A NOTE ON RANDOM MATRIX INTEGRALS, MOMENT IDENTITIES, AND CATALAN NUMBERS

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1. Introduction

In their paper "Random Matrix Theory and L-Functions at s = 1/2", Keating and Snaith give explicit formulas [Ke-Sn, (10) on page 94] for the matrix integrals

$$\int_{USp(2n)} \det(1-A)^s dA.$$

Here $USp(2n)$ is the compact symplectic group of size $2n$, dA is its Haar measure of total mass one, and $\det(1-A)$ is computed for the standard representation of $A \in USp(2n)$ as a matrix of size $2n$. Because the group $USp(2n)$ contains the scalar matrix -1, and because Haar measure is translation invariant, we have

$$\int_{USp(2n)} \det(1-A)^s dA = \int_{USp(2n)} \det(1+A)^s dA.$$

Their formula, valid for $s \in \mathbb{C}$ with $\Re(s) > -3/2$, is

$$\int_{USp(2n)} \det(1+A)^s dA = 2^{2ns} \prod_{j=1}^{n} \frac{\Gamma(n+j+1)\Gamma(1/2+s+j)}{\Gamma(1/2+j)\Gamma(1+s+n+j)}.$$

We were particularly interested in the case when s is an integer $r \geq -1$. Out of idle curiosity, we looked what their formula gave for the case $n = 1$, when $USp(2)$ is the group $SU(2)$, and for integer values of $r \geq -1$. For $r = -1, 0, 1, ..., 9$, we found the sequence

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796,$$

which is the start of the sequence of Catalan numbers C_n, indexed by integers $n \geq 0$:

$$C_n := \frac{1}{n+1}\binom{2n}{n}.$$

This made it seem likely that for every integer $r \geq -1$, we had the relation

$$\int_{SU(2)} \det(1+A)^r dA = C_{r+1}.$$

We will show

Theorem 1.1. *For every integer $r \geq -1$, we have the relation*

$$\int_{SU(2)} \det(1+A)^r dA = C_{r+1}.$$

The Catalan numbers are themselves matrix integrals over $SU(2)$. For integers $r \geq 0$ we have

$$C_r = \int_{SU(2)} \mathrm{Tr}(A)^{2r} dA, \quad 0 = \int_{SU(2)} \mathrm{Tr}(A)^{2r+1} dA.$$

So from Theorem 1.1, we have the identity, for integers $r \geq 0$,

$$\int_{SU(2)} \mathrm{Tr}(A)^{2r+2} dA = \int_{SU(2)} \det(1+A)^r dA.$$

For $SU(2)$, we have the identity

$$\det(1+A) = 2 + \mathrm{Tr}(A).$$

When we expand $(2 + \mathrm{Tr}(A))^r$ by the binomial theorem, we find the identity

Corollary 1.2.

$$C_{r+1} = \sum_{0 \leq d \leq r/2} 2^{r-2d} \binom{r}{2d} C_d.$$

This identity is presumably (?) known to Catalan experts.

We then looked at what the Keating-Snaith formula gave for the case $n = 2$, i.e. for the group $USp(4)$, and for integer values of $r \geq -1$. For $r = -1, 0, 1, ..., 7$, we found the sequence

$$1, 1, 3, 14, 84, 594, 4719, 40898, 379236.$$

Inspired by what had happened in the $SU(2)$ case, we computed (using Mathematica) the integrals

$$\int_{USp(4)} \mathrm{Tr}(A)^{2r+2} dA$$

for $r = -1, 0, 1, ..., 7$, and found this same sequence. This led us to suspect that we had the identity

$$\int_{USp(4)} \det(1+A)^r dA = \int_{USp(4)} \mathrm{Tr}(A)^{2r+2} dA$$

for every $r \geq 0$. We will show

Theorem 1.3. *For every integer $r \geq -1$, we have*

$$\int_{USp(4)} \det(1+A)^r dA = \int_{USp(4)} \mathrm{Tr}(A)^{2r+2} dA$$

.

However, this identity failed for every $n \geq 3$. Already for $r = 1$, the Keating-Snaith formula gives

$$\int_{USp(2n)} \det(1+A) dA = n + 1.$$

However, one knows that for every $n \geq 2$, one has

$$\int_{USp(2n)} \mathrm{Tr}(A)^4 dA = 3.$$

What is to be done?

It turns out that in order to understand the Keating-Snaith integrals

$$\int_{USp(2n)} \det(1+A)^r dA$$

along these lines, we must introduce the compact Spin group $USpin(2n+1)$ (the universal covering of the group $SO(2n+1,\mathbb{R})$ for the sum of squares quadratic form) and its 2^n-dimensional spin representation.

The general result is this.

Theorem 1.4. *For $n \geq 1$ and $r \geq 0$, we have the identity*

$$\int_{USp(2n)} \det(1+A)^r dA = \int_{USpin(2n+1)} \mathrm{Tr}(\mathrm{spin}(A))^{2r+2} dA.$$

This result includes the identities for $USp(2) = SU(2)$ and for $USp(4)$. Indeed, for $n = 1$ and $n = 2$, we have the accidents that $USpin(2n+1)$ is the group $USp(2n)$ and that the spin representation of $USpin(2n+1)$ is the standard representation of $USp(2n)$.

2. Proof of Theorem 1.4, via the Weyl integration formula

For a group G, we denote by $G^{\#}$ its space of conjugacy classes. When G is a topological group, we topologize $G^{\#}$ so that continuous functions on $G^{\#}$ are precisely the continuous central (invariant by conjugation) functions on G. The function $\det(1+A)$ is a continuous central function on $USp(2n)$ with values in $\mathbb{R}_{\geq 0}$, and the function $\mathrm{Tr}(\mathrm{spin}(A))$ is a continuous central function on $USpin(2n+1)$ with values in $\mathbb{R}$.

An element $A \in USp(2n)$ has n pairs of eigenvalues $e^{\pm i\theta_j}, j = 1, ..., n$, with angles $\theta_j \in [0, \pi]$, and A is determined up to conjugacy by the unordered n-tuple of its θ_j's. So a continuous (respectively Borel measurable and $\mathbb{R}_{\geq 0}$-valued) central function $A \mapsto f(A)$ on $USp(2n)^{\#}$ is a continuous (respectively Borel measurable and $\mathbb{R}_{\geq 0}$-valued) function

$$f(\theta_1, ..., \theta_n)$$

on $[0, \pi]^n$ which is invariant under the symmetric group S_n. For such a function f, the Weyl integration formula, cf. [Ka-Sar, 5.0.4] or [Weyl, p. 218, (7.8B)], asserts that

$$\int_{USp(2n)} f(A) dA = \int_{[0,\pi]^n} f(\theta_1, ..., \theta_n) \mu_{USp(2n)},$$

for $\mu_{USp(2n)}$ the measure on $[0, \pi]^n$ given by

$$\mu_{USp(2n)} = (1/n!)\Big(\prod_{1 \leq i < j \leq n} (2\cos(\theta_i) - 2\cos(\theta_j))^2\Big) \prod_{i=1}^{n} (2/\pi) \sin(\theta_i)^2 d\theta_i.$$

An element $A \in SO(2n+1, \mathbb{R})$ has the eigenvalue 1, and in addition it has n pairs of eigenvalues $e^{\pm i\theta_j}, j = 1, ..., n$, with angles $\theta_j \in [0, \pi]$, and A is determined up to conjugacy by the unordered n-tuple of its θ_j's. So a continuous (respectively Borel measurable and $\mathbb{R}_{\geq 0}$-valued) central function $A \mapsto f(A)$ on $USp(2n)^{\#}$ is a continuous (respectively Borel measurable and $\mathbb{R}_{\geq 0}$-valued) function

$$f(\theta_1, ..., \theta_n)$$

on $[0, \pi]^n$ which is invariant under the symmetric group S_n. For such a function f, the Weyl integration formula, cf. [Ka-Sar, 5.0.5] or [Weyl, p. 224, (9.7)], asserts that

$$\int_{SO(2n+1,\mathbb{R})} f(A) dA = \int_{[0,\pi]^n} f(\theta_1, ..., \theta_n) \mu_{SO(2n+1,\mathbb{R})},$$

for $\mu_{SO(2n+1,\mathbb{R})}$ the measure on $[0,\pi]^n$ given by

$$\mu_{SO(2n+1,\mathbb{R})} = (1/n!)(\prod_{1\le i<j\le n}(2\cos(\theta_i) - 2\cos(\theta_j))^2)\prod_{i=1}^{n}(2/\pi)\sin(\theta_i/2)^2 d\theta_i.$$

Notice the similarity between the formulas for the measures $\mu_{USp(2n)}$ and $\mu_{SO(2n+1,\mathbb{R})}$. The only difference is that each factor $\sin(\theta_i)^2$ in the first is replaced by $\sin(\theta_i/2)^2$ in the second.

The key lemma is this.

Lemma 2.1. *The measures $\mu_{USp(2n)}$ and $\mu_{SO(2n+1,\mathbb{R})}$ on $[0,\pi]^n$ are related by the identity*

$$\mu_{USp(2n)} = (\prod_{i=1}^{n}(2+2\cos(\theta_i))\mu_{SO(2n+1,\mathbb{R})}.$$

Proof. This is immediate from the trig identity

$$(2+2\cos(\theta))\sin(\theta/2)^2 = \sin(\theta)^2,$$

whose verification is left to the reader. □

The factor $\prod_{i=1}^{n}(2+2\cos(\theta_i))$ has the following two interpretations..

Lemma 2.2. *We have the following identities.*

(1) *For $A \in SO(2n+1,\mathbb{R})$ with eigenvalues 1 and n pairs of eigenvalues $e^{\pm i\theta_j}, j = 1,...,n$, with angles $\theta_j \in [0,\pi]$,*

$$(1/2)\det(1+A) = \prod_{i=1}^{n}(2+2\cos(\theta_i)).$$

(2) *For $A \in USp(2n)$ with n pairs of eigenvalues $e^{\pm i\theta_j}, j = 1,...,n$, with angles $\theta_j \in [0,\pi]$,*

$$\det(1+A) = \prod_{i=1}^{n}(2+2\cos(\theta_i)).$$

The spin representation of $USpin(2n+1)$ does not descend to $SO(2n+1,\mathbb{R})$, but its tensor square $\text{spin}^{\otimes 2}$ does. In terms of the standard representation std_{2n+1} of $SO(2n+1,\mathbb{R})$, and the double covering projection map

$$p : USpin(2n+1) \to SO(2n+1,\mathbb{R}),$$

one knows [Var, Lemma 6.6.2] that

$$\text{spin}^{\otimes 2} = (\sum_{i=0}^{n}\Lambda^i(std_{2n+1}))\circ p.$$

Each representation $\Lambda^i(std_{2n+1})$ is self-dual, and hence isomorphic to $\Lambda^{2n+1-i}(std_{2n+1})$. So we have

$$2\text{spin}^{\otimes 2} = (\sum_{i=0}^{2n+1}\Lambda^i(std_{2n+1}))\circ p.$$

For $A \in SO(2n+1,\mathbb{R})$ (indeed for $A \in GL(2n+1,\mathbb{C})$) we have the identity

$$\text{Tr}(\sum_{i=0}^{2n+1}\Lambda^i(A)) = \det(1+A).$$

So for any $B \in USpin(2n+1)$ lying over A, we have

$$\mathrm{Tr}(\mathrm{spin}(B))^2 = (1/2)\det(1+A).$$

For any continuous function g on $SO(2n+1, \mathbb{R})$, with pullback function $G := g \circ p$ on $USpin(2n+1)$, we have

$$\int_{USpin(2n+1)} G(A)dA = \int_{SO(2n+1,\mathbb{R})} g(A)dA,$$

simply because the direct image of Haar measure is Haar measure for any surjective homomorphism of compact groups, cf. [Ka-Sar, Lemma 1.3.1].

Putting this all together, we have, for $r \geq -1$,

$$\int_{USp(2n)} \det(1+A)^r dA = \int_{[0,\pi]^n} (\prod_{i=1}^{n}(2+2\cos(\theta_i)))^r \mu_{USp(2n)}$$

$$= \int_{[0,\pi]^n} (\prod_{i=1}^{n}(2+2\cos(\theta_i)))^{r+1} \mu_{SO(2n+1,\mathbb{R})}$$

$$= \int_{SO(2n+1,\mathbb{R})} ((1/2)\det(1+A))^{r+1} dA$$

$$= \int_{USpin(2n+1)} \mathrm{Tr}(\mathrm{spin}(A))^{2r+2} dA.$$

This concludes the proof of Theorem 1.4.

Remark 2.3. The odd moments of the spin representation all vanish:

$$\int_{USpin(2n+1)} \mathrm{Tr}(\mathrm{spin}(A))^{2r+1} dA = 0$$

for all $r \geq 0$. Indeed, this integral is the multiplicity of spin in the representation $\mathrm{spin}^{\otimes 2r}$, a representation which descends to $SO(2n+1, \mathbb{R})$. Hence all of its irreducible constituents also descend to $SO(2n+1, \mathbb{R})$. But none of these irreducible components can be isomorphic to spin, because the spin representation does not descend to $SO(2n+1, \mathbb{R})$.

3. A Catalan determinant interpretation

Theorem 3.1. *For integers $n \geq 1$ and $r \geq 0$, we have the (equivalent) identities*

$$\int_{USpin(2n+1)} \mathrm{Tr}(\mathrm{spin}(A))^{2r} dA = \det_{\substack{n\times n \\ 0\leq i,j\leq n-1}} (C_{r+i+j}),$$

$$\int_{SO(2n+1,\mathbb{R})} 2^{-r}\det(1+A))^r dA = \det_{\substack{n\times n \\ 0\leq i,j\leq n-1}} (C_{r+i+j}),$$

$$\int_{USp(2n)} \det(1+A)^r dA = \det_{\substack{n\times n \\ 0\leq i,j\leq n-1}} (C_{r+1+i+j}).$$

Proof. This is simply a matter of comparing the Keating-Snaith formula for

$$\int_{USp(2n)} \det(1+A)^r dA$$

with the formula, cf. [Krat, Theorem 3] and [Ge-Vi, page 21, line 6], of Gessel-Viennot for the Catalan determinant, and checking that the two formulas give the same answer. Each is a product of n terms. The individual terms don't quite match, but their ratios turn out to have product one. □

Remark 3.2. For $n = 1, 2, 3, 4, 5$, these sequences of $n \times n$ Catalan determinants indexed by r appear in the Online Encyclopedia of Integer Sequences [OEIS] as the sequences A000108, A005700, A006149, A006150, A006151 respectively.The interpretation of A000108, the sequence of Catalan numbers, as the even moments of the standard representation of $SU(2)$ is classical.The interpretation of A005700 as the sequence of even moments of the standard representation of $USp(4)$ occurs in [Ked-Suth, (11) on page 131]. For higher n, the moment interpretation of these determinants seems to be new. Is it?

4. A question

Our proof of Theorem 1.4,

$$\int_{USp(2n)} \det(1 + A)^r dA = \int_{USpin(2n+1)} \mathrm{Tr}(\mathrm{spin}(A))^{2r+2} dA,$$

via the Weyl integration formula, comes down to the trig identity of Lemma 2.1. From the point of view of representation theory, the first integral is, for $r \geq 0$, the multiplicity of the trivial representation in the r'th tensor power of the exterior algebra $\Lambda(std_{2n}) := \bigoplus_{k=0}^{2n} \Lambda^k(std_{2n}))$ as a representation of $USp(2n)$. The second integral is the multiplicity of the trivial representation in the $2r+2$'nd tensor power of the spin representation of $USpin(2n+1)$. Is there a representation theoretic proof that they are equal?

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