ORDERING COMPLEX NUMBERS... NOT*

If you have already begun studying complex numbers at school, you have probably been taught that it makes no sense to say that one complex number is less than another. However, there are various plausible ways in which we might attempt to do just that. Is it really true that none of them works? And if not, why not?

In fact, the question, "Can we order the complex numbers?" is far too easy! It is not at all difficult to suggest what an order for the complex numbers might mean. However, our first attempt will turn out to be rather unsatisfactory, and we are faced with the problem of asking a more precise question. By considering various ways in which we might order the complex numbers we shall try to discover what properties ought to hold in a "sensible" ordering.

Before we start, a very *very* brief introduction to complex numbers for readers who have not yet met them at school. Complex numbers have the form $a+bi$, where a and b are real numbers and i is a special symbol with the property that $i^2=-1$. That is, i is "the square root of minus one". We can add and multiply complex numbers just by expanding in the "obvious" way, collecting terms, and when possible simplifying by remembering that $i^2=-1$. For example,

$$(2+3i)+(4+5i)=2+4+3i+5i=6+8i$$

and

$$\begin{aligned}(2+3i)(4+5i)&=8+10i+12i+15i^2\\&=8+10i+12i-15=-7+22i\ .\end{aligned}$$

Complex numbers can be subtracted and divided too, but we won't need these for the present article so please ask your teacher. As usual, if we add 0 to something, or multiply by 1, we normally don't write the 0 or 1: thus, for example, $6+i$ has the form $a+bi$ with $a=6$ and $b=1$, while $-i$ is $a+bi$ with $a=0$ and $b=-1$. Any algebraic equality that is true for all real numbers is also true for all complex numbers. For example

$$z_1(z_2+z_3)=z_1z_2+z_1z_3\ ,$$

as you can very easily check by substituting $z_1=a+bi$, $z_2=c+di$, $z_3=e+fi$ and expanding both sides. Let me emphasize that we are talking about *equalities* here: as we shall show in the present article, the situation is very different if we wish to talk about *inequalities* where $\leq$ and $\geq$ are used in attempting to compare complex numbers.

If you haven't seen complex numbers before you may be – in fact, probably should be – seriously worried: how can i^2 possibly be negative? For the present, let me just suggest: don't worry! It works. As you know, the square of a *real number* cannot be negative; but complex numbers are not real numbers and some of their properties are different. You might be reassured if you realise that in the fifteenth century – perhaps even more recently – many people refused to accept the idea of negative numbers: how can "something" be less than "nothing"? Today this seems like a silly argument, and we understand that we can calculate with negatives just as well as with positive numbers.

So, let's try to come up with some way of defining one complex number to be less than or equal to another. As a **first attempt** we'll define $a+bi\leq c+di$ to mean that

$$a\leq c\quad\text{or}\quad b\leq d\ .$$

Note that here I'm using the symbol $\leq$ in two different senses. When I use it in connection with real numbers, for example, "$a\leq c$", it just means that a is less than or equal to c in the usual sense for real numbers which you are very familiar with. But when I'm talking about complex numbers, for example, "$a+bi\leq c+di$", I am defining a hypothetical order which may or may not turn out to be any good. Remember too that in mathematics, "or" always means "this or that *or both*", as in the statement, "You can get a concession ticket at the movies if you are a member of our Movie Club or under 15 years old" – they won't knock you back if you are a member *and* under 15.

Exercise. Mark the following statements true or false, if the meaning of $\leq$ is as we have just defined it. (Answers are at the end of the article).

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Before reading further, can you find anything unsatisfactory with this way of ordering complex numbers?

$2+5i \leq 8-i$	$2+5i \leq 1+3i$	$8-i \leq 8+7i$	$8-i \leq 1+3i$

Looking at the above example, you can see that $2+5i \leq 8-i$ and $8-i \leq 1+3i$. So you would expect that $2+5i \leq 1+3i$. But this is false! Therefore this is not at all a satisfactory way to define an ordering between complex numbers. The idea is that we would like to analyse our familiar concept of ordering real numbers, find out what are its important features, and duplicate these features, if possible, in the complex numbers. So let's try again.

A second attempt. Say that $a+bi \leq c+di$ whenever

$$a+b \leq c+d\ .$$

Exercise. Mark the following statements true or false. Is there still anything unsatisfactory about this attempt?

$2+5i \leq 8-i$	$2+5i \leq 1+3i$	$8-i \leq 8+7i$	$8-i \leq 1+3i$

As far as we can see from the present examples, this ordering does not have the defect of our first attempt; and in fact, it is not hard to show that there will never be any problems in this respect, no matter which examples we choose. However, there is a problem of a different kind. We already know that $2+5i \leq 8-i$; but it is also easy to check that $8-i \leq 2+5i$. Because of these two facts we would expect that $2+5i$ and $8-i$ must be the same number: but obviously they are not, and once again we have to regard this attempt as a failure.

The two properties that we would like, together with a third, are summed up in the following definition. A relation $\leq$ on some set of numbers (whether it is a well–known relation or one we have just made up) is called a **partial order** if it has the following three properties.

(1) It is *reflexive*: for any a in the set, $a \leq a$.

(2) It is *transitive*: for any a, b and c in the set, if $a \leq b$ and $b \leq c$ then $a \leq c$.

(3) It is *antisymmetric*: for any a and b in the set, if $a \leq b$ and $b \leq a$ then $a = b$.

Examples.

- The standard order relation of $\leq$ among real numbers. The three requirements are well–known facts about the real numbers.
- The divisibility relation on the positive integers: $a \mid b$ means that a is a factor of b. If you go through the definition and replace "$a \leq b$" everywhere you see it by "$a \mid b$", you will find that all three properties are still true.
- Our first attempt at defining an order on the complex numbers is not transitive, and therefore not a partial order.
- Our second attempt is transitive, but is not a partial order because it is not antisymmetric.

So, we are looking for a partial order on the complex numbers. As a **third attempt**, define $a+bi \leq c+di$ to mean that

$$a \leq c \quad \text{and} \quad b \leq d\ .$$

Exercise. True or false?

$2+5i \leq 8-i$	$2+5i \leq 1+3i$	$8-i \leq 8+7i$	$8-i \leq 1+3i$

It is possible to prove that this definition does give a partial order; unfortunately it is still not good enough! We have seen above that $2+5i \leq 8-i$ is false, and it is easy to check that $8-i \leq 2+5i$ is also false. This means that we simply cannot say which of $2+5i$ and $8-i$ is the smaller, which is bad since we would like to be able to compare *any* two complex numbers.

What we want is a **total order**: that is, a partial order which also satisfies a fourth property.

(4) For any a and b in the set, either $a \leq b$ or $b \leq a$ (or both).

Examples.

- The order $\leq$ on the real numbers is a total order.
- Divisibility is not a total order because, for example, the statements $5 \mid 7$ and $7 \mid 5$ are both *false*.
- Alphabetical order (if defined carefully!) among the words of the English language is a total order.
- It is possible to define a total order for complex numbers...

...in a **fourth attempt**. Say that $a + bi \leq c + di$ when

$$\textbf{either} \quad a < c \quad \textbf{or} \quad a = c \text{ and } b \leq d \, .$$

This is known as the *lexicographic* order, and if you think about it you can see that it is basically the same idea as alphabetical order. To compare two words we look at their first letters: if they are different we know which word comes first, while if they are the same then we look at the second letters, and so on.

Exercise. With the lexicographic order, are the following true or false?

$2 + 5i \leq 8 - i$	$2 + 5i \leq 1 + 3i$	$8 - i \leq 8 + 7i$	$8 - i \leq 1 + 3i$

It is possible to prove that this is a total order – so far, so good, but regrettably still not good enough! From our knowledge of the real numbers we expect an order relation to interact with addition and multiplication in a suitable way.

We say that the order $\leq$ on some set of numbers is **compatible with addition** when the following holds true:

(5) for all a, b and c in the set, if $a \leq b$ then $a + c \leq b + c$.

The order is **compatible with multiplication** when

(6) for all a, b and c in the set, if $a \leq b$ and $0 \leq c$ then $ac \leq bc$.

In fact, the lexicographic order is compatible with addition, but not with multiplication. For example, $2 + 5i \leq 8 - i$, and $0 \leq 1 - 2i$, so we should expect

$$(2 + 5i)(1 - 2i) \leq (8 - i)(1 - 2i) \ ,$$

that is, $12 + i \leq 6 - 17i$; but this is not true.

By now you're probably getting fed up and thinking this is never going to work; and in fact it isn't, which is why we say that it's impossible to order complex numbers. What we actually mean is that if we insist on properties (1) to (6) all being true, then it can't be done. Let's prove this.

Theorem. There is no way of defining an order relation on the complex numbers so that properties (1) to (6) are all true.

Comment. Before starting, let's give an outline of the method. The proof will be by contradiction. We shall assume that there is a way of defining $\leq$ on the complex numbers for which (1) to (6) hold, and we shall show that this leads to an impossible result. We'll begin by proving two preliminary facts (called "lemmas"). Note that we cannot assume that we are dealing with any *specific* order. For example, if we used facts about the lexicographic order, then in the end we would have proved that the lexicographic order doesn't work. But we know this already, and in any case we want to prove much more. So, we must not assume anything about our hypothetical order, other than that it obeys laws (1) to (6).

Proof of the theorem. Assume that $\leq$ is an ordering on the complex numbers for which facts (1) to (6) are true.

Lemma A. For any complex number a we have

$$0 \leq a \quad \text{if and only if} \quad -a \leq 0 \, .$$

Proof. Firstly, assume that $0 \leq a$. Property (5) allows us to add $-a$ to both sides of this inequality. Therefore $-a \leq a + (-a)$, that is, $-a \leq 0$.

Conversely, suppose that $-a \le 0$; using property (5) again, we add a to both sides to find $0 \le a$.

Lemma B. If z is any complex number then $0 \le z^2$.

Proof. Since $\le$ satisfies property (4) we have either $0 \le z$ or $z \le 0$.

In the first case, apply property (6) and take $a = 0$, $b = z$, $c = z$. Then it is true that $a \le b$ and $0 \le c$, and so $ac \le bc$, that is, $0 \le z^2$.

In the second case take $a = -z$ in Lemma A. Then $-a \le 0$ and so $0 \le a$, that is, $0 \le -z$. Use property (6) again, but this time take $a = 0$, $b = -z$ and $c = -z$. Again we have $a \le b$ and $0 \le c$, so $ac \le bc$, which again shows that $0 \le z^2$.

Thus, in each case we have $0 \le z^2$.

Proof of the theorem, concluded. First, by taking $z = 1$ in Lemma B we have $0 \le 1^2$, that is, $0 \le 1$; then from Lemma A we conclude that $-1 \le 0$. On the other hand taking $z = i$ in Lemma B tells us that $0 \le i^2$, that is, $0 \le -1$. So we have

$$-1 \le 0 \quad \text{and} \quad 0 \le -1 \ ;$$

but now property (3) shows us that $-1 = 0$, which is definitely not true! So the assumption that $\le$ obeys properties (1) to (6) has led us to an impossible conclusion, and we deduce that in fact $\le$ cannot be defined so as to satisfy these properties. More briefly: "it is impossible to (sensibly) order complex numbers".

Answers to exercises. (1) TFTT; (2) TFTF; (3) FFTF; (4) TFTF.

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