On the parity of $p(n)$

1. Introduction. About twenty years ago, the second author in [4] made the conjecture that for any given integer $m \geq 1$ and every r, $0 \leq r \leq m-1$, the partition function $p(mn+r)$ takes even values, as well as odd values, each for infinitely many n. In the case $m=1$, the result is due, independently, to O. Kolberg [2] and Morris Newman [3]. The case $m=2$ is settled in [4]. We have a proof of the conjecture for $m=4$, but are suppressing it because we prove here that the conjecture holds for $m=16$. Note that if the conjecture holds for a positive integer m, it also holds for all divisors of m.

2. The main result. We now prove the

THEOREM 2.1. *For each* r, $0 \leq r \leq 15$, $p(16n+r)$ *is infinitely often even, infinitely often odd.*

Proof. We have, modulo 2,

(2.1)

$$\begin{aligned} \sum p(n)x^n &= \frac{1}{\phi(x)} = \prod_{n\geq 1} \frac{1}{(1-x^n)} \\ &\equiv \prod_{n\geq 1} \frac{1-x^{2n}}{1-x^{2n-1}} \cdot \frac{1}{1-x^{4n}} \\ &= \sum_{n\geq 0} x^{\Delta(n)}/\phi(x^4), \quad \Delta(n) = n(n+1)/2. \\ &\equiv \sum_{n\geq 0} x^{\Delta(n)} \sum_{n\geq 0} x^{4\Delta(n)}/\phi(x^{16}). \end{aligned}$$

(We note in passing that continuing the iteration in (2.1) leads to a direct proof of the more important part of Theorem 1 of [1].)

It follows from (2.1) that

(2.2)

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

$$\phi(x^{16})\sum_{n\geq 0}p(n)x^n \equiv \sum_{n_1,n_2\geq 0} x^{\Delta(n_1)+4\Delta(n_2)} = \sum_{n\geq 0} c(n)x^n, \text{ say.}$$

Now, $\Delta(n) \equiv 0,\ 1,\ 3$ or $6 \pmod 9$, $4\Delta(n) \equiv 0,\ 3,\ 4$ or $6 \pmod 9$, so that $c(n) = 0$ if $n \equiv 2$ or $8 \pmod 9$.

Write $p_r(n) = p(16n+r)$. Define k_r to be the smallest k for which $p_r(k)$ is odd. A table of k_r is given below:

r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
k_r	0	0	1	0	0	0	0	0	1	2	5	2	0	0	0	3

Next let l_r be given by the following table:

r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
l_r	4	5	4	7	8	3	1	2	1	3	4	2	7	8	3	4

Suppose $p_r(n)$ is odd (alternatively even) for $n \geq n_0(r)$. We can suppose $n_0 \equiv l_r \pmod 9$ and that $2n_0 + 1 > k_r$.

Now let $N = N_r = (3n_0^2 + n_0)/2 + k_r$. Note that

$$16N + r \equiv 16((3l_r^2 + l_r)/2 + k_r) + r \equiv 2 \pmod 9, \quad \text{so} \quad c(16N + r) = 0.$$

It follows from (2.2) that, modulo 2,

(2.3)

$$p_r(N) + p_r(N-1) + p_r(N-2) + p_r(N-5) + p_r(N-7) + \cdots$$
$$\cdots + p_r(n_0 + k_r) + p_r(k_r) \equiv 0.$$

(The condition $2n_0 + 1 > k_r$ guarantees that $p_r(k_r)$ is indeed the last non-zero term on the left of (2.3))

But the left hand side of (2.3) is odd (there are an odd number $(2n_0 + 1)$ of terms, the last is odd, the others are all odd (alternatively even)). So we have a contradiction, and our theorem is proved.

References

[1] M. D. Hirschhorn, *On the residue* mod 2 *and* mod 4 *of* $p(n)$, Acta Arith. 38(1980), pp. 105-109.

[2] O. Kolberg, *Note on the parity of the partition function*, Math. Scand. 7(1959), pp. 377-378.

[3] Morris Newman, *Advanced Problem No. 4944*, Amer. Math. Monthly 69(1962), p. 175.

[4] M. V. Subbarao, *Some remarks on the partition function*, ibid. 73(1966), pp. 851-854.