' \cup B')$ is mixed on B'. Since G is unfriendly, it follows that (A', B') is not tame homogeneous pair in G, and therefore $A = A'$ and $B = B'$.

From the symmetry, we may assume that A is not a strongly stable set. By (1), a_1 is adjacent to a_2. Consider the partition $\{a_1\}, A \setminus \{a_1\}$ of A. Since (A, B) is matching connected, it follows that there exist $a \in A \setminus \{a_1\}$ and $b, b' \in B$ such that the pairs $a_1 b, ab'$ are adjacent, and the pairs $a_1 b', ab$ are antiadjacent. But since a_1 is adjacent to a_2 and b is adjacent to a_1, it follows that b is strongly complete to $\{a_2, \ldots, a_n\}$, a contradiction. This proves (2).

(3) There do not exist $x, y, z, w \in A$ such that the pairs xy, zw are adjacent, and the pairs yz, xw are antiadjacent.

Suppose that such x, y, z, w exist. By 6.3 and the symmetry, we may assume that x is adjacent to z, and y is antiadjacent to w. But now y-x-z-w is a path, and a_0 is a center for it, contrary to the fact that G is unfriendly. This proves (3).

For $i \in \{1, \ldots, n\}$, let i_0 be minimum such that a_{i_0} is strongly anticomplete to $\{a_{i_0+1}, \ldots, a_n\}$. If for some $j \in \{i_0+1, \ldots, n\}$ and $k \in \{j+1, \ldots, n\}$, a_j is adjacent to a_k, then a_k is strongly adjacent to a_{i_0}, a contradiction. This proves that for every $j \in \{i_0+1, \ldots, n\}$, a_j is strongly anticomplete to $\{a_{j+1}, \ldots, a_n\}$. Let $A' = \{a_{i_0}, \ldots, a_n\}$. Then A' is a strongly stable set. Let $a \in A'$ be such that $N(a) \cap B$ is maximal, subject to that $S(a) \cap B$ is maximal, subject to that $N(a) \cap A$ is minimal, and subject to that $S(a) \cap A$ minimal.

(4) There do not exist $q < i_0 \leq r$ such that a is adjacent to a_q, $a_r \neq a$, and a_r is antiadjacent to a_q.

Suppose that such q, r exist. We claim that a_r is strongly complete to $N(a) \cap B$. Suppose $b \in N(a) \cap B$ is anti-adjacent to a_r. Since $q < i_0$ and a_q is adjacent to a, and $a_1, \ldots, a_n$ is a laminar ordering of A, it follows that b is strongly anti-adjacent to a_q. But now, a_q is adjacent to a, a_r is strongly anticomplete to $\{a, a_q\}$, and b is adjacent to a, and anticomplete to $\{a_q, a_r\}$, contrary to 6.2.1. This proves that a_r is strongly complete to $N(a) \cap B$. Now, since a is adjacent to a_q, and a_r is antiadjacent to a_q, it follows from the choice of a that some vertex of A is adjacent to a_r and antiadjacent to a, contrary to (3). This proves (4).

(5) We can order the vertices of A as $a'_1, \ldots, a'_n$, such that $a'_i = a_i$ for $i \in \{1, \ldots, i_0 - 1\}$ and so that

1. *for $i \in \{1, \ldots, n\}$ if a'_i is adjacent to $b \in B$, then b is strongly complete to $\{a'_{i+1}, \ldots, a'_n\}$, and*

2. *for $i, j \in \{1, \ldots, n\}$, with $i < j$, if a'_i is adjacent to a'_j, then a'_j*

is strongly complete to $\{a'_1, \ldots, a'_{i-1}\}$*, and* a'_i *is strongly complete to* $\{a'_{i+1}, \ldots, a'_{j-1}\}$.

The proof it by induction on $|A|$. Since $a_n \in A'$, it follows that $A' \neq \emptyset$. If $|A'| = 1$, then $A' = \{a_n\}$, and (5) follows from the fact that $a_1, \ldots, a_n$ is a laminar ordering. So we may assume that $A' \neq \{a_n\}$.

It follows from the inductive hypothesis that (5) holds for $A \setminus \{a\}$. Let j_0 be minimum such that a_{j_0} is strongly anticomplete to $\{a_{j_0+1}, \ldots, a_n\} \setminus \{a\}$, and let $A'' = \{a_{j_0}, \ldots, a_n\} \setminus \{a\}$. Then A'' is a strongly stable set.

We claim that $j_0 \geq i_0$. Suppose not. Then, by the minimality of i_0, it follows that a_{j_0} is adjacent to a. Since a_{j_0} is anticomplete to $\{a_{j_0+1}, \ldots, a_n\} \setminus \{a\}$, it follows that $a = a_{j_0+1}$. Since $a \in A'$, we deduce that $a = a_{i_0}$. Since $A' \neq \{a_n\}$, it follows that that $n \geq i_0 + 1$. But now a_{j_0} is adjacent to a and antiadjacent to $a_{i_0} + 1$, contrary to (4). This proves that $j_0 \geq i_0$.

Now it follows from the definition of i_0 that either $j_0 = i_0$, or $j_0 = i_0 + 1$ and $a = a_{i_0}$; and in both cases $A' = A'' \cup \{a\}$. Let $a'_1, \ldots, a'_{n-1}$ be an ordering of the vertices of $A \setminus \{a\}$ satisfying (5). Then $a_i = a'_i$ for $i \in \{1, \ldots, i_0 - 1\}$. Let $a'_n = a$.

To prove that the first condition of (5) is satisfied, it is enough to show that if $b \in B$ has a neighbor in $A \setminus \{a\}$, then b is strongly adjacent to a. Suppose that $b \in B$ is adjacent to a'_i for some $i \in \{1, \ldots, n-1\}$ and antiadjacent to a. Since $a_1, \ldots, a_n$ is a laminar ordering of A, it follows that $a'_i \in A'$. Now it follows from the choice of a that there exists $b' \in B \setminus \{b\}$ such that b' is adjacent to a and antiadjacent to a'_i, contrary to (2). Thus the first condition of (5) is satisfied.

Next we show that the second condition of (5) is satisfied. It is enough to show that if a is adjacent to a'_i for some $i \in \{1, \ldots, n-1\}$, then a is strongly complete to $\{a'_1, \ldots, a'_{i-1}\}$, and a'_i is strongly complete to $\{a'_{i+1}, \ldots, a'_{n-1}\}$. Since A' is a strongly stable set, it follows that $i < i_0$, and therefore $a'_i = a_i$. Consequently, the fact that (A, B) is a laminar pair implies that a is strongly complete to $\{a_1, \ldots, a_{i-1}\} = \{a'_1, \ldots, a'_{i-1}\}$, and that a_i is strongly complete to $\{a_{i+1}, \ldots, a_{i_0-1}\} = \{a'_{i+1}, \ldots, a'_{i_0-1}\}$. Also, by (4), a'_i is strongly complete to $A' \setminus \{a\} = \{a'_{i_0}, \ldots, a_{n-1}\}$. This proves that the second condition of (5) is satisfied, and completes the proof of (5).

By (5) and the symmetry, we can order the vertices of B as $b'_1, \ldots, b'_m$ so that

1. for $j \in \{1, \ldots, m\}$, if b'_j is adjacent to $a \in A$, then a is strongly complete to $\{b'_{j+1}, \ldots, b'_m\}$, and

2. for $i, j \in \{1, \ldots, m\}$, with $i < j$, if b'_i is adjacent to b'_j, then b'_j is strongly complete to $\{b'_1, \ldots, b'_{i-1}\}$, and b'_i is strongly complete to $\{b'_{i+1}, \ldots, b'_{j-1}\}$.

Therefore G is 1-thin with base (a_0, b_0). This proves 6.9. ∎

Before we describe all trigraphs with a doubly dominating homogeneous pair, we need two more preliminary results.

6.10 *Let G be a 1-thin trigraph with base (a_0, b_0) and let $x, y \in V(G) \setminus \{a_0, b_0\}$ be such that (x, y) is a doubly dominating semi-adjacent pair. Then (possibly exchanging the roles of x and y)*

- *$x \in A$, $y \in B$, and*
- *G is 1-thin with base (x, y), and*
- *G is 2-thin with base (a_0, b_0, x, y).*

Proof. Since G is 1-thin, $V(G) = A \cup B \cup \{a_0, b_0\}$ and the vertices of A can be numbered $a_1, \dots, a_n$ and the vertices of B can be numbered $b_1, \dots, b_m$ such that

- a_0 is strongly complete to A and strongly anticomplete to B.
- b_0 is strongly complete to B and strongly anticomplete to A.
- a_0 is semi-adjacent to b_0.
- If $i, j \in \{1, \dots, n\}$, and $i < j$, and a_i is adjacent to a_j, then a_i is strongly complete to $\{a_{i+1}, \dots, a_{j-1}\}$, and a_j is strongly complete to $\{a_1, \dots, a_{i-1}\}$.
- If $i, j \in \{1, \dots, m\}$, and $i < j$, and b_i is adjacent to b_j, then b_i is strongly complete to $\{b_{i+1}, \dots, b_{j-1}\}$, and b_j is strongly complete to $\{b_1, \dots, b_{i-1}\}$.
- If $p \in \{1, \dots, n\}$ and $q \in \{1, \dots, m\}$, and a_p is adjacent to b_q, then a_p is strongly complete to $\{b_{q+1}, \dots, b_m\}$, and b_q is strongly complete to $\{a_{p+1}, \dots, a_n\}$.

Since (x, y) is a doubly dominating semi-adjacent pair, it follows that (using symmetry) $x \in A$ and $y \in B$. This proves the first assertion of the theorem.

Let $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$ such that $a_i = x$ and $b_j = y$. Since a_i is not strongly adjacent to b_j, it follows that $\{b_1, \dots, b_j\}$ is strongly anticomplete to $\{a_1, \dots, a_{i-1}\}$, and since (a_i, b_j) is strongly dominating, it follows that a_i is strongly complete to $\{a_1, \dots, a_{i-1}\}$. Similarly, $\{a_1, \dots, a_i\}$. is strongly anticomplete to $\{b_1, \dots, b_{j-1}\}$, and b_j is strongly complete to $\{b_1, \dots, b_{j-1}\}$. Since a_i is adjacent to b_j, it follows that $\{b_j, \dots, b_m\}$ is strongly complete to $\{a_{i+1}, \dots, a_n\}$, and since (a_i, b_j) is doubly dominating, it follows that a_i is strongly anticomplete to $\{a_{i+1}, \dots, a_n\}$. Similarly, a_i is strongly complete to $\{b_{j+1}, \dots, b_m\}$, and b_j is strongly anticomplete to $\{b_{j+1}, \dots, b_m\}$.

Let $s = m - j + i$ and let $t = n - i + j$. Let $s_p = a_{i-p}$ if $p \in \{1, \ldots, i\}$ and let $s_p = b_{m-p+i+1}$ for $p \in \{i+1, \ldots, s\}$. Let $t_p = b_{j-p}$ for $p \in \{1, \ldots, j\}$ and let $t_p = a_{n-p+j+1}$ for $p \in \{j+1, \ldots, t\}$. Let $S = \{s_1, \ldots, s_s\}$ and $T = \{t_1, \ldots, t_t\}$. Then $S \cup T = V(G) \setminus \{x, y\}$, $S \cap T = \emptyset$, x is strongly complete to S, and y is strongly complete to T. Thus the first three conditions of the definition of a 1-thin trigraph are satisfied.

We observe that since a_i is strongly complete to $\{a_1, \ldots, a_{i-1}\}$, it follows that $\{s_1, \ldots, s_i\}$ is a strong clique, and since b_j is strongly anticomplete to $\{b_{j+1}, \ldots, b_m\}$, it follows that $\{s_{i+1}, \ldots, s_s\}$ is a strongly stable set. Similarly, $\{t_1, \ldots, t_j\}$ is a strong clique, and $\{t_{j+1}, \ldots, t_t\}$ is a strongly stable set.

Let us check that the last three conditions of the definition of a 1-thin trigraph are satisfied. Let $p, q \in \{1, \ldots, s\}$, such that $p < q$ and s_p is adjacent to s_q. We claim that s_p is strongly complete to $\{s_{p+1}, \ldots, s_{q-1}\}$, and s_q is strongly complete to $\{s_1, \ldots, s_{p-1}\}$. Since $\{s_{i+1}, \ldots, s_s\}$ is a strongly stable set, it follows that $p \leq i$. Since $\{s_1, \ldots, s_i\}$ is a strong clique, we may assume that $q > i$. Since $s_i = a_0$, it follows that $p < i$. Thus $s_p = a_{i-p}$ and $s_q = b_{m-q+i+1}$. It follows that s_q is strongly complete to $\{a_{i-p+1}, \ldots, a_n\}$, and in particular, s_q is strongly complete to $\{a_{i-p+1}, \ldots, a_{i-1}\} = \{s_1, \ldots, s_{p-1}\}$; and s_p is strongly complete to $\{b_{m-q+i+2}, \ldots, b_m\} = \{s_{q-1}, \ldots, s_{i+1}\}$. Since $\{s_1, \ldots, s_i\}$ is a strong clique, it follows that s_p is strongly complete to $\{s_i, \ldots, s_{p-1}\}$, and the claim holds.

Form the symmetry, if $p, q \in \{1, \ldots, t\}$, and $p < q$, and t_p is adjacent to t_q, then t_p is strongly complete to $\{t_{p+1}, \ldots, t_{q-1}\}$, and t_q is strongly complete to $\{t_1, \ldots, t_{p-1}\}$.

To check the last condition, let $p \in \{1, \ldots, s\}$ and $q \in \{1, \ldots, t\}$, such that s_p is adjacent to t_q. We claim that s_p is strongly complete to $\{t_{q+1}, \ldots, t_t\}$, and t_q is strongly complete to $\{s_{p+1}, \ldots, s_s\}$. Suppose first that $p < i$. Since $\{a_1, \ldots, a_i\}$ is strongly anticomplete to $\{b_0, b_1, \ldots, b_{j-1}\}$, it follows that $q > j$. Thus $s_p = a_{i-p}$ and $t_q = a_{n-q+j+1}$. Then t_q is strongly complete to $\{a_1, \ldots, a_{i-p-1}\} = \{s_{p+1}, \ldots, s_{i-1}\}$; and s_p is strongly complete to $\{a_{i-p+1}, \ldots, a_{n-q+j}\}$, and, in particular, s_p is strongly complete to $\{a_{i+1}, \ldots, a_{n-q+j}\} = \{t_{q+1}, \ldots, t_t\}$. Since a_0 is strongly complete to A, and since $\{a_i, \ldots, a_n\}$ is strongly complete to $\{b_{j+1}, \ldots, b_m\}$, it follows that t_q is strongly complete to $\{s_i, \ldots, s_s\}$, and the claim follows. Thus we may assume that $p \geq i$, and, from the symmetry, $q \geq j$. But $\{s_i, \ldots, s_s\} = \{a_0, b_{j+1}, \ldots, b_m\}$ is strongly complete to $\{t_{j+1}, \ldots, t_t\} = \{a_{i+1}, \ldots, a_n\}$, and $\{t_j, \ldots, t_t\} = \{b_0, a_{i+1}, \ldots, a_n\}$ is strongly complete to $\{s_{i+1}, \ldots, s_s\} = \{b_{j+1}, \ldots, b_m\}$, and again, the claim holds. Therefore G is a 1-thin trigraph with base (x, y). This proves the second assertion of the theorem.

Let $K = \{a_1, \ldots, a_{i-1}\}$, $X = \{a_{i+1}, \ldots, a_n\}$, $M = \{b_1, \ldots, b_{j-1}\}$ and $Y = \{b_{j+1}, \ldots, b_m\}$. Now K is strongly anticomplete to M, and X is strongly complete to Y, and since G is 1-thin with base (a_0, b_0) it follows that K and M are strong cliques, and X and Y are strongly stable sets. Since G is 1-thin with base a_0, b_0, it follows that the remaining conditions of the definition of a 2-thin trigraph are satisfied, and so G is 2-thin with base (a_0, b_0, x, y), and (X, Y, K, M) is the canonical partition of G with respect to (a_0, b_0, x, y). This proves the last assertion of the theorem and completes the proof of 6.10. ∎

6.11 *Let G be a trigraph, and assume that $V(G) = A \cup B \cup C \cup D$ where A, B, C, D are all non-empty and pairwise disjoint, A is strongly complete to C and strongly anticomplete to D, and B is strongly complete to D and strongly anticomplete to C. Let G_1 be the trigraph obtained from $G|(A \cup B)$ by adding two new vertices c, d such that c is strongly complete to A and strongly anticomplete to B, d is strongly complete to B and strongly anticomplete to A, and c is semi-adjacent to d. Let G_2 be the trigraph obtained from $G|(C \cup D)$ by adding two new vertices a, b such that a is strongly complete to C and strongly anticomplete to D, b is strongly complete to D and strongly anticomplete to C, and a is semi-adjacent to b. Assume that G does not admit a homogeneous set decomposition, and that every tame homogeneous pair in G is doubly dominating. Then for $i = 1, 2$ G_i does not admit a homogeneous set decomposition, and every tame homogeneous pair in G_i is doubly dominating.*

Proof. Suppose first that G_1 admits a homogeneous set decomposition, and let X be a homogeneous set in G_1 with $1 < |X| < |V(G_1)|$. Let $Y = (X \setminus \{c\}) \cup C$ if $c \in X$, and let $Y = X$ if $c \notin X$. Let $Z = (Y \setminus \{d\}) \cup D$ if $d \in Y$, and let $Z = Y$ if $d \notin Y$. Then Z is a homogeneous set in G, $|Z| \geq |X| > 1$, and $|V(G_1) \setminus X| \leq |V(G) \setminus Z|$. Thus G admits a homogeneous set decomposition, a contradiction. This proves that G_1, and similarly G_2, does not admit a homogeneous set decomposition.

Next suppose that there is a tame homogeneous pair (P, Q) in G_1 that is not doubly dominating. We observe that cd is a doubly dominating semi-adjacent pair in G_1. Let S be the set of vertices of $V(G_1) \setminus (P \cup Q)$ that are strongly complete to P and strongly anticomplete to Q, T be the set of vertices of $V(G_1) \setminus (P \cup Q)$ that are strongly complete to Q and strongly anticomplete to P, U be the set of vertices of $V(G_1) \setminus (P \cup Q)$ that are strongly complete to $P \cup Q$ and V be the set of vertices of $V(G_1) \setminus (P \cup Q)$ that are strongly anticomplete to $P \cup Q$. Since (P, Q) is a homogeneous pair in G_1, it follows that $V(G_1) = P \cup Q \cup S \cup T \cup U \cup V$. Since (P, Q) is not doubly dominating, it follows that $U \cup V \neq \emptyset$.

Suppose first that $c \in P$. If $d \in P$, then, since every vertex of $A \cup B$ is mixed on $\{c, d\}$, it follows that $V(G_1) \subseteq P \cup Q$, contrary to the fact that (P, Q) is tame. Since d is semi-adjacent to c, it follows that $d \in Q$. But now U is strongly complete to $\{c, d\}$ and V is strongly anticomplete to $\{c, d\}$, contrary to the fact that the semi-adjacent pair cd is doubly dominating in G_1. This proves that $c \notin P$, and so, from the symmetry, $\{c, d\} \cap (P \cup Q) = \emptyset$.

Since c is strongly complete to A and strongly anticomplete to B, it follows that either $P \subseteq A$ or $P \subseteq B$. Similarly, either $Q \subseteq A$ or $Q \subseteq B$. It follows that (P, Q) is a tame homogeneous pair in G. If $P \cup Q \subseteq A$, then D is strongly anticomplete to $P \cup Q$, contrary to the fact that every tame homogeneous pair in G is doubly dominating. Thus, from the symmetry, $P \subseteq A$ and $Q \subseteq B$. It follows that there exists a vertex $x \in (A \cup B) \cap (U \cup V)$. But then $x \in V(G)$, and, again (P, Q) is not doubly dominating in G, a contradiction. So every tame homogeneous pair in G_1, and similarly in G_2 is doubly dominating. This proves 6.11. ∎

We are now ready to prove the main theorem of this section.

Proof of 6.1 Suppose that 6.1 is false, and let G be a counterexample to 6.1 with $|V(G)|$ minimum. Then G does not admit a homogeneous set decomposition. Let (A, B) be a tame homogeneous pair in G. Let C be the set of vertices of G that are strongly complete to A and strongly anticomplete to B, and let D be the set of vertices of G that are strongly complete to B and strongly anticomplete to A. Since (A, B) is a doubly dominating homogeneous pair in G, it follows that $V(G) = A \cup B \cup C \cup D$. Since (A, B) is a tame homogeneous pair in G, it follows that $A \neq \emptyset$ and $B \neq \emptyset$, $|A \cup B| > 2$, and $|C \cup D| > 2$. Since G does not admit a homogeneous set decomposition, it follows that $C \neq \emptyset$ and $D \neq \emptyset$. Let G_1 be the trigraph obtained from $G|(A \cup B)$ by adding two new vertices c, d such that c is strongly complete to A and strongly anticomplete to B, d is strongly complete to B and strongly anticomplete to A, and c is semi-adjacent to d. Let G_2 be the trigraph obtained from $G|(C \cup D)$ by adding two new vertices a, b such that a is strongly complete to C and strongly anticomplete to D, b is strongly complete to D and strongly anticomplete to C, and a is semi-adjacent to b.

Let $i \in \{1, 2\}$. Then $|V(G_i)| < |V(G)|$. By 6.11, G_i does not admit a homogeneous set decomposition, and every tame homogeneous pair in G_i is doubly dominating. We claim that G_i belongs to $\mathcal{T}_2$. If there is a tame doubly dominating homogeneous pair in G_i, then the claim follows from the minimality of G. So we may assume that there is no tame doubly dominating homogeneous pair in G_i, and therefore G_i does not admit a homogeneous pair decomposition, and there is a doubly dominating semi-adjacent pair in G_i. If one of $G_i, \overline{G_i}$ is unfriendly, then the claim follows from 6.9 and 3.6, so we may assume not. Now by 5.5, one of $G_i, \overline{G_i}$ belongs to $\mathcal{T}_0$, but ab is a doubly dominating semi-adjacent pair in G_2 and cd is a doubly dominating semi-adjacent pair on G_1, a contradiction. This proves the claim.

Since G is obtained from G_1 and G_2 by composing along (a, b, c, d), 6.8 implies that $G \in \mathcal{T}_2$. This proves 6.1. ∎

7 Understanding other homogeneous pairs

In this section we study tame homogeneous pairs in elementary bull-free trigraphs. We remind the reader that homogeneous pairs of types zero, one and two are defined in Section 3. Let (A, B) be a tame homogeneous pair in G. Let C be the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, D the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to B and strongly anticomplete to A, E the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to $A \cup B$, and F the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly anticomplete to $A \cup B$. We say that (A, B) is *a homogeneous pair of type three in* G if

- A is a strongly stable set, and
- B is a strong clique, and
- C is not strongly anticomplete to F, and
- C is not strongly complete to E.

We observe that the pair (A, B) is a of type three in G if and only if (B, A) is of type three in $\overline{G}$.

Our goal is to prove the following:

7.1 *Let G be an elementary bull-free trigraph. Assume that G does not admit a homogeneous set decomposition. Let (A, B) be a tame homogeneous pair in G that is not doubly dominating. Then one of $G, \overline{G}$ admits a 1-join, or a homogeneous pair of type one, two or three.*

First, given a tame homogeneous pair (A, B), we study the behavior of $G \setminus (A \cup B)$.

7.2 *Let G be an elementary bull-free trigraph, and let (A, B) be a tame homogeneous pair in G. Let C be the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, D the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to B and strongly anticomplete to A, E the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to $A \cup B$, and F the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly anticomplete to $A \cup B$. Assume that $E \cup F \neq \emptyset$. Then either*

1. *G admits a homogeneous set decomposition, or*
2. *one of $G, \overline{G}$ admits a 1-join, or*
3. *(possibly with the roles of C and D switched) each of the sets C, D, F is non-empty, $E = \emptyset$, D is strongly anticomplete to F, and C is not strongly anticomplete to F, or*

4. *(possibly with the roles of C and D switched) each of the sets C, D, E is non-empty, $F = \emptyset$, D is strongly complete to E, and C is not strongly complete to E, or*

5. *both of the following two statements hold:*

    - *D is not strongly complete to E, or C is not strongly anticomplete to F, and*

    - *C is not strongly complete to E, or D is not strongly anticomplete to F.*

Proof. First we observe that G satisfies the hypotheses of 7.2 if and only if $\overline{G}$ does, and G satisfies the conclusions of 7.2 if and only if $\overline{G}$ does. Moreover, passing to $\overline{G}$ exchanges the roles of C and D, and the roles of E and F; we may assume that neither of $G, \overline{G}$ admits 1-join, and that G (and therefore $\overline{G}$) does not admit a homogeneous set decomposition.

(1) If $F \neq \emptyset$, then F is not strongly anticomplete to $C \cup D$.

Suppose $F \neq \emptyset$, and F is strongly anticomplete to $C \cup D$. Since G does not admit a homogeneous set decomposition, it follows that $E \neq \emptyset$, and there exist vertices $e \in E$ and $f \in F$ such that e is adjacent to f. Choose $a \in A$ and $b \in B$ adjacent. Since $\{f, e, b, a, c\}$ is not a bull for any $c \in C$, it follows that e is strongly complete to C, and similarly e is strongly complete to D. Let E_0 be the set of vertices of E with a neighbor in F. Then E_0 is strongly complete to $C \cup D$. Let E' be the union of anticomponents X of E such that $X \cap E_0 \neq \emptyset$. We claim that E' is strongly complete to $C \cup D$. First we observe that if e_1-e_2-e_3 is an antipath with $e_1 \in E_0$, $e_2 \in E \setminus E_0$ and $e_3 \in C \cup D \cup (E \setminus E_0)$, then, choosing $f_1 \in F$ adjacent to e_1, we get that one of $\{f_1, e_1, e_3, b, e_2\}$ and $\{f_1, e_1, e_3, a, e_2\}$ is a bull, a contradiction. So no such antipath e_1-e_2-e_3 exists. This implies that every vertex of $E' \setminus E_0$ has an antineighbor in E_0, and, consequently, that E' is strongly complete to $C \cup D$. But now, since $E \setminus E'$ is strongly complete to E' and strongly anticomplete to F, it follows that $X = A \cup B \cup C \cup D \cup (E \setminus E')$ is a homogeneous set in G, and $e, f \in V(G) \setminus X$, contrary to the fact that G does not admit a homogeneous set decomposition. This proves (1).

Passing to the complement if necessary, we may assume that $F \neq \emptyset$. By (1), we may assume that some vertex $c \in C$ is adjacent to some vertex $f \in F$. Now we may assume that C is strongly complete to E, and that D is strongly anticomplete to F, for otherwise the fifth outcome of 7.2 holds.

(2) If $E \neq \emptyset$, then 7.2 holds.

Suppose $E \neq \emptyset$. Since C is strongly complete to E, (1) applied in $\overline{G}$ implies that there exists a vertex $d \in D$ antiadjacent to a vertex $e \in E$. Passing to $\overline{G}$ if necessary, we may assume that f is antiadjacent to e. But now, choosing $a \in A$ and $b \in B$ antiadjacent, we observe that $\{f, c, a, e, b\}$ is a bull, a contradiction. This proves (2).

In view of (2) we may assume that $E = \emptyset$. Now, since G does not admit a 1-join, it follows that $D \neq \emptyset$, and the fourth outcome of 7.2 holds. This proves 7.2. ∎

Next we prove two useful lemmas about the structure of the sets A and B of a homogeneous pair (A, B).

7.3 *Let G be a bull-free trigraph, and let (A, B) be a homogeneous pair in G. Let C be the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, D the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, E the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to $A \cup B$, and F the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly anticomplete to $A \cup B$. Assume that G does not admit a homogeneous set decomposition. Then:*

1. *If some vertex of C is adjacent to some vertex of F, then A is strongly stable.*
2. *If some vertex of D is antiadjacent to some vertex of E, then A is a strong clique.*

Proof. Since the second assertion of 7.3 follows from the first by passing to $\overline{G}$, it is enough to prove the first assertion. Let $c \in C$ be adjacent to $f \in F$. Suppose A is not strongly stable, and let X be a component of A with $|X| > 1$. Since G does not admit a homogeneous set decomposition, it follows that some vertex $v \in V(G) \setminus X$ is mixed on X. Since (A, B) is a homogeneous pair in G, and X is a component of A, it follows that $v \in B$. By 2.2, there exist vertices $x, y \in X$ such that x is adjacent to y, and v is adjacent to x and antiadjacent to y. But now $\{v, x, y, c, f\}$ is a bull, a contradiction. This proves 7.3. ∎

7.4 *Let G be a bull-free trigraph, and let (A, B) be a homogeneous pair in G. Let C be the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, D the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to B and strongly anticomplete to A, and F the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly anticomplete to $A \cup B$. Assume that $V(G) = A \cup B \cup C \cup D \cup F$, and that G does not admit a homogeneous set decomposition. Suppose that each of the sets C, D, F is non-empty, D is strongly anticomplete to F, and C is not strongly anticomplete to F. Then (A, B) is a homogeneous pair of type two in G.*

Proof. The proof is by induction on $|B|$. Since C is not strongly anticomplete to F, 7.3 implies that A is strongly stable.

(1) There do not exist vertices $a, a' \in A$ and $b, b' \in B$ such that a is adjacent to b and antiadjacent to b', a' is adjacent to b' and antiadjacent to b, and b is adjacent to b'.

If such a, a', b, b' exist, then $\{a, b, d, b', a'\}$ is a bull for every $d \in D$, a contradiction. This proves (1).

In order to prove that (A, B) is a homogeneous pair of type two, it remains to show that

1. there exists a rooted forest $(T, r_1, \ldots, r_k)$ such that $G|B$ is the closure of $(T, r_1, \ldots, r_k)$, and
2. if $b, b' \in B$ are semi-adjacent, then, possibly with the roles of b and b' exchanged, b is a leaf of T and a child of b', and
3. if $a \in A$ is adjacent to $b \in B$, then a is strongly adjacent to every descendant of b in T, and
4. let $u, v \in B$ and assume that u is a child of v. Let $i \in \{1, \ldots, k\}$ and let T_i be the component of T such that $u, v \in V(T_i)$. Let P be the unique path of T_i from v to r_i, and let X be the component of $T_i \setminus (V(P) \setminus \{v\})$ containing v (and therefore u). Let Y be the set of vertices of X that are semi-adjacent to v. Let $a \in A$ be adjacent to u and antiadjacent to v. Then a is strongly complete Y and to $B \setminus (V(X) \cup V(P))$, and a is strongly anticomplete to $V(P) \setminus \{v\}$.

We will refer to the conditions above as $(F1), \ldots, (F4)$.

(2) If B is not anticonnected, then (A, B) is a homogeneous pair of type two in G.

Suppose B is not anticonnected, and let $B_1, \ldots, B_k$ be the anticomponents of B. Suppose that there exist two distinct integers $i, j \in \{1, \ldots, k\}$ such that $|B_i| > 1$ and $|B_j| > 1$. Since G does not admit a homogeneous set decomposition, and since (A, B) is a homogeneous pair in G, and B_i, B_j are anticomponents of B, it follows that there exist vertices $a_i, a_j \in A$ such that a_i is mixed on B_i, and a_j is mixed on B_j. By 6.2.2, it follows that a_i is strongly anticomplete to B_j, and a_j to B_i. Thus a_i and a_j are distinct. Let $b_i \in B_i$ be a neighbor of a_i, and let $b_j \in B_j$ be a neighbor of a_j. Then b_i is adjacent to b_j, contrary to (1). This proves that $|B_i| > 1$ for at most one value of i, and so we may assume that $|B_1| = \ldots = |B_{k-1}| = 1$.

Suppose $|B_k| = 1$. Then B is a strong clique, and so, by 6.4, the vertices of B can be ordered $b_1, \ldots, b_k$, so that if $a \in A$ is adjacent to b_i, then a

is strongly complete to $\{b_{i+1}, \ldots, b_k\}$. Let T be the path b_1-$\ldots$-b_k. Then (T, b_i) is a rooted forest, and $G|B$ is the closure of (T, b_1), thus $(F1)$ holds. Since B is a strong clique, $(F2)$ holds. By the choice of the order $b_1, \ldots, b_k$, $(F3)$ holds, and, consequently, since F is a path, $(F4)$ holds. Thus (A, B) is a homogeneous pair of type two, and so we may assume that $|B_k| > 1$.

First we claim that if $a \in A$ has neighbor $b_i \in B \setminus B_k$, then a is strongly complete to B_k. Suppose a has an antineighbor in B_k. Since B_k is anticonnected, 2.2 applied in $\overline{G}$ and 6.2.2 imply that a is strongly anticomplete to B_k. Since B_k is not a homogeneous set in G, and since (A, B) is a homogeneous pair and B_k is an anticomponent of B, it follows that there exists $a' \in A$ mixed on B_k. By 6.2.2, a' is strongly antiadjacent to b_i. Let $b_k \in B_k$ be adjacent to a'. Then b_i is adjacent to b_k, a is adjacent to b_i and antiadjacent to b_k, and a' is adjacent to b_k and antiadjacent to b_i, contrary to (1). This proves the claim.

Let A_0 be the set of vertices of A that are mixed on B_k, A_1 the set of vertices of A that are strongly complete to B_k, and A_2 the set of vertices of A that are strongly anticomplete to B_k. Then, by the claim, $A_0 \cup A_2$ is strongly anticomplete to $B \setminus B_k$. Let $D' = D \cup A_1 \cup (B \setminus B_k)$ and let $F' = F \cup A_2$. Then (A_0, B_k) is a homogeneous pair in G, C is the set of vertices of $V(G) \setminus (A_0 \cup B_k)$ that are strongly complete to A_0 and strongly anticomplete to B_k, D' is the set of vertices of $V(G) \setminus (A_0 \cup B_k)$ that are strongly complete to B_k and strongly anticomplete to A_0, F' is the set of vertices of $V(G) \setminus (A_0 \cup B_k)$ that are strongly anticomplete to $A_0 \cup B_k$, $V(G) = A_0 \cup B_k \cup C \cup D' \cup F'$, each of the sets C, D', F' is non-empty, D' is strongly anticomplete to F', and C is not strongly anticomplete to F'. Now, since G does not admit a homogeneous set decomposition, it follows from the inductive hypothesis that (A_0, B_k) is a homogeneous pair of type two in G. Consequently,

1. there exists a rooted forest $(T_0, r_1, \ldots, r_p)$ such that $G|B_k$ is the closure of $(T_0, r_1, \ldots, r_p)$, and

2. if $b, b' \in B_k$ are semi-adjacent, then, possibly with the roles of b and b' exchanged, b is a leaf of T_0 and a child of b', and

3. if $a \in A_0$ is adjacent to $b \in B_k$, then a is strongly adjacent to every descendant of b in T_0, and

4. let $u, v \in B_k$ and assume that u is a child of v. Let $j \in \{1, \ldots, s\}$ and let Q_j be the component of T_0 such that $u, v \in V(Q_j)$. Let P be the unique path of Q_j from v to r_j, and let X be the component of $Q_j \setminus (V(P) \setminus \{v\})$ containing v (and therefore u). Let Y be the set of vertices of X that are semi-adjacent to v. Let $a \in A_0$ be adjacent to u and antiadjacent to v. Then a is strongly complete Y and to $B_k \setminus (V(X) \cup V(P))$, and a is strongly anticomplete to $V(P) \setminus \{v\}$.

$B \setminus B_k$ is a strong clique complete to B_k. By 6.4, the vertices of $B \setminus B_k$ can be ordered $b_1, \dots, b_{k-1}$ such that if $a \in A$ is adjacent to b_i for some $i \in \{1, \dots, k-1\}$, then a is strongly complete to $\{b_{i+1}, \dots, b_{k-1}\}$. Let T be the tree with vertex set $V(T) = V(T_0) \cup \{b_1, \dots, b_{k-1}\}$, such that $T|V(T_0) = T_0$, for $i \in \{1, \dots, k-2\}$, b_i is strongly adjacent to b_{i+1}, b_{k-1} is strongly adjacent to $r_1, \dots, r_p$, and all other vertex pairs are strongly antiadjacent. Then $G|B$ is the closure of (T, b_1), and so $(F1)$ holds. If $b, b' \in B$ are semi-adjacent in G, then $b, b' \in B_k$, and so, possibly with the roles of b and b' exchanged, b is a leaf of T_0 and a child of b', thus $(F2)$ holds. Next we check that $(F3)$ holds. If $a \in A$ is adjacent to $b \in B_k$, then $a \in A_0 \cup A_1$, and so a is strongly adjacent to all the descendants of b in T_0 and therefore in T. Suppose $a \in A$ is adjacent to b_i for some $i \in \{1, \dots, k-1\}$. Then a is strongly complete to B_k, and, from the choice of the order $b_1, \dots, b_{k-1}$, to $\{b_{i+1}, \dots, b_{k-1}\}$. This proves that $(F3)$ holds.

To check that $(F4)$ holds, let $u, v \in T$ such that u is a child of v, and let $a \in A$ be adjacent to u and antiadjacent to v. If $u \in \{b_1, \dots, b_{k-1}\}$, then the assertion of $(F4)$ follows from the assertion of $(F3)$, so we may assume that $u \in B_k$. Let Q_j be the component of T_0 such that $u \in V(Q_j)$. We may assume without loss of generality that $j = 1$ and $r_1 \in V(Q_j)$. Then either $v \in V(Q_1)$, or $v = b_{k-1}$ and $u = r_1$. Suppose $v = b_{k-1}$ and $u = r_1$. Then no vertex of B is semi-adjacent to v. Let P be the path b_{k-1}-$\dots$-b_1 of T. Then P is the unique path of T from v to b_1. By the choice of the order $b_1, \dots, b_{k-1}$, it follows that a is strongly anticomplete to $V(P) \setminus \{v\}$. Since b_{k-1} is strongly complete to $\{r_1, \dots, r_p\}$, it follows that $B \setminus (V(P) \setminus \{v\})$ is the vertex set of the component of $T \setminus (V(P) \setminus \{v\})$ containing v, and so the assertion of $(F4)$ holds.

Thus we may assume that $u \neq r_1$, and $v \in Q_j$. Then $a \in A_0$. Let P' be the unique path of Q_j from v to r_1, and let P be the path v-P'-r_1-b_{k-1}-$\dots$-b_1. Then P is the unique path of T from v to b_1. Since $(F4)$ is satisfied for (A_0, B_k) and $(T_0, r_1, \dots, r_p)$, it follows that a is strongly anticomplete to $V(P') \setminus \{v\}$. Since a is antiadjacent to v, and v is a descendant of each of $b_1, \dots, b_{k-1}$, $(F3)$ implies that a is strongly anticomplete to $\{b_1, \dots, b_{k-1}\}$, and so a is strongly anticomplete to $V(P) \setminus \{v\}$. Let X be the component of $T \setminus (V(P) \setminus \{v\})$ containing v, and let Y be the set of vertices of X that are semi-adjacent to v. Then X is also the component of $T_0 \setminus (V(P') \setminus \{v\})$ containing v, and so a is strongly complete to Y and to $B_k \setminus (V(X) \cup V(P'))$. But $B \setminus (V(X) \cup V(P)) = B_k \setminus (V(X) \cup V(P'))$, and so the assertion of $(F4)$ holds. Thus (A, B) is a homogeneous pair of type two in G. This proves (2).

(3) If there exist $x, y \in B$ such that x is semi-adjacent to y, x is strongly complete to $B \setminus \{x, y\}$ and y is strongly anticomplete to $B \setminus \{x, y\}$, then (A, B) is a homogeneous pair of type two in G.

Suppose such x, y exist. Let $B_0 = B \setminus \{x, y\}$. Let A_0 be the set of vertices of A that are mixed on B_0. Let $a \in A_0$, and let $b_1 \in B_0$ be a neighbor of a, and $b_2 \in B_0$ an antineighbor of a. Then a is mixed on one of $\{x, b_1\}$, $\{x, b_2\}$, and so, by 6.2.1, a is strongly adjacent to y. Also, a is mixed on one of $\{y, b_1\}$, $\{y, b_2\}$, and so by 6.2.2, a is strongly antiadjacent to x. Thus y is strongly complete to A_0, and x is strongly anticomplete to A_0. Let A_1 be the set of vertices of A that are strongly complete to B_0, and let A_2 be the set of vertices of A that are strongly anticomplete to B_0.

Suppose first that $B_0 = \emptyset$. Then there is symmetry between x and y, and by (1), we may assume that every vertex of A that is adjacent to x is strongly adjacent to y. Now setting T be the tree with vertex set $\{x, y\}$ such that x is semi-adjacent to y, we observe that (A, B) with the rooted tree (T, x) satisfies $(F1)$—$(F4)$. Thus we may assume that $B_0 \neq \emptyset$.

Suppose that some vertex $a_2 \in A_2$ is adjacent to x. Then, by 6.2.1, a_2 is strongly adjacent to y, contrary to 6.2.2. Thus x is strongly anticomplete to A_2. By 6.2.2, if $a \in A_1$ is adjacent to x, then a is strongly adjacent to y.

Next suppose that $|B_0| = 1$. Let $B_0 = \{b_0\}$. Then $|A_0| \leq 1$, and if $A_0 \neq \emptyset$, then the unique vertex of A_0 is semi-adjacent to b_0. Now, setting T be the tree with vertex set $\{x, y, b_0\}$ where b_0 is strongly adjacent to x and strongly antiadjacent to y, and y is semi-adjacent to x, we observe that (A, B) with the rooted tree (T, x) satisfies $(F1)$—$(F4)$.

Thus we may assume that $|B_0| > 1$, and so, since G does not admit a homogeneous set decomposition, it follows that $A_0 \neq \emptyset$. Let $C' = C \cup \{y\}$, $D' = D \cup A_1 \cup \{x\}$, and $F' = F \cup A_2$. Then (A_0, B_0) is a homogeneous pair in G, C' is the set of vertices of $V(G) \setminus (A_0 \cup B_0)$ that are strongly complete to A_0 and strongly anticomplete to B_0, D' is the set of vertices of $V(G) \setminus (A_0 \cup B_0)$ that are strongly complete to B_0 and strongly anticomplete to A_0, F' is the set of vertices of $V(G) \setminus (A_0 \cup B_0)$ that are strongly anticomplete to $A_0 \cup B_0$, $V(G) = A_0 \cup B_0 \cup C' \cup D' \cup F'$, each of the sets C', D', F' is non-empty, D' is strongly anticomplete to F, and C' is not strongly anticomplete to F'. Since G does not admit a homogeneous set decomposition, it follows from the inductive hypothesis that (A_0, B_0) is a homogeneous pair of type two in G. Thus

1. there exists a rooted forest $(T_0, r_1, \ldots, r_k)$ such that $G|B_0$ is the closure of $(T_0, r_1, \ldots, r_k)$, and

2. if $b, b' \in B_0$ are semi-adjacent, then, possibly with the roles of b and b' exchanged, b is a leaf of T_0 and a child of b', and

3. if $a \in A_0$ is adjacent to $b \in B_0$, then a is strongly adjacent to every descendant of b in T_0, and

4. let $u, v \in B_0$ and assume that u is a child of v. Let $j \in \{1, \ldots, s\}$ and let Q_j be the component of T_0 such that $u, v \in V(Q_j)$. Let P be the unique path of Q_j from v to r_i, and let X be the component of $Q_j \setminus (V(P) \setminus \{v\})$ containing v (and therefore u). Let Y be the set

of vertices of X that are semi-adjacent to v. Let $a \in A_0$ be adjacent to u and antiadjacent to v. Then a is strongly complete Y and to $B_k \setminus (V(X) \cup V(P))$, and a is strongly anticomplete to $V(P) \setminus \{v\}$.

Let T be the tree with vertex set B such that $T|V(T_0) = T_0$, x is strongly adjacent to $r_1, \ldots, r_k$, y is semi-adjacent to x, and all other vertex pairs are strongly antiadjacent in T. Then (T, x) is a rooted forest. We claim that (A, B) and (T, x) satisfy $(F1)$—$(F4)$. Since $G|B$ is the closure of (T, x), it follows that $(F1)$ is satisfied. If $b, b' \in B$ are semi-adjacent, then either $\{b, b'\} = \{x, y\}$, or $b, b' \in B_0$; and in both cases, possibly with the roles of b and b' exchanged, b is a leaf of T and a child of b', and so $(F2)$ is satisfied. Next we check that $(F3)$ is satisfied. Suppose first that $a \in A$ is adjacent to x. Then $a \in A_1$, and so a is strongly complete to $B_0 \cup \{y\}$, which means that a is strongly adjacent to all the descendants of x. Next suppose that $a \in A$ is adjacent to a vertex $b \in B_0$. Then $a \in A_0 \cup A_1$. Since all descendants of b in T are descendants of b in T_0, and since A_1 is strongly complete to B_0, it follows that a is strongly adjacent to all descendants of b. But now, since y has no descendants in T, it follows that (A, B) and (T, x) satisfy $(F3)$. Finally, to check $(F4)$, let $u, v \in B$, such that u is a child of v in F, and a is adjacent to u and antiadjacent to v. Suppose first that $v \in B_0$. Then $u \in B_0$, and $a \in A_0$. Let Q_j be the component of T_0 such that $u, v \in V(Q_j)$. Let P' be the unique path of Q_j from v to r_j. Let P be the path v-P'-r_j-x. Then P is the unique path of T from v to x. Now a is strongly anticomplete to $V(P') \setminus \{v\}$, and a is strongly antiadjacent to x, and thus a is strongly anticomplete to $V(P) \setminus \{v\}$. Let X be the component of $T \setminus (V(P) \setminus \{v\})$ containing v, and let Y be the set of vertices of X that are semi-adjacent to v. Then X is also the component of $T_0 \setminus (V(P') \setminus \{v\})$ containing v, and so a is strongly complete to Y and $B_0 \setminus (V(X) \cup V(P'))$. Since $a \in A_0$, it follows that a is strongly adjacent to y, and so a is strongly complete to $B \setminus (V(X) \cup V(P))$, and $(F4)$ holds. So we may assume that $v \notin B_0$. Since y has no children in T, it follows that $v = x$, and $(F4)$ holds. Thus (A, B) is a homogeneous pair of type two in G. This proves (3).

Since D is strongly complete to B and C is strongly anticomplete to B, and since G is elementary, it follows that there is no path of length three in B. Thus, (2),(3) and 5.3 imply that B is not connected. Let $B_1, \ldots, B_k$ be the components of B. We may assume that there exists $t \in \{0, 1, \ldots, k\}$ such that $|B_i| = 1$ for $i > t$, and $|B_i| > 1$ for $i \leq t$ (thus if B is a strongly stable set, then $t = 0$). For $i > t$, let $B_i = \{b_i\}$.

Let $i \in \{1, \ldots, t\}$. Let A_0^i be the set of vertices of A that are mixed on B_i, A_1^i the set of vertices of A that are strongly complete to B_i, and A_2^i the set of vertices of A that are strongly anticomplete to B_i. By 6.2.1, A_0^i is strongly complete to $B \setminus B_i$. Let $C_i = C \cup (B \setminus B_i)$, $D_i = D \cup A_1^i$ and $F_i = F \cup A_2^i$. Then (A_0^i, B_i) is a homogeneous pair in G, C_i is the set of vertices of $V(G) \setminus (A_0^i \cup B_i)$ that are strongly complete to A_0^i and strongly anticomplete to B_i, D_i is the set of vertices of $V(G) \setminus (A_0^i \cup B_i)$ that are strongly complete to B_i and strongly anticomplete to A_0^i, F_i is the set of vertices of $V(G) \setminus (A_0^i \cup B_i)$ that are strongly anticomplete to $A_0^i \cup B_i$, $V(G) = A_0^i \cup B_i \cup C_i \cup D_i \cup F_i$, each of the sets C_i, D_i, F_i is non-empty, D_i is strongly anticomplete to F_i, and C_i is not strongly anticomplete to F_i. Then, since G does not admit a homogeneous set decomposition, it follows from the inductive hypothesis that (A_0^i, B_i) is a homogeneous pair of type two in G. Thus

1. there exists a rooted forest $(T_i, r_1^i, \dots, r_{k_i}^i)$ such that $G|B_i$ is the closure of $(T_i, r_1^i, \dots, r_{k_i}^i)$, and

2. if $b, b' \in B_i$ are semi-adjacent, then, possibly with the roles of b and b' exchanged, b is a leaf of T_i and a child of b', and

3. if $a \in A_0^i$ is adjacent to $b \in B_i$, then a is strongly adjacent to every descendant of b in T_i, and

4. let $u, v \in B_i$ and assume that u is a child of v. Let $j \in \{1, \dots, s\}$ and let Q_j be the component of T_i such that $u, v \in V(Q_j)$. Let P be the unique path of Q_j from v to r_j, and let X be the component of $Q_j \setminus (V(P) \setminus \{v\})$ containing v (and therefore u). Let Y be the set of vertices of X that are semi-adjacent to v. Let $a \in A_0^i$ be adjacent to u and antiadjacent to v. Then a is strongly complete Y and to $B_i \setminus (V(X) \cup V(P))$, and a is strongly anticomplete to $V(P) \setminus \{v\}$.

Since B_i is connected, and $G|B_i$ is the closure of $(T_i, r_1^i, \dots, r_{k_i}^i)$, it follows that T_i is connected, and has a unique root, say b_i.

Let T be the forest with vertex set B, such that $T|B_i = F_i$ for $i \in \{1, \dots, t\}$, all other vertex pairs of T are strongly antiadjacent. Then $(T, b_1 \dots, b_k)$ is a rooted forest. We claim that (A, B) and $(T, b_1, \dots, b_k)$ satisfy $(F1)$—$F(4)$. Since $G|B_i$ is the closure of (T_i, b_i) for every $i \in \{1, \dots, t\}$, it follows that $G|B$ is the closure of $(T, b_1, \dots, b_k)$, and so $(F1)$ is satisfied. $(F2)$ is satisfied, since if b, b' are semi-adjacent then both b and b' belong to B_i for some $i \in \{1, \dots, t\}$. Since for every $i \in \{1, \dots, k\}$ and $b \in B_i$, all the descendants of b in T belong B_i, and since if $a \in A$ has a neighbor in B_i, then $a \in A_0^i \cup A_1^i$, the fact that $(F3)$ is satisfied for (A_0^i, B_i) and (T_i, b_i) implies that (A, B) and $(T, b_1, \dots, b_k)$ satisfy $(F3)$. Finally, to check $(F4)$ let $u, v \in B$ such that u is a child of v, and suppose that $a \in A$ is adjacent to u and antiadjacent to v. Then there exists $i \in \{1, \dots, t\}$ such that $u, v \in B_i$, $a \in A_0^i$ and T_i is the component of T containing v. Let P be the unique path of T_i from v to b_i. Then, P is the unique path of T from v to b_i, and since (A_0^i, B_i) and (T_i, b_i) satisfy $(F4)$, it follows that a is strongly anticomplete to $V(P) \setminus \{v\}$. Let X be the component of $T_i \setminus (V(P) \setminus \{v\})$ containing v, and let Y be the set of vertices of X that are semi-adjacent to v. Then X is the component of $T \setminus (V(P) \setminus \{v\})$ containing v. Since (A_0^i, B_i)

and (T_i, b_i) satisfy $(F4)$, it follows that a is strongly complete to Y and to $B_i \setminus (V(C) \cup V(P))$, and since $a \in A_0^i$, it follows that a is strongly complete to $B \setminus B_i$. Thus (A, B) and $(T, b_1, \ldots, b_k)$ satisfy $(F4)$, and so (A, B) is a homogeneous pair of type two in G. This proves 7.4. ∎

We can now prove 7.1

Proof of 7.1. Let C be the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, D the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to B and strongly anticomplete to A, E the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to $A \cup B$, and F the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly anticomplete to $A \cup B$. We may assume that neither of $G, \overline{G}$ admits a 1-join. Since G does not admit a homogeneous set decomposition, it follows that one of the last three outcomes of 7.2 holds. Passing to $\overline{G}$ if necessary, we may assume that $F \neq \emptyset$ and C is not strongly anticomplete to F. Since $F \neq \emptyset$, we deduce that either the third, or the fifth outcome of 7.2 holds. If the third outcome of 7.2 holds, then by 7.4 G admits a homogeneous pair of type two, so we may assume that the fifth outcome of 7.2 holds. Since C is not strongly anticomplete to F, 7.2 implies that either C is not strongly complete to E, or D is not strongly anticomplete to F.

Since C is not strongly anticomplete to F, 7.3 implies that A is a strongly stable set. If C is not strongly complete to E, then, by 7.3 applied in $\overline{G}$, we deduce that B is a strong clique and (A, B) is a homogeneous pair of type three in G. So we may assume that D is not strongly anticomplete to F. But then, again by 7.3, B is a strongly stable set, and (A, B) is a homogeneous pair of type one in G. This proves 7.1. ∎

8 Dealing with homogeneous pairs of type three

Let us first summarize what we know about the structure of elementary bull-free trigraphs so far:

8.1 *Let G be an elementary bull-free trigraph. Then either*

- *one of $G, \overline{G}$ belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$, or*
- *G admits a homogeneous set decomposition, or*
- *one of $G, \overline{G}$ admits a 1-join, or*
- *one of $G, \overline{G}$ admits a homogeneous pair decomposition of type one, two or three.*

Proof. By 3.8, one of the following holds:

- one of $G, \overline{G}$ belongs to $\mathcal{T}_1$, or
- G admits a homogeneous set decomposition, or

- G admits a homogeneous pair decomposition.

We may assume that G admits a homogeneous pair decomposition, for otherwise one of the outcomes of 8.1 holds. Thus there is a tame homogeneous pair in G. If every tame homogeneous pair in G is doubly dominating, then by 6.1, either G admits a homogeneous set decomposition, or one of $G, \overline{G}$ belongs to $\mathcal{T}_2$, and again 8.1 holds. Thus we may assume that there is a tame homogeneous pair in G which is not doubly dominating. Now, by 7.1, one of $G, \overline{G}$ admits a 1-join, or a homogeneous pair of type one, two or three. This proves 8.1. ∎

In this section we prove that one of the outcomes of 8.1, namely a homogeneous pair decomposition of type three, is unnecessary. We prove the following:

8.2 *Let G be an elementary bull-free trigraph. Then either*

- *one of $G, \overline{G}$ belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$, or*
- *G admits a homogeneous set decomposition, or*
- *one of $G, \overline{G}$ admits a 1-join, or*
- *one of $G, \overline{G}$ admits a homogeneous pair decomposition of type one or two.*

Proof. Suppose 8.2 is false, and let G be a counterexample to 8.2 with $|V(G)|$ minimum. It follows from 8.1 that one of $G, \overline{G}$ admits a homogeneous pair decomposition of type three, and therefore both G and $\overline{G}$ admit a homogeneous pair decomposition of type three. Let (P, Q) be a tame homogeneous pair of type three in G (and so (Q, P) is a homogeneous pair of type three in $\overline{G}$). Let C be the set of vertices of $V(G) \setminus (P \cup Q)$ that are strongly complete to P and strongly anticomplete to Q, D the set of vertices of $V(G) \setminus (P \cup Q)$ that are strongly complete to Q and strongly anticomplete to P, E the set of vertices of $V(G) \setminus (P \cup Q)$ that are strongly complete to $P \cup Q$, and F the set of vertices of $V(G) \setminus (P \cup Q)$ that are strongly anticomplete to $P \cup Q$. Since (P, Q) is of type three, it follows that

- P is strongly stable, and
- Q is a strong clique, and
- C is not strongly anticomplete to F, and
- C is not strongly anticomplete to E.

Let G' be the trigraph obtained from $G \setminus (P \cup Q)$ by adding two new vertices a and b such that a is strongly complete to $C \cup E$ and strongly anticomplete to $D \cup F$, b is strongly complete to $D \cup E$ and strongly anticomplete to $C \cup F$, and a is semi-adjacent to b. Then G' is an elementary bull-free trigraph. From the minimality of $|V(G)|$, it follows that one of the outcomes of 8.2 holds for G'. Since so far we have preserved the symmetry between G and $\overline{G}$, we may assume that either:

- $G' \in \mathcal{T}_1 \cup \mathcal{T}_2$, or
- G' admits a homogeneous set decomposition, or
- G' admits a 1-join, or
- G' admits a homogeneous pair decomposition of type one or two.

For a $X \subseteq V(G')$, we define the *lift of X to G*, $L(X)$, as follows:

$$L(X) = \begin{cases} X & if \quad a, b \notin X \\ (X \setminus a) \cup P & if \quad a \in X \; and \; b \notin X \\ (X \setminus b) \cup Q & if \quad b \in X \; and \; a \notin X \\ (X \setminus \{a, b\}) \cup P \cup Q & if \quad a, b \in X \end{cases}$$

Thus if X is a homogeneous set in G' with $1 < |X| < |V(G')|$, then then $L(X)$ is homogeneous set in G, and $1 < |L(X)| < |V(G)|$, and so G admits a homogeneous set decomposition, a contradiction. This proves that G' does not admit a homogeneous set decomposition. Also, if (M, N) is a tame homogeneous pair G', then, since a is semi-adjacent to b in G', it follows that either $\{a, b\} \subseteq M \cup N$, or $\{a, b\} \cap (M \cup N) = \emptyset$, and $(L(M), L(N))$ is a tame homogeneous pair in G. Moreover, if (M, N) is a homogeneous pair of type one in G', then, by 7.3, $(L(M), L(N))$ is a homogeneous pair of type one in G, and if (M, N) is a homogeneous pair of type two in G', then, by 7.4, $(L(M), L(N))$ is a homogeneous pair of type two in G, in both cases a contradiction to the fact that G is a counterexample to 8.2. Thus G' does not admit a homogeneous pair decomposition of type one or two.

Next suppose that G' admits a 1-join. Then $V(G')$ is the disjoint union of four non-empty sets M, N, R, S such that

- N is strongly complete to R, M is strongly anticomplete to $R \cup S$, and N is strongly anticomplete to S;
- $|M \cup N| > 2$ and $|R \cup S| > 2$, and
- M is not strongly complete and not strongly anticomplete to N, and
- R is not strongly complete and not strongly anticomplete to S.

Since a is semi-adjacent to b in G', we may assume that $a, b \in M \cup N$. But now $V(G) = L(M) \cup L(N) \cup R \cup S$ and

- $L(N)$ is strongly complete to R, $L(M)$ is strongly anticomplete to $R \cup S$, and $L(N)$ is strongly anticomplete to S;

- $|L(M) \cup L(N)| > 2$ and $|R \cup S| > 2$, and
- $L(M)$ is not strongly complete and not strongly anticomplete to $L(N)$, and
- R is not strongly complete and not strongly anticomplete to S.

Thus G admits a 1-join, a contradiction. This proves that G' does not admit a 1-join, and therefore $G' \in \mathcal{T}_1 \cup \mathcal{T}_2$.

(1) The vertices of P, Q can be ordered as $p_1, \dots, p_{|P|}$ and $q_1, \dots, q_{|Q|}$ such that if p_i is adjacent to q_j, then p_i is strongly complete to $\{q_{j+1}, \dots, q_{|Q|}\}$, and q_j is strongly complete to $\{p_{i+1}, \dots, p_{|P|}\}$.

Since P is a strongly stable set and Q is a strong clique, (1) follows from 6.4 applied in G and in $\overline{G}$. This proves (1).

Suppose $G' \in \mathcal{T}_1$. Then either there is a loopless graph H with $maxdeg(H) \leq 2$, such that G' admits an H-structure, or G' is a double melt. If G' admits an H-structure, we use the notation from the definition of an H-structure. Let $c_0 \in C$ and $e_0 \in E$ be antiadjacent.

(2) G' is not a double melt, and $\{a, b\} \cap (h(e) \cup h(e, v)) = \emptyset$ for every $e \in E(H)$ and $v \in V(H)$.

Let $\{x, y\} = \{a, b\}$ and suppose that either G' is a double melt or $x \in h(e) \cup h(e, v) \cup h(e, u)$ for some $e \in E(H)$ with ends u, v. In the latter case, since y is semi-adjacent to x, it follows that $y \in h(e) \cup h(e, v) \cup h(e, u)$. If G' is a double melt, let A, B, K, M be as in the definition of a double melt. If G' admits an H-structure, recall that $G'|(h(e) \cup h(e, v) \cup h(e, u))$ is an $h(e)$-melt, and let (K, M, A, B) be as in the definition of a melt, such that $K \subseteq h(e, v)$, $M \subseteq h(e, u)$, and either $h(e) = A$ or $h(e) = B$.

Since x is semi-adjacent to y, and K is strongly antiadjacent to M, we may assume that $x \in A$. From the symmetry, and since x is semi-adjacent to y and A is strongly stable, we may assume that $y \in K \cup M \cup B$. Assume first that $y \in B$. Now there is symmetry between x and y, and we may assume that if G' admits an H-structure, then $A = h(e)$. Since x is semi-adjacent to y, it follows from the definition of a melt that one of x and y is strongly anticomplete to K, and the other one is strongly anticomplete to M. We may assume that x is strongly anticomplete to M, and y to K. Thus no vertex of $A \cup B \cup K \cup M$ is adjacent to both x and y. But e_0 is strongly adjacent to both x, y, and so $V(G') \neq A \cup B \cup K \cup M$ and G' admits an H-structure. It follows that $y \in h(e, u)$. But now $e_0 \in V(G) \setminus (h(e) \cup h(e, v) \cup h(e, u))$ has both a neighbor in $h(e)$ and a neighbor in $h(e, u)$, a contradiction. This proves that $y \notin B$, and therefore $y \in K \cup M$, and from the symmetry we may assume that $y \in K$. Thus if G' admits an H-structure, then $y \in h(e,v)$ and $x \in h(e) \cup h(e,v)$.

If $x = a$ and $y = b$, then using (1), if G' is a double melt then so is G, and if G' admits an H-structure, then setting $h'(z) = L(h(z))$ for every $z \in V(H) \cup E(H) \cup (E(H) \times V(H))$, we observe that $G|(L(h(e)) \cup L(h(e,v)) \cup h(e,u))$ is an $L(h(e))$-melt, and G admits and H-structure, contrary to the fact that $G \notin \mathcal{T}_1$. This proves that $x = b$ and $y = a$.

We claim that $e_0 \in A \cup B \cup K \cup M$. This is clear if G' is a double melt, so we may assume that G' admits an H-structure. If $x \in h(e)$, then, since no vertex of $V(G) \setminus (h(e) \cup h(e,v) \cup h(e,u))$ has both a neighbor in $h(e)$ and a neighbor in $h(e,v)$, it follows that $e_0 \in A \cup B \cup K \cup M$. Thus, we may assume that $x \in h(e,v)$, and so $B = h(e)$. Since c_0 is adjacent to a and antiadjacent to b, it follows that $c_0 \in h(e) \cup h(e,v)$, and the definition of a melt implies that $c_0 \notin h(e)$. Thus $c_0 \in h(e,v)$. Now, since e_0 is adjacent to a and antiadjacent to c_0, it follows that $e_0 \in h(e) \cup h(e,v) \subseteq A \cup B \cup K$. This proves that claim.

Since K is strongly anticomplete to M and A is a strongly stable set, we deduce that $e_0 \in K \cup B$, and thus, if G' admits an H-structure, then $e_0 \in h(e) \cup h(e,v)$. Since c_0 is adjacent to a, it follows that $c_0 \in A \cup B \cup K$. Suppose $e_0 \in K$. Then, since c_0 is antiadjacent to e_0, we deduce that $c_0 \in A \cup B$. Since c_0 is antiadjacent to b and adjacent to a, it follows from the definition of a melt that $c_0 \notin B$. But then $c_0 \in A$, and c_0 is adjacent to a and antiadjacent to e_0, and b is adjacent to e_0 and antiadjacent to a, again contrary to the definition of a melt. Thus $e_0 \in B$. Since c_0 is antiadjacent to b and adjacent to a, the definition of a melt implies that $c_0 \notin B$, and, similarly, since c_0 is antiadjacent to e_0, it follows that $c_0 \notin A$. Thus $c_0 \in K$. But a is adjacent to both e_0 and b, and c_0 is antiadjacent to both e_0 and b, contrary to the definition of a melt. This proves (2).

By (2), G' admits an H-structure.

(3) $a \notin h(v)$ for any $v \in V(H)$.

Suppose that $a \in h(v)$ for some $v \in V(H)$. Then we may assume that $b \in A_v$. Since e_0 is strongly adjacent to both a and b, it follows that $e_0 \in B_v \cup h(v) \cup (\bigcup_{e \in E(H)} h(e,v))$. Since c_0 is adjacent to a, it follows that $c_0 \in h(v) \cup (\bigcup_{e \in E(H)} h(e,v)) \cup A_v \cup B_v$. Suppose $e_0 \in B_v$. Then, by the definition of an H-structure and since $G|(h(v) \cup S_v \cup T_v)$ is an $(h(v), A_v, B_v, C_v, D_v)$-clique connector, and since c_0 is antiadjacent to both e_0 and b, it follows that $c_0 \notin h(v) \cup (\bigcup_{e \in E(H)} h(e,v))$. So $c_0 \in A_v \cup B_v$, and, since b, c_0, e_0 are all adjacent to a, it follows (from the fact that $G|(h(v) \cup S_v \cup T_v)$ is an $(h(v), A_v, B_v, C_v, D_v)$-clique connector) that c_0 is strongly adjacent to one of b, e_0, a contradiction. This proves that $e_0 \notin B_v$. Next suppose that $e_0 \in h(v)$. Since e_0 is antiadjacent to c_0, it follows that

$c_0 \in A_v \cup B_v$. Since c_0 is strongly antiadjacent to b and strongly adjacent to a, it follows that $c_0 \notin B_v$, and so $c_0 \in A_v$. But now both b, c_0 are in A_v, the pairs c_0a and be_0 are adjacent, and the pairs c_0e_0 and ba are antiadjacent, contrary to the fact that $G|(h(v) \cup S_v \cup T_v)$ is an $(h(v), A_v, B_v, C_v, D_v)$-clique connector. This proves that $e_0 \notin h(v)$. Thus $e_0 \in h(e, v))$ for some edge $e \in E(H)$ incident with v. Since e_0 is strongly adjacent to b, it follows that $h(e, v)$ is strongly complete to A_v. Since c_0 is antiadjacent to e_0, it follows that $c_0 \in B_v$. But now, since $G|(h(v) \cup S_v \cup T_v)$ is an $(h(v), A_v, B_v, C_v, D_v)$-clique connector, it follows that c is strongly adjacent to b, a contradiction. This proves (3).

(4) $b \notin h(v)$ for any $v \in V(H)$.

Suppose $b \in h(v)$ for some $v \in V(H)$. We may assume that $a \in A_v$. Now, setting $h'(x) = L(h(x))$ for every $x \in V(H) \cup E(H) \cup (E(H) \times V(H))$ we observe (using (1)) that $G|(L(h(v) \cup L(A_v) \cup B_v \cup C_v \cup D_v)$ is an $(L(h(v)), L(A_v), B_v, C_v, D_v)$-clique connector, and G admits an H-structure, contrary to the fact that $G \notin \mathcal{T}_2$. This proves (4).

Since $\{a, b, e_0\}$ is a triangle, it follows that one of a, b, e_0 belongs to $h(v) \cup h(e, v)$ for some $e \in E(H)$ and $v \in V(H)$. By (2),(3) and (4), it follows that $e_0 \in h(v) \cup h(e, v)$.

Suppose first that $e_0 \in h(e, v)$ for some $e \in E(H)$ and $v \in V(H)$. Then v is an end of e. We may assume that $h(e, v)$ is strongly complete to A_v and strongly anticomplete to B_v. Since e_0 is strongly complete to $\{a, b\}$, it follows from (2),(3) and (4) that both a and b belong to A_v. But A_v is a strongly stable set, a contradiction. This proves that $e_0 \in h(v)$. Then, since a is semi-adjacent to b, and they are both strongly adjacent to e_0, it follows that (possibly switching the roles of A_v and B_v) $a \in A_v$ and $b \in B_v$. Since a is not strongly adjacent to b, and since $G'|(h(v) \cup S_v \cup T_v)$ is a clique-connector, it follows that $h(v) = \{e_0\}$, $A_v = \{a\}$ and $B_v = \{b\}$. Since a is semi-adjacent to b and e_0 is adjacent to both of a, b, it follows that v has degree zero in H. This implies that $C \cup D \cup F$ is strongly antiadjacent to e_0 and $E = \{e_0\}$. Suppose first that $D \neq \emptyset$. By 7.3, P is a strong clique, and therefore $|P| = 1$. Since (P, Q) is a tame homogeneous pair in G, we deduce that $|Q| > 1$. It follows from 7.3 that D is strongly anticomplete to F. Since G does not admit a homogeneous set decomposition, there exist $p_1 \in P$ and $q_1, q_2 \in Q$, such that p_1 is adjacent to q_1 and antiadjacent to q_2. But now p_1-e_0-q_2-d is a path, q_1 is a center for it, and every vertex of F is an anticenter for it, contrary to the fact that G is elementary. This proves that $D = \emptyset$. Now setting $h'(x) = h(x)$ for every $x \in (V(H) \cup E(H) \cup E(H) \times V(H)) \setminus \{v\}$, and $h'(v) = Q \cup \{e_0\}$ we observe that $G|(h'(v) \cup A \cup \{e_0\} \cup C)$ is an $(h'(v), A, \{e_0\}, C, \emptyset)$-clique connector, and G admits an H-structure. Therefore $G \in \mathcal{T}_1$, a contradiction. This proves that $G' \notin \mathcal{T}_1$.

Thus $G' \in \mathcal{T}_2$. Consequently, there exists a skeleton G'_0, such that either $G' = G'_0$ or for $i \in \{1, \ldots, k\}$, (a_i, b_i) is a doubly dominating semi-adjacent pair in G'_0, and G'_i is a trigraph such that

- $V(G'_i) = A_i \cup B_i \cup \{a'_i, b'_i\}$, and
- the sets $A_i, B_i, \{a'_i, b'_i\}$ are all non-empty and pairwise disjoint, and
- a'_i is strongly complete to A_i and strongly anticomplete to B_i, and
- b'_i is strongly complete to B_i and strongly anticomplete to A_i, and
- a'_i is semi-adjacent to b'_i, and either
  - both A_i, B_i are strong cliques, and there do not exist $a' \in A_i$ and $b' \in B_i$, such that a' is strongly anticomplete to $B_i \setminus \{b'\}$, b' is strongly anticomplete to $A_i \setminus \{a'\}$, and a' is semi-adjacent to b', or
  - both A_i, B_i are strongly stable sets, and there do not exist $a' \in A_i$ and $b' \in B_i$, such that a' is strongly complete to $B_i \setminus \{b'\}$, b' is strongly complete to $A_i \setminus \{a'\}$, and a' is semi-adjacent to b', or
  - one of $G'_i, \overline{G'_i}$ is a 1-thin trigraph with base (a'_i, b'_i), and G'_i is not a 2-thin trigraph

and G' is obtained from $G'_0, G'_1, \ldots G'_k$ as in the definition of the class $\mathcal{T}_2$. Since (a, b) is a semi-adjacent pair of G, it follows that (a, b) is a semi-adjacent pair of G'_i for some $i \in \{0, \ldots, k\}$.

Suppose first that $i = 0$ and (a, b) is a semi-adjacent pair in G'_0. Then for some integer $t \geq 1$ there exist trigraphs $F_1, \ldots, F_t$, each of which is a triad pattern, a triangle pattern or 2-thin, and G'_0 is obtained from $F_1, \ldots F_t$ as in the definition of a skeleton. Since (a, b) is semi-adjacent pair in G'_0, it follows that (a, b) is a semi-adjacent pair in F_j for some $j \in \{1, \ldots, t\}$; and since (a, b) is not doubly dominating, we deduce that F_j is a 2-thin trigraph. Let $A, B, K, M, x_{AK}, x_{AM}, x_{BK}, x_{BM}$ be as in the definition of a 2-thin trigraph. Let $\{x, y\} = \{a, b\}$. Since (a, b) is a semi-adjacent pair of F_i, and (a, b) is not doubly dominating in G', we may assume from the symmetry that $x \in A$ and $y \in K$.

Since G'_0 is obtained from $F_1, \ldots F_t$ by repeatedly composing along doubly dominating semi-adjacent pairs, it follows that $A \cup B \cup K \cup M \subseteq V(G'_0)$, and there exist non-empty pairwise-disjoint subsets $X_{AK}, X_{AM}, X_{BK}, X_{BM}$ of $V(G'_0) \setminus (A \cup B \cup K \cup M)$ such that

- A is strongly complete to $X_{AK} \cup X_{AM}$ and strongly anticomplete to $X_{BK} \cup X_{BM}$
- B is strongly complete to $X_{BK} \cup X_{BM}$ and strongly anticomplete to $X_{AK} \cup X_{AM}$

- K is strongly complete to $X_{AK} \cup X_{BK}$ and strongly anticomplete to $X_{AM} \cup X_{BM}$
- M is strongly complete to $X_{AM} \cup X_{BM}$ and strongly anticomplete to $X_{AK} \cup X_{BK}$
- X_{AK} is strongly complete to X_{BK} and strongly anticomplete to X_{AM}
- X_{BM} is strongly complete to X_{AM} and strongly anticomplete to X_{BK}.

We claim that $x = a$ and $y = b$. Suppose not, then $x = b$ and $y = a$. Then $X_{AK} \subseteq E$, $X_{AM} \subseteq D$, and $X_{BM} \subseteq F$. Thus D is not strongly complete to E and not strongly anticomplete to F, and by 7.3, P is a strong clique and Q is a strongly stable set. Since (P, Q) is a homogeneous pair of type three, it follows that P is a strongly stable set and Q is a strong clique, and so $|P| = |Q| = 1$, contrary to the fact that (P, Q) is tame. This proves the claim that $x = a$ and $y = b$.

Let F'_j be the trigraph obtained from $G|(L(A) \cup B \cup L(K) \cup M)$ by adding four new vertices $x_{AK}, x_{AM}, x_{BK}, x_{BM}$ such that

- $L(A)$ is strongly complete to $\{x_{AK}, x_{AM}\}$ and strongly anticomplete to $\{x_{BK}, x_{BM}\}$
- B is strongly complete to $\{x_{BK}, x_{BM}\}$ and strongly anticomplete to $\{x_{AK}, x_{AM}\}$
- $L(K)$ is strongly complete to $\{x_{AK}, x_{BK}\}$ and strongly anticomplete to $\{x_{AM}, x_{BM}\}$
- M is strongly complete to $\{x_{AM}, x_{BM}\}$ and strongly anticomplete to $\{x_{AK}, x_{BK}\}$
- x_{AK} is semi-adjacent to x_{BM}
- x_{AM} is semi-adjacent to x_{BK}
- the pairs $x_{AK}x_{BK}$ and $x_{AM}x_{BM}$ are strongly adjacent, and the pairs $x_{AK}x_{AM}$ and $x_{BK}x_{BM}$ are strongly antiadjacent.

It is now easy to see, using (1), that F'_j is a 2-thin trigraph with base $(x_{AK}, x_{BM}, x_{BK}, x_{AM})$.

Let G_0 be the skeleton obtained from $F_1, \dots, F'_j, \dots, F_t$ using the same pairs as G'_0 for composition. Then G is obtained from $G_0, G'_1, \dots G'_k$ as in the definition of the class $\mathcal{T}_2$, and so $G \in \mathcal{T}_2$, a contradiction. This proves that $i > 0$, and so (a, b) is a semi-adjacent pair of G'_i for some $i \in \{1, \dots, k\}$.

Since by 3.6, $G' \in \mathcal{T}_2$ if and only if $\overline{G'} \in \mathcal{T}_2$, we may assume that either both A_i, B_i are strongly stable sets, or G'_i is a 1-thin trigraph with base (a'_i, b'_i). If A_i, B_i are strongly stable sets, then, form the symmetry we may assume that $a \in A_i$ and $b \in B_i$; but no vertex of $V(G')$ has a neighbor in both A_i, B_i, contrary to the fact that E is strongly complete to $\{a, b\}$ and $E \neq \emptyset$. Thus G'_i is 1-thin with base (a'_i, b'_i). Let $|A_i| = n$ and $|B_i| = m$, and let the vertices of A_i and B_i be numbered $a''_1, \ldots, a''_n$ and $b''_1, \ldots, b''_m$, respectively, as in the definition of a 1-thin trigraph. Let $a''_0 = a'_i$ an $b''_0 = b'_i$.

(5) Either $\{a, b\} \subseteq A_i$, or $\{a, b\} \subseteq B_i$.

Suppose not. Then we may assume that $a \in A_i$ and $b \in B_i$. Say $a = a''_s$ and $b = b''_t$ for some $s \in \{1, \ldots, n\}$ and $t \in \{1, \ldots, m\}$. Since a is semi-adjacent to b, the fact that G'_i is 1-thin implies that a is strongly complete to $\{b''_{t+1}, \ldots, b''_m\}$ and strongly anticomplete to $\{b''_1, \ldots, b''_{t-1}\}$, and b is strongly complete to $\{a''_{s+1}, \ldots, a''_n\}$, and strongly anticomplete to $\{a''_1, \ldots, a''_{s-1}\}$. Let S be the set of vertices of G'_i that are strongly adjacent to a and strongly antiadjacent to b, T be the set of vertices of G'_i that are strongly adjacent to b and strongly antiadjacent to a, M be the set of vertices of G'_i that are strongly complete to $\{a, b\}$, and N be the set of vertices of G'_i that are strongly anticomplete to $\{a, b\}$. Since G'_i is 1-thin, it follows that there exist $p \in \{0, \ldots, s-1\}$, $q \in \{s, \ldots, n\}$, $x \in \{0, \ldots, t-1\}$ and $y \in \{t, \ldots, n\}$ such that

$$S = \{a''_0, \ldots, a''_p\} \cup \{b''_{y+1}, \ldots, b''_m\}$$
$$T = \{b''_0, \ldots, b''_x\} \cup \{a''_{q+1}, \ldots, a''_n\}$$
$$M = \{a''_{s+1}, \ldots, a''_q\} \cup \{b''_{t+1}, \ldots, b''_y\}$$

and

$$N = \{a''_{p+1}, \ldots, a''_{s-1}\} \cup \{b''_{x+1}, \ldots, b''_{t-1}\}.$$

We observe that if $t \geq x + 2$, then 7.3 implies that $|Q| = 1$, and if $q \geq s + 1$, then 7.3 implies that $|P| = 1$. Let $p_1, \ldots, p_{|P|}$ and $q_1, \ldots, q_{|Q|}$ be an ordering of the vertices of P and Q, respectively, as in (1). Let G''_i be the trigraph obtained from $G|(L(A_i) \cup L(B_i))$ by adding vertices a'_i and b'_i such that a'_i is strongly complete to $L(A_i)$ and strongly anticomplete to $L(B_i)$, b'_i is strongly complete to $L(B_i)$ and strongly anticomplete to $L(A_i)$, and a'_i is semi-adjacent to b'_i. It is easy to check that G''_i is a 1-thin trigraph with base (a'_i, b'_i), ordering the vertices of $L(A_i)$

$$a''_1, \ldots, a''_{s-1}, p_1, \ldots, p_{|P|}, a''_{s+1}, \ldots, a''_n$$

and the vertices of $L(B_i)$

$$b''_1, \ldots, b''_{t-1}, q_1, \ldots, q_{|Q|}, b''_{t+1}, \ldots, b''_m.$$

Now, using G''_i instead of G'_i, we observe that $G \in \mathcal{T}_2$, a contradiction. This proves (5).

From (5), we may assume that both a and b are in A_i. Let $\{x, y\} = \{a, b\}$. We may assume that $x = a''_s$ and $y = a''_t$ and $s < t$. Let S be the set of vertices of G'_i that are strongly adjacent to x and strongly antiadjacent to y, T be the set of vertices of G'_i that are strongly adjacent to y and strongly antiadjacent to x, M be the set of vertices of G'_i that are strongly complete to $\{x, y\}$, and N be the set of vertices of G'_i that are strongly anticomplete to $\{x, y\}$. Since G'_i is a 1-thin trigraph, x is semi-adjacent to y and (A_i, B_i) is a homogeneous pair in G', it follows that either there exist $p, q \in \{1, \dots, m\}$ with $p < q$ such that

$$S = \{a''_{s+1}, \dots, a''_{t-1}\}$$
$$T = \{b''_p, \dots, b''_{q-1}\}$$
$$N = \{a''_{t+1}, \dots, a''_n\} \cup \{b''_0, \dots, b''_{p-1}\}$$
$$M = \{a''_0, \dots, a''_{s-1}\} \cup \{b''_q, \dots, b''_m\}$$

or x is strongly anticomplete to B_i and there exists $p \in \{1, \dots, m\}$ such that

$$S = \{a''_{s+1}, \dots, a''_{t-1}\}$$
$$T = \{b''_p, \dots, b''_m\}$$
$$N = \{a''_{t+1}, \dots, a''_n\} \cup \{b''_0, \dots, b''_{p-1}\}$$
$$M = \{a''_0, \dots, a''_{s-1}\}$$

or both x and y are strongly anticomplete to B_i and

$$S = \{a''_{s+1}, \dots, a''_{t-1}\}$$
$$T = \emptyset$$
$$N = \{a''_{t+1}, \dots, a''_n\} \cup \{b''_0, \dots, b''_m\}$$
$$M = \{a''_0, \dots, a''_{s-1}\}.$$

Since G'_i is 1-thin, it follows that S is strongly anticomplete to N. Since (A_i, B_i) is a homogeneous pair of G'_i, it follows that every vertex of $F \setminus N$ is strongly anticomplete to A_i, and $C \cup D \subseteq A_i \cup B_i$. Since C is not strongly anticomplete to F we deduce that $x = b$, $y = a$, $S = D$ and $T = C$. Let G''_i be the trigraph obtained from $G|(L(A_i) \cup B_i)$ by adding vertices a'_i and b'_i such that a'_i is strongly complete to $L(A_i)$ and strongly anticomplete to B_i, b'_i is strongly complete to B_i and strongly anticomplete to $L(A_i)$, and a'_i is semi-adjacent to b'_i. Now it is easy to check that G''_i is a 1-thin trigraph with base (a'_i, b'_i), ordering the vertices of $L(A_i)$

$$a''_1, \dots, a''_{s-1}, q_{|Q|}, \dots, q_1, a''_{s+1}, \dots, a''_{t-1}, p_{|P|}, \dots, p_1, a''_{t+1}, \dots, a''_n$$

and keeping the order of the vertices of B_i unchanged. Now, using G''_i instead of G'_i, we observe that $G \in \mathcal{T}_2$, a contradiction. This completes the proof of 8.2. ∎

9 The proof of 3.9

In this section we finish the proof of 3.9, which we restate.

9.1 *Let G be an elementary bull-free trigraph. Then either*

- *one of $G, \overline{G}$ belongs to $\mathcal{T}_1 \cup \mathcal{T}_2$, or*
- *one of $G, \overline{G}$ contains a homogeneous pair of type one or two, or*
- *G admits a homogeneous set decomposition.*

Proof. Suppose 9.1 is false, and let G be a counterexample of 9.1 with $|V(G)|$ minimum. Then $\overline{G}$ is also a counterexample to 9.1, and $|V(G)| = |V(\overline{G})|$. By 8.2, and since both G and $\overline{G}$ are counterexamples to 9.1, we may assume that G admits a 1-join. Therefore, $V(G)$ is the disjoint union of fours non-empty sets A, B, C, D such that

- B is strongly complete to C, A is strongly anticomplete to $C \cup D$, and B is strongly anticomplete to D;
- $|A \cup B| > 2$ and $|C \cup D| > 2$, and
- A is not strongly complete and not strongly anticomplete to B, and
- C is not strongly complete and not strongly anticomplete to D.

Let G_1 be the trigraph obtained from $G|(A \cup B)$ by adding two new vertices c and d, such that c is strongly complete to B and strongly anticomplete to A, and d is semi-adjacent to c and strongly anticomplete to $A \cup B$. Let G_2 be the trigraph obtained from $G|(C \cup D)$ by adding two new vertices a and b, such that b is strongly complete to C and strongly anticomplete to D, and a is semi-adjacent to b and strongly anticomplete to $C \cup D$.

(1) G_1 does not admit a homogeneous set decomposition.

Suppose (1) is false. Then there is a homogeneous set $X \subseteq V(G_1)$ with $1 < |X| < |V(G_1)|$. Suppose first that $X \cap \{c, d\} \neq \emptyset$. Then, since c is semi-adjacent to d, it follows that $\{c, d\} \subseteq X$; and, since B is strongly complete to c and strongly anticomplete to d, we deduce that $B \subseteq X$. Moreover, since A is strongly anticomplete to d, it follows that $A \setminus X$ is strongly anticomplete to X. But now $(X \setminus \{c, d\}) \cup C \cup D$ is a homogeneous set in G, and G admits a homogeneous set decomposition, contrary to the fact that G is a counterexample to 9.1. This proves that $X \cap \{c, d\} = \emptyset$. Since c is strongly complete to B and strongly anticomplete to A, it follows that either $X \subseteq A$, or $X \subseteq B$. Now, since C is strongly complete to B and strongly anticomplete to A, and since D is strongly anticomplete to $A \cup B$, it follows that X is a homogeneous set in G, again contrary to the fact that G is a counterexample to 9.1. This proves (1).

(2) Neither of $G_1, \overline{G_1}$ admits a homogeneous pair of type one or two.

Suppose (2) is false, and let (X, Y) be a homogeneous pair of type one or two in G_1 or $\overline{G_1}$. If $X \cup Y \subseteq A \cup B$, then (X, Y) is a homogeneous pair of the same type in G or $\overline{G}$, contrary to the fact that G is a counterexample to 9.1. So $\{c, d\} \cap (X \cup Y) \neq \emptyset$. Since c is semi-adjacent to d, it follows that $\{c, d\} \subseteq X \cup Y$. We may assume that $d \in X$. Since (X, Y) is a homogeneous pair of type one or two in G_1 or $\overline{G_1}$, it follows that some vertex $v \in V(G_1) \setminus (X \cup Y)$ is strongly complete to X. But no vertex of $V(G_1)$ is strongly adjacent to d, a contradiction. This proves (2).

(3) Neither of $G_1, \overline{G_1}$ belongs to $\mathcal{T}_2$.

We observe that for every trigraph $H \in \mathcal{T}_2$, every vertex of H has both a strong neighbor and a strong antineighbor in H. This implies that if one of $G_1, \overline{G_1}$ belongs to $\mathcal{T}_2$, then every vertex of G_1 has a strong neighbor in G_1. But d does not have a strong neighbor in G_1, and (3) follows.

(4) $\overline{G_1} \notin \mathcal{T}_1$.

We observe that if H is a melt, then every vertex of H has a strong antineighbor in H. Since in $\overline{G_1}$, d is complete to $V(\overline{G_1}) \setminus \{d\}$, it follows that $\overline{G_1}$ is not a double melt. Therefore, there exist a graph H with $maxdeg(G) \leq 2$ such that $\overline{G_1}$ admits an H-structure. We use the notation from the definition of an H-structure. Since every vertex of a melt has a strong antineighbor in the melt, and since for every edge $e = uv$ of H, $G_1|(h(e) \cup h(e, v) \cup h(e, u))$ is an $h(e)$-melt, it follows that $d \notin h(e) \cup h(e, v)$ for any $e \in E(H)$, $v \in V(H)$. Suppose that $d \in h(v)$ for some $v \in V(H)$. Then, since d is complete to $V(\overline{G_1}) \setminus \{d\}$, it follows that $V(H) = \{v\}$, and $V(\overline{G_1}) \setminus h(v)$ can be partitioned into two sets A_v, B_v such that $\overline{G_1}$ is an $(h(v), A_v, B_v, \emptyset, \emptyset)$-clique connector. Since $h(v)$ is a strong clique, and since d is semi-adjacent to c, it follows that $c \in A_v \cup B_v$, say $c \in A_v$. Since $|V(G_1)| > 3$, it follows that A_v is strongly complete to B_v. Let the vertices of $h(v)$ be numbered as $k_1, \ldots k_k$, and let $k_i = d$. Then A_v is strongly complete to $\{k_1, \ldots, k_{i-1}\}$, and B_v is strongly complete to $\{k_i, \ldots, k_k\}$. Consequently, $A_v \cup \{k_i, \ldots, k_k\}$ is strongly complete to $B_v \cup \{k_1, \ldots, k_{i-1}\}$. Since by (1) G_1 does not admit a homogeneous set decomposition, it follows that $|A_v \cup \{k_i, \ldots, k_k\}| = 1$ or $B_v \cup \{k_1, \ldots, k_{i-1}\} = \emptyset$, contrary to the fact that $c, d \in \{k_i\} \cup A_v$, and $B_v \neq \emptyset$. This proves that $d \notin h(v)$ for any $v \in V(H)$. Consequently, $d \in L$. Since every vertex of L has a neighbor in $h(v)$ for at most one $v \in V(H)$, it follows that $|V(H)| \leq 1$. Since $A \cup \{c, d\}$ contains a triangle, it follows that $V(G_1) \neq L$, and so $|V(H)| = 1$, say $V(H) = \{v\}$, and we may assume that $d \in A_v$. Since A_v is a strongly stable set, it follows that $A_v = \{d\}$, and $A \cup B \cup \{c\}$ can be partitioned into disjoint subsets $B_v, C_v, h(v)$ such that $\overline{G_1}$ is an $(h(v), A_v, B_v, C_v, \emptyset)$-clique connector. If $c \in C_v$ then d is strongly complete to $h(v) \cup B_v$, and C_v is strongly anticomplete to $h(v) \cup B_v$, and so $h(v) \cup B_v$ is a homogeneous set in G_1, contrary to (1). Thus d is strongly complete to C_v, and so C_v is a homogeneous set in G_1. Now (1) implies that $|C_v| \leq 1$. If $c \in B_v$, then, since $\{c, d\}$ is contained in a triangle, it follows that c and d have a common neighbor in $h(v)$. Since c is semi-adjacent to d, this implies that $|h(v)| = |B_v| = 1$, and $|V(G_1)| = 4$, contrary to the fact that $|A \cup B| > 2$. This proves that $c \in h(v)$. Since d is strongly complete to $h(v) \setminus \{c\}$, and semi-adjacent to c, it follows that c is strongly complete to B_v. Now C_v is strongly anticomplete to $B_v \cup (h(v) \setminus \{c\})$ in $\overline{G_1}$, and $\{c, d\}$ is strongly complete to $B_v \cup (h(v) \setminus \{c\})$ in $\overline{G_1}$. Since by (1) G_1 does not admit a homogeneous set decomposition, it follows that $|B_v \cup (h(v) \setminus \{c\})| = 1$, and $|V(G_1)| = 4$, contrary to the fact that $|A \cup B| > 2$. This proves (4).

Now, since $|V(G_1)| < |V(G)|$, it follows that one of the outcomes of 9.1 holds for G_1, and therefore $G_1 \in \mathcal{T}_1$. From the symmetry, we deduce that $G_2 \in \mathcal{T}_1$. Since every vertex in a double melt has a strong neighbor in the melt, it follows that G_1, G_2 are not double melts. Therefore, there exist graphs H_1, H_2 each with maximum degree at most two, such that for $i = 1, 2$ G_i admits an H_i-structure. Let $L_i \subseteq V(G_i)$ and

$$h_i : V(H_i) \cup E(H_i) \cup (E(H_i) \times V(H_i)) \to 2^{V(G_i) \setminus L_i}$$

be as in the definition of an H_i-structure. Since for every $e \in E(H_i)$ with $e = \{u, v\}$, $G_i|(h(e) \cup h(e, v) \cup h(e, u))$ is an $h(e)$-melt, and since every vertex of a melt has a strong neighbor in the melt, it follows that $d \notin h_1(e) \cup h_1(e, v)$ for any $e \in E(H_1)$, $v \in V(H_1)$. Similarly, $a \notin h_2(e) \cup h_2(e, v)$ for any $e \in E(H_2)$, $v \in V(H_2)$. Since every vertex of $h_i(v)$ has a strong neighbor in $V(G_i)$ it follows that $d \notin h_1(v)$ for any $v \in V(H_1)$, and $a \notin h_2(v)$ for any $v \in V(H_2)$. Consequently, $d \in L_1$ and $a \in L_2$. Since d has no strong neighbors in $V(G_1) \setminus \{d\}$, and d is semi-adjacent to c, it follows that $c \in L_1$ and similarly $b \in L_2$.

By 7.3, B is a strongly stable set. We claim that $B \subseteq L_1 \cup (\bigcup_{e \in E(H_1)} h_1(e))$. Suppose not, then some vertex $b \in B$ belongs to $h_1(v) \cup h_1(e, v)$ for some $v \in V(H_1)$ and $e \in E(H_1)$. But that means that every neighbor of b in G_1 is adjacent to some other neighbor of b in G_1, contrary to the fact that $N(c) = B \cup \{d\}$. This proves that $B \subseteq L_1 \cup (\bigcup_{e \in E(H_1)} h_1(e))$, and similarly, $C \subseteq L_2 \cup (\bigcup_{e \in E(H_2)} h_2(e))$.

Let $L = (L_1 \cup L_2) \setminus \{a, b, c, d\}$, let H to be the disjoint union of H_1 and H_2. Now, defining

$$h : V(H) \cup E(H) \cup (E(H) \times V(H)) \to 2^{V(G) \setminus L}$$

as $h(x) = h_i(x)$ for $x \in V(H_i) \cup E(H_i) \cup (E(H_i) \times V(H_i))$, we observe that G admits an H-structure, and therefore $G \in \mathcal{T}_1$, contrary to the fact that G is a counterexample to 9.1. This proves 9.1. ∎

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