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Arithmetic Properties of Overpartitions into Odd Parts

Michael D. Hirschhorn[1] and James A. Sellers[2]

[1]School of Mathematics, University of New South Wales, Sydney 2052, Australia
m.hirschhorn@unsw.edu.au

[2]Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA
sellersj@math.psu.edu

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Abstract. In this article, we consider various arithmetic properties of the function $\overline{p_o}(n)$ which denotes the number of overpartitions of n using only odd parts. This function has arisen in a number of recent papers, but in contexts which are very different from overpartitions. We prove a number of arithmetic results including several Ramanujan-like congruences satisfied by $\overline{p_o}(n)$ and some easily-stated characterizations of $\overline{p_o}(n)$ modulo small powers of two. For example, it is proven that, for $n \geq 1$, $\overline{p_o}(n) \equiv 0 \pmod 4$ if and only if n is neither a square nor twice a square.

Keywords: congruence, overpartition, odd parts

1. Introduction

Throughout this work, we let $\overline{p}(n)$ be the number of overpartitions of the nonnegative integer n and $\overline{p_o}(n)$ be the number of overpartitions of n in which only odd parts are used. Here an *overpartition* of the nonnegative integer n is a nonincreasing sequence of natural numbers whose sum is n where the first occurrence of parts of each size may be overlined. For example, the overpartitions of 3 are

$$3,\ \overline{3},\ 2+1,\ \overline{2}+1,\ 2+\overline{1},\ \overline{2}+\overline{1},\ 1+1+1,\ \overline{1}+1+1.$$

Thus, from this example, we see that $\overline{p}(3) = 8$ and $\overline{p_o}(3) = 4$.

The function $\overline{p}(n)$ has been considered recently by a number of mathematicians including Corteel, Lovejoy, Mahlburg, Yee, and the authors. See [4, 5] and [8–12]. In [8] and [12], several Ramanujan-like congruences modulo small powers of two were proven for $\overline{p}(n)$. Indeed, all of these (and many more) congruences modulo small powers of two follow from a functional equation satisfied by the generating function for $\overline{p}(n)$.

Our goal in this note is to focus on similar results which hold for $\overline{p_o}(n)$. Interestingly enough, the generating function for $\overline{p_o}(n)$, which we will denote by $\overline{P_o}(q)$, has appeared quite recently in the works of Ardonne, Kedem, and Stone [2], Bessenrodt [3], Santos and Sills [13]. However, in none of these cases do the authors connect their work to overpartitions into odd parts. Moreover, they do not consider arithmetic properties satisfied by $\overline{p_o}(n)$; this was simply not their focus. But given the wealth of results now known for $\overline{p}(n)$, it is natural to consider whether $\overline{p_o}(n)$ satisfies similar properties.

In Section 2, we prove functional equations involving the generating functions for $\overline{p}(n)$ and $\overline{p_o}(n)$, respectively. We then utilize these to develop characterizations of $\overline{p_o}(n)$ modulo small powers of 2. In particular, we prove the following two theorems:

Theorem 1.1. *For all $n \geq 1$,*

$$\overline{p_o}(n) \equiv \begin{cases} 2 \pmod{4}, & \text{if } n \text{ is a square or } n \text{ is twice a square}, \\ 0 \pmod{4}, & \text{otherwise}. \end{cases}$$

Theorem 1.2. *Assume the prime factorization of n is given by*

$$n = 2^{\alpha} \prod p_i^{\alpha_i} q_i^{\beta_i} r_i^{\gamma_i} s_i^{\delta_i},$$

where each

$$p_i \equiv 1 \pmod{8}, \quad q_i \equiv 3 \pmod{8}, \quad r_i \equiv 5 \pmod{8}, \quad s_i \equiv 7 \pmod{8}.$$

Then $\overline{p_o}(n) \equiv 0 \pmod{8}$ if and only if one of the following holds:

- *at least one δ_i is odd,*
- *all δ_i are even and at least one γ_i is odd,*
- *all δ_i are even, all γ_i are even, at least one β_i is odd, and*

$$\prod (\alpha_i + 1)(\beta_i + 1) \equiv 0 \pmod{4},$$

- *all δ_i are even, all γ_i are even, all β_i are even, and*

$$\prod (\alpha_i + 1) \equiv 0 \pmod{4}.$$

We then state a number of corollaries of Theorems 1.1 and 1.2 which yield nice Ramanujan-like congruence results modulo powers of 2. We close Section 2 by proving a number of generating function identities for $\overline{p_o}(n)$ for specific arithmetic progressions.

In Section 3, we consider two infinite families of results culminating in the proof of the following theorem:

Theorem 1.3. *For all $n \geq 0$ and all $\alpha \geq 0$,*

$$\overline{p_o}(9^{\alpha}(9n+6)) \equiv 0 \pmod{12}, \quad \text{and}$$
$$\overline{p_o}(9^{\alpha}(27n+9)) \equiv 0 \pmod{6}.$$

The techniques employed below are elementary and involve a number of generating function manipulations. Moreover, Jacobi's Triple Product Identity [1, Theorem 2.8] is used often, so we state it here:

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \left(-zq; q^2\right)_\infty \left(-z^{-1}q; q^2\right)_\infty \left(q^2; q^2\right)_\infty, \tag{1.1}$$

where $(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots$. We will, at times, shorten $(q;q)_\infty$ by simply writing $(q)_\infty$.

2. Results Modulo Powers of Two

We begin with the generating function for $\overline{p}(n)$ which is given by

$$\overline{P}(q) = \sum_{n\geq 0} \overline{p}(n) q^n = \prod_{n\geq 1} \frac{1+q^n}{1-q^n} = \frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{\left(q^2;q^2\right)_\infty}{(q;q)_\infty^2}.$$

Our first goal is to prove a functional equation for $\overline{P}(q)$ which involves

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n\geq 1} q^{n^2}.$$

Theorem 2.1.

$$\overline{P}(q) = \phi(q)\overline{P}\left(q^2\right)^2.$$

Proof. From (1.1), it is clear that

$$\phi(q) = \left(-q;q^2\right)_\infty^2 \left(q^2;q^2\right)_\infty = \frac{\left(q^2;q^2\right)_\infty^5}{(q;q)_\infty^2 \left(q^4;q^4\right)_\infty^2},$$

after straightforward manipulations. Therefore

$$\begin{aligned}\phi(q)\overline{P}\left(q^2\right)^2 &= \frac{\left(q^2;q^2\right)_\infty^5}{(q;q)_\infty^2 \left(q^4;q^4\right)_\infty^2} \times \left(\frac{\left(q^4;q^4\right)_\infty}{\left(q^2;q^2\right)_\infty^2}\right)^2 \\ &= \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2} \\ &= \overline{P}(q).\end{aligned}$$

∎

Iteration of Theorem 2.1 yields the following theorem.

Theorem 2.2. $\sum_{n\geq 0} \overline{p}(n) q^n = \phi(q)\,\phi\left(q^2\right)^2 \phi\left(q^4\right)^4 \phi\left(q^8\right)^8 \cdots.$

We note that various congruences modulo small powers of two can easily be proved using Theorem 2.2 and we refer the reader to [8] and [12] for more information. (Note that the methods employed in [8] are quite different from those used in [12].)

We now obtain a theorem similar to Theorem 2.2 for $\overline{p_o}(n)$.

Theorem 2.3.

$$\overline{P_o}(q) = \phi(q)\overline{P}(q^2).$$

Proof. We begin with the generating function for $\overline{p_o}(n)$ which is

$$\overline{P_o}(q) = \sum_{n\geq 0} \overline{p_o}(n)q^n = \prod_{n\geq 1} \frac{1+q^{2n-1}}{1-q^{2n-1}} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} = \frac{\left(q^2;q^2\right)_\infty^3}{(q;q)_\infty^2\,(q^4;q^4)_\infty},$$

after some manipulations. Therefore

$$\begin{aligned}\phi(q)\overline{P}\left(q^2\right) &= \frac{\left(q^2;q^2\right)_\infty^5}{(q;q)_\infty^2\,(q^4;q^4)_\infty^2} \times \frac{\left(q^4;q^4\right)_\infty}{(q^2;q^2)_\infty^2} \\ &= \frac{\left(q^2;q^2\right)_\infty^3}{(q;q)_\infty^2\,(q^4;q^4)_\infty} \\ &= \overline{P_o}(q).\end{aligned}$$

∎

We can combine Theorem 2.3 with Theorem 2.1 to obtain the following:

Theorem 2.4. $\sum_{n\geq 0} \overline{p_o}(n)q^n = \phi(q)\,\phi\left(q^2\right)\,\phi\left(q^4\right)^2\,\phi\left(q^8\right)^4 \cdots.$

Theorem 2.4 can be utilized to obtain characterizations of $\overline{p_o}(n)$ modulo small powers of two in a very straightforward way. We do this below for the moduli 4 and 8.

For completeness' sake, we note that for all $n > 0$, $\overline{p_o}(n) \equiv 0 \pmod{2}$. This follows since, modulo 2,

$$\sum_{n\geq 0} \overline{p_o}(n)q^n = \prod_{n\geq 1} \frac{1+q^{2n-1}}{1-q^{2n-1}} \equiv \prod_{n\geq 1} \frac{1-q^{2n-1}}{1-q^{2n-1}} = 1.$$

We turn our attention to a characterization of $\overline{p_o}(n)$ modulo 4 by proving Theorem 1.1 mentioned above.

Proof of Theorem 1.1. Note that $(\phi(q))^{2^k} \equiv 1 \pmod{4}$ for $k \geq 1$. This is clear when one writes

$$\phi(q)^{2^k} = \left(1 + 2\sum_{n\geq 1} q^{n^2}\right)^{2^k},$$

and then expands via the binomial theorem. Therefore we have

$$\begin{aligned}\sum_{n\geq 0} \overline{p_o}(n)q^n &= \phi(q)\,\phi\left(q^2\right)\,\phi\left(q^4\right)^2\,\phi\left(q^8\right)^4 \cdots \\ &\equiv \phi(q)\phi\left(q^2\right) \pmod{4} \\ &= \left(1 + 2\sum_{n\geq 1} q^{n^2}\right)\left(1 + 2\sum_{n\geq 1} q^{2n^2}\right) \\ &\equiv 1 + 2\sum_{n\geq 1} q^{n^2} + 2\sum_{n\geq 1} q^{2n^2} \pmod{4}.\end{aligned}$$

The result follows. ∎

From Theorem 1.1, we see that numerous Ramanujan-like congruences modulo 4 are satisfied by $\overline{p_o}(n)$. All that is necessary is to find arithmetic progressions which fail to contain squares and doubles of squares. For example, we know that, for all $n \geq 0$, $\overline{p_o}(2^j n + (2r+1)) \equiv 0 \pmod 4$ for all $j \geq 2$ and $1 \leq r \leq 2^{j-1} - 1$ where r is not a multiple of 4. Many additional examples could be stated.

We next consider a characterization of $\overline{p_o}(n)$ modulo 8 by proving Theorem 1.2. But before we prove Theorem 1.2, we state one important lemma.

Lemma 2.5. *Let $r_{\{\square+\square\}}(n)$ be the number of ways to represent the nonnegative integer n as the sum of two (possibly equal) squares and let $r_{\{\square+2\square\}}(n)$ be the number of ways to represent the integer n as the sum of a square and twice a square. Also, assume the prime factorization of n is given by*

$$n = 2^{\alpha} \prod p_i^{\alpha_i} q_i^{\beta_i} r_i^{\gamma_i} s_i^{\delta_i},$$

where each

$$p_i \equiv 1 \pmod 8, \quad q_i \equiv 3 \pmod 8, \quad r_i \equiv 5 \pmod 8, \quad s_i \equiv 7 \pmod 8.$$

Then

$$r_{\{\square+\square\}}(n) = \begin{cases} 0, & \text{if any } \beta_i \text{ or } \delta_i \text{ is odd,} \\ 4\prod(\alpha_i+1)(\gamma_i+1), & \text{if all } \beta_i, \delta_i \text{ are even,} \end{cases}$$

and

$$r_{\{\square+2\square\}}(n) = \begin{cases} 0, & \text{if any } \gamma_i \text{ or } \delta_i \text{ is odd,} \\ 2\prod(\alpha_i+1)(\beta_i+1), & \text{if all } \gamma_i,\ \delta_i \text{ are even.} \end{cases}$$

Proof. These results follow from the results of Jacobi and Dirichlet. For the proofs of these see [6]. ∎

With Lemma 2.5 in hand, we can now prove Theorem 1.2.

Proof of Theorem 1.2. We consider the following set of equalities and congruences:

$$\begin{aligned}
\sum_{n\geq 0} \overline{p_o}(n) q^n &= \phi(q)\phi\left(q^2\right)\phi\left(q^4\right)^2\phi\left(q^8\right)^4\cdots \\
&\equiv \phi(q)\phi\left(q^2\right)\phi\left(q^4\right)^2 \pmod 8 \\
&\equiv \phi(q)\Big(\phi\left(q^2\right)+\phi\left(q^4\right)^2-1\Big) \pmod 8 \\
&= \phi(q)\phi\left(q^2\right)+\phi(q)\phi\left(q^4\right)^2-\phi(q) \\
&\equiv \phi(q)\phi\left(q^2\right)+\Big(\phi(q)+\phi\left(q^4\right)^2-1\Big)-\phi(q) \pmod 8 \\
&= \phi(q)\phi\left(q^2\right)+\phi\left(q^4\right)^2-1 \\
&= \sum_{n\geq 0} r_{\{\square+2\square\}}(n)q^n + \sum_{n\geq 0} r_{\{\square+\square\}}(n)q^{4n} - 1.
\end{aligned}$$

It follows that, modulo 8,

$$\overline{p_o}(n) \equiv \begin{cases} r_{\{\Box+2\Box\}}(n), & \text{if } 4 \nmid n, \\ r_{\{\Box+2\Box\}}(n) + r_{\{\Box+\Box\}}(n), & \text{if } 4 | n. \end{cases}$$

It is now an easy matter to use Lemma 2.5 to complete the proof of the theorem. ∎

Indeed, we can do better. We can provide a full characterization of $\overline{p_o}(n)$ modulo 8 as follows:

Theorem 2.6. *For all* $n \geq 1$, $\overline{p_o}(n) \equiv 0 \pmod 8$ *if and only if one of the following holds*:

- *at least one* δ_i *is odd,*
- *all* δ_i *are even and at least one* γ_i *is odd,*
- *all* δ_i *are even, all* γ_i *are even, at least one* β_i *is odd, and*

$$\prod(\alpha_i + 1)(\beta_i + 1) \equiv 0 \pmod 4,$$

- *all* δ_i *are even, all* γ_i *are even, all* β_i *are even, and*

$$\prod(\alpha_i + 1) \equiv 0 \pmod 4;$$

$\overline{p_o}(n) \equiv 4 \pmod 8$ *if and only if one of the following holds*:

- *all* δ_i *are even, all* γ_i *are even, any* β_i *is odd, and*

$$\prod(\alpha_i + 1)(\beta_i + 1) \equiv 2 \pmod 4,$$

- *all* δ_i *are even, all* γ_i *are even, all* β_i *are even, and*

$$\prod(\alpha_i + 1) \equiv 2 \pmod 4;$$

$\overline{p_o}(n) \equiv 2 \pmod 8$ *if and only if one of the following holds*:

- n *is an odd square or twice an odd square with*

$$\sum(\alpha_i + \beta_i) \equiv 0 \pmod 4,$$

- n *is an even square or twice an even square with*

$$\sum(\alpha_i + \beta_i) \equiv 2 \pmod 4;$$

$\overline{p_o}(n) \equiv 6 \pmod 8$ *if and only if one of the following holds*:

- n *is an odd square or twice an odd square with*

$$\sum(\alpha_i + \beta_i) \equiv 2 \pmod 4,$$

- *n is an even square or twice an even square with*

$$\sum(\alpha_i + \beta_i) \equiv 0 \pmod 4.$$

At this point, it is worthwhile to highlight a number of corollaries of Theorems 1.2 and 2.6, as numerous Ramanujan-like congruences can readily be seen from the theorems.

Corollary 2.7. *For all $n \geq 0$,*

$$\overline{p_o}(8n+5) \equiv 0 \pmod 8, \textit{ and}$$
$$\overline{p_o}(8n+7) \equiv 0 \pmod 8.$$

Proof. In the case of numbers of the form $8n+5$ or $8n+7$, either at least one δ_i or one γ_i is odd (using the notation employed in Theorem 1.2 and Theorem 2.6). This implies the result. ∎

Corollary 2.8. *For all $n \geq 0$,*

$$\overline{p_o}(18n+15) \equiv 0 \pmod 8, \quad \textit{and}$$
$$\overline{p_o}(36n+21) \equiv 0 \pmod 8.$$

Proof. Note that $18n+15 = 3(6n+5)$, $3 \nmid 6n+5$ and $6n+5$ is not a square. It follows that either at least one δ_i, γ_i or α_i is odd, or at least two of the β_i are odd. In any case the result follows. Similarly, we have $36n+21 = 3(12n+7)$, $3 \nmid 12n+7$ and $12n+7$ cannot be a square. The proof proceeds as above. ∎

We next consider a corollary which produces infinitely many congruences modulo 8 for an infinite subset of primes.

Corollary 2.9. *Let p be a prime such that $p \equiv 5 \pmod 8$ or $p \equiv 7 \pmod 8$ and let $r \in \{1, \ldots, p-1\}$. Then, for all $n \geq 0$, $\overline{p_o}(p^2n + pr) \equiv 0 \pmod 8$.*

Proof. Note that $p^2n + pr = p(pn+r)$ and $p \nmid pn+r$, so at least one δ_i or γ_i is odd. The result follows. ∎

Corollary 2.10. *For all $n \geq 0$,*

$$\overline{p_o}(2n) \equiv \begin{cases} -\overline{p_o}(n) \pmod 8, & \textit{if } n \textit{ is twice an odd square}, \\ \overline{p_o}(n) \pmod 8, & \textit{otherwise.} \end{cases}$$

In particular, if $\overline{p_o}(n) \equiv 0 \pmod 8$, then $\overline{p_o}(2n) \equiv 0 \pmod 8$.

Thus, for example, for all $n \geq 0$ and all $\alpha \geq 0$, we have

$$\overline{p_o}\left(2^\alpha(8n+5)\right) \equiv 0 \pmod 8, \qquad \text{and}$$
$$\overline{p_o}\left(2^\alpha(8n+7)\right) \equiv 0 \pmod 8.$$

Because the analysis of characterizations of $\overline{p_o}(n)$ becomes difficult modulo higher powers of 2, we now consider a different strategy. Using elementary generating function manipulation techniques, we can prove a number of generating function identities for $\overline{p_o}(n)$ for certain arithmetic progressions. We first provide a lemma which will prove useful within the intermediate steps of the proof.

First, we define four functions of q, one of which has already been used above:

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \prod_{n\geq 1} \left(1+q^{2n-1}\right)^2 \left(1-q^{2n}\right) = \frac{\left(q^2\right)_\infty^5}{(q)_\infty^2 \left(q^4\right)_\infty^2},$$

$$\psi(q) = \sum_{n=-\infty}^{\infty} q^{2n^2-n} = \prod_{n\geq 1} \left(1+q^{4n-3}\right)\left(1+q^{4n-1}\right)\left(1-q^{4n}\right) = \frac{\left(q^2\right)_\infty^2}{(q)_\infty},$$

$$D(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{n\geq 1} \left(1-q^{2n-1}\right)^2 \left(1-q^{2n}\right) = \prod_{n\geq 1} \frac{1-q^n}{1+q^n} = \frac{(q)_\infty^2}{\left(q^2\right)_\infty}, \text{ and}$$

$$Y(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2-2n} = \prod_{n\geq 1} \left(1-q^{6n-5}\right)\left(1-q^{6n-1}\right)\left(1-q^{6n}\right) = \frac{(q)_\infty \left(q^6\right)_\infty^2}{\left(q^2\right)_\infty \left(q^3\right)_\infty}.$$

Next, we state without proof a lemma which contains a number of elementary results relating these four functions. The proofs of each of these identities are relatively straightforward.

Lemma 2.11.

$$\phi(q) = \phi\left(q^4\right) + 2q\psi\left(q^8\right),$$

$$\phi(q)^2 = \phi\left(q^2\right)^2 + 4q\psi\left(q^4\right)^2,$$

$$\frac{1}{D(q)} = \prod_{n\geq 1} \frac{1+q^n}{1-q^n} = \frac{\left(q^2\right)_\infty}{(q)_\infty^2} = \frac{\phi(q)}{D\left(q^2\right)^2}.$$

With this lemma in hand, we can now proceed to prove a number of generating function identities for particular arithmetic progressions. We combine these in the statement of one theorem.

Theorem 2.12.

$$\sum_{n\geq 0} \overline{p_o}(2n+1)q^n = 2\frac{\left(q^2\right)_\infty \left(q^8\right)_\infty^2}{(q)_\infty^2 \left(q^4\right)_\infty},$$

$$\sum_{n\geq 0} \overline{p_o}(4n+3)q^n = 4\frac{\left(q^2\right)_\infty \left(q^4\right)_\infty \left(q^8\right)_\infty^2}{(q)_\infty^4},$$

$$\sum_{n\geq 0} \overline{p_o}(8n+7)q^n = 16\frac{\left(q^2\right)_\infty^3 \left(q^4\right)_\infty^6}{(q)_\infty^9}.$$

Proof. The proof follows from generating function manipulations that rely heavily on the lemmas mentioned above. First, note that

$$\begin{aligned}\sum_{n\geq 0}\overline{p_o}(n)q^n &= \prod_{n\geq 1}\frac{1+q^{2n-1}}{1-q^{2n-1}}\\ &= \prod_{n\geq 1}\frac{1+q^n}{1-q^n}\Big/\prod_{n\geq 1}\frac{1+q^{2n}}{1-q^{2n}}\\ &= \frac{D\left(q^2\right)}{D(q)}\\ &= \frac{\phi(q)}{D\left(q^2\right)}\quad \text{(using Lemma 2.11)}\\ &= \frac{\phi\left(q^4\right)+2q\psi\left(q^8\right)}{D\left(q^2\right)}\quad \text{(using Lemma 2.11)}.\end{aligned}$$

It follows that

$$\sum_{n\geq 0}\overline{p_o}(2n+1)q^n = 2\frac{\psi\left(q^4\right)}{D(q)} = 2\frac{\left(q^2\right)_\infty\left(q^8\right)_\infty^2}{(q)_\infty^2\left(q^4\right)_\infty}.$$

This proves the first identity in the theorem. Next, we can use Lemma 2.11 once again to write the generating function for $\overline{p_o}(2n+1)$ in a different way to obtain the generating function result for $\overline{p_o}(4n+3)$. Doing so yields

$$\sum_{n\geq 0}\overline{p_o}(2n+1)q^n = 2\frac{\psi\left(q^4\right)\phi(q)}{D\left(q^2\right)^2} = 2\frac{\psi\left(q^4\right)\left(\phi\left(q^4\right)+2q\psi\left(q^8\right)\right)}{D\left(q^2\right)^2}.$$

Then we see that

$$\sum_{n\geq 0}\overline{p_o}(4n+3)q^n = 4\frac{\psi\left(q^2\right)\psi\left(q^4\right)}{D(q)^2} = 4\frac{\left(q^4\right)_\infty^2}{\left(q^2\right)_\infty}\frac{\left(q^8\right)_\infty^2}{\left(q^4\right)_\infty}\frac{\left(q^2\right)_\infty^2}{(q)_\infty^4} = 4\frac{\left(q^2\right)_\infty\left(q^4\right)_\infty\left(q^8\right)_\infty^2}{(q)_\infty^4}.$$

We can apply this principle once again to obtain the result for $\overline{p_o}(8n+7)$. Notice that

$$\sum_{n\geq 0}\overline{p_o}(4n+3)q^n = 4\frac{\psi\left(q^2\right)\psi\left(q^4\right)\phi(q)^2}{D\left(q^2\right)^4} = 4\frac{\psi\left(q^2\right)\psi\left(q^4\right)\left(\phi\left(q^2\right)^2+4q\psi\left(q^4\right)^2\right)}{D\left(q^2\right)^4}.$$

Thus,

$$\sum_{n\geq 0}\overline{p_o}(8n+7)q^n = 16\frac{\psi(q)\psi\left(q^2\right)^3}{D(q)^4} = 16\frac{\left(q^2\right)_\infty^2}{(q)_\infty}\frac{\left(q^4\right)_\infty^6}{\left(q^2\right)_\infty^3}\frac{\left(q^2\right)_\infty^4}{(q)_\infty^8} = 16\frac{\left(q^2\right)_\infty^3\left(q^4\right)_\infty^6}{(q)_\infty^9}.$$

This completes the proof. ∎

One immediate corollary of Theorem 2.12 follows:

Corollary 2.13. *For all $n\geq 0$,*

$$\overline{p_o}(8n+7)\equiv 0 \pmod{16}.$$

3. Results Modulo Multiples of Three

We now consider a number of results modulo multiples of three. Before proving the main results of the section, we state a few important lemmas. The first lemma, a series of three short results, is stated without proof, while outlines of the proofs of the other lemmas are given.

Lemma 3.1.

$$D(q) = D\left(q^9\right) - 2qY\left(q^3\right),$$

$$\frac{D\left(q^3\right)^4}{D\left(q^9\right)} = D(q)D(\omega q)D\left(\omega^2 q\right), \quad \textit{where } \omega = e^{2\pi i/3}, \quad \textit{and}$$

$$\frac{D(q)^4}{D\left(q^3\right)} = D\left(q^3\right)^3 - 8qY(q)^3.$$

Lemma 3.2.

$$\frac{\left(q^3\right)_\infty^3}{(q)_\infty} - q\frac{\left(q^{12}\right)_\infty^3}{\left(q^4\right)_\infty} = \frac{\left(q^4\right)_\infty^3\left(q^6\right)_\infty^2}{\left(q^2\right)_\infty^2\left(q^{12}\right)_\infty}.$$

Proof. Let

$$c(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n}.$$

In [7], it was shown that

$$c(q) = 3\frac{\left(q^3\right)_\infty^3}{(q)_\infty} \quad \text{and} \quad c(q) - qc\left(q^4\right) = 3\frac{\left(q^4\right)_\infty^3\left(q^6\right)_\infty^2}{\left(q^2\right)_\infty^2\left(q^{12}\right)_\infty}.$$

The result follows. ∎

Lemma 3.3.

$$\frac{\left(q^4\right)_\infty^3}{\left(q^{12}\right)_\infty} - 3q\frac{\left(q^2\right)_\infty^2\left(q^{12}\right)_\infty^3}{\left(q^4\right)_\infty\left(q^6\right)_\infty^2} = \frac{(q)_\infty^3}{\left(q^3\right)_\infty}.$$

Proof. Let

$$b(q) = \sum_{m,m=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2},$$

where $\omega = e^{2\pi i/3}$. In [7], it was shown that

$$b(q) = \frac{(q)_\infty^3}{\left(q^3\right)_\infty} \quad \text{and} \quad b(q) = b\left(q^4\right) - 3q\frac{\left(q^2\right)_\infty^2\left(q^{12}\right)_\infty^3}{\left(q^4\right)_\infty\left(q^6\right)_\infty^2}.$$

The result follows. ∎

Lemma 3.4.

$$\frac{\left(q^3\right)_\infty^3}{(q)_\infty} - 4q\frac{\left(q^{12}\right)_\infty^3}{\left(q^4\right)_\infty} = \frac{(q)_\infty^3\left(q^6\right)_\infty^2}{\left(q^2\right)_\infty^2\left(q^3\right)_\infty}.$$

Proof. From Lemmas 3.2 and 3.3, we have

$$\begin{aligned}\frac{\left(q^3\right)_\infty^3}{(q)_\infty}-4q\frac{\left(q^{12}\right)_\infty^3}{(q^4)_\infty} &= \left(\frac{\left(q^3\right)_\infty^3}{(q)_\infty}-q\frac{\left(q^{12}\right)_\infty^3}{(q^4)_\infty}\right)-3q\frac{\left(q^{12}\right)_\infty^3}{(q^4)_\infty}\\
&=\frac{\left(q^4\right)_\infty^3\left(q^6\right)_\infty^2}{(q^2)_\infty^2(q^{12})_\infty}-3q\frac{\left(q^{12}\right)_\infty^3}{(q^4)_\infty}\\
&=\frac{\left(q^6\right)_\infty^2}{(q^2)_\infty^2}\left(\frac{\left(q^4\right)_\infty^3}{(q^{12})_\infty}-3q\frac{\left(q^2\right)_\infty^2\left(q^{12}\right)_\infty^3}{(q^4)_\infty(q^6)_\infty^2}\right)\\
&=\frac{(q)_\infty^3\left(q^6\right)_\infty^2}{(q^2)_\infty^2(q^3)_\infty}.\end{aligned}$$

∎

We now state and prove a theorem in the same vein as Theorem 2.12.

Theorem 3.5.

$$\sum_{n\geq 0}\overline{p_o}(3n)q^n = \frac{\left(q^2\right)_\infty^2\left(q^3\right)_\infty^2\left(q^6\right)_\infty}{(q)_\infty^4(q^{12})_\infty}, \quad \textit{and}$$

$$\sum_{n\geq 0}\overline{p_o}(9n+6)q^n = 12\frac{\left(q^2\right)_\infty^7\left(q^3\right)_\infty^6}{(q)_\infty^{12}(q^4)_\infty}.$$

Proof.

$$\begin{aligned}\sum_{n\geq 0}\overline{p_o}(n)q^n &= \frac{D\left(q^2\right)}{D(q)}\\
&=\frac{D\left(q^2\right)D(\omega q)D\left(\omega^2 q\right)}{D(q)D(\omega q)D\left(\omega^2 q\right)}\\
&=\frac{D\left(q^9\right)}{D\left(q^3\right)^4}\left(D\left(q^{18}\right)-2q^2Y(q^6)\right)\left(D\left(q^9\right)-2\omega qY\left(q^3\right)\right)\left(D\left(q^9\right)-2\omega^2 qY\left(q^3\right)\right)\\
&\quad\text{(using Lemma 3.1)}\\
&=\frac{D(q^9)}{D\left(q^3\right)^4}\left(D(q^{18})-2q^2Y(q^6)\right)\left(D\left(q^9\right)^2+2qD\left(q^9\right)Y\left(q^3\right)+4q^2Y\left(q^3\right)^2\right).\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n\geq 0}\overline{p_o}(3n)q^n &= \frac{D\left(q^3\right)}{D(q)^4}\left(D(q^6)D\left(q^3\right)^2-4qD\left(q^3\right)Y(q)Y\left(q^2\right)\right)\\ &= \frac{\left(q^2\right)_\infty^4\left(q^3\right)_\infty^2}{(q)_\infty^8\left(q^6\right)_\infty}\left(\frac{\left(q^3\right)_\infty^4}{\left(q^{12}\right)_\infty}-4q\frac{(q)_\infty\left(q^3\right)_\infty\left(q^{12}\right)_\infty^2}{\left(q^4\right)_\infty}\right)\\ &= \frac{\left(q^2\right)_\infty^4\left(q^3\right)_\infty^3}{(q)_\infty^7\left(q^6\right)_\infty\left(q^{12}\right)_\infty}\left(\frac{\left(q^3\right)_\infty^3}{(q)_\infty}-4q\frac{\left(q^{12}\right)_\infty^3}{\left(q^4\right)_\infty}\right)\\ &= \frac{\left(q^2\right)_\infty^2\left(q^3\right)_\infty^2\left(q^6\right)_\infty}{(q)_\infty^4\left(q^{12}\right)_\infty}\quad \text{(using Lemma 3.4).}\end{aligned}$$

Now we wish to dissect the generating function for $\overline{p_o}(3n)$ in order to obtain the generating function for $\overline{p_o}(9n+6)$ and complete our proof. We do so by the following manipulations:

$$\begin{aligned}\sum_{n\geq 0}\overline{p_o}(3n)q^n &= \frac{\left(q^2\right)_\infty^2\left(q^3\right)_\infty^2\left(q^6\right)_\infty}{(q)_\infty^4\left(q^{12}\right)_\infty}\\ &= \frac{D\left(q^3\right)D\left(q^6\right)}{D(q)^2}\\ &= D\left(q^3\right)D\left(q^6\right)\left(\frac{D(\omega q)D\left(\omega^2 q\right)}{D\left(q^3\right)^4/D\left(q^9\right)}\right)^2\\ &= \frac{D\left(q^6\right)D\left(q^9\right)^2}{D\left(q^3\right)^7}\left(D\left(q^9\right)^2+2qD\left(q^9\right)Y\left(q^3\right)+4q^2Y\left(q^3\right)^2\right)^2.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n\geq 0}\overline{p_o}(9n+6)q^n &= \frac{D\left(q^2\right)D\left(q^3\right)^2}{D(q)^7}\left(12D\left(q^3\right)^2Y(q)^2\right)\\ &= 12\frac{D\left(q^2\right)D\left(q^3\right)^4Y(q)^2}{D(q)^7}\\ &= 12\frac{\left(q^2\right)_\infty^7\left(q^3\right)_\infty^6}{(q)_\infty^{12}\left(q^4\right)_\infty}.\end{aligned}$$

∎

Two corollaries of Theorem 3.5 are worthy of note here.

Corollary 3.6.

$$\overline{p_o}(9n+6)\equiv\begin{cases}12 \pmod{24}, & \text{if } n=6k^2\pm 4k \text{ for some } k\geq 0,\\ 0 \pmod{24}, & \text{otherwise.}\end{cases}$$

Proof. Notice that

$$\begin{aligned}\sum_{n\geq 0}\frac{\overline{p_o}(9n+6)}{12}q^n &= \frac{(q^2)_\infty^7 (q^3)_\infty^6}{(q)_\infty^{12}(q^4)_\infty} \\ &\equiv \frac{(q^2)_\infty (q^{12})_\infty^2}{(q^4)_\infty (q^6)_\infty} \pmod 2 \\ &= Y\left(q^2\right) \\ &\equiv 1+\sum_{k\geq 1}\left(q^{6k^2-4k}+q^{6k^2+4k}\right) \pmod 2.\end{aligned}$$

The result follows. ∎

Corollary 3.7. *For all $n \geq 0$,*

$$\overline{p_o}(72n+15) \equiv 0 \pmod{48}.$$

Proof. This follows from Corollary 2.13 and Corollary 3.6. ∎

We close this article by proving the two infinite families of congruences stated in Theorem 1.3. These follow as corollaries of the following theorem:

Theorem 3.8. *For all $n \geq 0$,*

$$\begin{aligned}\overline{p_o}(27n+9) &\equiv 0 \pmod 6, \quad \textit{and} \\ \overline{p_o}(27n) &\equiv \overline{p_o}(3n) \pmod{12}.\end{aligned}$$

Proof. Using what has already been proven above regarding the generating function for $\overline{p_o}(3n)$, we have

$$\begin{aligned}\sum_{n\geq 0}\overline{p_o}(9n)q^n &= \frac{D\left(q^2\right)D\left(q^3\right)^2}{D(q)^7}\left(D\left(q^3\right)^4+16qD\left(q^3\right)Y(q)^3\right) \\ &= \frac{D\left(q^2\right)D\left(q^3\right)^3}{D(q)^7}\left(D\left(q^3\right)^3+16qY(q)^3\right) \\ &\equiv \frac{D\left(q^2\right)D\left(q^3\right)^3}{D(q)^7}\left(D\left(q^3\right)^3-8qY(q)^3\right) \pmod{24} \\ &= \frac{D\left(q^2\right)D\left(q^3\right)^3}{D(q)^7}\left(\frac{D(q)^4}{D\left(q^3\right)}\right) \\ &= \frac{D\left(q^2\right)D\left(q^3\right)^2}{D(q)^3} \\ &= D\left(q^2\right)D\left(q^3\right)^2\left(\frac{D(\omega q)D\left(\omega^2 q\right)}{D\left(q^3\right)^4/D\left(q^9\right)}\right)^3\end{aligned}$$

$$= \frac{D\left(q^9\right)^3}{D\left(q^3\right)^{10}} D\left(q^2\right)\left(D\left(q^9\right)^2+2qD\left(q^9\right)Y\left(q^3\right)+4q^2Y\left(q^3\right)^2\right)^3$$

$$= \frac{D\left(q^9\right)^3}{D\left(q^3\right)^{10}} D\left(q^2\right)\Big(D\left(q^9\right)^6+6qD\left(q^9\right)^5Y\left(q^3\right)+24q^2D\left(q^9\right)^4Y\left(q^3\right)^2$$

$$+56q^3D\left(q^9\right)^3Y\left(q^3\right)^3+96q^4D\left(q^9\right)^2Y\left(q^3\right)^4$$

$$+96q^5D\left(q^9\right)Y\left(q^3\right)^5+64q^6Y\left(q^3\right)^6\Big)$$

$$\equiv \frac{D\left(q^9\right)^3}{D\left(q^3\right)^{10}} D\left(q^2\right)\Big(D\left(q^9\right)^6-16q^3D\left(q^9\right)^3Y\left(q^3\right)^3$$

$$+64q^6Y\left(q^3\right)^6+6qD\left(q^9\right)^5Y\left(q^3\right)\Big) \pmod{24}$$

$$= \frac{D\left(q^9\right)^3}{D\left(q^3\right)^{10}} D\left(q^2\right)\left(\left(D\left(q^9\right)^3-8q^3Y\left(q^3\right)^3\right)^2+6qD\left(q^9\right)^5Y\left(q^3\right)\right)$$

$$= \frac{D\left(q^9\right)^3}{D\left(q^3\right)^{10}} D\left(q^2\right)\left(\left(\frac{D\left(q^3\right)^4}{D\left(q^9\right)}\right)^2+6qD\left(q^9\right)^5Y\left(q^3\right)\right)$$

$$= \frac{D\left(q^9\right)}{D\left(q^3\right)^{10}} D\left(q^2\right)\left(D\left(q^3\right)^8+6qD\left(q^9\right)^7Y\left(q^3\right)\right)$$

$$= \frac{D\left(q^9\right)}{D\left(q^3\right)^{10}} \left(D\left(q^{18}\right)-2q^2Y\left(q^6\right)\right)\left(D\left(q^3\right)^8+6qD\left(q^9\right)^7Y\left(q^3\right)\right).$$

It follows that, modulo 24,

$$\sum_{n\geq 0}\overline{p_o}(27n+9)q^n \equiv 6\frac{D\left(q^3\right)^8D\left(q^6\right)Y(q)}{D(q)^{10}} = 6\frac{\left(q^2\right)_\infty^9\left(q^3\right)_\infty^{15}}{(q)_\infty^{19}\left(q^6\right)_\infty^4\left(q^{12}\right)_\infty},$$

and, modulo 12,

$$\sum_{n\geq 0}\overline{p_o}(27n)q^n \equiv \frac{D\left(q^3\right)D\left(q^6\right)}{D(q)^2} = \sum_{n\geq 0}\overline{p_o}(3n)q^n.$$

This completes the proof. ∎

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